

Harsányi's Utilitarian Theorem

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1 Introduction

Harsányi's Utilitarian Theorem relates the decision-making of a collective of rational agents to the decision-makings of its constituent individuals. Harsányi originally framed the theorem in the context of welfare economics and social choice theory, in which case the theorem says that a rational group of agents' decision-making is given by maximizing an affine function of the welfares of the individuals. Two primary drivers of my interest in the theorem are links to agent foundations (hierarchical agency) and to ethics (moral uncertainty). This note is a self-contained¹ presentation of the theorem, including a succinct proof.

Before we get to this proof of Harsányi's Utilitarian Theorem, we first spend some time introducing terminology (Section 2) and proving the [Von Neumann]-Morgenstern Utility Theorem (Section 3). Readers familiar with the vNM theorem are encouraged to skip ahead to the discussion of Harsányi's Utilitarian Theorem in Section 4. Some philosophical context complements the math along the way.

2 Setup

Here, we set out the language of the vNM and Harsányi theorems. Our treatment applies to the case where there is a finite set of outcomes.² Let this set be $\mathcal{A} = \{A_1, A_2, \dots, A_n\}$. Let us, say, think of each A_i as a complete deterministic world history, i.e. a 4D block of spacetime. Our agents have preferences between probability distributions on \mathcal{A} , i.e. preferences are defined for pairs of elements in $\Delta(\mathcal{A})$.³ It will be convenient to represent the data of a probability distribution μ in vector form as

$$\boldsymbol{\mu} := (\mu(\{A_1\}), \mu(\{A_2\}), \dots, \mu(\{A_n\})).$$

¹except for assuming some probability theory and linear algebra

²I believe an analogous story is true in more generality, but I have not investigated the more general variants in detail. For instance, I refer the interested reader to [3] for a treatment of the case with infinitely many outcomes.

³ $\Delta(\mathcal{A})$ is the set of probability measures on the discrete σ -algebra $2^{\mathcal{A}}$, i.e. the set of all functions $\mu: 2^{\mathcal{A}} \rightarrow \mathbb{R}$ which satisfy $\mu(\emptyset) = 0$ and $\mu(2^{\mathcal{A}}) = 1$ and are additive under disjoint union.

⁴In this context, probability distributions are also often called *lotteries* or *prospects*.

Convex combinations of probability distributions are also probability distributions. For example, given $\mu, \nu \in \Delta(\mathcal{A})$, for any $p \in [0, 1]$, we have that $\lambda = p\mu + (1 - p)\nu$ is also a probability distribution. One can sample from λ by first flipping a biased coin with probability p of landing on heads. If it lands on heads, then sample from μ ; if it lands on tails, then sample from ν .

Definition 2.1. A preference specification \preceq over pairs of distributions is a subset of $\Delta(\mathcal{A}) \times \Delta(\mathcal{A})$, where we write $\mu \preceq \nu$ to mean that (μ, ν) is in the subset, and to intuitively mean that μ is no better than ν . We will take $\mu \succeq \nu$ to mean $\nu \preceq \mu$, we will take $\mu \sim \nu$ to mean that both $\mu \preceq \nu$ and $\nu \preceq \mu$, and we will take $\mu \prec \nu$ to mean that $\mu \preceq \nu$ but $\nu \not\preceq \mu$, and analogously define $\mu \succ \nu$.

Definition 2.2. We say a preference specification is *vNM-rational* if it satisfies the following axioms:

Completeness: For any $\mu, \nu \in \Delta(\mathcal{A})$, we have that $\mu \preceq \nu$ or $\nu \preceq \mu$.

Transitivity: For any $\lambda, \mu, \nu \in \Delta(\mathcal{A})$, if $\lambda \preceq \mu$ and $\mu \preceq \nu$, then $\lambda \preceq \nu$.

Continuity: For any $\lambda, \mu, \nu \in \Delta(\mathcal{A})$, if $\lambda \preceq \mu \preceq \nu$, then there is a $p \in [0, 1]$ such that $\mu \sim p\lambda + (1 - p)\nu$.

