TO: Professor Andreas Linninger / Grant Hartung

FROM: Kehinde Abioye

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SUBJECT: Equation for systems

The purpose of this assignment is to use knowledge of the matrix equation to solve for unknown variables of biological networks, such as blood flow.

Introduction

Part#1: Generation of Mass Conservation Balances Using Network Analysis

The flow and pressures of the Circle of Willis was analyzed. At any given point of pressure, the flow going in will be positive and any flow going out will be negative (Eq. 2). The Hagen-Poiseuille equation (eq. 1) described the resistance and flow between two points of pressures. The corresponding resistance will be matched with the flow between that point. The pressures at the outermost part of the Circle of Willis will have boundary conditions.

$$\Delta P = \alpha F, \quad \text{Eq. 1}$$

Resistance vector and Boundary Conditions, Eq. 1a

α = [0.3 0.09 0.3 0.12 0.3 0.3 0.09 0.3 0.12 0.3 0.06 0.3 0.09 0.06 0.3 0.3 0.09 0.09]; (mmHg·min/mL)

$$F_{in} - F_{out} = 0, \quad \text{Eq. 2}$$

Boundary Conditions for the system pictured in Figure 1

P1, P13, P7=100 (mmHg), Eq. 2a

P4, P6, P12 , P10, P17, P18=50 (mmHg), Eq. 2b

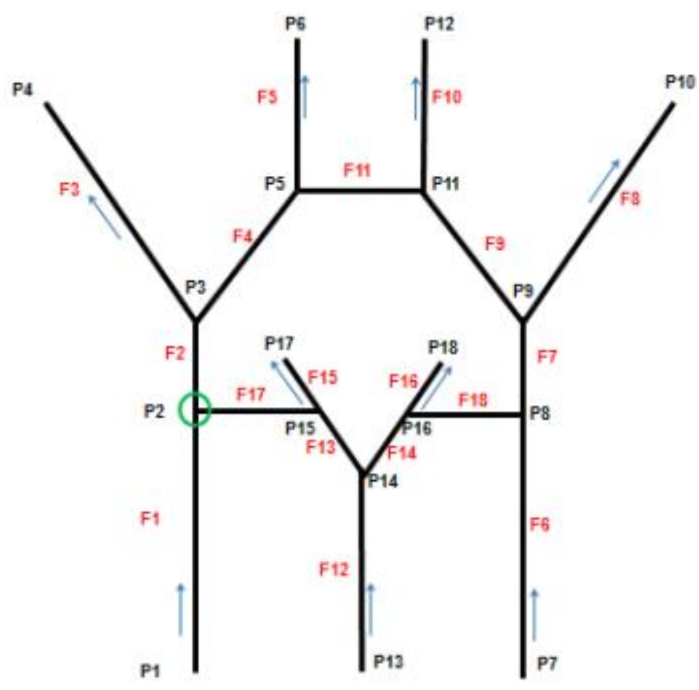

Fig 1. Schematic Network of Circle of Willis

Part#2: Writing the Network Problem in Matrix Format

Now that we have a set of equations from part 1, we can use the equation Ax=b (eq. 4) to find the flows and pressures of the Circle of Willis. The equations will be assembled in matrix form in the format presented in equation 3a, 3b, and 3c to find the solution x. The pressures will represent C1, the resistance will represent –R, the flows will be the C2 matrix and the boundary conditions will be the C3 matrix. The flows and pressures will be solved using the "A\b" command. The solutions will be checked using the residual error equation r = Ax-b. If r equals a null vector, then the solution is accurate. Three nodes will be selected at random to test the solutions. This will be done by plugging in the corresponding solution to the corresponding equation.

$$C_1\, p = R\, f, \quad \text{Eq. 3a}$$

$$C_2\, f = 0, \quad \text{Eq. 3b}$$

$$C_3 p = \bar{p}, \quad \text{Eq. 3c}$$

$$\begin{bmatrix} C1 & -R \\ 0 & C2 \\ C3 & 0 \end{bmatrix} \begin{bmatrix} p \\ f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \bar{p} \end{bmatrix}$$

$$A \quad * \quad x \quad = \quad b, \text{ Eq. 4}$$

Part#3

Pressure can be solved for by removing flows with matrix substitution. A new equation C_4 is created by multiplying C_2 and C_1 by the inverse of A. C_4 and C_3 make up the new A matrix and x accounts for the pressures. The solution will be inputted in equation 3a to get flows.

$$C4 = C2*R^{-}1*C1, \text{ Eq. 5a}$$

$$\begin{bmatrix} C4 \\ C3 \end{bmatrix} [p] = \begin{bmatrix} 0 \\ \bar{p} \end{bmatrix}, \text{ Eq. 5b}$$

Part#4

Element 11 in the resistance matrix will be replaced with 10e15 to see what will happen when there is a very high resistance, or blockage. The new R will be simulated in Part 2 and 3.

