IE675 Game Theory

Lecture Note Set 1

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Friday, March 30, 2001

1 INTRODUCTION

1.1 What is game theory?

Game theory is the study of problems of conflict and cooperation among independent decision-makers.

Game theory deals with *games of strategy* rather than *games of chance*.

The ingredients of a game theory problem include

- players (decision-makers)
- choices (feasible actions)
- payoffs (benefits, prizes, or awards)
- preferences to payoffs (objectives)

We need to know when one choice is better than another for a particular player.

1.2 Classification of game theory problems

Problems in game theory can be classified in a number of ways.

1.2.1 Static vs. dynamic games

In dynamic games, the order of decisions are important.

Question 1.1. Is it ever *really* possible to implement static decision-making in practice?

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1.2.2 Cooperative vs. non-cooperative

In a non-cooperative game, each player pursues his/her own interests. In a cooperative games, players are allowed to form coalitions and combine their decision-making problems.

	Non-cooperative	*Cooperative*
Static	Math programming Non-cooperative Game Theory	Cooperative Game Theory
Dynamic	Control Theory	Cooperative Dynamic Games

Note 1.1. This area of study is distinct from multi-criteria decision making.

Flow of information is an important element in game theory problems, but it is sometimes explicitly missing.

- noisy information
- deception

1.2.3 Related areas

- differential games
- optimal control theory
- mathematical economics

1.2.4 Application areas

- corporate decision making
- defense strategy
- market modelling
- public policy analysis
- environmental systems
- distributed computing
- telecommunications networks

1.2.5 Theory vs. simulation

The mathematical theory of games provides the fundamental laws and problem structure. Games can also be simulated to assess complex economic systems.

1.3 Solution concepts

The notion of a “solution” is more tenuous in game theory than in other fields.

Definition 1.1. *A* **solution** *is a systematic description of the outcomes that may emerge from the decision problem.*

- optimality (for whom??)
- feasibility
- equilibria

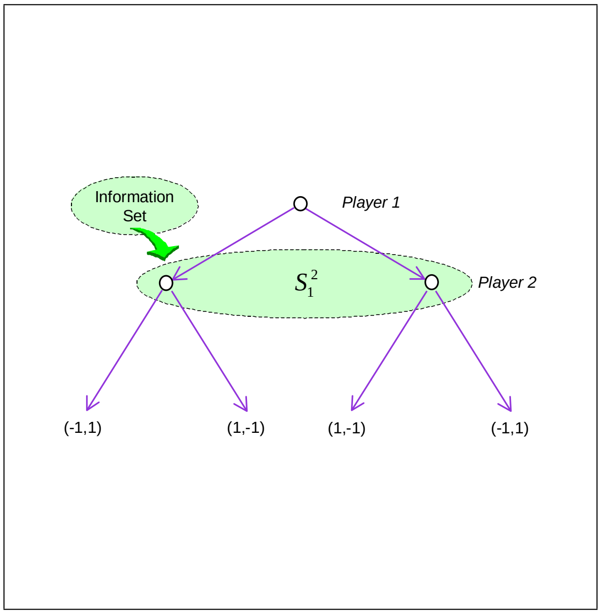
1.4 Games in extensive form

1.4.1 Example: Matching Pennies

- **Player 1:** Choose H or T
- **Player 2:** Choose H or T (not knowing Player 1’s choice)

- If the coins are alike, Player 2 wins 1 cent from Player 1
- If the coins are different, Player 1 wins 1 cent from Player 2

Written in extensive form, the game appears as follows

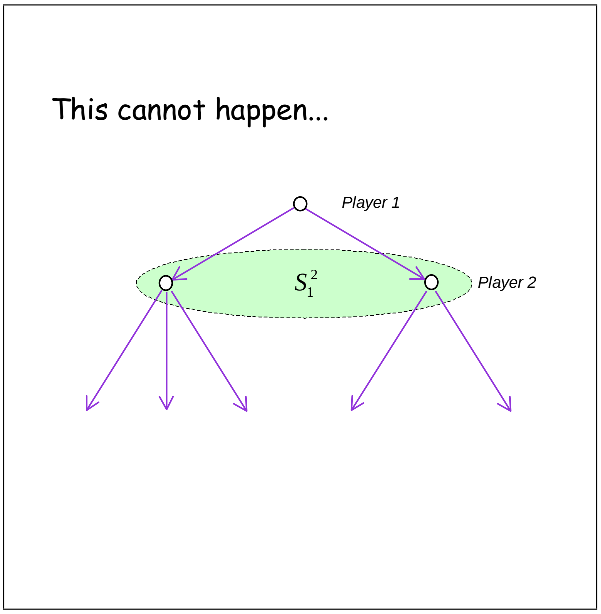


In order to deal with the issue of Player 2's knowledge about the game, we introduce the concept of an *information set*. When the game's progress reaches Player 2's time to move, Player 2 is supposed to know Player 1's choice. The set of nodes, S_1^2, is an information set for Player 2.

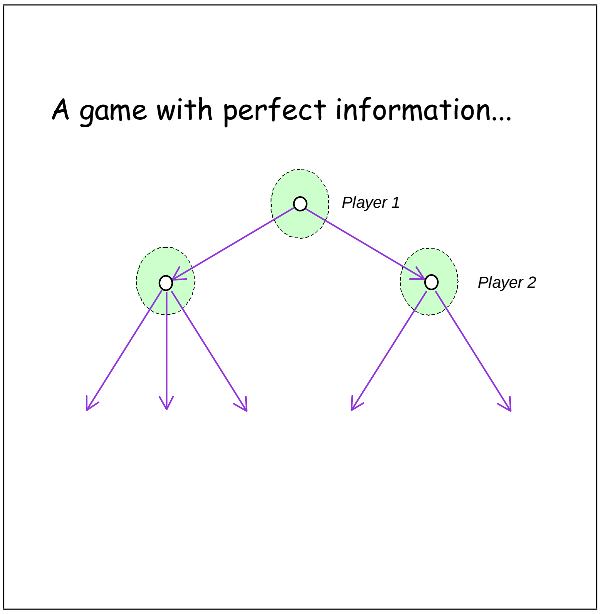
A player only knows the possible options emanating from an information. A player does not know which node within the information set is the actual node at which the progress of play resides.

There are some obvious rules about information sets that we will formally describe later.

For example, the following cannot occur. . .



Definition 1.2. *A player is said to have* **perfect information** *if all of his/her information sets are singletons.*



Definition 1.3. *A* **strategy** *for Player i is a function which assigns to each of Player i's information sets, one of the branches which follows a representative node from that set.*

Example 1.1. A strategy Ψ which has $\Psi(S_1^2) = \text{H}$ would tell Player 2 to select *Heads* when he/she is in information set S_1^2.

1.5 Games in strategic form

1.5.1 Example: Matching Pennies

Consider the same game as above and the following matrix of payoffs:

		Player 2	
		H	T
Player 1	H	(-1,1)	(1,-1)
	T	(1,-1)	(-1,1)

The rows represent the strategies of Player 1. The columns represent the strategies of Player 2.

Distinguish *actions* from *strategies*.

The matching pennies game is an example of a *non-cooperative game*.

1.6 Cooperative games

Cooperative games allow players to form coalitions to share decisions, information and payoffs.

For example, if we have player set

$$N = \{1, 2, 3, 4, 5, 6\}$$

A possible coalition structure would be

$$\{\{1, 2, 3\}, \{4, 6\}, \{5\}\}$$

Often these games are described in *characteristic function form.* A *characteristic function* is a mapping

$$v : 2^N \to \mathbb{R}$$

and $v(S)$ (where $S \subseteq N$) is the "worth" of the coalition S.

1.7 Dynamic games

Dynamic games can be cooperative or non-cooperative in form.

One class of cooperative dynamic games are hierarchical (organizational) games.

Consider a hierarchy of players, for example,

$$\begin{array}{c}\text{President}\\ \text{Vice-President}\\ \vdots\\ \text{Workers}\end{array}$$

Each level has control over a subset of all decision variables. Each may have an objective function that depends on the decision variables of other levels. Suppose that the top level makes his/her decision first, and then passes that information to the next level. The next level then makes his/her decision, and the play progresses down through the hierarchy.

The key ingredient in these problems is preemption, i.e., the "friction of space and time."

Even when everything is linear, such systems can produce *stable*, *inadmissible* solutions (Chew).

stable: No one wants to unilaterally move away from the given solution point.

inadmissible: There are feasible solution points that produce better payoffs for all players than the given solution point.

IE675 Game Theory

Lecture Note Set 2

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2 TWO-PERSON GAMES

2.1 Two-Person Zero-Sum Games

2.1.1 Basic ideas

Definition 2.1. *A game (in extensive form) is said to be* **zero-sum** *if and only if, at each terminal vertex, the payoff vector* $(p_1, \ldots, p_n)$ *satisfies* $\sum_{i=1}^{n} p_i = 0$.

Two-person zero sum games in *normal form*. Here's an example...

$$A = \begin{bmatrix} -1 & -3 & -3 & -2 \\ 0 & 1 & -2 & -1 \\ 2 & -2 & 0 & 1 \end{bmatrix}$$

The rows represent the strategies of Player 1. The columns represent the strategies of Player 2. The entries a_{ij} represent the payoff vector $(a_{ij}, -a_{ij})$. That is, if Player 1 chooses row i and Player 2 chooses column j, then Player 1 wins a_{ij} and Player 2 loses a_{ij}. If $a_{ij} < 0$, then Player 1 pays Player 2 $|a_{ij}|$.

Note 2.1. We are using the term *strategy* rather than *action* to describe the player's options. The reasons for this will become evident in the next chapter when we use this formulation to analyze games in extensive form.

Note 2.2. Some authors (in particular, those in the field of control theory) prefer to represent the outcome of a game in terms of *losses* rather than *profits*. During the semester, we will use both conventions.

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How should each player behave? Player 1, for example, might want to place a bound on his profits. Player 1 could ask “For each of my possible strategies, what is the least desirable thing that Player 2 could do to minimize my profits?” For each of Player 1’s strategies i, compute

$$\alpha_i = \min_j a_{ij}$$

and then choose that i which produces $\max_i \alpha_i$. Suppose this maximum is achieved for $i = i^*$. In other words, Player 1 is guaranteed to get at least

$$\underline{V}(A) = \min_j a_{i^*j} \geq \min_j a_{ij} \quad i = 1, \ldots, m$$

The value $\underline{V}(A)$ is called the *gain-floor* for the game A.

In this case $\underline{V}(A) = -2$ with $i^* \in \{2, 3\}$.

Player 2 could perform a similar analysis and find that j^* which yields

$$\overline{V}(A) = \max_i a_{ij^*} \leq \max_i a_{ij} \quad j = 1, \ldots, n$$

The value $\overline{V}(A)$ is called the *loss-ceiling* for the game A.

In this case $\overline{V}(A) = 0$ with $j^* = 3$.

Now, consider the joint strategies (i^*, j^*). We immediately get the following:

Theorem 2.1. *For every (finite) matrix game* $A = [a_{ij}]$

1. *The value* $\underline{V}(A)$ *and* $\overline{V}(A)$ *are unique.*
2. *There exists at least one security strategy for each player given by* (i^*, j^*).
3. $\min_j a_{i^*j} = \underline{V}(A) \leq \overline{V}(A) = \max_i a_{ij^*}$

Proof: (1) and (2) are easy. To prove (3) note that for any k and ℓ,

$$\min_j a_{kj} \leq a_{k\ell} \leq \max_i a_{i\ell}$$

and the result follows. ■

2.1.2 Discussion

Let's examine the decision-making philosophy that underlies the choice of (i^*, j^*). For instance, Player 1 appears to be acting as if Player 2 is trying to do as much harm to him as possible. This seems reasonable since this is a zero-sum game. Whatever, Player 1 wins, Player 2 loses.

As we proceed through this presentation, note that this same reasoning is also used in the field of statistical decision theory where Player 1 is the statistician, and Player 2 is "nature." Is it reasonable to assume that "nature" is a malevolent opponent?

2.1.3 Stability

Consider another example

$$A = \begin{bmatrix} -4 & 0 & 1 \\ 0 & 1 & -3 \\ -1 & -2 & -1 \end{bmatrix}$$

Player 1 should consider $i^* = 3$ ($\underline{V} = -2$) and Player 2 should consider $j^* = 1$ ($\overline{V} = 0$).

However, Player 2 can continue his analysis as follows

- Player 2 will choose strategy 1
- So Player 1 should choose strategy 2 rather than strategy 3
- But Player 2 would predict that and then prefer strategy 3

and so on.

Question 2.1. When do we have a stable choice of strategies?

The answer to the above question gives rise to some of the really important early results in game theory and mathematical programming.

We can see that if $\underline{V}(A) = \overline{V}(A)$, then both Players will settle on (i^*, j^*) with

$$\min_j a_{i^*j} = \underline{V}(A) = \overline{V}(A) = \max_i a_{ij^*}$$

Theorem 2.2. *If $\underline{V}(A) = \overline{V}(A)$ then*

1. *A has a saddle point*

2. *The saddle point corresponds to the security strategies for each player*

3. *The* value *for the game is* $V = \underline{V}(A) = \overline{V}(A)$

Question 2.2. Suppose $\underline{V}(A) < \overline{V}(A)$. What can we do? Can we establish a "spy-proof" mechanism to implement a strategy?

Question 2.3. Is it ever sensible to use expected loss (or profit) as a performance criterion in determining strategies for "one-shot" (non-repeated) decision problems?

2.1.4 Developing Mixed Strategies

Consider the following matrix game...

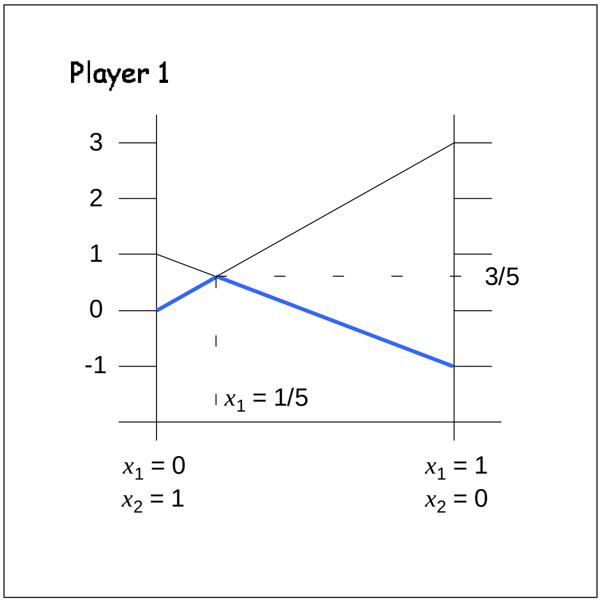
$$A = \begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix}$$

For Player 1, we have $\underline{V}(A) = 0$ and $i^* = 2$. For Player 2, we have $\overline{V}(A) = 1$ and $j^* = 2$. This game does not have a saddle point.

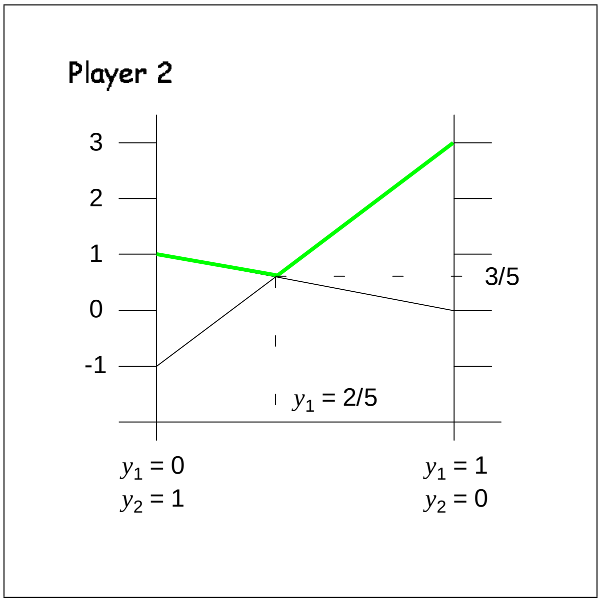
Let's try to create a "spy-proof" strategy. Let Player 1 randomize over his two *pure strategies*. That is Player 1 will pick the vector of probabilities $x = (x_1, x_2)$ where $\sum_i x_i = 1$ and $x_i \geq 0$ for all i. He will then select strategy i with probability x_i.

Note 2.3. When we formalize this, we will call the probability vector x, a *mixed strategy*.

To determine the "best" choice of x, Player 1 analyzes the problem, as follows...



Player 2 might do the same thing using probability vector $y = (y_1, y_2)$ where $\sum_i y_i = 1$ and $y_i \geq 0$ for all i.



If Player 1 adopts mixed strategy (x_1, x_2) and Player 2 adopts mixed strategy (y_1, y_2), we obtain an expected payoff of

$$\begin{aligned} V &= 3x_1y_1 + 0(1-x_1)y_1 - x_1(1-y_1) \\ &\quad +(1-x_1)(1-y_1) \\ &= 5x_1y_1 - y_1 - 2x_1 + 1 \end{aligned}$$

Suppose Player 1 uses $x_1^* = \frac{1}{5}$, then

$$V = 5\left(\frac{1}{5}\right)y_1 - y_1 - 2\left(\frac{1}{5}\right) + 1 = \frac{3}{5}$$

which doesn't depend on y! Similarly, suppose Player 2 uses $y_1^* = \frac{2}{5}$, then

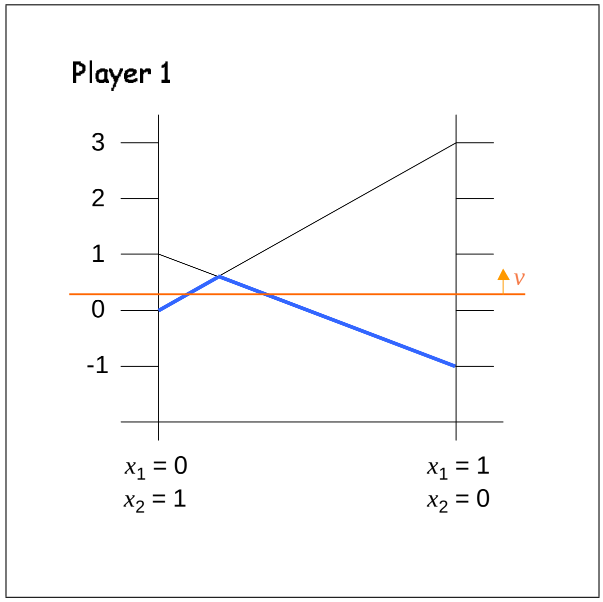
$$V = 5x_1\left(\frac{2}{5}\right) - \left(\frac{2}{5}\right) - 2x_1 + 1 = \frac{3}{5}$$

which doesn't depend on x!

Each player is solving a constrained optimization problem. For Player 1 the problem is

$$\begin{array}{lllll} & \max\{v\} & & & \\ \text{st:} & +3x_1 + 0x_2 & \geq & v & \\ & -1x_1 + 1x_2 & \geq & v & \\ & x_1 + x_2 & = & 1 & \\ & x_i & \geq & 0 & \forall\, i \end{array}$$

which can be illustrated as follows:



This problem is equivalent to

$$\max_x \min\{(3x_1 + 0x_2), (-x_1 + x_2)\}$$

For Player 2 the problem is

$$\begin{array}{rlcl} & \min\{v\} & & \\ \text{st:} & +3y_1 - 1y_2 & \leq & v \\ & +0y_1 + 1y_2 & \leq & v \\ & y_1 + y_2 & = & 1 \\ & y_j & \geq & 0 \quad \forall\, j \end{array}$$

which is equivalent to

$$\min_y \max\{(3y_1 - y_2), (0y_1 + y_2)\}$$

We recognize these as dual linear programming problems.

Question 2.4. We now have a way to compute a "spy-proof" mixed strategy for each player. Modify these two mathematical programming problems to produce the *pure* security strategy for each player.

In general, the players are solving the following pair of dual linear programming problems:

$$\begin{array}{llcl} & \max\{v\} & & \\ \text{st:} & \sum_i a_{ij} x_i & \geq & v \quad \forall\, j \\ & \sum_i x_i & = & 1 \\ & x_i & \geq & 0 \quad \forall\, i \end{array}$$

and

$$\begin{array}{llcl} & \min\{v\} & & \\ \text{st:} & \sum_j a_{ij} y_j & \leq & v \quad \forall\, i \\ & \sum_i y_i & = & 1 \\ & y_i & \geq & 0 \quad \forall\, j \end{array}$$

Note 2.4. Consider, once again, the example game

$$A = \begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix}$$

If Player 1 (the maximizer) uses mixed strategy $(x_1, (1 - x_1))$, and if Player 2 (the minimizer) uses mixed strategy $(y_1, (1 - y_1))$ we get

$$E(x, y) = 5x_1 y_1 - y_1 - 2x_1 + 1$$

and letting $x^* = \frac{1}{5}$ and $y^* = \frac{2}{5}$ we get $E(x^*, y) = E(x, y^*) = \frac{3}{5}$ for any x and y. These choices for x^* and y^* make the expected value independent of the opposing strategy. So, if Player 1 becomes a minimizer (or if Player 2 becomes a maximizer) the resulting mixed strategies would be the same!

Note 2.5. Consider the game

$$A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$$

By "factoring" the expression for $E(x, y)$, we can write

$$\begin{array}{rcl} E(x, y) & = & x_1 y_1 + 3x_1(1 - y_1) + 4(1 - x_1)y + 2(1 - x_1)(1 - y_1) \\ & = & -4x_1 y_1 + x_1 + 2y_1 + 2 \\ & = & -4(x_1 y_1 - \dfrac{x_1}{4} - \dfrac{y_1}{2} + \dfrac{1}{8}) + 2 + \dfrac{1}{2} \\ & = & -4(x_1 - \dfrac{1}{2})(y_1 - \dfrac{1}{4}) + \dfrac{5}{2} \end{array}$$

It's now easy to see that $x_1^* = \frac{1}{2}$, $y_1^* = \frac{1}{4}$ and $v = \frac{5}{2}$.

2.1.5 A more formal statement of the problem

Suppose we are given a matrix game $A_{(m \times n)} \equiv [a_{ij}]$. Each row of A is a pure strategy for Player 1. Each column of A is a pure strategy for Player 2. The value of a_{ij} is the payoff from Player 1 to Player 2 (it may be negative).

For Player 1 let

$$\underline{V}(A) = \max_i \min_j a_{ij}$$

For Player 2 let

$$\overline{V}(A) = \min_j \max_i a_{ij}$$

$\{$**Case 1**$\}$ (Saddle Point Case where $\underline{V}(A) = \overline{V}(A) = V$)

Player 1 can assure himself of getting at least V from Player 2 by playing his maximin strategy.

$\{$**Case 2**$\}$ (Mixed Strategy Case where $\underline{V}(A) < \overline{V}(A)$)

Player 1 uses probability vector

$$x = (x_1, \ldots, x_m) \quad \sum_i x_i = 1 \quad x_i \geq 0$$

Player 2 uses probability vector

$$y = (y_1, \ldots, y_n) \quad \sum_j y_j = 1 \quad y_j \geq 0$$

If Player 1 uses x and Player 2 uses strategy j, the expected payoff is

$$E(x, j) = \sum_i x_i a_{ij} = xA_j$$

where A_j is column j from matrix A.

If Player 2 uses y and Player 1 uses strategy i, the expected payoff is

$$E(i, y) = \sum_j a_{ij} y_j = A^i y^{\mathrm{T}}$$

where A^i is row i from matrix A.

Combined, if Player 1 uses x and Player 2 uses y, the expected payoff is

$$E(x,y) = \sum_i \sum_j x_i a_{ij} y_j = xAy^{\mathrm{T}}$$

The players are solving the following pair of dual linear programming problems:

$$\begin{array}{rlcll} & \max\{v\} & & & \\ \text{st:} & \sum_i a_{ij} x_i & \geq & v & \forall\, j \\ & \sum_i x_i & = & 1 & \\ & x_i & \geq & 0 & \forall\, i \end{array}$$

and

$$\begin{array}{rlcll} & \min\{v\} & & & \\ \text{st:} & \sum_j a_{ij} y_j & \leq & v & \forall\, i \\ & \sum_i y_i & = & 1 & \\ & y_i & \geq & 0 & \forall\, j \end{array}$$

The *Minimax Theorem* (von Neumann, 1928) states that there exists mixed strategies x^* and y^* for Players 1 and 2 which solve each of the above problems with equal objective function values.

2.1.6 Proof of the minimax theorem

Note 2.6. (From Başar and Olsder [1]) The theory of finite zero-sum games dates back to Borel in the early 1920's whose work on the subject was later translated into English (Borel, 1953). Borel introduced the notion of a conflicting decision situation that involves more than one decision maker, and the concepts of pure and mixed strategies, but he did not really develop a complete theory of zero-sum games. Borel even conjectured that the minimax theorem was false.

It was von Neumann who first came up with a proof of the minimax theorem, and laid down the foundations of game theory as we know it today (von Neumann 1928, 1937).

The following proof of the minimax theorem does not use the powerful tools of duality in linear programming problems. It is provided here for historical purposes.

Theorem 2.3. **Minimax Theorem** *Let $A = [a_{ij}]$ be an $m \times n$ matrix of real numbers. Let Ξ^r denote the set of all r-dimensional probability vectors, that is,*

$$\Xi^r = \{x \in \mathbb{R}^r \mid \textstyle\sum_{i=1}^r x_i = 1 \text{ and } x_i \geq 0\}$$

We sometimes call Ξ^r the **probability simplex.**

Let $x \in \Xi^m$ and $y \in \Xi^n$. Define

$$\begin{aligned}
\underline{V}(A) &= \max_x \min_y xAy^{\mathrm{T}} \\
\overline{V}(A) &= \min_y \max_x xAy^{\mathrm{T}}
\end{aligned}$$

Then $\underline{V}(A) = \overline{V}(A)$.

Proof: (By finite induction on $m+n$.)

The result is clearly true when A is a 1×1 matrix (i.e., $m+n=2$).