Independence: For any $\lambda, \mu, \nu \in \Delta(\mathcal{A})$ and $p \in (0, 1]$, we have $\mu \preceq \nu$ if and only if $p\mu + (1 - p)\lambda \preceq p\nu + (1 - p)\lambda$.

Here are some immediate corollaries:

Transitivity': If $\lambda \sim \mu$ and $\mu \sim \nu$, then $\lambda \sim \nu$.

Lifting: If $\mu \sim \mu'$ and $\nu \sim \nu'$, then $\mu \preceq \nu \iff \mu' \preceq \nu'$.

Independence': For $p \in [0, 1]$, if $\mu \sim \nu$, then $p\mu + (1 - p)\lambda \sim p\nu + (1 - p)\lambda$.

Interpolation: If $\mu \prec \nu$, then $p \leq q \iff (1 - p)\mu + p\nu \preceq (1 - q)\mu + q\nu$.

3 The vNM theorem

The vNM theorem tells us that a rational agent chooses as if it were maximizing the expectation of some quantity:

Theorem 1 ([Von Neumann]-Morgenstern Utility Theorem, 1947 [4]). *If an agent's preferences \preceq are vNM-rational, then there is a function $u: \mathcal{A} \rightarrow \mathbb{R}$ which serves as its utility function,⁵ by which we mean that*

$$\mu \preceq \nu \iff \mathbb{E}_\mu[u(A)] \leq \mathbb{E}_\nu[u(A)].$$

⁵Note that this is a purely behavioral condition – in particular, the theorem does not tell us that the agent has to internally be structured like a utility maximizer, or even that it has a concept of utility.

Proof. (From [5].) To construct such a function $u: \mathcal{A} \rightarrow \mathbb{R}$, we will first bring all distributions to a common scale. Let α_i be the probability distribution that assigns probability 1 to A_i , and assume WLOG that $\alpha_1 \preceq \dots \preceq \alpha_n$. For all α_i , we have $\alpha_1 \preceq \alpha_i \preceq \alpha_n$, so by the Continuity Axiom, there is a $p_i \in [0, 1]$ such that $\alpha_i \sim (1 - p_i)\alpha_1 + p_i\alpha_n$.

Consider a distribution μ ; we can write it as $\mu = \sum_{i=1}^n q_i \alpha_i$. Since $\alpha_i \cong (1 - p_i)\alpha_1 + p_i\alpha_n$, Independence' and Transitivity' give that

$$\mu \sim \sum_{i=1}^n q_i ((1 - p_i)\alpha_1 + p_i\alpha_n) = \left(\sum_{i=1}^n q_i (1 - p_i) \right) \alpha_1 + \left(\sum_{i=1}^n q_i p_i \right) \alpha_n =: \mu'.$$

We are now ready to construct the desired $u: \mathcal{A} \rightarrow \mathbb{R}$. If $\alpha_1 \sim \alpha_n$, then by Independence', $\mu' \sim \alpha_1$, in which case it follows from Transitivity' that $\mu \sim \nu$ for any μ, ν , and thus the function $u = 0$ satisfies the property in the theorem statement. The remaining case is $\alpha_1 \prec \alpha_n$. In this case, define $u(A_i) = p_i$, and note that

$$\mu' = (1 - \mathbb{E}_\mu[u(A)]) \alpha_1 + \mathbb{E}_\mu[u(A)] \alpha_n.$$

Analogously, for another distribution ν , we have

$$\nu \sim (1 - \mathbb{E}_\nu[u(A)]) \alpha_1 + \mathbb{E}_\nu[u(A)] \alpha_n = \nu'.$$

By Interpolation, $\mathbb{E}_\mu[u(A)] \leq \mathbb{E}_\nu[u(A)] \iff \mu' \preceq \nu'$. And by Lifting, $\mu' \preceq \nu' \iff \mu \preceq \nu$, completing the proof. \square

