Linear methods of classification

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Linear discriminant functions

- Classification of two classes ω_1 and ω_2.
- Linear discriminant function:

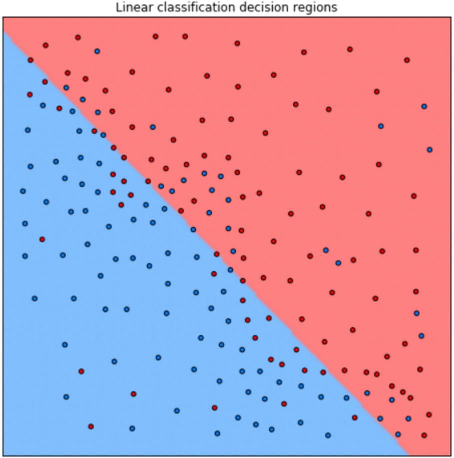
$$g(x) = w^T x + w_0$$

- Decision rule:

$$\widehat{y}(x) = \begin{cases} +1, & g(x) \geq 0 \\ -1, & g(x) < 0 \end{cases}$$

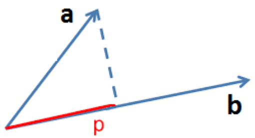
- Decision boundary $B = \{x : g(x) = 0\}$ is linear.

Example: decision regions



Reminder

1. $a=[a^1,...a^D]^T,\ b=[b^1,...b^D]^T$
2. Scalar product $\langle a,b\rangle = a^T b = \sum_{d=1}^{D} a_d b_b$
3. $a\perp b$ means that $\langle a,b\rangle = 0$
4. Norm $\|a\| = \sqrt{\langle a,a\rangle}$
5. Distance $\rho(a,b) = \|a-b\| = \sqrt{\langle a-b,a-b\rangle}$



- $p=\langle a,\frac{b}{\|b\|}\rangle$ - signed projection
- $|p| = \left|a,\frac{b}{\|b\|}\right|$- unsigned projection length

Properties

- Consider arbitrary
$$x_A, x_B \in B \Rightarrow \begin{cases} g(x_A) = w^T x_A + w_0 = 0 \\ g(x_B) = w^T x_B + w_0 = 0 \end{cases}$$
so $w^T(x_A - x_B) = 0$ and $w \perp B$.

Distance form origin

- Distance from the origin to B is equal to absolute value of the projection of $x \in B$ on $\frac{w}{\|w\|}$:

$$\langle x, \frac{w}{\|w\|} \rangle = \frac{\langle x, w \rangle}{\|w\|} = \{w^T x + w_0 = 0\} = -\frac{w_0}{\|w\|}$$

- So $\rho(0, B) = \frac{w_0}{\|w\|}$, and w_0 determines the offset from the origin.

Distance from x to B

Denote p - the projection of x on B, and $r = \langle \frac{w}{\|w\|}, x - p \rangle$ - the signed length of the orthogonal complement of x on B:

$$x = p + r\frac{w}{\|w\|}$$

After multiplication by w and addition of w_0:

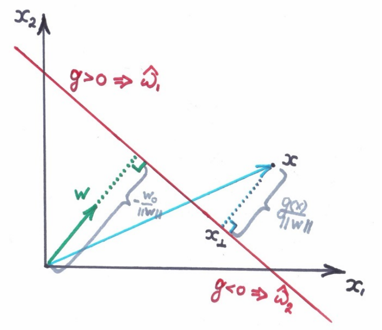
$$w^T x + w_0 = w^T p + w_0 + r\frac{\langle w, w \rangle}{\|w\|}$$

Using $w^T x + w_0 = g(x)$ and $w^T p + w_0 = 0$, we obtain:

$$r = \frac{g(x)}{\|w\|}$$

So from one side of the hyperplane $r > 0 \Leftrightarrow g(x) > 0$, and from the other side of the hyperplane $r < 0 \Leftrightarrow g(x) < 0$.