Assume that the theorem is true for all $p \times q$ matrices such that $p+q < m+n$ (i.e., for any submatrix of an $m \times n$ matrix). We will now show the result is true for any $m \times n$ matrix.

First note that xAy^{T}, $\max_x xAy^{\mathrm{T}}$ and $\min_y xAy^{\mathrm{T}}$ are all continuous functions of (x,y), x and y, respectively. Any continuous, real-valued function on a compact set has an extermum. Therefore, there exists x^0 and y^0 such that

$$\begin{aligned}
\underline{V}(A) &= \min_y x^0 Ay^{\mathrm{T}} \\
\overline{V}(A) &= \max_x xAy^{0\mathrm{T}}
\end{aligned}$$

It is clear that

(1) $$\underline{V}(A) \le x^0 Ay^{0\mathrm{T}} \le \overline{V}(A)$$

So we only need to show that $\underline{V}(A) \ge \overline{V}(A)$.

Let's assume that $\underline{V}(A) < \overline{V}(A)$. We will show that this produces a contradiction.

From Equation 1, either $\underline{V}(A) < x^0 Ay^{0\mathrm{T}}$ or $\overline{V}(A) > x^0 Ay^{0\mathrm{T}}$. We'll assume $\underline{V}(A) < x^0 Ay^{0\mathrm{T}}$ (the other half of the proof is similar).

Let $\underline{A}$ be an $(m-1) \times n$ matrix obtained from A by deleting a row. Let $\overline{A}$ be an $m \times (n-1)$ matrix obtained from A by deleting a column. We then have, for all $x \in \Xi^{m-1}$ and $y \in \Xi^{n-1}$,

$$\begin{aligned}
\underline{V}(\underline{A}) &= \max_{x'} \min_y x'\underline{A}y^{\mathrm{T}} \le \underline{V}(A) \\
\overline{V}(\underline{A}) &= \min_y \max_{x'} x'\underline{A}y^{\mathrm{T}} \le \overline{V}(A) \\
\underline{V}(\overline{A}) &= \max_x \min_{y'} x\overline{A}y'^{\mathrm{T}} \ge \underline{V}(A) \\
\overline{V}(\overline{A}) &= \min_{y'} \max_x x\overline{A}y'^{\mathrm{T}} \ge \overline{V}(A)
\end{aligned}$$

We know that $\underline{V}(\underline{A}) = \overline{V}(\underline{A})$ and $\underline{V}(\overline{A}) = \overline{V}(\overline{A})$. Thus

$$\begin{aligned}\underline{V}(\underline{A}) = \overline{V}(\underline{A}) \leq \overline{V}(A)\\ \underline{V}(\underline{A}) \leq \underline{V}(A) \leq \overline{V}(A)\end{aligned}$$

and

$$\begin{aligned}\underline{V}(\overline{A}) = \overline{V}(\overline{A}) \geq \overline{V}(A) \geq \underline{V}(A)\\ \underline{V}(\overline{A}) \geq \underline{V}(A)\end{aligned}$$

So, if it is really true that $\underline{V}(A) < \overline{V}(A)$ then, for all $\underline{A}$ and $\overline{A}$, we have

$$\begin{aligned}\overline{V}(A) &> \overline{V}(\underline{A})\\ \underline{V}(A) &> \underline{V}(\overline{A})\end{aligned}$$

Now, if is true that $\underline{V}(A) > \underline{V}(\overline{A})$ we will show that we can construct a vector Δx such that

$$\min_y (x^0 + \epsilon \Delta x) A y^{\mathrm{T}} > \underline{V}(A) = \max_x \min_y x A y^{\mathrm{T}}$$

for some $\epsilon > 0$. This would be clearly false and would yield our contradiction.

To construct Δx, assume

$$\underline{V}(A) > \underline{V}(\overline{A}) \tag{2}$$

For some k there is a column A_k of A such that

$$x^0 A_k > \underline{V}(A)$$

This must be true because if $x^0 A_k \leq \underline{V}(A)$ for all columns A_j, then $x^0 A y^{0\mathrm{T}} \leq \underline{V}(A)$ violating Equation 2.

For the above choice of k, let $\overline{A^k}$ denote the $m \times (n-1)$ matrix obtained by deleting the k^{th} column from A. From calculus, there must exist $x' \in \Xi^m$ such that

$$\begin{aligned}\underline{V}(\overline{A^k}) &= \max_x \min_{y'} x A y'^{\mathrm{T}}\\ &= \min_{y'} x' A y'^{\mathrm{T}}\end{aligned}$$

where $y' \in \Xi^n$ but with $y'_k = 0$ (because we deleted the k^{th} column from A).

Now, let $\Delta x = x' - x^0 \neq 0$ to get

$$\underline{V}(\overline{A^k}) = \min_{y'} (x^0 + \Delta x) A y'^{\mathrm{T}} > \underline{V}(A)$$

To summarize, by assuming Equation 2, we now have the following:

$$\begin{aligned}\underline{V}(A) &= \min_y x^0 A y^{\mathrm{T}} \\ \underline{V}(\overline{A^k}) &= \min_{y'} (x^0 + \Delta x) A y'^{\mathrm{T}} > \underline{V}(A)\end{aligned}$$

Therefore,

$$\min_{y'} (x^0 + \epsilon\Delta x) A y'^{\mathrm{T}} > \underline{V}(A) = \min_y x^0 A y'^{\mathrm{T}}$$

for all $0 < \epsilon \leq 1$.

In addition, xA_k is continuous, and linear in x, so

$$x^0 A_k > \underline{V}(A)$$

implies

$$(x^0 + \epsilon\Delta x) A_k > \underline{V}(A)$$

for some small $\epsilon > 0$. Let $x^\star = x^0 + \epsilon\Delta x$ for that small ϵ. This produces,

$$\begin{aligned}\min_y x^\star A y^{\mathrm{T}} &= \min_{0 \leq y_k \leq 1} \left\{ (1 - y_k) \left(\min_{y'} x^\star A y'^{\mathrm{T}} \right) + y_k \left(x^\star A_k \right) \right\} \\ &= \min \left\{ \min_{y'} x^\star A y'^{\mathrm{T}}, x^\star A_k \right\} > \underline{V}(A)\end{aligned}$$

which is a contradiction. ■

2.1.7 The minimax theorem and duality

The following theorem provides a modern proof of the minimax theorem, using duality:[1]

Theorem 2.4. *Consider the matrix game A with mixed strategies x and y for Player 1 and Player 2, respectively. Then*

1. **minimax statement**

$$\max_x \min_y E(x, y) = \min_y \max_x E(x, y)$$

[1]This theorem and proof is from my own notebook from a Game Theory course taught at Cornell in the summer of 1972. The course was taught by Professors William Lucas and Louis Billera. I believe, but I cannot be sure, that this particular proof is from Professor Billera.

2. **saddle point statement (mixed strategies)** *There exists x^* and y^* such that*

$$E(x, y^*) \le E(x^*, y^*) \le E(x^*, y)$$

for all x and y.

2a. **saddle point statement (pure strategies)** *Let $E(i, y)$ denote the expected value for the game if Player 1 uses pure strategy i and Player 2 uses mixed strategy y. Let $E(x, j)$ denote the expected value for the game if Player 1 uses mixed strategy x and Player 2 uses pure strategy j. There exists x^* and y^* such that*

$$E(i, y^*) \le E(x^*, y^*) \le E(x^*, j)$$

for all i and j.

3. **LP feasibility statement** *There exists x^*, y^*, and $v' = v''$ such that*

$$\boxed{\begin{array}{lcll} \sum_i a_{ij} x_i^* & \ge & v' & \forall\, j \\ \sum_i x_i^* & = & 1 & \\ x_i^* & \ge & 0 & \forall\, i \end{array}} \qquad \boxed{\begin{array}{lcll} \sum_j a_{ij} y_j^* & \le & v'' & \forall\, i \\ \sum_j y_j^* & = & 1 & \\ y_j^* & \ge & 0 & \forall\, j \end{array}}$$

4. **LP duality statement** *The objective function values are the same for the following two linear programming problems:*

$$\boxed{\begin{array}{llcll} & \max\{v\} & & & \\ \text{st:} & \sum_i a_{ij} x_i^* & \ge & v & \forall\, j \\ & \sum_i x_i^* & = & 1 & \\ & x_i^* & \ge & 0 & \forall\, i \end{array}} \qquad \boxed{\begin{array}{llcll} & \min\{v\} & & & \\ \text{st:} & \sum_j a_{ij} y_j^* & \le & v & \forall\, i \\ & \sum_i y_j^* & = & 1 & \\ & y_j^* & \ge & 0 & \forall\, j \end{array}}$$

Proof: We will sketch the proof for the above results by showing that

$$(4) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1) \Rightarrow (3) \Rightarrow (4)$$

and

$$(2) \Leftrightarrow (2a)$$

.

$\{(4) \Rightarrow (3)\}$ (3) is just a special case of (4).

$\{(3) \Rightarrow (2)\}$ Let 1_n denote a column vector of n ones. Then (3) implies that there exists x^*, y^*, and $v' = v''$ such that

$$\begin{aligned} x^*A &\geq v'1_n \\ x^*Ay^{\mathrm{T}} &\geq v'(1_n y^{\mathrm{T}}) = v' \qquad \forall\, y \end{aligned}$$

and

$$\begin{aligned} Ay^{*\mathrm{T}} &\leq v''1_m \\ xAy^{*\mathrm{T}} &\leq xv''1_m = v''(x1_m) = v'' \qquad \forall\, x \end{aligned}$$

Hence,

$$E(x^*, y) \geq v' = v'' \geq E(x, y^*) \quad \forall\, x, y$$

and

$$E(x^*, y^*) = v' = v'' = E(x^*, y^*)$$

$\{(2) \Rightarrow (2a)\}$ (2a) is just a special case of (2) using mixed strategies x with $x_i = 1$ and $x_k = 0$ for $k \neq i$.

$\{(2a) \Rightarrow (2)\}$ For each i, consider all convex combinations of vectors x with $x_i = 1$ and $x_k = 0$ for $k \neq i$. Since $E(i, y^*) \leq v$, we must have $E(x^*, y^*) \leq v$.

$\{(2) \Rightarrow (1)\}$

- $\{$**Case** $\geq\}$

$$\begin{aligned} E(x, y^*) &\leq E(x^*, y) \qquad \forall\, x, y \\ \max_x E(x, y^*) &\leq E(x^*, y) \qquad \forall\, y \\ \max_x E(x, y^*) &\leq \min_y E(x^*, y) \\ \min_y \max_x E(x, y) \leq \max_x E(x, y^*) &\leq \min_y E(x^*, y) \leq \max_x \min_y E(x, y) \end{aligned}$$

- $\{$**Case** $\leq\}$

$$\begin{aligned} \min_y E(x, y) &\leq E(x, y) \qquad \forall\, x, y \\ \max_x \left[\min_y E(x, y)\right] &\leq \max_x E(x, y) \qquad \forall\, y \\ \max_x \left[\min_y E(x, y)\right] &\leq \min_y \left[\max_x E(x, y)\right] \end{aligned}$$

$\{(1) \Rightarrow (3)\}$

$$\max_x \left[\min_y E(x,y)\right] = \min_y \left[\max_x E(x,y)\right]$$

Let $f(x) = \min_y E(x,y)$. From calculus, there exists x^* such that $f(x)$ attains its maximum value at x^*. Hence

$$\min_y E(x^*,y) = \max_x \left[\min_y E(x,y)\right]$$

$\{(3) \Rightarrow (4)\}$ This is direct from the duality theorem of LP. (See Chapter 13 of Dantzig's text.)

■

Question 2.5. Can the LP problem in section (4) of Theorem 2.4 have alternate optimal solutions. If so, how does that affect the choice of (x^*, y^*)?[2]

2.2 Two-Person General-Sum Games

2.2.1 Basic ideas

Two-person general-sum games (sometimes called "bi-matrix games") can be represented by two $(m \times n)$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ where a_{ij} is the "payoff" to Player 1 and b_{ij} is the "payoff" to Player 2. If $A = -B$ then we get a two-person zero-sum game, A.

Note 2.7. These are non-cooperative games with no side payments.

Definition 2.2. *The (pure) strategy* (i^*, j^*) *is a* **Nash equilibrium solution** *to the game* (A, B) *if*

$$\begin{array}{rcll} a_{i^*,j^*} & \geq & a_{i,j^*} & \forall\, i \\ b_{i^*,j^*} & \geq & b_{i^*,j} & \forall\, j \end{array}$$

Note 2.8. If both players are placed on their respective Nash equilibrium strategies (i^*, j^*), then each player cannot unilaterally move away from that strategy and improve his payoff.

[2]Thanks to Esra E. Aleisa for this question.

Question 2.6. Show that if $A = -B$ (zero-sum case), the above definition of a Nash solution corresponds to our previous definition of a saddle point.

Note 2.9. Not every game has a Nash solution using pure strategies.

Note 2.10. A Nash solution need not be the best solution, or even a reasonable solution for a game. It's merely a stable solution against unilateral moves by a single player. For example, consider the game

$$(A, B) = \begin{bmatrix} (4,0) & (4,1) \\ (5,3) & (3,2) \end{bmatrix}$$

This game has two Nash equilibrium strategies, $(4, 1)$ and $(5, 3)$. Note that both players prefer $(5, 3)$ when compared with $(4, 1)$.

Question 2.7. What is the solution to the following simple modification of the above game:[3]

$$(A, B) = \begin{bmatrix} (4,0) & (4,1) \\ (4,2) & (3,2) \end{bmatrix}$$

Example 2.1. (Prisoner's Dilemma) Two suspects in a crime have been picked up by police and placed in separate rooms. If both confess (C), each will be sentenced to 3 years in prison. If only one confesses, he will be set free and the other (who didn't confess (NC)) will be sent to prison for 4 years. If neither confesses, they will both go to prison for 1 year.

This game can be represented in strategic form, as follows:

	C	NC
C	(-3,-3)	(0,-4)
NC	(-4,0)	(-1,-1)

This game has one Nash equilibrium strategy, $(-3, -3)$. When compared with the other solutions, note that it represents one of the worst outcomes for both players.

2.2.2 Properties of Nash strategies

[3] Thanks to Esra E. Aleisa for this question.

Definition 2.3. *The pure strategy pair* (i_1, j_1) **weakly dominates** (i_2, j_2) *if and only if*

$$\begin{aligned} a_{i_1,j_1} &\geq a_{i_2,j_2} \\ b_{i_1,j_1} &\geq b_{i_2,j_2} \end{aligned}$$

and one of the above inequalities is strict.

Definition 2.4. *The pure strategy pair* (i_1, j_1) **strongly dominates** (i_2, j_2) *if and only if*

$$\begin{aligned} a_{i_1,j_1} &> a_{i_2,j_2} \\ b_{i_1,j_1} &> b_{i_2,j_2} \end{aligned}$$

Definition 2.5. (Weiss [3]) *The pure strategy pair* (i, j) *is* **inadmissible** *if there exists some strategy pair* $(i_\prime, j_\prime)$ *that weakly dominates* (i, j).

Definition 2.6. (Weiss [3]) *The pure strategy pair* (i, j) *is* **admissible** *if it is not* **inadmissible**.

Example 2.2. Consider again the game

$$(A, B) = \begin{bmatrix} (4,0) & (4,1) \\ (5,3) & (3,2) \end{bmatrix}$$

With Nash equilibrium strategies, $(4, 1)$ and $(5, 3)$. Only $(5, 3)$ is admissible.

Note 2.11. If there exists multiple admissible Nash equilibria, then side-payments (with collusion) may yield a "better" solution for all players.

Definition 2.7. *Two bi-matrix games* $(A.B)$ *and* (C, D) *are* **strategically equivalent** *if there exists* $\alpha_1 > 0$*,* $\alpha_2 > 0$ *and scalars* β_1*,* β_2 *such that*

$$\begin{aligned} a_{ij} &= \alpha_1 c_{ij} + \beta_1 \qquad \forall\, i, j \\ b_{ij} &= \alpha_2 d_{ij} + \beta_2 \qquad \forall\, i, j \end{aligned}$$

Theorem 2.5. *If bi-matrix games* $(A.B)$ *and* (C, D) *are strategically equivalent and* (i^*, j^*) *is a Nash strategy for* (A, B)*, then* (i^*, j^*) *is also a Nash strategy for* (C, D).

Note 2.12. This was used to modify the original matrices for the Prisoners' Dilemma problem in Example 2.1.

2.2.3 Nash equilibria using mixed strategies

Sometimes the bi-matrix game (A, B) does not have a Nash strategy using pure strategies. As before, we can use mixed strategies for such games.

Definition 2.8. *The (mixed) strategy (x^*, y^*) is a* **Nash equilibrium solution** *to the game (A, B) if*

$$\begin{aligned} x^* A y^{*\mathrm{T}} &\geq x A y^{*\mathrm{T}} \qquad \forall\, x \in \Xi^m \\ x^* B y^{*\mathrm{T}} &\geq x^* B y^{\mathrm{T}} \qquad \forall\, y \in \Xi^n \end{aligned}$$

where Ξ^r is the r-dimensional probability simplex.

Question 2.8. Consider the game

$$(A, B) = \begin{bmatrix} (-2, -4) & (0, -3) \\ (-3, 0) & (1, -1) \end{bmatrix}$$

Can we find mixed strategies (x^*, y^*) that provide a Nash solution as defined above?

Theorem 2.6. *Every bi-matrix game has at least one Nash equilibrium solution in mixed strategies.*

Proof: (This is the sketch provided by the text for Proposition 33.1; see Chapter 3 for a complete proofs for $N \geq 2$ players.)

Consider the sets Ξ^n and Ξ^m consisting of the mixed strategies for Player 1 and Player 2, respectively. Note that $\Xi^n \times \Xi^m$ is non-empty, convex and compact. Since the expected payoff functions xAy^{T} and xBy^{T} are linear in (x, y), the result follows using Brouwer's fixed point theorem, ■

2.2.4 Finding Nash mixed strategies

Consider again the game

$$(A, B) = \begin{bmatrix} (-2, -4) & (0, -3) \\ (-3, 0) & (1, -1) \end{bmatrix}$$

For Player 1

$$\begin{aligned} xAy^{\mathrm{T}} &= -2x_1y_1 - 3(1 - x_1)y_1 + (1 - x_1)(1 - y_1) \\ &= 2x_1y_1 - x_1 - 4y_1 + 1 \end{aligned}$$

For Player 2

$$xBy^{\mathrm{T}} = -2x_1y_1 - 2x_1 + y_1 - 1$$

In order for (x^*, y^*) to be a Nash equilibrium, we must have for all $0 \leq x_1 \leq 1$

(3) $$x^*Ay^{*\mathrm{T}} \geq xAy^{*\mathrm{T}} \qquad \forall\, x \in \Xi^m$$

(4) $$x^*By^{*\mathrm{T}} \geq x^*By^{\mathrm{T}} \qquad \forall\, y \in \Xi^n$$

For Player 1 this means that we want (x^*, y^*) so that for all x_1

$$\begin{aligned} 2x_1^*y_1^* - x_1^* - 4y_1^* + 1 &\geq 2x_1y_1^* - x_1 - 4y_1^* + 1 \\ 2x_1^*y_1^* - x_1^* &\geq 2x_1y_1^* - x_1 \end{aligned}$$

Let's try $y_1^* = \frac{1}{2}$. We get

$$\begin{aligned} 2x_1^*\left(\frac{1}{2}\right) - x_1^* &\geq 2x_1\left(\frac{1}{2}\right) - x_1 \\ 0 &\geq 0 \end{aligned}$$

Therefore, if $y^* = (\frac{1}{2}, \frac{1}{2})$ then any x^* can be chosen and condition (3) will be satisfied.

Note that only condition (3) and Player 1's matrix A was used to get Player 2's strategy y^*.

For Player 2 the same thing happens if we use $x_1^* = \frac{1}{2}$ and condition (4). That is, for all $0 \leq y_1 \leq 1$

$$\begin{aligned} -2x_1^*y_1^* - 2x_1^* + y_1^* - 1 &\geq -2x_1y_1^* - 2x_1 + y_1^* - 1 \\ -2x_1^*y_1^* + y_1^* &\geq -2x_1y_1^* + y_1 \\ -2\left(\frac{1}{2}\right)y_1^* + y_1^* &\geq -2\left(\frac{1}{2}\right)y_1^* + y_1 \\ 0 &\geq 0 \end{aligned}$$

How can we get the values of (x^*, y^*) that will work? One suggested approach from (Başar and Olsder [1]) uses the following:

Theorem 2.7. *Any mixed Nash equilibrium solution* (x^*, y^*) *in the interior of* $\Xi^m \times \Xi^n$ *must satisfy*

$$\sum_{j=1}^{n} y_j^*(a_{ij} - a_{1j}) = 0 \qquad \forall\, i \neq 1 \tag{5}$$

$$\sum_{i=1}^{m} x_i^*(b_{ij} - b_{i1}) = 0 \qquad \forall\, j \neq 1 \tag{6}$$

Proof: Recall that

$$\begin{aligned} E(x,y) = xAy^{\mathrm{T}} &= \sum_{i=1}^{m}\sum_{j=1}^{n} x_i y_j a_{ij} \\ &= \sum_{j=1}^{n}\sum_{i=1}^{m} x_i y_j a_{ij} \end{aligned}$$

Since $x_1 = 1 - \sum_{i=2}^{m} x_i$, we have

$$\begin{aligned} xAy^{\mathrm{T}} &= \sum_{j=1}^{n}\left[\sum_{i=2}^{m} x_i y_j a_{ij} + \left(1 - \sum_{i=2}^{m} x_i\right) y_j a_{1j}\right] \\ &= \sum_{j=1}^{n}\left[y_j a_{1j} + y_j \sum_{i=2}^{m} x_i (a_{ij} - a_{1j})\right] \\ &= \sum_{j=1}^{n}\left[y_j a_{1j} + \sum_{i=2}^{m} x_i \sum_{j=1}^{n} y_j (a_{ij} - a_{1j})\right] \end{aligned}$$

If (x^*, y^*) is an interior maximum (or minimum) then

$$\frac{\partial}{\partial x_i} xAy^{\mathrm{T}} = \sum_{j=1}^{n} y_j (a_{ij} - a_{1j}) = 0 \qquad \text{for } i = 2, \ldots, m$$

Which provide the Equations 5.

The derivation of Equations 6 is similar. ■

Note 2.13. In the proof we have the equation

$$xAy^{\mathrm{T}} = \sum_{j=1}^{n} y_j a_{1j} + \sum_{i=2}^{m} x_i \left[\sum_{j=1}^{n} y_j (a_{ij} - a_{1j})\right]$$

Any Nash solution (x^*, y^*) in the interior of $\Xi^m \times \Xi^n$ has

$$\sum_{j=1}^{n} y_j^*(a_{ij} - a_{1j}) = 0 \qquad \forall\, i \neq 1$$

So this choice of y^* produces

$$xAy^{\mathrm{T}} = \sum_{j=1}^{n} y_j a_{1j} + \sum_{i=2}^{m} x_i\,[0]$$

making this expression independent of x.

Note 2.14. Equations 5 and 6 only provide necessary (not sufficient) conditions, and only characterize solutions on the interior of the probability simplex (i.e., where every component of x and y are strictly positive).

For our example, these equations produce

$$\begin{aligned} y_1^*(a_{21} - a_{11}) + y_2^*(a_{22} - a_{12}) &= 0 \\ x_1^*(b_{12} - b_{11}) + x_2^*(b_{22} - b_{21}) &= 0 \end{aligned}$$

Since $x_2^* = 1 - x_1^*$ and $y_2^* = 1 - y_1^*$, we get

$$\begin{aligned} y_1^*(-3 - (-2)) + (1 - y_1^*)(1 - 0) &= 0 \\ -y_1^* + (1 - y_1^*) &= 0 \\ y_1^* &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} x_1^*(-3 - (-4)) + (1 - x_1^*)(-1 - 0) &= 0 \\ x_1^* - (1 - x_1^*) &= 0 \\ x_1^* &= \frac{1}{2} \end{aligned}$$

But, in addition, one must check that $x_1^* = \frac{1}{2}$ and $y_1^* = \frac{1}{2}$ are actually Nash solutions.

2.2.5 The Lemke-Howson algorithm

Lemke and Howson [2] developed a quadratic programming technique for finding mixed Nash strategies for two-person general sum games (A, B) in strategic form. Their method is based on the following fact, provided in their paper:

Let e_k denote a column vector of k ones, and let x and y be row vectors of dimension m and n, respectively. Let p and q denote scalars. We will also assume that A and B are matrices, each with m rows and n columns.

A mixed strategy is defined by a pair (x, y) such that

$$xe_m = ye_n = 1, \quad \text{and} \quad x \geq 0, y \geq 0 \tag{7}$$

with expected payoffs

$$xAy^{\mathrm{T}} \quad \text{and} \quad xBy^{\mathrm{T}}. \tag{8}$$

A Nash equilibrium solution is a pair $(\bar{x}, \bar{y})$ satisfying (7) such that for all (x, y) satisfying (7),

$$xA\bar{y}^{\mathrm{T}} \leq \bar{x}A\bar{y}^{\mathrm{T}} \quad \text{and} \quad \bar{x}By^{\mathrm{T}} \leq \bar{x}B\bar{y}^{\mathrm{T}}. \tag{9}$$

But this implies that

$$A\bar{y}^{\mathrm{T}} \leq \bar{x}A\bar{y}^{\mathrm{T}}e_m \quad \text{and} \quad \bar{x}B \leq \bar{x}B\bar{y}^{\mathrm{T}}e_n^{\mathrm{T}}. \tag{10}$$

Conversely, suppose (10) holds for $(\bar{x}, \bar{y})$ satisfying (7). Now choose an arbitrary (x, y) satisfying (7). Multiply the first expression in (10) on the left by x and second expression in (10) on the right by y^{T} to get (9). Hence, (7) and (10) are, together, equivalent to (7) and (9).

