4 Definite Integral

4.1 Definition, Necessary & sufficient conditions

Let $f : [a, b] \to \mathbb{R}$ be a bounded real valued function on the closed, bounded interval $[a, b]$. Also let m, M be the infimum and supremum of $f(x)$ on $[a, b]$, respectively.

Definition 4.1.1. *A partition P of $[a, b]$ is an ordered set $P = \{a = x_0, x_1, x_2, ..., x_n = b\}$ such that $x_0 < x_1 < ... < x_n$.*

Let m_k and M_k be the infimum and supremum of $f(x)$ on the subinterval $[x_{k-1}, x_k]$, respectively.

Definition 4.1.2. ***Lower sum:*** *The Lower sum, denoted with $L(P, f)$ of $f(x)$ with respect to the partition P is given by*

$$L(P, f) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}).$$

Definition 4.1.3. ***Upper sum:*** *The Upper sum, denoted with $U(P, f)$ of $f(x)$ with respect to the partition P is given by*

$$U(P, f) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}).$$

For a given partition P, $U(P, f) \geq L(P, f)$. In fact the same inequality holds for any two partitions. (see Lemma (4.1.6) below.)

Definition 4.1.4. ***Refinement of a Partition:*** *A partition Q is called a refinement of the partition P if $P \subseteq Q$.*

The following is a simple observation.

Lemma 4.1.5. *If Q is a refinement of P, then*

$$L(P, f) \leq L(Q, f) \text{ and } U(P, f) \geq U(Q, f).$$

Proof. Let $P = \{x_0, x_1, x_2, ..., x_{k-1}, x_k, ..., x_n\}$ and $Q = \{x_0, x_1, x_2, ..., x_{k-1}, z, x_k, ..., x_n\}$.

Then

$$\begin{aligned} L(P,f) &= m_0(x_1-x_0)+...+m_k(x_k-x_{k-1})+...+m_{n-1}(x_n-x_{n-1}) \\ &\leq m_0(x_1-x_0)+...+m_k^{'}(x_k-z)+m_k^{''}(z-x_{k-1})+...+m_{n-1}(x_n-x_{n-1}) \\ &= L(Q,f) \end{aligned}$$

where $m_k^{'} = \inf\limits_{[z,x_k]} f(x)$ and $m_k^{''} = \inf\limits_{[x_{k-1},z]} f(x)$.

Lemma 4.1.6. *If P_1 and P_2 be any two partitions, then*

$$L(P_1,f) \leq U(P_2,f).$$

Proof. Let $Q = P_1 \cup P_2$. Then Q is a refinement of both P_1 and P_2. So by Lemma (4.1.8),

$$L(P_1,f) \leq L(Q,f) \leq U(Q,f) \leq U(P_2,f).$$

Definition 4.1.7. *Let $\mathcal{P}$ be the collection of all possible partitions of $[a,b]$. Then the upper integral of f is*

$$\overline{\int_a^b} f = \inf\{U(P,f) : P \in \mathcal{P}\}$$

and lower integral of f is

$$\underline{\int_a^b} f = \sup\{L(P,f) : P \in \mathcal{P}\}.$$

An immediate consequence of Lemma (4.1.6) is

Lemma 4.1.8. *For a bounded function $f : [a,b] \to I\!R$,*

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f.$$

Definition 4.1.9. ***Riemann integrability:*** *$f : [a,b] \to \mathbb{R}$ is said to be Riemann integrable if*

$$\underline{\int_a^b} f = \overline{\int_a^b} f$$

and the value of the limit is denoted with $\displaystyle\int_a^b f(x)dx$. We say $f \in \mathcal{R}[a,b]$.

Example 1: Consider $f(x) = x$ on $[0,1]$ and the sequence of partitions $P_n = \{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}, \frac{n}{n}\}$.

Then

$$\begin{aligned} L(P_n, f) &= 0 \cdot \frac{1}{n} + \frac{1}{n} \cdot \frac{1}{n} + ... + \frac{n-1}{n}\frac{1}{n} \\ &= \frac{1}{n^2}[1 + 2 + ... + (n-1)] \\ &= \frac{n(n-1)}{2n^2} \end{aligned}$$

Thus $\lim_{n\to\infty} L(P_n, f) = \dfrac{1}{2}$. Hence from the definition $\underline{\int_0^1} f(x)dx \geq \frac{1}{2}$. Similarly

$$\begin{aligned} U(P_n, f) &= \frac{1}{n} \cdot \frac{1}{n} + \frac{2}{n} \cdot \frac{1}{n} + ... + \frac{n}{n}\frac{1}{n} \\ &= \frac{1}{n^2}[1 + 2 + ... + n] \\ &= \frac{n(n+1)}{2n^2} \end{aligned}$$

Hence $\lim_{n\to\infty} U(P_n, f) = \dfrac{1}{2}$. Again from the definition $\overline{\int_0^1} f(x)dx \leq \frac{1}{2}$.

Example 2: Consider $f(x) = x^2$ on $[0, 1]$ and the sequence of partitions $P_n = \{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}, \frac{n}{n}\}$. Then

$$\begin{aligned} U(P_n, f) &= \frac{1}{n^2} \cdot \frac{1}{n} + \left(\frac{2}{n}\right)^2 \cdot \frac{1}{n} + ... + \left(\frac{n}{n}\right)^2 \frac{1}{n} \\ &= \frac{1}{n^3}[1 + 2^2 + ... + n^2] \\ &= \frac{n(n+1)(2n+1)}{6n^3} \end{aligned}$$

Thus $\lim_{n\to\infty} U(P_n, f) = \dfrac{1}{3}$. Similarly

$$\begin{aligned} L(P_n, f) &= 0 \cdot \frac{1}{n} + \left(\frac{1}{n}\right)^2 \cdot \frac{1}{n} + ... + \left(\frac{n-1}{n}\right)^2 \frac{1}{n} \\ &= \frac{1}{n^3}[1 + 2^2 + ... + (n-1)^2] \\ &= \frac{n(n-1)(2n-1)}{6n^3} \end{aligned}$$

Therefore, $\lim_{n\to\infty} L(P_n, f) = \dfrac{1}{3}$.