Illustration



Linear decision rule:

$$\widehat{y}(x) = \begin{cases} +1, & g(x) > 0 \\ -1, & g(x) < 0 \end{cases}$$

Decision boundary: $g(x) = 0$, confidence of decision: $|g(x)| / \|w\| = \frac{w^T x + w_0}{\|w\|}$.

Multiple classes classification

- Classification among $\omega_1, \omega_2, ...\omega_C$.
- Use C discriminant functions $g_c(x) = w_c^T x + w_{c0}$
- Decision rule:

$$\widehat{c}(x) = \arg\max_c g_c(x)$$

- Decision boundary between classes ω_i and ω_j is linear:

$$(w_i - w_j)^T x + (w_{i0} - w_{j0}) = 0$$

- Decision regions are convex[1].

[1] why? prove that.

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Linear discriminant functions

- Consider binary classification of classes ω_1 and ω_2.
- Denote classes ω_1 and ω_2 with $y = +1$ and $y = -1$.
- Linear discriminant function: $g(x) = w^T x + w_0$,

$$\widehat{\omega} = \begin{cases} \omega_1, & g(x) \geq 0 \\ \omega_2, & g(x) < 0 \end{cases}$$

- Decision rule: $y = \text{sign}\, g(x)$.
- Define constant feature $x_0 \equiv 1$, then $g(x) = w^T x = \langle w, x \rangle$ for $w = [w_0, w_1, ..w_D]^T$.
- Define the margin $M(x, y) = g(x)y$
  - $M(x, y) \geq 0 <=>$ object x is correctly classified as y
  - $|M(x, y)|$ - confidence of decision

Weights selection

- Target: minimization of the number of misclassifications Q:

$$Q(w|X) = \sum_{n} \mathbb{I}[M(x_n, y_n|w) < 0] \to \min_{w}$$

- Problem: standard optimization methods are inapplicable, because $Q(w, X)$ is discontinuous.

Weights selection

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$$Q(w|X) = \sum_n \mathbb{I}[M(x_n, y_n|w) < 0] \to \min_w$$

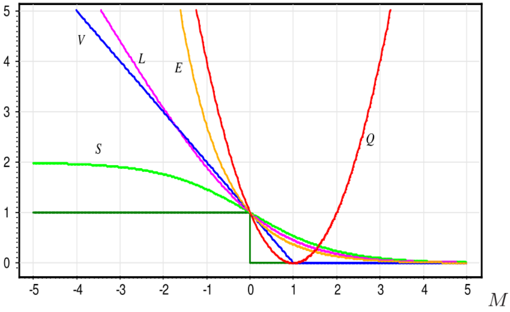
- Problem: standard optimization methods are inapplicable, because $Q(w, X)$ is discontinuous.
- Idea: approximate loss function with smooth function $\mathcal{L}$:

$$\mathbb{I}[M(x_n, y_n|w) < 0] \leq \mathcal{L}(M(x_n, y_n|w))$$

Approximation of the target criteria

We obtain the upper boundary on the empirical risk:

$$
\begin{aligned}
Q(w|X) &= \sum_n \mathbb{I}[M(x_n, y_n|w) < 0] \\
&\leq \sum_n \mathcal{L}(M(x_n, y_n|w)) = F(w)
\end{aligned}
$$



$Q(M) = (1-M)^2$
$V(M) = (1-M)_+$
$S(M) = 2(1+e^M)^{-1}$
$L(M) = \log_2(1+e^{-M})$
$E(M) = e^{-M}$

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Directional derivative

Definition 1

Consider differentiable function $f : \mathbb{R}^D \to \mathbb{R}$. A derivative along direction d, $\|d\| = 1$ is defined as

$$f'(x, d) = \lim_{\lambda \to 0} \frac{f(x + \lambda d) - f(x)}{\lambda}$$

Theorem 2

$f'(x, d) = \nabla f(x)^T d$

Proof. Using 1-st order taylor expansion we have

$$f(x + \lambda d) = f(x) + \nabla f(x)^T (\lambda d) + o(\|\lambda d\|)$$

$$\frac{f(x + \lambda d) - f(x)}{\lambda} = \nabla f(x)^T d + o(\|d\|) \xrightarrow{\lambda \to 0} \nabla f(x)^T d$$