This serves as the foundation for the proof of the following theorem:

Theorem 2.8. *Any mixed strategy (x^*, y^*) for bi-matrix game (A, B) is a Nash equilibrium solution if and only if x^*, y^*, p^* and q^* solve problem (LH):*

$$\begin{array}{rlcl}
\textit{(LH):} & \max_{x,y,p,q}\{xAy^{\mathrm{T}} + xBy^{\mathrm{T}} - p - q\} & & \\
\textit{st:} & Ay^{\mathrm{T}} & \leq & pe_m \\
& B^{\mathrm{T}}x^{\mathrm{T}} & \leq & qe_n \\
& x_i & \geq & 0 \quad \forall\ i \\
& y_j & \geq & 0 \quad \forall\ j \\
& \sum_{i=1}^{m} x_i & = & 1 \\
& \sum_{j=1}^{n} y_j & = & 1
\end{array}$$

Proof: $(\Rightarrow)$

Every feasible solution (x, y, p, q) to problem (LH) must satisfy the constraints

$$\begin{array}{rcl}
Ay^{\mathrm{T}} & \leq & pe_m \\
xB & \leq & qe_n^{\mathrm{T}}.
\end{array}$$

Multiply both sides of the first constraint on the left by x and multiply the second constraint on the right by y^{T}. As a result, we see that a feasible (x, y, p, q) must satisfy

$$\begin{aligned} xAy^{\mathrm{T}} &\leq p \\ xBy^{\mathrm{T}} &\leq q. \end{aligned}$$

Hence, for any feasible (x, y, p, q). the objective function must satisfy

$$xAy^{\mathrm{T}} + xBy^{\mathrm{T}} - p - q \leq 0.$$

Suppose (x^*, y^*) is any Nash solution for (A, B). Let

$$\begin{aligned} p^* &= x^*Ay^{*\mathrm{T}} \\ q^* &= x^*By^{*\mathrm{T}}. \end{aligned}$$

Because of (9) and (10), this implies

$$\begin{aligned} Ay^{*\mathrm{T}} &\leq x^*Ay^{*\mathrm{T}}e_m &= p^*e_m \\ x^*B &\leq x^*By^{*\mathrm{T}}e_n^{\mathrm{T}} &= q^*e_n^{\mathrm{T}}. \end{aligned}$$

So this choice of (x^*, y^*, p^*, q^*) is feasible, and results in the objective function equal to zero. Hence it's an optimal solution to problem (LH)

$(\Leftarrow)$

Suppose $(\bar{x}, \bar{y}, \bar{p}, \bar{q})$ solves problem (LH). From Theorem 2.6, there is at least one Nash solution (x^*, y^*). Using the above argument, (x^*, y^*) must be an optimal solution to (LH) with an objective function value of zero. Since $(\bar{x}, \bar{y}, \bar{p}, \bar{q})$ is an optimal solution to (LH), we must then have

$$\bar{x}A\bar{y}^{\mathrm{T}} + \bar{x}B\bar{y}^{\mathrm{T}} - \bar{p} - \bar{q} = 0 \tag{11}$$

with $(\bar{x}, \bar{y}, \bar{p}, \bar{q})$ satisfying the constraints

$$A\bar{y}^{\mathrm{T}} \leq \bar{p}e_m \tag{12}$$

$$\bar{x}B \leq \bar{q}e_n^{\mathrm{T}}. \tag{13}$$

Now multiply (12) on the left by $\bar{x}$ and multiply (13) on the right by $\bar{y}^{\mathrm{T}}$ to get

$$\bar{x}A\bar{y}^{\mathrm{T}} \leq \bar{p} \tag{14}$$

$$\bar{x}B\bar{y}^{\mathrm{T}} \leq \bar{q}. \tag{15}$$

Then (11), (14), and (15) together imply

$$\begin{aligned} \bar{x}A\bar{y}^{\mathrm{T}} &= \bar{p} \\ \bar{x}B\bar{y}^{\mathrm{T}} &= \bar{q}. \end{aligned}$$

So (12), and (13) can now be rewritten as

$$A\bar{y}^{\mathrm{T}} \leq \bar{x}A\bar{y}^{\mathrm{T}}e_m \tag{16}$$

$$\bar{x}B \leq \bar{x}B\bar{y}^{\mathrm{T}}e_n. \tag{17}$$

Choose an arbitrary $(x, y) \in \Xi^m \times \Xi^n$ and, this time, multiply (16) on the left by x and multiply (17) on the right by y^{T} to get

$$xA\bar{y}^{\mathrm{T}} \leq \bar{x}A\bar{y}^{\mathrm{T}} \tag{18}$$

$$\bar{x}By^{\mathrm{T}} \leq \bar{x}B\bar{y}^{\mathrm{T}} \tag{19}$$

for all $(x, y) \in \Xi^m \times \Xi^n$. Hence $(\bar{x}, \bar{y})$ is a Nash equilibrium solution. ■

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IE675 Game Theory

Lecture Note Set 3

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Monday, April 16, 2001

3 N-PERSON GAMES

3.1 N-Person Games in Strategic Form

3.1.1 Basic ideas

We can extend many of the results of the previous chapter for games with $N > 2$ players.

Let $M_i = \{1, \ldots, m_i\}$ denote the set of m_i pure strategies available to Player i.

Let $n_i \in M_i$ be the strategy actually selected by Player i, and let $a^i_{n_1,n_2,\ldots,n_N}$ be the payoff to Player i if

Player 1 chooses strategy n_1
Player 2 chooses strategy n_2
$\vdots$
Player N chooses strategy n_N

Definition 3.1. *The strategies* $(n_1^*, \ldots, n_N^*)$ *with* $n_i^* \in M_i$ *for all* $i \in N$ *form a Nash equilibrium solution if*

$$a^1_{n_1^*,n_2^*,\ldots,n_N^*} \;\geq\; a^1_{n_1,n_2^*,\ldots,n_N^*} \quad \forall\, n_i \in M_1$$

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$$
\begin{aligned}
a^2_{n_1^*,n_2^*,\dots,n_N^*} &\geq a^2_{n_1^*,n_2,\dots,n_N^*} \quad \forall\, n_2 \in M_2 \\
&\vdots \\
a^N_{n_1^*,n_2^*,\dots,n_N^*} &\geq a^N_{n_1^*,n_2^*,\dots,n_N} \quad \forall\, n_N \in M_N
\end{aligned}
$$

Definition 3.2. *Two N-person games with payoff functions $a^i_{n_1,n_2,\dots,n_N}$ and $b^i_{n_1,n_2,\dots,n_N}$ are* **strategically equivalent** *if there exists $\alpha_i > 0$ and scalars β_i for $i = 1, \dots, n$ such that*

$$
a^i_{n_1,n_2,\dots,n_N} = \alpha_i b^i_{n_1,n_2,\dots,n_N} + \beta_i \quad \forall\, i \in N
$$

3.1.2 Nash solutions with mixed strategies

Definition 3.3. *The mixed strategies $(y^{*1}, \dots, y^{*N})$ with $y^{*i} \in \Xi^{M_i}$ for all $i \in N$ form a Nash equilibrium solution if*

$$
\begin{aligned}
\sum_{n_1}\cdots\sum_{n_N} y^{*1}_{n_1} y^{*2}_{n_2} \cdots y^{*N}_{n_N} a^1_{n_1,\dots,n_N} &\geq \sum_{n_1}\cdots\sum_{n_N} y^{1}_{n_1} y^{*2}_{n_2} \cdots y^{*N}_{n_N} a^1_{n_1,\dots,n_N} \quad \forall\, y^1 \in \Xi^{M_1} \\
\sum_{n_1}\cdots\sum_{n_N} y^{*1}_{n_1} y^{*2}_{n_2} \cdots y^{*N}_{n_N} a^2_{n_1,\dots,n_N} &\geq \sum_{n_1}\cdots\sum_{n_N} y^{*1}_{n_1} y^{2}_{n_2} \cdots y^{*N}_{n_N} a^2_{n_1,\dots,n_N} \quad \forall\, y^2 \in \Xi^{M_2} \\
&\vdots \\
\sum_{n_1}\cdots\sum_{n_N} y^{*1}_{n_1} y^{*2}_{n_2} \cdots y^{*N}_{n_N} a^N_{n_1,\dots,n_N} &\geq \sum_{n_1}\cdots\sum_{n_N} y^{*1}_{n_1} y^{*2}_{n_2} \cdots y^{N}_{n_N} a^N_{n_1,\dots,n_N} \quad \forall\, y^N \in \Xi^{M_N}
\end{aligned}
$$

Note 3.1. Consider the function

$$
\begin{aligned}
\psi^i_{n_i}(y^1, \dots, y^n) = &\sum_{n_1}\cdots\sum_{n_N} y^1_{n_1} y^2_{n_2} \cdots y^N_{n_N} a^i_{n_1,\dots,n_N} \\
&- \sum_{n_1}\cdots\sum_{n_{i-1}}\sum_{n_{i+1}}\cdots\sum_{n_N} y^1_{n_1} \cdots y^{i-1}_{n_{i-1}} y^{i+1}_{n_{i+1}} \cdots y^N_{n_N} a^i_{n_1,\dots,n_N}
\end{aligned}
$$

This represents the difference between the following two quantities:

1. the expected payoff to Player i if all players adopt mixed strategies $(y^1, \dots, y^N)$:

$$
\sum_{n_1}\cdots\sum_{n_N} y^1_{n_1} y^2_{n_2} \cdots y^N_{n_N} a^i_{n_1,\dots,n_N}
$$

2. the expected payoff to Player i if all players except Player i adopt mixed strategies $(y^1, \ldots, y^N)$ and Player i uses pure strategy n_i:

$$\sum_{n_1} \cdots \sum_{n_{i-1}} \sum_{n_{i+1}} \cdots \sum_{n_N} y^1_{n_1} \cdots y^{i-1}_{n_{i-1}} y^{i+1}_{n_{i+1}} \cdots y^N_{n_N} a^i_{n_1,\ldots,n_N}$$

Remember that the mixed strategies include the pure strategies. For example, $(0, 1, 0, \ldots, 0)$ is a mixed strategy that implements pure strategy 2.

For example, in a two-player game, for each $n_1 \in M_1$ we have

$$\begin{aligned} \psi^1_{n_1}(y^1, y^2) &= \left[y^1_1 y^2_1 a^1_{11} + y^1_1 y^2_2 a^1_{12} + y^1_2 y^2_1 a^1_{21} + y^1_2 y^2_2 a^1_{22}\right] \\ &\quad - \left[y^2_1 a^1_{n_1 1} + y^2_2 a^1_{n_1 2}\right] \end{aligned}$$

The first term

$$y^1_1 y^2_1 a^1_{11} + y^1_1 y^2_2 a^1_{12} + y^1_2 y^2_1 a^1_{21} + y^1_2 y^2_2 a^1_{22}$$

is the expected value if Player 1 uses mixed strategy y_1. The second term

$$y^2_1 a^1_{n_1 1} + y^2_2 a^1_{n_1 2}$$

is the expected value if Player 1 uses pure strategy n_1. Player 2 uses mixed strategy y_2 in both cases.

The next theorem (Theorem 3.1) will guarantee that every game has at least one Nash equilibrium in mixed strategies. Its proof depends on things that can go wrong when $\psi^i_{n_i}(y^1, \ldots, y^n) < 0$. So we will define

$$c^i_{n_i}(y^1, \ldots, y^n) = \min\{\psi^i_{n_i}(y^1, \ldots, y^n), 0\}$$

The proof of Theorem 3.1 then uses the expression

$$\overline{y}^i_{n_i} = \frac{y^i_{n_i} + c^i_{n_i}}{1 + \sum_{j \in M_i} c^i_j}$$

Note that the denominator is the sum (taken over n_i) of the terms in the numerator. If all of the c^i_j vanish, we get

$$\overline{y}^i_{n_i} = y^i_{n_i}.$$

Theorem 3.1. *Every N-person finite game in normal (strategic) form has a Nash equilibrium solution using mixed strategies.*

Proof: Define $\psi^i_{n_i}$ and $c^i_{n_i}$, as above. Consider the expression

$$\overline{y}^i_{n_i} = \frac{y^i_{n_i} + c^i_{n_i}}{1 + \sum_{j \in M_i} c^i_j} \tag{1}$$

We will try to find solutions $y^i_{n_i}$ to Equation 1 such that

$$\overline{y}^i_{n_i} = y^i_{n_i} \quad \forall\, n_i \in M_i \text{ and } \forall\, i = 1, \ldots, N$$

The Brouwer fixed point theorem[1] guarantees that at least one such solution exists.

We will show that every solution to Equation 1 is a Nash equilibrium solution and that every Nash equilibrium solution is a solution to Equation 1.

To show that every Nash equilibrium solution is a solution to Equation 1, note that if $(y^{*1}, \ldots, y^{*N})$ is a Nash solution then, from the definition of a Nash solution,

$$\psi^i_{n_i}(y^{1*}, \ldots, y^{n*}) \geq 0$$

which implies

$$c^i_{n_i}(y^{1*}, \ldots, y^{n*}) = 0$$

and this holds for all $n_i \in M_i$ and all $i = 1, \ldots, N$. Hence, $(y^{1*}, \ldots, y^{n*})$ solves Equation 1.

> **Remark:** Will show that every solution to Equation 1 is a Nash equilibrium solution by contradiction. That is, we will assume that a mixed strategy $(y^1, \ldots, y^N)$ is a solution to Equation 1 but is not a Nash solution. This will lead us to conclude that $(y^1, \ldots, y^N)$ is not a solution to Equation 1, a contradiction.

Assume $(y^1, \ldots, y^N)$ is a solution to Equation 1 but is not a Nash solution. Then there exists a $i \in \{1, \ldots, N\}$ (say $i = 1$) with $\tilde{y}^1 \in \Xi^{M_1}$ such that

$$\begin{aligned} &\sum_{n_1} \cdots \sum_{n_N} y^1_{n_1} y^2_{n_2} \cdots y^N_{n_N} a^i_{n_1,\ldots,n_N} \\ &\qquad < \sum_{n_1} \cdots \sum_{n_N} \tilde{y}^1_{n_1} y^2_{n_2} \cdots y^N_{n_N} a^i_{n_1,\ldots,n_N} \end{aligned}$$

[1]The Brouwer fixed point theorem states that if S is a compact and convex subset of $\mathbb{R}^n$ and if $f : S \to S$ is a continuous function onto S, then there exists at least one $x \in S$ such that $f(x) = x$.

Rewriting the right hand side,

$$\sum_{n_1} \cdots \sum_{n_N} y^1_{n_1} y^2_{n_2} \cdots y^N_{n_N} a^i_{n_1,\dots,n_N}$$
$$< \sum_{n_1} \tilde{y}^1_{n_1} \left[\sum_{n_2} \cdots \sum_{n_N} y^2_{n_2} \cdots y^N_{n_N} a^i_{n_1,\dots,n_N} \right]$$

Now the expression

$$\left[\sum_{n_2} \cdots \sum_{n_N} y^2_{n_2} \cdots y^N_{n_N} a^i_{n_1,\dots,n_N} \right]$$

is a function of n_1. Suppose this quantity is maximized when $n_1 = \tilde{n}_1$. We then get,

$$\sum_{n_1} \cdots \sum_{n_N} y^1_{n_1} y^2_{n_2} \cdots y^N_{n_N} a^i_{n_1,\dots,n_N}$$
$$< \sum_{n_1} \tilde{y}^1_{n_1} \left[\sum_{n_2} \cdots \sum_{n_N} y^2_{n_2} \cdots y^N_{n_N} a^i_{\tilde{n}_1,\dots,n_N} \right]$$

which yields

$$\sum_{n_1} \cdots \sum_{n_N} y^1_{n_1} y^2_{n_2} \cdots y^N_{n_N} a^i_{n_1,\dots,n_N} \tag{2}$$
$$< \sum_{n_2} \cdots \sum_{n_N} y^2_{n_2} \cdots y^N_{n_N} a^i_{\tilde{n}_1,\dots,n_N} \tag{3}$$

> **Remark:** After this point we don't really use $\tilde{y}$ again. It was just a device to obtain $\tilde{n}_1$ which will produce our contradiction. Remember, throughout the rest of the proof, the values of $(y^1, \dots, y^N)$ claim be a fixed point for Equation 1. If $(y^1, \dots, y^N)$ is, in fact, not Nash (as was assumed), then we have just found a player (who we are calling Player 1) who has a *pure* strategy $\tilde{n}_1$ that can beat strategy y^1 when Players $2, \dots, N$ use mixed strategies $(y^2, \dots, y^N)$.

Using $\tilde{n}_1$, Player 1 obtains

$$\psi^1_{\tilde{n}_1}(y^1, \dots, y^n) < 0$$

which means that

$$c^1_{\tilde{n}_1}(y^1, \dots, y^n) < 0$$

which implies that

$$\sum_{j \in M_1} c_j^i < 0$$

since one of the indices in M_1 is $\tilde{n}_1$ and the rest of the c_j^i cannot be positive.

> **Remark:** Now the values $(y^1, \ldots, y^N)$ are in trouble. We have determined that their claim of being "non-Nash" produces a denominator in Equation 1 that is less than 1. All we need to do is find some pure strategy (say $\hat{n}_1$) for Player 1 with $c^i_{\hat{n}_i}(y^1, \ldots, y^n) = 0$. If we can, $(y^1, \ldots, y^N)$ will fail to be a fixed-point for Equation 1, and it will be y^1 that causes the failure. Let's see what happens...

Recall expression 2:

$$\sum_{n_1} \cdots \sum_{n_N} y^1_{n_1} y^2_{n_2} \cdots y^N_{n_N} a^i_{n_1,\ldots,n_N}$$

rewritten as

$$\sum_{n_1} y^1_{n_1} \left[\sum_{n_2} \cdots \sum_{n_N} y^2_{n_2} \cdots y^N_{n_N} a^i_{n_1,\ldots,n_N} \right]$$

and consider the term

$$\left[\sum_{n_2} \cdots \sum_{n_N} y^2_{n_2} \cdots y^N_{n_N} a^i_{n_1,\ldots,n_N} \right] \tag{4}$$

as a function of n_1 There must be some $n_1 = \hat{n}_1$ that minimizes expression 4, with

$$\sum_{n_1} \cdots \sum_{n_N} y^1_{n_1} y^2_{n_2} \cdots y^N_{n_N} a^i_{n_1,\ldots,n_N} \geq \left[\sum_{n_2} \cdots \sum_{n_N} y^2_{n_2} \cdots y^N_{n_N} a^i_{\hat{n}_1,n_2\ldots,n_N} \right]$$

For that particular strategy we have

$$\psi^1_{\hat{n}_1}(y^1, \ldots, y^n) \geq 0$$

which means that

$$c^1_{\hat{n}_1}(y^1, \ldots, y^n) = 0$$

Therefore, for Player 1, we get

$$\overline{y}^1_{\hat{n}_1} = \frac{y^1_{\hat{n}_1} + 0}{1 + [\text{something} < 0]} > y^1_{\hat{n}_1}$$

Hence, y^1 (which claimed to be a component of the non-Nash solution $(y^1, \ldots, y^N)$) fails to solve Equation 1. A contradiction. ■

The following theorem is an extension of a result for $N = 2$ given in Chapter 2. It provides necessary conditions for any interior Nash solution for N-person games.

Theorem 3.2. *Any mixed Nash equilibrium solution $(y^{*1}, \ldots, y^{*N})$ in the interior of $\Xi^{M_1} \times \cdots \times \Xi^{M_N}$ must satisfy*

$$\begin{aligned}
\sum_{n_2}\sum_{n_3}\cdots\sum_{n_N} y^{*2}_{n_2} y^{*3}_{n_3}\cdots y^{*N}_{n_N}(a^1_{n_1,n_2,n_3\ldots,n_N} - a^1_{1,n_2,n_3\ldots,n_N}) &= 0 \quad \forall\, n_1 \in M_1 - \{1\} \\
\sum_{n_1}\sum_{n_3}\cdots\sum_{n_N} y^{*1}_{n_1} y^{*3}_{n_3}\cdots y^{*N}_{n_N}(a^2_{n_1,n_2,n_3\ldots,n_N} - a^2_{n_1,1,n_3\ldots,n_N}) &= 0 \quad \forall\, n_2 \in M_2 - \{1\} \\
&\vdots \\
\sum_{n_1}\sum_{n_2}\cdots\sum_{n_{N-1}} y^{*1}_{n_1} y^{*2}_{n_2}\cdots y^{*N}_{n_N}(a^N_{n_1,n_2,n_3\ldots,n_N} - a^N_{n_1,n_2,n_3\ldots,1}) &= 0 \quad \forall\, n_N \in M_N - \{1\}
\end{aligned}$$

Proof: Left to the reader.

Question 3.1. Consider the 3-player game with the following values for

$$(a^1_{n_1,n_2,n_3}, a^2_{n_1,n_2,n_3}, a^3_{n_1,n_2,n_3}) :$$

For $n_3 = 1$

	$n_2 = 1$	$n_2 = 2$
$n_1 = 1$	$(1, -1, 0)$	$(0, 1, 0)$
$n_1 = 2$	$(2, 0, 0)$	$(0, 0, 1)$

For $n_3 = 2$

	$n_2 = 1$	$n_2 = 2$
$n_1 = 1$	$(1, 0, 1)$	$(0, 0, 0)$
$n_1 = 2$	$(0, 3, 0)$	$(-1, 2, 0)$

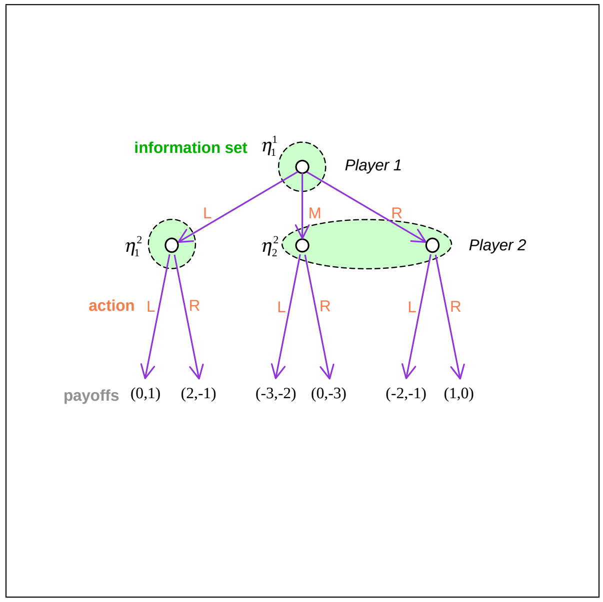
For example $a^2_{212} = 3$. Use the above method to find an interior Nash solution.

3.2 N-Person Games in Extensive Form

3.2.1 An introductory example

We will use an example to illustrate some of the issues associates with games in extensive form.

Consider a game with two players described by the following tree diagram:



Player 1 goes first and chooses an action among $\{\text{Left}, \text{Middle}, \text{Right}\}$. Player 2 then follows by choosing an action among $\{\text{Left}, \text{Right}\}$.

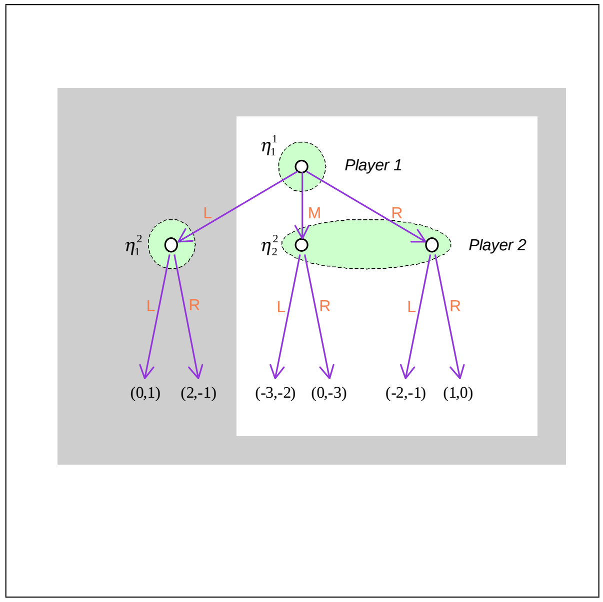
The payoff vectors for each possible combination of actions are shown at each terminating node of the tree. For example, if Player 1 chooses action $u_1 = L$ and Player 2 chooses action $u_2 = R$ then the payoff is $(2, -1)$. So, Player 1 gains 2 while Player 1 loses 1.

Player 2 does not have complete information about the progress of the game. His nodes are partitioned among two information sets $\{\eta_2^1, \eta_2^2\}$. When Player 2 chooses his action, he only knows which information set he is in, not which node.

Player 1 could analyze the game as follows:

- If Player 1 chooses $u_1 = L$ then Player 2 would respond with $u_2 = L$ resulting in a payoff of $(0, 1)$.
- If Player 1 chooses $u_1 \in \{M, R\}$ then the players are really playing the

following subgame:



which can be expressed in normal form as

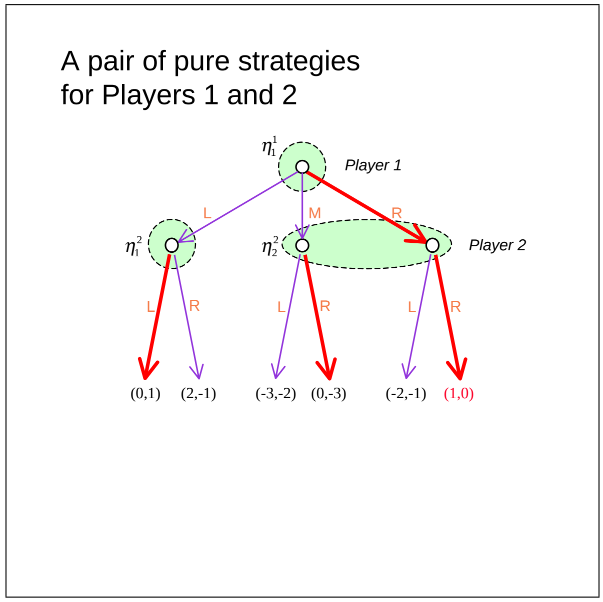
	L	R
M	(-3,-2)	(0,-3)
R	(-2,-1)	(1,0)

in which (R, R) is a Nash equilibrium strategy in pure strategies.

