χ^2 Test.

Statistical model:

$$y_i = \mu_i + z_i, \quad z_i \sim N(0,1), \quad 1 \le i \le n$$

$$\Longleftrightarrow \quad y \sim N(\mu, I), \quad \mu = (\mu_1 \dots \mu_n)$$

H_0: $\mu = 0$ vs H_1: at least one $\mu_i \neq 0$

Test statistic for χ^2 test:

$$T = \sum_{i=1}^{n} y_i = \|y\|^2$$

Under H_0: $T = \sum_{i=1}^{n} z_i^2 \quad \Longrightarrow$

iid $\sim N(0,1)$

$$\Rightarrow T \sim \chi^2_n \Rightarrow$$

$\Rightarrow$ we reject H_0 on significance level α:

$$T > \chi^2_n(1-\alpha)$$

Note: 1) $E z_i^2 = 1$ $\quad$ $\operatorname{var} z_i^2 = 2 \Rightarrow$

$\Rightarrow ET = n$; $\operatorname{var} T = 2n \Rightarrow$ CLT

$$\frac{T-n}{\sqrt{2n}} \xrightarrow{\mathcal{D}} N(0,1)$$

2) $P_0(T > \chi^2_n(1-\alpha)) = 1-\alpha \Rightarrow$ for large n

$$\Rightarrow P_0\Big(\underbrace{\frac{T-n}{\sqrt{2n}}}_{\sim N(0,1)} > \underbrace{\frac{\chi^2_n(1-\alpha) - n}{\sqrt{2n}}}_{\approx z(1-\alpha)}\Big) = 1-\alpha$$

$$\Rightarrow \chi^2_n(1-\alpha) \approx n + \sqrt{2n}\, z(1-\alpha)$$

Under H_1: $T \sim \chi^2(n, \tau)$ - non-central χ^2

non-centrality parameter

$$E(z_i+\mu_i)^2 = E z_i^2 + 2\mu_i E z_i + \mu_i^2 = \mu_i^2 + 1$$

$$\operatorname{var}(z_i+\mu_i)^2 = \overset{?}{E}(z_i+\mu_i)^4 - (\mu_i^2+1)^2 =$$

$$= E z_i^4 + 4\mu_i \underbrace{E z_i^3}_{0} + 6\mu_i^2 E z_i^2 + 4\mu_i^3 \underbrace{E z_i}_{0} + \mu_i^4 -$$

$$-(\mu_i^4 + 2\mu_i^2 + 1) = \left/ E z_i^k = \begin{cases} (k-1)!!, & k - \text{even} \\ 0, & k - \text{odd} \end{cases} \right/ =$$

$$= 3 + 6\mu_i^2 - 2\mu_i^2 - 1 = 4\mu_i^2 + 2$$

$$\Rightarrow ET = \|\mu\|^2 + n; \qquad \operatorname{var} T = 4\|\mu\|^2 + 2n$$

$$\overset{CLT}{\Longrightarrow} \quad \frac{T - (n + \|\mu\|^2)}{\sqrt{2n + 4\|\mu\|^2}} \sim N(0,1)$$

$$Z = \frac{T-n}{\sqrt{2n}} \text{ - test Statistic}$$

$$\theta = \frac{\|\mu\|^2}{\sqrt{2n}} \propto SNR = \frac{\text{signal}}{\text{noise}}$$ - a measure that shows how strong is a signal with respect to noise

$$H_0: Z \sim N(0,1) \quad \text{vs} \quad H_1: Z \sim N\left(\theta, 1 + \frac{\theta}{\sqrt{n/8}}\right)$$

$$\Gamma \quad \frac{T - (n + \|\mu\|)^2}{\sqrt{2n + 4\|\mu\|^2}} = \frac{\frac{T-n}{\sqrt{2n}} - \frac{\|\mu\|^2}{\sqrt{2n}}}{\sqrt{1 + \frac{2\|\mu\|^2}{n}}} = \frac{Z - \theta}{\sqrt{1 + \frac{\theta}{\sqrt{n/8}}}} \sim N(0,1) \quad \lrcorner$$

$\Rightarrow$ test statistic Z can distinguish hypotheses when θ is large enough.