$\square$

Direction of maximal growth/decrease

Theorem 3

For differentiable function $f(x)$ locally at point x:

- $\frac{\nabla f(x)}{\|\nabla f(x)\|}$ *is the direction of maximum growth*
- $-\frac{\nabla f(x)}{\|\nabla f(x)\|}$ *is the direction of maximal decrease.*

Proof. From Cauchi-Schwartz inequality, using that $\|d\| = 1$:

$$\left|\nabla f(x)^T d\right| \leq \|\nabla f(x)\| \, \|d\| = \|\nabla f(x)\|$$

Equality is achieved when $d \propto \nabla f(x)$, i.e. $d = \pm \nabla f(x) / \|\nabla f(x)\|$.
Theorem follows from 1-st order Taylor expansion

$$f(x + \lambda d) = f(x) + \nabla f(x)^T (\lambda d) + o(\|\lambda d\|)$$

□

Optimization

- Optimization task to obtain the weights:

$$F(w) = \sum_{i=1}^{N} \mathcal{L}(\langle w, x_i \rangle y_i) \to \min_{w}$$

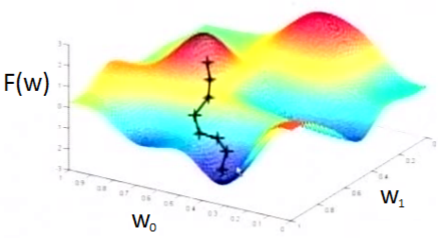
- Gradient descend algorithm:

INPUT:
η - parameter, controlling the speed of convergence
stopping rule

ALGORITHM:
initialize w_0 randomly
while stopping rule is not satisfied:
  $w_{n+1} \leftarrow w_n - \eta \frac{\partial F(w_n)}{\partial w}$
  $n \leftarrow n + 1$

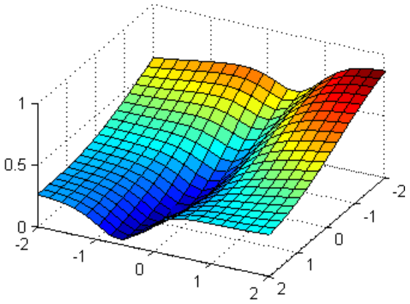
Gradient descend

- Possible stopping rules:
  - $|w_{n+1} - w_n| < \varepsilon$
  - $|F(w_{n+1}) - F(w_n)| < \varepsilon$
  - $n > n_{max}$
- Suboptimal method of minimization in the direction of the greatest reduction of $F(w)$:



Recommendations for use

- Convergence is faster for normalized features
  - feature normalization solves the problem of «elongated valleys»



Convergence acceleration

Stochastic gradient descend method

set the initial approximation w_0
calculate $\widehat{F} = \sum_{i=1}^{n} \mathcal{L}(M(x_i, y_i | w_0))$
iteratively until convergence $\widehat{Q}_{approx}$:

1. select random pair (x_i, y_i)
2. recalculate weights: $w_{n+1} \leftarrow w_n - \eta_n \mathcal{L}'(\langle w_n, x_i \rangle y_i) x_i y_i$
3. estimate the error: $\varepsilon_i = \mathcal{L}(\langle w_{n+1}, x_i \rangle y_i)$
4. recalculate the loss $\widehat{F} = (1 - \alpha)\widehat{F} + \alpha \varepsilon_i$
5. $n \leftarrow n + 1$

Variants for selecting initial weights

- $w_0 = w_1 = ... = w_D = 0$
- For logistic $\mathcal{L}$ (because the horizontal asymptotes):
  - randomly on the interval $[-\frac{1}{2D}, \frac{1}{2D}]$
- For other functions $\mathcal{L}$:
  - randomly
- $w_i = \frac{cov[x^i, y]}{var[x^i]}$ (these are regression weights, given that x^i are uncorrelated[2]).

