Transportation and Assignment Problems

The transportation model is a special class of linear programs. It received this name because many of its applications involve determining how to optimally transport goods. However, some of its important applications (eg production scheduling) actually have nothing to do with transportation.

The second type of problem is assignment problem. It involves such applications as assigning people to tasks. Although its applications appear to be quite different from those for the transportation, we shall see the assignment problem can be viewed as a special type of transportation problem.

Application of the transportation and assignment problem tend to require a very large number of constraints and variables, so straightforward computer applications of simplex method may require an exorbitant computational effort. Fortunately, a key characteristic of these problems is that most of the aij coefficient in the constraints are zeros, and the relatively few non zero coefficient appear in a distinctive pattern. As a result, it has been possible to develop special streamlined algorithms that achieve dramatic computational savings by exploiting this special structure of the problem. Therefore, it is important to become sufficiently familiar with these special types of problems that you can recognise them when they arises and apply the proper computational procedure.

Transportation problem

As pointed out above that many applications of TP deals with transportation of a commodity from source to destination. Therefore I will use prototype example to explain the TP.

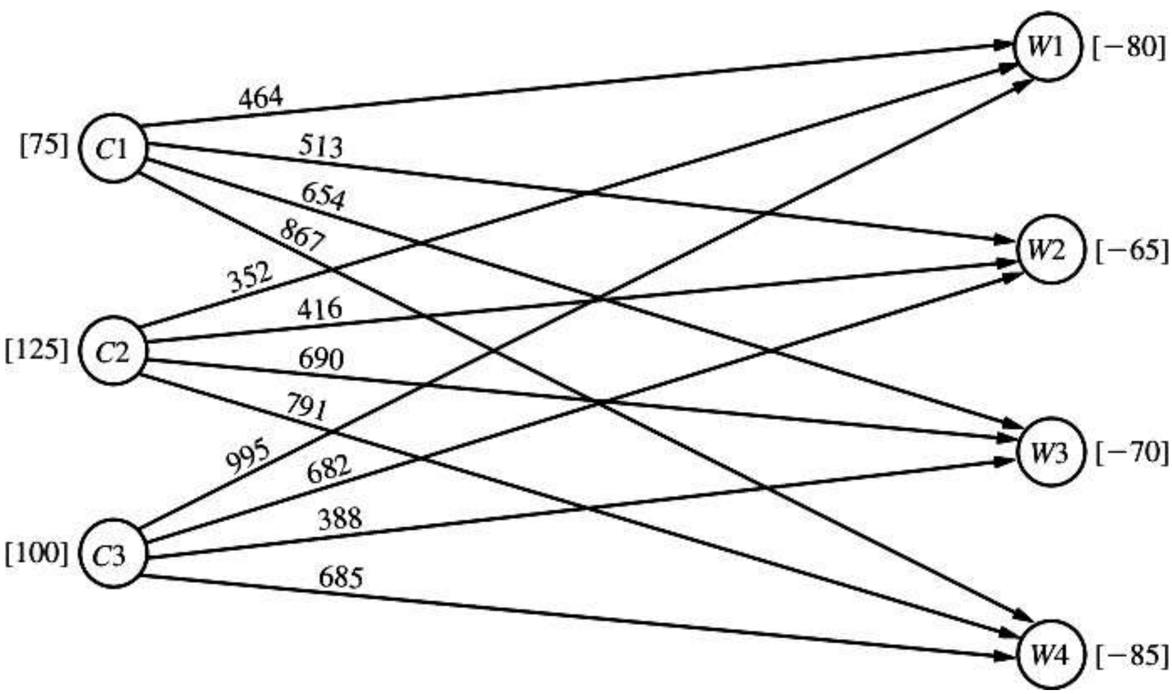
Example- Suppose XYZ Pvt Ld produce good X. The production take place at 3 place- S1, S2 and S3. From these production places, company supplies the X to its warehouses, which are located near its demand centre. The shipping cost or transportation cost from different production place to warehouses is given table 1

Table 1

	Shpping cost per unit				
	W1	W2	W3	W4	Output
S1	464	513	654	867	75
S2	352	416	690	791	125
S3	995	682	388	685	100
Allocation	80	65	70	85	

Si represents the production place(sources) and Wi represents the warehouses (destinations).

