Chapter 4

Calculus

4.1 The First Differential

Linear functions have such pleasant properties relative to non-linear functions that one is tempted to erase the latter wherever they occur and replace them with the former. Under certain circumstances, such replacement is legitimate. Note, to begin with that the number a and the linear function ax provide a good linear approximation of the function $f\colon \mathbb{R} \to \mathbb{R}$ at $\overline{x} \in \mathbb{R}$ if the error involved in the approximation,

$$\epsilon(v) \equiv f(\overline{x} + v) - f(\overline{x}) - av$$

is small relative to v:

$$\lim_{\|v\| \to 0} \frac{\epsilon(v)}{\|v\|} = \frac{f(\overline{x} + v) - f(\bar{x}) - av}{v} = 0$$

Similarly, a vector $\vec{a}$ and the associated linear function, $\vec{a} \cdot v$, provide a good linear approximation of $f\colon \mathbb{R}^n \to \mathbb{R}$ at the point $\overline{x} \in \mathbb{R}^n$ if the error involved in the approximation,

$$\epsilon(v) \equiv f(\overline{x} + v) - f(\overline{x}) - \vec{a} \cdot v$$

is small relative to the norm of v:

$$\lim_{\|v\| \to 0} \frac{\epsilon(v)}{\|v\|} = \frac{f(\overline{x} + v) - f(\bar{x}) - \vec{a} \cdot v}{\|v\|} = 0$$

Generalizing slightly we have:

Definition 4.1*.* The m by n matrix A is a good linear approximation of $f\colon \mathbb{R}^n \to \mathbb{R}^m$ if

$$\lim_{\|v\| \to 0} \frac{\epsilon(v)}{\|v\|} = \frac{\|f(\overline{x} + v) - f(\bar{x}) - Av\|}{\|v\|} = 0$$

where the ith row of A provides the approximation of f^i

This definition subsumes as special cases the circumstances in which $m = 1$ and $A = \vec{a}$ or $m = n = 1$ and $A = a$.

Two questions remain regarding the use of linear approximations. Which functions have good linear approximations? When such approximation exists, how can they be identified? Differential calculus was invented, in no small part, to provide the answers to these questions.

Definition 4.2. Suppose $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is a vector-valued function with the real-valued components $f = (f^1, \dots, f^m)$, $f^i\colon \mathbb{R}^n \to \mathbb{R}$. The function f is *differentiable at a point* $\overline{x}$ iff there is a matrix A s.t.

$$\lim_{\|v\|\to 0} \frac{\|f(\overline{x}+v) - f(\overline{x}) - Av\|}{\|v\|} = 0$$

in which case the matrix A is called the *derivative at* $\overline{x}$ and the linear transformation Av is called the *differential at* $\overline{x}$. The function is *differentiable* iff it is differentiable at every point in the domain.

The requirement for f to be differentiable at $\overline{x}$ is, of course, precisely the same as the requirement for the linear transformation Av to provide a good approximation of f at $\overline{x}$. “Differentiable” is thus synonymous with “has a good linear approximation” and “differential” is synonymous with “best linear approximation”.

As special cases note that when $m = n = 1$, the matrix A has but a single element

$$a = f'(\overline{x}) = \frac{df(\overline{x})}{dx}$$

the “derivative of f evaluated at $\overline{x}$”. Geometrically, a is the slope of $f(x)$ at $\overline{x}$ and av is the equation of the “tangent hyperplane”.

□ *Problem* 4.1. Using *Mathematica*, plot the tangent to the graph of

$$f(x) = (x+2)(x^2+1)x(x-1)(x-2)$$

at $x = 1.6$.

□ *Problem* 4.2. Using *Mathematica*, determine the *critical points* of the function specified in Problem 4.1, i.e., the points on the graph of f where $df/dx = 0$.

