Aspects of Quantum Information in Curved Spacetimes

A Project Report Submitted
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by

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Thiruvananthapuram - 695 551 **[Type Your Name]**

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ABSTRACT

The main aim of the project ....

Keywords: [Insert Keywords]

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Chapter 1

Introduction

General relativity has been an enormously successful theory of gravity. Discovered by Albert Einstein more than a century ago, general relativity has withstood the test of time. Not only has it successfully predicted minute corrections to the earlier accepted Newton's law of gravity, but has also predicted a plethora of new physical phenomena. These include the well tested gravitational lensing, gravitational time dilation, and the recently confirmed gravitational waves. The theory is a radical departure from how we previously conceived nature. Instead of a direct "force" of gravity between two objects, general relativity puts forth the idea that mass and energy curve the background spacetime and naturally move in geodesics dictated by this curvature.

However, whenever ill-defined things in the theory, such as singularities, are discovered, the faith in the theory's complete validity that it instills in us through its accurate predictions and stunning insights slips away. This largely happens because general relativity is a classical theory. It is incompatible with the even more strongly accepted quantum mechanical view of nature. Thus, it is natural that the theory breaks down at distances where one expects quantum gravitational effects become significant (as is the case near singularities when one approaches the Planck scale).

In the absence of a complete theory of quantum gravity, one can still use semi-classical methods to make predictions as long as one stays away from Planck scales. In these methods, one uses the usual quantum mechanics or quantum field theory assuming that the background is described classically. Although not fully correct, semi-classical formulations offer a good starting point.

On the other hand, quantum information theory has lately become a popular area of study. The increasing interest in developing quantum computing hardware and manipulating quantum information to perform tasks that are classically difficult/impossible is one of the key factors driving interest in this subject. Besides its pragmaticity, it would be anyway rather interesting to study whether gravity, being a fundamental force in nature, has any role to play in the manipulation of quantum information. It is the inquiry of aspects of this question that form the contents of this thesis.

It was recently shown that gravity indeed causes decoherence in center of mass superpositions of composite systems (like atoms, molecules, etc) [1]. In the former part of thesis, we work to cement the role of gravity in causing this decoherence using a fully-covariant formulation. In the latter of the thesis, we turn to a more fundamental issue of the black hole information problem. We introduce how the problem originally came about starting with a detailed derivation of Hawking radiation. Following this, we discuss the Page curve that forms an important development in this problem. We end by briefly alluding to recent directions in this area of research.

The thesis is organized as follows. In chapter 2, we present a simple and instructive model of decoherence put forth by De-Witt. In chapter 3, we turn to a review of ref. [1] and the works that followed discussing it [2, 3, **?**]. Chapters 4 and 5 form the bulk of the original work put forth by this thesis. They deal with incorporating tidal effects into to the phenomena of decoherence. Following this, we shift gears to the black hole information problem in chapters 6 and 7. These chapters are purely an appraisal of the relevant literature and contain detailed proofs of Hawking radiation and Page's theorem. We finally conclude in chapter 8.

Chapter 2

Decoherence: A Simple Model

2.1 Introduction

We present a simplified decoherence model first given by De-Witt [4]. Consider a massive body moving in one dimension under an arbitrary potential V. Let it collide with a light body that is moving freely. One could model the collision as a δ-function interaction. The Hamiltonian of the combined system becomes

$$\hat{H} = \frac{\hat{P}^2}{2M} + \hat{V}(X) + \frac{\hat{p}^2}{2m} + g\delta(x - X) \tag{2.1}$$

where m and M are the masses with canonical variables (x, p) and (X, P) respectively. g quantifies the strength of the interaction between them. As stated, we consider the case with $M >> m$. Naturally, we expect the state of the massive body to be undisturbed after the collision with the light body. The state of the massive body can be quite arbitrary. However, we assume that the velocity states into which it can be decomposed correspond to velocities that are small compared to the velocity of the light body. We can write the Schrödinger equation for the combined system. Let $\Psi(x, X, t)$ be the wave function of the combined system in the position basis.

Since the massive body is virtually unaffected, this can be decomposed into

$$\Psi(x, X, t) = \phi(x, X, t)\psi(X, t) \tag{2.2}$$

where ϕ refes to the wave function for the light body and ψ for the massive body. Note that there is an effect of the interaction on the light body. The schrödinger equation then reads

$$\frac{\psi}{2m}\frac{\partial^2\phi}{\partial x^2} + \frac{1}{2M}\frac{\partial^2\phi\psi}{\partial X^2} + V(X)\phi\psi + g\delta(x - X)\phi\psi = E'\phi\psi \tag{2.3}$$

with E' being the energy of the combined system. It is easy to see that E'= $\frac{p^2}{2m} + \frac{P^2}{2M} + V$ where p and P are eigenvalues of the momentum operator. Multiplying both sides by m and neglecting the terms with $m/M << 1$, we get

$$\frac{1}{2m}\frac{\partial^2\phi}{\partial x^2} + g\delta(x - X)\phi = E\phi \tag{2.4}$$

where $E = p^2/2m$. The above equation simply means that the motion of the light body can be described in terms of scattering by a fixed δ-function potential. This can be easily solved using standard methods.
For $x < X$, the wave function ϕ can be taken as

$$\phi = Ae^{ip(x-X)} + Be^{-ip(x-X)} \tag{2.5}$$

and for $x > X$,

$$\phi = Fe^{ip(x-X)} \tag{2.6}$$

with

$$\frac{B}{A} = R = \frac{-img}{p(1 + i\frac{mg}{p})} \qquad\qquad \frac{F}{A} = T = \frac{1}{1 + i\frac{mg}{p}} \tag{2.7}$$

where R and T are reflection and transmission coefficients for the δ-function potential. They satisfy $|R|^2 + |T|^2 = 1$. We can now write the wave function of the combined system Ψ in the position basis as

$$\langle X, x, t|\Psi\rangle = L^{-1/2}\big\{\theta(X-x)(e^{ipx}+Re^{ip(2X-x)})+\theta(x-X)Te^{ipx}\big\}e^{-itp^2/2m}\otimes\langle X, t|\psi\rangle \tag{2.8}$$

where $\theta(x)$ is the usual step function and L is the length of an effective "box" controlling the normalization of the momentum wave functions of the light body.

2.2 Density Operator and Localization

We note that the first part of the combined system wave function that is attributed to the light body has in it the coordinate dependent e^{i2pX} referring to the massive body. This represents the effect of the collision (momentum transfer) to the massive body. Although it seems like the massive body is unaffected since the massive body continues to be described by its native wave function $\psi(X, t)$, its role in decoherence is not as trivial. To see this, we construct the density operator $\hat{\rho}$ of the massive body by tracing out the light body degrees of freedom:

$$\langle X, t|\,\hat{\rho}\,|X', t\rangle = \lim_{L\to\infty}\int_{-L/2}^{L/2}\langle X, x, t|\Psi\rangle\,\langle\Psi|X', x, t\rangle\,dx \tag{2.9}$$

We can expand this out as

$$\begin{aligned}\langle X, t|\,\hat{\rho}\,|X', t\rangle = \lim_{L\to\infty}L^{-1}\int_{-L/2}^{L/2}\big\{\theta(X-x)(e^{ipx}+Re^{ip(2X-x)})+\theta(x-X)Te^{ipx}\big\}\\ \times\big\{\theta(X'-x)(e^{-ipx}+R^{*}e^{-ip(2X-x)})+\theta(x-X')T^{*}e^{-ipx}\big\}\otimes\langle X, t|\psi\rangle\,\langle\psi|X', t\rangle\end{aligned} \tag{2.10}$$

We drop the $\langle X,t|\psi\rangle\langle\psi|X',t\rangle$. It can be restored later. Multiplying the terms gives

$$\lim_{L\to\infty} L^{-1}\int_{-L/2}^{L/2}\Big\{\theta(X-x)\theta(X'-x)(e^{ipx}+Re^{ip(2X-x)})(e^{-ipx}+R^{*}e^{-ip(2X-x)})+$$
$$\theta(X-x)\theta(x-X')(e^{ipx}+Re^{ip(2X-x)})T^{*}e^{-ipx}+\theta(x-X)\theta(X'-x)Te^{ipx}(e^{-ipx}+R^{*}e^{-ip(2X-x)})+$$
$$\theta(x-X)\theta(x-X')|T|^{2}\Big\}dx \quad (2.11)$$

Notice that $\theta(X-x)\theta(X'-x)$ has support on $[-L/2,\min(X,X')]$. It is easy to similarly identify the support domain for the other θ compositions. As $L\to\infty$, only the first and the fourth term in the above expansion are non-zero. It is easy to see that the rest go to 0 since they give a convergent integral divided by L which diverges. The non-zero contributions add to give

$$\langle X,t|\,\hat{\rho}\,|X',t\rangle = \frac{1}{2}\left(1+|R|^{2}e^{2ip(X-X')}+|T|^{2}\right)\langle X,t|\psi\rangle\langle\psi|X',t\rangle \quad (2.12)$$

This can be simplified by substituting the values of R and T from (2.7) giving

$$\langle X,t|\,\hat{\rho}\,|X',t\rangle = \langle X,t|\psi\rangle\langle\psi|X',t\rangle\, M_p(X-X') \quad (2.13)$$

where

$$M_p(X-X') = \left[1+\left(\frac{gm}{p}\right)^{2}\right]^{-1}\left[1+\left(\frac{gm}{p}\right)^{2}e^{ip(X-X')}\cos p(X-X')\right] \quad (2.14)$$

Therefore, with the light body traced out, the density operator is no longer that of a pure state. It is multiplied by a modulation function (see figure()). Suppose now the massive body is allowed to collide with N identical bodies, all in the same momentum state and having identical δ-function interactions with the massive body. The final wave function will have terms for each light body and when these are traced out as above, we will get

$$\langle X,t|\,\hat{\rho}\,|X',t\rangle \xrightarrow[L\to\infty]{} \langle X,t|\psi\rangle\langle\psi|X',t\rangle\, E(X-X') \quad (2.15)$$

where

$$E(X - X') = \left[M_p(X - X')\right]^N \tag{2.16}$$

$E(X - X')$ is the *environmental modulation function*. For $N = 20$, we can see the function in figure. As one notices, localization begins to show itself. However, the function repeats itself and the localization is only to modulo π/p. A more honest environmental modulation function is when the massive body collides with N_1 light bodies with momentum p_1, N_2 light bodies with momentum p_2, and so on where we can choose p's to be incommensurable. Expectedly, the environmental modulation function now looks like

$$E(X - X') = \left[M_{p_1}(X - X')\right]^{N_1}\left[M_{p_2}(X - X')\right]^{N_2}... \tag{2.17}$$

We can see the absolute value of the function in the figure. As N grows, $E(X - X')$ becomes a narrow function and one can begin to see localization of the massive body to a point. This is essentially because the cosines are all of different frequencies and amplitude. They roughly cancel out except around the origin where they all have the same value close to unity.

2.3 Emergence of Classicality

There is a more sophisticated approach using decoherence functional which not only gives rise to localization but also accounts for the emergence of classicality. This formalism can be applied to the model we have above. We begin by giving a few definitions. Details can be found in the reference[5] .

We first introduce a set of projection operators that define a *coarse graining* of the possible dynamical histories of the massive body. For example,

$$P_\epsilon(\bar{X}, t) := \int_{\bar{X}-\frac{1}{2}\epsilon}^{\bar{X}+\frac{1}{2}\epsilon} |X, t\rangle \langle X, t| \, dX \tag{2.18}$$

where ϵ defines the coarseness of the graining. $\bar{X}$ is chosen from a discrete set of points, separated by intervals ϵ and then these projectors, at a given time, are mutually orthogonal.

Using the projection operators one may define the *decoherence functional* as

$$\begin{aligned} &D(...\bar{X}_1\bar{X}_2\bar{X}_3|\bar{X}_1'\bar{X}_2'\bar{X}_3'...) \\ &:= \mathrm{tr}\big[...P_\epsilon(\bar{X}_1,t_1)P_\epsilon(\bar{X}_2,t_2)P_\epsilon(\bar{X}_3,t_3)\hat{\rho}P_\epsilon(\bar{X}_1',t_1)P_\epsilon(\bar{X}_2',t_2)P_\epsilon(\bar{X}_3',t_3)...\big] \end{aligned} \tag{2.19}$$

$$= \int_\infty^\infty dX\,\langle X,t|\,...P_\epsilon(\bar{X}_1,t_1)P_\epsilon(\bar{X}_2,t_2)P_\epsilon(\bar{X}_3,t_3)\hat{\rho}P_\epsilon(\bar{X}_1',t_1)P_\epsilon(\bar{X}_2',t_2)P_\epsilon(\bar{X}_3',t_3)...\,|X,t\rangle \tag{2.20}$$

The time t in the final equation is arbitrary. However, we assume that the times $t_1, t_2,$ and t_3 are in chronological or anti-chronological order and fixed *a priori.*

The diagonal elements are real and positive and have a convenient interpretation. They give the probability that the trajectory of the massive body will be observed, at times $t_1, t_2,$ and t_3, to be within $\epsilon/2$ distance of the points $\bar{X}_1, \bar{X}_2, \bar{X}_3$,...at those times.

We now take only three instants of time $t_1, t_2,$ and t_3 for the sake of demonstration of locality and emergence of classicality. We consider the semi-classical limit of the Feynmann propogator

$$\langle X,t|X',t'\rangle \approx \left[\frac{i}{2\pi}\frac{\partial^2 S(X,t|X',t')}{\partial X\partial X'}\right]e^{iS(X,t|X',t')} \tag{2.21}$$

where $S(X,t|X',t')$ is the action along the classical trajectory defined by $X, t, X',$ and t' and we have set $\hbar = 1$. Evaluating the diagonal elements $D(\bar{X}_3,\bar{X}_2,\bar{X}_1|\bar{X}_1,\bar{X}_2,\bar{X}_3)$

using the definitions (2.18)-(2.20), we get

$$D(\bar{X}_3, \bar{X}_2, \bar{X}_1|\bar{X}_1, \bar{X}_2, \bar{X}_3)$$
$$= \int_{\infty}^{\infty} \int_{\bar{X}_1-\epsilon/2}^{\bar{X}_1+\epsilon/2} \int_{\bar{X}_2-\epsilon/2}^{\bar{X}_2+\epsilon/2} \int_{\bar{X}_3-\epsilon/2}^{\bar{X}_3+\epsilon/2} dX dX_1 dX_2 dX_3 \left(\frac{i}{2\pi}\right)^3 \frac{\partial^2 S(X,t|X_3,t_3)}{\partial X \partial X_3}$$
$$\times \frac{\partial^2 S(X_3,t_3|X_2,t_2)}{\partial X_3 \partial X_2} \frac{\partial^2 S(X_2,t_2|X_1,t_1)}{\partial X_2 \partial X_1} \langle X_1, t_1| \hat{\rho} |X_1, t_1\rangle \quad (2.22)$$

The phases cancel out as we get square of the absolute values of the inner products $|\langle X_i, t_i|X_j, t_j\rangle|^2$. We see that the diagonal elements vanish unless $\bar{X}_2$ at time t_2 lies within the distance of order ϵ from the classical trajectory defined by X_1, t_1 and X_3, t_3. Therefore, the probability that the trajectory of the massive body departs from the classical is negligibly small.

