Game Theory and Strategy

Introduction

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Game Theory: Definition and Assumptions

- Game theory studies strategic interactions within a group of individuals
  - Actions of each individual have an effect on the outcome
  - Individuals are aware of that fact
- Individuals are rational
  - have well-defined objectives over the set of possible outcomes
  - implement the best available strategy to pursue them
- Rules of the game and rationality are common knowledge

Example

- 10 people go to a restaurant for dinner
- Order expensive or inexpensive fish?
  - Expensive fish: value = 18, price = 20
  - Inexpensive fish: value = 12, price = 10
- Everbody pays own bill
  - What do you do?
  - Single person decision problem
- Total bill is shared equally
  - What do you do?
  - It is a GAME

Example: A Single Person Decision Problem

- Ali is an investor with $\$100$

	State	
	Good	Bad
Bonds	10%	10%
Stocks	20%	0%

- Which one is better?
- Probability of the good state p
- Assume that Ali wants to maximize the amount of money he has at the end of the year.
- Bonds: $\$110$
- Stocks: average (or expected) money holdings:

$$p \times 120 + (1 - p) \times 100 = 100 + 20 \times p$$

- If $p > 1/2$ invest in stocks
- If $p < 1/2$ invest in bonds

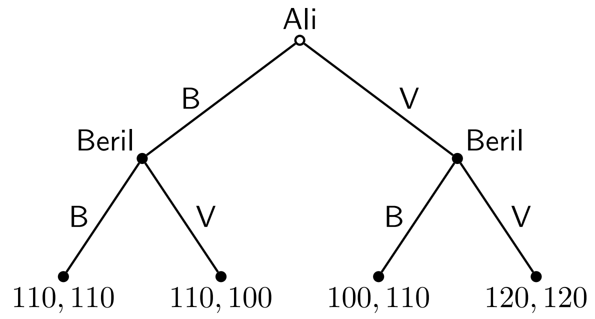
An Investment Game

- Ali again has two options for investing his $\$100$:
  - invest in bonds
    - certain return of 10%
  - invest it in a risky venture
    - successful: return is 20%
    - failure: return is 0%
  - venture is successful if and only if total investment is at least $\$200$
- There is one other potential investor in the venture (Beril) who is in the same situation as Ali
- They cannot communicate and have to make the investment decision without knowing the decisions of each other

		Beril	
		Bonds	Venture
Ali	Bonds	$110, 110$	$110, 100$
	Venture	$100, 110$	$120, 120$

Entry Game

- Strategic (or Normal) Form Games
  - used if players choose their strategies without knowing the choices of others
- Extensive Form Games
  - used if some players know what others have done when playing



Investment Game with Incomplete Information

- Some players have private (and others have incomplete) information
- Ali is not certain about Beril's preferences. He believes that she is
  - Normal with probability p
  - Crazy with probability $1 - p$

Normal (p):

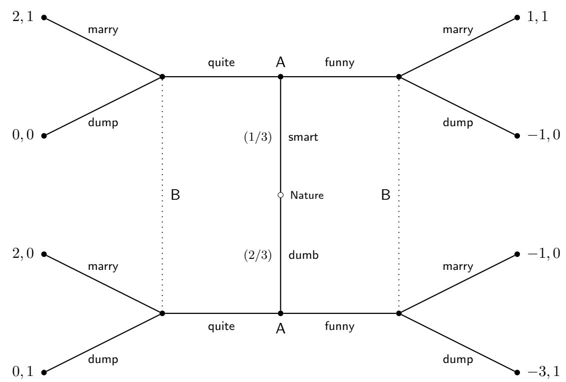
		Beril	
		Bonds	Venture
Ali	Bonds	$110, 110$	$110, 100$
	Venture	$100, 110$	$120, 120$

Crazy ($1 - p$):

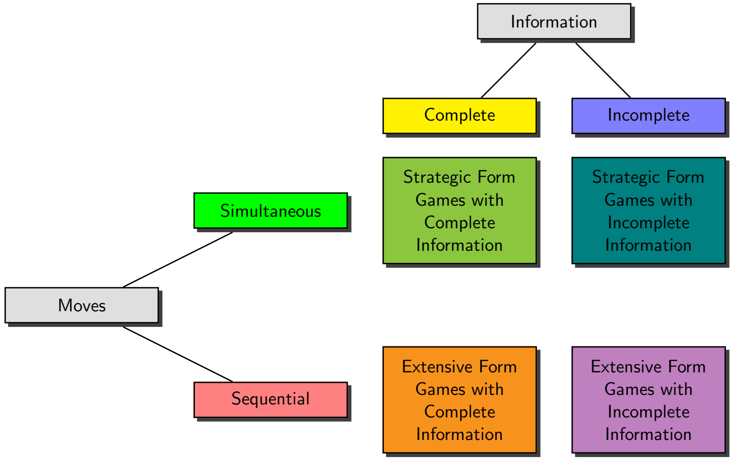
		Beril	
		Bonds	Venture
Ali	Bonds	$110, 110$	$110, 120$
	Venture	$100, 110$	$120, 120$

The Dating Game

- Ali takes Beril out on a date
- Beril wants to marry a smart guy but does not know whether Ali is smart
- She believes that he is smart with probability $1/3$
- Ali decides whether to be funny or quite
- Observing Ali's demeanor, Beril decides what to do



Game Forms



Outline of the Course

1. Strategic Form Games
2. Dominant Strategy Equilibrium and Iterated Elimination of Dominated Actions
3. Nash Equilibrium: Theory
4. Nash Equilibrium: Applications
    4.1 Auctions
    4.2 Buyer-Seller Games
    4.3 Market Competition
    4.4 Electoral Competition
5. Mixed Strategy Equilibrium
6. Games with Incomplete Information and Bayesian Equilibrium
7. Auctions
8. Extensive Form Games: Theory
    8.1 Perfect Information Games and Backward Induction Equilibrium
    8.2 Imperfect Information Games and Subgame Perfect Equilibrium
9. Extensive Form Games: Applications
    9.1 Stackelberg Duopoly
    9.2 Bargaining
    9.3 Repeated Games
10. Extensive Form Games with Incomplete Information
    10.1 Perfect Bayesian Equilibrium
    10.2 Signaling Games

Game Theory

Strategic Form Games

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Split or Steal

Split or Steal

Split or Steal

Van den Assem, Van Dolder, and Thaler, "Split or Steal? Cooperative Behavior When the Stakes Are Large" *Management Science*, 2012.

- Individual players on average choose "split" 53 percent of the time
- Propensity to cooperate is surprisingly high for consequential amounts
- Less likely to cooperate if opponent has tried to vote them off previously
  - Evidence for reciprocity
- Young males are less cooperative than young females
- Old males are more cooperative than old females

Split or Steal

		Sarah	
		Steal	Split
Steve	Steal	$0, 0$	$100, 0$
	Split	$0, 100$	$50, 50$

- Set of Players $N = \{Sarah, Steve\}$
- Set of actions: $A_{Sarah} = A_{Steve} = \{Steal, Split\}$
- Payoffs

Strategic Form Games

- It is used to model situations in which players choose strategies without knowing the strategy choices of the other players
- Also known as normal form games

A strategic form game is composed of

1. Set of players: N
2. A set of strategies: A_i for each player i
3. A payoff function: $u_i : A \to \mathbf{R}$ for each player i

$G = (N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N})$

- An outcome $a = (a_1, ..., a_n)$ is a collection of actions, one for each player
  - Also known as an action profile or strategy profile
- outcome space

$$A = \{(a_1, ..., a_n) : a_i \in A_i, i = 1, ..., n\}$$

Prisoners' Dilemma

		Player 2	
		c	n
Player 1	c	$-5, -5$	$0, -6$
	n	$-6, 0$	$-1, -1$

- $N = \{1, 2\}$
- $A_1 = A_2 = \{c, n\}$
- $A = \{(c, c), (c, n), (n, c), (n, n)\}$
- $u_1(c, c) = -5, u_1(c, n) = 0$, etc.

Contribution Game

- Everybody starts with 10 TL
- You decide how much of 10 TL to contribute to joint fund
- Amount you contribute will be doubled and then divided equally among everyone
- I will distribute slips of paper that looks like this

Name:______________

Your Contribution:______________

- Write your name and an integer between 0 and 10
- We will collect them and enter into Excel
- We will choose one player randomly and pay her
  Click here for the EXCEL file

Example: Price Competition

- Toys "R" Us and Wal-Mart have to decide whether to sell a particular toy at a high or low price
- They act independently and without knowing the choice of the other store
- We can write this game in a bimatrix format

Wal-Mart

Toys "R" Us	$High$	Low
$High$	$10, 10$	$2, 15$
Low	$15, 2$	$5, 5$

Example: Price Competition

W

T	H	L
H	$10, 10$	$2, 15$
L	$15, 2$	$5, 5$

- $N = \{T, W\}$
- $A_T = A_W = \{H, L\}$
- $u_T(H, H) = 10$
  $u_W(H, L) = 15$
  etc.

- What should Toys "R" Us play?
- Does that depend on what it thinks Wal-Mart will do?
- Low is an example of a dominant strategy
- it is optimal independent of what other players do
- How about Wal-Mart?
- (L, L) is a dominant strategy equilibrium

Dominant Strategies

- a_{-i} = profile of actions taken by all players other than i
- A_{-i} = the set of all such profiles

An action a_i strictly dominates b_i if

$$u_i(a_i, a_{-i}) > u_i(b_i, a_{-i}) \quad \text{for all } a_{-i} \in A_{-i}$$

a_i weakly dominates action b_i if

$$u_i(a_i, a_{-i}) \geq u_i(b_i, a_{-i}) \quad \text{for all } a_{-i} \in A_{-i}$$

and

$$u_i(a_i, a_{-i}) > u_i(b_i, a_{-i}) \quad \text{for some } a_{-i} \in A_{-i}$$

An action a_i is strictly dominant if it strictly dominates every action in A_i. It is called weakly dominant if it weakly dominates every action in A_i.

Dominant Strategy Equilibrium

If every player has a (strictly or weakly) dominant strategy, then the corresponding outcome is a (strictly or weakly) dominant strategy equilibrium.

W

T	H	L
H	$10, 10$	$2, 15$
L	$15, 2$	$5, 5$

- L strictly dominates H
- (L,L) is a strictly dominant strategy equilibrium

W

T	H	L
H	$10, 10$	$5, 15$
L	$15, 5$	$5, 5$

- L weakly dominates H
- (L,L) is a weakly dominant strategy equilibrium

Dominant Strategy Equilibrium

- A reasonable solution concept
- It only demands the players to be rational
- It does not require them to know that the others are rational too
- But it does not exist in many interesting games

Guess the Average

- We will play a game
- I will distribute slips of paper that looks like this

Name:_____________

Your guess:_____________

- Write your name and a number between 0 and 100
- We will collect them and enter into Excel
- The number that is closest to half the average wins
- Winner gets 6TL (in case of a tie we choose randomly)
  Click here for the EXCEL file

Beauty Contest

The Beauty Contest Thats Shaking Wall St., ROBERT J. SHILLER, NYT 3/9/2011

> *John Maynard Keynes supplied the answer in 1936, in "The General Theory of Employment Interest and Money," by comparing the stock market to a beauty contest. He described a newspaper contest in which 100 photographs of faces were displayed. Readers were asked to choose the six prettiest. The winner would be the reader whose list of six came closest to the most popular of the combined lists of all readers.*
> *The best strategy, Keynes noted, isn't to pick the faces that are your personal favorites. It is to select those that you think others will think prettiest. Better yet, he said, move to the "third degree" and pick the faces you think that others think that still others think are prettiest. Similarly in speculative markets, he said, you win not by picking the soundest investment, but by picking the investment that others, who are playing the same game, will soon bid up higher.*

Beauty Contest

New York Times online version

Price Matching

- Toys "R" Us web page has the following advertisement

- Sounds like a good deal for customers
- How does this change the game?

Price Matching

		Wal-Mart		
		High	*Low*	*Match*
Toys "R" us	*High*	10, 10	2, 15	10, 10
	Low	15, 2	5, 5	5, 5
	Match	10, 10	5, 5	10, 10

- Is there a dominant strategy for any of the players?
- There is no dominant strategy equilibrium for this game
- So, what can we say about this game?

Price Matching

		Wal-Mart		
		High	*Low*	*Match*
Toys "R" us	*High*	10, 10	2, 15	10, 10
	Low	15, 2	5, 5	5, 5
	Match	10, 10	5, 5	10, 10

- High is weakly dominated and Toys "R" us is rational
  - Toys "R" us should not use High
- High is weakly dominated and Wal-Mart is rational
  - Wal-Mart should not use High
- Each knows the other is rational
  - Toys "R" us knows that Wal-Mart will not use High
  - Wal-Mart knows that Toys "R" us will not use High
  - This is where we use common knowledge of rationality

Price Matching

- Therefore we have the following "effective" game

		Wal-Mart	
		Low	*Match*
Toys "R" us	*Low*	5, 5	5, 5
	Match	5, 5	10, 10

- Low becomes a weakly dominated strategy for both
- Both companies will play Match and the prices will be high
- The above procedure is known as Iterated Elimination of Dominated Strategies (IEDS)

To be a good strategist try to see the world from the perspective of your rivals and understand that they will most likely do the same

Dominated Strategies

- A "rational" player should never play an action when there is another action that gives her a higher payoff irrespective of how the others play
- We call such an action a dominated action

An action a_i is strictly dominated by b_i if

$$u_i(a_i, a_{-i}) < u_i(b_i, a_{-i}) \quad \text{for all } a_{-i} \in A_{-i}.$$

a_i is weakly dominatedby b_i if

$$u_i(a_i, a_{-i}) \leq u_i(b_i, a_{-i}) \quad \text{for all } a_{-i} \in A_{-i}$$

while

$$u_i(a_i, a_{-i}) < u_i(b_i, a_{-i}) \quad \text{for some } a_{-i} \in A_{-i}.$$

Iterated Elimination of Dominated Strategies

- Common knowledge of rationality justifies eliminating dominated strategies iteratively
- This procedure is known as Iterated Elimination of Dominated Strategies
- If every strategy eliminated is a strictly dominated strategy
  - Iterated Elimination of Strictly Dominated Strategies
- If IESDS leads to a unique outcome, we call the game dominance solvable
- If at least one strategy eliminated is a weakly dominated strategy
  - Iterated Elimination of Weakly Dominated Strategies

IESDS vs. IEWDS

- Order of elimination does not matter in IESDS
- It matters in IEWDS

	L	R
U	$3,1$	$2,0$
M	$4,0$	$1,1$
D	$4,4$	$2,4$

- Start with U
- Start with M

Effort Game

- You choose how much effort to expend for a joint project
  - An integer between 1 and 7
- The quality of the project depends on the smallest effort: $\underline{e}$
  - Weakest link
- Effort is costly
- If you choose e your payoff is

$$6 + 2\underline{e} - e$$

- We will randomly choose one round and one student and pay her
- Enter your name and effort choice
  Click here for the EXCEL file

Effort Game: 2 people 2 effort level

	L	H
L	$7,7$	$7,1$
H	$1,7$	$13,13$

- Is there a dominant strategy?
- What are the likely outcomes?
- If you expect the other to choose L, what is your best strategy (best response)?
- If you expect the other to choose H, what is your best strategy (best response)?
- (L, L) is an outcome such that
  - Each player best responds, given what she believes the other will do
  - Their beliefs are correct
- It is a Nash equilibrium

Nash Equilibrium

- Nash equilibrium is a strategy profile (a collection of strategies, one for each player) such that each strategy is a best response (maximizes payoff) to all the other strategies

An outcome $a^* = (a_1^*, ..., a_n^*)$ is a Nash equilibrium if for each player i

$$u_i(a_i^*, a_{-i}^*) \geq u_i(a_i, a_{-i}^*) \quad \text{for all } a_i \in A_i$$

- Nash equilibrium is self-enforcing: no player has an incentive to deviate unilaterally
- One way to find Nash equilibrium is to first find the best response correspondence for each player
  - Best response correspondence gives the set of payoff maximizing strategies for each strategy profile of the other players
- ... and then find where they "intersect"

Nash Equilibrium

	L	H
L	$\underline{7},\underline{7}$	$7,1$
H	$1,7$	$\underline{13},\underline{13}$

Set of Nash equilibria $= \{(L, L), (H, H)\}$

Best Response Correspondence

- The best response correspondence of player i is given by

$$B_i(a_{-i}) = \{a_i \in A_i : u_i(a_i, a_{-i}) \geq u_i(b_i, a_{-i}) \text{ for all } b_i \in A_i\}.$$

- $B_i(a_{-i})$ is a set and may not be a singleton
- In the effort game

	L	H
L	$\underline{7},\underline{7}$	$7,1$
H	$1,7$	$\underline{13},\underline{13}$

$$B_1(L) = \{L\} \quad B_1(H) = \{H\}$$
$$B_2(L) = \{L\} \quad B_2(H) = \{H\}$$

The Bar Scene

The Bar Scene

	Blonde	Brunette
Blonde	$0, 0$	$2, 1$
Brunette	$1, 2$	$1, 1$

- See S. Anderson and M. Engers: Participation Games: Market Entry, Coordination, and the Beautiful Blonde, Journal of Economic Behavior and Organization, 2007

Stag Hunt

Jean-Jacques Rousseau in A Discourse on Inequality

> *If it was a matter of hunting a deer, everyone well realized that he must remain faithful to his post; but if a hare happened to pass within reach of one of them, we cannot doubt that he would have gone off in pursuit of it without scruple...*

	$Stag$	$Hare$
$Stag$	$2, 2$	$0, 1$
$Hare$	$1, 0$	$1, 1$

How would you play this game?