We can represent the above problem with network diagram as following



In above fig, the arrows shows the possible routes for transportation, where the number next to each arrow is the shipping cost per unit for that route. A square bracket net to each location gives the number of units to be shipped out of that location (so that the allocation into each warehouse is given as a negative number)

We can also represent the above problem in term of linear programming.

With above cost structure, the linear programming problem of the firm will be

$$
\begin{aligned}
\text{Minimize} \qquad Z = {} & 464x_{11} + 513x_{12} + 654x_{13} + 867x_{14} + 352x_{21} + 416x_{22} \\
& + 690x_{23} + 791x_{24} + 995x_{31} + 682x_{32} + 388x_{33} + 685x_{34},
\end{aligned}
$$

subject to the constraints

$$
\begin{array}{cccccccccccccccccccccccc}
x_{11} & + & x_{12} & + & x_{13} & + & x_{14} & & & & & & & & & & & & & & & & = & 75 \\
 & & & & & & & & x_{21} & + & x_{22} & + & x_{23} & + & x_{24} & & & & & & & & = & 125 \\
 & & & & & & & & & & & & & & & & x_{31} & + & x_{32} & + & x_{33} & + x_{34} & = & 100 \\
x_{11} & & & & & & & + & x_{21} & & & & & & & + & x_{31} & & & & & & = & 80 \\
 & & x_{12} & & & & & & & + & x_{22} & & & & & & & + & x_{32} & & & & = & 65 \\
 & & & & x_{13} & & & & & & & + & x_{23} & & & & & & & + & x_{33} & & = & 70 \\
 & & & & & & x_{14} & & & & & & & + & x_{24} & & & & & & & + x_{34} & = & 85
\end{array}
$$

and

$$x_{ij} \geq 0 \qquad (i = 1, 2, 3;\ j = 1, 2, 3, 4).$$

The table 2 shows the constraint coefficients. As I will explain in class, it is the special structure in the pattern of these coefficients that distinguishes this problem as a transportation problem, not its context.

Table 2

Coefficient of:

$$
\begin{array}{ccc}
 & \begin{array}{cccccccccccc} x_{11} & x_{12} & x_{13} & x_{14} & x_{21} & x_{22} & x_{23} & x_{24} & x_{31} & x_{32} & x_{33} & x_{34} \end{array} & \\
\mathbf{A} = & \left[\begin{array}{cccccccccccc}
1 & 1 & 1 & 1 & & & & & & & & \\
 & & & & 1 & 1 & 1 & 1 & & & & \\
 & & & & & & & & 1 & 1 & 1 & 1 \\
1 & & & & 1 & & & & 1 & & & \\
 & 1 & & & & 1 & & & & 1 & & \\
 & & 1 & & & & 1 & & & & 1 & \\
 & & & 1 & & & & 1 & & & & 1
\end{array}\right] & \begin{array}{l} \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} \text{Supply consts} \\ \left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right\} \text{Demand Consts.} \end{array}
\end{array}
$$

The supply constraints also called as Source constraints and the Demand constraints are also called as Warehouse constraints.

It becomes very clear from above example that coefficient table of transportation problem have very special structure. We can show the same pattern for general transportation problem with source i=1,2,…m and destination j=1,2,… n.

The unit cost of this general transportation problem will be

		cost per unit distributed				supply
		destination				
		1	2	...	n	
	1	c11	c12		c1n	S1
source	2	c21	c22		c2n	S2
	...					
	m	cm1	cm2		cmn	Sm
	demand	d1	d2		dn	

In above table c11 represents the unit cost of transportation from source 1 to destination 1. In the same way, c12 repesents the unit cost of transportation from source 1 to destination 2.

