Optimal Tiered Pricing

immediate

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Abstract

This report serves as a documentation of the process to define and solve a closed-form optimization problem for a seller selling a tiered product to a heterogeneous customer base. The goal is to build this problem in such a way it can be solved analytically, and use the analytical expression as the foundation for a feedback-based dynamic pricing controller that reaches optimal pricing while being blind to the population distribution.

1 INTRODUCTION

When selling a singular product, a seller must calculate the price that maximizes their profits - not too low such that their margins are tiny and not too high where customers don't buy their product. This is the ECON 101, basic pricing problem that most students have encountered at least once.

However, when a seller is marketing multiple products that compete with each other (such as in the example of most subscription services, or a coffee shop that sells multiple sizes) the seller needs to price their products where low end products are purchased by more frugal customers, and vice-versa. This is a nontrivial problem.

My goal is to consider a monopolistic seller offering multiple levels of a service who seeks to maximize their profit, selling to a heterogeneous customer base, with the ability to vary their prices over time.

2 Formulation

For starters, the company sells a product at b different tiers. For each of these tiers i, there is an associated cost to the company c_i, and a price they charge to the consumer p_i. A customer will be choosing a tier i to purchase, so we also can account for the decision to not purchase where the customer chooses tier 0. In total, the company manages a list of costs $C = c_0, c_1 \cdots c_b$ and a list of prices $P = p_0, p_1 \cdots p_b$, where c_0 and p_0 are 0.

To approach a formalization for this problem, we also should at the service from the perspective of a customer. In this representation, N customers make decisions based on their 'valuation parameter,' - a customer n is represented by the random variable v_n given by a probability distribution. This formulation is inspired directly by work in tiered ISP pricing by Lee, Jeong and Seo, where they use a Gaussian random variable to model their heterogeneous population base [1].

We can represent a customer's utility as benefits minus costs (to them). If we assume that the cost of the service to the company is proportional to the utility perceived by customers, the utility of customer n from service i can be represented by $u_{n,i} = c_i v_n - p_i$. However, customers tend to perceive a diminishing marginal return for services, proportional to the price of the base (in this case, the lowest) tier. Thus, we can redefine utility as $u_{n,i} = c_1 v_n (\frac{c_i}{c_n})^\lambda - p_i$, where λ is a constant between 0 and 1 representing the strength of diminishing marginal returns.

A customer will choose the tier that maximizes their utility, so the choice a customer makes can be given by $arg \max_{i=0}^{b} u_{n,i}$.

The company's profit from each customer for a given tier i is $p_i - c_i$, so the total profit for that tier is $(p_i - c_i) \sum_{n=0}^{N} \mathbb{1}_{i=arg \max_{j=0}^{b} u_{n,j}}$ where $\mathbb{1}$ is an indicator function that is 1 if the event is true and 0 otherwise.

From here, we can come to our expression for the total profit of the company:

$F = \sum_{i=0}^{B} (p_i - c_i) \sum_{n=0}^{N} \mathbb{1}_{i=arg \max_{j=0}^{b} u_{n,j}}$

This is the expression we need to maximize.

2.1 Simplification

It is very difficult to optimize the equation through the layer of abstraction in the argmax. So, we can instead seek to determine the probability that a user chooses a tier given a random variable v and a set of prices and costs. This can be used to calculate profit expectation, which can give us a theoretical expected profit for any number of users.

We can examine the utility function for each cost as a function of 'v', the valuation parameter.

We have:

$u_0(v) = 0$, $u_1(v) = c_1 v - p_1$, and generally $u_i(v) = c_1 v (\frac{c_i}{c_1})^\lambda - p_i$

These are linear equations, so let's graph them for three tiers, with $\lambda = \frac{4}{5}$, $c_1 = 1, c_2 = 3, c_3 = 5$, $p_1 = 2, p_2 = 5, p_3 = 8$.

Here is the plot for those coefficients:

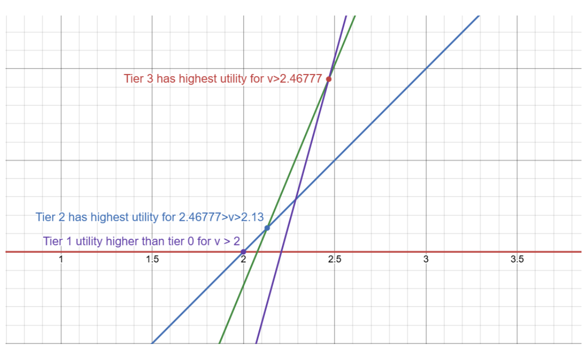

Figure 1: Utility vs Valuation Parameter. Red line graphs tier 0, Blue line graphs tier 1, Green line graphs tier 2, Purple line graphs tier 3

As we can see, as long as tiers are ordered in terms of increasing costs (thus increasing slopes), the intersection points between two tiers will mark the point where every valuation parameter higher than that intersection's valuation parameter belongs to the higher tier. We can use this to formulate a nearly complete generalization on a way to partition the valuation parameter space:

$t_0 = -\infty, t_i = \frac{p_i - p_{i-1}}{c_1(\frac{c_i}{c_1})^\lambda - c_1(\frac{c_{i-1}}{c_1})^\lambda}$ for $B >= i > 1$

To define an upper bound that can come in handy later, we could also claim that $t_{B+1} = \infty$

This construction, however, is wrong. This is because it fails to consider the edge case where a higher tier intersects a lower tier before the tier before it. For instance, in the previous scenario if p_1 was 2.1, the graph would look like this:

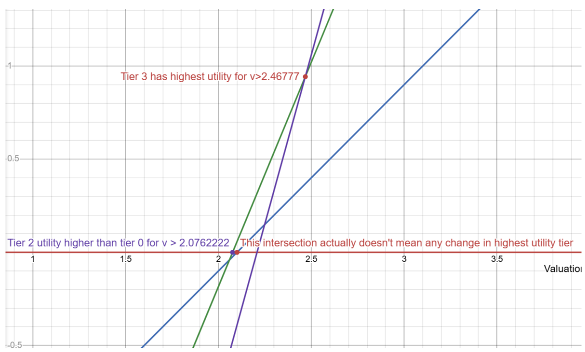

Figure 2: Utility vs Valuation Parameter with a slightly shifted tier 1. Red line graphs tier 0, Blue line graphs tier 1, Green line graphs tier 2, Purple line graphs tier 3

We can see here that the intersections don't necessarily mean a change in maximum, which leads to a frustrating problem of mathematically defining thresholds in a general case (this can easily be done with programming in O(n) time, but how to do it in a closed-form fashion?

Regardless, if we could calculate $t_{0 \cdots n}$, we could define the expectation of profit with a random variable v of a known distribution like so:

$E[F] = N \sum_{i=0}^{B} (p_i - c_i) Pr\{t_i < v < t_{i+1}\}$

3 Computerized Optimization

Even without a closed-form expression, it is trivial to write simulation for this problem to visualize and understand its scope. To do this,

References

1. Lee SH, Jeong HY, and Seo SW. Optimal pricing and capacity partitioning for tiered access service in virtual networks. Computer Networks 2013;57:3941–56.