Stag Hunt

- Set of Nash equilibria:

$$\mathbf{N}(SH) = \{(S, S), (H, H)\}$$

- What do you think?

Nash Demand Game

- Each of you will be randomly matched with another student
- You are trying to divide 10 TL
- Each writes independently how much she wants (in multiples of 1 TL)
- If two numbers add up to greater than 10 TL each gets nothing
- Otherwise each gets how much she wrote
- Write your name and demand on the slips
- I will match two randomly
- Choose one pair randomly and pay them
  Click here for the EXCEL file

Optimization

Let $f : \mathbb{R}^n \to \mathbb{R}$ and $\mathcal{D} \subset \mathbb{R}^n$. A constrained optimization problem is

$$\max f(x) \quad \text{subject to } x \in \mathcal{D}$$

- f is the objective function
- $\mathcal{D}$ is the constraint set
- A solution to this problem is $x \in \mathcal{D}$ such that

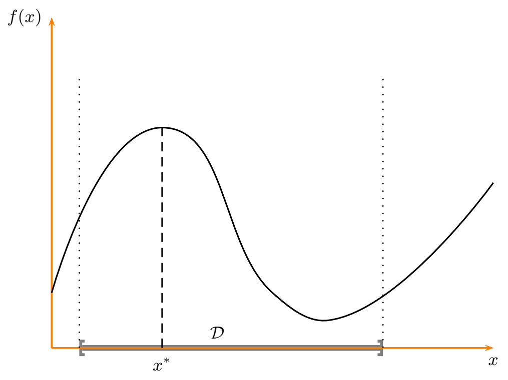
$$f(x) \geq f(y) \quad \text{for all } y \in \mathcal{D}$$

  Such an x is called a maximizer
- The set of maximizers is denoted

$$\text{argmax}\{f(x)|x \in \mathcal{D}\}$$

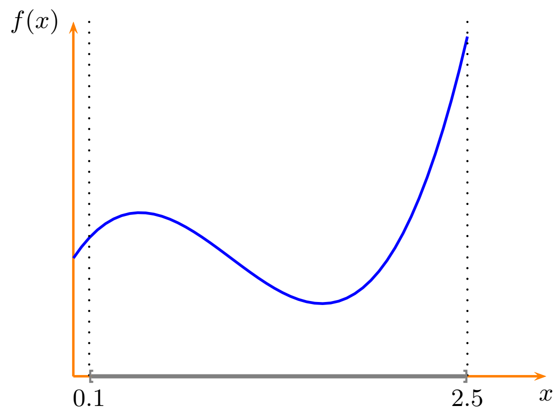
- Similarly for minimization problems

A Graphical Example



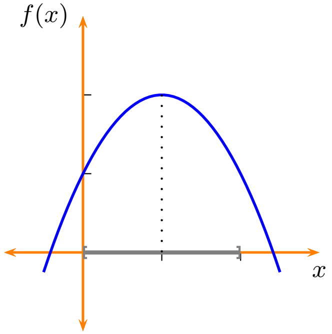
Example

$\max x^3 - 3x^2 + 2x + 1$ subject to $0.1 \leq x \leq 2.5$



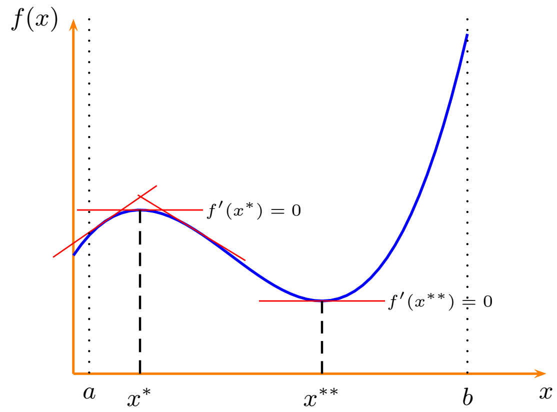
Example

$\max -(x-1)^2 + 2$ s.t. $x \in [0,2]$.



A Simple Case

Let $f : \mathbb{R} \to \mathbb{R}$ and consider the problem $\max_{x \in [a,b]} f(x)$.



We call a point x^* such that $f'(x^*) = 0$ a critical point.

Interior Optima

Theorem

Let $f : \mathbb{R} \to \mathbb{R}$ and suppose $a < x^* < b$ is a local maximum (minimum) of f on $[a,b]$. Then, $f'(x^*) = 0$.

- Known as first order conditions
- Only necessary for interior local optima
  - Not necessary for global optima
  - Not sufficient for local optima.
- To distinguish between interior local maximum and minimum you can use second order conditions

Theorem

Let $f : \mathbb{R} \to \mathbb{R}$ and suppose $a < x^* < b$ is a local maximum (minimum) of f on $[a,b]$. Then, $f''(x^*) \le 0$ ($f''(x^*) \ge 0$).

Recipe for solving the simple case

Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function and consider the problem $\max_{x \in [a,b]} f(x)$. If the problem has a solution, then it can be found by the following method:

1. Find all critical points: i.e., $x^* \in [a,b]$ s.t. $f'(x^*) = 0$
2. Evaluate f at all critical points and at boundaries a and b
3. The one that gives the highest f is the solution

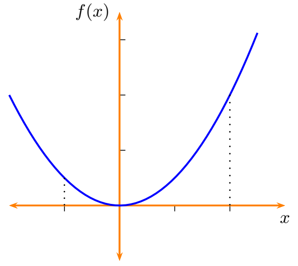
- We can use Weierstrass theorem to determine if there is a solution
- Note that if $f'(a) > 0$ (or $f'(b) < 0$), then the solution cannot be at a (or b)

Example

$\max x^2$ s.t. $x \in [-1, 2]$.

Solution

x^2 is continuous and $[-1, 2]$ is closed and bounded, and hence compact. Therefore, by Weierstrass theorem the problem has a solution.
$f'(x) = 2x = 0$ is solved at $x = 0$, which is the only critical point. We have $f(0) = 0, f(-1) = 1, f(2) = 4$. Therefore, 2 is the global maximum.

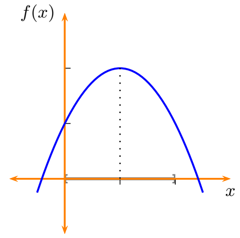


Example

$\max -(x-1)^2 + 2$ s.t. $x \in [0, 2]$.

Solution

f is continuous and $[0, 2]$ is compact. Therefore, the problem has a solution. $f'(x) = -2(x-1) = 0$ is solved at $x = 1$, which is the only critical point. We have $f(1) = 2, f(0) = 1, f(2) = 1$. Therefore, 1 is the global maximum. Note that $f'(0) > 0$ and $f'(2) < 0$ and hence we could have eliminated 0 and 2 as candidates.



What is the solution if the constraint set is $[-1, 0.5]$?

Recipe for general problems

- Generalizes to $f : \mathbb{R}^n \to \mathbb{R}$ and the problem is

$$\max f(x) \quad \text{subject to } x \in \mathcal{D}$$

  - Find critical points $x^* \in \mathcal{D}$ such that $Df(x^*) = 0$
  - Evaluate f at the critical points and the boundaries of $\mathcal{D}$
  - Choose the one that give the highest f
- Important to remember that solution must exist for this method to work
- In more complicated problems evaluating f at the boundaries could be difficult
- For such cases we have the method of the Lagrangean (for equality constraints) and Kuhn-Tucker conditions (for inequality constraints)

Cournot Duopoly

- Two firms competing by choosing how much to produce
- Augustine Cournot (1838)

Inverse demand function

$$p(q_1 + q_2) = \begin{cases} a - b(q_1 + q_2), & q_1 + q_2 \leq a/b \\ 0, & q_1 + q_2 > a/b \end{cases}$$

Cost function of firm $i = 1, 2$

$$c_i(q_i) = cq_i$$

where $a > c \geq 0$ and $b > 0$
Therefore, payoff function of firm $i = 1, 2$ is given by

$$u_i(q_1, q_2) = \begin{cases} (a - c - b(q_1 + q_2))q_i, & q_1 + q_2 \leq a/b \\ -cq_i, & q_1 + q_2 > a/b \end{cases}$$

Claim

Best response correspondence of firm $i \neq j$ is given by

$$B_i(q_j) = \begin{cases} \frac{a-c-bq_j}{2b}, & q_j < \frac{a-c}{b} \\ 0, & q_j \geq \frac{a-c}{b} \end{cases}$$

Proof.

- If $q_2 \geq \frac{a-c}{b}$, then $u_1(q_1, q_2) < 0$ for any $q_1 > 0$. Therefore, $q_1 = 0$ is the unique payoff maximizer.
- If $q_2 < \frac{a-c}{b}$, then the best response cannot be $q_1 = 0$ (why?). Furthermore, it must be the case that $q_1 + q_2 \leq \frac{a-c}{b} \leq \frac{a}{b}$, for otherwise $u_1(q_1, q_2) < 0$. So, the following first order condition must hold
$$\frac{\partial u_1(q_1, q_2)}{\partial q_1} = a - c - 2bq_1 - bq_2 = 0$$

Similarly for firm 2. □

Claim

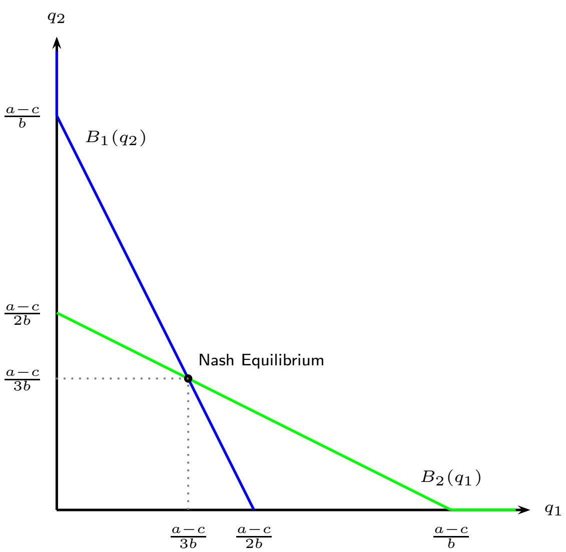
The set of Nash equilibria of the Cournot duopoly game is given by

$$\mathbf{N}(G) = \left\{ \left(\frac{a-c}{3b}, \frac{a-c}{3b} \right) \right\}$$

Proof.

Suppose (q_1^*, q_2^*) is a Nash equilibrium and $q_i^* = 0$. Then, $q_j^* = (a-c)/2b < (a-c)/b$. But, then $q_i^* \notin B_i(q_j^*)$, a contradiction. Therefore, we must have $0 < q_i^* < (a-c)/b$, for $i = 1, 2$. The rest follows from the best response correspondences. □

Cournot Nash Equilibrium



Cournot Oligopoly

In equilibrium each firm's profit is

$$\frac{(a-c)^2}{9b}$$

- Is there a way for these two firms to increase profits?
- What if they form a cartel?
- They will maximize
$$U(q_1 + q_2) = (a - c - b(q_1 + q_2))(q_1 + q_2)$$
- Optimal level of total production is
$$q_1 + q_2 = \frac{a-c}{2b}$$
- Half of the maximum total profit is
$$\frac{(a-c)^2}{8b}$$
- Is the cartel stable?

Game Theory

Strategic Form Games: Applications

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Outline

Auctions

Many economic transactions are conducted through auctions

- treasury bills
- foreign exchange
- publicly owned companies
- mineral rights
- airwave spectrum rights
- art work
- antiques
- cars
- houses
- government contracts

Also can be thought of as auctions

- takeover battles
- queues
- wars of attrition
- lobbying contests

Auction Formats

1. Open bid auctions
   1.1 ascending-bid auction
      - ⋆ aka English auction
      - ⋆ price is raised until only one bidder remains, who wins and pays the final price

   1.2 descending-bid auction
      - ⋆ aka Dutch auction
      - ⋆ price is lowered until someone accepts, who wins the object at the current price
2. Sealed bid auctions
   2.1 first price auction
      - ⋆ highest bidder wins; pays her bid

   2.2 second price auction
      - ⋆ aka Vickrey auction
      - ⋆ highest bidder wins; pays the second highest bid

Auction Formats

Auctions also differ with respect to the valuation of the bidders

1. Private value auctions
   - each bidder knows only her own value
   - artwork, antiques, memorabilia
2. Common value auctions
   - actual value of the object is the same for everyone
   - bidders have different private information about that value
   - oil field auctions, company takeovers

Strategically Equivalent Formats

Open Bid		Sealed Bid
English Auction	$\leftrightsquigarrow$	Second Price
Dutch Auction	$\leftrightsquigarrow$	First Price

We will study sealed bid auctions

- For now we will assume that values are common knowledge
  - value of the object to player i is v_i dollars
- For simplicity we analyze the case with only two bidders
- Assume $v_1 > v_2 > 0$

Second Price Auctions

- Highest bidder wins and pays the second highest bid
- In case of a tie, the object is awarded to player 1

Strategic form:

1. $N = \{1, 2\}$
2. $A_1 = A_2 = \mathbf{R}_+$
3. Payoff functions: For any $(b_1, b_2) \in \mathbf{R}^2_+$

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_2, & \text{if } b_1 \geq b_2, \\ 0, & \text{otherwise.} \end{cases}$$

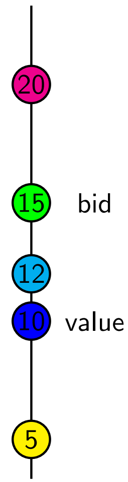
$$u_2(b_1, b_2) = \begin{cases} v_2 - b_1, & \text{if } b_2 > b_1, \\ 0, & \text{otherwise.} \end{cases}$$

Second Price Auctions

I. Bidding your value weakly dominates bidding higher

Suppose your value is \$10 but you bid \$15. Three cases:

1. The other bid is higher than \$15 (e.g. \$20)
   - You loose either way: no difference
2. The other bid is lower than \$10 (e.g. \$5)
   - You win either way and pay \$5: no difference
3. The other bid is between \$10 and \$15 (e.g. \$12)
   - You loose with \$10: zero payoff
   - You win with \$15: loose \$2

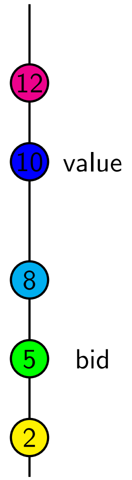


Second Price Auctions

II. Bidding your value weakly dominates bidding lower

Suppose your value is \$10 but you bid \$5. Three cases:

1. The other bid is higher than \$10 (e.g. \$12)
   - You loose either way: no difference
2. The other bid is lower than \$5 (e.g. \$2)
   - You win either way and pay \$2: no difference
3. The other bid is between \$5 and \$10 (e.g. \$8)
   - You loose with \$5: zero payoff
   - You win with \$10: earn \$2



Weakly dominant strategy equilibrium $= (v_1, v_2)$

There are many Nash equilibria. For example $(v_1, 0)$

First Price Auctions

- Highest bidder wins and pays her own bid
- In case of a tie, the object is awarded to player 1

Strategic form:

1. $N = \{1, 2\}$
2. $A_1 = A_2 = \mathbf{R}_+$
3. Payoff functions: For any $(b_1, b_2) \in \mathbf{R}^2_+$

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_1, & \text{if } b_1 \geq b_2, \\ 0, & \text{otherwise.} \end{cases}$$

$$u_2(b_1, b_2) = \begin{cases} v_2 - b_2, & \text{if } b_2 > b_1, \\ 0, & \text{otherwise.} \end{cases}$$

Nash Equilibria of First Price Auctions

- There is no dominant strategy equilibrium
- How about Nash equilibria?
- We can compute the best response correspondences
- or we can adopt a direct approach
  - You first find the necessary conditions for a Nash equilibrium
    - ★ If a strategy profile is a Nash equilibrium then it must satisfy these conditions
  - Then you find the sufficient conditions
    - ★ If a strategy profile satisfies these conditions, then it is a Nash equilibrium

Necessary Conditions

Let (b_1^*, b_2^*) be a Nash equilibrium. Then,

1. Player 1 wins: $b_1^* \geq b_2^*$

Proof

Suppose not: $b_1^* < b_2^*$. Two possibilities:

1.1 $b_2^* \leq v_2$: Player 1 could bid v_2 and obtain a strictly higher payoff

1.2 $b_2^* > v_2$: Player 2 has a profitable deviation: bid zero

Contradicting the hypothesis that (b_1^*, b_2^*) is a Nash equilibrium.