This is illustrated in Figure 4.1 for the case in which $f(x) = \ln(x)$, $\overline{x} = 2$ and

$$\begin{aligned} f(\overline{x}+v) &= f(\overline{x}) + f'(\overline{x})v + \mu(v) \text{ or} \\ \ln(2+v) &= \ln(2) + \frac{1}{2}v + \mu(v) \end{aligned}$$

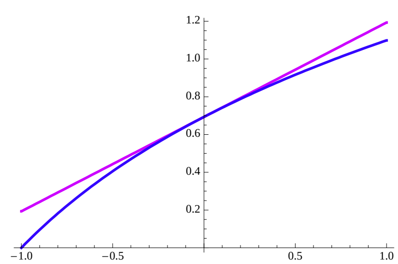

Figure 4.1: Best Linear Approximation

When $m = 1$ but $n > 1$, the matrix A has but a single row

$$\vec{a} = f_x(\overline{x}) = \left(\frac{\partial f(\overline{x})}{\partial x_1}, \dots, \frac{\partial f(\overline{x})}{\partial x_n}\right)$$

which is called the *gradient* of $f\colon \mathbb{R}^n \to \mathbb{R}$ evaluated at $\overline{x}$.

Consider, for example, the case in which $f(x) = {x_1}^2 + x_2$, $\overline{x} = (2,3)$, $f(2,3) = 7$ and $f_x(2,3) = (4,1)$. The graph of this function would be a surface in $\mathbb{R}^3$ and the differential would be a hyperplane tangent to this surface at the point $(2,3,7)$.

The gradient itself is best illustrated in $\mathbb{R}^2$ — see Figure 4.2. The parabola represents a *level contour* for f, i.e., the inverse image of a particular point in the range. The gradient at the point $\overline{x} = (2, 3)$ illustrates three *very* important facts about the gradient: (i) It is orthogonal to a tangent to the level contour at the point at which it is evaluated. (ii) It points in the direction in which the function increases at the greatest rate, i.e., the direction of steepest ascent. (iii) Its length is equal to the slope of the function in the direction of steepest ascent.

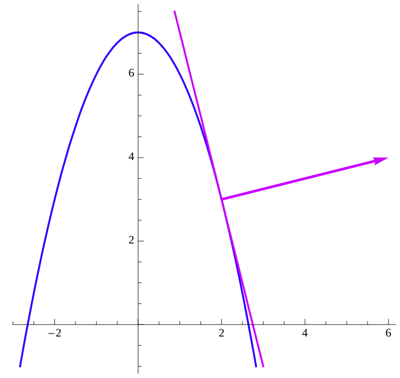

Figure 4.2: The Gradient of $x_1{}^2 + x_2$ at $\overline{x} = (2, 3)$

The latter fact follows from the fact that $f(x + v) \approx f(x) + f_x \cdot v$. Now take v to be a unit length movement in the direction f_x so that $v = f_x/\|f_x\|$, substitute for v and rearrange to get

$$f\left(x + \frac{f_x}{\|f_x\|}\right) - f(x) \approx f_x \cdot \frac{f_x}{\|f_x\|} = \|f_x\|$$

Recall now that when you graph the function $f \colon \mathbb{R} \mapsto \mathbb{R}$ corresponding to $y = x^2$, for example, you are drawing a picture of the set $\{(x, y) \in \mathbb{R}^2 \mid y = x^2\}$. Similarly

Definition 4.3. The *graph* of $f \colon \mathbb{R}^n \mapsto \mathbb{R}$ is the set $\{(x, y) \in \mathbb{R}^{n+1} \mid y = f(x)\}$.

□ *Problem* 4.3. Describe the graph of $y = a \cdot x$ where $a \in \mathbb{R}^n$, i.e., the set $G \equiv \{(y, x) \in \mathbb{R}^{n+1} \mid y = a \cdot x\}$. What is the gradient of this map? What is the value of y when $x = a/\|a\|$? Is G a linear subspace? Is G^C open and dense in the usual topology?

□ *Problem* 4.4. Suppose $f(x_1, x_2) = x_1 + \ln(1 + x_2)$. Calculate and illustrate $f_x(\overline{x})$ for $\overline{x} = (0, 0), (1, 0), (0, 1)$ and $(0, -1)$. Hint: In *Mathematica* `Log[x]` is used to compute the natural log of x.

In general, when $f \colon \mathbb{R}^n \to \mathbb{R}^m$:

$$A = f_x(\overline{x}) = \begin{bmatrix} \partial f^1(\bar{x})/\partial x_1 & \cdots & \partial f^1(\overline{x})/\partial x_n \\ \vdots & \ddots & \vdots \\ \partial f^m(\overline{x})/\partial x_1 & \cdots & \partial f^m(\overline{x})/\partial x_n \end{bmatrix}$$

is called the *Jacobian* of f. Notice that the ith row of the Jacobian is simply the gradient of the ith component of f.