Part#5

Element 11 in the resistance matric will be replaced with 0 to see what will happen when there is a very low resistance, such as if the blood vessel was short. The new –R will be simulated in Part 2 and 3.

Methods

Part#1: Generation of Mass Conservation Balances Using Network Analysis

Equations were created for the flows and pressures of a Circle of Willis using the Hagen-Poiseuille and conservation balances equations. A vector α was given and inserted into the Hagen-Poiseuille equation accordingly. The boundaries of the Circle of Willis had no flow, therefore, a conservation balances equation could not be made for those nodes. Boundary conditions were provided.

Part#2: Writing the Network Problem in Matrix Format

Using the equations found in part 1, matrices were created. The Hagen-Poiseuille equations were assigned to a 18x18 Matrix C_1. This matrix represented the pressures. Any instance where a pressure was not used was assigned a value of zero. A 18x18 Matrix -R represented the α vector multiplied by one. A diagonal matrix of α values was created. The conservation balances were assigned to a 9x18 matrix C_2. Any place in the matrix missing value was given the value 0. This matrix represented the flows. Matrix C_3 is a 9x18 matrix that is the boundary conditions. The 36x36 matrix A was assembled by combing the matrices accordingly, inserting zero matrices when necessary. The vector x is a 36x1 matric and is the unknown variables p(pressures) and f (flows). The "\" symbol in MATLAB was used to find the values of the pressure and flows by multiplying the b vector with the inverse matrix of A. The values were checked using the residual equation r=Ax-b. When r equals 0 then the solution satisfies the equations. The larger r is less accurate. For loops were used to obtain the r values.

Part#3:

A matrix C_4 was created by using the inverse function in MATLAB "inv" to find the inverse of R and multiplying the matrix by C_2 and C_1. The matrix was then combined using ";" to C_3 and pressures were found using "\" to multiply the inverse of the new A matrix with C_4 and C_3 by b which had the values of the boundary conditions.

Part#4:

Element 11 of the α matrix was replaced with 10e15. The R matrix was modified to reflect this and used as the new R matrix for part 2 and 3.

Part#5:

Element 11 of the α matrix was replaced with 0. The R matrix was modified to reflect this and used as the new R matrix for part 2 and 3.

Results

Part#1: Generation of Mass Conservation Balances Using Network Analysis

The Hagan-Poiseuille equations, boundary condition equations, and conservation balance equations were formed as in equation 1. There are a total of 18 Hagen-Poiseuille equations, 9 boundary condition equations, and 9 conservation balance equations with a total of 32 equations.

Hagen-Poiseuille

1: $P_1-P_2=\alpha_1 F_1$

2: $P_2-P_3=\alpha_2 F_2$

3: $P_3-P_4=\alpha_3 F_3$

4: $P_3-P_5=\alpha_4 F_4$

5: $P_5-P_6=\alpha_5 F_5$

6: $P_5-P_{11}=\alpha_{11} F_{11}$

7: $P_{11}-P1_2=\alpha_{10} F_{10}$

8: $P_{11}-P_9=\alpha_9 F_9$

9: $P_9-P_{10}=\alpha_8 F_8$

10: $P_8-P_9=\alpha_7 F_7$

11: $P_7-P_8=\alpha_6 F_6$

12: $P_8-P_{16}=\alpha_{18} F_{18}$

13: $P_{16}-P_{18}=\alpha_{16} F_{16}$

14: $P_{13}-P_{14}=\alpha_{12} F_{12}$

15: $P_{15}-P_{17}=\alpha_{15} F_{15}$

16: $P_2-P_{15}=\alpha_{17} F_{17}$

17: $P_{15}-P_{14}=\alpha_{13} F_{13}$

18: $P_{14}-P_{16}=\alpha_{14} F_{14}$

Conservation Balances

1: $F_1-F_2-F_{17}=0$

2: $F_2-F_3-F_4=0$

3: $F_4-F_5-F_{11}=0$

4: $F_6-F_7-F_{18}=0$

5: $F_7+F_9-F_8=0$

6: $F_{11}-F_{10}-F_9=0$

7: $F_{12}-F_{14}-F_{13}=0$

8: $F_{17}-F_{15}=0$

9: $F_{18}-F_{16}=0$

Boundary Conditions

1: P_1=100

2: P_4=50

3: P_6=50

4: P_7=100

5: P_{10}=50

6: P_{12}=50

7: P_{13}=100

8: P_{17}=50

9: P_{18}=50

Part#2: Writing the Network Problem in Matrix Format

C1, -R, C2 and C3 were created and concatenated in matrix A. The solution x was found using “A\b.” The solution was tested for Flow 1, 2, and 3 and pressure 1, 2. And 3, as shown in appendix A.