2. $b_1^* = b_2^*$

Proof

Suppose not: $b_1^* > b_2^*$. Player 1 has a profitable deviation: bid b_2^*

3. $v_2 \leq b_1^* \leq v_1$

Proof

Exercise

Sufficient Conditions

So, any Nash equilibrium (b_1^*, b_2^*) must satisfy

$$v_2 \leq b_1^* = b_2^* \leq v_1.$$

Is any pair (b_1^*, b_2^*) that satisfies these inequalities an equilibrium?

Set of Nash equilibria is given by

$$\{(b_1, b_2) : v_2 \leq b_1 = b_2 \leq v_1\}$$

Price Competition Models

- Quantity (or capacity) competition: Cournot Model
  - Augustine Cournot (1838)
- Price Competition: Bertrand Model
  - Joseph Bertrand (1883)

Two main models:

1. Bertrand Oligopoly with Homogeneous Products
2. Bertrand Oligopoly with Differentiated Products

Bertrand Duopoly with Homogeneous Products

- Two firms, each with unit cost $c \geq 0$
- They choose prices
  - The one with the lower price captures the entire market
  - In case of a tie they share the market equally
- Total market demand is equal to one (not price sensitive)

Strategic form of the game:

1. $N = \{1, 2\}$
2. $A_1 = A_2 = \mathbf{R}_+$
3. Payoff functions: For any $(P_1, P_2) \in \mathbf{R}_+^2$

$$u_1(P_1, P_2) = \begin{cases} P_1 - c, & \text{if } P_1 < P_2, \\ \frac{P_1 - c}{2}, & \text{if } P_1 = P_2, \\ 0, & \text{if } P_1 > P_2. \end{cases}$$

$$u_2(P_1, P_2) = \begin{cases} P_2 - c, & \text{if } P_2 < P_1, \\ \frac{P_2 - c}{2}, & \text{if } P_2 = P_1, \\ 0, & \text{if } P_2 > P_1. \end{cases}$$

Nash Equilibrium

Suppose P_1^*, P_2^* is a Nash equilibrium. Then

1. $P_1^*, P_2^* \geq c$. Why?
2. At least one charges c
   - $P_1^* > P_2^* > c$?
   - $P_2^* > P_1^* > c$?
   - $P_1^* = P_2^* > c$?
3. $P_2^* > P_1^* = c$?
4. $P_1^* > P_2^* = c$?

The only candidate for equilibrium is $P_1^* = P_2^* = c$, and it is indeed an equilibrium.

The unique Nash equilibrium of the Bertrand game is $(P_1^*, P_2^*) = (c, c)$

- What if unit cost of firm 1 exceeds that of firm 2?
- What if prices are discrete?

Bertrand Duopoly with Differentiated Products

- Two firms with products that are imperfect substitutes
- The demand functions are

$$Q_1(P_1, P_2) = 10 - \alpha P_1 + P_2$$

$$Q_2(P_1, P_2) = 10 + P_1 - \alpha P_2$$

- Assume that $\alpha > 1$
- Unit costs are c

Exercise

Formulate as a strategic form game and find its Nash equilibria.

A Model of Election

Spatial Voting Models

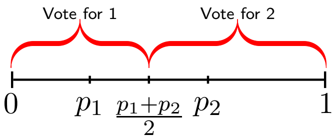
- Candidates choose a policy
  - 10% tax rate vs. 25% tax rate
  - pro-EU vs anti-EU
- Only goal is to win the election
  - preferences: win $\succ$ tie $\succ$ lose
- Voters have ideal positions over the issue
  - one voter could have 15% as ideal tax rate, another 45%
- One-dimensional policy space: $[0, 1]$
- Identify each voter with her ideal position $t \in [0, 1]$
- Voters' preferences are single peaked
  - They vote for that candidate whose position is closest to their ideal point
- Society is a continuum and voters are distributed uniformly over $[0, 1]$

Strategic Form of the Game

1. $N = \{1, 2\}$
2. $A_1 = A_2 = [0, 1]$
3.

$$u_i(p_1, p_2) = \begin{cases} 1, & \text{if } i \text{ wins} \\ \frac{1}{2}, & \text{if there is a tie} \\ 0, & \text{if } i \text{ loses} \end{cases}$$

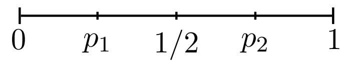
Say the two candidates choose $0 < p_1 < p_2 < 1$



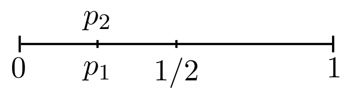
Nash Equilibrium

Suppose p_1^*, p_2^* is a Nash equilibrium. Then

1. Outcome must be a tie
   - Whatever your opponent chooses you can always guarantee a tie
2. $p_1^* \neq p_2^*$?



3. $p_1^* = p_2^* \neq 1/2$?



The only candidate for equilibrium is $p_1^* = p_2^* = 1/2$, which is indeed an equilibrium.

The unique Nash equilibrium of the election game is $(p_1^*, p_2^*) = (1/2, 1/2)$

Other Election Models

- This result generalizes to models with more general distributions
- Equilibrium is for each party to choose the median position
  - Known as the median voter theorem

Other Models

- Models with participation costs
- Models with more than two players
- Models with multidimensional policy space
- Models with ideological candidates

Game Theory

Mixed Strategies

Levent Koçkesen

Koç University

Matching Pennies

Player 2

Player 1	H	T
H	$-1,1$	$1,-1$
T	$1,-1$	$-1,1$

- How would you play?
- No solution?
- You should try to be unpredictable
- Choose randomly

Mixed Strategy Equilibrium

Player 2

Player 1	H	T
H	$-1,1$	$1,-1$
T	$1,-1$	$-1,1$

A mixed strategy is a probability distribution over the set of actions.

- Suppose Player 2 chooses H with probability 1/2 and T with probability 1/2
- What should Player 1 do?

Mixed Strategy Equilibrium

Player 2

Player 1	H	T
H	$-1,1$	$1,-1$
T	$1,-1$	$-1,1$

- If she chooses H
- She gets -1 with prob. $1/2$ and 1 with prob. $1/2$
- What is the value of this to player 1?
- We assume the value is the expected payoff:

$$\frac{1}{2} \times (-1) + \frac{1}{2} \times 1 = 0$$

- What is the expected payoff to T?

$$\frac{1}{2} \times 1 + \frac{1}{2} \times (-1) = 0$$

- She is indifferent between H and T
- She is also indifferent between H and T with any probability

Mixed Strategy Equilibrium

Player 2

		H	T
Player 1	H	$-1, 1$	$1, -1$
	T	$1, -1$	$-1, 1$

- Similarly, if player 1 plays H and T with equal probabilities
- Player 2 is indifferent between playing H and T with any probability
- Player 1's strategy is a best response to player 2's strategy and conversely
- We have a Mixed Strategy Equilibrium

Mixed Strategy Equilibrium

Player 2

		H	T
Player 1	H	$-1, 1$	$1, -1$
	T	$1, -1$	$-1, 1$

In a mixed strategy equilibrium every action played with positive probability must be a best response to other players' mixed strategies

- In particular players must be indifferent between actions played with positive probability

Mixed Strategy Equilibrium

Player 2

		H	T
Player 1	H	$-1, 1$	$1, -1$
	T	$1, -1$	$-1, 1$

- Suppose player 1 chooses H with probability p and player 2 chooses H with probability q
- Player 1's expected payoff to
  - H is $q \times (-1) + (1-q) \times 1 = 1 - 2q$
  - T is $q \times 1 + (1-q) \times (-1) = -1 + 2q$
- Indifference condition

$$1 - 2q = -1 + 2q$$

implies $q = 1/2$

Mixed Strategy Equilibrium

Player 2

		H	T
Player 1	H	$-1, 1$	$1, -1$
	T	$1, -1$	$-1, 1$

- Player 2's expected payoff to
  - H is $p \times 1 + (1-p) \times (-1) = -1 + 2p$
  - T is $p \times (-1) + (1-p) \times 1 = 1 - 2p$
- Indifference condition

$$1 - 2p = -1 + 2p$$

implies $p = 1/2$

Mixed Strategy Equilibrium

$p = 1/2$ is a best response to $q = 1/2$ and $q = 1/2$ is a best response to $p = 1/2$

$(p = 1/2, q = 1/2)$ is a mixed strategy equilibrium

Since there is no pure strategy equilibrium, this is also the unique Nash equilibrium

Hawk-Dove

Player 2

Player 1		H	D
	H	$0,0$	$6,1$
	D	$1,6$	$3,3$

- How would you play?
- What is the stable population composition?
- Nash equilibria?
  - (H, D)
  - (D, H)
- How about $3/4$ hawkish and $1/4$ dovish?
  - On average a dovish player gets $(3/4) \times 1 + (1/4) \times 3 = 3/2$
  - A hawkish player gets $(3/4) \times 0 + (1/4) \times 6 = 3/2$
  - No type has an evolutionary advantage
- This is a mixed strategy equilibrium

Mixed and Pure Strategy Equilibria

- How do you find the set of all (pure and mixed) Nash equilibria?
- In 2×2 games we can use the best response correspondences in terms of the mixed strategies and plot them
- Consider the Battle of the Sexes game

Player 2

Player 1		m	o
	m	$2,1$	$0,0$
	o	$0,0$	$1,2$

- Denote Player 1's strategy as p and that of Player 2 as q (probability of choosing m)

	m	o
m	$2,1$	$0,0$
o	$0,0$	$1,2$

- What is Player 1's best response?
- Expected payoff to
  - m is $2q$
  - o is $1 - q$
- If $2q > 1 - q$ or $q > 1/3$
  - best response is m (or equivalently $p = 1$)
- If $2q < 1 - q$ or $q < 1/3$
  - best response is o (or equivalently $p = 0$)
- If $2q = 1 - q$ or $q = 1/3$
  - he is indifferent
  - best response is any $p \in [0, 1]$

Player 1's best response correspondence:

$$B_1(q) = \begin{cases} \{1\}, & \text{if } q > 1/3 \\ [0,1], & \text{if } q = 1/3 \\ \{0\}, & \text{if } q < 1/3 \end{cases}$$

	m	o
m	$2,1$	$0,0$
o	$0,0$	$1,2$

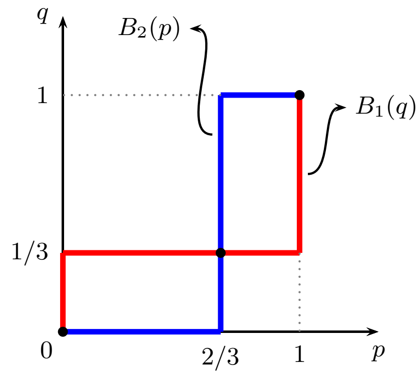
- What is Player 2's best response?
- Expected payoff to
  - m is p
  - o is $2(1-p)$
- If $p > 2(1-p)$ or $p > 2/3$
  - best response is m (or equivalently $q = 1$)
- If $p < 2(1-p)$ or $p < 2/3$
  - best response is o (or equivalently $q = 0$)
- If $p = 2(1-p)$ or $p = 2/3$
  - she is indifferent
  - best response is any $q \in [0,1]$

Player 2's best response correspondence:

$$B_2(p) = \begin{cases} \{1\}, & \text{if } p > 2/3 \\ [0,1], & \text{if } p = 2/3 \\ \{0\}, & \text{if } p < 2/3 \end{cases}$$

$$B_1(q) = \begin{cases} \{1\}, & \text{if } q > 1/3 \\ [0,1], & \text{if } q = 1/3 \\ \{0\}, & \text{if } q < 1/3 \end{cases}$$

$$B_2(p) = \begin{cases} \{1\}, & \text{if } p > 2/3 \\ [0,1], & \text{if } p = 2/3 \\ \{0\}, & \text{if } p < 2/3 \end{cases}$$



Set of Nash equilibria

$\{(0,0),(1,1),(2/3,1/3)\}$

Dominated Actions and Mixed Strategies

- Up to now we tested actions only against other actions
- An action may be undominated by any other action, yet be dominated by a mixed strategy
- Consider the following game

	L	R
T	$1,1$	$1,0$
M	$3,0$	$0,3$
B	$0,1$	$4,0$

- No action dominates T
- But mixed strategy $(\text{prob}(M) = 1/2, \text{prob}(B) = 1/2)$ strictly dominates T

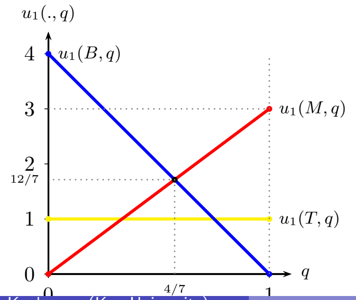
A strictly dominated action is never used with positive probability in a mixed strategy equilibrium

Dominated Actions and Mixed Strategies

- An easy way to figure out dominated actions is to compare expected payoffs
- Let player 2's mixed strategy given by $q = prob(L)$

	L	R
T	$1,1$	$1,0$
M	$3,0$	$0,3$
B	$0,1$	$4,0$

$$\begin{aligned} u_1(T,q) &= 1 \\ u_1(M,q) &= 3q \\ u_1(B,q) &= 4(1-q) \end{aligned}$$

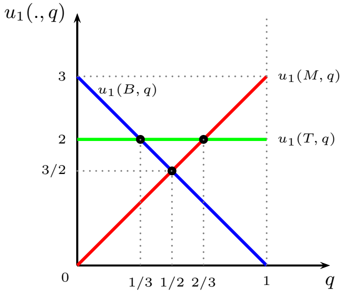


- An action is a never best response if there is no belief (on A_{-i}) that makes that action a best response
- T is a never best response
- An action is a NBR iff it is strictly dominated

What if there are no strictly dominated actions?

	L	R
T	$2,0$	$2,1$
M	$3,3$	$0,0$
B	$0,1$	$3,0$

- Denote player 2's mixed strategy by $q = prob(L)$
- $u_1(T,q) = 2, u_1(M,q) = 3q, u_1(B,q) = 3(1-q)$



- Pure strategy Nash eq. (M,L)
- Mixed strategy equilibria?
  - Only one player mixes? Not possible
  - Player 1 mixes over $\{T,M,B\}$? Not possible
  - Player 1 mixes over $\{M,B\}$? Not possible
  - Player 1 mixes over $\{T,B\}$? Let $p = prob(T)$
    $q = 1/3, 1-p = p \rightarrow p = 1/2$
  - Player 1 mixes over $\{T,M\}$? Let $p = prob(T)$
    $q = 2/3, 3(1-p) = p \rightarrow p = 3/4$

Real Life Examples?

- Ignacio Palacios-Huerta (2003): 5 years' worth of penalty kicks
- Empirical scoring probabilities

	L	R
L	$58,42$	$95,5$
R	$93,7$	$70,30$

R is the natural side of the kicker

- What are the equilibrium strategies?

Penalty Kick

	L	R
L	$58,42$	$95,5$
R	$93,7$	$70,30$

- Kicker must be indifferent

$$58p + 95(1-p) = 93p + 70(1-p) \Rightarrow p = 0.42$$

- Goal keeper must be indifferent

$$42q + 7(1-q) = 5q + 30(1-q) \Rightarrow q = 0.39$$

	Theory	Data
Kicker	39%	40%
Goallie	42%	42%

- Also see Walker and Wooders (2001): Wimbledon

Game Theory

Strategic Form Games with Incomplete Information

Levent Koçkesen

Koç University

Games with Incomplete Information

- Some players have incomplete information about some components of the game
  - Firm does not know rival's cost
  - Bidder does not know valuations of other bidders in an auction
- We could also say some players have private information
- What difference does it make?
- Suppose you make an offer to buy out a company
- If the value of the company is V it is worth $1.5V$ to you
- The seller accepts only if the offer is at least V
- If you know V what do you offer?
- You know only that V is uniformly distributed over $[0, 100]$. What should you offer?
- Enter your name and your bid

  Click here for the EXCEL file

Bayesian Games

- We will first look at incomplete information games where players move simultaneously
  - Bayesian games
- Later on we will study dynamic games of incomplete information

What is new in a Bayesian game?