The mathematical formation for this problem will be

Min Z = $\sum_i \sum_j c_{ij} x_{ij}$

Subject to

$\sum_i x_{ij} = s_i$

$\sum_j x_{ij} = d_j$

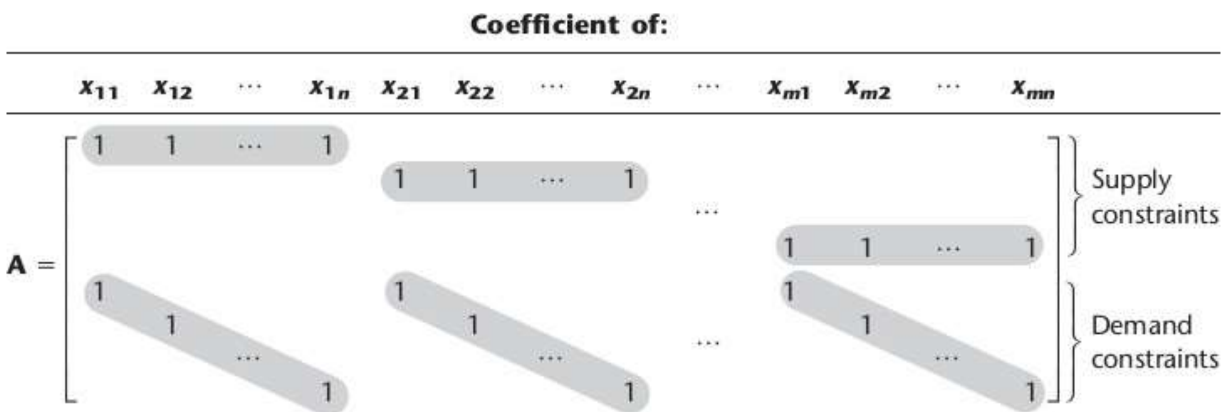
$x_{ij} \geq 0$ for all i and j.

In transportation model we make following two assumption.

- The requirement assumption- Each sources has a fixed supply of units, where this entire supply must be distributed to the destinations. Similarly, each destination has a fixed demand for units, where this entire demand must be received from the sources.
- The cost assumptions- the cost of distributing units from any particular source to any particular destination is directly proportional to the number of units distributed. Therefore, the cost is just the unit cost of distribution times the number of units distributed.

Note that the resulting table of constraint coefficients has the special structure shown in following table.

Table 3



So any problem (whether involving transportation or not) fits the model for a transportation problem if it can be described completely in terms of a parameter table like above table and it satisfies both the requirement assumptions and the cost assumption. The objective is to minimise the total cost of distributing the units. All the parameters of the model are included in this table.

Any linear programming problem that fits this special formulation is of the transportation problem type, regardless of its physical context.

<u>Solution Properties</u>

The feasible solution property- a transportation problem will have feasible solutions if and only if

$\sum si=\sum dj$ $\Rightarrow$ model is balanced.

i.e. total supply must be equal to total demand.

Integer solution property- for transportation problems where every si and dj have an integer value, all the basic variables (allocations) in every basic feasible solution (including an optimal one) also have integer values.

<u>Basic Terminology</u>

1. Rim Requirement- the supply and demand requirements at various source and destinations is called Rim requirement.
2. Feasible Solution- A feasible solution to a transportation problem is a set of non-negative allocations, xij, that satisfies the rim (row and column) restrictions.
3. Basic Feasible solution- A feasible solution is called a basic feasible solution if it contains no more than m+n-1 non negative allocations where m is the number of rows and n is the number of columns of the transportation problem.
4. Degenerated basic feasible- A basic solution that contains less than m+n-1 non-negative allocations in independent positions. The allocation are said to be independent positions, if it is not possible to form a closed path (loop) means by allowing horizontal and vertical lines and if all the corner cells are occupied. (explanation in class)
5. Non-degenerated basic feasible- A basic solution that contains exactly m+n-1 non-negative allocations is called degenerated basic feasible solution.
6. Optimal Solution- A feasible solution (non necessarily basic) is said to be optimal if it minimises (maximises) the transportation cost (profit).
7. Balance and Unbalance Transportation Problem-If the total demand is equal to total supply then TP is called balance TP, otherwise it is unbalance TP
8. Occupied and Unoccupied cells- the allocated cells in the transportation cells occupied cells and empty cells are called non-occupied cells.

The Transportation Algorithm

Algorithm- A process or set of rules to be followed in calculations or other problem solving operations

The transportation algorithm follows the exact steps of the simplex method. However, instead of using the regular simplex tableau, we take advantage of the special structure of the transportation model to organise the computations in a more convenient form.