Now consider $D(\bar{X}_1\bar{X}_2\bar{X}_3|\bar{X}_1'\bar{X}_2'\bar{X}_3')$, which are the off-diagonal elements of the decoherence functional. We demonstrate localization by showing that the off-diagonal terms go to 0. We begin by expanding the trace definition of the decoherence functional

$$D(\bar{X}_1\bar{X}_2\bar{X}_3|\bar{X}_1'\bar{X}_2'\bar{X}_3') = \mathrm{tr}\big[P_\epsilon(\bar{X}_3,t_3)P_\epsilon(\bar{X}_2,t_2)P_\epsilon(\bar{X}_1,t_1)\hat{\rho}P_\epsilon(\bar{X}_1',t_1)P_\epsilon(\bar{X}_2',t_2)P_\epsilon(\bar{X}_3',t_1)\big] \quad (2.23)$$

By the cyclicity of trace and the projection operators being (approximately) orthogonal, we get

$$D(\bar{X}_1\bar{X}_2\bar{X}_3|\bar{X}_1'\bar{X}_2'\bar{X}_3') = \mathrm{tr}\big[P_\epsilon(\bar{X}_2,t_2)P_\epsilon(\bar{X}_1,t_1)\hat{\rho}P_\epsilon(\bar{X}_1',t_1)P_\epsilon(\bar{X}_2',t_2)P_\epsilon(\bar{X}_3,t_3)\big]\delta_\epsilon(X_3-X_3'). \quad (2.24)$$

where $\delta_\epsilon(X_3 - X_3')$ is an approximate delta function with a vanishingly small spread of ϵ resulting from the coarse graining definition of the projection operators. Thus we see that $X_3 = X_3'$ in order for the off-diagonal elements to not vanish. Moreover, if we now use the integral expansion of the above, we get the expression $\langle X_1, t_1| \rho |X_1', t_1\rangle$. Using equation (2.15) we see that this will go to zero unless $X_1 = X_1'$ because the environmental modulation function has support only at the origin in the limiting case of large particles. Furthermore, it is also easy to see that $X_2 = X_2'$ for the integral to be non-zero.

Therefore, the matrix defined by the decoherence functional is essentially diagonal. This is a true sign of decoherence. Even the diagonal terms vanish unless the points $(\bar{X}_1, t_1), (\bar{X}_2, t_2), (\bar{X}_3, t_3)$... lie within a distance ϵ of a classical trajectory. This is a result of the coarse graining we observe in the real world which consists of dividing the configuration space of the massive body into intervals of certain widths and tracing out the light body degrees of freedom.

Chapter 3

Decoherence due to Gravitational Time Dilation

3.1 Introduction to PZCB 2015

The interplay between the background spacetime and quantum systems they harbour is typically studied in extreme physical conditions, namely, at high energies and strong gravitational fields. However, it has become evident in the past decade or so that the interplay between general relativity and low-energy quantum systems is interesting in its own right [6, 7, 8, 9, 10, 1, 11]. This regime can be well approximated by the framework of *relativistic* quantum mechanics in first quantization. General relativistic effects have been discussed in these works within this framework. However, since general relativity is fundamentally classical, one has to be careful while including its effects in quantum frameworks.

Of the works that have featured in this area, Pikovski et al.'s 2015 paper [1] is of interest to us considering the conceptual subtleties involved in their proposal. We shall briefly review their work along with some of the ensuing comments and debates in this chapter. In this paper, PZCB[1] [1] consider low-energy quantum phenomena in the presence of gravitational time dilation and show that the latter causes decoherence of quantum superpositions. For this, they begin by considering a simplified model of a composite particle having $N/3$ constituents that are modelled as independent three-dimensional harmonic oscillators. Such a model then describes N internal harmonic modes of the particle. The total Hamiltonian for the system is taken as $H = H_{\rm cm} + H_0 + H_{\rm int}$, where $H_{\rm cm}$ is some Hamiltonian for the centre of mass of the particle, H_0 is the Hamiltonian associated to the internal modes, and $H_{\rm int}$ is the Hamiltonian for the gravitational interaction of internal modes of the composite particle. They take $H_{\rm int}$ to be

$$H_{\rm int} = \Phi(x)\frac{H_0}{c^2} = \hbar\frac{gx}{c^2}\left(\sum_{i=1}^{N}\omega_i n_i\right) \tag{3.1}$$

where $\Phi(x) = gx$ is the gravitational potential. The above expression is reasoned as follows. According to Einstein's gravity, any measure of energy couples to gravity. If one considers a particle of mass m with H_0 being the Hamiltonian that generates the time evolution of internal energy modes, gravity couples to the total mass (energy) $m + H_0/c^2$. They clarify that this separation between rest mass and rest energy is dictated by the scale one is working at. At a given scale, the degrees of freedom that do not contribute to the dynamics or are "frozen" can be included in m.

Following this, they write the interaction with the gravitational potential Φ is as $(m + H_0/c^2)\Phi$ (to lowest order in c^{-2}). The $m\Phi$ term is the usual 'mgh'-type term that we absorb in $H_{\rm cm}$ leaving out H_{int} as written in equation (3.1). They consider the particle to be at rest with its center of mass in superposition at two vertically distinct positions x_1 and x_2 with a difference $\Delta x = x_2 - x_1$. Thus, the center of mass at time $t = 0$ is in the state

$$|\psi_{cm}(0)\rangle = \frac{1}{\sqrt{2}}(|x_1\rangle + |x_2\rangle) \tag{3.2}$$

[1] This is the authors' collective abbreviation used in papers commenting on their work

The internal degrees of freedom are in thermal equilibrium at local temperature T. Consequently, each i^{th} constituent is described by a thermal density matrix

$$\hat{\rho}_i = (\pi \bar{n}_i)^{-1} \int \mathrm{d}^2\alpha \, \exp(-|\alpha_i|^2/\bar{n}_i) \, |\alpha_i\rangle \langle\alpha_i| \, , \tag{3.3}$$

written in the coherent state representation with average excitation $\bar{n}_i = (e^{\hbar\omega_i/k_B T} - 1)$. The total initial state can then be given as

$$\hat{\rho}_0 = |\psi_{cm}\rangle \langle\psi_{cm}| \otimes \prod_{i=1}^{N} \hat{\rho}_i \tag{3.4}$$

Note that there is no entanglement between the internal degrees of freedom and the center of mass degrees of freedom to begin with. The time evolution operator can be written as $\hat{U} = -it/\hbar(H_{\text{cm}} + H_0 + H_{\text{int}})$. Since we are interested in interference of the system, we focus on the cross term ρ_{12} of the density matrix. After evolving the system for time t, we get

$$\rho_{12}(t) = (\hat{U}\rho_0\hat{U}^{\dagger})_{12} \tag{3.5}$$

$$= \frac{1}{2\pi\bar{n}_i} e^{imgΔxt/\hbar} \prod_{i=1}^{N} \int \mathrm{d}^2\alpha \, \exp(-|\alpha_i|^2/\bar{n}_i) \left|\alpha_i e^{-i\omega_i(x_1)t}\right\rangle \left\langle\alpha_i e^{-i\omega_i(x_2)t}\right| \tag{3.6}$$

where $\omega_i(x) = \omega_i(1+gx/c^2)$. The above expressions elicits the positional dependence of the frequencies of the internal modes. Since our system of interest is that of the center of mass, we trace out the internal degrees of freedom to get $\rho_{12}^{cm}(t)$, the cross term for the density matrix for the center of mass degrees of freedom. A good measure for quantum coherence is visibility defined as $V(t) = 2|\rho_{12}(t)|$. Evaluating visbility in this case gives

$$V(t) = \prod_{i=1}^{N} \left(1 + \bar{n}_i(1 - e^{-i\omega_i gt\Delta x/c^2})\right) \tag{3.7}$$

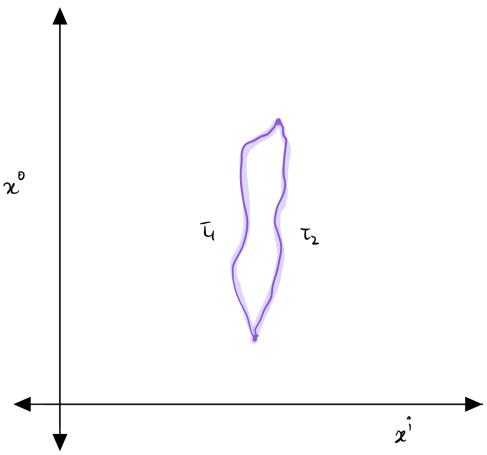

Figure 3.1: A composite particle in superposition of two semi-classical paths in spacetime with an arbitrary metric. Each path accumulates its own proper time. The paths interfere when the world lines meet, with the visibility being a function of the accumulated proper time difference.

Simplifying for the case $\omega_i g\Delta x/c^2 \ll 1$ and $\bar{n}_i \approx k_B T/\hbar\omega_i$ (in the high-termperature limit), we get

$$V(t) \approx e^{-(t/\tau_{\mathrm{dec}})^2} \tag{3.8}$$

for times $t^2 \ll N\tau_{dec}^2$ with

$$\tau_{dec} = \sqrt{\frac{2}{N}}\frac{\hbar c^2}{k_B T g \Delta x}$$

defining the decoherence time scale. Note that the above time scale contains the constants $\hbar, c, k_B,$ and g. Therefore, this can be considered a quantum, relativistic, gravitational, and thermodynamic effect.

Following this, PZCB go on to show that the effect is general and originates from the total proper time difference between superposed world lines. If we consider a particle moving with its center of mass in superposition along two world lines, each of the superposed paths accumulate their own proper times. This causes a difference in evolution along the two paths and causes them to interfere once the world lines meet (see figure). The world lines here are assumed to be semi-classical, that is,

having coordinates $\{\bar{x}_1, \bar{p}_1\}$ and $\{\bar{x}_2, \bar{p}_2\}$.
Mathematically, the amplitude of each path evolves according to $U = \exp[i \int dt(H_{\rm cm} + H_0(1 + \Gamma/c^2))]$, where $\Gamma = \Phi(x) - p^2/2m^2$ accounts for both special relativistic and general relativistic time dilation. Visibility is then written as

$$V = \mathrm{Tr}\left[e^{-i\int dt(H_{\rm cm}+H_0(1+\Gamma(\bar{x_1},\bar{p_1})/c^2))}\rho_0 e^{-i\int dt(H_{\rm cm}+H_0(1+\Gamma(\bar{x_2},\bar{p_2})/c^2))}\right] \tag{3.9}$$

Since $d\tau = dt\sqrt{g_{\mu\nu}\dot{x}_\mu\dot{x}_\nu} \approx= dt(1 + \Gamma/c^2)$, the visibility is evaluated as

$$V = \left|\langle e^{-iH_0\Delta\tau}\rangle\right| \tag{3.10}$$

where $\Delta\tau = \tau_1 - \tau_2$ is the proper time difference between the two world lines of the center of mass modes in superposition. The expectation value is taken with respect to the initial state.
Following this, PZCB go on to derive a master equation that describes the dynamics of the composite system in a background spacetime (to lowest order in c^{-2}). The rest of the paper then talks about the nature of the decoherence caused by this mechanism. As per the work in this thesis, our limited review until the derivation of visibility is sufficient. We now look at certain debates and comments that ensued this work.

3.2 Comments on PZCB 2015

There were numerous comments questioning the validity of the proposition put forth by PZCB. For the purposes of this synopsis, we briefly describe a few relevant ones. Bonder, Okon, and Sudarski [**?**, 12] pointed out that the results cannot be right in the light of the equivalence principle since in an accelerated frame with no gravitational field, there is nothing to attribute the decoherence to. PZCB do not include any curvature effects in their analysis. PZCB responded by stating that the decoherence studied in their work is attributed to time dilation — which does not contradict the equivalence principle [13].

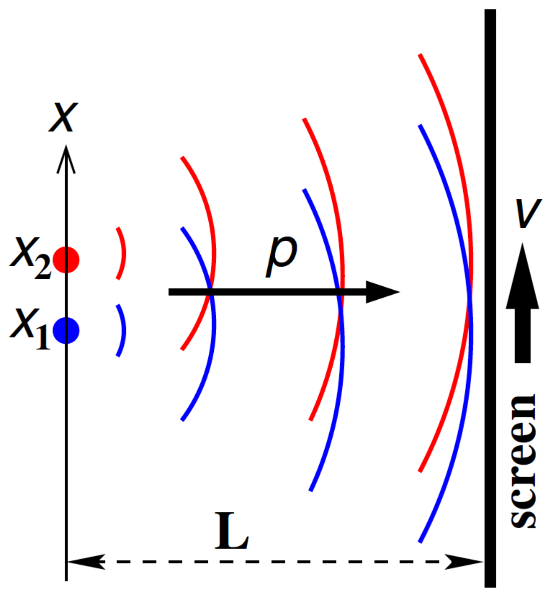

Figure 3.2: Diagram representing the experiment considered in [3]. Figure credits: [2]

Next, Pang et al. [3] (see figure 3.3) questioned the physical interpretation of the effect when one considers the equivalence principle. If one accepts the effect in [1], then they are lead to believe that according to the equivalence principle, the motion of an accelerated observer affects the evolution of the system so as to cause decoherence. This requires clarification that [3] provide. They work in a Lorentz frame and separate out the effects due to propagation and due to the acceleration of the detector frame (note that in [1], the laboratory was accelerating). They consider the propagation of a wave packet along one direction and a screen which detects the wave packet and has an arbitrary trajectory in the perpendicular direction. They go on to show that the dephasing that occurs is purely a result of the trajectory of the detector. Due to the dispersion in internal energies ΔE, the packets have different velocities (as shown by the de Broglie wave dispersion relationship $\omega_k \approx k^2/2m$), and thus arrive at the detector at different times. They describe the measurement process in terms of a quantity σ_m, which gives the number of particles captured by the detector (screen) per area of over the lifetime of the experiment. Assuming the initial visibility (when the detector was at rest with respect to the wave packets) to be V, they show that the new visibility V' as a result of the relatively moving detector is given as

$$\frac{V'}{V} = 1 - \frac{\alpha^2 \dot{z}_{\bar{m}} t_{\bar{m}}^2}{2} \left(\frac{\Delta m}{\bar{m}} \right) \tag{3.11}$$

where α is the wave number of the spatial oscillation of the initial interference pattern, $\dot{z}_{\bar{m}}$ is the transverse velocity of the screen at time t_m, Δm^2 is the variance of the mass distribution, and $\bar{m}$ is the average mass. Specializing their results to the case in [1] with $\dot{z}_{\bar{m}} = g t_{\bar{m}}$, $\alpha = k_0(z_1 - z_2)/L, V = 1, \Delta m = k_B T \sqrt{N}$, they get

$$V' = 1 - \frac{N}{2} \big[g(z_1 - z_2) k_B T) \big]^2 t_{\bar{m}}^2 \tag{3.12}$$

where k_0 is where the momentum space distribution of the initial wave function is centered, and $z_{1,2}$ denote the heights of the center of mass modes in superposition in the reference frame of the emitter. The above result is exactly same as (4.16). In essence, this goes to show that the loss of visibility that PZCB calculate is a *kinematic effect* and not a gravitational one. In ref. [11], PZCB acknowledge the above work. However, they distance their setup from the one in [3] since in the case of [1], the interferometric paths perfectly overlap at the end of the experiment whereas that is not the case in [3].