- Each player has a type: summarizes a player's private information
  - Type set for player i: Θ_i
    - A generic type: θ_i
  - Set of type profiles: $\Theta = \times_{i \in N} \Theta_i$
    - A generic type profile: $\theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$
- Each player has beliefs about others' types
  - $p_i : \Theta_i \to \triangle(\Theta_{-i})$
  - $p_i(\theta_{-i}|\theta_i)$
- Players' payoffs depend on types
  - $u_i : A \times \Theta \to \mathbf{R}$
  - $u_i(a|\theta)$
- Different types of same player may play different strategies
  - $a_i : \Theta_i \to A_i$
  - $\alpha_i : \Theta_i \to \triangle(A_i)$

Bayesian Games

- Incomplete information can be anything about the game
  - Payoff functions
  - Actions available to others
  - Beliefs of others; beliefs of others' beliefs of others'...
- Harsanyi showed that introducing types in payoffs is adequate

Bayesian Equilibrium

Bayesian equilibrium is a collection of strategies (one for each type of each player) such that each type best responds given her beliefs about other players' types and their strategies

Also known as Bayesian Nash or Bayes Nash equilibrium

Bank Runs

- You (player 1) and another investor (player 2) have a deposit of \$100 each in a bank
- If the bank manager is a good investor you will each get \$150 at the end of the year. If not you loose your money
- You can try to withdraw your money now but the bank has only \$100 cash
  - If only one tries to withdraw she gets \$100
  - If both try to withdraw they each can get \$50
- You believe that the manager is good with probability q
- Player 2 knows whether the manager is good or bad
- You and player 2 simultaneously decide whether to withdraw or not

Bank Runs

The payoffs can be summarized as follows

	W	N
W	$50, 50$	$100, 0$
N	$0, 100$	$150, 150$

Good q

	W	N
W	$50, 50$	$100, 0$
N	$0, 100$	$0, 0$

Bad $(1-q)$

Two Possible Types of Bayesian Equilibria

1. Separating Equilibria: Each type plays a different strategy
2. Pooling Equilibria: Each type plays the same strategy

- How would you play if you were Player 2 who knew the banker was bad?
- Player 2 always withdraws in bad state

Separating Equilibria

	W	N
W	$50, 50$	$100, 0$
N	$0, 100$	$150, 150$

Good q

	W	N
W	$50, 50$	$100, 0$
N	$0, 100$	$0, 0$

Bad $(1-q)$

1. (Good: W, Bad: N)
   - Not possible since W is a dominant strategy for Bad
2. (Good: N, Bad: W)
   Player 1's expected payoffs

   W: $q \times 100 + (1-q) \times 50$
   N: $q \times 150 + (1-q) \times 0$

   Two possibilities

   2.1 $q < 1/2$: Player 1 chooses W. But then player 2 of Good type must play W, which contradicts our hypothesis that he plays N
   2.2 $q \geq 1/2$: Player 1 chooses N. The best response of Player 2 of Good type is N, which is the same as our hypothesis

Separating Equilibrium

$q < 1/2$: No separating equilibrium
$q \geq 1/2$: Player 1: N, Player 2: (Good: N, Bad: W)

Pooling Equilibria

	W	N
W	$50, 50$	$100, 0$
N	$0, 100$	$150, 150$

Good q

	W	N
W	$50, 50$	$100, 0$
N	$0, 100$	$0, 0$

Bad $(1-q)$

1. (Good: N, Bad: N)
   - Not possible since W is a dominant strategy for Bad
2. (Good: W, Bad: W)
   Player 1's expected payoffs

   W: $q \times 50 + (1-q) \times 50$

   N: $q \times 0 + (1-q) \times 0$

   Player 1 chooses W. Player 2 of Good type's best response is W.
   Therefore, for any value of q the following is the unique

Pooling Equilibrium

Player 1: W, Player 2: (Good: W, Bad: W)

If $q < 1/2$ the only equilibrium is a bank run

Cournot Duopoly with Incomplete Information about Costs

- Two firms. They choose how much to produce $q_i \in \mathbb{R}_+$
- Firm 1 has high cost: c_H
- Firm 2 has either low or high cost: c_L or c_H
- Firm 1 believes that Firm 2 has low cost with probability $\mu \in [0, 1]$
- payoff function of player i with cost c_j

$$u_i(q_1, q_2, c_j) = (a - (q_1 + q_2))\, q_i - c_j q_i$$

- Strategies:

$$q_1 \in \mathbb{R}_+ \qquad q_2 : \{c_L, c_H\} \to \mathbb{R}_+$$

Complete Information

- Firm 1

$$\max_{q_1} (a - (q_1 + q_2))\, q_1 - c_H q_1$$

- Best response correspondence

$$BR_1(q_2) = \frac{a - q_2 - c_H}{2}$$

- Firm 2

$$\max_{q_2} (a - (q_1 + q_2))\, q_2 - c_j q_2$$

- Best response correspondences

$$BR_2(q_1, c_L) = \frac{a - q_1 - c_L}{2}$$
$$BR_2(q_1, c_H) = \frac{a - q_1 - c_H}{2}$$

Complete Information

Nash Equilibrium

- If Firm 2's cost is c_H

$$q_1 = q_2 = \frac{a - c_H}{3}$$

- If Firm 2's cost is c_L

$$q_1 = \frac{a - c_H - (c_H - c_L)}{3}$$
$$q_2 = \frac{a - c_H + (c_H - c_L)}{3}$$

Incomplete Information

- Firm 2

$$\max_{q_2} \left(a - (q_1 + q_2)\right) q_2 - c_j q_2$$

- Best response correspondences

$$BR_2(q_1, c_L) = \frac{a - q_1 - c_L}{2}$$
$$BR_2(q_1, c_H) = \frac{a - q_1 - c_H}{2}$$

- Firm 1 maximizes

$$\mu\{[a - (q_1 + q_2(c_L))]\, q_1 - c_H q_1\} + (1-\mu)\{[a - (q_1 + q_2(c_H))]\, q_1 - c_H q_1\}$$

- Best response correspondence

$$BR_1(q_2(c_L), q_2(c_H)) = \frac{a - [\mu q_2(c_L) + (1-\mu) q_2(c_H)] - c_H}{2}$$

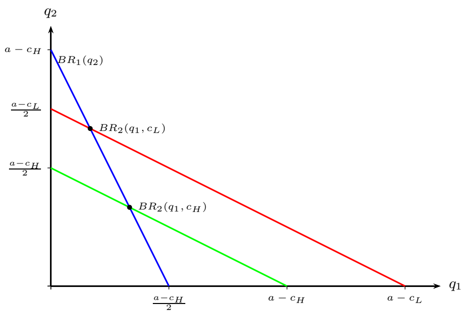
Bayesian Equilibrium

$$q_1 = \frac{a - c_H - \mu(c_H - c_L)}{3}$$
$$q_2(c_L) = \frac{a - c_L + (c_H - c_L)}{3} - (1-\mu)\frac{c_H - c_L}{6}$$
$$q_2(c_H) = \frac{a - c_H}{3} + \mu\frac{c_H - c_L}{6}$$

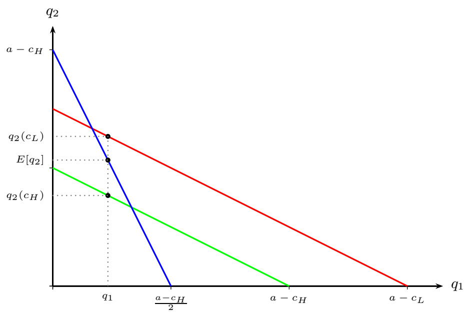
- Is information good or bad for Firm 1?
- Does Firm 2 want Firm 1 to know its costs?

Complete vs. Incomplete Information

Complete Information



Incomplete Information



Game Theory

Auctions

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Koç University

Auctions

Many economic transactions are conducted through auctions

- treasury bills
- foreign exchange
- publicly owned companies
- mineral rights
- airwave spectrum rights
- art work
- antiques
- cars
- houses
- government contracts

Also can be thought of as auctions

- takeover battles
- queues
- wars of attrition
- lobbying contests

Auction Formats

1. Open bid auctions
   1.1 ascending-bid auction
      - ★ aka English auction
      - ★ price is raised until only one bidder remains, who wins and pays the final price
   1.2 descending-bid auction
      - ★ aka Dutch auction
      - ★ price is lowered until someone accepts, who wins the object at the current price
2. Sealed bid auctions
   2.1 first price auction
      - ★ highest bidder wins; pays her bid
   2.2 second price auction
      - ★ aka Vickrey auction
      - ★ highest bidder wins; pays the second highest bid

Auction Formats

Auctions also differ with respect to the valuation of the bidders

1. Private value auctions
   - each bidder knows only her own value
   - artwork, antiques, memorabilia
2. Common value auctions
   - actual value of the object is the same for everyone
   - bidders have different private information about that value
   - oil field auctions, company takeovers

Independent Private Values

- Each bidder knows only her own valuation
- Valuations are independent across bidders
- Bidders have beliefs over other bidders' values
- Risk neutral bidders
  - If the winner's value is v and pays p, her payoff is $v - p$

Second Price Auctions

- Only one item will be sold
- Your value is the last 2 digits of your KU ID
  - This is the max you are willing to pay
- You choose an integer between 0 and 100
- Highest bidder wins, pays the second highest bid
- Write your name, value, and bid
- Your payoff: value - price
- More than one winners $\rightarrow$ I will pick one randomly

Auctions as a Bayesian Game

- set of players $N = \{1, 2, \ldots, n\}$
- type set $\Theta_i = [\underline{v}, \bar{v}], \underline{v} \geq 0$
- action set, $A_i = \mathbf{R}_+$
- beliefs
  - opponents' valuations are independent draws from a distribution function F
  - F is strictly increasing and continuous
- payoff function

$$u_i(a, v) = \begin{cases} \frac{v_i - P(a)}{m}, & \text{if } a_j \leq a_i \text{ for all } j \neq i, \text{and } |\{j : a_j = a_i\}| = m \\ 0, & \text{if } a_j > a_i \text{ for some } j \neq i \end{cases}$$

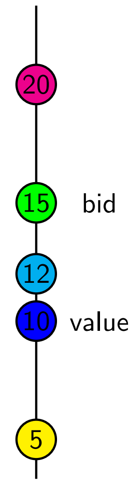
  - $P(a)$ is the price paid by the winner if the bid profile is a

Second Price Auctions

I. Bidding your value weakly dominates bidding higher

Suppose your value is \$10 but you bid \$15. Three cases:

1. There is a bid higher than \$15 (e.g. \$20)
   - You loose either way: no difference
2. 2nd highest bid is lower than \$10 (e.g. \$5)
   - You win either way and pay \$5: no difference
3. 2nd highest bid is between \$10 and \$15 (e.g. \$12)
   - You loose with \$10: zero payoff
   - You win with \$15: loose \$2

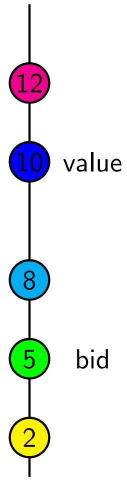


Second Price Auctions

II. Bidding your value weakly dominates bidding lower

Suppose your value is \$10 but you bid \$5. Three cases:

1. There is a bid higher than \$10 (e.g. \$12)
   - You loose either way: no difference
2. 2nd highest bid is lower than \$5 (e.g. \$2)
   - You win either way and pay \$2: no difference
3. 2nd highest bid is between \$5 and \$10 (e.g. \$8)
   - You loose with \$5: zero payoff
   - You win with \$10: earn \$2



English Auction

- Suppose you value the item at 100 TL
- What is your optimal strategy?
- Stay in bidding until the price exceeds 100 TL
- This is a dominant strategy
- If everyone plays this strategy what happens?
  - The bidder with highest value wins
  - Pays something close to second highest value

First Price Auctions

- Only one item will be sold
- Your value is the second from last 2 digits of your KU ID
  - This is the max you are willing to pay
- You choose an integer between 0 and 100
- Highest bidder wins, pays her bid
- Write your name, value, and bid
- Your payoff: value - price
- More than one winners → I will pick one randomly

First Price Auctions

- Would you bid your value?
- What happens if you bid less than your value?
  - You get a positive payoff if you win
  - But your chances of winning are smaller
  - Optimal bid reflects this tradeoff
- Bidding less than your value is known as bid shading
- Choose your bid b to maximize

$$\pi = (v - b)\,\mathrm{prob}(win)$$

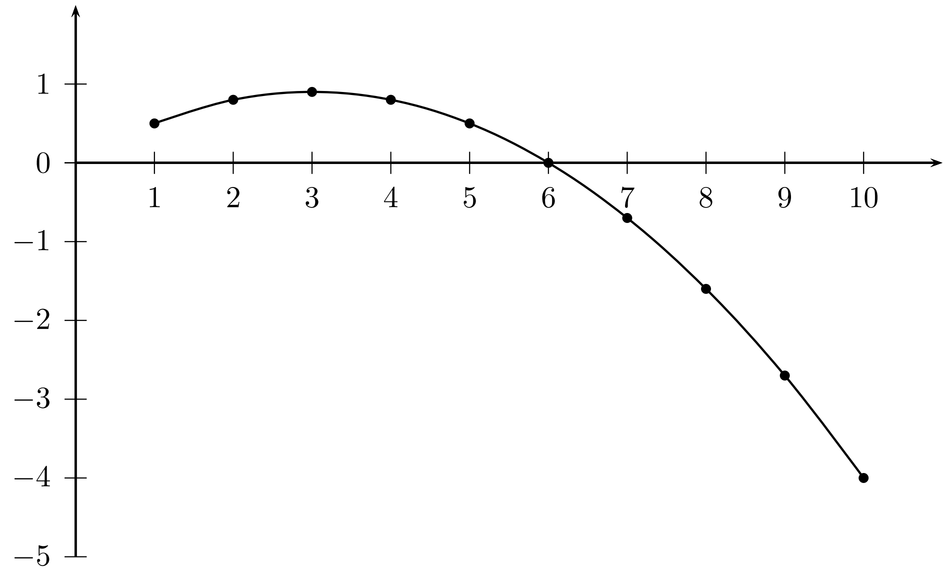
- Probability of winning depends on your bid and others' bids
- Given what you believe about the others, it is increasing in your bid

Example

Suppose your value is 6 and the highest possible value is 10

b	$v-b$	prob	π
10	-4	1	-4
9	-3	0.9	-2.7
8	-2	0.8	-1.6
7	-1	0.7	-0.7
6	0	0.6	0
5	1	0.5	0.5
4	2	0.4	0.8
3	3	0.3	0.9
2	4	0.2	0.8
1	5	0.1	0.5

Example



Bayesian Equilibrium of First Price Auctions

- Only 2 bidders
- You are player 1 and your value is v
- Values are independently and uniformly distributed over $[0,1]$
- Is there an equilibrium in which both bidders use the same bidding strategy $\beta_i(v) = av$, $i = 1,2$, where $a > 0$?
- Highest possible bid by the other $= a \Rightarrow$ optimal bid $\leq a$
- Your expected payoff if you bid b

$$\begin{aligned}(v-b)\,\text{prob(you win)} &= (v-b)\,\text{prob}(b > aV_2) \\ &= (v-b)\,\text{prob}(V_2 < b/a) \\ &= (v-b)\frac{b}{a}\end{aligned}$$

Bayesian Equilibrium of First Price Auctions

- Best response can be found by solving:

$$\max_{0 \leq b \leq a} (v-b)\frac{b}{a}$$

- The critical value is found by using FOC:

$$-\frac{b}{a} + \frac{v-b}{a} = 0 \Rightarrow b = \frac{v}{2}$$

- So, if there is an equilibrium in which $\beta_i(v) = av$, $i = 1,2$, where $a > 0$, it must be

$$\beta_i(v) = \frac{1}{2}v$$

Bayesian Equilibrium of First Price Auctions

- Is this an equilibrium?
- Given that the other player bids half her value
- it is never optimal to bid more than $1/2$
- Your expected payoff to bidding $b \in [0, 1/2]$ is

$$\begin{aligned}(v-b)\,\mathrm{prob}(b > V_2/2) &= (v-b)\,\mathrm{prob}(V_2 < 2b) \\ &= 2(v-b)b\end{aligned}$$

- Is $b = v/2$ a solution to the following problem?

