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LOCATION MODELS OF HORIZONTAL DIFFERENTIATION: A SPECIAL CASE OF VERTICAL DIFFERENTIATION MODELS*

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We study the relationship that exists between two families of models of product differentiation: the class of location or Hotelling-type models of horizontal differentiation, and models of vertical differentiation. Our main result is that every model belonging to a very large class of Hotelling-type models (including all the commonly used specifications) is actually a special case of a vertical product differentiation model. Formally speaking this means that the Hotelling type-model and the corresponding vertical product differentiation model are equivalent. Specifically, we show that the equilibria emerging in the two categories of models are identical in a well defined sense.

I. INTRODUCTION

IN THE literature on product differentiation it is commonplace to distinguish between models of vertical differentiation and models of horizontal differentiation (see, for example, Waterson [1989]).

The defining characteristic of vertical product differentiation is that all consumers have the same ranking of the variants of a product. One can think, for instance, of variants of a product differing in quality, with everyone agreeing that higher quality is preferable. Hence, if prices are identical, all consumers buy the same variant.

In the case of horizontal product differentiation, there is no such "natural ranking of the variants. Accordingly, if all the variants of a product are sold at the same price, there is a positive demand for each of them. The most popular models within this category are probably the location or Hotelling-type models. Consumers are characterized by their location that corresponds to their ideal product. A variant of the product is defined by its location in the characteristics space. If prices are identical, consumers buy from the firm that is closest to them. Two consumers who differ in location may thus have different rankings over the variants that are offered.

These two definitions are quite different: at first one would think that vertical and horizontal product differentiation are essentially different phenomena. And indeed, most of the existing studies have concentrated on either of these two cases of product differentiation.

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However, some authors have noticed that horizontal and vertical product differentiation models can yield quite similar results. This was first observed by Shaked and Sutton [1983]. They show that the model of vertical product differentiation presents some features that are "... reminiscent of the 'location' paradigm ...". Champsaur and Rochet [1989] argue along the same lines when they claim that some particular models of vertical differentiation and horizontal differentiation produce results of the same nature (see also Neven [1986]).

In spite of these observations, it seems fair to say that vertical and horizontal differentiation models are generally considered as being at least "somewhat different". The purpose of this note is to provide some further insight into the relationship that exists between these two families of models. It turns out that the results go well beyond the above-mentioned similarities. Our main result is that every model belonging to a very large class of Hotelling-type models (including all the commonly used specifications) is actually a special case of a vertical product differentiation model. In these cases, the distinction between vertical and horizontal differentiation appears to be merely a matter of interpretation. Formally speaking the Hotelling type-model and the corresponding vertical product differentiation model are equivalent. Specifically, we show that the equilibria emerging in the two categories of models are identical in a well-defined sense.

The paper is organized as follows. We first provide a formal definition of the two types of models in section II. The main result is then established in section III. Finally, in section IV, a discussion of the results and some concluding remarks are given.

II. MODELS

We start by formally describing the two types of models. Whatever the model considered, it is assumed that each consumer buys one unit of the product, irrespective of its price.[1]

II(i). *Model H: horizontal product differentiation*

- There are n firms, indexed by $i = 1, \ldots, n$, selling a homogenous product that is produced at zero marginal cost. Firms are located at $q_i \in [0, 1]$ and charge the (mill) price $p_i \geqslant 0$. Let $\underline{q} = (q_1, \ldots, q_n)$ and $\underline{p} = (p_1, \ldots, p_n)$ be the corresponding vectors. Firm i's profits are denoted by $\pi_i^H(\underline{p}, \underline{q}) \equiv p_i D_i^H(\underline{p}, \underline{q})$ where D_i^H is demand for product i.
- Consumers are identified by their locations $\theta \in [0, 1]$ and are distributed according to the density function $f(\theta)$.

[1] This assumption is made to simplify presentation. Our proposition remains true in presence of an outside alternative providing a utility level of zero.

—The utility derived by a consumer θ buying from firm i is given by

$$u^H(\theta, q_i) - p_i \tag{1}$$

where

$$u^H(\theta, q_i) = U_0 - t(|\theta - q_i|), \tag{2}$$

U_0 being a positive constant and $t(\cdot)$ a strictly increasing function of its argument with $t(0) = 0$.

—Firms play a two-stage game. In the first stage they simultaneously choose their locations and in the second, they simultaneously choose their prices. The solution concept is subgame-perfect Nash equilibrium.