So it seems reasonable for the players to use the following strategies:

- For Player 1
  - If Player 1 is in information set η_1^1 choose R.
- For Player 2
  - If Player 2 is in information set η_1^2 choose L.
  - If Player 2 is in information set η_2^2 choose R.

These strategies can be displayed in our tree diagram as follows:



For games in strategic form, we denote the set of pure strategies for Player i by $M_i = \{1, \ldots, m_i\}$ and let $n_i \in M_i$ denote the strategy actually selected by Player i. We will now consider a strategy γ^i as a function whose domain is the set of information sets of Player i and whose range is the collection of possible actions for Player i. For the strategy shown above

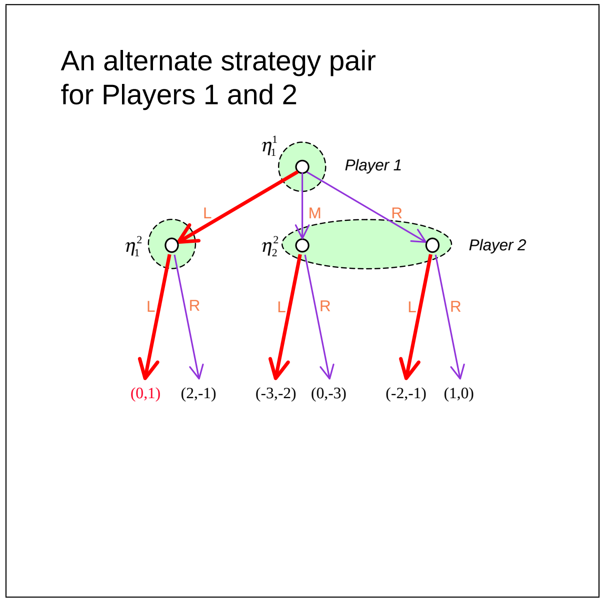
$$\gamma^1(\eta_1^1) = R$$

$$\gamma^2(\eta^2) = \begin{cases} L & \text{if } \eta^2 = \eta_1^2 \\ R & \text{if } \eta^2 = \eta_2^2 \end{cases}$$

The players' task is to choose the best strategy from those available. Using the notation from Section 3.1.1, the set $M_i = \{1, \ldots, m_i\}$ now represents the indices of the possible strategies, $\{\gamma_1^i, \ldots, \gamma_{m_i}^i\}$, for Player i.

Notice that if either player attempts to change his strategy unilaterally, he will not improve his payoff. The above strategy is, in fact, a Nash equilibrium strategy as we will formally define in the next section.

There is another Nash equilibrium strategy for this game, namely

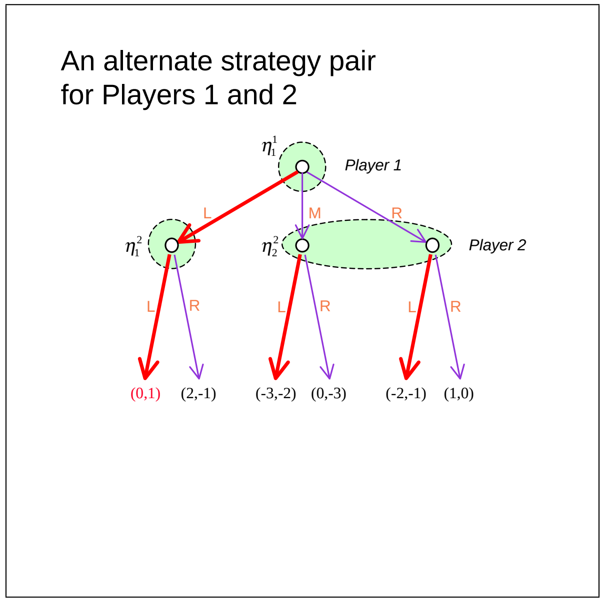


$$\gamma^1(\eta_1^1) = L$$

$$\gamma^2(\eta^2) = \begin{cases} L & \text{if } \eta^2 = \eta_1^2 \\ L & \text{if } \eta^2 = \eta_2^2 \end{cases}$$

There is another Nash equilibrium strategy for this game, namely the strategy pair

(γ_1^1, γ_1^2).



But this strategy did not arise from the recursive procedure described in Section 3.2.1. But (γ_1^1, γ_1^2) is, indeed, a Nash equilibrium. Neither player can improve his payoff by a unilateral change in strategy. Oddly, there is no reason for Player 1 to implement this strategy. If Player 1 chooses to go Left, he can only receive 0. But if Player 1 goes Right, Player 2 will go Right, not Left, and Player 1 will receive a payoff of 1. This example shows that games in extensive form can have Nash equilibria that will never be considered for implementation,

3.2.2 Basic ideas

Definition 3.4. *An N-player* **game in extensive form** *is a directed graph with*

1. *a specific vertex indicating the starting point of the game.*
2. *N cost functions each assigning a real number to each terminating node of the graph. The i^{th} cost function represents the gain to Player i if that node is reached.*
3. *a partition of the nodes among the N players.*
4. *a sub-partition of the nodes assigned to Player i into information sets $\{\eta_k^i\}$. The number of branches emanating from each node of a given information*

set is the same, and no node follows another node in the same information set.

We will use the following notation:

η^i information sets for Player i.

u^i actual actions for Player i emanating from information sets.

$\gamma^i(\cdot)$ a function whose domain is the set of all information sets $\{\eta^i\}$ and whose range is the set of all possible actions $\{u^i\}$.

The set of $\gamma^i(\cdot)$ is the collection of possible (pure) strategies that Player i could use. In the parlance of economic decision theory, the γ^i are *decision rules.* In game theory, we call them (pure) *strategies.*

For the game illustrated in Section 3.2.1, we can write down all possible strategy pairs (γ^1, γ^2). The text calls these *profiles.*

Player 1 has 3 possible pure strategies:

$$\begin{aligned}\gamma^1(\eta^1_1) &= L \\ \gamma^1(\eta^1_1) &= M \\ \gamma^1(\eta^1_1) &= R\end{aligned}$$

Player 2 has 8 possible pure strategies which can be listed in tabular form, as follows:

	γ^2_1	γ^2_2	γ^2_3	γ^2_4
η^2_1 :	L	R	L	R
η^2_2 :	L	L	R	R

Each strategy pair (γ^1, γ^2), when implemented, results in payoffs to both players which we will denote by $(J^1(\gamma^1, \gamma^2), J^2(\gamma^1, \gamma^2))$. These payoffs produce a game in strategic (normal) form where the rows and columns correspond to the possible pure strategies of Player 1 and Player 2, respectively.

	γ^2_1	γ^2_2	γ^2_3	γ^2_4
γ^1_1	(0,1)	(2,-1)	(0,1)	(2,-1)
γ^1_2	(-3,-2)	(-3,-2)	(0,-3)	(0,-3)
γ^1_3	(-2,-1)	(-2,-1)	(1,0)	(1,0)

Using Definition 3.1, we have two Nash equilibria, namely

$$\begin{array}{lll} (\gamma_1^1, \gamma_1^2) & \text{with} & J(\gamma_1^1, \gamma_1^2) = (0,1) \\ (\gamma_3^1, \gamma_3^2) & \text{with} & J(\gamma_3^1, \gamma_3^2) = (1,0) \end{array}$$

This formulation allows us to

- focus on identifying "good" decision rules even for complicated strategies
- analyze games with different information structures
- analyze multistage games with players taking more than one "turn"

3.2.3 The structure of extensive games

The general definition of games in extensive form can produce a variety of different types of games. This section will discuss some of the approaches to classifying such games. These classification schemes are based on

1. the topology of the directed graph
2. the information structure of the games
3. the sequencing of the players

This section borrows heavily from Başar and Olsder [1]. We will categorize multi-stage games, that is, games where the players take multiple turns. This classification scheme extends to differential games that are played in continuous time. In this section, however, we will use it to classify multi-stage games in extensive form.

Define the following terms:

η_k^i information for Player i at stage k.

x_k state of the game at stage k. This completely describes the current status of the game at any point in time.

$y_k^i = h_k^i(x_k)$ is the state measurement equation, where

$h_k^i(\cdot)$ is the state measurement function

y_k^i is the observation of Player i at state k.

u_k^i decision of Player i at stage k.

The purpose of the function h_k^i is to recognize that the players may not perfect information regarding the current state of the game. The information available to Player i at stage k is then

$$\eta_k^i = \{y_1^1, \ldots, y_k^1; y_1^2, \ldots, y_k^2; \cdots; y_1^N, \ldots, y_k^N\}$$

Based on these ideas, games can be classified as

open loop

$$\eta_k^i = \{x_1\} \quad \forall\, k \in K$$

closed loop, perfect state

$$\eta_k^i = \{x_1, \ldots, x_k\} \quad \forall\, k \in K$$

closed loop, imperfect state

$$\eta_k^i = \{y_1^i, \ldots, y_k^i\} \quad \forall\, k \in K$$

memoryless, perfect state

$$\eta_k^i = \{x_1, x_k\} \quad \forall\, k \in K$$

feedback, perfect state

$$\eta_k^i = \{x_k\} \quad \forall\, k \in K$$

feedback, imperfect state

$$\eta_k^i = \{y_k^i\} \quad \forall\, k \in K$$

Example 3.1. **Princess and the Monster.** This game is played in complete darkness. A princess and a monster know their starting positions in a cave. The game ends when they bump into each other. Princess is trying to maximize the time to the final encounter. The monster is trying to minimize the time. (Open Loop)

Example 3.2. **Lady in the Lake.** This game is played using a circular lake. The lady is swimming with maximum speed v_ℓ. A man (who can't swim) runs along the shore of the lake at a maximum speed of v_m. The lady wins if she reaches shore and the man is not there. (Feedback)

3.3 Structure in extensive form games

I am grateful to Pengfei Yi who provided significant portions of Section 3.3.

The solution of an arbitrary extensive form game may require enumeration. But under some conditions, the structure of some games will permit a recursive solution procedure. Many of these results can be found in Başar and Olsder [1].

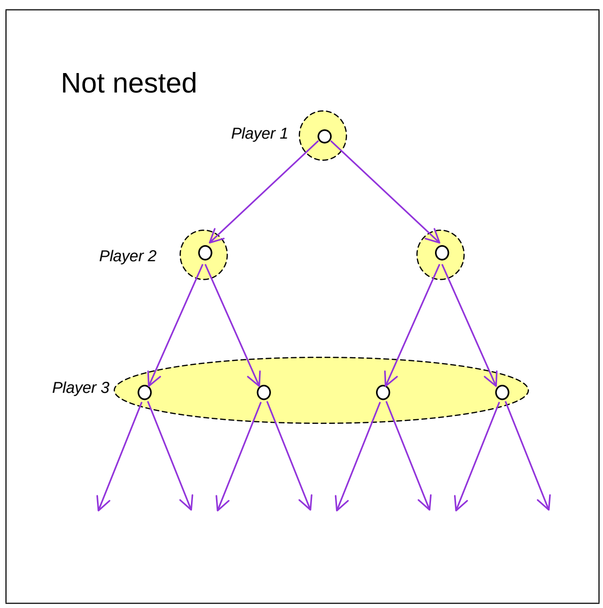
Definition 3.5. *Player i is said to be a* **predecessor** *of Player j if Player i is closer to the initial vertex of the game's tree than Player j.*

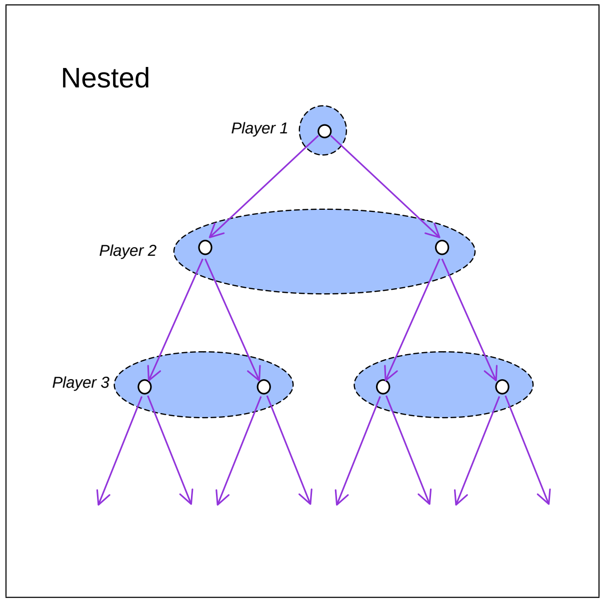
Definition 3.6. *An extensive form game is* **nested** *if each player has access to the information of his predecessors.*

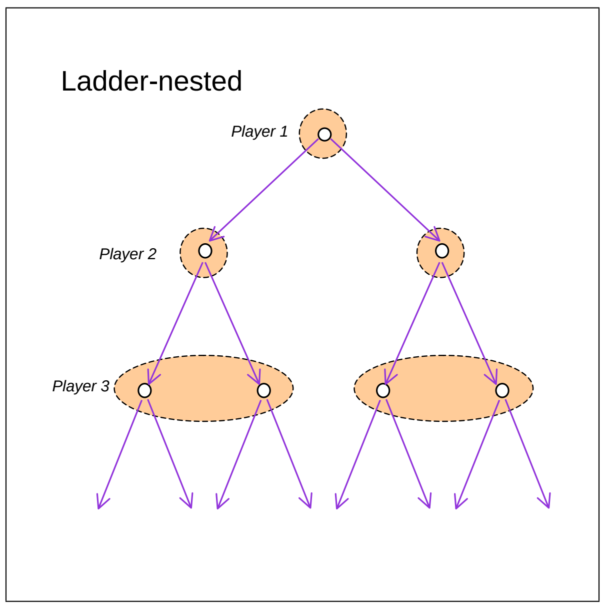
Definition 3.7. (Başar and Olsder [1]) *A nested extensive form game is* **ladder-nested** *if the only difference between the information available to any player (say Player i) and his immediate predecessor (Player $(i-1)$) involves only the actions of Player $(i-1)$, and only at those nodes corresponding to the branches emanating from singleton information sets of Player $(i-1)$.*

Note 3.2. Every 2-player nested game is ladder-nested

The following three figures illustrate the distinguishing characteristics among non-nested, nested, and ladder-nested games.

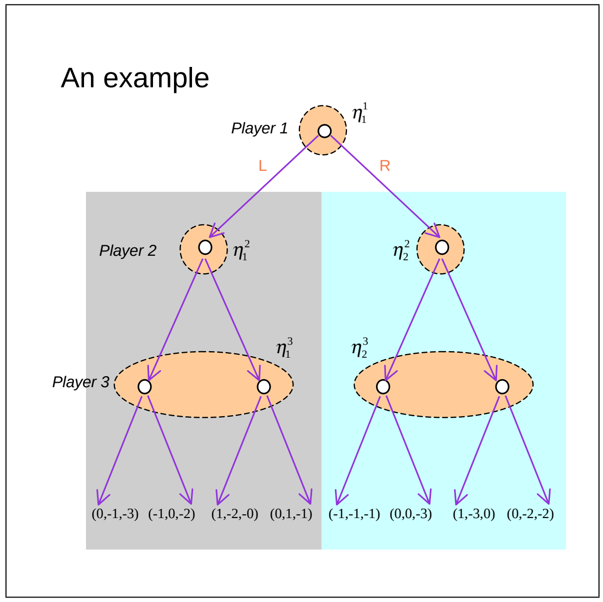






The important feature of ladder-nested games is that the tree can be decomposed in to sub-trees using the singleton information sets as the starting vertices of the sub-trees. Each sub-tree can then be analyzed as game in strategic form among those players involved in the sub-tree.

As an example, consider the following ladder-nested game:



This game can be decomposed into two bimatrix games involving Player 2 and Player 3. Which of these two games are actually played by Player 2 and Player 3 depends on the action (**L** or **R**) of Player 1.

If Player 1 chooses $u^1 =$ **L** then Player 2 and Player 3 play the game

		Player 3	
		L	**R**
Player 2	**L**	$(-1,-3)$	$(0,-2)$
	R	$(-2,0)$	$(1,-1)$

Suppose Player 2 uses a mixed strategy of choosing **L** with probability 0.5 and **R** with probability 0.5. Suppose Player 3 also uses a mixed strategy of choosing **L** with probability 0.5 and **R** with probability 0.5. Then these mixed strategies are a Nash equilibrium solution for this sub-game with an expected payoff to all three players of $(0,-0.5,-1.5)$.

If Player 1 chooses $u^1 =$ **R** then Player 2 and Player 3 play the game

		Player 3	
		L	**R**
Player 2	**L**	$(-1,-1)$	$(0,-3)$
	R	$(-3,0)$	$(0,-2)$

This subgame has a Nash equilibrium in pure strategies with Player 2 and Player 3 both choosing **L**. The payoff to all three players in this case is of $(-1, -1, -1)$.

To summarize the solution for all three players we will introduce the concept of a *behavioral strategy:*

Definition 3.8. *A* **behavioral strategy** *(or* **locally randomized strategy***) assigns for each information set a probability vector to the alternatives emanating from the information set.*

When using a behavioral strategy, a player simply randomizes over the alternatives from each information set. When using a mixed strategy, a player randomizes his selection from the possible pure strategies for the entire game.

The following behavioral strategy produces a Nash equilibrium for all three players:

$$
\begin{aligned}
\gamma^1(\eta_1^1) &= \mathsf{L} \\
\gamma^2(\eta_1^2) &= \begin{cases} \mathsf{L} & \text{with probability } 0.5 \\ \mathsf{R} & \text{with probability } 0.5 \end{cases} \\
\gamma^2(\eta_2^2) &= \begin{cases} \mathsf{L} & \text{with probability } 1 \\ \mathsf{R} & \text{with probability } 0 \end{cases} \\
\gamma^3(\eta_1^3) &= \begin{cases} \mathsf{L} & \text{with probability } 0.5 \\ \mathsf{R} & \text{with probability } 0.5 \end{cases} \\
\gamma^3(\eta_2^3) &= \begin{cases} \mathsf{L} & \text{with probability } 1 \\ \mathsf{R} & \text{with probability } 0 \end{cases}
\end{aligned}
$$

with an expected payoff of $(0, -0.5, -1.5)$.

Note 3.3. When using a behavioral strategy, a player, at each information set, must specify a probability distribution over the alternatives for that information set. It is assumed that the choices of alternatives at different information sets are made independently. Thus it might be reasonable to call such strategies “uncorrelated” strategies.

Note 3.4. For an arbitrary game, not all mixed strategies can be represented by using behavioral strategies. Behavioral strategies are easy to find and represent. We would like to know when we can use behavioral strategies instead of enumerating all pure strategies and randomizing among those pure strategies.

Theorem 3.3. *Every single-stage, ladder-nested N-person game has at least one Nash equilibrium using behavioral strategies.*

3.3.1 An example by Kuhn

One can show that every behavioral strategy can be represented as a mixed strategy. But an important question arises when considering mixed strategies vis-à-vis behavioral strategies: Can a mixed strategy always be represented by a behavioral strategy?

The following example from Kuhn [2] shows a remarkable result involving behavioral strategies. It shows what can happen if the players do not have a property called *perfect recall*. Moreover, the property of *perfect recall* alone is a necessary and sufficient condition to obtain a one-to-one mapping between behavioral and mixed strategies for any game.

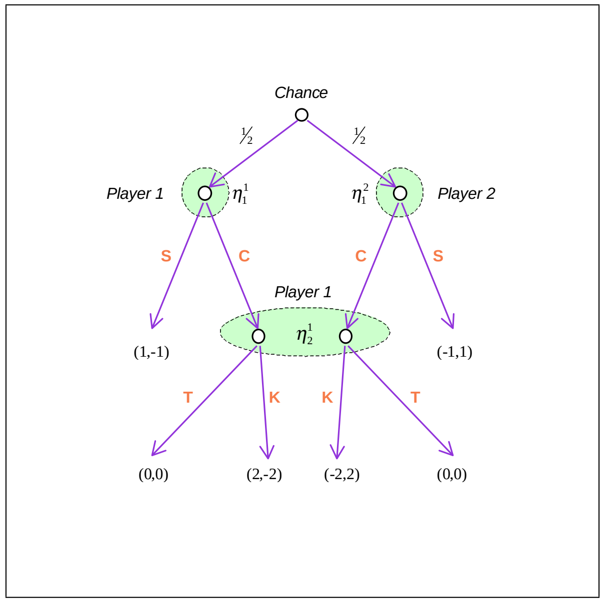
In a game with **perfect recall,** each player remembers everything he knew at previous moves and all of his choices at these moves.

A zero-sum game involves two players and a deck of cards. A card is dealt to each player. If the cards are not different, two more cards are dealt until one player has a higher card than the other.

The holder of the high card receives $1 from his opponent. The player with the high card can choose to either stop the game or continue.

If the game continues, Player 1 (who forgets whether he has the high or low card) can choose to leave the cards as they are or trade with his opponent. Another $1 is then won by the (possibly different) holder of the high card.

The game can be represented with the following diagram:



where

S	Stop the game
C	Continue the game
T	Trade cards
K	Keep cards

At information set η_1^1, Player 1 makes the critical decision that causes him to eventually lose perfect recall at η_2^1. Moreover, it is Player 1's own action that causes this loss of information (as opposed to Player 2 causing the loss). This is the reason why behavioral strategies fail for Player 1 in this problem.

Define the following pure strategies for Player 1:

$$\gamma_1^1(\eta^1) = \begin{cases} \textsf{S} & \text{if } \eta^1 = \eta_1^1 \\ \textsf{T} & \text{if } \eta^1 = \eta_2^1 \end{cases} \qquad \gamma_2^1(\eta^1) = \begin{cases} \textsf{S} & \text{if } \eta^1 = \eta_1^1 \\ \textsf{K} & \text{if } \eta^1 = \eta_2^1 \end{cases}$$

$$\gamma_3^1(\eta^1) = \begin{cases} \textsf{C} & \text{if } \eta^1 = \eta_1^1 \\ \textsf{T} & \text{if } \eta^1 = \eta_2^1 \end{cases} \qquad \gamma_4^1(\eta^1) = \begin{cases} \textsf{C} & \text{if } \eta^1 = \eta_1^1 \\ \textsf{K} & \text{if } \eta^1 = \eta_2^1 \end{cases}$$

and for Player 2:

$$\gamma_1^2(\eta_1^2) = \textsf{S} \qquad \gamma_2^2(\eta_1^2) = \textsf{C}$$

This results in the following strategic (normal) form game:

	γ_1^2	γ_2^2
γ_1^1	$(1/2, -1/2)$	$(0,0)$
γ_2^1	$(-1/2, 1/2)$	$(0,0)$
γ_3^1	$(0,0)$	$(-1/2, 1/2)$
γ_4^1	$(0,0)$	$(1/2, -1/2)$

Question 3.2. Show that the mixed strategy for Player 1:

$$(\tfrac{1}{2}, 0, 0, \tfrac{1}{2})$$

and the mixed strategy for Player 2:

$$(\tfrac{1}{2}, \tfrac{1}{2})$$

result in a Nash equilibrium with expected payoff $(\frac{1}{4}, -\frac{1}{4})$.

Question 3.3. Suppose that Player 1 uses a behavioral strategy (x, y) defined as follows: Let $x \in [0, 1]$ be the probability Player 1 chooses **S** when he is in information set η_1^1, and let $y \in [0, 1]$ be the probability Player 1 chooses **T** when he is in information set η_2^1.

Also suppose that Player 2 uses a behavioral strategy (z) where $z \in [0, 1]$ is the probability Player 2 chooses **S** when he is in information set η_1^2.

Let $E^i((x, y), z)$ denote the expected payoff to Player $i = 1, 2$ when using behavioral strategies (x, y) and (z). Show that,

$$E^1((x, y), z) = (x - z)(y - \tfrac{1}{2})$$

and $E^1((x, y), z) = -E^2((x, y), z)$ for any x, y and z.

Furthermore, consider

$$\max_{x,y} \min_{z} (x - z)(y - \tfrac{1}{2})$$

and show that the every equilibrium solution in behavioral strategies must have $y = \frac{1}{2}$ where

$$E^1((x, \tfrac{1}{2}), z) = -E^2((x, \tfrac{1}{2}), z) = 0.$$

Therefore, using only behavioral strategies, the expected payoff will be $(0, 0)$. If Player 1 is restricted to using only behavioral strategies, he can guarantee, at most, an expected gain of 0. But if he randomizes over all of his pure strategies and stays with that strategy throughout the game, Player 1 can get an expected payoff of $\frac{1}{4}$.

Any behavioral strategy can be expressed as a mixed strategy. But, without perfect recall, not all mixed strategies can be implemented using behavioral strategies.