$$\max_{0 \le b \le 1/2} 2(v-b)b$$

- The critical value is

$$b = \frac{v}{2}$$

Bayesian Equilibrium of First Price Auctions

- Payoff to $b = v/2$ is $v^2/2$
- Payoff to $b = 0$ is 0 and to $b = 1/2$ is $v - 1/2$
- Check that

$$\frac{v^2}{2} \ge 0 \text{ and } \frac{v^2}{2} \ge v - \frac{1}{2}$$

- We conclude that $\beta_i(v) = v/2$, $i = 1, 2$ is a Bayesian equilibrium

Bayesian Equilibrium of First Price Auctions

- n bidders
- You are player 1 and your value is $v > 0$
- Values are independently and uniformly distributed over $[0, 1]$
- Is there an equilibrium in which both bidders use the same bidding strategy $\beta_i(v) = av$, $i = 1, 2, \ldots, n$, where $a > 0$?
- Optimal bid $b \le a$
- Your expected payoff to bidding $b \in [0, a]$

$$(v-b)\,\mathrm{prob}(\text{you win})$$

$$(v-b)\,\mathrm{prob}(b > aV_2 \text{ and } b > aV_3 \ldots \text{ and } b > aV_n)$$

- This is equal to

$$(v-b)\,\mathrm{prob}(b > aV_2)\,\mathrm{prob}(b > aV_3)\ldots\mathrm{prob}(b > aV_n) = (v-b)(b/a)^{n-1}$$

Bayesian Equilibrium of First Price Auctions

- Best response can be found by solving:

$$\max_{0 \le b \le a} (v-b)(b/a)^{n-1}$$

- The critical value is found by using FOC:

$$-(b/a)^{n-1} + (n-1)\frac{v-b}{a}(b/a)^{n-2} = 0 \Rightarrow b = \frac{n-1}{n}v$$

- So, if there is an equilibrium in which $\beta_i(v) = av$, $i = 1, 2, \ldots, n$, where $a > 0$, it must be

$$\beta_i(v) = \frac{n-1}{n}v$$

Bayesian Equilibrium of First Price Auctions

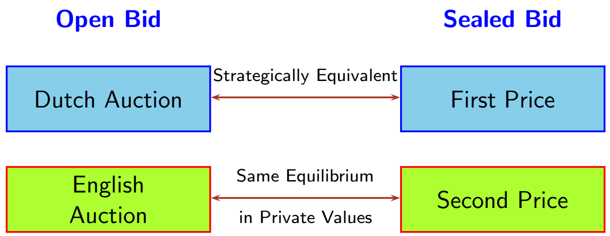
- Payoff to $b = \frac{n-1}{n}v$ is v^n/n
- Payoff to $b = 0$ is 0 and to $b = (n-1)/n$ is $v - (n-1)/n$
- Check that

$$v^n/n \geq 0 \text{ and } v^n/n \geq v - (n-1)/n$$

- We conclude that $\beta_i(v) = \frac{n-1}{n}v$, $i = 1, 2, \ldots, n$ is a Bayesian equilibrium

Equivalent Formats

- English auction has the same equilibrium as Second Price auction
- This is true only if values are private
- Stronger equivalence between Dutch and First Price auctions



Which One Brings More Revenue?

- Second Price
  - Bidders bid their value
  - Revenue = second highest bid
- First Price
  - Bidders bid less than their value
  - Revenue = highest bid
- Which one is better?
- Turns out it doesn't matter

Which One Brings More Revenue?

Revenue Equivalence Theorem

Any auction with independent private values with a common distribution in which

1. the number of the bidders are the same and the bidders are risk-neutral,
2. the object always goes to the buyer with the highest value,
3. the bidder with the lowest value expects zero surplus,

yields the same expected revenue.

Common Value Auctions and Winner's Curse

- Suppose you are going to bid for an offshore oil lease
- Value of the oil tract is the same for everybody
- But nobody knows the true value
- Each bidder obtains an independent and unbiased estimate of the value
- Your estimate is $100 million
- Suppose everybody, including you, bids their estimate and you are the winner
- What did you just learn?
- Your estimate must have been larger than the others
- The true value must be smaller than $100 million
- You overpaid

Common Value Auctions and Winner's Curse

- If everybody bids her estimate, then winning is bad news
- This is known as Winner's Curse
- Optimal strategies are complicated
- Bidders bid much less than their value to prevent winner's curse

To prevent winner's curse

Base your bid on expected value conditional on winning

Common Value Auctions

Auction formats are not equivalent in common value auctions

- Open bid auctions provide information and ameliorates winner's curse
  - Bids are more aggressive
- Sealed bid auctions do not provide information
  - Bids are more conservative

Auction Design: Failures

- New Zeland Spectrum Auction (1990)
  - Used second price auction with no reserve price
  - Estimated revenue NZ$ 240 million
  - Actual revenue NZ$36 million
- Some extreme cases

Winning Bid	Second Highest Bid
NZ$100,000	NZ$6,000
NZ$7,000,000	NZ$5,000
NZ$1	None

Source: John McMillan, "Selling Spectrum Rights," *Journal of Economic Perspectives*, Summer 1994

- Problems
  - Second price format politically problematic
    - Public sees outcome as selling for less than its worth
  - No reserve price

Auction Design: Failures

- Australian TV Licence Auction (1993)
  - Two satellite-TV licences
  - Used first price auction
  - Huge embarrasment
- High bidders had no intention of paying
- They bid high just to guarantee winning
- They also bid lower amounts at A$5 million intervals
- They defaulted
  - licences had to be re-awarded at the next highest bid
  - those bids were also theirs
- Outcome after a series of defaults

Initial Bid	Final Price
A$212 mil.	A$117 mil.
A$177 mil.	A$77 mil.

Source: John McMillan, "Selling Spectrum Rights," *Journal of Economic Perspectives*, Summer 1994

- Problem: No penalty for default

Auction Design: Failures

Turkish GSM licence auction

- April 2000: Two GSM 1800 licences to be auctioned
- Auction method:
  1. Round 1: First price sealed bid auction
  2. Round 2: First price sealed bid auction with reserve price
     - ★ Reserve price is the winning bid of Round 1
- Bids in the first round

Bidder	Bid Amount
Is-Tim	$2.525 bil.
Dogan+	$1.350 bil.
Genpa+	$1.224 bil.
Koc+	$1.207 bil.
Fiba+	$1.017 bil.

- Bids in the second round: NONE!
- Problem: Facilitates entry deterrence

Auction Design

- Good design depends on objective
  - Revenue
  - Efficiency
  - Other

One common objective is to maximize expected revenue

- In the case of private independent values with the same number of risk neutral bidders format does not matter
- Auction design is a challenge when
  - values are correlated
  - bidders are risk averse
- Other design problems
  - collusion
  - entry deterrence
  - reserve price

Auction Design

- Correlated values: Ascending bid auction is better
- Risk averse bidders
  - Second price auction: risk aversion does not matter
  - First price auction: higher bids
- Collusion: Sealed bid auctions are better to prevent collusion
- Entry deterrence: Sealed bid auctions are better to promote entry

A hybrid format, such as Anglo-Dutch Auction, could be better.

Anglo-Dutch auction has two stages:

1. Ascending bid auction until only two bidders remain
2. Two remaining bidders make offers in a first price sealed bid auction

Game Theory

Extensive Form Games

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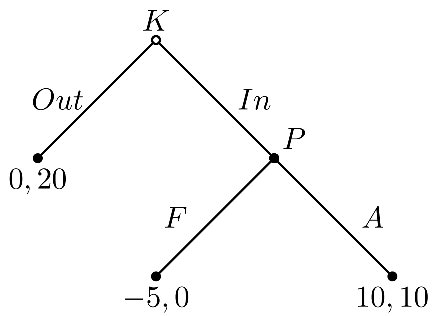
Extensive Form Games

- Strategic form games are used to model situations in which players choose strategies without knowing the strategy choices of the other players
- In some situations players observe other players' moves before they move
- Removing Coins:
  - There are 21 coins
  - Two players move sequentially and remove 1, 2, or 3 coins
  - Winner is who removes the last coin(s)
  - We will determine the first mover by a coin toss
  - Volunteers?

Entry Game

- Kodak is contemplating entering the instant photography market and Polaroid can either fight the entry or accommodate



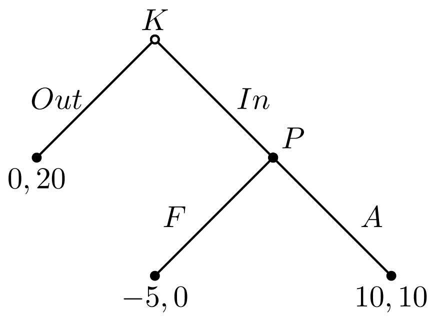
Extensive Form Games

- Strategic form has three ingredients:
  - set of players
  - sets of actions
  - payoff functions
- Extensive form games provide more information
  - order of moves
  - actions available at different points in the game
  - information available throughout the game
- Easiest way to represent an extensive form game is to use a game tree

Game Trees

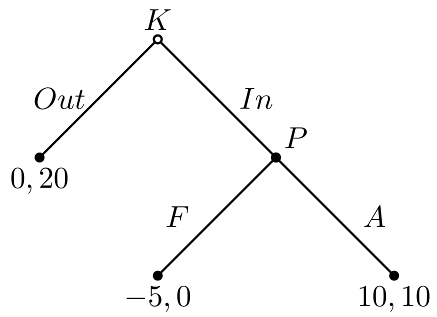
What's in a game tree?

- nodes
  - decision nodes
  - initial node
  - terminal nodes
- branches
- player labels
- action labels
- payoffs
- information sets
  - to be seen later



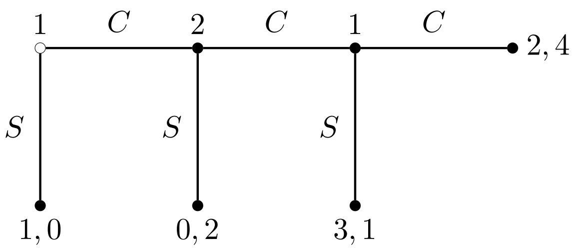
Extensive Form Game Strategies

A pure strategy of a player specifies an action choice at each decision node of that player



- Kodak's strategies
  - $S_K = \{Out, In\}$
- Polaroid's strategies
  - $S_P = \{F, A\}$

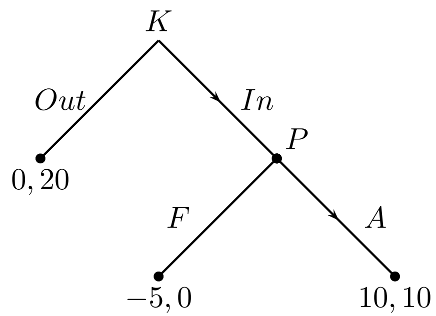
Extensive Form Game Strategies



- $S_1 = \{SS, SC, CS, CC\}$
- $S_2 = \{S, C\}$

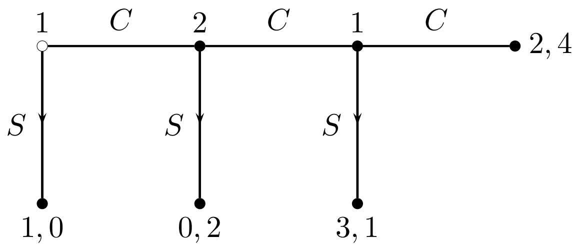
Backward Induction Equilibrium

- What should Polaroid do if Kodak enters?
- Given what it knows about Polaroid's response to entry, what should Kodak do?
- This is an example of a backward induction equilibrium



- At a backward induction equilibrium each player plays optimally at every decision node in the game tree (i.e., plays a sequentially rational strategy)
- (In, A) is the unique backward induction equilibrium of the entry game

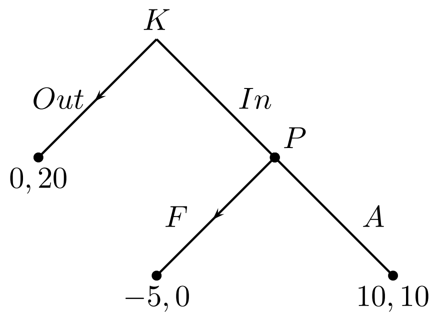
Backward Induction Equilibrium



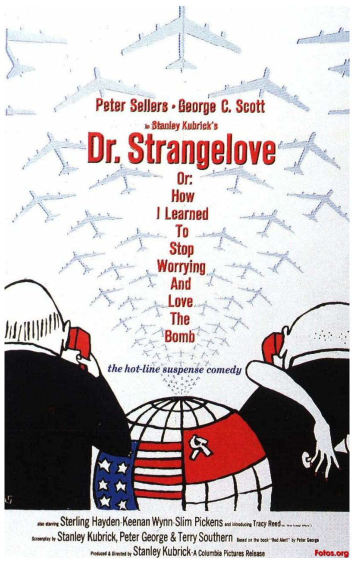
- What should Player 1 do if the game reaches the last decision node?
- Given that, what should Player 2 do if the game reaches his decision node?
- Given all that what should Player 1 do at the beginning?
- Unique backward induction equilibrium (BIE) is (SS, S)
- Unique backward induction outcome (BIO) is (S)

Power of Commitment

- Remember that (In, A) is the unique backward induction equilibrium of the entry game. Polaroid'd payoff is 10.
- Suppose Polaroid commits to fight (F) if entry occurs.
- What would Kodak do?



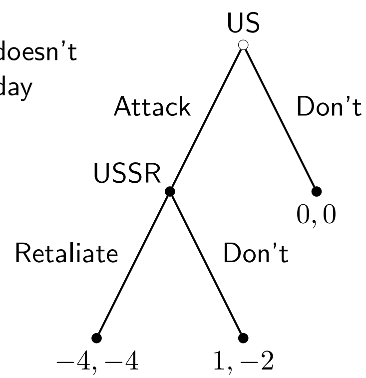
- Outcome would be Out and Polaroid would be better off
- Is this commitment credible?



- A U.S. air force base commander orders thirty four B-52's to launch a nuclear attack on Soviet Union
- He shuts off all communications with the planes and with the base
- U.S. president invites the Russian ambassador to the war room and explains the situation
- They decide to call the Russian president Dimitri

Dr. Strangelove

- What is the outcome if the U.S. doesn't know the existence of the doomsday device?
- What is the outcome if it does?
- Commitment must be observable
- What if USSR can un-trigger the device?
- Commitment must be irreversible



Thomas Schelling

The power to constrain an adversary depends upon the power to bind oneself.

Credible Commitments: Burning Bridges

- In non-strategic environments having more options is never worse
- Not so in strategic environments
- You can change your opponent's actions by removing some of your options
- 1066: William the Conqueror ordered his soldiers to burn their ships after landing to prevent his men from retreating
- 1519: Hernn Corts sank his ships after landing in Mexico for the same reason

Sun-tzu in *The Art of War*, 400 BC

At the critical moment, the leader of an army acts like one who has climbed up a height, and then kicks away the ladder behind him.

Strategic Form of an Extensive Form Game

- If you want to apply a strategic form solution concept
  - Nash equilibrium
  - Dominant strategy equilibrium
  - IEDS
- Analyze the strategic form of the game

Strategic form of an extensive form game

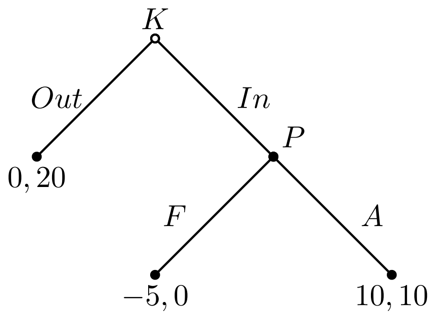
1. Set of players: N
   and for each player i
2. The set of strategies: S_i
3. The payoff function:

$$u_i : S \to \mathbf{R}$$

where $S = \times_{i \in N} S_i$ is the set of all strategy profiles.