Steps- the steps of the transportation algorithm are exact parallels of the simplex algorithm,

Step 1- Determine a starting basic feasible solution, and

Step 2- use the optimality condition of the simplex method to determine the entering variable from among all the non basic variables. If the optimality condition is satisfied, stop. Otherwise go to step 3

Step 3- use the feasibility condition of the simplex method to determine the leaving variable from among all the current basic variables, and find the new basic solution. Return to step 2.

Step 1: Determination of the starting solution

A general transportation model with m sources and n destinations has m+n constraint equations, one for each source and each destination. However, because the transportation model is always balanced (sum of the supply=sum of the demad), one of these equations is redundant. Thus, the model has m+n-1 independent constraint equations, which means that the starting solution basic solution consists of m+n-1 independent equations, which means that the starting basic solution consists of m+n-1 basic variables.

The special structure of the transportation problem allows securing a nonartificial starting basic solution using one of three methods

(1) Least cost method
(2) Vogel approximation method.

Above methods differ in the quality of the starting basic solution they produce, in the sense that a better starting solution yields a smaller objective value.

Least cost method

Step I write the transportation problem in a tabular form

Step II Assigns as much as possible to the cell with the smallest unit cost.

Step III the satisfied row or column is crossed out and the amount of supply and demand are adjusted accordingly. If both a row and a column are satisfied simultaneously, only one is crossed out.

Step IV Look for the uncrossed out cell with the smallest unit cost and repeat the process until exactly one row or column is left uncrossed out.

Vogel's approximation method (VAM)

VAM is an improved version of the least cost method that generally, but not always, produces more efficient starting solutions.

Step I Write the TP in the tabular form

Step II For each row (column), determine a penalty measure by subtracting the smallest unit cost element in the row (column) from the next smallest unit cost element in the same row (column)

Step III Identify the row or column with the largest penalty. Break ties arbitrarily. Allocate as much as possible to the variable with the least unit cost in the selected row or column. Adjust the supply and demand, and cross out the satisfied row or column. If a row and a column are satisfied simultaneously, only one of the two is crossed out, and the remaining row (column) is assigned zero supply (demand).

Step IV (a) if exactly one row or column with zero supply or demand remains uncrossed out, stop.

(b) if one row (column) with positive supply (demand) remains uncrossed out, determine the basic variable in the row (column) by the least cost method. Stop

(c) if all the uncrossed out rows and columns have (remaining) zero supply and demand,

Determine the zero basic variables by the least cost method. Stop

(d) Otherwise, go to step II

VAM is popular because it is relatively easy to implement by hand.

The penalty in VAM represents the minimum extra unit cost incurred by failing to make an allocation to the cell having the smallest unit in that row or column, this criterion does take costs into account in an effective way.

Numerical Example-----------------

Unbalanced Transportation Model

A necessary and sufficient condition for the existence of feasible solution to the general transportation problem is that the total demand must equal the total supply. However, sometimes there may be more demand than the supply and vice versa in which case the problem is said to be unbalanced. If may occur in the following situation

1. SS > DD
2. DD > SS

In case I, we introduce a dummy destination in the transportation table. The cost of transportating to this destination are all set equal to zero. The requirement at this dummy destination is then assumed to be equal to

SS – DD

In case 2, we introduce a dummy source in the transportation table. The cost of transporting from this source to any destinations is all set equal to zero. The availability at this dummy source is assumed to be equal to DD – SS.

<u>Step 2 and 3: Test of optimality and optimal solution</u>

Once an initial solution is obtained, the next step is to check its optimality. An optimal solution is one where there is no other set of transportation route (allocations) that will further reduce the total transportation cost. Thus we have to evaluate each unoccupied cell (represents unused route) in the transportation table in terms of an opportunity of reducing total transportation cost.

There are two methods to check optimality-

1. Stepping Stone Method
2. Modified distribution (MODI)

Stepping Stone Method

This is a procedure for determining the potential of improving upon each of the non-basic variables in terms of the objective function. To determine this potential, each of the non-basic variables is considered one by one. For each such cell, we find what effect on the total cost would be if one unit is assigned to this cell. With this information, then, we come to know whether the solution is optimal or not. If not, we improve that solution.

