ALA DA

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2) Let $u_1 = (1,1,1,1)$, $u_2 = (1,-1,2,2)$ and
$u_3 = (1,2,-3,-4)$

$\therefore v_1 = u_1 = (1,1,1,1)$

$$v_2 = u_2 - \frac{\langle v_2 , v_1 \rangle}{\|v_1\|^2} v_1 = (1,-1,2,2) - \frac{4}{4}(1,1,1,1)$$

$$= (0,-2,1,1)$$

$$v_3 = u_3 - \frac{\langle v_3 , v_1 \rangle}{\|v_1\|^2} \cdot v_1 - \frac{\langle v_2 \cdot u_2 \rangle}{\|v_2\|^2} \cdot v_2$$

$$= (1,2,-3,-4) - \frac{(-4)}{4}(1,1,1,1) - \frac{6}{6}(0,-2,1,1)$$

$$= (2,3,-2,-3) - (0,-2,1,1)$$

$$= (2,5,-3,-4)$$

$$\frac{v_1}{\|v_1\|} = \frac{1}{2}(1,1,1,1) \quad \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{6}}(0,-2,1,1)$$

$$\frac{v_3}{\|v_3\|} = \frac{1}{3\sqrt{6}}(2,5,-3,-4)$$

$\{(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (0, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}),$
$(\frac{2}{3\sqrt{6}}, \frac{5}{3\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{-4}{\sqrt{6}})\}$ are the orthonormal basis.

Orthogonal projection of $v = (1, 2, -3, 4)$, over U,

$$\text{proj}_u(v) = \left(\frac{u_1 \cdot v}{\|u_1\|^2}\right) u_1 + \left(\frac{u_2 \cdot v}{\|u_2\|^2}\right) \cdot u_2 + \left(\frac{u_3 \cdot v}{\|u_3\|^2} \cdot v_3\right)$$

$$= \frac{4}{4}(1,1,1,1) + \frac{1}{10}(1,-1,2,2) + \frac{(-2)}{30}(1,2,-3,-4)$$

$$= (1,1,1,1) + \left(\frac{1}{10}, \frac{-1}{10}, \frac{1}{5}, \frac{1}{5}\right) - \left(\frac{1}{15}, \frac{2}{15}, \frac{-1}{5}, \frac{4}{15}\right)$$

$$\text{proj}_u(v) = \begin{bmatrix} \frac{31}{30} \\ \frac{23}{30} \\ 7/5 \\ 22/15 \end{bmatrix}$$

(3) $f(x) = x + k$, $g(x) = x^2$

$$\langle f, g \rangle = \int_0^1 f(x)\, g(x) \, dx$$

$$= \int_0^1 (x+k)\, x^2 \, dx$$

$$= \int_0^1 (x^3 + kx^2)\, dx$$

$$= \left[\frac{x^4}{4} + \frac{kx^3}{3}\right]_0^1$$

$$= \left[\left(\frac{1}{4} + \frac{k}{3} \right) \cdot (0 + 0) \right]$$

$$0 = \left(\frac{3 + 4k}{12} \right)$$

$$3 + 4k = 0$$

$$k = \frac{-3}{4}$$

$$k = -3/4$$

(4) $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ The basis of the $C(A) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

Null space of A is orthogonal compliment of column space of A.

$$\sim \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \to \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \to \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore x_1 + x_2 = 0$

$x_3 = 0$

Let $x_2 = t$

$$N(A) = \begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Basis of the orthogonal compliment of column space of A is $(-1, 1, 0)$

(5) $U = \text{span}\left[(1,1,0,0), (1,0,1,0)\right]$

$W = \text{span}\left[(0,1,0,1), (0,0,1,1)\right]$

(ii) $U^{\perp}$

$$U = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$x_1 + x_3 = 0$

$x_2 - x_3 = 0$

$\boxed{x_2 = x_3}$

$\boxed{x_1 = -x_3} = -x_2$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_3 \\ x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ x_4 \end{bmatrix}$$

$$= x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

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$\therefore$ The basis for $U^{\perp}$ is $= \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