Strategic Form of an Extensive Form Game

1. $N = \{K, P\}$
2. $S_K = \{Out, In\}, S_P = \{F, A\}$
3. Payoffs in the bimatrix



K \ P	F	A
Out	$0, 20$	$0, 20$
In	$-5, 0$	$10, 10$

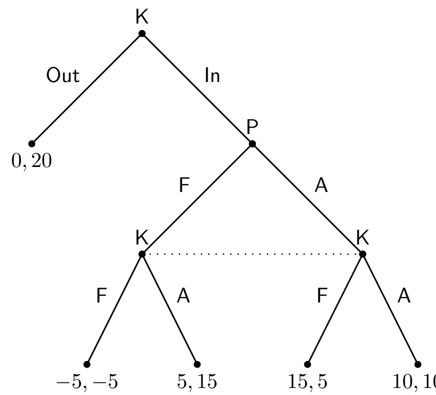
- Set of Nash equilibria $= \{(In, A), (Out, F)\}$
- (Out, F) is sustained by an incredible threat by Polaroid
- Backward induction equilibrium eliminates equilibria based upon incredible threats
- Nash equilibrium requires rationality
- Backward induction requires sequential rationality
  - Players must play optimally at every point in the game

Extensive Form Games with Imperfect Information

- We have seen extensive form games with perfect information
  - Every player observes the previous moves made by all the players
- What happens if some of the previous moves are not observed?
- We cannot apply backward induction algorithm anymore

Consider the following game between Kodak and Polaroid

- Kodak doesn't know whether Polaroid will fight or accommodate
- The dotted line is an information set:
  - a collection of decision nodes that cannot be distinguished by the player
- We cannot determine the optimal action for Kodak at that information set



Subgame Perfect Equilibrium

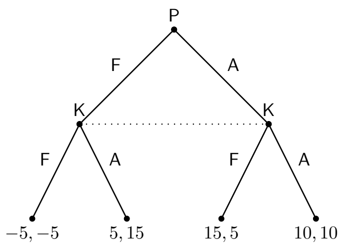
We will introduce another solution concept: Subgame Perfect Equilibrium

Definition

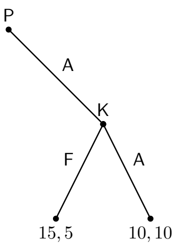
A subgame is a part of the game tree such that

1. it starts at a single decision node,
2. it contains every successor to this node,
3. if it contains a node in an information set, then it contains all the nodes in that information set.

This is a subgame



This is not a subgame



Subgame Perfect Equilibrium

Extensive form game strategies

A pure strategy of a player specifies an action choice at each information set of that player

Definition

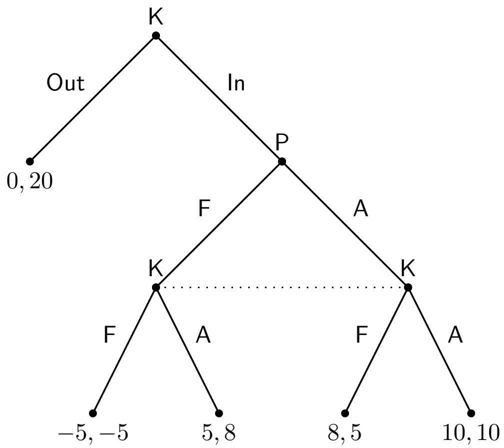
A strategy profile in an extensive form game is a subgame perfect equilibrium (SPE) if it induces a Nash equilibrium in every subgame of the game.

To find SPE

1. Find the Nash equilibria of the "smallest" subgame(s)
2. Fix one for each subgame and attach payoffs to its initial node
3. Repeat with the reduced game

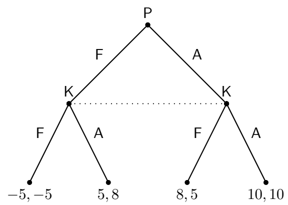
Subgame Perfect Equilibrium

Consider the following game



Subgame Perfect Equilibrium

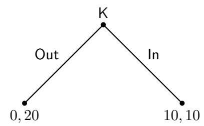
The "smallest" subgame



Its strategic form

		P: F	P: A
K	F	$-5,-5$	$8,5$
	A	$5,8$	$10,10$

- Nash equilibrium of the subgame is (A,A)
- Reduced subgame is



- Its unique Nash equilibrium is (In)
- Therefore the unique SPE of the game is ((In,A),A)

Game Theory

Extensive Form Games: Applications

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A Simple Game

- You have 10 TL to share
- A makes an offer
  - x for me and $10 - x$ for you
- If B accepts
  - A's offer is implemented
- If B rejects
  - Both get zero
- Half the class will play A (proposer) and half B (responder)
  - Proposers should write how much they offer to give responders
  - I will distribute them randomly to responders
    - They should write Yes or No

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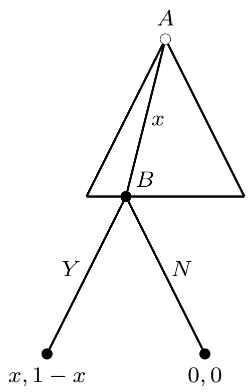
Ultimatum Bargaining

- Two players, A and B, bargain over a cake of size 1
- Player A makes an offer $x \in [0, 1]$ to player B
- If player B accepts the offer (Y), agreement is reached
  - A receives x
  - B receives $1 - x$
- If player B rejects the offer (N) both receive zero



Subgame Perfect Equilibrium of Ultimatum Bargaining

We can use backward induction

- B's optimal action
  - $x < 1 \rightarrow$ accept
  - $x = 1 \rightarrow$ accept or reject

1. Suppose in equilibrium B accepts any offer $x \in [0, 1]$
   - What is the optimal offer by A? $x = 1$
   - The following is a SPE

$$x^* = 1$$
$$s_B^*(x) = Y \text{ for all } x \in [0, 1]$$

2. Now suppose that B accepts if and only if $x < 1$
   - What is A's optimal offer?
     - $x = 1$?
     - $x < 1$?

Unique SPE

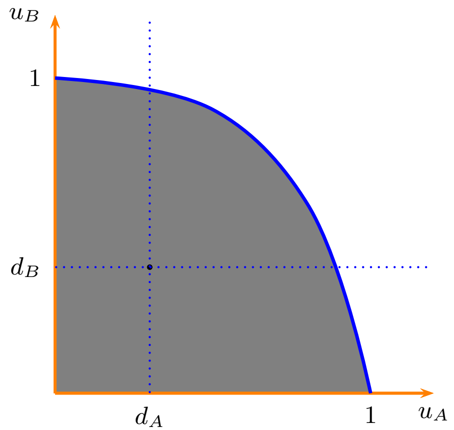
$x^* = 1, s_B^*(x) = Y$ for all $x \in [0, 1]$

Bargaining

- Bargaining outcomes depend on many factors
  - Social, historical, political, psychological, etc.
- Early economists thought the outcome to be indeterminate
- John Nash introduced a brilliant alternative approach
  - Axiomatic approach: A solution to a bargaining problem must satisfy certain "reasonable" conditions
    - These are the axioms
  - How would such a solution look like?
  - This approach is also known as cooperative game theory
- Later non-cooperative game theory helped us identify critical strategic considerations

Bargaining

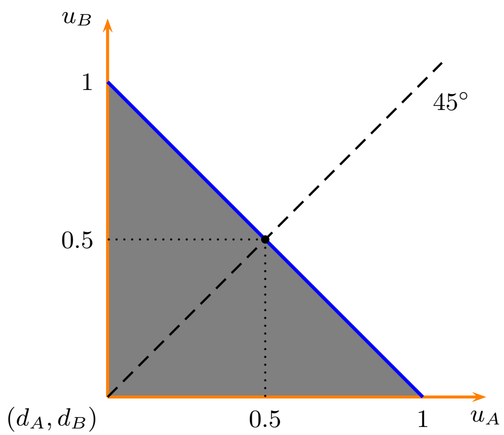
- Two individuals, A and B, are trying to share a cake of size 1
- If A gets x and B gets y,utilities are $u_A(x)$ and $u_B(y)$
- If they do not agree, A gets utility d_A and B gets d_B
- What is the most likely outcome?



Bargaining

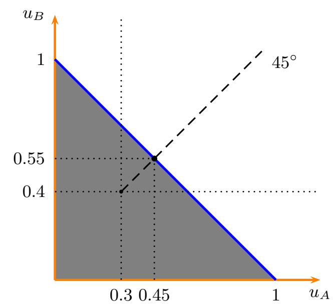
Let's simplify the problem

- $u_A(x) = x$, and $u_B(x) = x$
- $d_A = d_B = 0$
- A and B are the same in every other respect
- What is the most likely outcome?



Bargaining

How about now? $d_A = 0.3, d_B = 0.4$



- Let x be A's share. Then

$$\text{Slope } = 1 = \frac{1 - x - 0.4}{x - 0.3}$$

  or $x = 0.45$
- So A gets 0.45 and B gets 0.55

Bargaining

- In general A gets

$$d_A + \frac{1}{2}(1 - d_A - d_B)$$

- B gets

$$d_B + \frac{1}{2}(1 - d_A - d_B)$$

- But why is this reasonable?
- Two answers:
  1. Axiomatic: Nash Bargaining Solution
  2. Non-cooperative: Alternating offers bargaining game

Bargaining: Axiomatic Approach

- John Nash (1950): The Bargaining Problem, Econometrica
  1. Efficiency
     - ★ No waste
  2. Symmetry
     - ★ If bargaining problem is symmetric, shares must be equal
  3. Scale Invariance
     - ★ Outcome is invariant to linear changes in the payoff scale
  4. Independence of Irrelevant Alternatives
     - ★ If you remove alternatives that would not have been chosen, the solution does not change

Nash Bargaining Solution

- What if parties have different bargaining powers?
- Remove symmetry axiom
- Then A gets

$$x_A = d_A + \alpha(1 - d_A - d_B)$$

- B gets

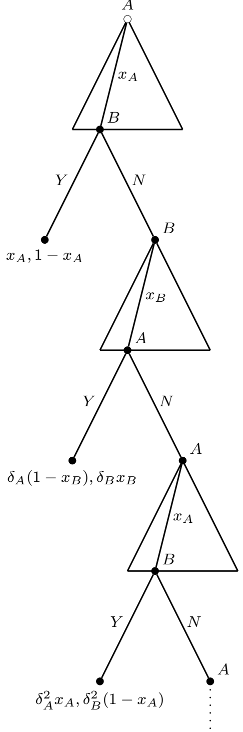
$$x_B = d_B + \beta(1 - d_A - d_B)$$

- $\alpha, \beta > 0$ and $\alpha + \beta = 1$ represent bargaining powers
- If $d_A = d_B = 0$

$$x_A = \alpha \quad \text{and} \quad x_B = \beta$$

Alternating Offers Bargaining

- Two players, A and B, bargain over a cake of size 1
- At time 0, A makes an offer $x_A \in [0, 1]$ to B
  - ▸ If B accepts, A receives x_A and B receives $1 - x_A$
  - ▸ If B rejects, then
- at time 1, B makes a counteroffer $x_B \in [0, 1]$
  - ▸ If A accepts, B receives x_B and A receives $1 - x_B$
  - ▸ If A rejects, he makes another offer at time 2
- This process continues indefinitely until a player accepts an offer
- If agreement is reached at time t on a partition that gives player i a share x_i
  - ▸ player i's payoff is $\delta_i^t x_i$
  - ▸ $\delta_i \in (0, 1)$ is player i's discount factor
- If players never reach an agreement, then each player's payoff is zero



Alternating Offers Bargaining

Stationary No-delay Equilibrium

1. No Delay: All equilibrium offers are accepted
2. Stationarity: Equilibrium offers do not depend on time

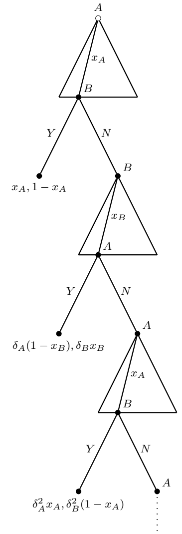
Let equilibrium offers be (x_A^*, x_B^*)

- What does B expect to get if she rejects x_A^*?
  - $\delta_B x_B^*$
- Therefore, we must have

$$1 - x_A^* = \delta_B x_B^*$$

- Similarly

$$1 - x_B^* = \delta_A x_A^*$$



Alternating Offers Bargaining

There is a unique solution

$$x_A^* = \frac{1-\delta_B}{1-\delta_A\delta_B}$$
$$x_B^* = \frac{1-\delta_A}{1-\delta_A\delta_B}$$

- There is at most one stationary no-delay SPE
- Still have to verify there exists such an equilibrium
- The following strategy profile is a SPE

Player A: Always offer x_A^*, accept any x_B with $1 - x_B \geq \delta_A x_A^*$
Player B: Always offer x_B^*, accept any x_A with $1 - x_A \geq \delta_B x_B^*$

Properties of the Equilibrium

Bargaining Power

Player A's share

$$\pi_A = x_A^* = \frac{1-\delta_B}{1-\delta_A\delta_B}$$

Player B's share

$$\pi_B = 1 - x_A^* = \frac{\delta_B(1-\delta_A)}{1-\delta_A\delta_B}$$

- Share of player i is increasing in δ_i and decreasing in δ_j
- Bargaining power comes from patience
- Example

$$\delta_A = 0.9, \delta_B = 0.95 \Rightarrow \pi_A = 0.35, \pi_B = 0.65$$

Properties of the Equilibrium

First mover advantage

If players are equally patient: $\delta_A = \delta_B = \delta$

$$\pi_A = \frac{1}{1+\delta} > \frac{\delta}{1+\delta} = \pi_B$$

First mover advantage disappears as $\delta \to 1$

$$\lim_{\delta \to 1} \pi_i = \lim_{\delta \to 1} \pi_B = \frac{1}{2}$$

Capacity Commitment: Stackelberg Duopoly

- Remember Cournot Duopoly model?
  - Two firms simultaneously choose output (or capacity) levels
  - What happens if one of them moves first?
    - or can commit to a capacity level?
- The resulting model is known as Stackelberg oligopoly
  - After the German economist Heinrich von Stackelberg in *Marktform und Gleichgewicht* (1934)
- The model is the same except that, now, Firm 1 moves first

Profit function of each firm is given by

$$u_i(Q_1, Q_2) = (a - b(Q_1 + Q_2))Q_i - cQ_i$$

Nash Equilibrium of Cournot Duopoly

Best response correspondences:

$$Q_1 = \frac{a - c - bQ_2}{2b}$$

$$Q_2 = \frac{a - c - bQ_1}{2b}$$

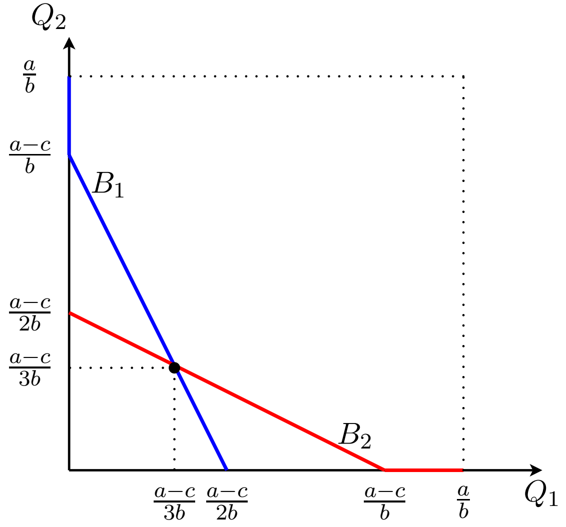
Nash equilibrium:

$$(Q_1^c, Q_2^c) = \left(\frac{a-c}{3b}, \frac{a-c}{3b}\right)$$

In equilibrium each firm's profit is

$$\pi_1^c = \pi_2^c = \frac{(a-c)^2}{9b}$$

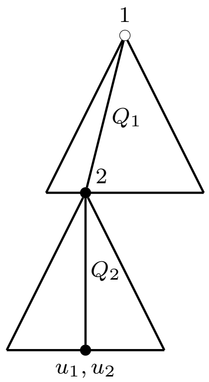
Cournot Best Response Functions



Stackelberg Model

The game has two stages:

1. Firm 1 chooses a capacity level $Q_1 \geq 0$
2. Firm 2 observes Firm 1's choice and chooses a capacity $Q_2 \geq 0$



$u_i(Q_1, Q_2) = (a - b(Q_1 + Q_2))Q_i - cQ_i$

Backward Induction Equilibrium of Stackelberg Game

- Sequential rationality of Firm 2 implies that for any Q_1 it must play a best response:
$$Q_2(Q_1) = \frac{a - c - bQ_1}{2b}$$
- Firms 1's problem is to choose Q_1 to maximize:
$$[a - b(Q_1 + Q_2(Q_1))]Q_1 - cQ_1$$
given that Firm 2 will best respond.
- Therefore, Firm 1 will choose Q_1 to maximize
$$[a - b(Q_1 + \frac{a - c - bQ_1}{2b})]Q_1 - cQ_1$$
This is solved as
$$Q_1 = \frac{a - c}{2b}$$

Backward Induction Equilibrium of Stackelberg Game

Backward Induction Equilibrium

$$Q_1^s = \frac{a - c}{2b}$$
$$Q_2^s(Q_1) = \frac{a - c - bQ_1}{2b}$$

Backward Induction Outcome

$$Q_1^s = \frac{a - c}{2b} > \frac{a - c}{3b} = Q_1^c$$
$$Q_2^s = \frac{a - c}{4b} < \frac{a - c}{3b} = Q_2^c$$

Firm 1 commits to an aggressive strategy

Equilibrium Profits

$$\pi_1^s = \frac{(a - c)^2}{8b} > \frac{(a - c)^2}{9b} = \pi_1^c$$
$$\pi_2^s = \frac{(a - c)^2}{16b} < \frac{(a - c)^2}{9b} = \pi_2^c$$

There is first mover advantage

Game Theory

Repeated Games

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Repeated Games

- Many interactions in the real world have an ongoing structure
  - Firms compete over prices or capacities repeatedly
- In such situations players consider their long-term payoffs in addition to short-term gains
- This might lead them to behave differently from how they would in one-shot interactions
- Consider the following pricing game in the DRAM chip industry

		Samsung	
		High	Low
Micron	High	$2,2$	$0,3$
	Low	$3,0$	$1,1$

- What happens if this game is played only once?
- What do you think might happen if played repeatedly?