We can summarise the Stepping Stone method in following steps

1. Construct a transportation table with a given unit cost of transportation along with the rim conditions
2. Determine a initial basic feasible solution (allocation) using a suitable method as discussed earlier
3. Evaluate all unoccupied cells for the effect of transferring one unit from an occupied cell to the unoccupied cell. This transfer is made by forming a closed path that retains the SS and DD condition of the problem
4. Check the sign of each of the net change in the unit transportation costs. If the net changes are plus or zero, then the an optimal solution has been arrived at, otherwise go to step 5
5. Select the unoccupied cell with most negative net change among all unoccupied cells.
6. Assign an many units as possible to unoccupied cell satisfying rim conditions. The maximum number of units to be assigned are equal to the smaller circled number among the occupied cells with the minus value in a closed path.
7. Go to step 3, and repeat the problem until all unoccupied cells are evaluated and the net change result in positive or zero.

Numerical Example – in Class

Modified Distribution (MODI) method

MODI method is based on the concept of duality. To explain this, we will use prototype example.

Suppose the unit cost of transportation of XYZ company is given in following table.

		Destination		Supply
		W1	W2	
Source	A1	c11	c12	S1
	A2	c21	c22	S2
Demand		d1	d2	

The simplex formulation of this problem will be

Min $Z = c_{11}x_{11} + c_{12}x_{12} + c_{21}x_{21} + c_{22}x_{22}$

$$\begin{aligned} \text{s.t} \quad & x_{11} + x_{12} = s_1 \\ & x_{21} + x_{22} = s_2 \\ & x_{11} + x_{21} = d_1 \\ & x_{12} + x_{22} = d_2 \end{aligned}$$

For duality, introduce u_i for supply constraint and v_j for demand constraint.

Dual

Max $Z = u_1 s_1 + u_2 s_2 + v_1 d_1 + v_2 d_2$

$$\begin{aligned} \text{s.t.} \quad & u_1 + v_1 \leq c_{11} \\ & u_1 + v_2 \leq c_{12} \\ & u_2 + v_1 \leq c_{21} \\ & u_2 + v_2 \leq c_{22} \end{aligned}$$

so in case of general transportation model:

Min $\qquad Z = \sum_i \sum_j c_{ij} x_{ij}$

Subject to

$\sum_i x_{ij} = s_i$

$\sum_j x_{ij} = d_j$

$.x_{ij} \geq 0$ for all i and j

The dual will be

Min $\qquad Z = \sum_i u_i s_i + \sum_j v_j d_j \; u_i + v_j \leq$

c_{ij} for all i and ju_i , $v_j \geq 0$ for

all i and j

from economic interpretation, $u_i + v_j$ = implied cost (u_i denotes the location rent and vj denotes the market price) and c_{ij} = actual cost

The variable xij form an optimal solution to the given transportation provided

i. Solution xij is feasible for all (i,j) with respect to original transportation model,
ii. $(c_{ij} - u_i - v_j)$ xij =0 for all i and j. This condition is known as complementary slackness for transportation problem.

The last condition indicates that

a. If xij > 0 and is feasible then $c_{ij} - u_i - v_j = 0$
   so $c_{ij} = u_i + v_j$ for each occupied cell
b. If xij = 0 and is feasible then $c_{ij} - u_i - v_j \neq 0$
   so $c_{ij} \neq u_i + v_j$ for each occupied cell.

If $c_{ij \geq}$ $u_i + v_j$ i.e. actual cost ≥ implied cost, then it is not desirable to have xij ≥ 0 in the solution mix because it will increase transportation cost

If $c_{ij} \leq u_i + v_j$ i.e. actual cost ≤ implied cost, then it is desirable to have xij ≥ 0 in the solution mix because it will decrease transportation cost.

Define dij = c_{ij} -($u_i + v_j$) for all i and j.