II(ii). *Model V: vertical product differentiation*

—There are n firms indexed by $i = 1, \ldots, n$. Firm i produces a good of quality $q_i \in [q^-, q^+]$ at the marginal cost $c(q_i)$, which is constant with respect to quantity, and charges the price $p_i \geqslant c(q_i)$. It is assumed that $c(q_i)$ is an increasing function of quality. Profits are given by $\pi_i^V(\underline{p}, \underline{q}) \equiv [p_i - c(q_i)] D_i^V(\underline{p}, \underline{q})$ where D_i^V is firm i's demand.

—Consumers are identified by a parameter $\theta \in [\underline{\theta}, \bar{\theta}]$ that characterizes their preferences over qualities. They are distributed according to the density function $f(\theta)$.

—The utility of a consumer θ patronizing firm i is given by

$$u^V(\theta, q_i) - p_i \tag{3}$$

where u^V is strictly increasing in q_i.

—Firms play a two-stage game in which they first choose qualities and then prices. The solution is given by a subgame perfect Nash equilibrium.

Model H is a model à la Hotelling [1929]. The term $t(|\theta - q|)$ in (2) corresponds to the transportation cost and is a function of the distance $|\theta - q|$, with $\theta, q \in [0,1]$. The most commonly used specifications of this model are the linear and quadratic transportation costs. Note that $u^H(\theta, q)$ is not monotonic in q so that, if all firms charge the same price, each has a positive demand. Furthermore, marginal production cost is equal across firms. Model V has been proposed by Mussa and Rosen [1978]. Notice that $u^V(\theta, q)$ is strictly increasing in q. Therefore, if all the goods are made available at the same price, only the highest quality firm has a positive demand. Marginal production costs are now different across firms. Finally, while all consumers agree that a higher quality is better, they differ in the willingness to pay for higher qualities. This is captured by the parameter θ characterizing an individual's preferences and thus his willingness to pay for quality. If for instance $u^V(\theta, q) = \theta q$, θ is the *marginal* willingness to pay for quality. More generally, if $u^V_{q\theta} > 0$ a higher θ reflects a higher marginal willingness to pay for quality (see Neven [1986] and Champsaur and Rochet [1989]).[2]

III. RESULTS

The next proposition states our main result.

Proposition. Consider any specification of Model H such that $t(\cdot)$ is continuously differentiable on $[0, 1]$. Then, there exists a specification of Model V such that $\pi_i^H(\underline{p}, \underline{q}) = \pi_i^V(\underline{\hat{p}}, \underline{q})$, where $\hat{p}_i \equiv p_i + c(q_i)$, holds for all i, $\underline{p}$ and $\underline{q}$. Therefore, $(\underline{p}^*, \underline{q}^*)$ is an equilibrium of Model H if and only if $(\underline{\hat{p}}^*, \underline{q}^*)$ is an equilibrium of the corresponding specification of Model V.

Proof. To establish our proposition, it is sufficient to identify a specification of Model V for which the equivalence of the profit functions holds. It then follows immediately that the two models give the same equilibrium. Consider the following model. First set $q^- = \underline{\theta} = 0$ and $q^+ = \bar{\theta} = 1$. Furthermore, let

$$u^V(\theta, q) = u_1(q) + u_2(\theta) - t(|\theta - q|) \tag{4}$$

where $u_1(q)$ is chosen for $u^V(\theta, q)$ to be strictly increasing in q over $[0, 1]$. To see that this is always possible, note that one can set $u_1(q) \equiv Aq$ with $A > \max_{x \in [o, 1]} t'(x)$. By construction (4) thus satisfies the condition imposed on the utility (3). Finally, define

$$c(q) = u_1(q) \tag{5}$$

We now show that (4) and (5) is one of the desired specifications of Model V. Consider the second stage of the game for any given $\underline{q}$. We have

$$u^H(\theta, q_i) - p_i \gtreqless u^H(\theta, q_j) - p_j \tag{6}$$

for $i, j = 1, \ldots, n$, if and only if

$$u^V(\theta, q_i) - \hat{p}_i \gtreqless u^V(\theta, q_j) - \hat{p}_j \tag{7}$$

where u^H is given by (2), u^V by (4), and $\hat{p}_k \equiv p_k + c(q_k)$ for $k = 1, \ldots, n$. Indeed, (6) can be written as

$$-t(|\theta - q_i|) - p_i \gtreqless -t(|\theta - q_j|) - p_j$$

[2] It is obvious from (3) that marginal utility of income is constant and the same for all individuals (quasi-linear preferences). With such a specification the income distribution has no impact on the results. An alternative approach to vertical differentiation, presented by Gabszewicz and Thisse [1979] and Shaked and Sutton [1983], uses a different family of preferences. In this model, all individuals have the same (Cobb-Douglas) utility function and differences in the willingness to pay for quality result from differences in income. However, it is worth noting that this model and the Mussa and Rosen model yield similar properties (see, for example, Tirole [1988]).