C_1 =

P1	P2	P3	P4	P5	P6	P7	P8	P9	P10	P11	P12	P13	P14	P15	P16	P17	P18
1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	-1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	-1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1	-1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	-1	0	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	-1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	-1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	1	-1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	-1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	-1
0	0	0	0	0	0	0	0	0	0	0	0	1	-1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	-1	0
0	1	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	-1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	-1	0	0

C_2 =

F1	F2	F3	F4	F5	F6	F7	F8	F9	F10	F11	F12	F13	F14	F15	F16	F17	F18
1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0
0	1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	-1	0	0	0	0	0	-1	0	0	0	0	0	0	0
0	0	0	0	0	1	-1	0	0	0	0	0	0	0	0	0	0	-1
0	0	0	0	0	0	1	-1	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	-1	-1	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	-1	-1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	1

C_3 =

P1	P2	P3	P4	P5	P6	P7	P8	P9	P10	P11	P12	P13	P14	P15	P16	P17	P18
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

R =

0.3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0.09	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0.3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0.12	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0.3	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0.3	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0.09	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0.3	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0.12	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0.3	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0.06	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0.3	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0.09	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0.06	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.3	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.3	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.09	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.09

Pressure and flow values were calculated in MATLAB and are shown in Table 1.

Table 1: Solution for pressures and flows

Pressures		**Flows**	
$P_1 = 100$	$P_{10} = 50$	$F_1 = 97.9051$	$F_{10} = 32.8876$
$P_2 = 70.6285$	$P_{11} = 59.8663$	$F_2 = 77.4793$	$F_{11} = -0.7717$
$P_3 = 63.6553$	$P_{12} = 50$	$F_3 = 45.5178$	$F_{12} = 91.3703$
$P_4 = 50$	$P_{13} = 100$	$F_4 = 31.9615$	$F_{13} = 42.2082$
$P_5 = 59.8200$	$P_{14} = 72.5889$	$F_5 = 32.7332$	$F_{14} = 49.1622$
$P_6 = 50$	$P_{15} = 68.7902$	$F_6 = 96.3122$	$F_{15} = 62.6339$
$P_7 = 100$	$P_{16} = 69.6392$	$F_7 = 80.0105$	$F_{16} = 65.4639$
$P_8 = 71.1063$	$P_{17} = 50$	$F_8 = 46.3513$	$F_{17} = 20.4257$
$P_9 = 63.9054$	$P_{18} = 50$	$F_9 = 33.6593$	$F_{18} = 16.3017$

Part#3:

Only the pressures were solved for by reducing matric A to C4 and C3.

Table 2: Solution for pressures only

$P_1 = 100$	$P_{10} = 50$
$P_2 = 70.6285$	$P_{11} = 59.8663$
$P_3 = 63.6553$	$P_{12} = 50$
$P_4 = 50$	$P_{13} = 100$
$P_5 = 59.8200$	$P_{14} = 72.5889$
$P_6 = 50$	$P_{15} = 68.7902$
$P_7 = 100$	$P_{16} = 69.6392$
$P_8 = 71.1063$	$P_{17} = 50$
$P_9 = 63.9054$	$P_{18} = 50$

Part#4:

Element 11 of the alpha matrix was replaced with 10e15 and the alpha matrix was used to find x in both methods from part 1 and 2.

Method 1:

Table 3: Solutions for both pressure and flow for method 1.

Pressures		**Flows**	
P_1 =100	P_{10} = 50	F_1 = 97.9938	F_{10} = 33.7222
P_2 = 70.6019	P_{11} = 59.9682	F_2 = 77.7429	F_{11} = 0
P_3 = 63.6050	P_{12} = 50	F_3 = 45.3500	F_{12} = 91.3651
P_4 = 50	P_{13} = 100	F_4 = 32.3929	F_{13} = 42.3467
P_5 = 59.7179	P_{14} = 72.5905	F_5 = 32.3929	F_{14} = 49.0184
P_6 = 50	P_{15} = 68.7793	F_6 = 96.2249	F_{15} = 62.5976
P_7 = 100	P_{16} = 69.6494	F_7 = 79.7454	F_{16} = 65.4979
P_8 = 71.1325	P_{17} = 50	F_8 = 46.5181	F_{17} = 20.2509
P_9 = 63.9554	P_{18} = 50	F_9 = 33.2272	F_{18} = 16.4795

Method 2:

Table 4: Solutions for both pressure and flow for method 2

Pressures		**Flows**	
P_1 =100	P_{10} = 50	F_1 = 97.9938	F_{10} = 33.7222
P_2 = 70.6019	P_{11} = 59.9682	F_2 = 77.7429	F_{11} = 0
P_3 = 63.6050	P_{12} = 50	F_3 = 45.3500	F_{12} = 91.3651
P_4 = 50	P_{13} = 100	F_4 = 32.3929	F_{13} = 42.3467
P_5 = 59.7179	P_{14} = 72.5905	F_5 = 32.3929	F_{14} = 49.0184
P_6 = 50	P_{15} = 68.7793	F_6 = 96.2249	F_{15} = 62.5976
P_7 = 100	P_{16} = 69.6494	F_7 = 79.7454	F_{16} = 65.4979
P_8 = 71.1325	P_{17} = 50	F_8 = 46.5181	F_{17} = 20.2509
P_9 = 63.9554	P_{18} = 50	F_9 = 33.2272	F_{18} = 16.4795

Part#5

Element 11 of the alpha matrix was replaced with 0 and the alpha matrix was used to find x in both methods from part 1 and 2.

Method 1:

When setting α_11 equal to 0, the flow decreased.

Table 5: Solutions for both pressure and flow for method 1

Pressures		**Flows**	
P_1 =100	P_{10} = 50	F_1 = 97.8849	F_{10} = 32.8105
P_2 = 70.6345	P_{11} = 59.8431	F_2 = 77.4195	F_{11} = -0.9469
P_3 = 63.6668	P_{12} = 50	F_3 = 45.5559	F_{12} = 91.3715
P_4 = 50	P_{13} = 100	F_4 = 31.8636	F_{13} = 42.1767
P_5 = 59.8431	P_{14} = 72.5885	F_5 = 32.8105	F_{14} = 49.1948
P_6 = 50	P_{15} = 68.7926	F_6 = 96.3321	F_{15} = 62.6421
P_7 = 100	P_{16} = 69.6369	F_7 = 80.0707	F_{16} = 65.4562
P_8 = 71.1004	P_{17} = 50	F_8 = 46.3134	F_{17} = 20.4654
P_9 = 63.8940	P_{18} = 50	F_9 = 33.7573	F_{18} = 16.2614

Method 2:

When setting α_11 equal to 0, x=[NaN]

Discussion

I have learned that the matrix equation can be used to solve for unknowns in a biological network. I also learned that when a resistance is close to or equal to zero, it creates a problem in the second method (part 2) since 0 can't be divided by 0 when finding the inverse of R. A result is a NaN matrix for x.

Conclusion

The matrix equation has various applications. When solving for a system of equations it proves to be beneficial. The residual error equation can be used to check the solution of the equation solved for using the matrix equation.

Appendix A

```
clc
clear
filename = 'C1.xlsx';
C1 = xlsread(filename);
filename = 'C2.xlsx';
C2 = xlsread(filename);
filename = 'C3.xlsx';
C3 = xlsread(filename);
filename = 'R.xlsx';
R = xlsread(filename);
resistance = [0.3 0.09 0.3 0.12 0.3 0.3 0.09 0.3 0.12 0.3 0.06 0.3 0.09 0.06
0.3 0.3 0.09 0.09]

% Part 2
A = [C1 R;(zeros(9,18)) C2;C3 (zeros(9,18))];
b =
[0;0;0;0;0;0;0;0;0;0;0;0;0;0;0;0;0;0;0;0;0;0;0;0;0;0;0;100;50;50;100;50;50;10
0;50;50];
x=A\b
r = (A*x)-b;
for i = 1:36
r(i)=(A(i,:)*x)-b(i);
end
test1=(x(1)-x(2))-(resistance(1)*x(19))
test2=(x(2)-x(3))-(resistance(2)*x(20))
test3=(x(3)-x(4))-(resistance(3)*x(21))

% Part 3
C4=C2*inv(R)*C1
A2 = [C4;C3];
b2 = [0;0;0;0;0;0;0;0;0;100;50;50;100;50;50;100;50;50];
x2 = A2\b2
r2 = (A2*x2)-b2;
for i = 1:18
r2(i)=(A2(i,:)*x2)-b2(i);
end

% Part 4
resistance(11)=10000000000000000;
res4=diag(resistance)*-1;
A = [C1 res4;(zeros(9,18)) C2;C3 (zeros(9,18))];
x=A\b
C4=C2*inv(res4)*C1;
A2 = [C4;C3];
b2 = [0;0;0;0;0;0;0;0;0;100;50;50;100;50;50;100;50;50];
x2 = A2\b2

% Part 5
resistance(11)=0;
res5=diag(resistance)*-1
A = [C1 res5;(zeros(9,18)) C2;C3 (zeros(9,18))];
x=A\b
C4=C2*inv(res5)*C1;
A2 = [C4;C3];
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b2 = [0;0;0;0;0;0;0;0;0;100;50;50;100;50;50;100;50;50];
x2 = A2\b2
```