□ *Problem* 4.5. Suppose that $f \colon \mathbb{R}^n \to \mathbb{R}^m$. Show that if f is differentiable at $\overline{x}$ then it must also be continuous at $\overline{x}$.

4.2 Quadratic Forms

Having done the best we can with a linear approximation, it is natural to ask what improvement might be possible with a quadratic approximation, i.e., a mapping $Q : \mathbb{R}^n \mapsto \mathbb{R}$ where

$$\begin{aligned} Q(x) &= x^T B x \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} x_i x_j \\ &= b_{11} x_1^2 + (b_{12} + b_{21}) x_1 x_2 + (b_{13} + b_{31}) x_1 x_3 + \cdots + b_{nn} x_n^2 \end{aligned}$$

and B is an n by n matrix.

□ *Problem* 4.6. Show that for any n by n matrix B

$$x^T Bx \equiv x^T \left[\frac{B + B^T}{2}\right] x, \quad \forall x$$

and thus that B can be taken to be a symmetric matrix, $b_{ij} = b_{ji}$, without any loss of generality.

Such *quadratic forms* have a number of interesting properties. Note first that if $\lambda \in \mathbb{R}$ then

$$\begin{aligned} Q(\lambda x) &= (\lambda x)^T B(\lambda x) \\ &= \lambda^2 x^T Bx \\ &= \lambda^2 Q(x) \end{aligned}$$

Quadratic forms are thus *homogeneous of the second degree* where

Definition 4.4. A function $f : \mathbb{R}^n \mapsto \mathbb{R}$ is *homogeneous of degree* ρ iff

$$f(tx) = t^\rho f(x), \quad \forall t \neq 0$$

The fact that quadratic forms are homogeneous of the second degree means that their graphs are symmetric about the y-axis

$$\begin{aligned} Q(-x) &= (-1)^2 Q(x) \\ &= Q(x) \end{aligned}$$

There are only a few possible types of quadratic forms.

- The quadratic form is *positive definite* iff $Q(x) > 0, \quad \forall x \neq 0$.
- The quadratic form is *negative definite* iff $Q(x) < 0, \quad \forall x \neq 0$.
- The quadratic form is *positive semi-definite* iff $Q(x) \geq 0, \quad \forall x \neq 0$.
- The quadratic form is *negative semi-definite* iff $Q(x) \leq 0, \quad \forall x \neq 0$.
- The quadratic form is *indefinite* iff $Q(x') > 0$ for some x' and $Q(x'') < 0$ for some x''.

These possibilities can be nicely illustrated for the case in which $n = 2$. The case of a positive definite quadratic form where

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

is illustrated in Figure 4.3. Note that the graph is symmetric about the y-axis and lies *strictly above* the x-plane *everywhere* other than the origin. Cross sections taken parallel to the x-plane are either ellipses when $y > 0$, a single point when $y = 0$ or empty when $y < 0$. Other positive definite quadratic forms will have similar graphs, i.e., “bowls” which touch the x-plane only at the origin and lie strictly above it everywhere else. Cross sections when $y > 0$ will either be ellipses or, in degenerate cases, circles.

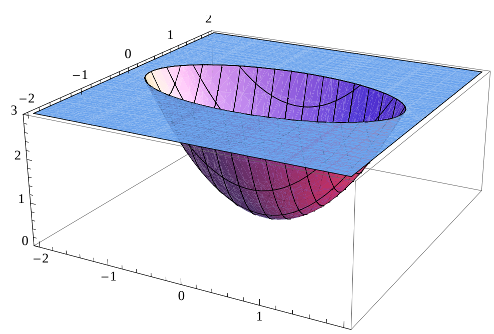

Figure 4.3: Positive Definite

Since $-Q(x)$ is a negative definite quadratic form iff $Q(x)$ is positive definite, we don't need a new picture for negative definite quadratic forms — simply turn Figure 4.3 upside down.