(i) $U+W$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \rightarrow \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \leftarrow \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\downarrow$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Basis of $U+W$ is $= \{(1,1,0,0), (0,1,-1,0), (0,0,1,1)\}$

$\dim(U+W) = 3$

$U^\perp + W^\perp$

$W^\perp$

$$W = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$x_2 + x_4 = 0 \rightarrow x_2 = -x_4$

$x_3 + x_4 = 0 \longrightarrow x_3 = -x_4$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} 0 \\ -1 \\ -1 \\ 1 \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ Basis $= \{(0,-1,-1,1), (1,0,0,0)\}$

$$V^\perp = \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \to \sim \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\to \sim \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \to \sim \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\to \sim \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \sim \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Basic of $V^\perp + W^\perp = \{(1,-1,-1,0), (0,1,1,-1), (0,0,0,1)\}$

$\dim(V^\perp + W^\perp) = 3$

$$\dim(V^\perp \cap W^\perp) = \dim(V^\perp) + \dim(W^\perp) - \dim(V^\perp + W^\perp)$$
$$= 2 + 2 - 3$$
$$= 1$$

(v) $U \cap W$:

Let $V = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \lambda + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \mu \;\middle|\; \lambda, \mu \in \mathbb{R} \right\}$

$W = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \lambda + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \mu \;\middle|\; \lambda, \mu \in \mathbb{R} \right\}$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \lambda_1 \\ \mu_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \lambda_2 \\ \mu_2 \end{bmatrix}$$

$\lambda_1 + \mu_1 = 0$

$\lambda_1 = \lambda_2$

$\mu_1 = \mu_2$

$\lambda_2 + \mu_2 = 0$

$\boxed{\lambda_2 = -\mu_2}$

$\therefore$

(1) (1) $\langle u,v\rangle = u_1v_1 + u_3v_3$

Given:

$u = (u_1, u_2, u_3) \quad v = (v_1, v_2, v_3)$

axiom 1:

$(u,v) = u_1v_1 + u_3v_3 = (v,u)$

axiom 2:

$\langle u+v, w\rangle = (u_1+v_1)w_1 + (u_3+v_3)w_3$

$= u_1w_1 + v_1w_1 + u_3w_3 + v_3w_3$

$= u_1w_1 + u_3w_3 + v_1w_1 + v_3w_3$

$= \langle u, w\rangle + \langle v, w\rangle$

axiom 3:

$\langle ku, v\rangle = k\langle u, v\rangle$

$\langle ku, v\rangle = ku_1v_1 + ku_3v_3$

$= k(u_1v_1 + u_3v_3)$

$= k(\langle u,v\rangle)$

axiom 4:

$\langle u,u\rangle \geq 0 \ \& \ \langle u,u\rangle = 0$

$\Leftrightarrow u = 0$

$\langle u, u \rangle = u_1^2 + u_3^3 \geq 0$

$\langle u, u \rangle = u_1^2 + u_3^2 = 0$

$\Longleftrightarrow u_1 = u_3 = 0$

(ii) $\langle u, v \rangle = u_1 v_1 - u_2 v_2 + u_3 v_3$

axiom 1:

$\langle u, v \rangle = \langle v, u \rangle$

axiom 2:

$\langle u+v, w \rangle = (u_1 + v_1) w_1 - (u_2 + v_2) w_2 + (u_3 + v_3) w_3$

$= u_1 w_1 + v_1 w_1 - u_2 w_2 - v_2 w_2 + u_3 w_3 + v_3 w_3$

$= (u_1 w_1 - u_2 w_2 + u_3 w_3) + (v_1 w_1 - v_2 w_2 + v_3 w_3)$

$= \cancel{(u, w)} \langle u, w \rangle + \langle v, w \rangle$

axiom 3:

$\langle ku, v \rangle = k u_1 v_1 - k u_2 v_2 + k u_3 v_3$

$= k(u_1 v_1 - u_2 k_2 + u_3 v_3)$

$= k \langle u, v \rangle$