Besides these, there were additional objections made in [2]. Here, Diósi argues that the center of mass superposition considered in [1] decoheres in the laboratory frame but does not in the case of a freely falling frame, going on to pit the result in [1] against the equivalence principle. PZCB acknowledge this in [11] and remark that the two experiments compared in [2] are different. They note that if one is interested in measuring the coherence in the laboratory frame *at the same time*, it does not compare to measuring the coherence *at the same time* in the freely falling

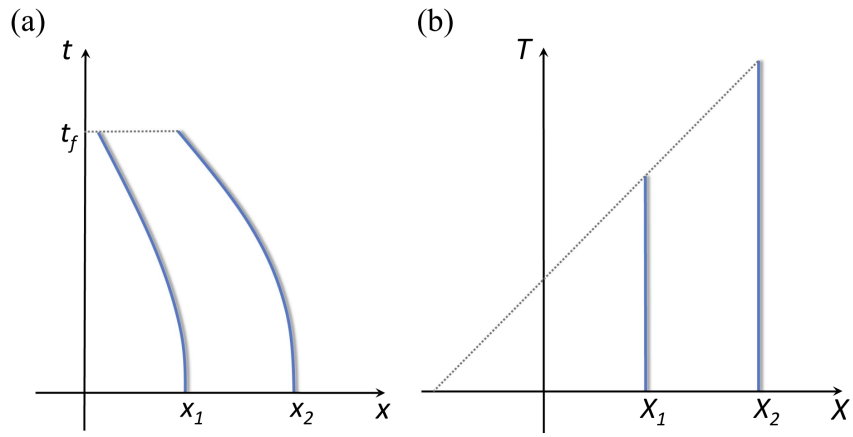

Figure 3.3: Two freely falling particles as seen in the a. laboratory frame b. freely falling frame. Credits: [13]

frame. Instead if one transforms the coordinates correctly, one must measure the coherence at *different times* in the freely falling case to reproduce the same results as in the case of the laboratory frame. As far as mathematical consistency is concerned, we believe that PZCB's argument is valid. However, since they do not present any operational measurement procedure, there is no way to tell how can one measure a superposition at different times, regardless of any reference frame.
There were other comments on PZCB's proposal including [14]. All in all, PZCB present valid responses to comments on their work. However, they are not completely satisfactory. In particular, as pointed out in [12], "The interface between gravitation and quantum theory is a fascinating subject. It has inspired novel and exiting ideas, many of them adventuring beyond standard quantum mechanics or general relativity. The topic is also riddled with subtleties and slight confusion can easily lead to questionable conclusions". More precisely, we believe that one has to be careful when attributing physical effects to gravity while restricting oneself to experiments on earth on a weak gravitational background, thereby neglecting curvature to begin with. Gravity's attribution to this phenomena can be cemented if one brings in curvature contributions to PZCB's expressions. This is precisely what we have done in our work.

Chapter 4

Genuine Gravitational Decoherence

4.1 PZCB Hamiltonian

In order to derive the Hamiltonian used in [1], we consider the metric in the post-Newtonian expansion (as in [1]) with $g_{00} = -(1 + 2\Phi(x)/c^2 + 2\Phi^2(x)/c^4)$ and $g_{ij} = \delta_{ij}(1 - 2\Phi/c^2)$. Using the above equation (4.7), the Hamiltonian becomes

$$H = \sqrt{-g_{00}(c^2 g^{\mu\nu} p_\mu p_\nu + m^2 c^4)} \tag{4.1}$$

PZCB identify the m^2c^4 term more generally as E_{rest}^2, or the rest-mass energy in the co-moving frame of reference of the particle. Since we are considering internal degrees of freedom as well, E_{rest} can be split as

$$E_{rest} = mc^2 + H_0, \tag{4.2}$$

which when substituted into the Hamiltonian in place of m^2c^4 gives

$$H = H_{cm} + H_0\left(1 + \frac{\Phi(x)}{c^2} - \frac{p^2}{2m^2c^2}\right) \tag{4.3}$$

which is the Hamiltonian used in [1] with $H_{cm} = mc^2 + m\Phi + p^2/2m$ being the Hamiltonian of the frozen degrees of freedom or more simply, the Hamiltonian of the center of mass. We could have alternatively derived (4.3) by just replacing $m \to m + H_0/c^2$ as reasoned in the previous chapter. We now move on to describe our work in formulating the above Hamiltonian. As described in the previous chapter, one of the major contentions with PZCB's work was the lack of genuine gravitational signatures (such as curvature) and confusion regarding the validity of equivalence principle in deriving their effect. it is clear that the Hamiltonian used by them was the root cause of the effect and thus any clarification would have to root from there.

4.2 Fermi Normal Coordinates and Their Construction

4.3 PZCB Hamiltonian with Tidal Effects

We utilize the framework of Fermi Normal Coordinates (FNC) to introduce curvature effects. Fermi Normal coordinates are a generalization of the Riemann Normal Coordinates adapted to any worldline $\gamma(\tau)$. In other words, these are locally flat coordinates that an observer performing an experiment naturally uses to report his/her results. Thus, this forms an appropriate framework for the problem at hand.
We begin by first re-deriving the Hamiltonian used in [1]. The Hamiltonian for a single particle interacting with a weak-field background in FNC can be identified to be $H = -\hat{p}_0$. This can be motivated as follows. Consider the familiar free-particle

Lagrangian

$$\mathcal{S} = -mc^2 \int d\tau \tag{4.4}$$

Noting that $u_i u^i = -c^2$, $u^i d\tau = dx^i$, and $p_i = mu_i$ we can re-write the above as

$$\mathcal{S} = \int p_i dx^i \tag{4.5}$$

Now, the time coordinate t_P assigned to an event P in a normal convex neighborhood of the trajectory is nothing but the proper time τ at the point on the trajectory connected to P by a spacelike geodesic. In other words, $t_P = \tau$. The above action can thus be split as

$$\mathcal{S} = \int \left(p_0 + p_\mu \frac{dx^\mu}{d\tau} \right) d\tau \tag{4.6}$$

which immediately identifies the Hamiltonian $H = -p_0$. This can further be written explicitly by using the relation $p.p = -m^2$ as[15]

$$H = \frac{g^{0\mu} p_\mu c}{g^{00}} + \sqrt{\frac{g^{\mu\nu} p_\mu p_\nu c^2 + m^2 c^4}{-g^{00}} + \left(\frac{g^{0\mu} p_\mu c}{g^{00}} \right)^2} \tag{4.7}$$

where greek indices denote spatial coordinates. The above must be interpreted as an operator equation and not an algebraic one.

In FNC, the metric takes the form

$$g_{00} = -\left[\left(1 + \frac{a_\mu y^\mu}{c^2} \right)^2 + R_{0\mu 0\nu} y^\mu y^\nu \right] + O(y^3) \tag{4.8}$$

$$g_{0\mu} = -(2/3) R_{0\rho\mu\sigma} y^\rho y^\sigma + O(y^3) \tag{4.9}$$

$$g_{\mu\nu} = \delta_{\mu\nu} - (1/3) R_{\mu\rho\nu\sigma} y^\rho y^\sigma + O(y^3) \tag{4.10}$$

The metric is valid in a local neighborhood of the considered trajectory. Note that setting $y^\mu = 0$ gives us the original trajectory with the Minkowski metric $diag(-1, 1, 1, 1)$. Using (4.7), (4.8), (4.9), and (4.10), we get the Hamiltonian for a particle in FNC to be

$$H-mc^2 = \frac{p^2}{2m}\left[\left((1+\frac{a_\mu y^\mu}{c^2}\right)^2+\frac{R_{0\mu0\nu}y^\mu y^\nu}{2}\right]+ma_\mu y^\mu+\frac{mc^2}{2}R_{0\mu0\nu}y^\mu y^\nu+\frac{m}{2}\frac{a_\mu a_\nu y^\mu y^\nu}{c^2}+ \frac{2}{3}R_{0\rho\mu\sigma}y^\rho y^\sigma p_\mu c-\frac{1}{6}\frac{p_\mu p_\nu}{m}R_{\nu\rho\nu\sigma}y^\rho y^\sigma. \quad (4.11)$$

Now, we take the non-relativistic limit $p \ll mc$. We assume that the acceleration term $a_\mu y^\mu \ll c^2$, and also ignore terms that go as $\mathcal{R}y^2 \times (p/mc)^2$ since they are of second order of smallness. Doing this, we get

$$H - mc^2 = \frac{p^2}{2m} + ma_\mu y^\mu + \frac{1}{2}mc^2 R_{0\mu0\nu}y^\mu y^\nu \quad (4.12)$$

Note the familiar rest-mass energy term mc^2 along with the redshift term $ma_\mu y^\mu$. The redshift term $ma_\mu y^\mu$ can be identified with $m\Phi(x)$ term in the center of mass Hamiltonian H_{cm}. It naturally comes about as an artefact of the accelarated reference frame stationary on earth. Additionally, note the curvature corrections that begin to show in the Hamiltonian that were previously absent. We now adopt the technique of replacing m with $m + H_0/c^2$ to obtain the final Hamiltonian for this setup:

$$H - mc^2 = \frac{p^2}{2m} + mc^2\left(\frac{R_{0\mu0\nu}y^\mu y^\nu}{2} + \frac{a_\mu y^\mu}{c^2}\right) + H_0\left(1 + \frac{R_{0\mu0\nu}y^\mu y^\nu}{2} + \frac{a_\mu y^\mu}{c^2} - \frac{p^2}{2m^2c^2}\right) \quad (4.13)$$

$$= H_{cm} + H_0\left(1 + \frac{R_{0\mu0\nu}y^\mu y^\nu}{2} + \frac{a_\mu y^\mu}{c^2} - \frac{p^2}{2m^2c^2}\right) \quad (4.14)$$

where we have absorbed the relevant center of mass terms in H_{cm}. The above Hamiltonian works for any path that we consider in spacetime. Notice that the above Hamiltonian has the same structure as (4.3).

Now consider a particle with its center of mass in superposition of two states with vertically distinct positions x_1 and x_2 with height difference $\Delta x = x_2 - x_1$. Now, let us say we pick a reference path in the lab for which $y_1^\mu = (s_1, 0, 0)$ and $y_2^\mu = (s_2, 0, 0)$ for the two superposed C.O.M paths respectively. The superposition of the center

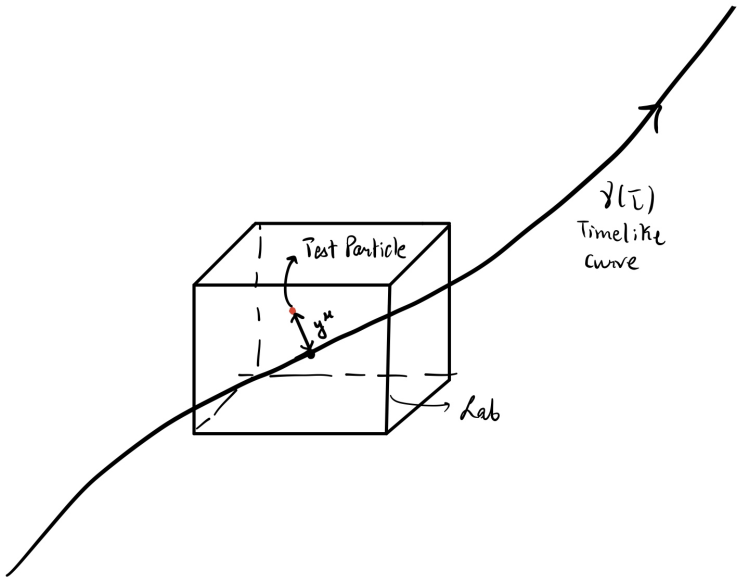

Figure 4.1: Lab with a test particle following an arbitrary path $\gamma(\tau)$ in spacetime

of masses can then be written as $|\Psi_{cm}\rangle = (1/\sqrt{2})(|s_1\rangle + |s_2\rangle)$. Each of the superposed path amplitude evolves according to $U(t) = \exp(-i/\hbar)\int dt[H_{cm} + (1 + (\Gamma(y^\mu, p)/c^2))H_0]$ along the respective world-lines where $\Gamma(y^\mu, p) = R_{0\mu 0\nu} y^\mu y^\nu c^2/2 + a_\mu y^\mu - p^2/2m^2$. The image that one should have in mind at this point is that of **??**. Now, following the methods in PZCB's paper, the interference visibility is then

$$V = \left|\mathrm{Tr}\left[e^{-i/\hbar\int dt(1+\Gamma(s_1,p_1)/c^2)H_0}\rho_0 e^{i/\hbar\int dt(1+\Gamma(s_2,p_2)/c^2)H_0}\right]\right| \tag{4.15}$$

where we are tracing out the internal degrees of freedom. As $d\tau = dt\sqrt{g_{ij}\dot{x}^i\dot{x}^j} \approx dt(1 + \Gamma/c^2)$, the above simplifies to

$$V = \left|\langle e^{iH_0\Delta\tau/\hbar}\rangle\right| \tag{4.16}$$

where $\Delta\tau = \tau_1 - \tau_2$ is the proper time difference between the two world lines and the expectation value is taken with respect to the initial state. In other words, we have reproduced the same result as in [1] using the framework of Fermi Normal coordinates incorporating curvature terms! This was not surprising since we immediately noticed the structural similarity of the resultant Hamiltonian (4.14) with the Hamiltonian (4.3). Another reason why this happens is because in the non-relativistic limit, only g_{00} decides the coupling of the internal degrees of freedom with gravity as well as the expansion of $d\tau$ in terms of the coordinate time differential dt. Thus, this always results in the neat simplification of $H_0(1+\Gamma)dt$ to $H_0 d\tau$.

