n hypothesis $H_{0,1} \dots H_{0,n}$
we test the global null: $H_0 = \bigcap_{i=1}^{n} H_{0,i}$

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single hypothesis

2 random samples:
$X_1 \dots X_k$ $\quad \mu_x, \sigma_x^2$; $\quad \hat{\mu}_x = \bar{X}$; $\hat{\sigma}_x^2$
$y_1 \dots y_l$ $\quad \mu_y, \sigma_y^2$; $\quad \hat{\mu}_y = \bar{y}$; $\hat{\sigma}_y^2$

$H_0: \mu_x = \mu_y$ vs two standard ways to formulate H_A:

(1) $H_A: \mu_x \neq \mu_y$
— two-sided test

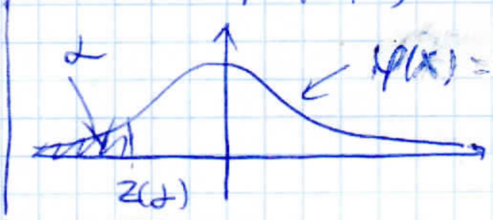
(2) $H_A: \mu_x > \mu_y$
- one sided test.

t-statistic (test function): $\quad T = \dfrac{\bar{X} - \bar{y}}{\sqrt{\frac{1}{k}\hat{\sigma}_x^2 + \frac{1}{l}\hat{\sigma}_y^2}} \xrightarrow[k,l \to \infty]{D} N(0,1)$

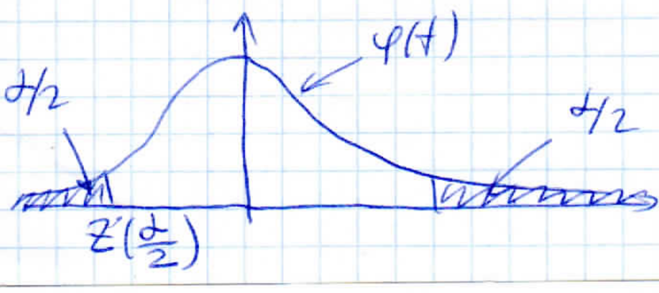
(1) two-sided test:
$H_0: \mu_x = \mu_y$ vs $H_A: \mu_x \neq \mu_y$.
we reject H_0 on significance level α
($\alpha = P(\text{Type I Er}) = P_{H_0}(\text{reject } H_0)$) when

$z(\alpha)$ - quantile of level α of standard normal distribution

$Z \sim N(0,1)$: $\quad z(\alpha)$: $P(Z < z(\alpha)) = \alpha$



$z(\alpha) = \Phi^{-1}(\alpha)$
inverse to cdf
$\Phi(x) = \int_{-\infty}^{x} \varphi(t)dt$



we reject H_0 when:
$|T| > |z(\frac{\alpha}{2})|$
$\iff$ p-value : $p < \alpha$
$p = 2(1 - \Phi(|T|))$

(2) One-sided test:

$H_0: \mu_x = \mu_y$ vs $H_A: \mu_x > \mu_y$

We reject H_0 when $T > |z(\alpha)| \iff$

$\iff$ p-value $p < \alpha$

$p = 1 - \Phi(T)$

Note: When H_0 is true then $T \underset{l,k \text{ large}}{\sim} N(0,1)$

when H_0 is not true then

$$T \underset{k,l \text{ large}}{\sim} N\Big(\underbrace{\frac{\mu_x - \mu_y}{\sqrt{\frac{\sigma_x^2}{k} + \frac{\sigma_y^2}{l}}}}_{\mu}, 1\Big)$$

$\Rightarrow$ we can reduce the problem:

We have $y_i \sim N(\mu_i, 1)$, $i = 1, \dots, n$ independent

We are interested in n hypotheses:

$H_{0,i}: \mu_i = 0$

$H_0 = \bigcap_{i=1}^{n} H_{0,i} \iff$ all $\mu_i = 0$, $i = 1, \dots, n$.

(H_A: some means $\mu_i \neq 0$).