which, in turn, amounts to

$$u_1(q_i)-t(|\theta-q_i|)-[p_i+c(q_i)] \gtreqless u_1(q_j)-t(|\theta-q_j|)-[p_j+c(q_j)].$$

Clearly, this expression is equivalent to (7). Hence we have shown that firm i's demand at $\underline{p}=(p_1,\ldots,p_n)$ in Model H is equal to firm i's demand at $\underline{\hat{p}}=(\hat{p}_1,\ldots,\hat{p}_n)$ in Model V. Since $\hat{p}_i-c(q_i)=p_i$, this implies that for any given $\underline{q}$, firm i's profit as a function of $\underline{p}$ in Model H is the same as firm i's profit as a function of $\underline{\hat{p}}$ in Model V. Formally we have $\pi_i^H(\underline{p},\underline{q})=\pi_i^V(\underline{\hat{p}},\underline{q})$, for all i, $\underline{p}$ and $\underline{q}$. It then follows that $\underline{p}^*$ is a Nash equilibrium of the price subgame in Model H if and only if $\underline{\hat{p}}^*$ is a Nash equilibrium of the price subgame in Model V. Consequently, it is readily verified that for every $\underline{q}$, the firms' first-stage payoffs are identical in the two models. Hence, the two games have the same equilibria in $\underline{q}$. Q.E.D.

In other words, the above proposition implies that, under very mild assumptions on the transportation cost, *any Hotelling-type model is a special case of vertical product differentiation*: the locations in an equilibrium in Model H are exactly the qualities in an equilibrium in Model V, while the equilibrium prices in Model H are equal to the equilibrium profit margins in Model V. In particular, this implies that the equilibrium profits are the same in both models.

As mentioned in the introduction, Shaked and Sutton [1983] have argued that vertical differentiation is similar to horizontal differentiation if consumers "differ in their ranking of products, at unit variable cost". Put in another way, this condition can be stated as follows: if all potential variants q are priced at marginal cost $c(q)$, then each consumer θ has a *specific* most-preferred quality. Our proposition sheds some further light on this observation since the most-preferred quality for a consumer θ is just equal to θ when u^V and c are given by (4) and (5), respectively.[3]

This argument also makes it clear that the converse of our proposition cannot be expected to hold. It is indeed obvious that any vertical differentiation model in which consumers agree about the ranking of products at unit variable cost (e.g. because cost does not increase sufficiently fast with quality) cannot be equivalent to a model of horizontal differentiation. On the other hand, it follows from our proposition that any vertical differentiation model that can be written as (4) and (5) is itself *equivalent* (in the sense of the above proposition) to a horizontal differentiation model à la Hotelling. An example showing the importance of this result is given below.

Finally, an immediate corollary of the above proposition is that a (pure

[3] Furthermore it may also suggest why the integration of horizontal and vertical differentiation undertaken by Shaked and Sutton [1987] retains several of the distinctive features of the vertical differentiation models.

strategy) equilibrium exists in Model H if and only if its Model V counterpart has a (pure strategy) equilibrium.

IV. DISCUSSION AND CONCLUSIONS

(i) One popular specification of the Mussa–Rosen model is given by

$$u^V(\theta, q) = \theta q \tag{8}$$

and

$$c(q) = \frac{q^2}{2} \tag{9}$$

Since $\theta q = q^2/2 + \theta^2/2 - (\theta - q)^2/2$, (8) and (9) are special cases of (4) and (5). Therefore, there exists a Hotelling-type model which is equivalent to (8)–(9). It is easy to see that this model is the Hotelling model with *quadratic transport costs*, i.e. $t(|\theta - q|) = (\theta - q)^2/2$. Our proposition then implies that both models must yield the same outcomes. For example, in the case of two firms, we have $q_1^* = 0$ and $q_2^* = 1$: both firms choose to maximize their differentiation in terms of location (Model H) or in terms of quality (Model V).

Champsaur and Rochet [1989] and Neven [1986] have pointed out the similarity existing between Model H with quadratic transport cost and Model V specified by (8) and (9). However, these authors have not provided a detailed comparative analysis of the two models. Therefore, our result supplements their work by showing that *both models necessarily yield equivalent solutions.*

(ii) Location models of product differentiation have been studied extensively and many results are available (see Gabszewicz and Thisse [1986a] for a recent survey). Our analysis implies that these results carry over to a particular class of vertical differentiation models, namely those defined by (4) and (5).