[2] why?

Discussion of SGD

Advantages

- Easy to implement
- Works online
- A small subset of learning objects may be sufficient for accurate estimation

Discussion of SGD

Advantages

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Drawbacks

- Needs selection of η_n:
  - too big: divergence
  - too small: very slow convergence
- Overfitting possible for large D and small N
- When $\mathcal{L}(u)$ has left horizontal asymptotes (e.g. sigmoid), the algorithm may «get stuck» for large values of $\langle w, x_i \rangle$.

Examples

Delta rule $\mathcal{L}(M) = (M-1)^2$

$$w \leftarrow w - \eta(\langle w, x_i \rangle - y_i)x_i$$

Perceptron of Rosenblatt $\mathcal{L}(M) = [-M]_+$

$$w \leftarrow w + \begin{cases} 0, & \langle w, x_i \rangle y_i \geq 0 \\ \eta x_i y_i & \langle w, x_i \rangle y_i < 0 \end{cases}$$

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Regularization for SGD[3]

L_2-regularization for upperbound approximation:

$$F^{regularized}(w) = F(w) + \lambda \sum_{d=1}^{D} w_d^2$$

L_1-regularization for upperbound approximation:

$$F^{regularized}(w) = F(w) + \lambda \sum_{d=1}^{D} |w_d|^2$$

λ is the parameter controlling strength of regularization = model complexity.

[3] how will SGD step change? Interpret.

Regularization

- General regularization.

$$F^{regularized}(w) = Q(w) + \lambda R(w)$$

- Examples:

$$
\begin{aligned}
R(w) &= \|w\|_1 = \sum_{d=1}^{D} |w_d| \\
R(w) &= \|w\|_2^2 = \sum_{d=}^{D} (w_d)^2 \\
R(w) &= \alpha \|w\|_1 + (1-\alpha) \|w\|_2^2 \,, \; \alpha \in [0,1]
\end{aligned}
$$

L_1 norm

- $||w||_1$ regularizer will do feature selection.
- Consider
$$Q(w) = \sum_{i=1}^{N} \mathcal{L}_i(w) + \lambda \sum_{d=1}^{D} |w_d|$$
- if $\lambda > \sup_w \left| \sum_{i=1}^{N} \frac{\partial \mathcal{L}(w)}{\partial w_i} \right|$, then it becomes optimal to set $w_i = 0$
- For smaller C more inequalities will become active.

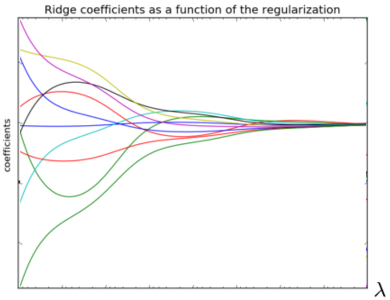
L_2 norm

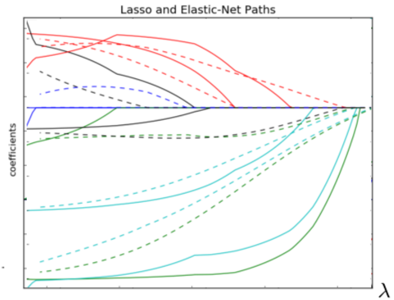
- $||w||_1$ regularizer will do feature selection.
- Consider $R(w) = \|w\|_2^2 = \sum_d w_d^2$

$$Q(w) = \sum_{i=1}^{n} \mathcal{L}_i(w) + \lambda \sum_{d=1}^{D} w_d^2$$

- $\frac{\partial R(w)}{\partial w_i} = 2w_i \to 0$ when $w_i \to 0$.