$\Leftrightarrow$ the power of the χ^2 test is not determined by the length of the largest needle but the $\|\mu\|^2$ compared to $\sqrt{n}$

<u>Definition of SNR</u>

$$y_i = \mu_i + \sigma z_i \quad ; \quad i = 1, \dots, n \qquad z_i \text{ iid} \sim N(0,1)$$

the detection power depends on

$$\theta = \sqrt{\frac{n}{2}} \; \frac{\|\mu\|^2}{\sigma^2 n}$$

$$SNR \stackrel{df}{=} \frac{\|\mu\|^2}{\sigma^2 n} = \frac{\text{total signal power}}{\text{total expected noise power}}$$

$$\Rightarrow \quad \theta \propto SNR \qquad \Longleftrightarrow \quad \theta = \sqrt{\tfrac{n}{2}}\, SNR$$

Comparison between Bonferroni's and χ^2 tests

- when the signal is more or less uniformly distributed there is no test better than the χ^2 test.
- when we have "needle in the haystack" problem there is no test better than Bonferroni test

Example 1.

$$\theta = \frac{\|\mu\|^2}{\sqrt{2n}}, \quad \text{when } \theta \to \infty \ \chi^2 \text{ test can identify } H_1.$$

Let $\mu_1 = \dots = \mu_n = \gamma_n \Rightarrow \|\mu\|^2 = n\gamma_n^2$

$$\Rightarrow \theta = \frac{n\gamma_n^2}{\sqrt{2n}} = \frac{1}{\sqrt{2}}\sqrt{n}\,\gamma_n^2$$

If $\gamma_n \to 0$ s.t. $\sqrt{n}\gamma_n^2 \to \infty$, $n \to \infty$

(for example, $\gamma_n = n^{-(\frac{1}{4}+\varepsilon)}$, $\varepsilon > 0$)

If $\gamma_n = const$

then H_1 can be detected by χ^2 test

But Bonferroni can identify $\mu^{(n)} > (1+\varepsilon)\sqrt{2\log n}$

Example 2

$\mu_1 = (1+\varepsilon)\sqrt{2\log n}$; $\mu_2 = \dots = \mu_n = 0$. (Bonf can identify H_1)

$\|\mu\|^2 = (1+\varepsilon)^2 \cdot 2\log n \Rightarrow$

$$\theta = \frac{\|\mu\|^2}{\sqrt{2n}} = 2(1+\varepsilon)^2 \frac{\log n}{\sqrt{2n}} \to 0, \ n \to \infty$$

$\Rightarrow \chi^2$ test can not identify Bonferroni needle.

Is there a test that does better than χ^2-test when $\theta \ll 1$?

(signal is $\approx$ uniformly distributed)

We show that the optimal test given by the Neyman-Pearson Lemma is powerless

Bayesian Problem:

$H_0: \mu = 0$ vs $H_1: \mu \sim \Pi_\rho$,

here $S^{n-1} = \{u \in \mathbb{R}^n : \|u\| = 1\}$ - sphere of radius 1

$\Pi = Unif(S^{n-1})$

$S^{n-1}_\rho = \{\rho u : u \in S^{n-1}\}$; $\Pi_\rho = Unif(S^{n-1}_\rho)$

$$f_1(y/u) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{\|y-\rho u\|^2}{2}} \quad \overset{\text{Law of Total Probability}}{\Longrightarrow}$$

$$f_1(y) = \int_{S^{n-1}} f_1(y/u)\, d\Pi(u)$$

$$f_0(y) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{\|y\|^2}{2}}$$

$$\Rightarrow L = \frac{f_1(y)}{f_0(y)} = \int_{S^{n-1}} \frac{e^{-\frac{\|y-\rho u\|^2}{2}}}{e^{-\|y\|^2/2}}\, d\Pi(u) =$$

$$-\|y-\rho u\|^2 + \|y\|^2 = -\|y\|^2 + 2\rho\langle y, u\rangle - \rho^2 \underbrace{\|u\|^2}_{=1} + \|y\|^2 =$$

$$= 2\rho \underbrace{\langle y, u\rangle}_{= u^T y} - \rho^2$$

$$= \int_{S^{n-1}} e^{-\frac{1}{2}\rho^2 + \rho u^T y}\, \Pi(du)$$

We show that optimal test cannot identify H_1 when $\theta_n = \frac{\rho^2}{\sqrt{2n}} \to 0$, $n \to \infty$.

Scheme proof:

(1) $E_0 L = 1$ (property of likelihood ratio)

$\Big/ \ L = \dfrac{dP_1}{dP_0} \Rightarrow E_0 L = \int \dfrac{dP_1}{dP_0}\, dP_0 = \int dP_1 = 1$

(2) $\mathrm{var}_0 L \to 0$??