[Figure 4.4](#) illustrates the graph of the positive semi-definite quadratic form associated with

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

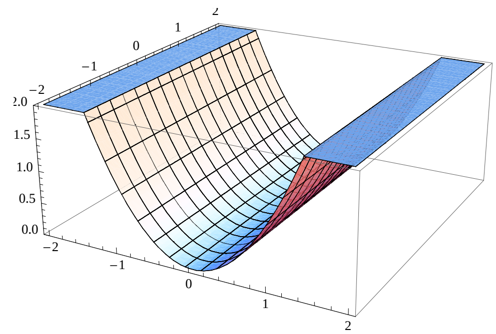

Figure 4.4: Positive Semi-Definite

Again the graph is symmetric about the y-axis and lies *on or above* the x-plane everywhere other than the origin. Here, however, the graph actually coincides with the x-plane along a line. Cross sections taken parallel to the x-plane are either pairs of parallel lines when $y > 0$, a single line when $y = 0$ or empty when $y < 0$. A degenerate possibility for a positive semi-definite quadratic form is that its graph coincides with the x-plane everywhere — the case in which every element of B is equal to zero. Other possibilities are similar to Figure 4.4 — "troughs" which coincide with the x-plane along a line and lie strictly above it elsewhere.

Once again we need not bother with a graph for the negative version. Since $-Q(x)$ is negative semi-definite iff $Q(x)$ is positive semi-definite, simply turn Figure 4.4 upside down for an illustration of a negative semi-definite quadratic form.

Only the case of an indefinite quadratic form remains and this is illustrated in Figure 4.5 for the case in which

$$B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

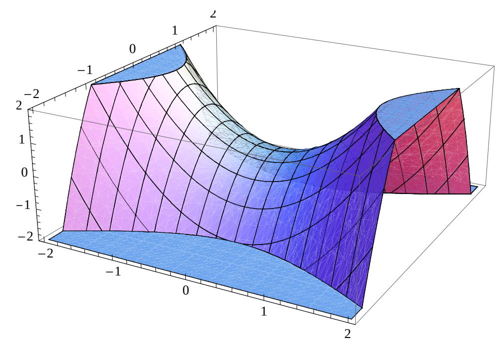

Figure 4.5: Indefinite

As always, the graph is symmetric about the y-axis. Here the graph sometimes lies above the x-plane and sometimes lies below it. Cross sections taken parallel to the x-plane are pairs of hyperbolas when $y > 0$ or when $y < 0$ or a pair of crossing lines when $y = 0$ [$x_2 = \pm x_1$].

Other indefinite quadratic forms are similar — "saddles" with regions both above and below the x-plane.

4.3 The Second Differential

Armed with quadratic forms we can now say that

Definition 4.5. A vector a and an n by n matrix B provide a good *quadratic approximation* of $f : \mathbb{R}^n \to \mathbb{R}$ at the point $\overline{x}$ if the error involved in the approximation,

$$\mu(v) \equiv f(\overline{x} + v) - f(\overline{x}) - a \cdot v - \frac{1}{2} v^T B v$$

is small relative to $\|v\|^2$:

$$\lim_{\|v\| \to 0} \frac{\mu(v)}{\|v\|^2} = \frac{f(\overline{x} + v) - f(\tilde{x}) - a \cdot v - \frac{1}{2} v^T B v}{\|v\|^2} = 0$$

This means that the quadratic form, $v^T Bv$, is a good (relative to $\|v\|^2$) approximation of the error left from the linear approximation. [A good quadratic approximation of $f\colon \mathbb{R}^n \to \mathbb{R}^m$ would similarly require a vector a^i and a matrix B^i for each component f^i.]

As might be suspected, the second derivative is the key to obtaining the quadratic approximation.

Definition 4.6. $f\colon \mathbb{R}^n \to \mathbb{R}$ is *twice differentiable at* $\overline{x}$ iff $\exists\ a \in \mathbb{R}^n$ and an n by n matrix B s.t.

$$\lim_{\|v\|\to 0} \frac{f(\overline{x}+v) - f(\overline{x}) - a \cdot v - \frac{1}{2} v^T Bv}{\|v\|^2} = 0$$

in which case B is called the *second derivative* and the quadratic form $1/2v^T Bv$ is called the *second differential*. The function is *twice differentiable* iff it is twice differentiable at every point in the domain.

As special cases note that when $n = 1$ the matrix B has but a single element

$$b = f''(\overline{x}) = \frac{d^2 f(\overline{x})}{dx^2}$$

the "second derivative of $f(x)$ evaluated at $\overline{x}$". Geometrically, b is the slope of $f'(x)$ at $\overline{x}$ — the rate at which the slope of f changes at $\overline{x}$ — or the *curvature* of f.

