There are a total of five problems. You have to solve all of them.

Problem 1 (CO5): Nonregular Language (25 points)

Use the pumping lemma to **demonstrate** that L_1, L_2, L_3, L_4 and L_5 is not regular.

(a) $L_1 = \{w \in \{0,1,2\}^* : 0^n 1^n 2^n \text{ where } n \geq 0\}$ (5 points)

(b) $L_2 = \{w \in \{0,1\}^* : 0^x 1^y 0^z \text{ where } z > x + y \text{ and } x, y \geq 0\}$ (5 points)

(c) $L_3 = \{w \in \{0,1\}^* : w \text{ is a palindrome}\}$ (5 points)

(d) $L_4 = \{w \in \{a,b\}^* : \text{ numbers of a in } w \text{ is a prime number}\}$ (5 points)

(e) $L_5 = \{w \in \{0\}^* : 0^{3^n} \text{ where } n \geq 0\}$ (5 points)

(a) Assume for the sake of contradiction that L_1 is regular. Then let p be the pumping length for L_1. Now we take the string

$$w = 0^p 1^p 2^p \in L_1.$$

Then the length of w is $|w| = 3p \geq p$. So w can be split into xyz such that $|y| > 0$, $|xy| \leq p$, and $xy^iz \in L_1$ for each $i \geq 0$. Since $|xy| \leq p$, and the first p characters of w are all 0s, we can conclude that y consists of only 0s.

Then, for $i = 2$, xy^2z will be

$$xy^2z = xyyz = 0^{p+|y|} 1^p 2^p \notin L_1.$$

We have excess 0s in $xyyz$. Thus we get a contradiction! Hence, L_1 is not a regular language.

(b) Assume for the sake of contradiction that L_2 is regular. Then let p be the pumping length for L_2. Now we take the string

$$w = 0^p 1^p 0^{2p+1} \in L_2.$$

Then the length of w is $|w| = 4p + 1 > p$. So w can be split into xyz such that $|y| > 0$, $|xy| \leq p$, and $xy^iz \in L_2$ for each $i \geq 0$. Since $|xy| \leq p$, and the first p characters of w are all 0s, we can conclude that y consists of only 0s.

Then, for $i = 2$, xy^2z will be

$$xy^2z = xyyz = 0^{p+|y|} 1^p 0^{2p+1}.$$

This string is not in L_2, since $p + |y| + p \geq 2p + 1$. Thus we get a contradiction! Hence, L_2 is not a regular language.

(c) Assume for the sake of contradiction that L_3 is regular. Then let p be the pumping length for L_3. Now we take the string

$$w = 0^p 1 0^p \in L_3.$$

Then the length of w is $|w| = 2p + 1 > p$. So w can be split into xyz such that $|y| > 0$, $|xy| \leq p$, and $xy^iz \in L_3$ for each $i \geq 0$. Since $|xy| \leq p$, and the first p characters of w are all 0s, we can conclude that y consists of only 0s.

Then, for $i = 2$, xy^2z will be

$$xy^2z = xyyz = 0^{p+|y|} 1 0^p = 0^p 0^{|y|} 1 0^p.$$

This string is not a palindrome, so it is not in L_3. Thus we get a contradiction! Hence, L_3 is not a regular language.

(d) Assume for the sake of contradiction that L_4 is regular. Then let p be the pumping length for L_4. Now we take the string

$$w = a^q \in L_4,$$

where q is a prime number greater than or equal to p. The length of w is $|w| = q \geq p$. So w can be split into xyz such that $|y| > 0$, $|xy| \leq p$, and $xy^iz \in L_4$ for each $i \geq 0$.

The length of $w = xyz$ is q. Then, for $i = 2$, the length of $xy^2z = xyyz$ will be $q + |y|$, which should be a prime. So, for any $i > 0$, the length of xy^iz will be $q + (i-1)|y|$.

So, for $i > 0$,

$$xy^iz = xyy^{i-1}z = a^{q+(i-1)|y|}.$$

Since this string is in L_4, $q + (i-1)|y|$ is a prime number for each $i > 0$. But this is clearly not true, since choosing $i = q + 1$ gives

$$q + (i-1)|y| = q + q|y|,$$

which is divisible by q. Thus we get a contradiction! Hence, L_4 is not a regular language.

(e) Assume for the sake of contradiction that L_5 is regular. Then let p be the pumping length for L_5. Now we take the string

$$w = 0^{3^p} \in L_5.$$

Then the length of w is $|w| = 3^p > p$. So w can be split into xyz such that $|y| > 0$, $|xy| \leq p$, and $xy^iz \in L_5$ for each $i \geq 0$.

Then, for $i = 2$, xy^2z will be

$$xy^2z = xyyz = 0^{3^p + |y|}.$$

Since this string is in L_5, $3^p + |y|$ is a power of 3 (which is, of course, larger than 3^p). The next power of 3 larger than 3^p is 3^{p+1}. So we have

$$3^p + |y| \geq 3^{p+1} \implies |y| \geq 3^{p+1} - 3^p = 2 \cdot 3^p.$$

On the other hand, $|xy| \leq p$ gives us that $|y| \leq p$. So

$$p \geq |y| \geq 2 \cdot 3^p.$$

This is clearly false since $3^p > p$. Thus we get a contradiction! Hence, L_5 is not a regular language.