4 Harsányi's Utilitarian Theorem

Harsányi's Utilitarian Theorem connects the rational decision-making of a collective of rational agents to the utilities of the individual agents. Before stating the theorem, here is a list of cases the theorem can be applied to — that is, a list of cases where, plausibly, a collective of rational agents is making decisions on behalf of the individual rational agents:⁶

- different intelligent (alien) species agreeing to pursue common goals;
- the EU making decisions on behalf of its member states;
- a country making decisions on behalf of its citizens;
- a family making decisions on behalf of its members;

⁶Beware though that there are quite a few items on this list for which it is quite nontrivial how or whether the theorem applies. For example, in the moral uncertainty case, it is unclear if deontology corresponds to a set of rational preferences over ways the universe could be in some interesting sense. The same concern also applies to particular virtues in the virtue ethics example. Additionally, I think there are some wholes which might be "more than sums of their subagents". That is, I think it might make sense for some collectives to hold some preferences which are not held on behalf of their subagents. For example, it seems likely that this would be the case for decompositions of the human mind into subagents.

- a person making decisions on behalf of their (future) timeselves⁷;
- a person making decisions combining verdicts of the different moral theories they are uncertain over;
- a person subscribing to virtue ethics making decisions that embody a set of virtues;
- an ethical theory grounding moral truth in concern for all rational agents, or for all sentient beings;
- an AI agent acting in the interest of a collection of humans.

Theorem 2 (Harsányi’s Utilitarian Theorem, 1956 [2]). *Let $I = \{1, 2, \dots, m\}$ index a set of agents, each with preferences; let \succeq_i denote the preferences of $i \in I$. Suppose the set of agents taken as a unit also has preferences denoted \succeq_I . Suppose that the following three properties all hold:*

Individual Rationality: For each $i \in I$, \succeq_i is vNM-rational.

Collective Rationality: Additionally, \succeq_I is vNM-rational.

Pareto Optimality: ⁸⁹ *If $\mu \sim_i \nu$ for every $i \in I$, then $\mu \sim_I \nu$.*

Then by the vNM theorem, there is a utility function $u_i: \mathcal{A} \rightarrow \mathbb{R}$ describing the preferences of each $i \in I$, as well as a utility function u_I describing the

⁷<https://www.lesswrong.com/posts/DdmScdukXBff5CbNS/a-gentle-primer-on-caring-including-in-strange-senses-with-slicing-people-up-across-time>

⁸I will now explain why I call this condition Pareto Optimality. Note that it is implied by the property that if $\mu \succeq_i \nu$ for every $i \in I$, then $\mu \succeq_I \nu$. Given some additional assumptions (for instance that there is a pair of lotteries λ, κ such that $\lambda \succeq_i \kappa$ for all $i \in I$ with at least one strict inequality), this condition follows from the condition that if everyone weakly prefers μ to ν with at least one $i \in I$ strictly preferring μ to ν , then $\mu \succ_I \nu$. The reason to call this last condition Pareto Optimality is that if it were not the case that $\mu \succ_I \nu$, then the “collective policy” \succ_I would not necessarily pick μ over ν when given the choice, and every $i \in I$ would weakly prefer, with one $i \in I$ strictly preferring, a collective policy which is the same except for definitely picking μ over ν . In other words, the collective policy would not be Pareto optimal. And conversely, if there is some other collective policy which every $i \in I$ would non-strictly prefer and some $i \in I$ would strictly prefer to \succeq_I regardless of the choice between two lotteries faced, then there must be some μ, ν such that $\mu \succeq_i \nu$ for all $i \in I$, $\mu \succ_i \nu$ for some $i \in I$, and $\mu \preceq_I \nu$. (However, if one assumes some probability distribution on the set of choices between lotteries faced, then one can write down a more stringent Pareto condition. One can also think of this distinction as the given condition being Pareto Optimality for agents with Knightian uncertainty over the decision problem that is to be faced.) The initial condition that $[\forall i, \mu \sim_i \nu] \implies \mu \sim_I \nu$ is not equivalent to this last Pareto condition, for instance because it is also satisfied when $u_I = -\sum_{i \in I} u_i$ which definitely does not in general satisfy the Pareto condition, but since it is (given some reasonable additional assumptions) implied by a genuine Pareto condition and conceptually not that far from it, I think it still makes sense to call it Pareto Optimality.