When $n > 1$ the n by n matrix B has the second partial derivatives of f evaluated at $\overline{x}$ as its elements

$$B = f_{xx}(\overline{x}) = \begin{bmatrix} \partial^2 f(\overline{x})/\partial x_1^2 & \cdots & \partial^2 f(\bar{x})/\partial x_1 \partial x_n \\ \vdots & \ddots & \vdots \\ \partial^2 f(\overline{x})/\partial x_n \partial x_1 & \cdots & \partial^2 f(\bar{x})/\partial x_n^2 \end{bmatrix}$$

and is called the *Hessian* of f.

The role of the second differential in approximating the error left from the linear (first differential) approximation can be illustrated in the context of the example of Figure 4.1 on page 48. The error from the linear approximation of $\ln(x)$, $\epsilon(v)$, is plotted in Figure 4.6 together with the best quadratic approximation of this error, namely, the second differential

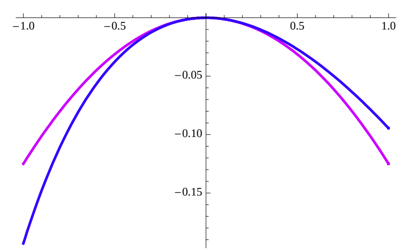

Figure 4.6: Best Quadratic Approximation

$$f(\overline{x}+v) - f(\overline{x}) - f'(\overline{x})v = \frac{1}{2!} f''(\overline{x})v^2 + \mu(v) \text{ or}$$

$$\ln(2+v) - \ln(2) - \frac{1}{2}v = -\frac{1}{2}\frac{1}{4}v^2 + \mu(v)$$

□ *Problem* 4.7. Suppose $g(x_1, x_2, x_3) = x_1^{1/6} x_2^{1/3} x_3^{1/2}$. Calculate $g_x(\overline{x})$ and $g_{xx}(\overline{x})$ for $\overline{x} = (1, 8, 9)$. What is the equation of the hyperplane that is tangent to $S = \{x \in \mathbb{R}^3 \mid g(x) = 6\}$ at $\overline{x} = (1, 8, 9)$? Hint: In *Mathematica* the gradient and hessian of a function can be computed using the following:

```
gradient[f_,x_List]:=Map[D[f, #] &, x]
hessian[f_,x_List]:=Outer[D,gradient[f,x],x]
f = x1^(1/6) x2^(1/3) x3^(1/2);
x = {x1,x2,x3}
gradient[f,x]
hessian[f,x]//MatrixForm
```

To be twice differentiable at a point is to have a good quadratic approximation at that point — the second derivative is the quadratic term of the approximation. Twice differentiable functions are also differentiable, of course, since the definition requires finding the first derivative as well. The first derivative not only exists if f is twice differentiable at $\overline{x}$, it is continuous as well. Putting matters somewhat loosely, continuity of a function implies an absence of breaks in its graph. To be differentiable implies an additional smoothness — an absence of kinks — since the graph must have unique tangents wherever it is differentiable. To be twice differentiable means that the graph of the derivative has no kinks and the function is smoother still. Pressing on, mathematicians use the term *smooth* to describe a function which is differentiable at all orders.

□ *Problem* 4.8. Suppose $f: \mathbb{R} \to \mathbb{R}$. Give an example of an f that is smooth. Give an example of an f that is differentiable but not smooth.

If you're guessing that a cubic approximation could be used to approximate the error left from the linear and quadratic approximations and that this would be based on the third derivative, you're exactly right. Continuing the example from Figures Figure 4.1 on page 48 and Figure 4.6 on the previous page, the error from the linear and quadratic approximations of $\ln(x)$, $\epsilon(v)$, is plotted in Figure 4.7 together with the best cubic approximation of this error, namely, the third differential

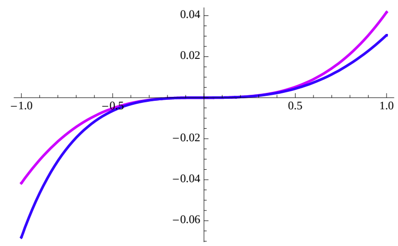

Figure 4.7: Best Cubic Approximation

$$f(\overline{x}+v) - f(\overline{x}) - f'(\overline{x})v - \frac{1}{2!}f''(\overline{x})v^2 = \frac{1}{3!}f'''(\overline{x})v^3 + \mu(v) \text{ or} \tag{4.1}$$

$$\ln(2+v) - \ln(2) - \frac{1}{2}v + \frac{1}{2}\frac{1}{4}v^2 = \frac{1}{6}\frac{1}{4}v^3 + \mu(v)$$

Rearranging Equation 4.1 gives the *Taylor series approximation* of order three:

$$f(\overline{x}+v) = f(\overline{x}) + f'(\overline{x})v + \frac{1}{2!}f''(\overline{x})v^2 + \frac{1}{3!}f'''(\overline{x})v^3 + \mu(v)$$

Taylor series approximations of higher orders are analogous.