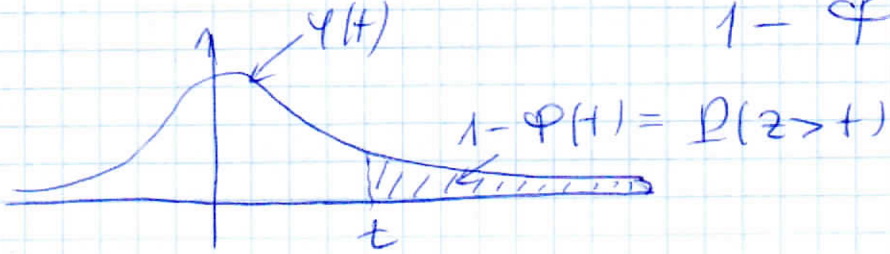
Bonferroni's method rejects H_0 if

$\min_{1 \le i \le n} p_i \le \frac{\alpha}{n} \iff \begin{cases} \max_{1 \le i \le n} |y_i| \ge |z(\frac{\alpha}{2n})| & \text{in the two-sided test} \\ \max_{1 \le i \le n} y_i \ge |z(\frac{\alpha}{n})| & \text{in the one-sided test} \end{cases}$

p-value for $H_{0,i}$

<u>Theorem</u> $\quad Z \sim N(0,1) \qquad \varphi(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$

$$\underset{②}{\frac{\varphi(t)}{t}\left(1-\frac{1}{t^2}\right) \leq} \; \underset{\underset{1-\Phi(t)}{\parallel}}{P(Z>t)} \; \underset{①}{\leq} \frac{\varphi(t)}{t}$$



Γ ①: $P(z>t) \leq \frac{\varphi(t)}{t}$

$$P(z>t) = \int_t^{+\infty} \varphi(u)\,du \leq \int_t^{+\infty} \frac{u}{t}\varphi(u)\,du =$$

$\frac{u}{t} \geq 1$, when $u \geq t$

$$= \Big/ (\varphi(t))' = -t\varphi(t) \Big/ = \frac{1}{t}\int_t^{+\infty} (-\varphi(u))'\,du =$$

$$= \frac{1}{t}\,\varphi(u)\Big|_{+\infty}^{t} = \frac{\varphi(t)}{t}$$

② $P(z>t) \geq \frac{\varphi(t)}{t}\left(1-\frac{1}{t^2}\right)$

$$L(t) = P(z>t) - \frac{\varphi(t)}{t}\left(1-\frac{1}{t^2}\right) = 1-\Phi(t) - \frac{\varphi(t)}{t} + \frac{\varphi(t)}{t^3}$$

(i) $L(0) = \frac{1}{2} - \lim\limits_{t\to 0+} \underset{\sim}{\varphi(t)}\, \frac{\overset{\sim -1}{t^2-1}}{\underset{\to 0+}{t^3}} = +\infty$

(ii) $L(+\infty) = 0 - \lim\limits_{t\to+\infty} \underset{\searrow 0}{\varphi(t)} \underbrace{\frac{t^2-1}{t^3}}_{\sim \frac{1}{t} \to 0} = 0.$

(iii) $L'(t) = -\varphi(t) - \frac{-t^2\varphi(t) - \varphi(t)}{t^2} + \frac{-t^4\varphi(t) - 3t^2\varphi(t)}{t^6} =$

$$= -\varphi(t) + \varphi(t) + \frac{\varphi(t)}{t^2} - \frac{\varphi(t)}{t^2} - \frac{3\varphi(t)}{t^4} = -\frac{3\varphi(t)}{t^4} < 0$$

$\Longrightarrow L(t) \downarrow\downarrow$

(i), (ii), (iii) $\Longrightarrow$ ② ⌋

Note: $\frac{1}{t^2} \to 0,\ t \to \infty \Rightarrow P(z > t) =$

$$= 1 - \Phi(t) \underset{t\to\infty}{\sim} \frac{\varphi(t)}{t}$$

For large t $\Longleftrightarrow$ $\begin{array}{l}\alpha - \text{fixed} \\ n \to \infty\end{array}$

$$P(z > t) = \frac{\alpha}{n} \Longleftrightarrow \frac{\varphi(t)}{t} \approx \frac{\alpha}{n}$$

$$\frac{\varphi(t)}{t} \approx \frac{\alpha}{n} \approx \frac{1}{t} e^{-t^2/2} \cdot \frac{1}{\sqrt{2\pi}} \approx \frac{\alpha}{n} \Longleftrightarrow$$

$$\Longleftrightarrow e^{-\frac{t^2}{2} - \log t} \approx \frac{\alpha}{n} \cdot \sqrt{2\pi}$$

$$\Longleftrightarrow -\frac{t^2}{2} - \log t \approx -\log n + const$$

$$\Longleftrightarrow \underbrace{-\frac{t^2}{2} - \log t}_{\sim -\frac{t^2}{2}} \approx \underbrace{-\log n + const}_{\sim -\log n}$$

$$\frac{t^2}{2} \approx \log n$$

$$\Longleftrightarrow t \approx \sqrt{2 \log n}$$

So $\frac{\varphi(t)}{t} \approx \frac{\alpha}{n} \Longleftrightarrow t = |z(\frac{\alpha}{n})| \approx \sqrt{2\log n}$