Theorem 3.4. (Kuhn [2]) *Perfect recall is a necessary and sufficient condition for all mixed strategies to be induced by behavioral strategies.*

A formal proof of this theorem is in [2]. Here is a brief sketch: We would like to know under what circumstances there is a 1-1 correspondence between behavioral and mixed strategies. Suppose a mixed strategy consists of the following mixture of three pure strategies:

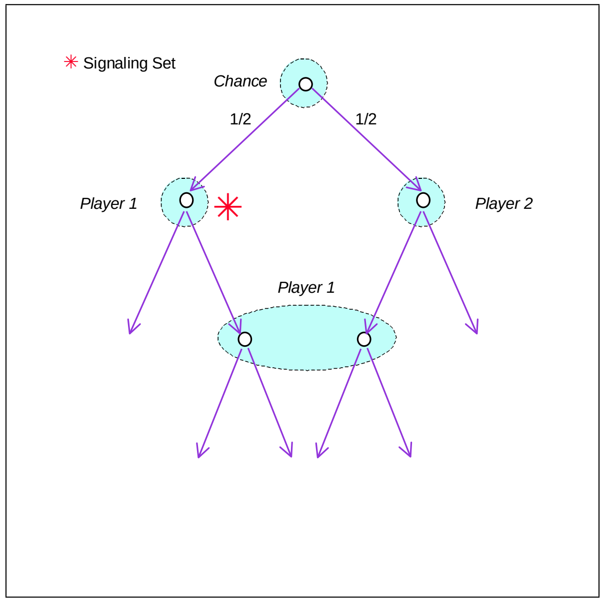
$$\begin{array}{lll} \text{choose} & \gamma_a & \text{with probability } \frac{1}{2} \\ \text{choose} & \gamma_b & \text{with probability } \frac{1}{3} \\ \text{choose} & \gamma_c & \text{with probability } \frac{1}{6} \end{array}$$

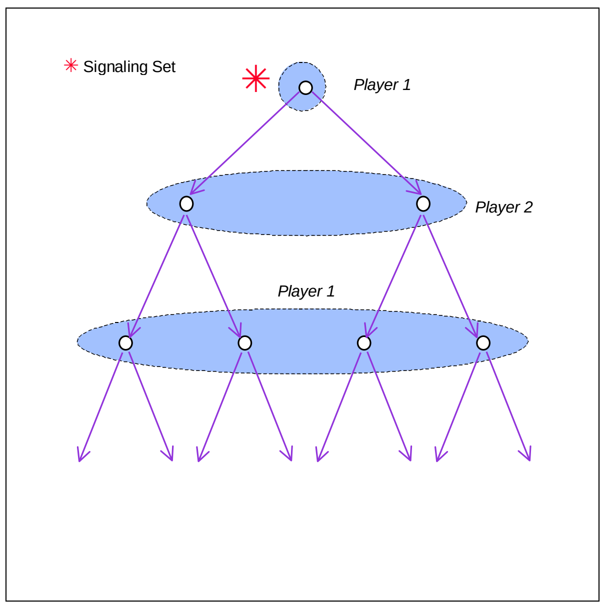
Suppose that strategies γ_b and γ_c lead the game to information set η. Suppose that strategy γ_a does not go to η. If a player is told he is in information η, he can use perfect recall to backtrack completely through the game to learn whether strategy γ_b or γ_c was used. Suppose $\gamma_b(\eta) = u_b$ and $\gamma_c(\eta) = u_c$. Then if the player is in η, he can implement the mixed strategy with the following behavioral strategy:

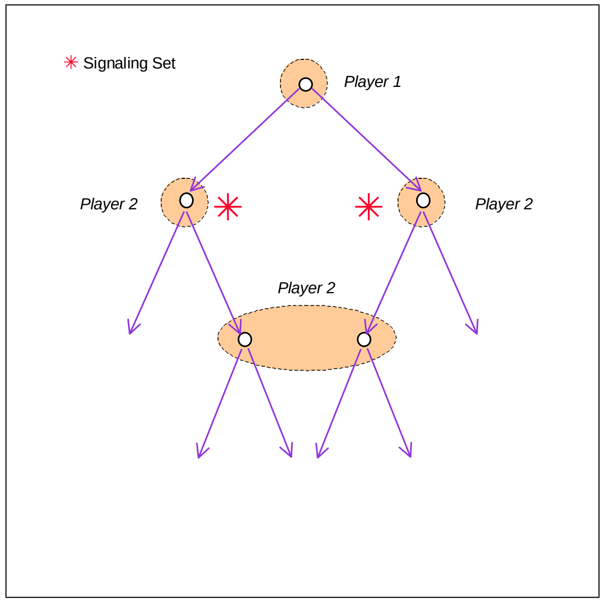
$$\begin{array}{lll} \text{choose} & u_b & \text{with probability } \frac{2}{3} \\ \text{choose} & u_c & \text{with probability } \frac{1}{3} \end{array}$$

A game may not have perfect recall, but some strategies could take the game along paths that, as sub-trees, have the property of perfect recall. Kuhn [2] and Thompson [4] employ the concept of **signaling information sets**. In essence, a signaling information set is that point in the game where a decision by a player could cause him to lose the property of perfect recall.

In the following three games, the signaling information sets are marked with (*):







3.4 Stackelberg solutions

3.4.1 Basic ideas

This early idea in game theory is due to Stackelberg [3]. Its features include:

- hierarchical ordering of the players
- strategy decisions are made and announced sequentially
- one player has the ability to enforce his strategy on others

This approach introduce is notion of a *rational reaction* of one player to another's choice of strategy.

Example 3.3. Consider the bimatrix game

	γ_1^2	γ_2^2	γ_3^2
γ_1^1	$(0,1)$	$(-2,-1)$	$(-\frac{3}{2},-\frac{2}{3})$
γ_2^1	$(-1,-2)$	$(-1,0)$	$(-3,-1)$
γ_2^1	$(1,0)$	$(-2,-1)$	$(-2,\frac{1}{2})$

Note that (γ_2^1, γ_2^2) is a Nash solution with value $(-1,0)$.

Suppose that Player 1 must "lead" by announcing his strategy, first. Is this an advantage or disadvantage? Note that,

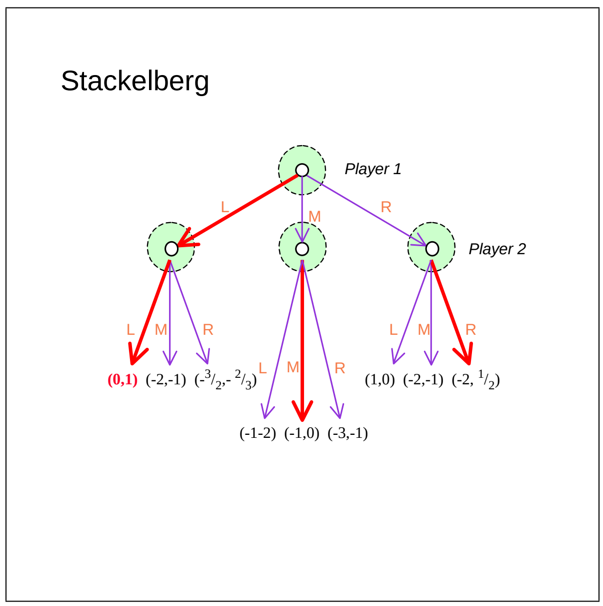
If Player 1 chooses	γ_1^1	Player 2 will respond with	γ_1^2
If Player 1 chooses	γ_2^1	Player 2 will respond with	γ_2^2
If Player 1 chooses	γ_3^1	Player 2 will respond with	γ_3^2

The best choice for Player 1 is γ_1^1 which will yield a value of $(0, 1)$. For this game, the **Stackelberg solution** is an improvement over the Nash solution for *both players.*

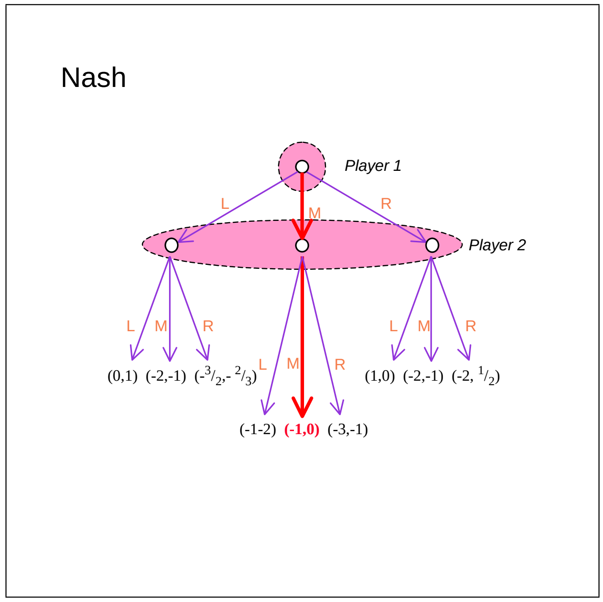
If we let

$$\begin{aligned} \gamma_1^1 &= \textbf{L} & \qquad \gamma_1^2 &= \textbf{L} \\ \gamma_2^1 &= \textbf{M} & \qquad \gamma_2^2 &= \textbf{M} \\ \gamma_3^1 &= \textbf{R} & \qquad \gamma_3^2 &= \textbf{R} \end{aligned}$$

we can implement the Stackelberg strategy by playing the following game in extensive form:



The Nash solution can be obtained by playing the following game:



There may not be a unique response to the leader's strategy. Consider the following example:

	γ_1^2	γ_2^2	γ_3^2
γ_1^1	$(0,0)$	$(-1,0)$	$(-3,-1)$
γ_2^1	$(-2,1)$	$(-2,0)$	$(1,1)$

In this case

If Player 1 chooses	γ_1^1	Player 2 will respond with	γ_1^2 or γ_2^2
If Player 1 chooses	γ_2^1	Player 2 will respond with	γ_1^2 or γ_3^2

One solution approach uses a minimax philosophy. That is, Player 1 should secure his profits against the alternative rational reactions of Player 2. If Player 1 chooses γ_1^1 the least he will obtain is -1, and he chooses γ_2^1 the least he will obtain is -2. So his (minimax) Stackelberg strategy is γ_1^1.

Question 3.4. In this situation, one might consider mixed Stackelberg strategies. How could such strategies be defined, when would they be useful, and how would they be implemented?

Note 3.5. When the follower's response is not unique, a natural solution approach would be to *side-payments*. In other words, Player 1 could provide an incentive to Player 2 to choose an action in Player 1's best interest. Let $\epsilon > 0$ be a small side-payment. Then the players would be playing the Stackelberg game

	γ^2_1	γ^2_2	γ^2_3
γ^1_1	$(-\epsilon, \epsilon)$	$(-1, 0)$	$(-3, -1)$
γ^1_2	$(-2, 1)$	$(-2, 0)$	$(1 - \epsilon, 1 + \epsilon)$

3.4.2 The formalities

Let Γ^1 and Γ^2 denote the sets of pure strategies for Player 1 and Player 2, respectively. Let $J^i(\gamma^1, \gamma^2)$ denote the payoff to Player i if Player 1 chooses strategy $\gamma^1 \in \Gamma^1$ and Player 2 chooses strategy $\gamma^2 \in \Gamma^2$. Let

$$R^2(\gamma^1) \equiv \{\xi \in \Gamma^2 \mid J^2(\gamma^1, \xi) \geq J^2(\gamma^1, \gamma^2) \; \forall \, \gamma^2 \in \Gamma^2\}$$

Note that $R^2(\gamma^1) \subseteq \Gamma^2$ and we call $R^2(\gamma^1)$ the **rational reaction** of Player 2 to Player 1's choice of γ^1. A **Stackelberg strategy** can be formally defined as the $\hat{\gamma}^1$ that solves

$$\min_{\gamma^2 \in R^2(\hat{\gamma}^1)} J^1(\hat{\gamma}^1, \gamma^2) = \max_{\gamma^1 \in \Gamma^1} \min_{\gamma^2 \in R^2(\gamma^1)} J^1(\gamma^1, \gamma^2) = J^{1*}$$

Note 3.6. If $R^2(\gamma^1)$ is a singleton for all $\gamma^1 \in \Gamma^1$ then there exists a mapping

$$\psi^2 : \Gamma^1 \to \Gamma^2$$

such that $R^2(\gamma^1) = \{\gamma^2\}$ implies $\gamma^2 = \psi^2(\gamma^1)$. In this case, the definition of a Stackelberg solution can be simplified to the $\hat{\gamma}^1$ that solves

$$J^1(\hat{\gamma}^1, \psi^2(\gamma^1)) = \max_{\gamma^1 \in \Gamma^1} J^1(\gamma^1, \psi^2(\gamma^1))$$

It is easy to prove the following:

Theorem 3.5. *Every two-person finite game has a Stackelberg solution for the leader.*

Note 3.7. From the follower's point of view, his choice of strategy in a Stackelberg game is always optimal (i.e., the best he can do).

Question 3.5. Let J^{1*} (defined as above) denote the Stackelberg value for the leader Player 1, and let J_N^1 denote any Nash equilibrium solution value for the same player. What is the relationship (bigger, smaller, etc.) between J^{1*} and J_N^1? What additional conditions (if any) do you need to place on the game to guarantee that relationship?

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Lecture Note Set 4

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Friday, March 30, 2001

4 UTILITY THEORY

4.1 Introduction

This section is really independent of the field of game theory, and it introduces concepts that pervade a variety of academic fields. It addresses the issue of quantifying the seemingly nonquantifiable. These include attributes such as quality of life and aesthetics. Much of this discussion has been borrowed from Keeney and Raiffa [1]. Other important references include Luce and Raiffa [2], Savage [4], and von Neumann and Morgenstern [5].

The basic problem of assessing value can be posed as follows: A decision maker must choose among several alternatives, say $W_1, W_2, \ldots, W_n$, where each will result in a consequence discernible in terms of a *single* attribute, say X. The decision maker does not know with certainty which consequence will result from each of the variety of alternatives. We would like to be able to quantify (in some way) our preferences for each alternative.

The literature on utility theory is extensive, both theoretical and experimental. It has been the subject of significant criticism and refinement. We will only present the fundamental ideas here.

4.2 The basic theory

Definition 4.1. *Given any two outcomes A and B we write $A \succ B$ if A is preferable to B. We will write $A \simeq B$ if $A \not\succ B$ and $B \not\succ A$.*

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4.2.1 Axioms

The relations $\succ$ and $\simeq$ must satisfy the following axioms:

1. Given any two outcomes A and B, exactly one of the following must hold:
   (a) $A \succ B$
   (b) $B \succ A$
   (c) $A \simeq B$
2. $A \simeq A$ for all A
3. $A \simeq B$ implies $B \simeq A$
4. $A \simeq B$ and $B \simeq C$ implies $A \simeq C$
5. $A \succ B$ and $B \succ C$ implies $A \succ C$
6. $A \succ B$ and $B \simeq C$ implies $A \succ C$
7. $A \simeq B$ and $B \succ C$ implies $A \succ C$

4.2.2 What results from the axioms

The axioms provide that $\simeq$ is an *equivalence relation* and $\succ$ produces a *weak partial ordering* of the outcomes.

Now assume that

$$A_1 \prec A_2 \prec \cdots \prec A_n$$

Suppose that the decision maker is indifferent to the following two possibilities:

$$\begin{array}{rl} \textbf{Certainty option:} & \text{Receive } A_i \text{ with probability } 1 \\ \textbf{Risky option:} & \left\{ \begin{array}{l} \text{Receive } A_n \text{ with probability } \pi_i \\ \text{Receive } A_1 \text{ with probability } (1-\pi_i) \end{array} \right. \end{array}$$

If the decision maker is consistent, then $\pi_n = 1$ and $\pi_1 = 0$, and furthermore

$$\pi_1 < \pi_2 < \cdots < \pi_n$$

Hence, the π's provide a numerical ranking for the A's.

Suppose that the decision maker is asked express his preference for probability distributions over the A_i. That is, consider mixtures, p' and p'', of the A_i where

$$\begin{array}{ll} p'_i \geq 0 & \sum_{i=1}^n p'_i = 1 \\ p''_i \geq 0 & \sum_{i=1}^n p''_i = 1 \end{array}$$

Using the π's, we can consider the question of which is better, p' or p'', by computing the following "scores":

$$\begin{aligned} \overline{\pi}' &= \sum_{i=1}^n p'_i \pi_i \\ \overline{\pi}'' &= \sum_{i=1}^n p''_i \pi_i \end{aligned}$$

We claim that the choice of p' versus p'' should be based on the relative magnitudes of $\overline{\pi}'$ and $\overline{\pi}''$.

Note 4.1. Suppose we have two outcomes A and B with the probability of getting each equal to p and $(1-p)$, respectively. Denote the *lottery* between A and B by

$$Ap \oplus B(1-p)$$

Note that this is not expected value, since A and B are not real numbers.

Suppose we choose p'. This implies that we obtain A_i with probability p'_i and this is indifferent to obtaining

$$A_n\pi_i \oplus A_1(1-\pi_i)$$

with probability p'_i. Now, sum over all i and consider the quantities

$$\begin{aligned} & A_n \sum_{i=1}^n \pi_i p'_i \oplus A_1 \sum_{i=1}^n (1-\pi_i) p'_i \\ \simeq \quad & A_n \sum_{i=1}^n \pi_i p'_i \oplus A_1 \left(1 - \sum_{i=1}^n \pi_i p'_i\right) \\ \simeq \quad & A_n \overline{\pi}' \oplus A_1(1-\overline{\pi}') \end{aligned}$$

So if $\overline{\pi}' > \overline{\pi}''$ then

$$A_n\overline{\pi}' \oplus A_1(1-\overline{\pi}') \succ A_n\overline{\pi}'' \oplus A_1(1-\overline{\pi}'')$$

This leads directly to the following. . .

Theorem 4.1. *If* $A \succ C \succ B$ *and*

$$pA \oplus (1-p)B \simeq C$$

then $0 < p < 1$ *and* p *is unique.*

Proof: See Owen [3]. ■

Theorem 4.2. *There exists a real-valued function* $u(\cdot)$ *such that*

1. (Monotonicity) $u(A) > u(B)$ *if and only if* $A \succ B$.
2. (Consistency) $u(pA \oplus (1-p)B) = pu(A) + (1-p)u(B)$
3. *the function* $u(\cdot)$ *is unique up to a linear transformation. In other words, if* u *and* v *are utility functions for the same outcomes then* $v(A) = \alpha u(A) + \beta$ *for some* α *and* β.

Proof: See Owen [3]. ■

Consider a lottery (L) which yields outcomes $\{A_i\}_{i=1}^n$ with probabilities $\{p_i\}_{i=1}^n$. Then let

$$\tilde{A} = A_1p_1 \oplus A_2p_2 \oplus \cdots \oplus A_np_n$$

Because of the properties of utility functions, we have

$$E[(u(\tilde{A})] = \sum_{i=1}^{n} p_i u(A_i)$$

Consider

$$u^{-1}\left(E[(u(\tilde{A})]\right)$$

This is an *outcome* that represents lottery (L)

Suppose we have two utility functions u_1 and u_2 with the property that

$$u_1^{-1}\left(E[(u_1(\tilde{A})]\right) \simeq u_2^{-1}\left(E[(u_2(\tilde{A})]\right) \quad \forall\, \tilde{A}$$

Then u^1 and u^2 will imply the same preference rankings for any outcomes. If this is true, we write $u^1 \sim u^2$. Note that some texts (such as [1]) say that u_1 and u_2 are *strategically equivalent.* We won't use that definition, here, because this term has been used for another property of strategic games.

4.3 Certainty equivalents

Definition 4.2. *A* **certainty equivalent** *of lottery (L) is an outcome $\hat{A}$ such that the decision maker is indifferent between (L) and the certain outcome $\hat{A}$.*

In other words, if $\hat{A}$ is a certainty equivalent of (L)

$$\begin{aligned} u(\hat{A}) &= E[u(\tilde{A})] \\ \hat{A} &\simeq u^{-1}\left(E[u(\tilde{A})]\right) \end{aligned}$$

You will also see the terms *cash equivalent* and *lottery selling price* in the literature.

Example 4.1. Suppose outcomes are measured in terms of real numbers, say $A = x$. For any a and $b > 0$

$$u(x) = a + bx \sim x$$

Suppose the decision maker has a lottery described by the probability density $f(x)$ then

$$E[\tilde{x}] = \int x f(x)\, dx$$

Note that

$$u(\hat{x}) = E[u(\tilde{x})] = E[a + b\tilde{x}] = a + bE[\tilde{x}]$$

Taking u^{-1} of both sides shows that $\hat{x} = E[\tilde{x}]$.

Hence, if the utility function is linear, the certainty equivalent for any lottery is the expected consequence of that lottery.

Question 4.1. Suppose $u(x) = a - be^{-cx} \sim -e^{-cx}$ where $b > 0$. Suppose the decision maker is considering a 50-50 lottery yielding either x_1 or x_2. So

$$E[\tilde{x}] = \frac{x_1 + x_2}{2}$$

Find the solution to $u(\hat{x}) = E[u(\tilde{x})]$ to obtaining the certainty equivalent for this lottery. In other words, solve

$$-e^{-c\hat{x}} = \frac{-(e^{-cx_1} + e^{-cx_2})}{2}$$

Question 4.2. This is a continuation of Question 4.1. If $u(x) = -e^{-cx}$ and $\hat{x}$ is the certainty equivalent for the lottery $\tilde{x}$, show that $\hat{x} + x_0$ is the certainty equivalent for the lottery $\widetilde{x + x_0}$.

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Lecture Note Set 5

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Friday, March 30, 2001

5 STATIC COOPERATIVE GAMES

5.1 Some introductory examples

Consider a game with three players 1, 2 and 3. Let $N = \{1, 2, 3\}$ Suppose that the players can freely form coalitions. In this case, the possible *coalition structures* would be

$$\{\{1\}, \{2\}, \{3\}\} \quad \{\{1, 2, 3\}\}$$

$$\{\{1, 2\}, \{3\}\} \quad \{\{1, 3\}, \{2\}\} \quad \{\{2, 3\}, \{1\}\}$$

Once the players form their coalition(s), they inform a referee who pays each coalition an amount depending on its membership. To do this, the referee uses the function $v : 2^N \to \mathbb{R}$. Coalition S receives $v(S)$. This is a game in **characteristic function form** and v is called the **characteristic function.**

For simple games, we often specify the characteristic function without using brackets and commas. For example,

$$v(12) \equiv v(\{1, 2\}) = 100$$

The function v may actually be based on another game or an underlying decision-making problem.

---

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[2]Much of the material for this section has been cultivated from the lecture notes of Louis J. Billera and William F. Lucas. The errors and omissions are mine.

An important issue is the division of the game's proceeds among the players. We call the vector $(x_1, x_2, \ldots, x_N)$ of these payoffs an **imputation**. In many situations, the outcome of the game can be expressed solely in terms of the resulting imputation.

Example 5.1. Here is a three-person, constant sum game:

$$\begin{aligned} &v(123) = 100 \\ &v(12) = v(13) = v(23) = 100 \\ &v(1) = v(2) = v(3) = 0 \end{aligned}$$

How much will be given to each player? Consider solutions such as

$$\begin{aligned} (x_1, x_2, x_3) &= (\tfrac{100}{3}, \tfrac{100}{3}, \tfrac{100}{3}) \\ (x_1, x_2, x_3) &= (50, 50, 0) \end{aligned}$$

Example 5.2. This game is similar to Example 5.1.

$$\begin{aligned} &v(123) = 100 \\ &v(12) = v(13) = 100 \\ &v(23) = v(1) = v(2) = v(3) = 0 \end{aligned}$$

Player 1 has veto power but if Player 2 and Player 3 form a coalition, they can force Player 1 to get nothing from the game. Consider this imputation as a solution:

$$(x_1, x_2, x_3) = (\tfrac{200}{3}, \tfrac{50}{3}, \tfrac{50}{3})$$

5.2 Cooperative games with transferable utility

Cooperative TU (transferable utility) games have the following ingredients:

1. a characteristic function $v(S)$ that gives a value to each subset $S \subset N$ of players
2. payoff vectors called *imputation* of the form $(x_1, x_2, \ldots, x_n)$ which represents a realizable distribution of wealth

3. a preference relation over the set of imputations
4. solution concepts

   Global: *stable sets*

   > solutions outside of the stable set can be blocked by some coalition, and nothing in the stable set can be blocked by another member of the stable set.

   Local: *bargaining sets*

   > any objection to an element of a bargaining set has a counter-objection.

   Single point: *Shapley value*

Definition 5.1. *A* **TU game in characteristic function form** *is a pair* (N, v) *where* $N = \{1, \dots, n\}$ *is the set of players and* $v : 2^N \to \mathbb{R}$ *is the characteristic function.*

Note 5.1. We often assume either that the game is

superadditive: $v(S \cup T) \geq v(S) + v(T)$ for all $S, T \subseteq N$, such that $S \cap T = \emptyset$

or that the game is

cohesive: $v(N) \geq v(S)$ for all $S \subseteq N$

We define the set of *imputations* as

$$A(v) = \{x \mid \textstyle\sum_{i=1}^{n} x_i = v(N) \text{ and } x_i \geq v(\{i\}) \; \forall\, i \in N\} \subset \mathbb{R}^N$$

If $S \subseteq N$, $S \neq \emptyset$ and $x, y \in A(v)$ then we say that x **dominates** y **via** S, ($x \;\mathsf{dom}_S\; y$) if and only if

1. $x_i > y_i$ for all $i \in S$
2. $\sum_{i \in S} x_i \leq v(S)$

If x dominates y via S, we write $x \;\mathsf{dom}_S\; y$.

If $x \;\mathsf{dom}_S\; y$ for some $S \subseteq N$ then we say that x **dominates** y and write $x \;\mathsf{dom}\; y$.

For $x \in A(v)$, we define the **dominion of** x **via** S as

$$\mathsf{Dom}_S\, x \equiv \{y \in A(v) \mid x \;\mathsf{dom}_S\; y\}$$

For any $B \subseteq A(v)$ we define

$$\mathsf{Dom}_S B \equiv \bigcup_{y \in B} \mathsf{Dom}_S\, y$$

and

$$\mathsf{Dom}\, B \equiv \bigcup_{T \subseteq N} \mathsf{Dom}_T\, B$$

We say that $K \subset A(v)$ is a **stable set** if

1. $K \cap \mathsf{Dom}\, K = \emptyset$
2. $K \cup \mathsf{Dom}\, K = A(v)$

In other words, $K = A(v) - \mathsf{Dom}\, K$

The **core** is defined as

$$\mathcal{C} \equiv \{x \in A(v) \mid \textstyle\sum_{i \in S} x_i \geq v(S)\ \forall\, S \subset N\}$$

Note 5.2. If the game is cohesive, the core is the set of undominated imputations.

Theorem 5.1. *The core of a cooperative TU game (N, v) has the following properties:*

1. *The core $\mathcal{C}$ is an intersection of half spaces.*
2. *If stable sets K_α exist, then $\mathcal{C} \subset \cap_\alpha K_\alpha$*
3. $(\cap_\alpha K_\alpha) \cap \mathsf{Dom}\,\mathcal{C} = \emptyset$

Note 5.3. For some games (e.g., constant sum games) the core is empty.