□ *Problem* 4.9. `Series[f[x],{x,a,n}]` is the *Mathematica* command for generating an nth order Taylor series expansion of $f[x]$ with respect to x at $x = a$. Use it to obtain the 4th order expansion of `1/Sqrt[1+x]` at $x = 0$.

4.4 Convex and Concave Functions

Convex and concave functions are particularly important in Microeconomics. A function is concave if and only if a line segment connecting any two points on the graph of the function lies on or below its graph:

Definition 4.7. The function $f: X \mapsto \mathbb{R}$ where X is a convex subset of $\mathbb{R}^n$ is *concave* iff $x, y \in \mathbb{R}^n$ and $0 < a < 1$ implies $f(ax + (1-a)y) \geq af(x) + (1-a)f(y)$

A function is strictly concave if a line segment connecting any two distinct points on the graph of the function lies below the graph at all points other than the end points. See the left-hand panel of Figure 4.8 on following page.

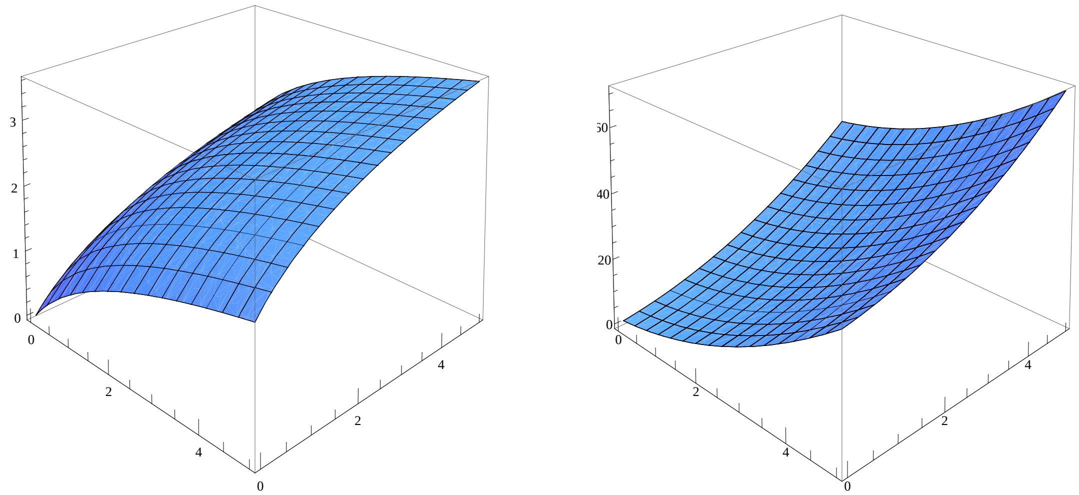

Figure 4.8: Strictly Concave (left) and Convex (right)

Definition 4.8. The function $f\colon X \mapsto \mathbb{R}$ where X is a convex subset of $\mathbb{R}^n$ is *strictly concave* iff $x, y \in \mathbb{R}^n$, $x \neq y$ and $0 < a < 1$ implies $f(ax + (1-a)y) > af(x) + (1-a)f(y)$.

The function f, on the other hand, is *convex* iff $g(x) = -f(x)$ is concave. A convex function is characterized by the fact that line segments connecting two points lie on or above the graph. Similarly, f is *strictly convex* iff $-f$ is strictly concave. A strictly convex function is illustrated in the right-hand panel of Figure 4.8.

□ *Problem* 4.10. Show that if $f\colon \mathbb{R}^n \mapsto \mathbb{R}$ is a linear transformation then f is both concave and convex.