Note that since a lab is an accelerating frame with $a_\mu = (0, 0, g)$, $a_\mu y^\mu$ can be identified with the gravitational potential Φ. However, PZCB were quick to term this precise coupling of $H_0\Phi/c^2$ having genuine gravitational footing. It is now clear that we can just as easily get rid of this coupling by choosing a freely falling frame. The coupling that we have obtained in (4.14) on the other hand is genuinely gravitational since it incorporates the Riemann curvature term into it. Therefore, even if we were to put the setup in a freely falling frame, we would observe the decoherence of center of mass modes, only this time due to tidal effects. The expression, however, is precisely what PZCB calculate as (4.16).

We can go on to do more with the expression (4.11) that we have painstakingly derived. One can wonder whether there exist genuine gravitational corrections to PZCB's expression for visibility at relativistic orders p/mc. Actually, the Hamiltonian (4.14) already has a $(p/mc)^2$ term and there seemed to be no problem in getting the visibility as (4.16). However, we did ignore a few terms along the way and we now show that these terms can in fact produce corrections, even to first order in p/mc.

Taking the expression (4.11), we use the familiar recipe of $m \to m + H_0/c^2$. Doing this, we get the Hamiltonian involving the coupling of gravity with internal degrees of freedom H_0 as

$$H = H_{cm} + H_0\left[1 + \frac{R_{0\mu0\nu}y^\mu y^\nu}{2} + \frac{a_\mu y^\mu}{c^2} - \frac{p^2}{2m^2c^2}\left\{\left(1 + \frac{a_\mu y^\mu}{c^2}\right)^2 + R_{0\mu0\nu}y^\mu y^\nu\right\} + \frac{1}{2}\frac{a_\mu a_\nu y^\mu y^\nu}{c^4} + \frac{1}{6}\frac{p^\mu p^\nu}{m^2c^2}R_{\mu\rho\nu\sigma}y^\rho y^\sigma\right] \tag{4.17}$$

where H_{cm} in this case is the Hamiltonian (4.11). Since we are after genuined gravitational corrections, we can set $a_\mu = 0$. Doing this, we get

$$H = H_{cm0} + H_0\left[1 + \frac{R_{0\mu0\nu}y^\mu y^\nu}{2} - \frac{p^2}{2m^2c^2} - \frac{p^2}{2m^2c^2}R_{0\mu0\nu}y^\mu y^\nu + \frac{1}{6}\frac{p_\mu p_\nu}{m^2c^2}R_{\mu\rho\nu\sigma}y^\rho y^\sigma\right] \tag{4.18}$$

$$= H_{cm0} + H_0(1 + \Gamma_{new}) \tag{4.19}$$

where $H_{cm0} = H_{cm}|_{a_\mu=0}$ and we have named the coupling to H_0 as Γ_{new}. Taking the metric (4.8-4.10), we can now calculate the proper time τ in terms of the coordinate time t using the relation $d\tau = dt\sqrt{g_{ij}x^i x^j}/c$. Doing this, we get

$$d\tau = dt\left[1 + \frac{R_{0\mu0\nu}y^\mu y^\nu}{2} - \frac{p^2}{2m^2c^2} + \frac{1}{6}\frac{p_\mu p_\nu}{m^2c^2}R_{\mu\rho\nu\sigma}y^\rho y^\sigma + \frac{2}{3}\frac{p^\mu}{mc}R_{0\rho\mu\sigma}y^\rho y^\sigma\right] \tag{4.20}$$

$$= dt(1 + \Gamma_\tau). \tag{4.21}$$

In the previous case, the Hamiltonian (4.14)'s Γ exactly matched the Γ that one gets when evaluating the proper time in terms of the coordinate time (see above equation (4.16)). However, once we consider the coupling $(p/mc)^n \times \mathcal{R}$, this does not hold true any more. Due to this reason, the visibility no longer remains (4.16) and we get extra terms. Calculating the new visibility with corrections of order $(p/mc)\mathcal{R}$, we get

$$V = |\langle e^{-(i/\hbar)H_0(\Delta\tau + \int dt\ \Delta R(t))}\rangle| \tag{4.22}$$

where

$$\Delta R(t) = \frac{2}{3mc}\Bigg(R_{0\rho\mu\sigma}(y_1(t))y_1^\rho(t)y_1^\sigma(t)p_1^\mu(t) - R_{0\rho\mu\sigma}(y_2(t))y_2^\rho(t)y_2^\sigma(t)p_2^\mu(t)\Bigg). \quad (4.23)$$

where we have restricted our analysis to semi-classical paths, that is, having coordinates $y_1(t), p_1(t), y_2(t), p_2(t)$ along the two world lines (as depicted in **??**). Therefore, we see that there are indeed corrections that arise out of tidal effects, a genuine signature of gravity.

4.4 Outcomes

To summarize, we have done the following in this chapter.

1. We have described a more rigorous formulation to derive the Hamiltonian in [1] incorporating tidal effects. The result is (4.12).
2. We have provided a resolution to certain questions ensuing PZCB's work. The Hamiltonians (4.14) and (4.18) explicitly incorporate curvature and go a step further to show that the proposed effect is genuinely gravitational. They predict decoherence of c.o.m superpositions even in free fall ($a^\mu = 0$) thereby showing that the effect was not simply a consequence of the frame of reference considered by PZCB. This needed to be clarified.
3. We have calculated genuine gravitational corrections to PZCB's expression for visibility, finally arriving at (4.22) .

Chapter 5

Hamiltonian of a Composite Particle in Curved Spacetime

5.1 Is gravity Unbiased?

In the previous chapters, the Hamiltonians we have written down (namely 4.3,4.13, and 4.17) have made an implicit but well motivated assumption about the coupling of gravity to internal energy modes. The assumption that gravity couples to all energies in an unbiased fashion allows us to replace $m \to m + H_0/c^2$. However, it would be better if it were systematically derivable from first principles by considering a toy model for an atom. This is our aim for this chapter; to give toy model to more convincingly justify the naive replacement of $m \to m+H_0/c^2$. Although not with the sole aim of doing this, ref. [16] showed that the "atom" behaves as composite point particle whose mass comprises of rest masses of the constituent particles as well as the internal energy. In other words, they have already done what we are setting out to do here. Before looking into what we add to the discussion, let us look at their model and framework for deriving the replacement $m \to m + H_0/c^2$. We shall only briefly review their methodology without delving into the equations here.

Ref. [16] consider the following model for an atom: A system of two particles without spin, with electric charges $e_1 = -e_2 = e$, masses m_1, m_2, and spatial positions $\mathbf{r}_1, \mathbf{r}_2$. They consider the mutual electromagnetic interaction but neglect the mutual gravitational interaction. They place this atom in an external electromagnetic field and an external weak gravitational field with potential ϕ.

To get the coupling of gravity to the particles, they begin with a single free point particle classical Lagrangian with mass m and position $\mathbf{x}$ given by

$$L = -mc^2 \sqrt{g_{\mu\nu} \dot{x}^\mu x^\nu / c^2} \tag{5.1}$$

where

$$g_{\mu\nu} = \begin{pmatrix} -1 - 2\frac{\phi}{c^2} - 2\beta\frac{\phi^2}{c^4} + \mathcal{O}(c^{-6}) & \mathcal{O}(c^{-5}) \\ \mathcal{O}(c^{-5}) & (1 - 2\gamma\frac{\phi}{c^2})\mathbf{I} + \mathcal{O}(c^{-4})) \end{pmatrix} \tag{5.2}$$

where β and γ are the "Eddington-Robertson parameters" to account for deviations to General Relativity. $\beta = \gamma = 1$ gives General Relativity. They consider a two particle system and write the combined Lagrangian as a sum for each particle individually. They then add in the coupling of the particles to the electromagnetic fields separately borrowing the expressions from [17]. Taking the Legendre transform w.r.t $\dot{\mathbf{r}}_i$ $(i = 1, 2)$, they obtain the Hamiltonian for a two-particle system interacting with an external gravitational and electromagnetic field. After this, they transform to the center of mass variables and interpret different parts of the Hamiltonian as contributions internal, c.o.m, and mixed (between internal and external) interactions. Following this, they write down the solutions to the full Maxwell's equations which are now gravitationally modified because of the non-Minkowski metric (5.2). Doing this gives them additional corrections to their earlier Hamiltonian. Re-interpreting the terms appropriately in their corrected new Hamiltonian, they exactly reproduce the relationship

$$H_{C,\text{final}} = H_{\text{point}}\Big(\mathbf{P}, \mathbf{R}; M + H_A/c^2\Big) \tag{5.3}$$

where $H_{C,\text{final}}$ is the Hamiltonian containing the central dynamics of the composite system, $H_{\text{point}}(\mathbf{P}, \mathbf{R}; M + H_A/c^2)$ is the Hamiltonian for a point particle of momentum $\mathbf{P}$, position $\mathbf{R}$, and mass $M + H_A/c^2$, with H_A being the Hamiltonian describing the internal dynamics of the system.

Their work provides a fairly rigorous justification to the unbiased coupling of gravity. But here we ask whether this holds true once we do not ignore the internal gravitational coupling between the masses. Given gravity's elusiveness, we cannot be certain whether the gravitational coupling as shown above holds good when we considering *gravitational* internal energies in the atom themselves. In this chapter, we construct a toy model to investigate this. Before that, we briefly review the results in ref. [18] that will be essential in our further discussion.

5.2 Precursor to a Toy Model

Before we begin to discuss our efforts to investigate the naive replacement of mass to include *gravitational* interaction energies, we need to make a slight digression discussing parts of [18] and the results therein. We only review the results relevant to our discussion and make the connections explicitly wherever necessary.

Taylor et al. [18] consider the problem of a black hole moving in an external spacetime and investigate the effects of the tidal moments created by the external spacetime on the black hole (and vice versa). In their analysis, they consider the gravity of the external spacetime to be sufficiently weak so as to be appropriately described by a post-Newtonian approximation to general relativity. They conduct their analysis by dividing the space into three regions. For our purposes, the metric in the region near the black hole $\mathcal{B} \cup \mathcal{O}$ suffices and thus we shall limit our discussion to it. $\mathcal{B} \cup \mathcal{O}$ forms the neighborhood of the black hole where the black hole's gravity is too strong to be ignored in comparison to the external background. The black hole is considered to have a mass m with gravitational radius $M := Gm/c^2$. Furthermore, they consider the strength of the tidal perturbations to be measured by the inverse length scale $\mathcal{R}^{-1}$, where $\mathcal{R}$ is the radius of curvature of the external spacetime. They assume that $M/\mathcal{R} \ll 1$; the tidal perturbations are weak. $M/\mathcal{R} \ll 1$ is also the limit

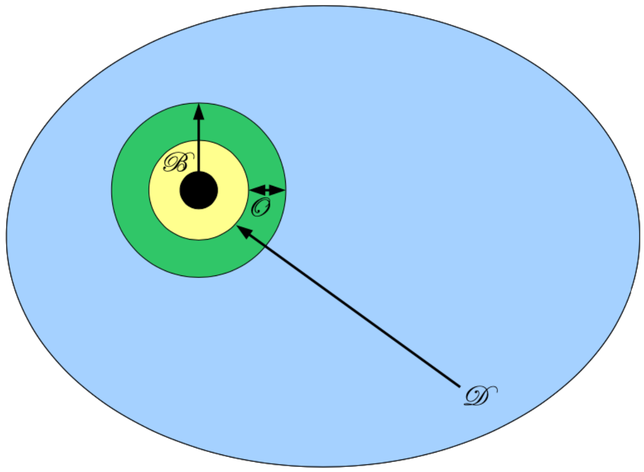

Figure 5.1: Post-Newtonian environment $\mathcal{D}$, the black hole neighborhood $\mathcal{B}$, and the overlap region $\mathcal{O}$. The black hole is represented by a black disc. Such a clean separation only makes sense in the limit $M/\mathcal{R} \ll 1$Credits: [18]

where a clean separation of the regions as in 5.1 can be made without too much trouble. So it is actually assumed *apriori* in their analysis. Finally, the black hole moves in an empty region of spacetime. The metric in this region is presented in co-moving harmonic coordinates of the black hole, $\bar{x}^\alpha = (\bar{x}^0, \bar{x}^a) = (c\bar{t}, \bar{x}, \bar{y}, \bar{z})$, where the harmonic condition reads as $\partial_\beta(\sqrt{-g}g^{\alpha\beta})$. The Greek indices run from 0 to 3, whereas Latin from 1 to 3. In their notation, the bar always indicates quantities written in the Harmonic co-moving coordinates.

The FNC we introduced in chapter 4 also describe the tidal environment along a geodesic. Once we place a black hole on this geodesic, the metric is modified due to the gravitational effects of the black hole. The resultant metric reads:

$$g_{\bar{0}\bar{0}} = -\frac{1 - M/\bar{r}}{1 + M/\bar{r}} - \bar{r}^2(1 - M/\bar{r})^2\mathcal{E}^q + \mathcal{O}(\frac{\bar{r}^3}{\mathcal{R}^2\mathcal{L}}) \tag{5.4}$$

$$g_{\bar{0}\bar{a}} = \frac{2}{3}\bar{r}^2(1 - M/\bar{r})(1 + M/\bar{r})^2\mathcal{B}^q_a + \mathcal{O}(\frac{v}{c}\frac{\bar{r}^3}{\mathcal{R}^2\mathcal{L}}) \tag{5.5}$$

$$\begin{aligned} g_{\bar{a}\bar{b}} &= \frac{1 + M/\bar{r}}{1 - M/\bar{r}}\Omega_a\Omega_b + (1 + M/\bar{r})^2\gamma_{ab} - r^2(1 + M/\bar{r})^2\mathcal{E}^q\Omega_a\Omega_b \\ &-M\bar{r}(1 + M/\bar{r})^2(1 + M^2/3\bar{r}^2)(\Omega_a\mathcal{E}^q_b + \Omega_b\mathcal{E}^q_a) - \bar{r}^2(1 - M/\bar{r})^2(1 + M/\bar{r})^3\gamma_{ab}\mathcal{E}^q \\ & M\bar{r}(1 + M/\bar{r})^2(1 - M^2/3\bar{r}^2)\mathcal{E}^q_{ab} + \mathcal{O}(\frac{\bar{r}^3}{\mathcal{R}^2\mathcal{L}}) \end{aligned} \tag{5.6}$$

where $\bar{r} = \sqrt{\delta_{ab}\bar{x}^a\bar{x}^b}$ (δ_{ab} being the Kronecker delta), $\Omega_a := \bar{x}^a/\bar{r}$ is a unit radial vector, and $\gamma_{ab} := \delta_{ab} - \Omega_a\Omega_b$. Furthermore the tidal potentials are given as

$$\mathcal{E}^q = \frac{1}{c^2}\bar{\mathcal{E}}_{cd}\Omega^c\Omega^d \tag{5.7}$$

$$\mathcal{E}^q_a = \frac{1}{c^2}\gamma^c_a\gamma^d_b\bar{\mathcal{E}}_{cd}\Omega^d \tag{5.8}$$

$$\mathcal{E}^q_{ab} = \frac{1}{c^2}\left(2\gamma^a_c\gamma^d_b\bar{\mathcal{E}}_{cd} + \gamma_{ab}\mathcal{E}^q\right) \tag{5.9}$$

$$\mathcal{B}^q_a = \frac{1}{c^3}\epsilon_{apq}\Omega^p\bar{\mathcal{B}}^q_c\Omega^c \tag{5.10}$$

where the label q stands for "quadrupole". The tensorial functions $\bar{\mathcal{E}}_{ab}(\bar{t})$ and $\bar{\mathcal{B}}_{ab}(\bar{t})$ are the tidal moments that characterize the black hole's tidal environment. They are a function of the Riemann tensor $R_{\mu\nu\rho\sigma}$.