$\Rightarrow$ quantile $|z(\frac{\alpha}{n})|$ grows as $\sqrt{2\log n}$ with a small correction factor

$\Longleftrightarrow$ Bonferroni reject when $\underline{\underline{\max|y_i| > \sqrt{2\log n}}}$

No dependence on α! $\Longleftrightarrow$ our rejection threshold for $\max|y_i| \underset{\text{asymptotically}}{=} \sqrt{2\log n}$

This is consequence of the fact

$$\frac{\max|y_i|}{\sqrt{2\log n}} \xrightarrow{P} 1$$

(known result from the Theory of Probability)

For finite samples it is possible to develop approximations to $z(\frac{\alpha}{n})$ which are more accurate than $\sqrt{2\log n}$

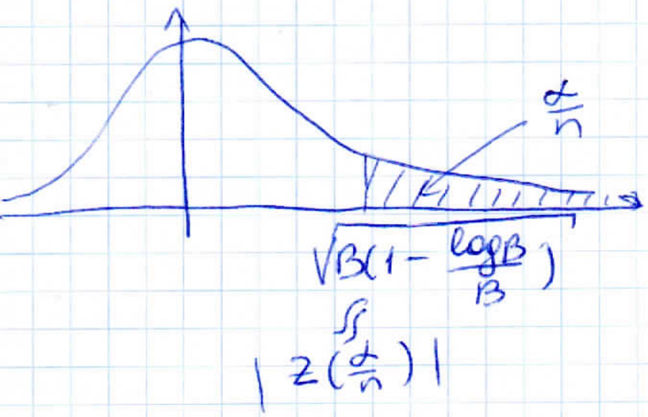
$$B := 2\log\frac{n}{\alpha} - \log(2\pi) \qquad \begin{matrix} B\to\infty \\ n\to\infty \end{matrix}$$

$$|z(\tfrac{\alpha}{n})| \approx \sqrt{B\cdot\left(1-\frac{\log B}{B}\right)} = \sqrt{B-\log B}$$

$$\ulcorner\ P(\underset{\sim N(0,1)}{Z} > |z(\tfrac{\alpha}{n})|) \approx P(Z > \sqrt{B-\log B}) = \left|\frac{\varphi(t)}{t}\right| =$$

$$= \frac{1}{\sqrt{B-\log B}}\,\frac{e^{-\frac{B-\log B}{2}}}{\sqrt{2\pi}} = \frac{e^{-B/2}}{\sqrt{2\pi}}\underbrace{\frac{\sqrt{B}}{\sqrt{B-\log B}}}_{\sim 1} =$$

$$= \cancel{\frac{1}{\sqrt{2\pi}}}\, e^{-\log\frac{n}{\alpha}+\frac{1}{2}\log(2\pi)} = \frac{\alpha}{n}\ \lrcorner$$



Note:

$$|z(\tfrac{\alpha}{n})| \approx \sqrt{B-\log B} = \sqrt{2\log\frac{n}{\alpha} - \log 2\pi - \log(2\log n - \log 2\pi)}$$

$$= \sqrt{2\log n - \log(2\pi\alpha^2) - \log(2\log n - \log(2\pi\alpha^2))} =$$

$$= \sqrt{2\log n - \log\log n + const} \approx \sqrt{2\log n - \log\log n}$$

$$= \sqrt{2\log n}\,\sqrt{1-\frac{\log\log n}{2\log n}} \approx \left|\ \sqrt{1-x} = 1-\frac{1}{2}x+\bar{o}(x^2),\ x\to 0\ \right|$$

$$\approx \sqrt{2\log n}\left(1-\frac{\log\log n}{4\log n}\right)$$

When you use Bonferroni correction, the threshold grows with the number of tests
⇑ critical value $\Rightarrow$ you may loose the power of the test