So if dij ≤ 0 then solution is not optimal. In sum, we follow following steps in MODI method

1. For an initial basic feasible solution with (m+n-1) occupied cells, calculate ui and vj for row and columns

   To start with, any one of ui's and vj is assigned the value zero. Then complete the calculation of ui's and vj's for other rows and columns by using the relation

   $c_{ij} = u_i + v_j$ for all occupied cells
2. For unoccupied cells, calculate opportunity cost dij by using the relationship dij = c_{ij} –($u_i + v_j$) for all i and j
3. Examine sign of each dij
   i. If dij > 0 then current basic feasible solution is optimal
   ii. If dij = 0 then current basic feasible solution will remain unaffected but an alternative solution exist
   iii. If one or more dij < 0 the solution is not optimal.
   iv.
4. Reallocation of units by allocating units to the unoccupied cell, and calculate the new transportation cost.
5. Test the revised solution for optimality. The procedure terminates when all dij ≥ 0 for unoccupied cells.

Numerical example---- in class

Assignment Problem

The AP is a special type of LPP where assignees are being assigned to perform task. For example, the assignees might be employees who need to be given work assignments. However, the assignees might not be people. They could be machines or vehicles or plants or even time slots to be assigned tasks.

To fit the definition of an assignment problem, the problem need to formulate in a way that satisfies the following assumptions

1. The number of assignees and the number of tasks are the same
2. Each assignees is to be assigned to exactly one task
3. Each task is to performed by exactly one assignee
4. There is a cost cij associated with assignee i performing task j.
5. The objective is to determine how all n assignments should be made to minimise the total cost.

Any problem satisfying all these assumptions can be solved extremely efficiently by algorithm designed specifically for assignment problem.

Mathematical form

The mathematical model for the assignment problem uses the following decision variables

Xij = 1 if assignee(worker) i perform task j

=0 if not.

Thus each xij is a binary variable (it has value 0 or 1). Lets Z denots the total cost, the assignment problem model is

Min Z= $\sum_i \sum_j c_{ij} x_{ij}$

Subject to

$\sum_i x_{ij}=1$ for i= 1. ....n

$\sum_j x_{ij}=1$ for j=1 ....n

$X_{ij} \geq 0$ for all i and j.

The first set of functional constraints specifies that each assignee is to perform exactly one task, whereas the second set requires each task to be performed by exactly one assignee.

TP & AP

In comparison to TP, in AP we have following restrictions

1. Number of sources (m) = number of destinations (n)
2. Each supply (S_i) = 1
3. Each demand (d_i) = 1

Because assignment model is special case of transportation model so we can solve assignment problem directly as regular transportation model. Nevertheless, the fact that all the supply and demand amounts equal 1 has led to the development of a simple solution algorithm called the **Hungarian Method**.

The use of transportation Model technique in assignment model has following drawbacks-

1. Wasted iterations- because in assignment model, the number of basic variable is n and all basic variables are binary variable so there always are n-1 degenerate basic variables. As we know that degenerate basic variable do not cause any major complication in the execution of the algorithm. However they do frequently cause wasted iterations.
2. Transportation simplex method is purely a general purpose algorithm for solving all transportation problems. Therefore, it does nothing to exploit the additional special structure in this special type of transportation problem (n=m, si=1 and di=1).

Hungarian Method

Hungarian Method is for assigning jobs by a one-for-one matching to identify the lowest-cost solution. Each job must be assigned to only one machine. It is assumed that every machine is capable of handling every job, and that the costs or values associated with each assignment combination are known and fixed. The number of rows and columns must be the same.

1. Subtract the smallest number in each row from every number in the row. This is called row reduction

2. Subtract the smallest number in each column of the new table from every number in the column. This is called column reduction.
3. Test whether an optimal assignment can be made. You do this by determining the minimum number of lines to cover all zeros. If the number of lines equals the number of rows, an optimal set of assignment is possible. Otherwise go on to step 4
4. If the number of lines is less than the number of rows, modify the table in the following way
   (a) Subtract the smallest uncovered number from every uncovered number in the table
   (b) Add the smallest uncovered number to the numbers at intersections of covering lines
   (c) Numbers crossed out but at the interactions of cross out lines carry over unchanged to the next table
5. Repeat step 3 and 4 until an optimal set of assignments is possible.
6. Make the assignments one at a time in positions that have zero elements. Begin with rows or columns that have only one zero. Since each row and each column needs to receive exactly one assignment, cross out both the row and the column involved after each assignment is made. Then move on to the rows and such row or column that are not yet crossed out to select the next assignment, with preference again given to any such row or column that has only one zero that is not crossed out. Continue until every row or column has exactly one assignment and so has been crossed out.