Dynamic Rivalry

- If a firm cuts its price today to steal business, rivals may retaliate in the future, nullifying the "benefits" of the original price cut
- In some concentrated industries prices are maintained at high levels
  - U.S. steel industry until late 1960s
  - U.S. cigarette industry until early 1990s
- In other similarly concentrated industries there is fierce price competition
  - Costa Rican cigarette industry in early 1990s
  - U.S. airline industry in 1992
- When and how can firms sustain collusion?
- They could formally collude by discussing and jointly making their pricing decisions
  - Illegal in most countries and subject to severe penalties

Implicit Collusion

- Could firms collude without explicitly fixing prices?
- There must be some reward/punishment mechanism to keep firms in line
- Repeated interaction provides the opportunity to implement such mechanisms
- For example Tit-for-Tat Pricing: mimic your rival's last period price
- A firm that contemplates undercutting its rivals faces a trade-off
  - short-term increase in profits
  - long-term decrease in profits if rivals retaliate by lowering their prices
- Depending upon which of these forces is dominant collusion could be sustained
- What determines the sustainability of implicit collusion?
- Repeated games is a model to study these questions

Repeated Games

- Players play a stage game repeatedly over time
- If there is a final period: finitely repeated game
- If there is no definite end period: infinitely repeated game
  - We could think of firms having infinite lives
  - Or players do not know when the game will end but assign some probability to the event that this period could be the last one
- Today's payoff of \$1 is more valuable than tomorrow's \$1
  - This is known as discounting
  - Think of it as probability with which the game will be played next period
  - ... or as the factor to calculate the present value of next period's payoff
- Denote the discount factor by $\delta \in (0,1)$
- In PV interpretation: if interest rate is r

$$\delta = \frac{1}{1+r}$$

Payoffs

- If starting today a player receives an infinite sequence of payoffs

$$u_1, u_2, u_3, \ldots$$

- The payoff consequence is

$$(1-\delta)(u_1 + \delta u_2 + \delta^2 u_3 + \delta^3 u_4 \cdots)$$

- Example: Period payoffs are all equal to 2

$$\begin{aligned}(1-\delta)(2+\delta 2+\delta^2 2+\delta^3 2+\cdots) &= 2(1-\delta)(1+\delta+\delta^2+\delta^3+\cdots) \\ &= 2(1-\delta)\frac{1}{1-\delta} \\ &= 2\end{aligned}$$

Repeated Game Strategies

Strategies may depend on history

		Samsung High	Samsung Low
Micron	High	$2,2$	$0,3$
	Low	$3,0$	$1,1$

- Tit-For-Tat
  - Start with High
  - Play what your opponent played last period
- Grim-Trigger (called Grim-Trigger II in my lecture notes)
  - Start with High
  - Continue with High as long as everybody always played High
  - If anybody ever played Low in the past, play Low forever
- What happens if both players play Tit-For-Tat?
- How about Grim-Trigger?

Equilibria of Repeated Games

- There is no end period of the game
- Cannot apply backward induction type algorithm
- We use One-Shot Deviation Property to check whether a strategy profile is a subgame perfect equilibrium

One-Shot Deviation Property

A strategy profile is an SPE of a repeated game if and only if no player can gain by changing her action after any history, keeping both the strategies of the other players and the remainder of her own strategy constant

- Take an history
- For each player check if she has a profitable one-shot deviation (OSD)
- Do that for each possible history
- If no player has a profitable OSD after any history you have an SPE
- If there is at least one history after which at least one player has a profitable OSD, the strategy profile is NOT an SPE

Grim-Trigger Strategy Profile

There are two types of histories

1. Histories in which everybody always played High
2. Histories in which somebody played Low in some period

Histories in which everybody always played High

- Payoff to G-T

$$\begin{aligned}(1-\delta)(2+\delta 2+\delta^2 2+\delta^3 2+\cdots) &= 2(1-\delta)(1+\delta+\delta^2+\delta^3+\cdots) \\ &= 2\end{aligned}$$

- Payoff to OSD (play Low today and go back to G-T tomorrow)

$$\begin{aligned}(1-\delta)(3+\delta+\delta^2+\delta^3+\cdots) &= (1-\delta)(3+\delta(1+\delta+\delta^2+\delta^3+\cdots \\ &= 3(1-\delta)+\delta\end{aligned}$$

We need

$$2 \geq 3(1-\delta)+\delta \quad \text{or} \quad \delta \geq 1/2$$

Histories in which somebody played Low in some period

- Payoff to G-T

$$(1-\delta)(1+\delta+\delta^2+\delta^3+\cdots) = 1$$

- Payoff to OSD (play High today and go back to G-T tomorrow)

$$\begin{aligned}(1-\delta)(0+\delta+\delta^2+\delta^3+\cdots) &= (1-\delta)\delta(1+\delta+\delta^2+\delta^3+\cdots) \\ &= \delta\end{aligned}$$

OSD is NOT profitable for any δ

For any $\delta \geq 1/2$ Grim-Trigger strategy profile is a SPE

Forgiving Trigger

- Grim-trigger strategies are very fierce: they never forgive
- Can we sustain cooperation with limited punishment
  - For example: punish for only 3 periods

Forgiving Trigger Strategy

- Cooperative phase: Start with H and play H if
  - everybody has always played H
  - or k periods have passed since somebody has played L
- Punishment phase: Play L for k periods if
  - somebody played L in the cooperative phase

- We have to check whether there exists a one-shot profitable deviation after any history
- or in any of the two phases

Forgiving Trigger Strategy

Cooperative Phase

- Payoff to F-T $= 2$
- Payoff to OSD Outcome after a OSD

$$(L,H),\underbrace{(L,L),(L,L),\ldots,(L,L)}_{k\text{ times}},(H,H),(H,H),\ldots$$

Corresponding payoff

$$(1-\delta)[3+\delta+\delta^2+\ldots+\delta^k+2\delta^{k+1}+2\delta^{k+2}+\ldots] = 3-2\delta+\delta^{k+1}$$

- No profitable one-shot deviation in the cooperative phase if and only if

$$3-2\delta+\delta^{k+1} \leq 2$$

or

$$\delta^{k+1}-2\delta+1 \leq 0$$

- It becomes easier to satisfy this as k becomes large

Forgiving Trigger Strategy

Punishment Phase

Suppose there are $k' \leq k$ periods left in the punishment phase.

- Play F-T

$$\underbrace{(L,L),(L,L),\ldots,(L,L)}_{k' \text{ times}},(H,H),(H,H),\ldots$$

- Play OSD

$$\underbrace{(H,L),(L,L),\ldots,(L,L)}_{k' \text{ times}},(H,H),(H,H),\ldots$$

- F-T is better

Forgiving Trigger strategy profile is a SPE if and only if

$$\delta^{k+1} - 2\delta + 1 \leq 0$$

Imperfect Detection

- We have assumed that cheating (low price) can be detected with absolute certainty
- In reality actions may be only imperfectly observable
  - Samsung may give a secret discount to a customer
- Your sales drop
  - Is it because your competitor cut prices?
  - Or because demand decreased for some other reason?
- If you cannot perfectly observe your opponent's price you are not sure
- If you adopt Grim-Trigger strategies then you may end up in a price war even if nobody actually cheats
- You have to find a better strategy to sustain collusion

Imperfect Detection

- If your competitor cuts prices it is more likely that your sales will be lower
- Adopt a threshold trigger strategy: Determine a threshold level of sales s and punishment length T
  - Start by playing High
  - Keep playing High as long as sales of both firms are above s
  - The first time sales of either firm drops below s play Low for T periods; and then restart the strategy
- p_H : probability that at least one firm's sales is lower than s even when both firms choose high prices
- p_L : probability that the other firm's sales are lower than s when one firm chooses low prices
- $p_L > p_H$
- p_H and p_L depend on threshold level of sales s
  - Higher the threshold more likely the sales will fall below the threshold
  - Therefore, higher the threshold higher are p_H and p_L

Imperfect Detection

For simplicity let's make payoff to (Low,Low) zero for both firms

		Samsung	
		High	Low
Micron	High	2,2	-1,3
	Low	3,-1	0,0

- Denote the discounted sum of expected payoff (NPV) to threshold trigger strategy by v

$$v = 2 + \delta\left[(1-p_H)v + p_H\delta^T v\right]$$

- We can solve for v

$$v = \frac{2}{1-\delta\left[(1-p_H) + p_H\delta^T\right]}$$

- Value decreases as
  - Threshold increases (p_H increases)
  - Punishment length increases
- You don't want to trigger punishment too easily or punish too harshly

Imperfect Detection

- What is the payoff to cheating?

$$3 + \delta\left[(1-p_L)v + p_L\delta^T v\right]$$

- Threshold grim trigger is a SPE if

$$2 + \delta\left[(1-p_H)v + p_H\delta^T v\right] \geq 3 + \delta\left[(1-p_L)v + p_L\delta^T v\right]$$

that is

$$\delta v(1-\delta^T)(p_L - p_H) > 1$$

- It is easier to sustain collusion with harsher punishment (higher T) although it reduces v
- The effect of the threshold s is ambiguous: an increase in s
  - decreases v
  - may increase $p_L - p_H$

How to Sustain Cooperation?

Main conditions

- Future is important
- It is easy to detect cheaters
- Firms are able to punish cheaters

What do you do?

1. Identify the basis for cooperation
   - Price
   - Market share
   - Product design
2. Share profits so as to guarantee participation
3. Identify punishments
   - Strong enough to deter defection
   - But weak enough to be credible
4. Determine a trigger to start punishment
5. Find a method to go back to cooperation

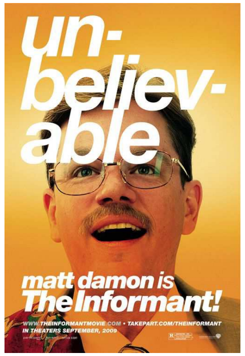
Lysine Cartel: 1992-1995

- John M. Connor (2002): Global Cartels Redux: The Amino Acid Lysine Antitrust Litigation (1996)
- This is a case of an explicit collusion - a cartel
- Archer Daniels Midland (ADM) and four other companies charged with fixing worldwide lysine (an animal feed additive) price
- Before 1980s: the Japanese duopoly, Ajinomoto and Kyowa Hakko
- Expansion mid 1970s to early 1980s to America and Europe
- In early 1980s, South Korean firm, Sewon, enters the market and expands to Asia and Europe
- 1986 - 1990: US market divided 55/45% btw. Ajinomoto and Kyowa Hakko
- Prices rose to \$3 per pound (\$1-\$2 btw 1960 and 1980)
- In early 1991 ADM and Cheil Sugar Co turned the lysine industry into a five firm oligopoly
- Prices dropped rapidly due to ADMs aggressive entry as a result of its excess capacity

Cartel Behavior

- April 1990: A, KH and S started meetings
- June 1992: five firm oligopoly formed a trade association
- Multiparty price fixing meetings amongst the 5 corporations
- Early 1993: a brief price war broke out
- 1993: establishment of monthly reporting of each company's sales
- Prices rose in this period from 0.68 to 0.98, fell to 0.65 and rose again to above 1\$

Cartel Meetings Caught on Tape



- Mark Whitacre, a rising star at ADM, blows the whistle on the companys price-fixing tactics at the urging of his wife Ginger
- In November 1992, Whitacre confesses to FBI special agent Brian Shepard that ADM executives including Whitacre himself had routinely met with competitors to fix the price of lysine

Cartel Meetings Caught on Tape

- Whitacre secretly gathers hundreds of hours of video and audio over several years to present to the FBI
- Documents here:
  http://www.usdoj.gov/atr/public/speeches/4489.htm
  http://www.usdoj.gov/atr/public/speeches/212266.htm
- Criminal investigation resulted in fines and prison sentences for executives of ADM
- Foreign companies settled with the United States Department of Justice Antitrust Division
- Whitacre was later charged with and pled guilty to committing a $9 million fraud that occurred during the same time period he was working for the FBI

Cartel Meetings Caught on Tape

1. Identify the basis for cooperation
   - Price
   - Market share
2. Share profits so as to guarantee participation
   - There is an annual budget for the cartel that allocates projected demand among the five
   - Prosecutors captured a scoresheet with all the numbers
   - Those who sold more than budget buy from those who sold less than budget
3. Identify punishments
   - Retaliation threat by ADM taped in one of the meetings
   - ADM has credibility as punisher: low-cost/high-capacity
   - Price cuts: 1993 price war?

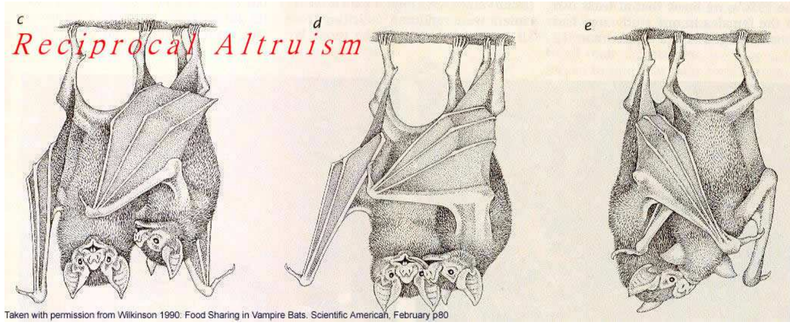
Stickleback Fish

- When a potential predator appears, one or more sticklebacks approach to check it out
- This is dangerous but provides useful information
  - If hungry predator, escape
  - Otherwise stay
- Milinski (1987) found that they use Tit-for-Tat like strategy
  - Two sticklebacks swim together in short spurts toward the predator
- Cooperate: Move forward
- Defect: Hang back

Stickleback Fish

- Milinski also run an ingenious experiment
- Used a mirror to simulate a cooperating or defecting stickleback
- When the mirror gave the impression of a cooperating stickleback
  - The subject stickleback move forward
- When the mirror gave the impression of a defecting stickleback
  - The subject stickleback stayed back

Vampire Bats



- Vampire bats (Desmodus rotundus) starve after 60 hours
- They feed each other by regurgitating
- Is it kin selection or reciprocal altruism?
  - Kin selection: Costly behavior that contribute to reproductive success of relatives
- Wilkinson, G.S. (1984), Reciprocal food sharing in the vampire bat, Nature.
  - Studied them in wild and in captivation

Vampire Bats

- If a bat has more than 24 hours to starvation it is usually not fed
  - Benefit of cooperation is high
- Primary social unit is the female group
  - They have opportunities for reciprocity
- Adult females feed their young, other young, and each other
  - Does not seem to be only kin selection
- Unrelated bats often formed a buddy system, with two individuals feeding mostly each other
  - Reciprocity
- Also those who received blood more likely to donate later on
- If not in the same group, a bat is not fed
  - If not associated, reciprocation is not very likely
- It is not only kin selection

Medieval Trade Fairs

- In 12th and 13th century Europe long distance trade took place in fairs
- Transactions took place through transfer of goods in exchange of promissory note to be paid at the next fair
- Room for cheating
- No established commercial law or state enforcement of contracts
- Fairs were largely self-regulated through Lex mercatoria, the "merchant law"
  - Functioned as the international law of commerce
  - Disputes adjudicated by a local official or a private merchant
  - But they had very limited power to enforce judgments
- Has been very successful and under lex mercatoria, trade flourished
- How did it work?

Medieval Trade Fairs

- What prevents cheating by a merchant?
- Could be sanctions by other merchants
- But then why do you need a legal system?
- What is the role of a third party with no authority to enforce judgments?