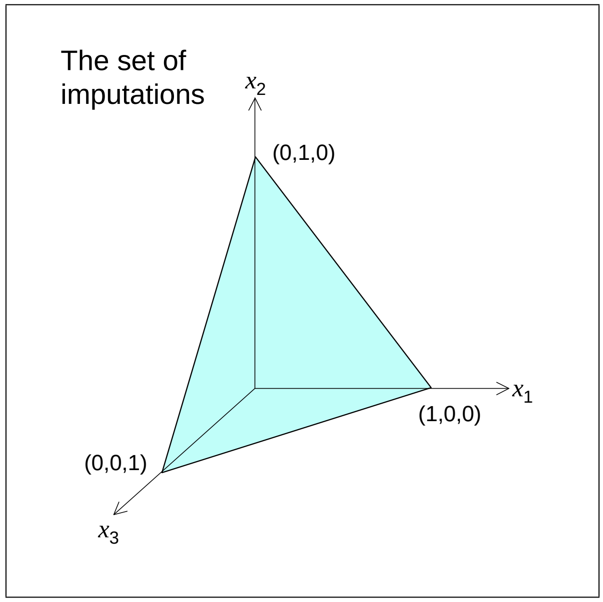
As an example consider the following constant sum game with $N = 3$:

$$\begin{aligned} &v(123) = 1 \\ &v(12) = v(13) = v(23) = 1 \\ &v(1) = v(2) = v(3) = 0 \end{aligned}$$

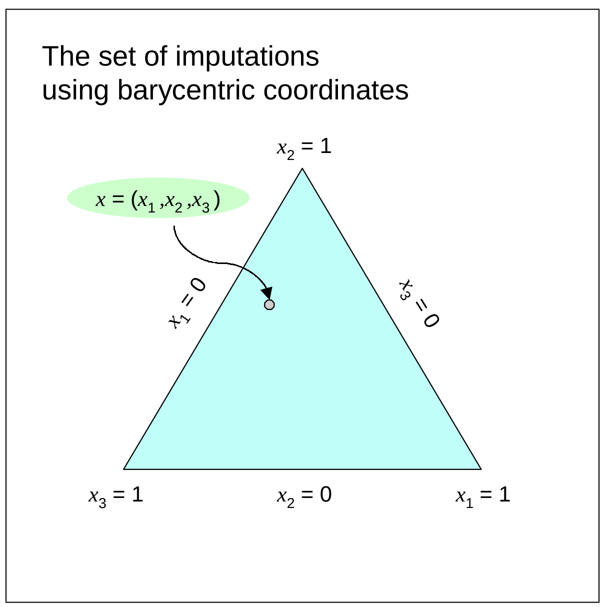
The set of imputations is

$$A(v) = \{x = (x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 1 \text{ and } x_i \geq 0 \text{ for } i = 1, 2, 3\}$$

This set can be illustrated as a subset in $\mathbb{R}^3$ as follows:

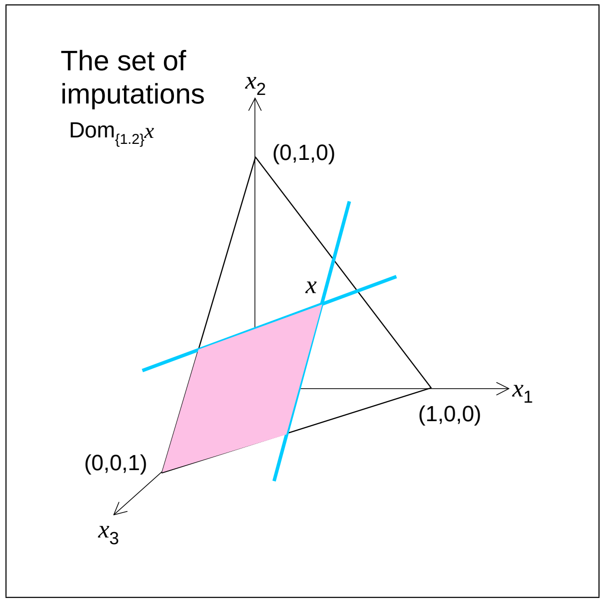


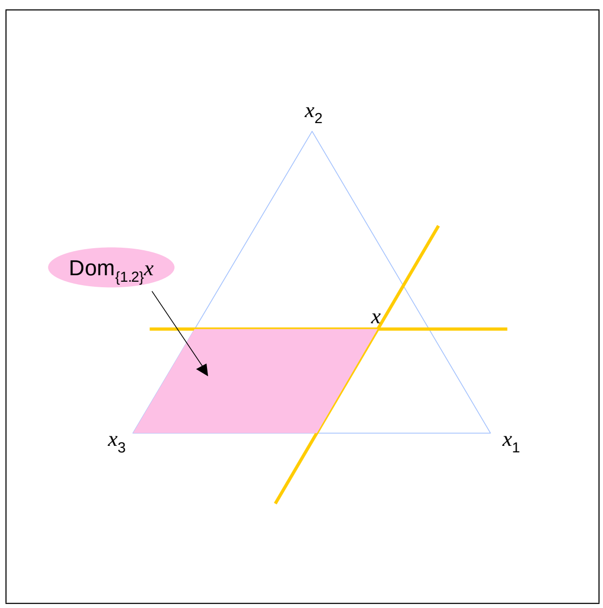
or alternatively, using barycentric coordinates



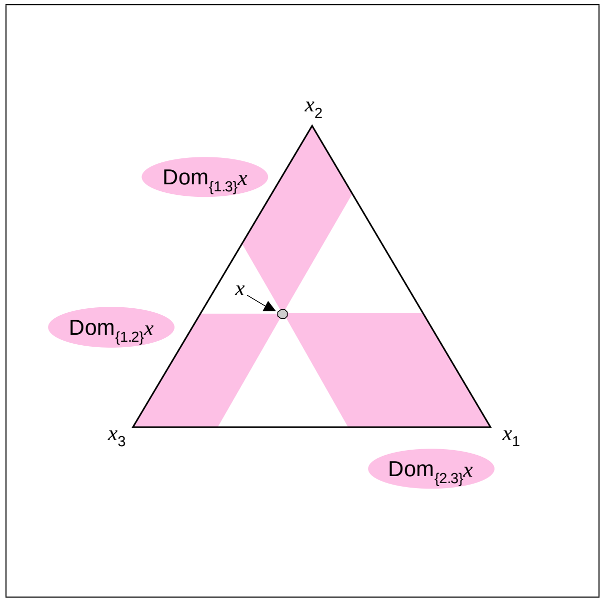
For an interior point x we get

$$\text{Dom}_{\{1,2\}}\, x = A(v) \cap \{y \mid y_1 < x_1 \text{ and } y_2 < x_2\}$$





And for all two-player coalitions we obtain



Question 5.1. Prove that

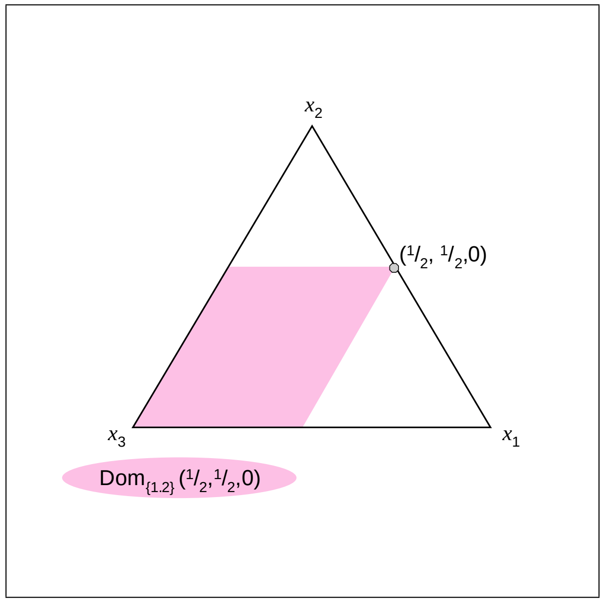
$$
\begin{array}{lrcll}
(1) & \mathrm{Dom}_N A(v) & = & \emptyset & \\
(2) & \mathrm{Dom}_{\{i\}} A(v) & = & \emptyset & \forall\, i \\
(3) & \mathcal{C} & = & \emptyset &
\end{array}
$$

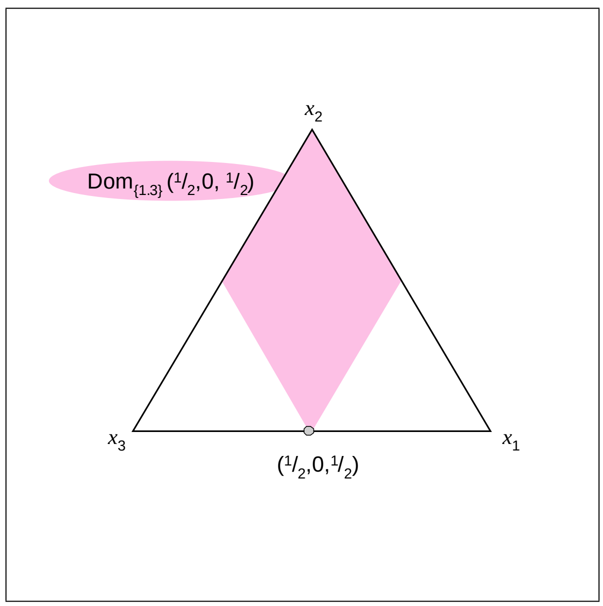
Note that (1) and (2) are general statements, while (3) is true for this particular game.

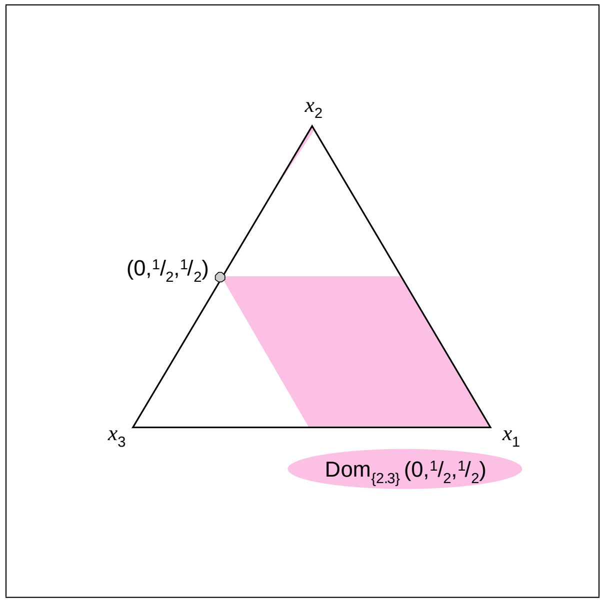
Now consider the set

$$K = \{(\tfrac{1}{2}, \tfrac{1}{2}, 0), (\tfrac{1}{2}, 0, \tfrac{1}{2}), (0, \tfrac{1}{2}, \tfrac{1}{2})\}$$

and note that the sets $\mathrm{Dom}_{\{1,2\}}\,(\frac{1}{2},\frac{1}{2},0)$, $\mathrm{Dom}_{\{1,3\}}\,(\frac{1}{2},0,\frac{1}{2})$, and $\mathrm{Dom}_{\{2,3\}}\,(0,\frac{1}{2},\frac{1}{2})$ can be illustrated as follows:







We will let you verify that

1. $K \cap \mathrm{Dom}\, K = \emptyset$
2. $K \cup \mathrm{Dom}\, K = A(v)$

so that K is a stable set.

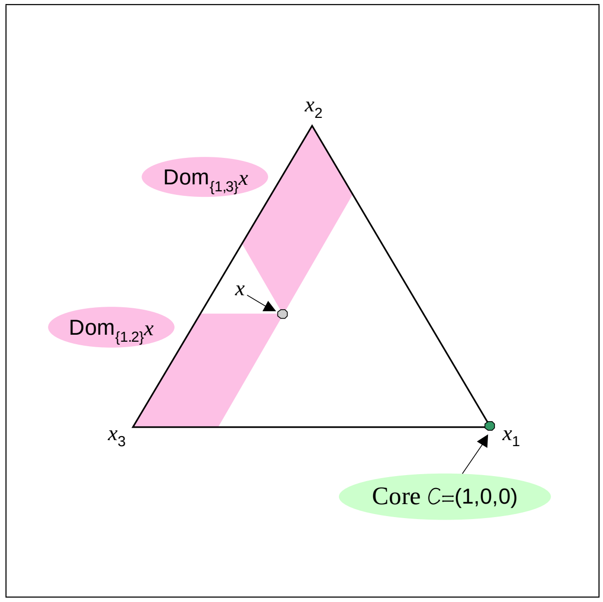
Question 5.2. There are more stable sets (an uncountable collection). Find them, and show that, for this example,

$$\begin{aligned} \cap_\alpha K_\alpha &= \emptyset \\ \cup_\alpha K_\alpha &= A(v) \end{aligned}$$

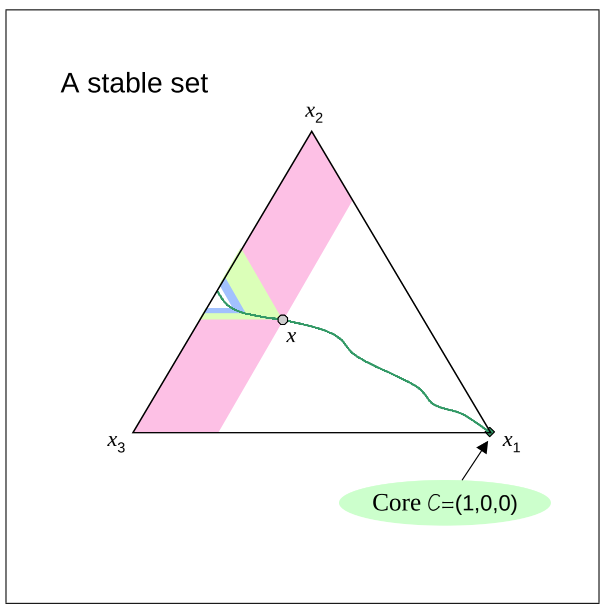
Now, let's look at the veto game:

$$v(123) = 1$$
$$v(12) = v(13) = 1$$
$$v(23) = v(1) = v(2) = v(3) = 0$$

This game has a core at $(1, 0, 0)$ as shown in the following diagram:



Question 5.3. Verify that any continuous curve from $\mathcal{C}$ to the surface $x_2 + x_3 = 1$ with a Lipshitz condition of 30° or less is a stable set.



Note that

$$\begin{aligned} \cap_\alpha K_\alpha &= \mathcal{C} \\ \cup_\alpha K_\alpha &= A(v) \end{aligned}$$

5.3 Nomenclature

Much of this section is from Willick [13].

5.3.1 Coalition structures

A *coalition structure* is any partition of the player set into coalitions Let $N = \{1, 2, \ldots, n\}$ denote the set of n players.

Definition 5.2. *A* **coalition structure***, $\mathcal{P}$, is a partition of N into non-empty sets such that $\mathcal{P} = \{R_1, R_2, \ldots, R_M\}$ where $R_i \cap R_j = \emptyset$ for all $i \neq j$ and $\cup_{i=1}^{M} R_i = N$.*

5.3.2 Partition function form

Let $\mathcal{P}_0 \equiv \{\{1\}, \{2\}, \ldots, \{n\}\}$ denote the singleton coalition structure. The coalition containing all players N is called the *grand coalition*. The coalition structure $\mathcal{P}_N \equiv \{N\}$ is called the *grand coalition structure*.

In *partition function form games*, the value of a coalition, S, can depend on the coalition arrangement of players in $N - S$ (See Lucas and Thrall [11]).

Definition 5.3. *The game (N, v) is a n-***person game in partition function form** *if $v(S, \mathcal{P})$ is a real valued function which assigns a number to each coalition $S \in \mathcal{P}$ for every coalition structure $\mathcal{P}$.*

5.3.3 Superadditivity

A game is superadditive if $v(S \cup T) \geq v(S) + v(T)$ for all $S, T \subseteq N$ such that $S \cap T = \emptyset$.

Most non-superadditive games can be mapped into superadditive games. The following reason is often given: Suppose there exist disjoint coalitions S and T such that

$$v(S \cup T) < v(S) + v(T)$$

Then S and T could secretly form the coalition $S \cup T$ and collect the value $v(S) + v(T)$. The coalition $S \cup T$ would then divide the amount among its total membership.

Definition 5.4. *The game v is said to be the* **superadditive cover** *of the game u if for all $P \subseteq N$,*

$$v(P) = \max_{\mathcal{P}_P^*} \sum_{R \in \mathcal{P}_P^*} u(R)$$

where $\mathcal{P}_P^*$ be a partition of P.

Note 5.4. $\mathcal{P}_P^*$ is a coalition structure restricted to members of P

Note 5.5. A problem with using a superadditive cover is that it requires the ingredient of secrecy. Yet all of the players are assumed to have perfect information.

It also requires a dynamic implementation process. The players need to first decide on their secret alliance, then collect the payoffs as S and T individually, and finally divide the proceeds as $S \cup T$. But characteristic function form games are assumed to be static.

Example 5.3. Consider this three-person game:

$$\begin{aligned}
&u(123) = 1\\
&u(12) = u(13) = u(23) = 1\\
&u(2) = u(3) = 0\\
&u(1) = 5
\end{aligned}$$

Note that (N, u) is not superadditive. The superadditive cover of (N, u) is

$$\begin{aligned}
&v(123) = 6\\
&v(12) = 5\\
&v(13) = 5\\
&v(23) = 1\\
&v(2) = v(3) = 0\\
&v(1) = 5
\end{aligned}$$

We can often relax the requirement of superadditivty and assume only that the grand coalition obtains a value at least as great as the sum of the values of any partition of the grand coalition. Such games are called *cohesive*.

Definition 5.5. *A characteristic function game is said to be* **cohesive** *if*

$$v(N) = \max_{\mathcal{P}} \sum_{P \in \mathcal{P}} v(P).$$

There are important examples of cohesive games. For instance, we will see later that some models of hierarchical organizations produce cohesive games that are not superadditive.

5.3.4 Essential games

Definition 5.6. *A game is* **essential** *if*

$$\sum_{i \in N} v(\{i\}) < v(N)$$

A game is **inessential** *if*

$$\sum_{i \in N} v(\{i\}) \geq v(N)$$

Note 5.6. If $\sum_{i \in N} v(i) > v(N)$ then A(v) = Ø. If $\sum_{i \in N} v(i) = v(N)$ then $A(v) = \{(v(1), v(2), \ldots, v(n))\}$

5.3.5 Constant sum games

Definition 5.7. *A game is a* **constant sum game** *if*

$$v(S) + v(N - S) = v(N) \quad \forall\, S \subset N$$

5.3.6 Strategic equivalence

Definition 5.8. *Two games* (N, v_1) *and* (N, v_2) *are* **strategically equivalent** *if and only if there exist* $c > 0$ *and scalars* $a_1, \ldots, a_n$ *such that*

$$v_1(S) = cv_2(S) + \sum_{i \in S} a_i \quad \forall\, M \subseteq N$$

Properties of strategic equivalence:

1. It's a linear transformation
2. It's an equivalence relation
   - reflexive
   - symmetric
   - transitive

   Hence it partitions the set of games into equivalence classes.
3. It's an isomorphism with respect to dom on $A(v_2) \to A(v_1)$. So, strategic equivalence preserves important solution concepts.

5.3.7 Normalization

Definition 5.9. *A game (N, v) is in $(0,1)$* **normal form** *if*

$$\begin{aligned} v(N) &= 1 \\ v(\{i\}) &= 0 \quad \forall\, i \in N \end{aligned}$$

The set $A(v)$ for a game in $(0,1)$ normal form is a "probability simplex."

Suppose a game is in $(0,1)$ normal form and superadditive, then $0 \leq v(S) \leq 1$ for all $S \subseteq N$.

An essential game (N, u) can be converted to $(0,1)$ normal form by using

$$v(S) = \frac{u(S) - \sum_{i \in S} u(\{i\})}{u(N) - \sum_{i \in N} u(\{i\})}$$

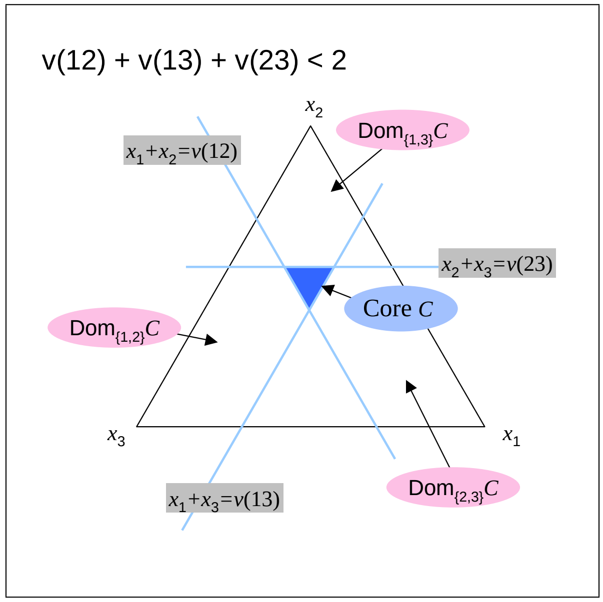
Note that the denominator must be positive for any essential game (N, u).

Note 5.7. For $N = 3$ a game in $(0,1)$ normal form can be completely defined by specifying $(v(12), v(13), v(23))$.

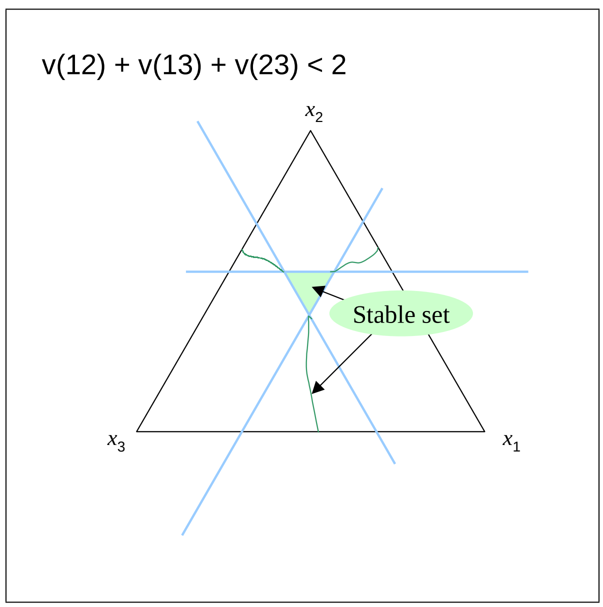
Question 5.4. Show that $\mathcal{C} \neq \varnothing$ for any three-person $(0,1)$ normal form game with

$$v(12) + v(13) + v(23) < 2$$

Here's an example:[2]



Show that stable sets are of the following form:



[2]My thanks to Ling Wang for her suggestions on this section.

Produce similar diagrams for the case $v(12) + v(13) + v(23) > 2$.

Is $\mathcal{C} = \text{Ø}$ for $v(12) + v(13) + v(23) = 2$?

5.4 Garbage game

There are N players. Each player produces one bag of garbage and dumps it in another's yard. The payoff for any player is

$$-1 \times (\text{the number of bags in his yard})$$

We get

$$\begin{aligned} v(N) &= -n \\ v(M) &= |M| - n \quad \text{for } |M| < n \end{aligned}$$

We have $\mathcal{C} = \text{Ø}$ when $n > 2$. To show this, note that $x \in \mathcal{C}$ implies

$$\sum_{i \in N - \{j\}} x_i \geq v(N - \{j\}) = -1 \quad \forall\, j \in N$$

Summing over all $j \in N$,

$$\begin{aligned} (n-1) \sum_{i \in N} x_i &\geq -n \\ (n-1) v(N) &\geq -n \\ (n-1)(-n) &\geq -n \\ n &\leq 2 \end{aligned}$$

5.5 Pollution game

There are n factories around a lake.

Input water is free, but if the lake is dirty, a factory may need to pay to clean the water. If k factories pollute the lake, the cost to a factory to clean the incoming water is kc.

Output water is dirty, but a factory might pay to treat the effluent at a cost of b.

Assume $0 < c < b < nc$.

If a coalition M forms, all of it's members could agree to pollute with a payoff of $|M|(-nc)$. Or, all of it's members could agree to clean the water with a payoff of $|M|(-(n-|M|)c) - |M|b$. Hence,

$$\begin{aligned} v(M) &= \max\{\{|M|(-nc)\}, \{|M|(-(n-|M|)c) - |M|b\}\} \quad \text{for } M \subset N \\ v(N) &= \max\left\{\{-n^2c\}, \{-nb\}\right\} \end{aligned}$$

Question 5.5. Show that $\mathcal{C} \neq \emptyset$ and $x = (-b, \ldots, -b) \in \mathcal{C}$.

5.6 Balanced sets and the core

The presentation in this section is based on Owen [9]

The core $\mathcal{C}$ can be defined as the set of all $(x_1, \ldots, x_n) \in A(V) \subset \mathbb{R}^n$ such that

$$\begin{aligned} \sum_{i \in N} x_i \equiv x(N) &= v(N) \quad \text{and} \\ \sum_{i \in S} x_i \equiv x(S) &\geq v(S) \quad \forall\, S \in 2^N \end{aligned}$$

If we further define an **additive set function** $x(\cdot)$ as any function such that

$$\begin{gathered} x : 2^N \to \mathbb{R} \\ x(S) = \sum_{i \in S} x(\{i\}) \end{gathered}$$

we get the following, equivalent, definition of a core:

Definition 5.10. *The core $\mathcal{C}$ of a game (N, v) is the set of additive $a : 2^N \to \mathbb{R}$ such that*

$$\begin{gathered} a(N) = v(N) \\ a \geq v \end{gathered}$$

The second condition means that $a(S) \geq v(S)$ for all $S \subset N$.

We would like to characterize those characteristic functions v for which the core is nonempty.

Note that $\mathcal{C} \neq \emptyset$ if and only if the linear programming problem

$$(4) \qquad \begin{array}{rl} \min & z = \sum_{i=1}^{n} x_i \\ \text{st:} & \sum_{i \in S} x_i \geq v(S) \quad \forall\, S \subset N \end{array}$$

has a minimum $z^* \leq v(N)$.