Illustration





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Binary classification

- Linear classifier:

$$score(\omega_1|x) = w^T x$$

- +relationship between score and class probability is assumed:

$$p(\omega_1|x) = \sigma(w^T x)$$

where $\sigma(z) = \frac{1}{1+e^{-z}}$ - sigmoid function

Binary classification: estimation

Using the property $1-\sigma(z)=\sigma(-z)$ obtain that

$$p(y=+1|x)=\sigma(w^Tx)\Longrightarrow p(y=-1|x)=\sigma(-w^Tx)$$

So for $y\in\{+1,-1\}$

$$p(y|x)=\sigma(y\langle w,x\rangle)$$

Therefore ML estimation can be written as:

$$\prod_{i=1}^{N}\sigma(\langle w,x_i\rangle y_i)\to\max_w$$

Loss function for 2-class logistic regression

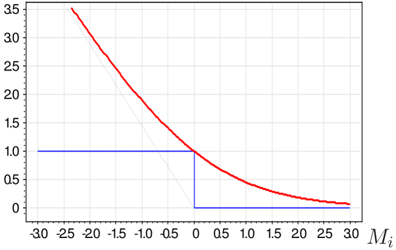
For binary classification $p(y|x) = \sigma(\langle w, x \rangle y)$ $w = [\beta_0', \beta]$, $x = [1, x_1, x_2, ...x_D]$.

Estimation with ML:

$$\prod_{i=1}^{n} \sigma(\langle w, x_i \rangle y_i) \to \max_w$$

which is equivalent to

$$\sum_{i}^{n} \ln(1 + e^{-\langle w, x_i \rangle y_i}) \to \min_w$$



It follows that logistic regression is linear discriminant estimated with loss function $\mathcal{L}(M) = \ln(1 + e^{-M})$.

SGD realization of logistic regression

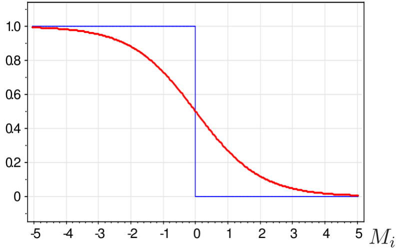
Substituting $\mathcal{L}(M) = \ln(1 + e^{-M})$ into update rule, we obtain that for each sample (x_i, y_i) weights should be adapted according to

$$w \leftarrow w + \eta\sigma(-M_i)x_i y_i$$

Perceptron of Rosenblatt update rule:

$$w \leftarrow w + \eta\mathbb{I}[M_i < 0]x_i y_i$$

- Logistic rule update is the smoothed variant of perceptron's update.
- The more severe the error (according to margin) - the more weights are adapted.



Multiple classes

Multiple class classification:

$$\begin{cases} score(\omega_1|x) = w_1^T x \\ score(\omega_2|x) = w_2^T x \\ \cdots \\ score(\omega_C|x) = w_C^T x \end{cases}$$

+relationship between score and class probability is assumed:

$$p(\omega_c|x) = softmax(w_c^T x | x_1^T x, ... x_C^T x) = \frac{exp(w_c^T x)}{\sum_i exp(w_i^T x)}$$

Multiple classes

Weights ambiguity:

w_c, $c = 1, 2, ...C$ defined up to shift v:

$$\frac{exp((w_c - v)^T x)}{\sum_i exp((w_i - v)^T x)} = \frac{exp(-v^T x) exp(w_c^T x)}{\sum_i exp(-v^T x) exp(w_i^T x)} = \frac{exp(w_c^T x)}{\sum_i exp(w_i^T x)}$$

To remove ambiguity usually $v = w_C$ is subtracted.

Estimation with ML:

$$\begin{cases} \prod_{n=1}^{N} softmax(w_{y_n}^T x_n | x_1^T x, ... x_C^T x) \to \max_{w_1, ... w_{C-1}} \\ w_C = \mathbf{0} \end{cases}$$