5.3 Toy Model for PZCB Hamiltonian in Tidal Fields

With the required expressions from [18] and their physical meanings discussed, we can now begin to justify the replacement $m \to m + H_0/c^2$ when the internal energies arise due to purely gravitational, that is, in terms of tidal effects determined by the functions $\bar{\mathcal{E}}_{ab}$ and $\bar{\mathcal{B}}_{ab}$. We start by considering two bodies of mass μ and m placed

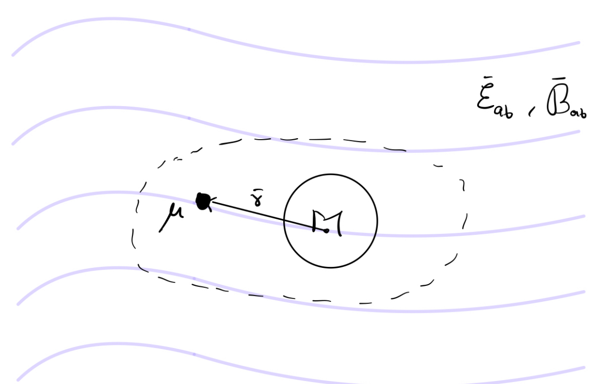

Figure 5.2: Two bodies in an esternal tidal background determined by the tidal moments $\mathcal{E}_{ab}$ and $\mathcal{B}_{ab}$

in an external tidal environment such as the one discussed in the previous section. Only their mutual gravitational interaction is of interest to us here since this case has not been solved before, at least not in the framework of FNC where tidal effects are taken into account. To simplify the problem, we work in the limit $\mu \ll m$. In this limit, we can think of a particle of mass μ orbiting around an object of mass m, both placed in an external tidal field. The fact that the center of mass is at m and that our metric (5.4-5.6) is written w.r.t the coordinates of the black hole (in this case the mass m), makes it easier for us to write the Hamiltonian in the center of mass coordinates. Essentially, the smaller mass μ moves in the combined background

caused by the mass m and the external tidal field. The metric (5.4-5.6) will then naturally bring out an interaction term into the Hamiltonian as we shall see. For simplicity, we restrict our analysis to the case that the mass μ has negligible to no momentum. This can be physically done by applying, say, a trapping potential of purely non-gravitational nature.
In that case, the total classical Hamiltonian for the system is given as

$$H = \sqrt{g_{\bar{0}\bar{0}}}\mu c^2 + H_m + V \tag{5.11}$$

where $g_{\bar{0}\bar{0}}$ is the metric from 5.4 and V is some non-gravitational trapping potential. We leave the Hamiltonian H_m for the mass m unspecified for now since the coordinates and metric at hand are not ideal for writing the Hamiltonian for the mass m. It suffices to show that gravity (tidal effects) couple to mass μ in the same way as they couple to the gravitational internal energies of masses m and μ. Before expanding out the above expression, it helps to introduce some notation that will help interpretation of the terms. Firstly,

$$\frac{\bar{M}}{\bar{r}} = \frac{Gm}{c^2\bar{r}} := \frac{-\Phi_{int}}{c^2}. \tag{5.12}$$

The above definition makes sense since the potential generated by the mass m considered as a monopole will be $-Gm/\bar{r}$. Now, a good candidate for the internal energy of the m and μ system is $H_0 = \mu\Phi_{int} = -GM\mu/r$. It fits right with our understanding of the gravitational internal energy between two masses. In the above language, (5.11) can be expanded as

$$H = \left[\left(1 + \frac{\Phi_{int}}{c^2}\right)\left(1 - \frac{\Phi_{int}}{c^2}\right)^{-1} + \left(1 + \frac{\Phi_{int}}{c^2}\right)^2 \frac{\bar{\mathcal{E}}_{ab}\bar{x}^a\bar{x}^b}{c^2}\right]^{1/2} \mu c^2 + H_m + V \tag{5.13}$$

where $\bar{x}_a$ are the spatial coordinates for the mass μ. Typically, $\Phi_{int}/c^2 \ll 1$ and so we can expand to first order in it giving

$$H = \left[\left(1 + \frac{\Phi_{int}}{c^2}\right) + \frac{1}{2}\left(1 + 2\frac{\Phi_{int}}{c^2}\right)\frac{\bar{\mathcal{E}}_{ab}\bar{x}^a\bar{x}^b}{c^2}\right]\mu c^2 + H_m \tag{5.14}$$

$$= \mu c^2 + H_0 + \mu\frac{\bar{\mathcal{E}}_{ab}\bar{x}^a\bar{x}^b}{2} + \frac{H_0}{c^2}\bar{\mathcal{E}}_{ab}\bar{x}^a\bar{x}^b + H_m + V. \tag{5.15}$$

Note the similarity of the above Hamiltonian with (4.13) with $a_\mu = 0$ and $p = 0$. A direct comparison between $R_{0\mu 0\nu}$ and $\bar{\mathcal{E}}_{ab}$ must *not* be made since the metric we use here is in harmonic coordinates and the metric in the previous chapter uses the standard FNC. That said, they both ultimately describe tidal effects which are of interest here. Now, staring at the above expression one notices an unexpected discrepancy of a factor of 2 in comparison to (4.13). Instead of coupling to $\mu + H_0/c^2$, the tidal terms (that form a genuine gravitational interaction) couple to $\mu + 2H_0/c^2$ violating our naive expectation that the coupling to all energies must be the same. Why has this happened? It would be rather bold to claim that gravity couples differently to gravitational potential energies. It turns out that this discrepancy is not novel and has been noticed in the literature previously, starting from the times of Eddington [19, 20]. It was also recently noted in [21, 16] and a resolution was provided. The solution to this problem is rather straightforward and lies in the interpretation of physical quantities involved; internal energy in this case. Specifically, it is incorrect to interpret $-Gm\mu/r$ as the internal energy. The $'r'$ in this expression is the *Euclidean* distance given by $\sqrt{\delta_{ab}x^a x^b}$ and is not a valid distance. With respect to the mass m, there is a background dependence to the distance as it is placed in a curved spacetime. Therefore, the right geometric distance given as $\sqrt{g_{\bar{a}\bar{b}}\bar{x}^a\bar{x}^b}$ needs to be used. Taking this into account, the internal energy $H_{0,\text{new}}$ is given as $-Gm\mu/\sqrt{g_{\bar{a}\bar{b}}\bar{x}^a\bar{x}^b}$. Evaluating the metric distance, we get

$$H_{0,\text{new}} = -\frac{Gm\mu}{r}\left(1 + \frac{1}{2}\bar{\mathcal{E}}_{ab}\bar{x}^a\bar{x}^b\right) \tag{5.16}$$

Note that despite the hairy expression (5.6), the result above turns out to be simple since only need the metric $g_{\bar{a}\bar{b}}$ upto $\mathcal{O}(1)$. This is because we are looking to write down the Hamiltonian to first order in m/r. Re-writing the Hamiltonian (5.15) with the re-interpreted internal energy gives

$$H = \mu c^2 + \frac{\mu}{2}\bar{\mathcal{E}}_{ab}\bar{x}^a\bar{x}^b + H_{0,\text{new}}\left(1 + \frac{1}{2}\bar{\mathcal{E}}_{ab}\bar{x}^a\bar{x}^b\right). \tag{5.17}$$

The above Hamiltonian has no discrepancy and we have successfully given, in a seemingly restricted but novel fashion, provided a justification for Pikovski et. al's replacement of $mm + H_0/c^2$. With this, we end our discussion on gravitational decoherence in tidal backgrounds. Moving forward, the rest of the thesis introduces a rather infamous problem relevant to the title of this thesis, namely, the black hole information problem.

Chapter 6

Shifting Gears — Towards the Black Hole Information Problem

6.1 Introduction

Starting this chapter, we shift gears from non-relativistic quantum mechanics in weak backgrounds to the other end of the spectrum, namely, relativistic quantum mechanics in strongly curved background spacetimes. The "aspect" of quantum information in gravity that we seek to discuss here inadvertently once again relates to information loss, namely, the *black hole information problem*. However, unlike decoherence, the information loss that gravity seems to cause is more fundamental, and not necessarily because we cannot keep track of the degrees of freedom as in the case of decoherence.

The realization that quantum mechanics in strongly curved classical spacetimes (like that of black holes) can lead to information loss was first made possible by Hawking [22, 23]. Hawking showed that black holes lose mass by emitting radiation. Once this process is complete, it seems as if the information contents of the matter that made up the black hole are lost since the radiation does not carry enough information for one to be able to "reconstruct" the black hole. If this were true, it would violate a core principle of physics, both classical and quantum, of time reversibility. This realization marks the beginning of the black hole information problem.

The underlying motivation to get to the root of this issue is the hope that it would lead to a better understanding of quantum gravity. Indeed, for the past 45 years or so, this problem has been the catalyst for several important developments in all of theoretical physics, let alone quantum gravity [24, 25, 26, 27, 28, 29, 30, 31, 32, 33]. Here, we aim to introduce the problem, beginning right from its inception with Hawking radiation. We then look at a subsequent development by Don Page that underlies all present-day discussions of the problem. Before we begin however, we briefly digress to introduce the framework of quantum field theory in curved spacetimes that underlies Hawking's calculation.

6.2 Brief Review of QFT in Curved Spacetimes

In this section, we shall briefly review and set up the basic formalism for quantum field theory (QFT) in curved spacetimes. In this formalism, the gravitational field is treated as a classical background on which matter fields are quantized. This is similar to the case of the light matter interaction explained by the Rabi model in quantum optics, where the background EM field is classical but the atom is treated as a quantum two-level system. In the absence of a complete theory of quantum gravity, we expect this formalism to be a good approximation to predict quantum effects involving gravity (while we remain far away from Planck scales). QFT in curved spacetimes is an extensive area of study and we will only review the basic results necessary to understand Hawking radiation in the following chapter. Our discussion here is mainly from ref. [34].

We begin by picking a globally hyperbolic spacetime $(\mathcal{M},\ g)$ and foliate it with Cauchy surfaces Σ. Cauchy surfaces can be thought of as planes of simultaneity that correspond to a "moment in time". Once we pick a Cauchy surface and specify our initial data of the quantum field on it, the solutions to the equations of motion are completely determined on the rest of the spacetime. To begin, we place a free scalar field onto this spacetime. This field satisfies the Klein-Gordon equation

$$\Box\phi - m^2\phi = 0 \tag{6.1}$$

where $\Box = \nabla_i \nabla^i$, is the covariant d'Alembertian operator. Given two solutions f_1 and f_2 to the above equation, the inner product on a spacelike hypersurface Σ_0 is naturally defined as

$$(f_1, f_2) = i \int_{\Sigma_0} (f_2^* \overleftrightarrow{\partial_j} f_1) \mathrm{d}\Sigma^j \tag{6.2}$$

where $\mathrm{d}\Sigma^i = \mathrm{d}\Sigma\, n^i$, with $\mathrm{d}\Sigma$ being the volume element of Σ_0 and n^i being the time-like normal to this spacelike hypersurface.

We can now quantize this scalar field using the standard methods of canonical quantization. We first define the canonically conjugate momentum as

$$\pi = \frac{\delta \mathcal{L}}{\delta \dot{\phi}} \tag{6.3}$$

where $\mathcal{L}$ is the standard scalar field Lagrangian that gives the equation (6.1). We then promote ϕ and π to operators and impose canonical equal time commutation relations

$$[\phi(t, \mathbf{x}), \phi(t, \mathbf{x}')] = 0 \tag{6.4}$$

$$[\pi(t, \mathbf{x}), \pi(t, \mathbf{x}')] = 0 \tag{6.5}$$

$$[\phi(t, \mathbf{x}), \pi(t, \mathbf{x}')] = i\delta(\mathbf{x} - \mathbf{x}') \tag{6.6}$$

Theorems in differential geometry guarantee that when one is working on a Cauchy surface, one can always find a complete set of orthonormal solutions $\{f_i, f_i^*\}$ to the equation (6.1) satisfying

$$(f_i, f_j) = \delta_{ij} \tag{6.7}$$

$$(f_i^*, f_j^*) = -\delta_{ij} \tag{6.8}$$

$$(f_i^*, f_j) = 0 \tag{6.9}$$

Following this, we can expand ϕ in terms of these solutions as

$$\phi = \sum_i [a_i f_i + a_i^\dagger f_i^*] \tag{6.10}$$

with (6.4) necessitating $[a_i, a_j^\dagger] = \delta_{ij}$. We can now consider the Fock representation of the Hilbert space of the relevant quantum states. With this, one can define a unique vacuum state $|0\rangle$ such that $a_i |0\rangle = 0 \; \forall \; i$. States with particles in them can be constructed by the operation of $a_i^\dagger$ in a similar manner.

In the case of the familiar Minkowski spacetime, there are a unique set of positive frequency modes that solve (6.1), namely, $f_k \propto e^{i\mathbf{k}.\mathbf{x} - i\omega_k t}$ with $\omega_k = \sqrt{k^2 + m^2}$. The action of the Poincaré group leaves the vacuum state invariant in this case. In a general curved spacetime, however, Poincaré group no longer remains the symmetry of the spacetime. Consequently, the choice of modes is not unique. As a result, the notion of a particle becomes ambiguous.