⁹I think there are reasons to only accept this axiom when individual preferences are based in personal welfare only. See the counterexample provided in footnote 7 here: <https://www.lesswrong.com/posts/DdmScdukXBff5CbNS>.

preferences of the group. But what's more, there are coefficients c_0, c_1, \dots, c_k such that $u_I = c_0 + \sum_{i \in I} c_i u_i$.¹⁰

Proof. (Inspired by [1].) For each $i \in I$, we think of u_i as a vector, $\mathbf{u}_i = (u_i(A_1), u_i(A_2), \dots, u_i(A_n))$, we let $\mathbf{u}_0 = (1, 1, \dots, 1) \in \mathbb{R}^n$, and consider the subspace $U = \text{span}\{\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_m\}$. Define $\mathbf{u}_I = (u_I(A_1), u_I(A_2), \dots, u_I(A_n))$, and consider $\mathbf{w} = \text{proj}_{U^\perp} \mathbf{u}_I$. In the rest of the argument, we will show that \mathbf{w} is a “direction in the space of measures” which I “would care about” but which no $i \in I$ “would care about”, implying that $\mathbf{w} = 0$ so as to not contradict Pareto Optimality.

Let μ be the uniform distribution on \mathcal{A} , i.e. $\mu = (\frac{1}{n}, \dots, \frac{1}{n})$, and consider the vector $\nu = \mu + \varepsilon \mathbf{w}$ where $\varepsilon > 0$ is chosen to be small enough to keep all entries positive. Since $\mathbf{w} \perp \mathbf{u}_0$, we have that ν also corresponds to a probability distribution ν . For all $i \in I$, since $\mathbf{u}_i \perp \mathbf{w}$, we have $\mu \sim_i \nu$, and therefore using property 3,

$$\mathbf{u}_I \cdot \mu = \mathbb{E}_\mu[u_I] = \mathbb{E}_\nu[u_I] = \mathbf{u}_I \cdot \nu = \mathbf{u}_I \cdot \mu + \varepsilon \mathbf{u}_I \cdot \mathbf{w},$$

whence $0 = \mathbf{u}_I \cdot \mathbf{w} = \|\text{proj}_{U^\perp} \mathbf{u}_I\|^2$, implying $\text{proj}_{U^\perp} \mathbf{u}_I = 0$, so $\mathbf{u}_I \in U$. Therefore, $\mathbf{u}_I = c_0 \mathbf{u}_0 + \sum_{i \in I} c_i \mathbf{u}_i$, and thus $u_I = c_0 + \sum_{i \in I} c_i u_i$. \square

Summary

In this paper, we discussed two canonical utility theorems of some relevance to ethics and to agent foundations. Firstly, we specified what it means for preferences to be rational. Secondly, we stated and proved the [Von Neumann]-Morgenstern Utility Theorem. Thirdly, we briefly discussed the possible philosophical significance of the Harsányi Utilitarian Theorem. Finally, we provided a semi-novel proof of the theorem.

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¹⁰A fun aside: note that the special case with $I = \{1\}$ says that any two utility functions u, v representing preferences which are indifferent between the same pairs of lotteries are affine functions of each other, i.e. $v = a + bu$. In particular, any two utility functions representing the same preferences are thus affine functions of each other; in fact, it is easy to see that we can always choose $b > 0$ in this case, and that given $b > 0$, we indeed have that $a + bu$ represents the same preferences. In other words, the utility function for a given set of preferences is unique up to positive affine transformation.

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