Consider the dual to the above linear programming problem (4)

$$(5) \qquad \begin{array}{rlcll} \max & \sum\limits_{S \subset N} y_S v(S) & = & q & \\ \text{st:} & \sum\limits_{S \ni i} y_S & = & 1 & \forall\, i \in N \\ & y_S & \geq & 0 & \forall\, S \subset N \end{array}$$

Both the linear program (4) and its dual (5) are always feasible. So

$$\min z = \max q$$

by the duality theorem. Hence, the core is nonempty if and only if

$$\max q \leq v(N)$$

This leads to the following:

Theorem 5.2. *A necessary and sufficient condition for the game (N, v) to have $\mathcal{C} \neq \emptyset$ is that for every nonnegative vector $(y_S)_{S \subset N}$ satisfying*

$$\sum_{S \ni i} y_S = 1 \quad \forall\, i$$

we have

$$\sum_{S \subset N} y_S v(S) \leq v(N)$$

To make this more useful, we introduce the concept of a *balanced collection* of coalitions.

Definition 5.11. $\mathcal{B} \subset 2^N$ *is* **balanced** *if there exists $y_S \in \mathbb{R}$ with $y_S > 0$ for all $S \in \mathcal{B}$ such that*

$$\sum_{S \ni i} y_S = 1 \quad \forall\, i \in N$$

y is called the **balancing vector** (or weight vector) for $\mathcal{B}$. The individual y_S's are called **balancing coefficients**.

Example 5.4. Suppose $N = \{1, 2, 3\}$

$\mathcal{B} = \{\{1\}, \{2\}, \{3\}\}$ is a balanced collection with $y_{\{1\}} = 1$, $y_{\{2\}} = 1$, and $y_{\{3\}} = 1$.

$\mathcal{B} = \{\{1, 2\}, \{3\}\}$ is a balanced collection with $y_{\{1,2\}} = 1$ and $y_{\{3\}} = 1$.

$\mathcal{B} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ is a balanced collection with $y_{\{1,2\}} = \frac{1}{2}$, $y_{\{1,3\}} = \frac{1}{2}$, and $y_{\{2,3\}} = \frac{1}{2}$.

Theorem 5.3. *The union of balanced collections is balanced.*

Lemma 5.1. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be balanced collections such that $\mathcal{B}_1 \subset \mathcal{B}_2$ but $\mathcal{B}_1 \neq \mathcal{B}_2$. Then there exists a balanced collection $\mathcal{B}_3 \neq \mathcal{B}_2$ such that $\mathcal{B}_3 \cup \mathcal{B}_1 = \mathcal{B}_2$.*

The above lemma leads us to define the following:

Definition 5.12. *A* **minimal balanced collection** *is a balanced collection for which no proper subcollection is balanced.*

Theorem 5.4. *Any balanced collection can be written as the union of minimal balanced collections.*

Theorem 5.5. *Any balanced collection has a unique balancing vector if and only if it is a minimal balanced collection.*

Theorem 5.6. *Each extreme point of the polyhedron for the dual linear programming problem (5) is the balancing vector of a minimal balanced collection.*

Corollary 5.1. *A minimal balanced collection has at most n sets.*

The result is the following theorem:

Theorem 5.7. (Shapley-Bondareva) *The core is nonempty if and only if for every minimal balanced collection $\mathcal{B}$ with balancing coefficients $(y_S)_{S\in\mathcal{B}}$ we have*

$$v(N) \geq \sum_{s\in\mathcal{B}} y_S v(S)$$

Example 5.5. Let $N = \{1, 2, 3\}$. Besides the partitions, such as $\{\{1, 2\}, \{3\}\}$, there is only one other minimal balanced collection, namely,

$$\mathcal{B} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$$

with

$$y = \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right)$$

Therefore a three-person game (N, v) has a nonempty core if and only if

$$\begin{aligned} \tfrac{1}{2}v(\{1,2\}) + \tfrac{1}{2}v(\{1,3\}) + \tfrac{1}{2}v(\{2,3\}) &\leq v(N) \\ v(\{1,2\}) + v(\{1,3\}) + v(\{2,3\}) &\leq 2v(N) \end{aligned}$$

Question 5.6. Use the above result and reconsider Question 5.4 on page 5-14.

Question 5.7. Suppose we are given $v(S)$ for all $S \neq N$. What is the smallest value of $v(N)$ such that $\mathcal{C} \neq \emptyset$?

5.7 The Shapley value

Much of this section is from Yang [14].

Definition 5.13. *A* **carrier** *for a game (N, v) is a coalition $T \subseteq N$ such that $v(S) \leq v(S \cap T)$ for any $S \subseteq N$.*

This definition is slightly different from the one given by Shapley [10]. Shapley uses $v(S) = v(S \cap T)$ instead of $v(S) \leq v(S \cap T)$. However, when the game (N, v) is superadditive, the definitions are equivalent.

A carrier is a group of players with the ability to benefit the coalitions they join. A coalition can remove any of its members who do not belong to the carrier and get the same, or greater value.

Let $\Pi(N)$ denote the set of all permutations on N, that is, the set of all one-to-one mappings from N onto itself.

Definition 5.14. (Owen [9]) *Let (N, v) be an n-person game, and let $\pi \in \Pi(N)$. Then, the game $(N, \pi v)$ is defined as the game (N, u), such that*

$$u(\{\pi(i_1), \pi(i_2), \ldots, \pi(i_{|S|})\}) = v(S)$$

for any coalition $S = \{i_1, i_2, \ldots, i_{|S|}\}$.

Definition 5.15. (Friedman [3]) *Let (N, v) be an n-person game. The* **marginal value,** *$c_S(v)$, for coalition $S \subseteq N$ is given by*

$$c_{\{i\}}(v) \equiv v(\{i\})$$

for all $i \in N$, and

$$c_S(v) \equiv v(S) - \sum_{L \subset S} c_L(v)$$

for all $S \subseteq N$ with $|S| \geq 2$.

The marginal value of S can also be computed by using the formula

$$c_S(v) = \sum_{L \subseteq S} (-1)^{|S|-1} v(L).$$

5.7.1 The Shapley axioms

Let $\phi(v) = (\phi_1(v), \phi_2(v), \ldots, \phi_n(v))$ be an n-dimensional vector satisfying the following three axioms:

Axiom S 1. (Symmetry) *For each $\pi \in \Pi(N)$, $\phi_{\pi(i)}(\pi v) = \phi_i(v)$.*

Axiom S 2. (Rationing) *For each carrier C of (N, v)*

$$\sum_{i \in C} \phi_i(v) = v(C).$$

Axiom S 3. (Law of Aggregation) *For any two games (N, v) and (N, w)*

$$\phi(v + w) = \phi(v) + \phi(w).$$

Theorem 5.8. (Shapley [10]) *For any superadditive game (N, v) there is a unique vector of values $\phi(v) = (\phi_1(v), \ldots, \phi_n(v))$ satisfying the above three axioms. Moreover, for each player i this value is given by*

$$\phi_i(v) = \sum_{\substack{S \subseteq N \\ S \ni i}} \frac{1}{|S|} c_S(v) \tag{6}$$

Note 5.8. The Shapley value can be equivalently written [9] as

$$\phi_i(v) = \sum_{\substack{T \subseteq N \\ T \ni i}} \left(\frac{(|T| - 1)!(n - |T|)!}{n!} \right) [v(T) - v(T - \{i\})] \tag{7}$$

This formula can be interpreted as follows: Suppose n players arrive one after the other into a room that will eventually contain the grand coalition. Consider all possible sequencing arrangements of the n players. Suppose that any sequence can occur with probability $\frac{1}{n!}$. If Player i arrives and finds coalition $T - \{i\}$ already in the room, his contribution to the coalition is $v(T) - v(T - \{i\})$. The Shapley value is the expected value of the contribution of Player i.

5.8 A generalization of the Shapley value

Suppose we introduce the concept of taxation (or resource redistribution) and relax just one of the axioms. Yang [14], has shown that the Shapley value and the **egalitarian value**

$$\phi_i^0(v) = \frac{v(N)}{n} \quad \forall\, i \in N$$

are then the extremes of an entire family of values for all cohesive (not necessarily superadditive) games.

Axiom Y 1. (Symmetry) *For each* $\pi \in \Pi(N)$, $\psi_{\pi(i)}(\pi v) = \psi_i(v)$.

Axiom Y 2. (Rationing) *For each carrier* C *of* (N, v)

$$\sum_{i \in C} \psi_i(v) = g(C)v(C) \quad \textit{with } \frac{|C|}{n} \leq g(C) \leq 1.$$

Axiom Y 3. (Law of Aggregation) *For any two games* (N, v) *and* (N, w)

$$\psi(v + w) = \psi(v) + \psi(w).$$

Note that Yang only modifies the second axiom. The function $g(C)$ is called the *rationing function*. It can be any real-valued function defined on attributes of the carrier C with range $\left[\frac{|C|}{n}, 1\right]$. If the game (N, v) is superadditive, then $g(C) = 1$ yields Shapley's original axioms.

A particular choice of the rationing function $g(C)$ produces a convex combination between the egalitarian value and the Shapley value. Let $N = \{1, \ldots, n\}$ and let $c \equiv |C|$ for $C \subseteq N$. Given the value of the parameter $r \in [\frac{1}{n}, 1]$ consider the real-valued function

$$g(C) \equiv g(c, r) = \frac{(n - c)r + (c - 1)}{n - 1}.$$

The function $g(C)$ specifies the distribution of revenue among the players of a game.

Note that this function can be rewritten as

$$g(c, r) = 1 - (1 - r)\left(\frac{n - c}{n - 1}\right).$$

For games with a large number of players,

$$\lim_{n \to \infty} g(c, r) = r \in (0, 1]$$

so that $(1 - r)$ can be regarded as a "tax rate" on carriers.

Using this form of the rationing function results in the following:[3]

Theorem 5.9. *Let (N, v) be a cohesive n-person cooperative transferable utility game. For each $r \in [\frac{1}{n}, 1]$, there exists a unique value, $\psi_{i,r}(v)$, for each Player i satisfying the three axioms. Moreover, this unique value is given by*

$$\psi_{i,r}(v) = (1 - p)\phi_i(v) + p\frac{v(N)}{n} \quad \forall\, i \in N \tag{8}$$

where $p = \dfrac{n - nr}{n - 1} \in (0, 1)$.

Note that the rationing function can be written[4] in terms of $p \in (0, 1)$ as

$$g(c, p) = p + (1 - p)\frac{c}{n}$$

Example 5.6. Consider a two-person game with

$$v(\{1\}) = 1, \quad v(\{2\}) = 0, \quad v(\{1, 2\}) = 2$$

Player 2 can contribute 1 to a coalition with Player 1. But, Player 1 can get 1 on his own, leaving Player 2 with nothing.

The family of values is

$$\psi_r(v) = \left(\frac{1}{2} + r, \frac{3}{2} - r\right)$$

for $\frac{1}{2} \leq r \leq 1$. The Shapley value (with $r = 1$) is $\left(\frac{3}{2}, \frac{1}{2}\right)$.

[3] We are indebted to an anonymous reviewer for the simplified version of this theorem.

[4] Once again, our thanks to the same anonymous reviewer for this observation.

Example 5.7. Consider a modification of the above game in Example (5.6) with

$$v(\{1\}) = 1, \quad v(\{2\}) = 0, \quad v(\{1,2\}) = 1$$

In this case, Player 2 is a dummy player.

The family of values is

$$\psi_r(v) = (r, 1-r)$$

for $\frac{1}{2} \leq r \leq 1$. The Shapley value (with $r = 1$) is $(1, 0)$.

Example 5.8. This solution approach can be applied to a problem suggested by Nowak and Radzik [8]. Consider a three-person game where

$$v(\{1\}) = v(\{2\}) = 0, \quad v(\{3\}) = 1,$$
$$v(\{1,2\}) = 3.5, \quad v(\{1,3\}) = v(\{2,3\}) = 0,$$
$$v(\{1,2,3\}) = 5.$$

The Shapley value for this game is

$$\phi(v) = \left(\tfrac{25}{12}, \tfrac{25}{12}, \tfrac{10}{12}\right).$$

Note that the Shapley value will not necessarily satisfy the condition of *individual rationality*

$$\phi_i(v) \geq v(\{i\})$$

when the characteristic function v is not superadditive. That is the case here since $\phi_3(v) < v(\{3\})$.

The *solidarity value* (Nowak and Radzik [8]) $\xi(v)$ of this game is

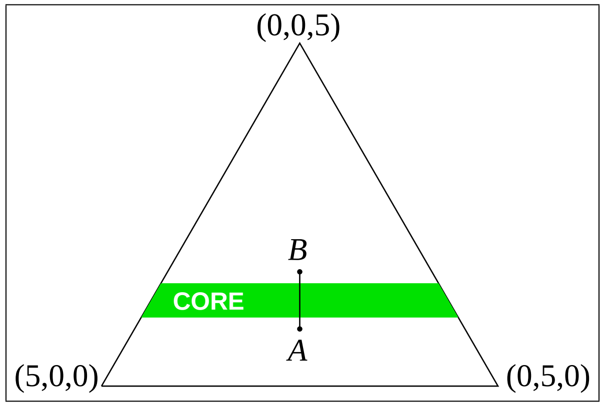
$$\xi(v) = \left(\tfrac{16}{9}, \tfrac{16}{9}, \tfrac{13}{9}\right)$$

and is in the core of (N, v).

For every $r \in [\frac{1}{n}, 1]$, the general form of the family of values is

$$\psi_r(v) = \left(\frac{35 + 15r}{24}, \frac{35 + 15r}{24}, \frac{50 - 30r}{24}\right).$$

The diagram in the following figure shows the relationship between the family of values and the core.



Note that, in the diagram,

$$
\begin{aligned}
A &= \left(\tfrac{25}{12}, \tfrac{25}{12}, \tfrac{10}{12}\right) \quad \text{(the Shapley value)} \\
B &= \left(\tfrac{5}{3}, \tfrac{5}{3}, \tfrac{5}{3}\right).
\end{aligned}
$$

Neither of these extreme values of the family of values is in the core for this game. However, those solutions for $\frac{7}{15} \leq r \leq \frac{13}{15}$ are elements of the core.

Example 5.9. Nowak and Radzik [8] offer the following example related to social welfare and income redistribution: Players 1, 2, and 3 are brothers living together. Players 1 and 2 can make a profit of one unit, that is, $v(\{1,2\}) = 1$. Player 3 is a disabled person and can contribute nothing to any coalition. Therefore, $v(\{1,2,3\}) = 1$. Also, $v(\{1,3\}) = v(\{2,3\}) = 0$ and $v(\{i\}) = 0$ for every Player i.

Shapley value of this game is

$$\phi(v) = \left(\tfrac{1}{2}, \tfrac{1}{2}, 0\right)$$

and for the family of values, we get

$$\psi_r(v) = \left(\frac{1+r}{4}, \frac{1+r}{4}, \frac{1-r}{2}\right)$$

for $r \in \left[\frac{1}{3}, 1\right]$. Every r yields a solution satisfying individual rationality, but, in this case, $\psi_r(v)$ belongs to the core only when it equals the Shapley value ($r = 1$).

For this particular game, the solidarity value is a member of the family when $r = \frac{5}{9}$. Nowak and Radzik propose this single value as a "better" solution for the game (N, v) than its Shapley value. They suggest that it could be used to include subjective social or psychological aspects in a cooperative game.

Question 5.8. Suppose game (N, v) has core $\mathcal{C} \neq \emptyset$. Let

$$\mathcal{F} \equiv \{\psi_r(v) \,|\, \frac{1}{n} \leq r \leq 1\}$$

denote the set of Yang's values when using rationing function $g(c, r)$. Under what conditions will $\mathcal{C} \cap \mathcal{F} \neq \emptyset$?

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IE675 Game Theory

Lecture Note Set 6

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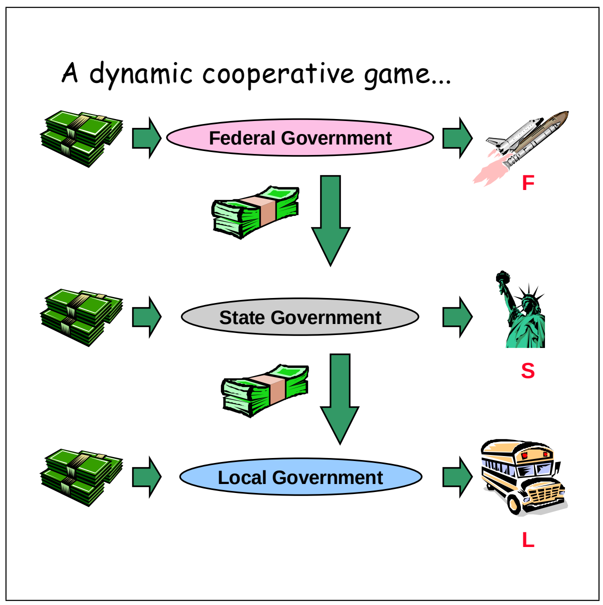
Tuesday, April 3, 2001

6 DYNAMIC COOPERATIVE GAMES

6.1 Some introductory examples

Consider the following hierarchical game:



In this particular example,

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1. The system has interacting players within a hierarchical structure
2. Each player executes his polices after, and with full knowledge of, the decisions of predecessors.
3. Players might form coalitions in order to improve their payoff.

What do we mean by (3)?

For examples (without coalitions) see Cassidy, *et al.* [12] and Charnes, *et al.* [13].

Without coalitions:

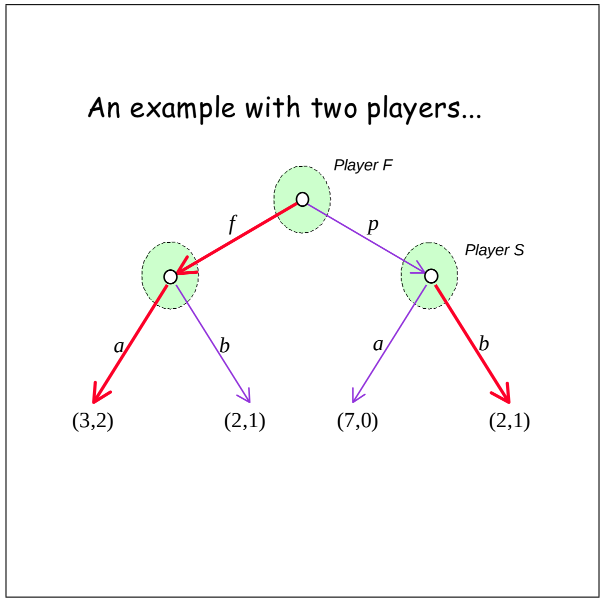
$$\begin{aligned} \text{Payoff to Federal government} &= g_F(F,S,L) \\ \text{Payoff to State government} &= g_S(F,S,L) \\ \text{Payoff to Local government} &= g_L(F,S,L) \end{aligned}$$

A coalition structure of $\{\{F,S\},\{L\}\}$ would result in the players maximizing the following objective functions:

$$\begin{aligned} \text{Payoff to Federal government} &= g_F(F,S,L) + g_S(F,S,L) \\ \text{Payoff to State government} &= g_F(F,S,L) + g_S(F,S,L) \\ \text{Payoff to Local government} &= g_L(F,S,L) \end{aligned}$$

The order of the play remains the same. Only the objectives change.

Here is a two-player game of the same type, but written in extensive form:



where

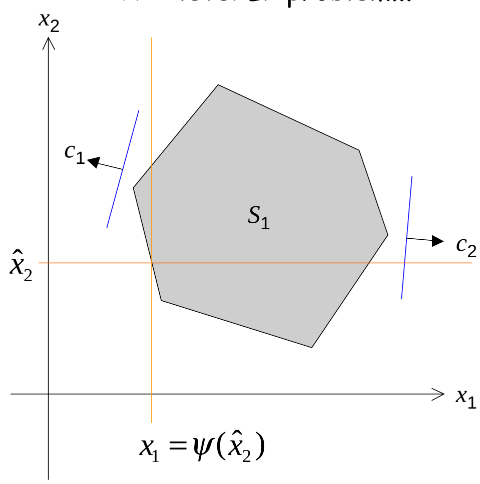
f	Full funding
p	Partial funding
a	Project a
b	Project b

The Stackelberg solution to this game is (f, a) with a payoff of $(3, 1)$. However, if the players cooperated, and utility was transferable, they could get 7 with strategy (p, a).

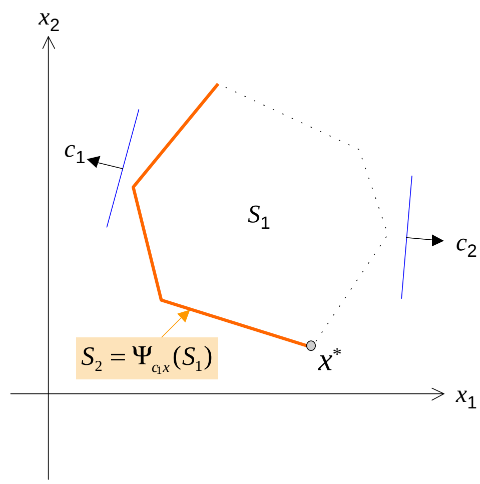
The key element causing this effect is preemption. A dynamic, cooperative model is needed.

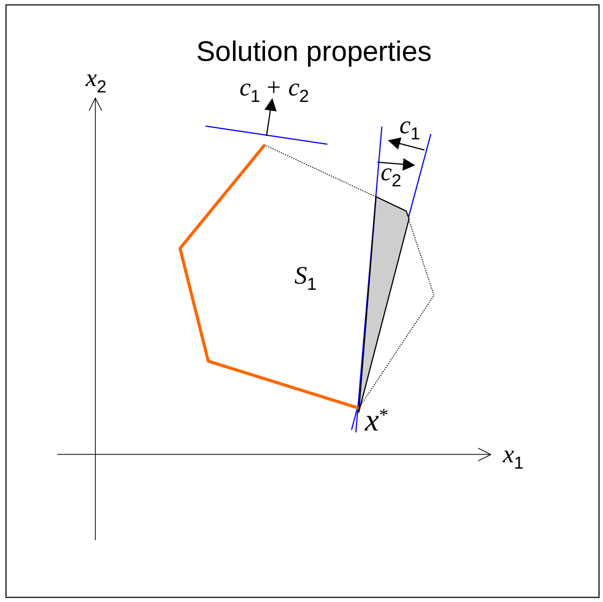
Chew [14] showed that even linear models can exhibit this behavior *and* he developed a dynamic cooperative game model.

A 2-level LP problem...



Rational reactions





6.1.1 Issues

See Bialas and Karwan [4] for details.

1. alternate optimal solutions
2. nonconvex feasible region

Note 6.1. The cause of inadmissible solutions is not the fault of the optimizers, but, rather, the sequential and preemptive nature of the decision process (i.e., the "friction of space and time").

6.2 Multilevel mathematical programming

The non-cooperative model in this section will serve as the foundation for our cooperative dynamic model. See also Bialas and Karwan [6].

Note 6.2. *Some history:* Sequential optimization problems arise frequently in many fields, including economics, operations research, statistics and control theory. The origin of this class of problems is difficult to trace since it is woven into the fabric of many scientific disciplines.

For the field of operations research, this topic arose as an extension to linear programming (see, for example, Bracken and McGill [8] Cassidy, *et al.* [12], Charnes, *et al.* [13])

In particular, Bracken, *et al.* [8, 7, 9] define a two-level problem where the constraints contain an optimization problem. However, the feasible region of the lower level planner does not depend on the decision variables to the upper-level planner. Removing this restriction, Candler and Norton [10] named this class of problems "multilevel programming." A number of researchers mathematically characterized the geometry of this problem and developed solution algorithms (see, for example, [1, 4, 5, 15]).

For a more complete bibliography, see Vicente and Calamai [18].

Let the decision variable space (Euclidean n-space), $\mathbb{R}^n \ni x = (x_1, x_2, \ldots, x_n)$, be partitioned among r levels,

$$\mathbb{R}^{n_k} \ni x^k = (x_1^k, x_2^k, \ldots, x_{n_k}^k) \quad \text{for } k = 1, \ldots, r,$$

where $\sum_{k=1}^{r} n_k = n$. Denote the maximization of a function $f(x)$ over $\mathbb{R}^n$ by varying only $x^k \in \mathbb{R}^{n_k}$ given fixed $x^{k+1}, x^{k+2}, \ldots, x^r$ in $\mathbb{R}^{n_{k+1}} \times \mathbb{R}^{n_{k+2}} \times \cdots \times \mathbb{R}^{n_r}$ by

$$\max\{f(x) : (x^k \,|\, x^{k+1}, x^{k+2}, \ldots, x^r)\}. \tag{1}$$

Note 6.3. The value of expression (1) is a function of $x^1, x^2, \ldots, x^{k-1}$.

Let the full set of system constraints for all levels be denoted by S. Then the problem at the lowest level of the hierarchy, level one, is given by

$$(P^1)\begin{cases} \max & \{f_1(x) : (x^1 \,|\, x^2, \ldots, x^r)\} \\ \text{st:} & x \in S^1 = S \end{cases}$$

Note 6.4. The problem for the level-one decision maker P^1 is simply a (traditional) mathematical programming problem dependent on the given values of $x^2, \ldots, x^r$. That is, P^1 is a parametric programming problem.

The feasible region, $S = S^1$, is defined as the **level-one feasible region.** The solutions to P^1 in $\mathbb{R}_1^n$ for each fixed $x^2, x^3, \ldots, x^r$ form a set,

$$S^2 = \{\hat{x} \in S^1 : f_1(\hat{x}) = \max\{f_1(x) : (x^1 \,|\, \hat{x}^2, \hat{x}^3, \ldots, \hat{x}^r)\}\},$$

called the **level-two feasible region** over which $f_2(x)$ is then maximized by varying x^2 for fixed $x^3, x^4, \ldots, x^r$.