In a general spacetime, one can then consider another complete orthonormal set of modes $\{g_j, g_j^*\}$ that solve the equation (6.1). Again, the field ϕ can be expanded in terms of these modes as

$$\phi = \sum_i [b_i g_i + b_i^\dagger g_i^*]. \tag{6.11}$$

where $[b_i, b_j^\dagger] = \delta_{ij}$. This then defines a new vacuum state $|0'\rangle$ such that $b_j |0'\rangle = 0$, and consequently a new Fock space. Due to the completeness of both sets of solutions, we can expand one in terms of the other as

$$g_i = \sum_j \alpha_{ij} f_j + \beta_{ij} f_j^* \tag{6.12}$$

where α_{ij} and β_{ij} are called the Bogoliubov coefficients. They are given as

$$\alpha_{ij} = (g_i, f_j) \qquad \beta_{ij} = -(g_i, f_j^*) \tag{6.13}$$

We could also write it the other way around expressing f_j in terms of g_j and g_j^* as

$$f_i = \sum_j \alpha_{ij}^* g_j - \beta_{ij} g_j^* \tag{6.14}$$

Next, we can get the transformation between $\{a_i, a_i^\dagger\}$ and $\{b_i, b_i^\dagger\}$ using the orthonormality of modes as

$$b_j = \sum_i \alpha_{ji}^* a_i - \beta_{ji}^* a_i^\dagger \tag{6.15}$$

$$a_i = \sum_j \alpha_{ji} b_j + \beta_{ji}^* b_j^\dagger \tag{6.16}$$

Furthermore, the Bogoliubov coefficients have the property

$$\sum_k [\alpha_{ik}^* \alpha_{jk} - \beta_{ik}^* \beta_{jk}] = \delta_{ij} \tag{6.17}$$

$$\sum_k [\alpha_{ik} \beta_{jk} - \beta_{ik} \alpha_{jk}] = 0 \tag{6.18}$$

by virtue of the commutation relations on the sets of annihilation and creation operators $\{a_i, a_i^\dagger\}$ and $\{b_i, b_i^\dagger\}$. As a result of the above properties, it is easy to see that the expectation value of the number operator $N_i = a_i^\dagger a_i$ in the vacuum state $|0'\rangle$ is given as

$$\langle 0'| N_i |0'\rangle = \sum_i |\beta_{ij}|^2 \tag{6.19}$$

Therefore, as long as $\beta_{ij} \neq 0$, vacuum state of g_i modes contains $\sum_i |\beta_{ij}|^2$ particles in the f_i modes.

Chapter 7

Hawking Radiation and Information Problem

7.1 Introduction

In the year of 1974, Stephen W. Hawking presented his seminal paper titled *Black Hole Explosions?* [22] in which he showed that black holes emit radiation with a Planck spectrum of temperature given as

$$T = \frac{\hbar c^3}{8\pi G k_B M} \tag{7.1}$$

where $\hbar, c, G$, and k_B are natural constants and M is the mass of the black hole. Hawking's result continued to provide important insights into the quantum nature of space-time till date and thus forms an important part of any discussion on quantum information and gravity. In this chapter, we will present one of its derivations using some of the tools introduced in the previous chapter. The derivation will mostly follow the one given in [34] combined with some other resources including Hawking's original paper [22, 35].

7.2 Derivation of Hawking Radiation

Consider a spherically symmetric ball that is sufficiently sparse such that the spacetime is Minkowski. One can then construct standard Minkowski vacuum states in this spacetime. If the conditions are right, the spherically symmetric ball might collapse to form a black hole. Once this happens, the metric outside the black hole will be the unique spherically symmetric solutions to Einsteins equations — the Schwarzschild metric. Therefore, after this collapse, the vacuum will not correspond to the initial vacuum states that were constructed in Minkowski spacetime. This is because the dynamical formation of a black hole leads to energies being exchanged between the quantum field and the background. Our aim is to calculate the Bogoliubov coefficients for this collapse and reproduce the result (7.1).

For simplicity, we consider the case of a massless scalar field ϕ in a Schwarzschild spacetime. On solving for the modes using the wave equation $\Box\phi = 0$, we obtain the form

$$\frac{1}{r}R_{\omega l}(r)Y_{lm}(\theta,\phi)e^{-i\omega t} \tag{7.2}$$

where Y_{lm} is a spherical harmonic, and the radial function $R_{\omega l}$ satisfies

$$\frac{d^2R_{\omega l}}{dr^{*2}} + \left\{\omega^2 - \left[\frac{l(l+1)}{r^2} + \frac{2M}{r^3}\right]\left[1 - \frac{2M}{r}\right]\right\} = 0 \tag{7.3}$$

with $r^* = r + 2M\ln|(r/2M) - 1|$ being the familiar tortoise coordinate. As $r \to \infty$ (equivalently, $r^* \to \infty$), the above equation reduces to a harmonic oscillator equation with frequency ω. The solutions to this equation are $e^{\pm i\omega r}$. Consequently, the modes can be written as

$$\frac{1}{r}e^{-i\omega u}Y_{lm} \tag{7.4}$$

and

$$\frac{1}{r}e^{-i\omega v}Y_{lm} \tag{7.5}$$

where $u = t - r^*$ and $v = t + r^*$ are the null coordinates. Modes in equation (7.4) are the outgoing modes and modes in equation (7.5) are the ingoing ones.

Let us now trace the path of an asymptotically incoming mode from $\mathcal{I}^-$ of the form

images/penrosediagrambh.jpeg

Figure 7.1: Penrose diagram for a ball of astrophysical matter collapsing to form a black hole

(7.5). As the mode approaches the surface of the ball, it gets blueshifted as it gains energy by going into the gravitational potential. While this mode goes through the center and is leaving the ball, it begins to get redshifted. However, note that since the ball is collapsing all this time, the gravitational potential is getting "deeper". In more formal terms, we say that the surface gravity of the ball is increasing. Therefore, the redshift of the mode while leaving the ball will be greater than the blueshift it suffered while entering it. For a ball undergoing collapse to a black hole, the time scale of the shrinkage is comparable to the transit time of the mode. When this happens, the net redshift becomes appreciable. However, note that not all modes will be able to escape. Modes that reach the surface of the ball late enough will not be able to escape since the black hole would have formed while they were passing through the ball. These modes will just fall into the singularity. One of these modes will just manage to escape and will form what one calls the event horizon of the black hole.

In order to get a quantitative hold on the redshift that the incoming modes suffer, we restrict to the two-dimensional model of a collapsing star, suppressing the angular variables. This captures the essential features and makes the problem simpler to solve. Outside the collapsing ball, we can take the line element to be generally of the form

$$ds^2 = C(r)dudv \tag{7.6}$$

where

$$u = t - r^* + R_0^* \tag{7.7}$$

$$v = t + r^* - R_0^* \tag{7.8}$$

and

$$r^* = \int C^{-1}dr \tag{7.9}$$

where R_0^* is a constant whose physical understanding will become clear soon. In the interior of the collapsing shell, we take the metric to be

$$ds^2 = A(U,V)dUdV \tag{7.10}$$

where A is an arbitrary smooth, non-singular function with

$$U = \tau - r + R_0 \tag{7.11}$$

$$V = \tau + r - R_0. \tag{7.12}$$

R_0 and R_0^* are related in the same way as r and r^*. $C(r)$ has not been specified yet but is chosen such that $C \to 1$ and $\partial C/\partial r \to 0$ as $r \to \infty$. This is done to make sure that the metric is asymptotically Minkowski. $C = 0$ marks the event horizon.

Different $C(r)$ give models for different black holes. For instance, $C(r) = 1 - 2Mr^{-1}$ is for the case of a Schwarzschild black hole and $C(r) = 1 - 2Mr^{-1} + e^2 r^{-2}$ is for the Reissner-Nordstorm black hole. Furthermore, we have assumed that the collapse begins at $\tau = 0$ and have thereby defined R_0 to be the radial coordinate of the boundary of the ball when the collapse begins. Note that when the collapse begins, the coordinates $u = v = V = U = 0$. For $\tau > 0$, we assume that the ball shrinks along the trajectory $r = R(\tau)$. Finally, $A(U, V)$ is added to keep the derivation general at this stage and the final result will not depend on it.

One can now imagine the incoming modes $e^{-i\omega v}$ to undergo distortion during the collapse and emerge as outgoing modes of the form $e^{-i\omega G(u)}$. $G(u) = u$ is the standard mode function in case of no change in the background field. In order to find $G(u)$, we need to use U and V coordinates as crutches since they cover the region inside the black hole. Therefore, we begin by assuming U and V to be some functions

$$U = \alpha(u) \tag{7.13}$$

and

$$v = \beta(V). \tag{7.14}$$

The relation between U and V can be determined at the center of the collapse, i.e. at $r = 0$ as $V = U - 2R_0$. As the radial coordinate $r \geq 0$, we model the collapse as a reflection of null rays about $r = 0$ by applying the boundary condition $\phi = 0$ at $r = 0$. Now, at $r = 0$

$$v = \beta(V) = \beta(U - 2R_0) = \beta[\alpha(u) - 2R_0]. \tag{7.15}$$

The above gives the "trajectory" for the center in terms of the coordinates u and v. The solution for a massless scalar field ϕ governed by the equation $\Box\phi = 0$ and reflecting boundary conditions with the boundary trajectory as (7.15) is given by

$$i(4\pi\omega)^{-1/2}(e^{-i\omega v} - e^{-i\omega\beta[\alpha(u)-2R_0]}). \tag{7.16}$$

The above tells us that an incoming mode $e^{-i\omega v}$ now has a rather complicated outgoing component $e^{-i\omega\beta[\alpha(u)-2R_0]}$. An explicit way to find the form for this outgoing wave is to match the metrics across the surface $r = R(\tau)$. This gives us

$$C(dt^2 - \frac{dR^2}{C^2}) = A(d\tau^2 - dR^2) \tag{7.17}$$

which finally results in the relation

$$\frac{dt}{d\tau} = \{CA(1-\dot{R}) + \dot{R}^2\}^{1/2}C^{-1}. \tag{7.18}$$

where $\dot{R} = dR/d\tau$. Using the above, we see that

$$\begin{aligned} \alpha'(u) = \frac{dU}{du} = \frac{dU}{d\tau}\left(\frac{du}{d\tau}\right)^{-1} &= (1-\dot{R})\left(\frac{dt}{d\tau} - \frac{\dot{R}}{C}\right)^{-1} \\ &= (1-\dot{R})C\{[AC(1-\dot{R}^2) + \dot{R}^2]^{\frac{1}{2}} - \dot{R}\}^{-1} \end{aligned} \tag{7.19}$$

In a similar fashion, we get

$$\beta'(V) = \frac{dv}{dV} = C^{-1}(1+\dot{R})^{-1}\{[AC(1-\dot{R}^2) + \dot{R}^2]^{\frac{1}{2}} + \dot{R}\}. \tag{7.20}$$

Note that all the functions are evaluated at the surface $r = R(\tau)$. Now, since we are interested in the experience of the observers at the timelike infinity (i^+), we focus our attention on the modes piled up in a small strip along the null ray γ which forms the even horizon (see figure 7.1). In other words, we are interested in the case $C \to 0$. As this happens, equations (7.19) and (7.20) take the form

$$\alpha'(u) \approx \frac{(\dot{R}-1)}{2\dot{R}}C \tag{7.21}$$

$$\beta'(V) \approx \frac{A(1-\dot{R})}{2\dot{R}}. \tag{7.22}$$

where we have used $(\dot{R}^2)^{1/2} = |\dot{R}| = -\dot{R}$. Furthermore, the ball approaches the event horizon, $R(\tau)$ can be taylor expanded as

$$R(\tau) = R_h + \nu(\tau_h - \tau) + \mathcal{O}[(\tau_h - \tau)^2)] \tag{7.23}$$

where $R = R_h$ at the horizon, $\tau = \tau_h$ when $R = R_h$, and $\nu = -\dot{R}$. With the above expansion, we can write equation (7.21) as

$$\begin{aligned} \alpha'(u) &\approx \frac{\nu+1}{2\nu}\left\{ \frac{\partial C}{\partial R}\bigg|_{R=R_h} (R - R_h) + \mathcal{O}[(\tau - \tau_h)^2] \right\} \\ &= (\nu + 1)\big\{\kappa(\tau_h - \tau) + \mathcal{O}[(\tau - \tau_h)^2]\big\} \end{aligned} \tag{7.24}$$

where the constant

$$\kappa = \frac{1}{2}\frac{\partial C}{\partial R}\bigg|_{R=R_h} \tag{7.25}$$

is defined to be the surface gravity of the black hole. Now, the coordinate $U = \tau - R_h - \nu(\tau - \tau_h) + R_0$. This gives us

$$U + R_h - R_0 - \tau_h = (\nu + 1)(\tau - \tau_h). \tag{7.26}$$

Substituting the above into (7.24), we get

$$\alpha'(u) = \frac{dU}{du} \approx -\kappa(U + R_h - R_0 - \tau_h). \tag{7.27}$$

Integrating the above, we obtain

$$U \propto e^{-\kappa u} + \text{constant} \tag{7.28}$$

for an asymptotic late time observer. Furthermore, the null ray $u \to \infty$ correspond to the event horizon. Therefore, for large values of u, the incoming modes from $\mathcal{I}^-$ can be traced back to a small strip of values of $v(V)$. Again, since we are interested in this late time observer's experience, we can take A as approximately constant in

(7.22) and integrate to get

$$v = -AV\left(\frac{\nu+1}{2\nu}\right) + \text{constant} \tag{7.29}$$

Finally, substituting (7.29) and (7.28) into the mode form (7.16), we get the late time asymptotic modes to be

$$i(4\pi\omega)^{-1/2}(e^{-i\omega v} - e^{i\omega(ce^{-\kappa u}+d)}) \tag{7.30}$$

where c and d are constants. The above mode form tells us that incoming modes undergo an exponential redshift while escaping, characterized by the time-scale κ^{-1}. We now move on to using these mode distortions due to the background to find the particle production. We expand the *quantum* field ϕ in terms of the positive frequency modes $g_{\omega lm}$

$$\phi = \sum_{lm}\int d\omega(a_{\omega lm}g_{\omega lm} + a^{\dagger}_{\omega lm}g^{*}_{\omega lm}) \tag{7.31}$$

where $g'_{\omega lm}$s are normalized as

$$(g_{\omega_1 l_1 m_1}, g_{\omega_2 l_2 m_2}) = \delta_{l_1 l_2}\delta_{m_1 m_2}\delta(\omega_1 - \omega_2). \tag{7.32}$$

The above modes reduce to the incoming spherical modes $\propto e^{-i\omega v}$ in the remote past. We choose the quantum state here to be the vacuum characterized by

$$a_{\omega lm}\left|0\right\rangle = 0 \quad \forall \omega, l, m \tag{7.33}$$

In other words, we take no incoming radiation from the past null infinity.
As per our ray tracking resulting in equation (7.30), a $v = constant$ ray emerges out of the collapse as a $u = constant$ ray with $u = h(v)$ or equivalently, $v = h^{-1}(u) \equiv H(u)$.