Medieval Trade Fairs

- If two merchants interact repeatedly honesty can be sustained by trigger strategy
- In the case of trade fairs, this is not necessarily the case
- Can modify trigger strategy
  - Behave honestly iff neither party has ever cheated anybody in the past
- Requires information on the other merchant's past
- There lies the role of the third party

Medieval Trade Fairs

- Milgrom, North, and Weingast (1990) construct a model to show how this can work
- The stage game:
  1. Traders may, at a cost, query the judge, who publicly reports whether any trader has any unpaid judgments
  2. Two traders play the prisoners' dilemma game
  3. If queried before, either may appeal at a cost
  4. If appealed, judge awards damages to the plaintiff if he has been honest and his partner cheated
  5. Defendant chooses to pay or not
  6. Unpaid judgments are recorded by the judge

Medieval Trade Fairs

- If the cost of querying and appeal are not too high and players are sufficiently patient the following strategy is a subgame perfect equilibrium:
  1. A trader querries if he has no unpaid judgments
  2. If either fails to query or if query establishes at least one has unpaid judgement play Cheat, otherwise play Honest
  3. If both queried and exactly one cheated, victim appeals
  4. If a valid appeal is filed, judge awards damages to victim
  5. Defendant pays judgement iff he has no other unpaid judgements
- This supports honest trade
- An excellent illustration the role of institutions
  - An institution does not need to punish bad behavior, it just needs to help people do so

Game Theory

Extensive Form Games with Incomplete Information

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Extensive Form Games with Incomplete Information

- We have seen extensive form games with perfect information
  - Entry game
- And strategic form games with incomplete information
  - Auctions
- Many incomplete information games are dynamic
- There is a player with private information
- Signaling Games: Informed player moves first
  - Warranties
  - Education
- Screening Games: Uninformed player moves first
  - Insurance company offers contracts
  - Price discrimination

Signaling Examples

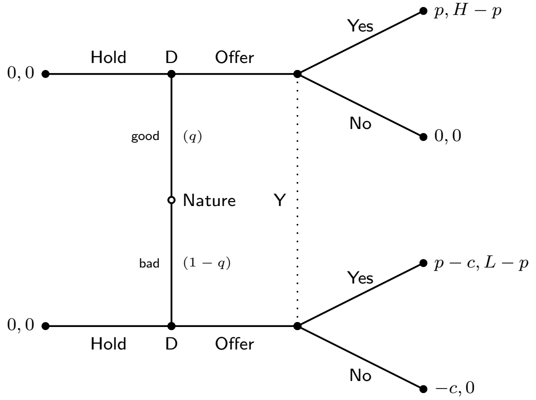
- Used-car dealer
  - How do you signal quality of your car?
  - Issue a warranty
- An MBA degree
  - How do you signal your ability to prospective employers?
  - Get an MBA
- Entrepreneur seeking finance
  - You have a high return project. How do you get financed?
  - Retain some equity
- Stock repurchases
  - Often result in an increase in the price of the stock
  - Manager knows the financial health of the company
  - A repurchase announcement signals that the current price is low
- Limit pricing to deter entry
  - Low price signals low cost

Signaling Games: Used-Car Market

- You want to buy a used-car which may be either good or bad (a lemon)
- A good car is worth H and a bad one L dollars
- You cannot tell a good car from a bad one but believe a proportion q of cars are good
- The car you are interested in has a sticker price p
- The dealer knows quality but you don't
- The bad car needs additional work that costs c to make it look like good
- The dealer decides whether to put a given car on sale or not
- You decide whether to buy or not
- Assume

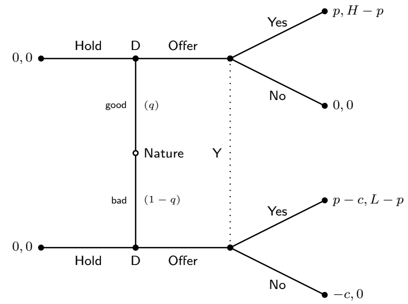
$$H > p > L$$

Signaling Games: Used-Car Market



- We cannot apply backward induction
  - No final decision node to start with
- We cannot apply SPE
  - There is only one subgame - the game itself
- We need to develop a new solution concept

Signaling Games: Used-Car Market



- Dealer will offer the bad car if you will buy
- You will buy if the car is good
- We have to introduce beliefs at your information set
- Given beliefs we want players to play optimally at every information set
  - sequential rationality
- We want beliefs to be consistent with chance moves and strategies
  - Bayes Law gives consistency

Bayes Law

Suppose a fair die is tossed once and consider the following events:
A: The number 4 turns up.
B: The number observed is an even number.
The sample space and the events are

$$\begin{aligned} S &= \{1,2,3,4,5,6\} \\ A &= \{4\} \\ B &= \{2,4,6\} \end{aligned}$$

$$P(A) = 1/6, P(B) = 1/2$$

Suppose we know that the outcome is an even number. What is the probability that the outcome is 4? We call this a conditional probability

$$P(A \mid B) = \frac{1}{3}$$

Bayes Law

Given two events A and B such that $P(B) \neq 0$ we have

$$P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)}$$

Note that since

$$P(A \text{ and } B) = P(B \mid A)\, P(A)$$

We have

$$P(A \mid B) = \frac{P(B \mid A)\, P(A)}{P(B)}$$

A^c: complement of A

$$P(B) = P(B \mid A)\, P(A) + P(B \mid A^c)\, P(A^c)$$

Therefore,

$$P(A \mid B) = \frac{P(B \mid A)\, P(A)}{P(B \mid A)\, P(A) + P(B \mid A^c)\, P(A^c)}$$

The probability $P(A)$ is called the prior probability and $P(A \mid B)$ is called the posterior probability.

Bayes Law: Example

- A machine can be in two possible states: good (G) or bad (B)
- It is good 90% of the time
- The item produced by the machine is defective (D)
  - 1% of the time if it is good
  - 10% of the time if it is bad
- What is the probability that the machine is good if the item is defective?

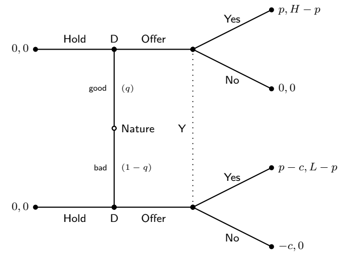
$P(G) = 0.9, P(B) = 1 - 0.9 = 0.1, P(D \mid G) = 0.01, P(D \mid B) = 0.1$

Therefore, by Bayes' Law

$$\begin{aligned} P(G \mid D) &= \frac{P(D \mid G)\, P(G)}{P(D \mid G)\, P(G) + P(D \mid B)\, P(B)} \\ &= \frac{0.01 \times 0.9}{0.01 \times 0.9 + 0.10 \times 0.1} \\ &= \frac{.009}{.019} \cong .47 \end{aligned}$$

In this example the prior probability that the machine is in a good condition is 0.90, whereas the posterior probability is 0.47.

Bayes Law



- Dealer's strategy: Offer if good; Hold if bad
- What is your consistent belief if you observe the dealer offer a car?

$$\begin{aligned} P(\text{good}|\text{offer}) &= \frac{P(\text{offer and good})}{P(\text{offer})} \\ &= \frac{P(\text{offer}|\text{good})P(\text{good})}{P(\text{offer}|\text{good})P(\text{good}) + P(\text{offer}|\text{bad})P(\text{bad})} \\ &= \frac{1 \times q}{1 \times q + 0 \times (1-q)} \\ &= 1 \end{aligned}$$

Strategies and Beliefs

A solution in an extensive form game of incomplete information is a collection of

1. A behavioral strategy profile
2. A belief system

We call such a collection an assessment

- A behavioral strategy specifies the play at each information set of the player
  - This could be a pure strategy or a mixed strategy
- A belief system is a probability distribution over the nodes in each information set

Perfect Bayesian Equilibrium

Sequential Rationality

At each information set, strategies must be optimal, given the beliefs and subsequent strategies

Weak Consistency

Beliefs are determined by Bayes Law and strategies whenever possible

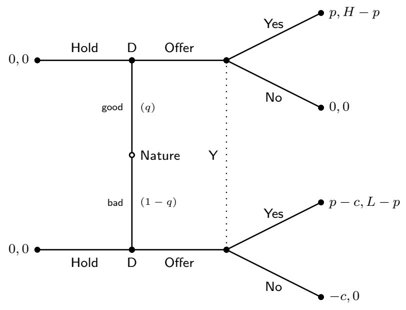
The qualification "whenever possible" is there because if an information set is reached with zero probability we cannot use Bayes Law to determine beliefs at that information set.

Perfect Bayesian Equilibrium

An assessment is a PBE if it satisfies

1. Sequentially rationality
2. Weak Consistency

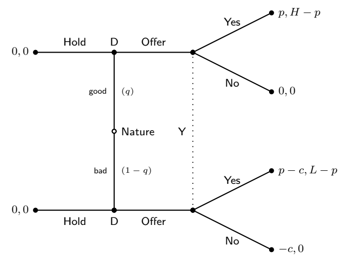
Back to Used-Car Example



As in Bayesian equilibria we may look for two types of equilibria:

1. Pooling Equilibria: Good and Bad car dealers play the same strategy
2. Separating Equilibrium: Good and Bad car dealers play differently

Pooling Equilibria



Both types Offer

- What does Bayes Law imply about your beliefs?

$$
\begin{aligned}
P(\text{good}|\text{offer}) &= \frac{P(\text{offer and good})}{P(\text{offer})} \\
&= \frac{P(\text{offer}|\text{good})P(\text{good})}{P(\text{offer}|\text{good})P(\text{good}) + P(\text{offer}|\text{bad})P(\text{bad})} \\
&= \frac{1 \times q}{1 \times q + 1 \times (1-q)} = q
\end{aligned}
$$

- Makes sense?

Pooling Equilibria: Both Types Offer

If you buy a car with your prior beliefs your expected payoff is

$$V = q \times (H-p) + (1-q) \times (L-p) \geq 0$$

- What does sequential rationality of seller imply?
- You must be buying and it must be the case that

$$p \geq c$$

- Under what conditions buying would be sequentially rational?

$$V \geq 0$$

Pooling Equilibrium I

If $p \geq c$ and $V \geq 0$ the following is a PBE

- Behavioral Strategy Profile: (Good: Offer, Bad: Offer),(You: Yes)
- Belief System: $P(good|offer) = q$

Pooling Equilibria: Both Types Hold

- You must be saying No
  - Otherwise Good car dealer would offer
- Under what conditions would you say No?

$$P(good|offer) \times (H-p) + (1 - P(good|offer)) \times (L-p) \leq 0$$

- What does Bayes Law say about $P(good|offer)$?
- Your information set is reached with zero probability
  - You cannot apply Bayes Law in this case
- So we can set $P(good|offer) = 0$

Pooling Equilibrium II

The following is a PBE

- Behavioral Strategy Profile: (Good: Hold, Bad: Hold),(You: No)
- Belief System: $P(good|offer) = 0$

This is complete market failure: a few bad apples (well lemons) can ruin a market

Separating Equilibria

Good:Offer and Bad:Hold

- What does Bayes Law imply about your beliefs?

$$P(good|offer) = 1$$

- What does you sequential rationality imply?
  - You say Yes
- Is Good car dealer's sequential rationality satisfied?
  - Yes
- Is Bad car dealer's sequential rationality satisfied?
  - Yes if $p \leq c$

Separating Equilibrium I

If $p \leq c$ the following is a PBE

- Behavioral Strategy Profile: (Good: Offer, Bad: Hold),(You: Yes)
- Belief System: $P(good|offer) = 1$

Separating Equilibria

Good:Hold and Bad:Offer

- What does Bayes Law imply about your beliefs?

$$P(good|offer) = 0$$

- What does you sequential rationality imply?
  - You say No
- Is Good car dealer's sequential rationality satisfied?
  - Yes
- Is Bad car dealer's sequential rationality satisfied?
  - No

There is no PBE in which Good dealer Holds and Bad dealer Offers

If $p > c$ and $V < 0$ only equilibrium is complete market failure: even the good cars go unsold.

Mixed Strategy Equilibrium

The following is a little involved so let's work with numbers

$$H = 3000, L = 0, q = 0.5, p = 2000, c = 1000$$

- Let us interpret player You as a population of potential buyers
- Is there an equilibrium in which only a proportion x, $0 < x < 1$, of them buy a used car?
- What does sequential rationality of Good car dealer imply?
  - Offer
- What does sequential rationality of buyers imply?
  - Bad car dealers must Offer with positive probability, say b
- Buyers must be indifferent between Yes and No

$$P(good|offer)(3000 - 2000) + (1 - P(good|offer))(0 - 2000) = 0$$

$$P(good|offer) = 2/3$$

Mixed Strategy Equilibrium

- What does Bayes Law imply?

$$P(good|offer) = \frac{0.5}{0.5 + (1 - 0.5)b} = \frac{2}{3}$$

$b = 0.5$

- Bad car dealers must be indifferent between Offer and Hold

$$x(2000 - 1000) + (1 - x)(-c) = 0$$

or $x = 0.5$

Mixed Strategy Equilibrium

The following is a PBE

- Behavioral Strategy Profile: (Good: Offer, Bad: Offer with prob. 1/2),(You: Yes with prob. 1/2)
- Belief System: $P(good|offer) = 2/3$

What is an MBA Worth?

- There are two types of workers
  - high ability (H): proportion q
  - low ability (L): proportion $1 - q$
- Output is equal to
  - H if high ability
  - L if low ability
- Workers can choose to have an MBA (M) or just a college degree (C)
- College degree does not cost anything but MBA costs
  - c_H if high ability
  - c_L if low ability
- Assume
$$c_L > H - L > c_H$$
- There are many employers bidding for workers
  - Wage of a worker is equal to her expected output
- MBA is completely useless in terms of worker's productivity!

What is an MBA Worth?

If employers can tell worker's ability wages will be given by

$$w_H = H, w_L = L$$

- Nobody gets an MBA
- Best outcome for high ability workers

If employers can only see worker's education, wage can only depend on education

- Employers need to form beliefs about ability in offering a wage

$$\begin{aligned} w_M &= p_M \times H + (1 - p_M) \times L \\ w_C &= p_C \times H + (1 - p_C) \times L \end{aligned}$$

where p_M (p_C) is employers' belief that worker is high ability if she has an MBA (College) degree

Separating Equilibria

Only High ability gets an MBA

- What does Bayes Law imply?
$$p_M = 1, p_C = 0$$
- What are the wages?
$$w_M = H, w_C = L$$
- What does High ability worker's sequential rationality imply?
$$H - c_H \geq L$$
- What does Low ability worker's sequential rationality imply?
$$L \geq H - c_L$$
- Combining
$$c_H \leq H - L \leq c_L$$
which is satisfied by assumption
- MBA is a waste of money but High ability does it just to signal her ability

Separating Equilibria

Only Low ability gets an MBA

- What does Bayes Law imply?
$$p_M = 0, p_C = 1$$
- What are the wages?
$$w_M = L, w_C = H$$
- What does High ability worker's sequential rationality imply?
$$H \geq L$$
which is satisfied
- High ability worker is quite happy: she gets high wages and doesn't have to waste money on MBA
- What does Low ability worker's sequential rationality imply?
$$L - c_L \geq H$$
which is impossible
- Too bad for High ability workers: Low ability workers want to mimic them
- No such equilibrium: A credible signal of high ability must be costly

Pooling Equilibria

Both get an MBA

- What does Bayes Law imply?

$$p_M = q, p_C = \text{indeterminate}$$

- What are the wages?

$$w_M = qH + (1-q)L, w_C = p_C H + (1-p_C)L$$

- High ability worker's sequential rationality imply

$$qH + (1-q)L - c_H \geq p_C H + (1-p_C)L$$

- Low ability worker's sequential rationality imply

$$qH + (1-q)L - c_L \geq p_C H + (1-p_C)L$$

- Since we assumed $c_L > H - L$ the last inequality is not satisfied
- No such equilibrium: It is not worth getting an MBA for low ability workers if they cannot fool the employers.

Pooling Equilibria

Neither gets an MBA

- What does Bayes Law imply?

$$p_C = q, p_M = \text{indeterminate}$$

- What are the wages?

$$w_C = qH + (1-q)L, w_M = p_M H + (1-p_M)L$$

- High ability worker's sequential rationality imply

$$qH + (1-q)L \geq p_M H + (1-p_M)L - c_H$$

- Low ability worker's sequential rationality imply

$$qH + (1-q)L \geq p_M H + (1-p_M)L - c_L$$

- These are satisfied if $c_H \geq (p_M - q)(H-L)$. If, for example, $p_M = q$
- High ability workers cannot signal their ability by getting an MBA because employers do not think highly of MBAs

Signaling Recap

- Signaling works only if
  - it is costly
  - it is costlier for the bad type
- Warranties are costlier for lemons
- MBA degree is costlier for low ability applicants
- Retaining equity is costlier for an entrepreneur with a bad project
- Stock repurchases costlier for management with over-valued stock
- Low price costlier for high cost incumbent