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Monotone convergence theorem

In the mathematical field of real analysis, the **monotone convergence theorem** is any of a number of related theorems proving the convergence of monotonic sequences (sequences that are non-decreasing or non-increasing) that are also bounded. Informally, the theorems state that if a sequence is increasing and bounded above by a supremum, then the sequence will converge to the supremum; in the same way, if a sequence is decreasing and is bounded below by an infimum, it will converge to the infimum.

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Convergence of a monotone sequence of real numbers

Lemma 1

If a sequence of real numbers is increasing and bounded above, then its supremum is the limit.

Proof

Let $(a_n)_{n\in\mathbb{N}}$ be such a sequence, and let $\{a_n\}$ be the set of terms of $(a_n)_{n\in\mathbb{N}}$. By assumption, $\{a_n\}$ is non-empty and bounded above. By the least-upper-bound property of real numbers, $c = \sup_n\{a_n\}$ exists and is finite. Now, for every $\varepsilon > 0$, there exists N such that $a_N > c - \varepsilon$, since otherwise $c - \varepsilon$ is an upper bound of $\{a_n\}$, which contradicts the definition of c. Then since $(a_n)_{n\in\mathbb{N}}$ is increasing, and c is its upper bound, for every $n > N$, we have $|c - a_n| \leq |c - a_N| < \varepsilon$. Hence, by definition, the limit of $(a_n)_{n\in\mathbb{N}}$ is $\sup_n\{a_n\}$.

Lemma 2

If a sequence of real numbers is decreasing and bounded below, then its infimum is the limit.

Proof

The proof is similar to the proof for the case when the sequence is increasing and bounded above,

Theorem

If $(a_n)_{n\in\mathbb{N}}$ is a monotone sequence of real numbers (i.e., if $a_n \leq a_{n+1}$ for every $n \geq 1$ or $a_n \geq a_{n+1}$ for every $n \geq 1$), then this sequence has a finite limit if and only if the sequence is bounded.[1]

Proof

- "If"-direction: The proof follows directly from the lemmas.
- "Only If"-direction: By (ε, δ)-definition of limit, every sequence $(a_n)_{n\in\mathbb{N}}$ with a finite limit L is necessarily bounded.

Convergence of a monotone series

Theorem

If for all natural numbers j and k, $a_{j,k}$ is a non-negative real number and $a_{j,k} \leq a_{j+1,k}$, then[2]:168

$$\lim_{j\to\infty}\sum_k a_{j,k} = \sum_k \lim_{j\to\infty} a_{j,k}.$$

The theorem states that if you have an infinite matrix of non-negative real numbers such that

1. the columns are weakly increasing and bounded, and
2. for each row, the series whose terms are given by this row has a convergent sum,

then the limit of the sums of the rows is equal to the sum of the series whose term k is given by the limit of column k (which is also its supremum). The series has a convergent sum if and only if the (weakly increasing) sequence of row sums is bounded and therefore convergent.

As an example, consider the infinite series of rows