Consequently, asymptotic modes look like

$$f_{\omega lm} \sim \frac{Y_{lm}(\theta,\phi)}{\sqrt{4\pi\omega}} \times \begin{cases} e^{-i\omega v} & \mathcal{I}^- \\ e^{-i\omega H(u)} & \mathcal{I}^+ \end{cases} \tag{7.34}$$

and

$$F_{\omega lm} \sim \frac{Y_{lm}(\theta,\phi)}{\sqrt{4\pi\omega}} \times \begin{cases} e^{-i\omega u} & \mathcal{I}^+ \\ e^{-i\omega h(v)} & \mathcal{I}^- \end{cases} \tag{7.35}$$

where we have already calculated $v = H(u)$ as

$$v = -ce^{-\kappa u} - d. \tag{7.36}$$

Let us call the limiting value of v as $u \to \infty$ (or as we approach the event horizon) as $v = v_0$. The above equation then becomes

$$v = v_0 - ce^{-\kappa u}. \tag{7.37}$$

Note that the above equation limits the values of v to be less than v_0, as it should since all the rays $v > v_0$ fall into the singularity. Next, we can get $u = h(v)$ by just inverting the above relation. This gives

$$u = h(v) = -\frac{1}{\kappa}\ln\left(\frac{v_0 - v}{c}\right). \tag{7.38}$$

We are ready to calculate the Bogoliubov coefficients between the modes $F_{\omega lm}$ and $f_{\omega lm}$. Tracing the $F_{\omega lm}$ modes on $\mathcal{I}^+$ to $\mathcal{I}^-$, we get

$$F_{\omega lm} \sim \frac{1}{\sqrt{4\pi\omega}} \begin{cases} e^{\frac{i\omega}{\kappa}\ln((v_0 - v)/c)} & v < v_0 \\ 0 & v > v_0 \end{cases} \tag{7.39}$$

where we have suppressed the angular dependence for convenience. On the null hypersurface $\mathcal{I}^-$, we can write the transformation between the modes as

$$F_{\omega lm} = \int_0^\infty d\omega' \{\alpha^*_{\omega\omega' lm} f_{\omega' lm} - \beta_{\omega\omega' lm} f^*_{\omega' lm}\}. \tag{7.40}$$

where

$$f_{\omega' lm} = \frac{1}{\sqrt{4\pi\omega}} e^{-\omega' v}, \tag{7.41}$$

and we use the notation $\alpha_{\omega\omega' lm} = \alpha_{\omega lm,\omega' lm}$ (and a similar one with $\beta_{\omega\omega' lm}$), since the angular dependence is the same in all the modes we consider. The above transformation equation is just a continuous variable generalization of (6.14).

Our aim is to obtain $\sum_{\omega'} |\beta_{\omega'\omega lm}|^2$, or equivalently, the expectation number of particles in the late time vacuum in the mode ωlm. We follow a clever set of tricks for this. First, we take the fourier transform of $F_{\omega_l m}$ as

$$\tilde{F}_{\omega lm}(\Omega) = \int_{-\infty}^\infty dv\; e^{i\Omega v} F_{\omega lm}(v) \tag{7.42}$$

$$\Leftrightarrow F_{\omega lm}(v) = \int_{-\infty}^\infty \frac{d\Omega}{2\pi}\; e^{-i\Omega v} \tilde{F}_{\omega lm}(\Omega) \tag{7.43}$$

We can think of Ω as playing the role of frequency and v the role of time in analogy with a standard time-frequency fourier transform. Next, we write (7.43) as

$$F_{\omega lm}(v) = \int_{-\infty}^0 \frac{d\Omega}{2\pi}\; e^{-i\Omega v} \tilde{F}_{\omega lm}(\Omega) + \int_0^\infty \frac{d\Omega}{2\pi}\; e^{-i\Omega v} \tilde{F}_{\omega lm}(\Omega) \tag{7.44}$$

$$= \int_0^\infty \frac{d\Omega}{2\pi} \Big\{ e^{-i\Omega v} \tilde{F}_{\omega lm}(\Omega) + e^{i\Omega v} \tilde{F}_{\omega lm}(-\Omega) \Big\}. \tag{7.45}$$

Comparing the above equation with (7.40), we see that

$$\frac{\tilde{F}_{\omega lm}(\omega')}{2\pi} = \frac{\alpha^*_{\omega\omega' lm}}{\sqrt{4\pi\omega'}} \tag{7.46}$$

which gives

$$\alpha^*_{\omega\omega' lm} = \frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}} \int_{-\infty}^{v_0} dv\ e^{i\omega' v} e^{\frac{i\omega}{\kappa}\ln((v_0 - v)/c)}. \tag{7.47}$$

In a similar manner, we can get

$$\beta_{\omega'\omega lm} = \frac{-1}{2\pi}\sqrt{\frac{\omega'}{\omega}} \int_{-\infty}^{v_0} dv\ e^{-i\omega' v} e^{\frac{i\omega}{\kappa}\ln((v_0 - v)/c)}. \tag{7.48}$$

Changing variables with $v' = v_0 - v$, the above coefficients become

$$\alpha^*_{\omega\omega' lm} = \frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}} e^{i\omega v_0} \int_{0}^{\infty} dv'\ e^{-i\omega' v'} e^{\frac{i\omega}{\kappa}\ln(v'/c)} \tag{7.49}$$

and

$$\beta_{\omega'\omega lm} = \frac{-1}{2\pi}\sqrt{\frac{\omega'}{\omega}} e^{i\omega v_0} \int_{0}^{\infty} dv'\ e^{i\omega' v'} e^{\frac{i\omega}{\kappa}\ln(v'/c)}. \tag{7.50}$$

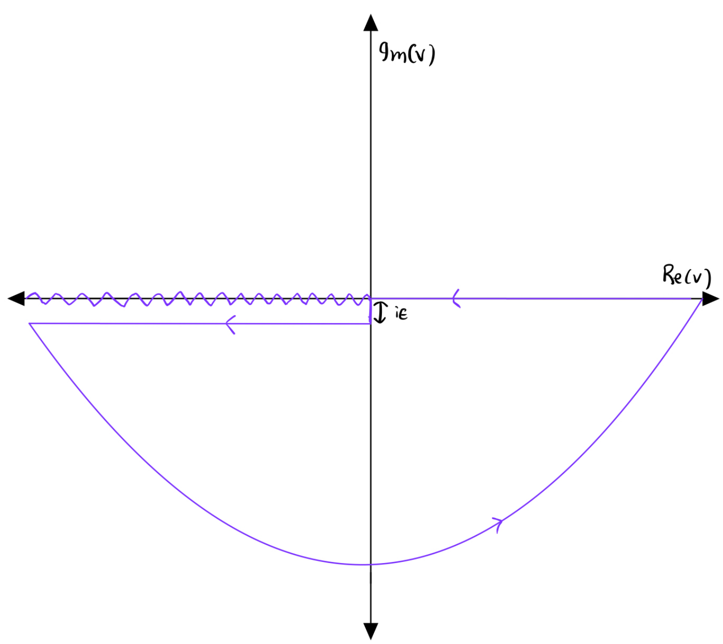

Figure 7.2: Contour for the integral (7.51)

The above functions are well behaved everywhere except a logarithmic branch cut along the negative real axis. Furthermore, since there are no poles located in the contour illustrated in figure 7.2,

$$\oint_0^\infty dv' \, e^{-i\omega' v'} e^{\frac{i\omega}{\kappa}\ln(v'/c)} = 0. \tag{7.51}$$

Expanding out the above integral, we have

$$\int_0^\infty dv' \, e^{-i\omega' v'} e^{\frac{i\omega}{\kappa}\ln(v'/c)} + \int_{-\infty}^0 dv' \, e^{-i\omega'(v'-i\epsilon)} e^{\frac{i\omega}{\kappa}\ln(v'/c-i\epsilon)} = 0. \tag{7.52}$$

Changing variables ($v' \to -v'$) in the second term and using $\ln(v'/c - i\epsilon) = -\pi i + \ln(v'/c)$, and taking $\epsilon \to 0$ we get

$$e^{\frac{\pi\omega}{\kappa}} \int_0^\infty dv' \, e^{i\omega' v'} e^{\frac{i\omega}{\kappa}\ln(v'/c)} = -\int_0^\infty dv' \, e^{-i\omega' v'} e^{\frac{i\omega}{\kappa}\ln(v'/c)}. \tag{7.53}$$

Noticing that the integrals are the α and β coefficients, we can write

$$e^{\frac{\pi\omega}{\kappa}} |\beta_{\omega'\omega lm}| = |\alpha_{\omega'\omega lm}|. \tag{7.54}$$

Furthermore, using the normalization of the coefficients

$$\sum_{\omega'} \left(|\alpha_{\omega'\omega lm}|^2 - |\beta_{\omega'\omega lm}|^2\right) = 1, \tag{7.55}$$

we get the mean number $\langle N_\omega \rangle$ of particles created in the mode ωlm as

$$\langle N_{\omega lm} \rangle = \sum_{\omega'} |\beta_{\omega'\omega lm}|^2 = \frac{1}{e^{\frac{2\pi\omega}{\kappa}} - 1}. \tag{7.56}$$

The above is a Planckian spectrum with the temperature

$$T_H = \frac{\kappa}{2\pi} \tag{7.57}$$

For the case of a non-rotating Schwarzschild black-hole $\kappa = 1/(4M)$, and the Hawking temperature reads as

$$T_H = \frac{1}{8\pi M}! \tag{7.58}$$

Let us take a minute to admire the above equation. It has been obtained by the four most culminating theories of modern physics: thermodynamics (k_B), gravity (G), quantum mechanics ($\hbar$), and special relativity (c). Our derivation, although painfully long and technical, gives us a physical understanding of the journey of a mode from the past infinity, all the way to its reception (demise) at the future infinity (singularity).

In the process of deriving the above, we have skipped some technical complications that come when talking about the full four dimensional problem. We have also not alluded to back-scattering of the modes, the adiabatic approximation, or the trans-Planckian problem and its resolution. It suffices to here say that these technicalities do not change the above result drastically, and the main features, like its thermal nature and the fact that black holes indeed radiate, persist. We refer the reader to ref. [22, 36] for these details.

7.3 Introduction to Information Loss

The above discussion indicates that an observer at late times will be bombarded with a flux of particles whose temperature is given by (7.58) in the case of a non-rotating Schwarzschild black hole of mass M. Corresponding to the positive frequency (energy) modes that a late time observer experiences coming out of a black hole, there are negative frequency (energy) modes that enter into the black hole as well. These "negative" energy modes reduce the mass of the black hole. In other words, the black hole evaporates and this phenomenon is thus termed Hawking radiation. It can be shown that the rate of evaporation increases as the black hole shrinks in size (mass) and as this continues, there comes a point when the black hole completely evaporates in a finite amount of time. This fact, a seemingly innocuous one, is a genesis of all the modern day issues that come under the umbrella term, "the black hole information problem/paradox". We will explain in this section how exactly a black hole completely radiating give rise to a problem/paradox. Our discussion here follows the treatment in ref. [37] and [38].

We know that entanglement between observables in a region of space is a pervasive feature of quantum field theory. This is even true if two regions we have considered are causally complementary. For instance, we know that the Minkowski vacuum is a highly entangled state. Even if we consider two Rindler wedges that are causally disconnected, the entanglement between the wedges (more specifically, the observables in the wedges) persists. This is made more precise by the Reeh-Schlieder theorem which states that one can construct *any* state $|\psi\rangle$ of the quantum field, no matter where its contents lie in space, by selecting *any* region $\mathcal{O}$ in space and acting on the vacuum by the elements of the local algebra $\mathcal{A}(\mathcal{O})$ [39]. The Reeh-Schlieder property applies not only to the vacuum, but to any state with bounded energy.

Now let us consider a moment in time after the black hole has formed, defined by the Cauchy hypersurface Σ_0. If we take any pure state $|\psi\rangle$ of the quantum field on this hypersurface, there will be entanglement between the observables inside and outside the horizon of the black hole. Although the full state is pure, the description for an outside observer is a mixed one since they have access to only a part of Σ_0 and have effectively "traced" over the region inside the horizon. Now once the black hole completely evaporates in a finite time, the entire state of the quantum field at a later time slice Σ_1 is entirely mixed! Quantum mechanics tells us that a mixed state, in this case a thermal state, carries less information in comparison to a pure state. For instance, the black hole might have been made up of all kinds of matter with a variety of properties but the thermal radiation is only characterized by its temperature and no more information is available to us once the evaporation completes. Thus, the evaporation of a black hole has led to a *loss of information* and risked one of the basic tenets of quantum mechanics: the principle of *unitarity*, by the virtue of which which pure states have to evolve to pure states.

One realizes that the situation is more gruesome when even the *equivalence principle* is put to questioning. Here is how it happens. The equivalence principle tells us

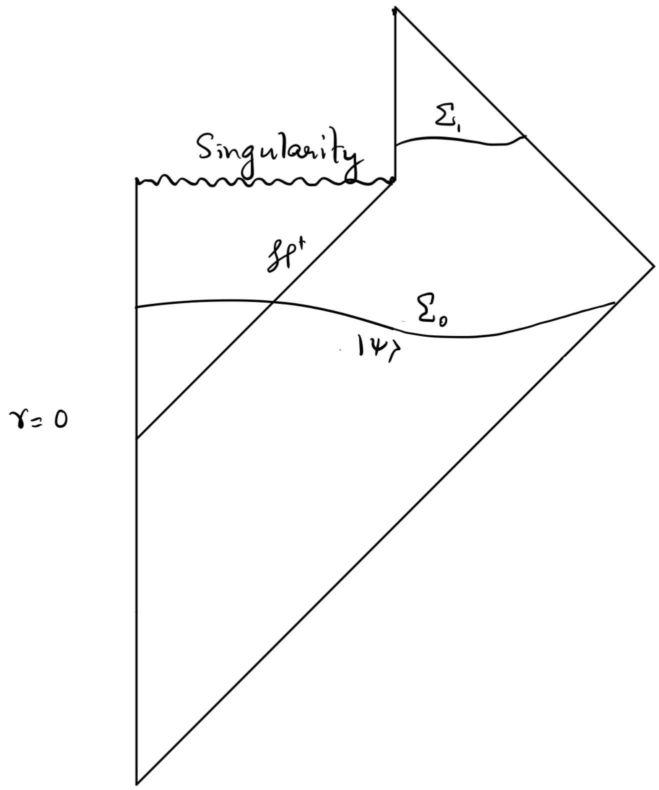

Figure 7.3: A rough sketch of a Penrose diagram for a black hole undergoing and completing its evaporation

that spacetime near the black hole horizon is locally Minkowski and therefore for an observer close to the horizon, the Rindler description is valid. Since the Rindler wedges are entangled, the black hole horizon must also hide correlations like the Rindler horizon. Unless we get rid of these correlations, the description outside the black hole remains mixed. Therefore, the equivalence principle necessitates a mixed state description for the exterior. Now in case we try to resolve the paradox by claiming that the effective description on the outside may not be effectively mixed (which seems like a way out), we are forced to question the equivalence principle. The story does not end here. The risk of validity of the equivalence principle is greatly increased in a further development to this paradox, namely in the AMPS paradox [28], which is out of the scope of this thesis.