$\xRightarrow{(1)+(2)}$ $L \xrightarrow{P} 1$

$\xRightarrow{\text{like in previous lecture}}$ $P_1(\text{Type II Error}) \to 1-\alpha$

<u>Prop</u> $\mathrm{var}_0 L \to 0$, $\theta_n = \dfrac{\rho^2}{\sqrt{2n}} \to 0$, $n \to \infty$

Γ

$$\left|\begin{array}{l} y \sim N(0, I) \\ E e^{\langle a, y\rangle} = e^{\frac{1}{2}\|a\|^2} \ \text{– mgf of Gaussian random vector} \\ \quad \| \\ E e^{a^T y} \end{array}\right.$$

$$E_0 L^2 = E_0 \iint_{S^{n-1}\, S^{n-1}} e^{-\frac{\rho^2}{2} + \rho\langle u, y\rangle} \cdot e^{-\frac{\rho^2}{2} + \rho\langle v, y\rangle}\, \pi(du)\pi(dv)$$

$$= e^{-\rho^2} E_0 \iint e^{\rho\langle u+v, y\rangle}\, \pi(du)\pi(dv) =$$

$$= e^{-\rho^2} \iint E_0 e^{\langle \rho(u+v), y\rangle}\, \pi(du)\pi(dv) =$$

$$= e^{-\rho^2} \iint e^{\frac{1}{2}\rho^2\|u+v\|^2}\, \pi(du)\pi(dv) =$$

$$= \Big/ \ \|u+v\|^2 = \underbrace{\|u\|^2}_{1} + \underbrace{\|v\|^2}_{1} + 2\langle u, v\rangle = 2 + 2\langle u, v\rangle \Big/ =$$

$$= e^{-\rho^2} \iint e^{\rho^2 + \rho^2\langle u, v\rangle}\, \pi(du)\pi(dv) = \iint e^{\rho^2\langle u, v\rangle}\, \pi(du)\pi(dv)$$

$$\left|\begin{array}{l} w = Sv, \ \text{where } S \text{ is rotation operator} \Leftrightarrow S \text{ is orthogonal matrix}: S^T S = I \\ \Rightarrow \|v\|^2 = 1 \Rightarrow \|w\|^2 = \|Sv\|^2 = \langle Sv, Sv\rangle = \\ \qquad = \langle \underbrace{S^T S}_{I} v, v\rangle = \langle v, v\rangle = \|v\|^2 \\ |\det S| = 1 \Rightarrow \text{Jacobian of this transformation} = 1 \end{array}\right.$$

$\Rightarrow v = S^T w \qquad S(S_{n-1}) = S_{n-1}$.

$$\int_{S_{n-1}} e^{\rho^2 \langle u, v\rangle} \pi(du) = \int_{S_{n-1}} e^{\rho^2\langle u, S^T w\rangle} \pi(du) =$$

$$= \int_{S_{n-1}} e^{\rho^2\langle Su, w\rangle} \pi(du) = |Su = z| = \int_{S_{n-1}} e^{\rho^2\langle z, w\rangle} \pi(dz)$$

$$= \int_{S_{n-1}} e^{\rho^2\langle u, w\rangle} \pi(du)$$

$w := (1\ 0 \dots 0) \Rightarrow$

$$\Rightarrow \iint_{S_{n-1}\,S_{n-1}} e^{\rho^2\langle u, v\rangle} \pi(du)\,\pi(dv) = \int_{S_{n-1}}\int_{S_{n-1}} e^{\rho^2\langle u, w\rangle} \pi(du)\pi(dv)$$

$$= \iint_{S_{n-1}\,S_{n-1}} e^{\rho^2 u_1} \pi(du)\,\pi(dv) = \left|\int_{S_{n-1}} \pi(dv) = 1\right| =$$

$$= \int_{S_{n-1}} e^{\rho^2 u_1} \pi(du)$$

$$e^{\rho^2 u_1} = 1 + \rho^2 u_1 + \frac{\rho^4 u_1^2}{2} + \dots$$

$E e^{\rho^2 u_1} \ominus$

$E u_1 = 0$ — due to symmetry

$u_1^2 + \dots + u_n^2 = 1 \Rightarrow E u_1^2 = \frac{1}{n}$

ident. distributed

$\ominus\ 1 + \frac{\rho^4}{2n} + \bar{o}\left(\frac{\rho^4}{2n}\right) = 1 + \theta_n^2 + \bar{o}(\theta_n^2) \longrightarrow 1$, when $\theta_n \to 0$.

We have that $E_0 L^2 \to 1$, $E_0 L = 1$

$\Rightarrow \operatorname{var} L \to 0$, $\theta_n \to 0$ $\lrcorner$