Concave functions have particularly nice properties:

Theorem 22. Suppose that X is a convex subset of $\mathbb{R}^n$. Then

1. If $f : X \mapsto \mathbb{R}$ is concave then it is also continuous in the interior of X.
2. If
$$f(x) = \sum_{i=1}^{k} \alpha_i f_i(x)$$
with $\alpha_i \geq 0$, $i = 1, \ldots, k$, and if each $f_i : X \mapsto \mathbb{R}$, $i = 1, \ldots, k$, is concave then f is also concave. [A non-negative linear combination of concave functions must also be concave.]

□ *Problem* 4.11. Illustrate the first part of Theorem 22 for $f\colon \mathbb{R} \to \mathbb{R}$ by showing that a discontinuous function cannot be concave.

□ *Problem* 4.12. Prove the second part of Theorem 22.

The second derivatives of concave functions also have intuitive properties:

Theorem 23. Hessians of Concave Functions. Suppose $f(x)$ is a continuously twice differentiable real valued function on an open convex set X in $\mathbb{R}^n$ with Hessian $f_{xx}(x)$. Then

1. The function f is concave (convex) if and only if $f_{xx}(x)$ is negative (positive) semi-definite for all $x \in X$.
2. The function f is strictly concave (convex) if $f_{xx}(x)$ is negative (positive) definite for all $x \in X$.

Note that the "and only if" is missing from the second proposition. The function $f(x) = -(x-1)^4$, for example, is strictly concave but its Hessian is only negative semi-definite at $x = 1$ since $f''(x) = -12(x-1)^2 = 0$ at $x = 1$ — see Figure 4.9.

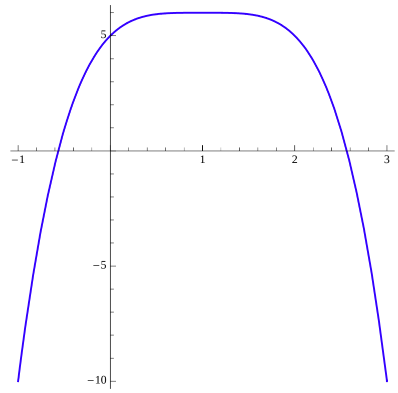

Figure 4.9: Strictly Concave but Negative Semi-Definite

Finally, and perhaps most important to the current discussion

Definition 4.9. Suppose the function $f\colon X \mapsto \mathbb{R}$, where X is a convex subset of $\mathbb{R}^n$. Then the set $L_f(r) \equiv \{z \in \mathbb{R}^n \mid f(z) \geq r\} \subset X$ is called a *level set* for f.

Theorem 24. If $f X \mapsto \mathbb{R}$, where X is a convex subset of $\mathbb{R}^n$, is concave then the level set $L_f(r)$ is a convex set for any $r \in \mathbb{R}$.

■ *Problem* 4.13. [Answer] Prove Theorem Theorem 24.

The converse of this proposition isn't true — a weaker condition than concavity, namely quasi-concavity, is sufficient to guarantee the convexity of $L_f(r)$.

Definition 4.10. The function $f\colon X \mapsto \mathbb{R}$, where X is a convex subset of $\mathbb{R}^n$, is *quasi-concave* iff $\forall r \in \mathbb{R}$, the *level set* $L_f(r)$ is a convex set and *strictly quasi-concave* iff $L_f(r)$ is a strictly convex set.

An equivalent definition which is sometimes useful is

Definition 4.11. The function $f\colon X \mapsto \mathbb{R}$ where X is a convex subset of $\mathbb{R}^n$ is

1. quasi-concave if

$$f(x') \geq f(x) \implies f\left(ax' + (1-a)x\right) \geq f(x), \quad \forall x', x \in X \text{ and } 0 \leq a \leq 1$$

2. strictly quasi-concave if

$$f(x') \geq f(x) \implies f\left(ax' + (1-a)x\right) > f(x), \quad \forall x' \neq x \in X \text{ and } 0 < a < 1$$

A strictly quasi-concave (but not concave) function is illustrated in Figure 4.10. As you would expect, f is *quasi-convex* iff $-f$ is quasi-concave and *strictly quasi-convex* iff $-f$ is strictly quasi-concave.