So what is the final verdict on the above problem? It is largely agreed upon that the information does indeed come out of a black hole. It is believed that the information lies in the exponentially small correlations between the radiation that comes out over the lifetime of a black hole. It was Don Page who first quantified how the entanglement entropy for the radiation looks over the lifetime of the black hole assuming a unitary evaporation [24, 25]. With this information, one can plot the *Page curve* that gives the structure of entanglement entropy of the radiation vs time. This curve is not just applicable to black holes, since the underlying quantum information theoretic result holds for any bipartite quantum system. In this sense, the result is quite general and is worth a discussion even if one believes that the black hole evaporation process is not unitary.

7.4 Page Curve

Before we move to describe the Page curve, we discuss a purely quantum information theoretic result given by Page [24] that helps us deduce the structure of the Page curve.

Consider two systems A and B with dimensions $|A|$ and $|B|$, where we use $|.|$ to denote the dimension of the system. Their respective Hilbert spaces are denoted as $\mathcal{H}_A$ and $\mathcal{H}_B$. The combined Hilbert space is then given as

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B \tag{7.59}$$

with dimension $N = |A||B|$. Consider a pure state $|\psi\rangle \in \mathcal{H}_{AB}$. The density matrices ρ_A and ρ_B for the subsystems A and B respectively can be obtained as

$$\rho_A = \mathrm{Tr}_B\, \rho \tag{7.60}$$

$$\rho_B = \mathrm{Tr}_A\, \rho \tag{7.61}$$

where $\rho = |\psi\rangle \langle\psi|$. The Von Neumann entropy of each subsystem is then given as

$$S_A = -\operatorname{Tr}(\rho_A \ln \rho_A) \tag{7.62}$$

$$S_B = -\operatorname{Tr}(\rho_B \ln \rho_B) \tag{7.63}$$

Page then asks what is the average of, say S_A, summed over all possible pure states ρ, denoting this quantity by $\langle S_A\rangle$. This average is taken over the unitarily invariant Haar measure. In his seminal 1993 paper [24], Page goes on to show that for $1 \ll |A| < |B|$

$$\langle S_A\rangle \simeq \ln|A| - \frac{1}{2}\frac{|A|}{|B|} \tag{7.64}$$

In other words, if $|B| \gg |A|$, the average entropy of system A is very close to the entropy of a maximally mixed state (whose Von Neumann entropy is $\ln|A|$). D. Harlow [36] provides a proof for this that is different from that originally given by Page. It is relatively cleaner and does not require taking the limit $|A| \ll |B|$. We shall expand upon the proof given in ref. [36], filling in the missing steps. Before elaborating on the proof, we shall quickly review certain distance measures between quantum states that one often comes across in quantum information theory, let alone in the following proof.
The proximity of two quantum states ρ and σ can be defined using the L_1 norm defined as

$$\|M\|_1 = \operatorname{Tr}\sqrt{M^\dagger M} \tag{7.65}$$

where $M = \rho - \sigma$. One could alternatively use the L_2 norm defined as

$$\|M\|_2 = \sqrt{\operatorname{Tr} M^\dagger M} \tag{7.66}$$

Furthermore, using the Cauchy-Schwartz inequality, it is not difficult to show that

$$\|M\|_2 \leq \|M\|_1 \leq \sqrt{N}\|M\|_2 \tag{7.67}$$

where M is any operator and N is the dimension of the Hilbert space. At this stage, ref. [36] states a version of Page's result (7.64), which is now commonly referred to as the Page's theorem. The theorem reads as follows:

Theorem 7.4.1 (Page's Theorem)**.** *Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ be any bipartite system. Let $|\psi\rangle \in \mathcal{H}_{AB}$ be a random state obtained by acting a random unitary U that is sampled uniformly from the Haar measure, on a particular state $|\psi_0\rangle \in \mathcal{H}_{AB}$. Let ρ_A and ρ_B be the reduced density matrices for systems A and B. Then, we have*

$$\int dU \|\rho_A(U) - \frac{I_A}{|A|}\|_1 \leq \sqrt{\frac{|A|^2 - 1}{|A||B| + 1}}, \tag{7.68}$$

where I_A is the identity matrix on A. In the limit $|A| \ll |B|$, the typical deviations of ρ_A from a maximally mixed state is extremely small.

Proof. Using Jensen's inequality and (7.67), we have

$$\left(\int dU \|\rho_A(U) - \frac{I_A}{|A|}\|_1\right)^2 \leq \int dU \left(\|\rho_A(U) - \frac{I_A}{|A|}\|_1\right)^2 \leq |A| \int dU \left(\|\rho_A(U) - \frac{I_A}{|A|}\|_2\right)^2 \tag{7.69}$$

We now evaluate the rightmost part of the inequality. Using the hermiticity of ρ and I_A, we have

$$|A| \int dU \left(\|\rho_A(U) - \frac{I_A}{|A|}\|_2\right)^2 = |A| \int dU \,\mathrm{Tr}\left(\rho_A^2(U) - 2\frac{\rho_A(U)}{|A|} + \frac{I_A}{|A|^2}\right) \tag{7.70}$$

$$= |A| \int dU \,\mathrm{Tr}\big(\rho_A(U)^2\big) - 1 \tag{7.71}$$

where we have used the fact that $\mathrm{Tr}(\rho_A) = 1$ and $\mathrm{Tr}(I_A) = |A|$ in the second step. We now move on to evaluate the quantity in the integral. For this, we would first like to express the integrand purely in terms of unitaries, which we can then integrate using certain identities.

Let us denote the basis of systems A and B as $|i_A\rangle$ and $|i_B\rangle$ where i_A and i_B range from 0 to the cardinality of the respective spaces. We begin by choosing $|\psi_0\rangle = |0_A 0_B\rangle$.

The choice of $|\psi_0\rangle$ does not affect the final result since the Haar random unitary U that gives us $|\psi\rangle$ samples uniformly from the whole space (by definition of the Haar measure).
In the case of a single quantum system, we would have written $|\psi\rangle = \sum_i U_{i0} |0\rangle$. In the bipartite case, each index of U_{ij} is simply a double index $i = (i_A, i_B)$ and $j = (j_A, j_B)$. With this, we can write

$$|\psi\rangle = \sum_{i_A,i_B} U_{(i_A,i_B),(0,0)} |i_A\rangle |i_B\rangle \tag{7.72}$$

The corresponding density matrix is given as

$$\rho = |\psi\rangle \langle\psi| = \sum_{i_A,i_B} \sum_{j_A,j_B} U_{(i_A,i_B),(0,0)} U^\dagger_{(0,0),(j_A,j_B)} |i_A, i_B\rangle \langle j_A, j_B| \,. \tag{7.73}$$

We can then write the reduced density matrix for system A as

$$\rho_A(U) = \sum_{i_A,i_B,j_A} U_{(i_A,i_B),(0,0)} U^\dagger_{(0,0),(j_A,j_B)} |i_A\rangle \langle j_A| \tag{7.74}$$

where we have used orthogonality of the basis elements. Now, we have

$$\rho_A^2(U) = \sum_{i_A,i_B,j_A,l_A,k_B} U_{(i_A,i_B),(0,0)} U^\dagger_{(0,0),(j_A,j_B)} U_{(j_A,k_B),(0,0)} U^\dagger_{(0,0),(l_A,k_B)} |i_A\rangle \langle l_A| \tag{7.75}$$

We now take the trace on both sides of the above equation. This gives

$$\mathrm{Tr}\big(\rho_A(U)^2\big) = \sum_{i_A,j_A,i_B,k_B} U_{(i_A,i_B),(0,0)} U_{(j_A,k_B),(0,0)} U^\dagger_{(0,0),(j_A,j_B)} U^\dagger_{(0,0),(i_A,k_B)} \tag{7.76}$$

At this stage we state without proof a useful result in the theory of integration of unitary polynomials over the Haar measure, given as [40]

$$\int dU\, U_{ij}U_{kl}U^\dagger_{mn}U^\dagger_{op} = \frac{1}{N^2-1}(\delta_{in}\delta_{kp}\delta_{jm}\delta_{lo} + \delta_{ip}\delta_{kn}\delta_{jo}\delta_{lm}) \\ + \frac{1}{N(N^2-1)}(\delta_{in}\delta_{kp}\delta_{jo}\delta_{lm} + \delta_{ip}\delta_{kn}\delta_{jm}\delta_{lo}), \quad (7.77)$$

where N is the dimensionality of the unitary operator U. Using the above result in (7.76) with each index above being a double index in this case (for example, $i = (i_A, j_A), k = (j_A, k_B)$, etc.), we get

$$|A|\int dU\, \mathrm{Tr}\big(\rho_A(U)^2\big) - 1 = |A|(|A|^2|B| + |B|^2|A|)\left(\frac{1}{N^2-1} - \frac{1}{N(N^2-1)}\right) - 1 \quad (7.78)$$

where $N = |A||B|$. Simplifying the above and taking a square root on both sides of (7.69), we have

$$\int dU \|\rho_A(U) - \frac{I_A}{|A|}\|_1 \leq \sqrt{\frac{|A|^2-1}{|A||B|+1}}. \quad (7.79)$$

□

Clearly, in the limit $|A| \ll |B|$, we see that the averaged trace norm distance between ρ_A and $I_A/|A|$ approaches 0, or in other words, deviations of ρ_A from the maximally mixed state are extremely small.

At this stage, the application of the above version of Page's theorem to construct the qualitative structure of the Page curve (figure 7.4) is straightforward. In this case, our bipartite system is that of the black hole (BH) and radiation (R), which we assume is in a pure state. Now, since the process of evaporation is complicated, it is tempting to consider this pure state in $\mathcal{H}_{BH} \otimes \mathcal{H}_R$ to be effectively random. This makes the previous theorem applicable to our radiation and black hole subspaces. During the early times of the radiation, the thermal entropy of the radiation $\log |R|$ is low since there is barely any radiation to begin with ($|R|$ is small). As more radiation comes out, we expect $|R|$ and hence the thermal entropy of the radiation to grow. We can also make an educated guess of how this growth looks. Considering

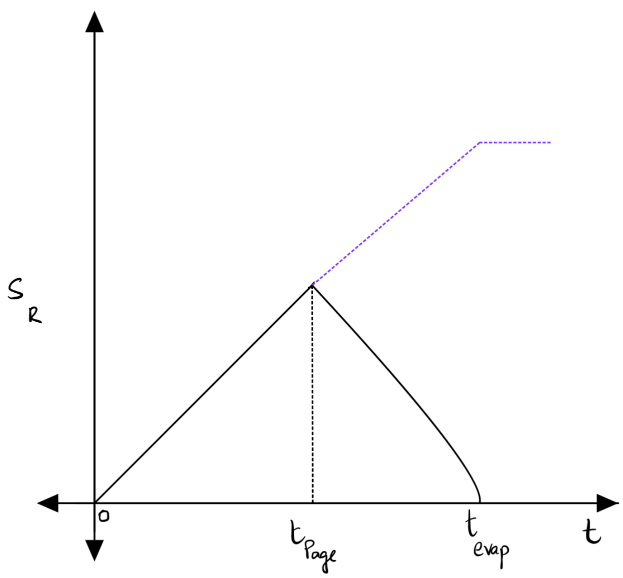

Figure 7.4: Page curve (black) depicting the entanglement entropy of the radiation S_R vs time t. Hawking's initial proposal (violet) depicting a linear increase in entropy until the end of the evaporation

the evaporation in terms of photons and modelling it as (1+1) dimensional photon gas (ignoring the angular momentum of the radiation), the thermal entropy goes as

$$S_R^{\text{Thermal}} \propto tT \tag{7.80}$$

where t is the time since the black hole began evaporating and $T(\propto M^{-1})$ is the temperature of the radiation. The linear dependence in T is reminiscent of the spacial dimension which is unity in this simplified model of radiation. On the other hand, the thermal entropy for the black hole is just the Beckenstein-Hawking entropy ($\propto A \propto M^2$). Therefore, for $t \ll M^3$, $S_R^{\text{Thermal}} \ll S_{BH}^{\text{Thermal}}$. During this early stage of evaporation, we can apply the Page's theorem by considering system A to be the radiation system R. Since during the early stages $|BH| \gg |R|$, we have

$$S_R \approx S_R^{\text{Thermal}} \propto tT \qquad t \ll M^3 \tag{7.81}$$

Thus, for some time, we expect the radiation entropy S_R grows linearly in time. In the mid-stages of the evaporation process, $|R| \approx |BH|$ and thus $S_R^{\text{Thermal}} \approx S_{BH}^{\text{Thermal}}$. This time in the black hole evaporation process is commonly referred to as the *Page time*. It occurs roughly halfway through the evaporation process. After the Page time, we can apply Page's theorem in the other direction considering system A to be that of the black hole. With this, we get that $S_{BH} \approx S_{BH}^{\text{Thermal}}$. Since the total state is pure, at all stages of the radiation we have $S_{BH} = S_R$. This is true of any subsystem of a pure state and can be easily seen using the Schmidt decomposition. Thus, towards the end, when $|BH|$ is small, or in other words when the radiation is close to complete, we expect $S_R \to 0$. The time dependence for this latter half looks like [41, 36]

$$S_R \approx S_{BH}^{\text{Thermal}} \propto S_0 \left(1 - \frac{t}{t_{\text{evap}}}\right)^{2/3} \qquad t_{\text{Page}} < t < t_{\text{evap}} \tag{7.82}$$

where S_0 is the original coarse grained entropy of the black hole and t_{evap} is the time taken for the black hole to evaporate. The above reasoning thus justifies the qualitative features of the black hole evaporation.

The shape of the Page curve indicates that during the initial times of the evaporation the information lies in correlations between the internal modes of the black hole and the radiation. The radiation itself, however, contains no information leading to a growth in S_R. As the black hole evaporates beyond the Page time, the earlier entanglement between the black hole and the radiation now manifests as entanglement between late and early time radiation. This then progressively starts to decrease the entropy in the radiation since information starts to "leak" out of the black hole. However, at any given instant, an observer cannot notice this temporal spread of information, and is led to believe the radiation is thermal at all times. By the end of the evaporation, all of the information is out in the form of correlations between early and late time radiation, thereby finally providing a template for the resolution to the paradox.

Page curve is an elegant proposal and is widely accepted to be true in the case of the unitary evaporation of the black hole. Some physicists believe that reproducing the Page curve is synonymous to having solved the black hole information problem and that there is nothing else left to do. Recent developments in AdS/CFT models and techniques to calculate entanglement entropy have led to a realization of the Page curve, thereby showing that Hawking's result is indeed compatible with unitarity [42, 43]. We refer the reader to [44] for a review of these developments, and end our discussion on the information problem.

Chapter 8

Conclusion

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