To see the interest of this equivalence, consider the following two examples. First, in the Hotelling model with quadratic transport costs and n firms, it has been shown that (for $n > 4$) equilibrium profits increase monotonically when one moves away from the central firm(s) to the peripheral firms (see Kats and Neven [1990]). In the model (8)–(9) this implies that *there is no monotonic relationship between the quality offered by a firm and its profit.* Specifically, the firms producing the top and bottom qualities earn the same profits; the firms offering the second highest and lowest quality make lower profits; and the firms offering the average quality have the lowest profits.

Second, the equivalence does not hold only for profit-maximizing firms. A straightforward argument shows that it remains valid for mixed oligopolies involving both profit-maximizing and welfare-maximizing firms. Using a result derived by Cremer *et al.* [1991] in the case of quadratic transport costs, it can be shown that for a sufficiently large number of firms (i.e. $n > 5$), *the total welfare is the same when the public firm chooses to produce either the top or the bottom quality in the vertical product differentiation model* (8)–(9). Furthermore, producing the average quality yields the lowest level of welfare. Other results obtained by Cremer *et al.* [1991] can be restated similarly.

(iii) As far as the Hotelling-type models are concerned, only very mild conditions on the transportation costs are required to establish our proposition. Nevertheless, it is worth looking at cases where these conditions are not met. A simple example is when $t(|\theta-q|) = |\theta-q|^{\alpha}$ with $0 < \alpha < 1$. Clearly, we have $\lim_{x \downarrow 0} t'(x) = \infty$. With such a specification, it is not possible to construct a function $u^V(\theta, q)$ like (4) which is strictly increasing in q. For q larger than, but sufficiently close to, θ the derivative of t always dominates the first term in (4). Hence, horizontal differentiation models of this kind could constitute a separate category in that they do not seem to be special cases of vertical product differentiation models.

To conclude, observe that (4) and (5) do not satisfy the condition for the "finiteness property" to hold, i.e. all consumers rank products similarly when sold at marginal cost.[4] This is not surprising, since it is known that this property cannot be verified in location models of product differentiation.

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[4] The finiteness property means that a finite number of firms only can have a positive demand at a price equilibrium with prices exceeding marginal cost, even when fixed costs are negligible. The vertical differentiation models giving rise to this property have been studied by Gabszewicz, Shaked, Sutton and Thisse in various publications (see Tirole [1988] for a brief overview). As shown by Gabszewicz and Thisse [1986b], one way to cast vertical differentiation models into the location framework is to assume that firms must locate in $[1, \infty[$. The condition found by Shaked and Sutton [1983, p. 1477] for the finiteness property to be satisfied may then be restated in terms of the transportation cost function.

References

Champsaur, P. and Rochet, J., 1989, 'Multiproduct Duopolists', *Econometrica*, 57, pp. 533–557.

Cremer, H., Marchand, M. and Thisse, J.-F., 1991, 'Mixed Oligopoly with Differentiated Products', *International Journal of Industrial Organization*, 9, forthcoming.

Gabszewicz, J. J. and Thisse, J.-F., 1979, 'Price Competition, Quality and Income Disparities', *Journal of Economic Theory*, 20, pp. 340–359.

Gabszewicz, J. J. and Thisse, J.-F., 1986a, 'Spatial Competition and the Location of Firms', in Lesourne, J. and Sonnenschein, H. (ed.), *Location Theory. Fundamentals of Pure and Applied Economics 5* (Harwood Academic Publishers, London), pp. 1–71.

Gabszewicz, J. J. and Thisse, J.-F., 1986b, 'On the Nature of Competition with Differentiated Products', *The Economic Journal*, 96, pp. 160–172.

Hotelling, H., 1929, 'Stability in Competition', *Economic Journal*, 39, pp. 41–57.

Kats, A. and Neven, D., 1990, 'Oligopolistic Equilibrium in Hotelling's Model', mimeo, Department of Economics, Virginia Polytechnic Institute & State University.

Neven, D., 1986, 'Address Models of Differentiation', in Norman, G. (ed.), *Spatial Pricing and Differentiated Markets* (Pion, London), pp. 5–18.

Mussa, M. and Rosen, S., 1978, 'Monopoly and Product Quality', *Journal of Economic Theory*, 18, pp. 301–317.

Shaked, A. and Sutton, J., 1983, 'Natural Oligopolies', *Econometrica*, 51, pp. 1469–1483.

Shaked, A. and Sutton, J., 1987, 'Product Differentiation and Industrial Structure', *Journal of Industrial Economics*, 36, pp. 131–146.

Tirole, J., 1988, *The Theory of Industrial Organization.* Cambridge, Massachusetts: The MIT Press.

Waterson, M., 1989, 'Models of Product Differentiation', *Bulletin of Economic Research*, 41, pp. 1–27.