Thus the problem at level two is given by

$$(P^2)\begin{cases} \max & \{f_2(x) \,:\, (x^2 \,|\, x^3, x^4, \ldots, x^r)\} \\ \text{st:} & x \in S^2 \end{cases}$$

In general, the **level-k feasible region** is defined as

$$S^k = \{\hat{x} \in S^{k-1} \,|\, f_{k-1}(\hat{x}) = \max\{f_{k-1}(x) \,:\, (x^{k-1} \,|\, \hat{x}^k, \ldots, \hat{x}^r)\}\},$$

Note that $\hat{x}^{k-1}$ is a function of $\hat{x}^k, \ldots, \hat{x}^r$. Furthermore, the problem at each level can be written as

$$(P^k)\begin{cases} \max & \{f_k(x) \,:\, (x^k \,|\, x^{k+1}, \ldots, x^r)\} \\ \text{st:} & x \in S^k \end{cases}$$

which is a function of $x^{k+1}, \ldots, x^r$, and

$$(P^r) :\ \max_{x \in S^r} f_r(x)$$

defines the entire problem. This establishes a collection of nested mathematical programming problems $\{P^1, \ldots, P^r\}$.

Question 6.1. P^k depends on given $x^{k+1}, \ldots, x^r$, and only x^k is varied. But $f^k(x)$ is defined over all $x^1, \ldots, x^r$. Where are the variables $x^1, \ldots, x^{k-1}$ in problem P^k?

Note that the objective at level k, $f_k(x)$, is defined over the decision space of all levels. Thus, the level-k planner may have his objective function determined, in part, by variables controlled at other levels. However, by controlling x^k, after decisions from levels $k+1$ to r have been made, level k may influence the policies at level $k-1$ and hence all lower levels to improve his own objective function.

6.2.1 A more general definition

See also Bialas and Karwan [5].

Let the vector $x \in \mathbb{R}^N$ be partitioned as (x^a, x^b). Then we can define the following set function over the collection of closed and bounded regions $S \subset \mathbb{R}^N$:

$$\Psi_f(S) = \{\hat{x} \in S \,:\, f(\hat{x}) = \max\{f(x) \,|\, (x^a \,|\, \hat{x}^b)\}\}$$

as the **set of rational reactions** of f over S. This set is also sometimes called the *inducible region.* If for a fixed $\hat{x}^b$ there exists a unique $\hat{x}^a$ which maximizes $f(x^a, \hat{x}^b)$ over all $(x^a, \hat{x}^b) \in S$, then there induced a mapping

$$\hat{x}^a = \psi_f(\hat{x}^b)$$

which provides the rational reaction for each $\hat{x}^b$, and we can then write

$$\boldsymbol{\Psi}_f(S) = S \cap \{(x^a, x^b) \,:\, x^a = \psi_f(x^b)\}$$

So if $S = S^1$ is the level-one feasible region, the level-two feasible region is

$$S^2 = \boldsymbol{\Psi}_{f_1}(S^1)$$

and the level-k feasible region is

$$S^k = \boldsymbol{\Psi}_{f_{k-1}}(S^{k-1})$$

Note 6.5. Even if S^1 is convex, $S^k = \boldsymbol{\Psi}_{f_{k-1}}(S^{k-1})$ for $k \geq 2$ are typically non-convex sets.

6.2.2 The two-level linear resource control problem

The two-level linear resource control problem is the multilevel programming problem of the form

$$\begin{array}{rl} \text{max} & c^2 x \\ \text{st:} & x \in S^2 \end{array}$$

where

$$S^2 = \{\hat{x} \in S^1 \,:\, c^1\hat{x} = \max\{c^1 x \,:\, (x^1 \,|\, \hat{x}^2)\}\}$$

and

$$S^1 = S = \{x \,:\, A^1 x^1 + A^2 x^2 \leq b,\ x \geq 0\}$$

Here, level 2 controls x^2 which, in turn, varies the resource space of level one by restricting $A^1 x^1 \leq b - A^2 x^2$.

The nested optimization problem can be written as:

$$(P^2)\left\{\begin{array}{ll} \text{max} & \{c^2 x = c^{21}x^1 + c^{22}x^2 \,:\, (x^2)\} \\ & \text{where } x^1 \text{ solves} \\ & \\ & (P^1)\left\{\begin{array}{rl} \text{max} & \{c^1 x = c^{11}x^1 + c^{12}x^2 \,:\, (x^1 \,|\, x^2)\} \\ \text{st:} & A^1 x^1 + A^2 x^2 \leq b \\ & x \geq 0 \end{array}\right. \end{array}\right.$$

Question 6.2. Suppose someone gives you a proposed solution x^* to problem P^2. Develop an "easy" way to test that x^* is, in fact, the solution to P^2.

Question 6.3. What is the solution to P^2 if $c^1 = c^2$. What happens if c^1 is substituted with $\alpha c^1 + (1-\alpha)c^2$ for some $0 \leq \alpha \leq 1$?

6.2.3 The two-level linear price control problem

The two-level linear price control problem is another special case of the general multilevel programming problem. In this problem, level two controls the cost coefficients of level one:

$$(P^2)\begin{cases} \max & \{c^2 x = c^{21}x^1 + c^{22}x^2 \;:\; (x^2)\} \\ \text{st:} & A^2x^2 \leq b^2 \\ & \text{where } x^1 \text{ solves} \\ & \\ & (P^1)\begin{cases} \max & \{(x^2)^{\mathsf{t}}x^1 \;:\; (x^1 \,|\, x^2)\} \\ \text{st:} & A^1x^1 \leq b^1 \\ & x^1 \geq 0 \end{cases} \end{cases}$$

In this problem, level two controls the cost coefficients of level one.

6.3 Properties of S^2

Theorem 6.1. *Suppose $S^1 = \{x \;:\; Ax = b,\; x \geq 0\}$ is bounded. Let*

$$S^2 = \{\hat{x} = (\hat{x}^1, \hat{x}^2) \in S^1 \;:\; c^1\hat{x}^1 = \max\{c^1x^1 \;:\; (x^1 \,|\, \hat{x}^2)\}\}$$

then the following hold:

(i) $S^2 \subseteq S^1$

(ii) Let $\{y_t\}_{t=1}^{\ell}$ be any ℓ points of S^1, such that $x = \sum_t \lambda_{t=1}^{\ell} y_t \in S^2$ with $\lambda_t \geq 0$ and $\sum_t \lambda_t = 1$. Then $\lambda_t > 0$ implies $y_t \in S^2$.

Proof: See Bialas and Karwan [4].

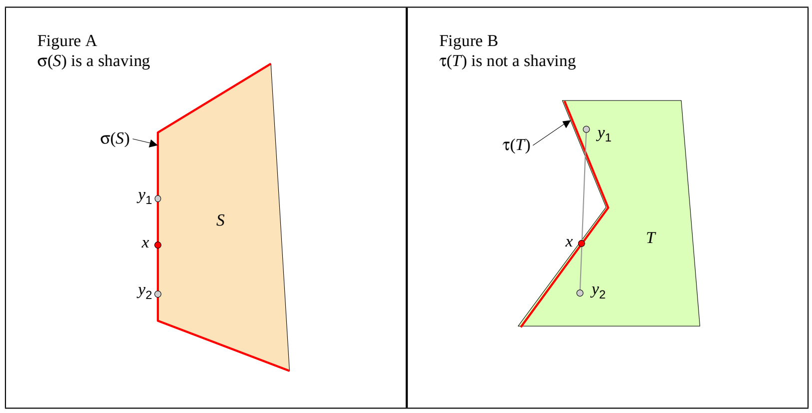
Note 6.6. The following results are due to Wen [19] (Chapter 2).

- a set S^2 with the above property is called a **shaving** of S^1
- shavings of shavings are shavings.
- shavings can be decomposed into convex sets that are shavings

- a convex set is always a shaving of itself.
- a relationship between shavings and the Kuhn-Tucker conditions for linear programming problems.

Definition 6.1. *Let $S \subseteq \mathbb{R}^n$. A set $\sigma(S) \subseteq S$ is a* shaving *of S if and only if for any $y_1, y_2, \ldots, y_\ell \in S$, and $\lambda_1 \geq 0,\ \lambda_2 \geq 0, \ldots, \lambda_\ell \geq 0$ such that $\sum_{t=1}^{\ell} \lambda_t = 1$ and $\sum_{t=1}^{\ell} \lambda_t y_t = x \in \sigma(S)$, the statement $\{\lambda_i > 0\}$ implies $y_i \in \sigma(S)$.*

The following figures illustrate the notion of a shaving.



The red region, $\sigma(S)$, in Figure A is a shaving of the set S. However in Figure B, the point $\lambda_1 y_1 + \lambda_2 y_2 = x \in \tau(T)$ with $\lambda_1 + \lambda_2 = 1$, $\lambda_1 > 0$, $\lambda_2 > 0$. But y_1 and y_2 do not belong to $\tau(T)$. Hence $\tau(T)$ is not a shaving.

Theorem 6.2. *Suppose $T = \sigma(S)$ is a shaving of S and $\tau(T)$ is a shaving of T. Let $\tau \circ \sigma$ denote the composition of the functions τ and σ. Then $\tau \circ \sigma(S)$ is a shaving of S.*

Proof: Let $y_1, y_2, \ldots, y_\ell \in S$, and $\lambda_1 \geq 0,\ \lambda_2 \geq 0, \ldots, \lambda_\ell \geq 0$ such that $\sum_{t=1}^{\ell} \lambda_t = 1$ and $\sum_{t=1}^{\ell} \lambda_t y_t = x \in \sigma(S) = T$.

Suppose $\lambda_i > 0$. Since $\sigma(S)$ is a shaving of S then $y_i \in \sigma(S) = T$. Since $\tau(T)$ is a shaving of T, $y_i \in T$, and $\lambda_i > 0$ then $y_i \in \tau(T)$. Therefore $y_i \in \tau(\sigma(S))$ so $\tau \circ \sigma(S)$ is a shaving of S. ■

It is easy to prove the following theorem:

Theorem 6.3. *If S is a convex set, the $\sigma(S) = S$ is a shaving of S.*

Theorem 6.4. *Let $S \subseteq \mathbb{R}^N$. Let $\sigma(S)$ be a shaving of S. If x is an extreme point of $\sigma(S)$, then x is an extreme point of S.*

Proof: See Bialas and Karwan [4].

Corollary 6.1. *An optimal solution to the two-level linear resource control problem (if one exists) occurs at an extreme point of the constraint set of all variables (S^1).*

Proof: See Bialas and Karwan [4].

These results were generalized to n-levels by Wen [19]. Using Theorems 6.2 and 6.4, if f_k is linear and S^1 is a bounded convex polyhedron then the extreme points of

$$S^k = \mathbf{\Psi}_{k-1}\mathbf{\Psi}_{k-2}\cdots\mathbf{\Psi}_2\mathbf{\Psi}_1(S^1)$$

are extreme points of S^1. This justifies the use of extreme point search procedures to finding the solution to the n-level linear resource control problem.

6.4 Cooperative Stackelberg games

This section is based on Chew [14], Bialas and Chew [3], and Bialas [2].

6.4.1 An Illustration

Consider a game with three players, named 1, 2 and 3, each of whom controls an unlimited quantity of a commodity, with a different commodity for each player. Their task is to fill a container of unit capacity with amounts of their respective commodities, never exceeding the capacity of the container. The task of filling will be performed in a sequential fashion, with player 3 (the player at the "top" of the hierarchy) taking his turn first. A player cannot remove a commodity placed in the container by a previous player.

At the end of the sequence, a referee pays each player one dollar (or fraction, thereof) for each unit of his respective commodity which has been placed in the container. It is easy to see that, since player 3 has preemptive control over the container, he will fill it completely with his commodity, and collect one dollar.

Suppose, however, that the rules are slightly changed so that, in addition, player 3 could collect five dollars for each unit of *player one's* commodity which is placed in the container. Since player 2 does not receive any benefit from player one's commodity, player 2 would fill the container with his own commodity on his turn, if given the opportunity. This is the *rational reaction* of player 2. For this reason, player 3 has no choice but to fill the container with his commodity and collect only one dollar.

6.4.2 Coalition Formation

In the previous example, there are six dollars available to the three players. Divided equally, each of the three players could improve their payoffs. However, because of the sequential and independent nature of the decisions, such a solution cannot be attained.

The solution to the above problem is, thus, not Pareto optimal (see Chew [14]). However, as suggested by the example, the formation of a coalition among subsets of the players could provide a means to achieve Pareto optimality. The members of each coalition act for the benefit of the coalition as a whole. The question immediately raised are:

- which coalitions will tend to form,
- are the coalitions enforceable, and
- what will be the resulting distribution of wealth to each of the players?

The game in partition function form (see Lucas and Thrall [16] and Shenoy [17]) provides a framework for answering these questions in this Stackelberg setting.

Definition 6.2. *An* **abstract game** *is a pair* (X, dom) *where* X *is a set whose members are called* **outcomes** *and* dom *is a binary relation on* X *called* **domination**.

Let $G = \{1, 2, \dots, n\}$ denote the set of n players. Let $\mathcal{P} = \{R_1, R_2, \dots, R_M\}$ denote a coalition structure or partition of G into nonempty coalitions, where $R_i \cap R_j = \emptyset$ for all $i \neq j$ and $\cup_{i=1}^{M} R_i = G$.

Let $\mathcal{P}_0 \equiv \{\{1\}, \{2\}, \dots, \{n\}\}$ denote the coalition structure where no coalitions have formed and let $\mathcal{P}_G \equiv \{G\}$ denote the **grand coalition**.

Consider $\mathcal{P} = \{R_1, R_2, \dots, R_M\}$, an arbitrary coalition structure. Assume that utility is additive and transferable. As a result of the coalition formation, the objective function of each player in coalition R_j becomes,

$$f'_{R_j}(x) = \sum_{i \in R_j} f_i(x).$$

Although the sequence of the players' decisions has not changed, their objective functions have. Let $R(i)$ denote the unique coalition $R_j \in \mathcal{P}$ such that player $i \in R_j$. Instead of maximizing $f_i(x)$, player i will now be maximizing $f'_{R(i)}(x)$. Let $\hat{x}(\mathcal{P})$ denote the solution to the resulting n-level optimization problem.

Definition 6.3. *Suppose that S^1 is compact and $\hat{x}(\mathcal{P})$ is unique. The value of (or payoff to) coalition $R_j \in \mathcal{P}$, denoted by $v(R_j, \mathcal{P})$, is given by*

$$v(R_j, \mathcal{P}) \equiv \sum_{i \in R_j} f_i(\hat{x}(\mathcal{P})).$$

Note 6.7. The function v need not be superadditive. Hence, one must be careful when applying some of the traditional game theory results which require superadditivity to this class of problems.

Definition 6.4. *A* **solution configuration** *is a pair $(r, \mathcal{P})$, where r is an n-dimensional vector (called an* **imputation***) whose elements r_i $(i = 1, \ldots, n)$ represent the payoff to each player i under coalition structure $\mathcal{P}$.*

Definition 6.5. *A solution configuration $(r, \mathcal{P})$ is a* **feasible solution configuration** *if and only if $\sum_{i \in R} r_i \leq v(R, \mathcal{P})$ for all $R \in \mathcal{P}$.*

Let Θ denote the set of all solution configurations which are feasible for the hierarchical decision-making problem under consideration. We can then define the binary relation dom, as follows:

Definition 6.6. *Let $(r, \mathcal{P}_r)$, $(s, \mathcal{P}_s) \in \Theta$. Then $(r, \mathcal{P}_r)$* **dominates** *$(s, \mathcal{P}_s)$ denoted by $(r, \mathcal{P}_r)\mathsf{dom}(s, \mathcal{P}_s)$, if and only if there exists an nonempty $R \in \mathcal{P}$, such that*

$$r_i > s_i \quad \textit{for all} \quad i \in R \quad \textit{and} \tag{2}$$

$$\sum_{i \in R} r_i \leq v(R, \mathcal{P}_r). \tag{3}$$

Condition (2) implies that each decision maker in R prefers coalition structure $\mathcal{P}_r$ to coalition structure $\mathcal{P}_s$. Condition (3) ensures that R is a feasible coalition in $\mathcal{P}_r$. That is, R must not demand more for the imputation r than its value $v(R, \mathcal{P}_r)$.

Definition 6.7. *The* **core***, $\mathcal{C}$, of an abstract game is the set of undominated, feasible solution configurations.*

When the core is nonempty, each of its elements represents an enforceable solution configuration within the hierarchy.

6.4.3 Results

We have now defined a model of the formation of coalitions among players in a Stackelberg game. Perfect information is assumed among the players, and coalitions are allowed to form freely. No matter which coalitions form, the order of the players' actions remains the same. Each coalition earns the combined proceeds that each individual coalition member would have received in the original Stackelberg game. Therefore, a player's rational decision may now be altered because he is acting for the joint benefit of the members of his coalition.

Using the above model, several results can be obtained regarding the formation of coalitions among the players. First, the distribution of wealth to any feasible coalition cannot exceed the value of the grand coalition. This is provided by the following lemma:

Lemma 6.1. *If solution configuration* $(z, \mathcal{P}) \in \Theta$ *then*

$$\sum_{i=1}^{n} z_i \leq \sum_{i=1}^{n} f_i(\hat{x}(\mathcal{P}_G)) = v(G, \mathcal{P}_G) \equiv V^*.$$

Theorem 6.5. *If* $(z, \mathcal{P}) \in \mathcal{C} \neq \emptyset$ *then* $\sum_{i=1}^{n} z_i = V^*$.

It is also possible to construct a simple sufficient condition for the core to be empty. This is provided in Theorem 6.6.

Theorem 6.6. *The abstract game* (Θ, dom) *has* $\mathcal{C} = \emptyset$ *if there exists coalition structures* $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_m$ *and coalitions* $R_j \in \mathcal{P}_j$ $(j = 1, \ldots, m)$ *with* $R_j \cap R_k = \emptyset$ *for all* $j \neq k$ *such that*

$$\sum_{j=1}^{m} v(R_j, \mathcal{P}_j) > V^*. \tag{4}$$

Finally, we can easily show that, in any 2-person game of this type, the core is always nonempty.

Theorem 6.7. *If* $n = 2$ *then* $\mathcal{C} \neq \emptyset$.

6.4.4 Examples and Computations

We will expand on the illustration given in Section 6.4.1. Let c_{ij} represent the reward to player i if the commodity controlled by player j is placed in the container. Let C represent the matrix $[c_{ij}]$ and let x be an n-dimensional vector with x_j representing the amount of commodity j placed in the container. Note that $\sum_{j=1}^{n} x_j \leq 1$

and $x_j \geq 0$ for $j = 1, \ldots, n$. For the illustration provided in Section 6.4.1,

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix}.$$

Note that Cx^{T} is a vector whose components represent the earnings to each player.

Chew [14] provides a simple procedure to solve this game. The algorithm requires $c_{11} > 0$.

Step 0: Initialize i=1 and j=1. Go to *Step 1.*

Step 1: If $i = n$, stop. The solution is $\hat{x}_j = 1$ and $\hat{x}_k = 0$ for $k \neq j$. If $i \neq n$, then go to *Step 2.*

Step 2: Set $i = i + 1$. If $c_{ii} > c_{ij}$, then set $j = i$. Go to *Step 1.*

If no ties occur in *Step 2* (i.e., $c_{ii} \neq c_{ij}$) then it can be shown that the above algorithm solves the problem (see Chew [14]).

Example 6.1. Consider the three player game of this form with

$$C = C_{\mathcal{P}_0} = \begin{bmatrix} 10 & 4 & 0 \\ 0 & 1 & 1 \\ 1 & 4 & 3 \end{bmatrix}.$$

With coalition structure $\mathcal{P}_0 = \{\{1\}, \{2\}, \{3\}\}$, the solution is $(x_1, x_2, x_3) = (0, 1, 0)$ and the coalition values are $v(\{1\}, \mathcal{P}_0) = 4$, $v(\{2\}, \mathcal{P}_0) = 1$ and $v(\{3\}, \mathcal{P}_0) = 4$.

Consider coalition structure $\mathcal{P} = \{\{1, 2\}, \{3\}\}$, The payoff matrix becomes

$$C_{\mathcal{P}} = \begin{bmatrix} 10 & 5 & 1 \\ 10 & 5 & 1 \\ 1 & 4 & 3 \end{bmatrix}$$

and a solution of $(0, 0, 1)$. The values of the coalitions in this case are $v(\{1, 2\}, \mathcal{P}) = 1$ and $v(\{3\}, \mathcal{P}) = 3$.

Note that coalition structure $\mathcal{P}$ is not superadditive since

$$v(\{1\}, \mathcal{P}_0) + v(\{2\}, \mathcal{P}_0) > v(\{1, 2\}, \mathcal{P}).$$

When Players 1 and 2 do not cooperate, Player 2 fills the container with a benefit of 4 to Player 3. Suppose the bottom two players form coalition $\{1,2\}$. Then if Player 2 is given an *empty* container, the coalition will have Player 1 fill it with his commodity, earning 10 for the coalition. So, if Player 3 does not fill the container, the formation of coalition $\{1,2\}$ reduces Player 3's benefit from 4 to 1. As a result, Player 3 fills the container himself, and earns 3. The coalition $\{1,2\}$ only earns 1 (not 10).

Remember that Chew's model assumes that all players have full knowledge of the coalition structure that has formed. Obvious natural extensions of this simple model would incorporate secret coalitions and delayed coalition formation (i.e., changes in the coalition structure while the container is being passed).

Example 6.2. Consider the three player game of this form with

$$C = C_{\mathcal{P}_0} = \begin{bmatrix} 4 & 1 & 4 \\ 1 & 0 & 3 \\ 2 & 5 & 1 \end{bmatrix}.$$

With coalition structure $\mathcal{P}_0 = \{\{1\},\{2\},\{3\}\}$, the solution is $(x_1,x_2,x_3) = (1,0,0)$ and the coalition values are $v(\{1\},\mathcal{P}_0) = 4$, $v(\{2\},\mathcal{P}_0) = 1$ and $v(\{3\},\mathcal{P}_0) = 2$.

Under the formation of coalition structure $\mathcal{P} = \{\{1\},\{2,3\}\}$, the resources of players 2 and 3 are combined. This yields a payoff matrix of

$$C_{\mathcal{P}} = \begin{bmatrix} 4 & 1 & 4 \\ 3 & 5 & 4 \\ 3 & 5 & 4 \end{bmatrix}$$

and a solution of $(0,1,0)$. The values of the coalitions in this case are $v(\{1\},\mathcal{P}) = 1$ and $v(\{2,3\},\mathcal{P}) = 5$.

Finally, if all of the players join to form the grand coalition, $\mathcal{P}_G$, the payoff matrix becomes

$$C_{\mathcal{P}_G} = \begin{bmatrix} 7 & 6 & 8 \\ 7 & 6 & 8 \\ 7 & 6 & 8 \end{bmatrix}$$

with a solution of $(0,0,1)$ and $v(\{1,2,3\},\mathcal{P}_G) = 8$. Note that

$$v(\{1\},\mathcal{P}_0) + v(\{2,3\},\mathcal{P}) > v(\{1,2,3\},\mathcal{P}_G).$$

From Theorem 6.6, we know that the core for this game is empty.

6.5 BIBLIOGRAPHY

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IE675 Game Theory — Spring 2001

Homework 1 — **DUE February 1, 2001**

1. Obtain the optimal mixed strategies for the following matrix games:

$$\begin{bmatrix} 0 & 4 \\ 4 & 2 \\ 1 & 3 \end{bmatrix} \qquad \begin{bmatrix} 3 & 0 \\ 1 & 5 \end{bmatrix}$$

2. Show that, if we treat an $m \times n$ matrix game as a point in mn-dimensional euclidean space, the value of the game is a continuous function of the game. (See below)

3. Revise the dual linear programming problems for determining optimal mixed-strategies so that they can find the optimal pure strategy. Show that your formulation works with an example.

Notes:

For question (2), recall the following from real analysis. . .

Definition 1.1 A *metric space* is a set E, together with a rule which associates with each pair $x, y \in E$ a real number $d(x, y)$ such that

a. $d(x, y) \geq 0$ for all $x, y \in E$

b. $d(x, y) = 0$ if and only if $x = y$

c. $d(x, y) = d(y, x)$ for all $x, y \in E$

d. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in E$

Definition 1.2 Let E and E' be metric spaces, with distances denoted d and d', respectively, let $f : E \to E'$, and let $x_0 \in E$. Then f is said to be *continuous at* x_0 if, given any real number $\epsilon > 0$, there exists a real number $\delta > 0$ such that if $x \in E$ and $d(x, x_0) < \delta$, then $d'(f(x), f(x_0)) < \epsilon$.

Definition 1.3 If E and E' are metric spaces and $f : E \to E'$ is a function, then f is said to be *continuous on* E, or, more briefly, *continuous*, if f is continuous at all points of E.

IE675 GAME THEORY SPRING 2001

Homework 2 **DUE February 8, 2001**

1. Obtain the optimal mixed strategies for the following matrix games:

$$\begin{bmatrix} 1 & 3 & -1 & 2 \\ -3 & -2 & 2 & 1 \\ 0 & 2 & -2 & 1 \end{bmatrix} \qquad \begin{bmatrix} -1 & -3 & 1 & -2 \\ 3 & 2 & -2 & -1 \\ 0 & -2 & 2 & -1 \end{bmatrix}$$

2. Let A be a matrix game and let $V = x_0 A y_0^{\mathrm{T}}$ denote the expected value of the game when using the mixed saddle-point strategies x_0 and y_0. Consider a revised game where Player 1 must announce his choice of row first, and then (knowing Player 1's choice) Player 2 announces his choice of column. Let V_S denote the (expected) value of the game under these rules. What can one say (if anything) about the relationship between V and V_S?

   Hint: We say that V_S is the value from a Stackelberg strategy.

Homework 3 **DUE February 15, 2001**

1. Theorem 2.8 in the lecture notes states that the Lemke-Howson quadratic programming problem can be used to find Nash equilibrium solutions for a general-sum strategic form game. Although it's the correct approach, the proof in the lecture notes is, to say the least, rather sloppy.

   Using the proof of Theorem 2.8 provided in the lecture notes as a starting point, develop an improved version of the proof. Try to make your proof clear and concise.