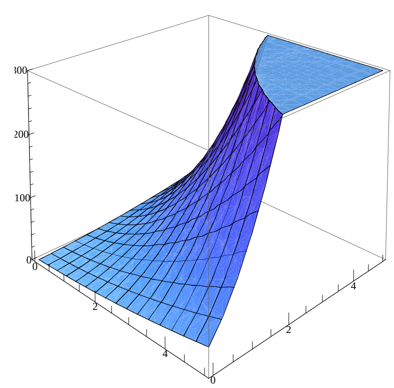

Figure 4.10: Strictly Quasi-Concave

□ *Problem* 4.14. Show that *any* monotone increasing (or decreasing) function is quasi-concave if $X \subset \mathbb{R}$. [The function $f\colon \mathbb{R} \mapsto \mathbb{R}$ is a *monotone increasing (decreasing) function* iff $x' \geq x$ implies $f(x') \geq (\leq) f(x)$.]

□ *Problem* 4.15. Show that *any* monotone increasing (or decreasing) function is quasi-convex if $X \subset \mathbb{R}$.

□ *Problem* 4.16. Suppose $f : \mathbb{R} \to \mathbb{R}$ is *single-peaked*, i.e., it is *either* monotone increasing, monotone decreasing or there exists an $x^* \in \mathbb{R}$ such that f is monotone increasing for $x \leq x^*$ and monotone decreasing for $x \geq x^*$. Is f quasi-concave? Is f concave?

It is an unfortunate fact that quasi-concave functions share few of the other nice properties of concave functions. Quasi-concave functions are not necessarily continuous — $f(x) = 1$ if $x = 0$ and $f(x) = 0$ if $x \neq 0$ with $x \in \mathbb{R}$ is, for example, quasi-concave but neither concave nor continuous. Non-negative linear combinations of quasi-concave functions, moreover, are *not* necessarily quasi-concave. The functions, x^2 and $(x-1)^2$ for $x \in X \equiv [0,1]$, for example, are both quasi-concave (but not concave) and yet $\frac{1}{2}x^2 + \frac{1}{2}(x-1)^2$ is not quasi-concave. This is illustrated in Figure 4.11 on following page.

Lastly, consider the *bordered Hessian*:

$$B(x) \equiv \begin{bmatrix} 0 & \partial f(x)/\partial x_1 & \cdots & \partial f(x)/\partial x_n \\ \partial f(x)/\partial x_1 & \partial^2 f(x)/\partial x_1^2 & \cdots & \partial^2 f(x)/\partial x_1 \partial x_n \\ \vdots & \vdots & \ddots & \vdots \\ \partial f(x)/\partial x_n & \partial^2 f(x)/\partial x_n \partial x_1 & \cdots & \partial^2 f(x)/\partial x_n^2 \end{bmatrix}$$

with $(k+1)$st successive *principal minor*

$$B_k(x) \equiv \begin{vmatrix} 0 & \partial f(x)/\partial x_1 & \cdots & \partial f(x)/\partial x_k \\ \partial f(x)/\partial x_1 & \partial^2 f(x)/\partial x_1^2 & \cdots & \partial^2 f(x)/\partial x_1 \partial x_k \\ \vdots & \vdots & \ddots & \vdots \\ \partial f(x)/\partial x_k & \partial^2 f(x)/\partial x_k \partial x_1 & \cdots & \partial^2 f(x)/\partial x_k^2 \end{vmatrix}$$

$k = 1, \ldots, n$. If $f: \mathbb{R}^n \mapsto \mathbb{R}$ is twice continuously differentiable and quasi concave then it can be shown that $B_2(x) \geq 0$, $B_3(x) \leq 0$, …, $(-1)^n B_n(x) \geq 0$ for all $x \in \mathbb{R}^n$ — not so nice! It is fortunate that second order characterizations of concave and convex functions prove to be much more important than those for their recalcitrant "quasi" cousins.

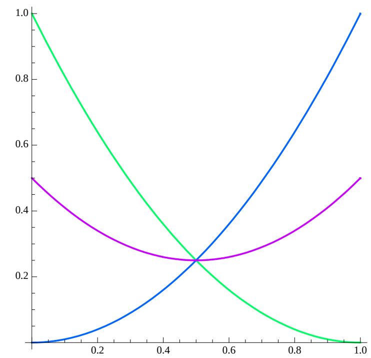

Figure 4.11: Quasi-Concave Combinations

4.5 Answers

Problem 4.13 on preceding page. Suppose $x, y \in L_f(r)$. Then $f(x), f(y) \geq r$. But concavity then implies that $f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda) f(y) \geq \lambda r + (1-\lambda) r = r$ and $\lambda x + (1-\lambda)y \in L_f(r)$.