**CIS 621 Algorithms and Complexity**
**Spring 2018**

# Assignment 4, Problem 3 Sample Solution

3. (exercise 6.23, p 196, from DPV) A mission-critical production system has n stages that have to be performed sequentially; stage $i$ is performed by machine $M_i$. Each machine $M_i$ has a probability $r_i$ of functioning reliably and a probability $1 - r_i$ of failing (and the failures are independent). Therefore, if we implement each stage with a single machine, the probability that the whole system works is $r_1 \cdot r_2 \cdots r_n$. To improve this probability we add redundancy by having $m_i$ copies of the machine $M_i$ that performs stage $i$. The probability that all $m_i$ copies fail simultaneously is only $(1 - r_i)^{m_i}$, so the probability that stage $i$ is completed correctly is $1 - (1 - r_i)^{m_i}$ and the probability that the whole system works is $\Pi_{i=1}^{n}[1 - (1 - r_i)^{m_i}]$. Each machine $M_i$ has a cost $c_i$, and there is a total budget $B$ to buy machines. (Assume that $B$ and the $c_i$ are positive integers.)

   Given the probabilities $r_1, r_2, \ldots, r_n$, the costs $c_1, c_2, \ldots, c_n$, and the budget $B$, find the maximum reliability that can be achieved within budget $B$.

   **solution:**
   This is a bit like the coin problem (and the 0-1 knapsack) without repetition. The subproblem will be denoted $REL[i, b]$, the most reliable configuration of machines $1, 2, \ldots, i$ (at least one of each machine) available within budget $b$. The desired answer will be $REL[n, B]$.

   For the recurrence, if we are under budget, we return reliability 0 (which basically says "not possible"). Similarly, if we are out of budget ($b = 0$), but still need to buy machines ($i > 0$), we return 0 (assume all $c_i > 0$). If $i = 0$, we have have no machines that we have to buy, so reliability is 1 (if it were 0, then everything would get multiplied by 0, which is no good).

   If there is budget left ($b > 0$) and machines left to buy ($i > 0$), we try all possibilities $m$ of machines of type $i$ to buy - we have to buy at least $m \geq 1$, and up to $m \leq \lfloor \frac{b}{c_i} \rfloor \leq b \leq B$, of them. In each case, the remaining budget will be $b - m \cdot c_i$. The best reliability for machines $1, \ldots, i - 1$ will be $REL[i - 1, b - m \cdot c_i]$, which needs to be multiplied by the contribution of the $m$ copies of $M_i$, $(1 - (1 - r_i)^m)$. Summarizing,

$$REL[i, b] = \begin{cases} 0 & \text{if } b < 0 \\ 0 & \text{if } b = 0, i > 0 \\ 1 & \text{if } b \geq 0, i = 0 \\ \max_{1 \leq m \leq \lfloor \frac{b}{c_i} \rfloor} [REL[i - 1, b - m \cdot c_i] \cdot (1 - (1 - r_i)^m)] & \text{otherwise.} \end{cases}$$