Hence from the definition $\underline{\int_a^b} f \geq 1/3$ and $\overline{\int_a^b} f \leq 1/3$.

Remark 4.1. *In the above two examples* $\underline{\int_0^1} f = \overline{\int_0^1} f$ *thanks to Lemma 4.1.8*

The following example illustrates the non-integrability.

Example 3: On $[0,1]$, define $f(x) = \begin{cases} 1, & x \in Q, \\ 0, & x \notin Q. \end{cases}$

Let P be a partition of $[0,1]$. In any sub interval $[x_{k-1}, x_k]$, there exists a rational number and irrational number. Then the supremum in any subinterval is 1 and infimum is 0. Therefore, $L(P,f) = 0$ and $U(P,f) = 1$. Hence $\underline{\int_0^1} f \neq \overline{\int_0^1} f$.

Necessary and sufficient condition for integrability

Theorem 4.1.10. *A bounded function* $f \in \mathcal{R}[a,b]$ *if and only if for every* $\epsilon > 0$, *there exists a partition* P_ϵ *such that*

$$U(P_\epsilon, f) - L(P_\epsilon, f) < \epsilon.$$

Proof. $\Leftarrow$: Let $\epsilon > 0$. Then from the definition of upper and lower integral we have

$$\overline{\int_a^b} f - \underline{\int_a^b} f \leq U(P_\epsilon, f) - L(P_\epsilon, f) < \epsilon(\text{ by hypothesis}).$$

Thus the conclusion follows as $\epsilon > 0$ is arbitrary.

$\Rightarrow$: Conversely, since $\overline{\int_a^b} f$ is the infimum, for any $\epsilon > 0$, there exists a partition P_1 such that

$$U(P_1, f) < \overline{\int_a^b} f + \frac{\epsilon}{2}.$$

Similarly there exists a partition P_2 such that

$$L(P_2, f) > \underline{\int_a^b} f - \frac{\epsilon}{2}.$$

Let $P_\epsilon = P_1 \cup P_2$. Then P_ϵ is a refinement of P_1 and P_2. Hence

$$\begin{aligned}
U(P_\epsilon, f) - L(P_\epsilon, f) &\leq U(P_1, f) - L(P_2, f) \\
&\leq \overline{\int_a^b} f + \frac{\epsilon}{2} - \underline{\int_a^b} f + \frac{\epsilon}{2} \\
&= \epsilon \text{ (as } f \text{ is integrable, } \overline{\int_a^b} f = \underline{\int_a^b} f)
\end{aligned}$$

This complete the theorem. ///

Now it is easy to see that the functions considered in Example 1 and Example 2 are integrable. For any $\epsilon > 0$, we can find n (large) and P_n such that $\frac{1}{n} < \epsilon$. Then

$$U(P_n, f) - L(P_n, f) = \frac{1}{2n^2}(n(n+1) - n(n-1)) = \frac{1}{n} < \epsilon.$$

Similarly one can choose n in Example 2.

Remark 4.2. *$f : [a, b] \to \mathbb{R}$ is integrable if and only if there exists a sequence $\{P_n\}$ of partitions of $[a, b]$ such that*

$$\lim_{n\to\infty} U(P_n, f) - L(P_n, f) = 0.$$

Remark 4.3. *Let $S(P, f) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}),\ \xi_i \in [x_{i-1}, x_i]$. Then we have the following*

$$m(b - a) \leq L(P, f \leq S(P, f) \leq U(P, f) \leq M(b - a).$$

In fact, one has the following Darboux theorem:

Theorem 4.1.11. *Let f $f : [a, b] \to I\!R$ be a Riemann integrable function. Then for a given $\epsilon > 0$, there exists $\delta > 0$ such that for any partition P with $\|P\| := \max_{1\leq i\leq n} |x_i - x_{i-1}| < \delta$, we have*

$$|S(P, f) - \int_a^b f(x)dx| < \epsilon.$$

Corollary: If $f \in \mathcal{R}[a, b]$, then for any sequence of partitions $\{P_n\}$ with $\|P_n\| \to 0$, we have $L(P_n, f) \to \int_a^b f(x)dx$ and $U(P_n, f) \to \int_a^b f(x)dx$.

Remark 4.4. *From the above theorem, we note that if there exists a sequence of partition $\{P_n\}$ such that $\|P_n\| \to 0$ and $U(P_n, f) - L(P_n, f) \not\to 0$ as $n \to \infty$, then f is not integrable.*

Problem: Show that the function $f : [0,1] \to I\!R$

$$f(x) = \begin{cases} 1+x & x \in \mathbb{Q} \\ 1-x & x \notin \mathbb{Q} \end{cases}$$

is not integrable.

Solution: Consider the sequence of partitions $P_n = \{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n}{n} = 1\}$. Then

$$\begin{aligned} U(P_n, f) &= (1 + \frac{1}{n})\frac{1}{n} + (1 + \frac{2}{n})\frac{1}{n} + ... + (1 + \frac{n}{n})\frac{1}{n} \\ &= 1 + \frac{1}{n^2}(1 + 2 + ... + n) \\ &\to \frac{3}{2} \text{ as } n \to \infty \end{aligned}$$

Now using the fact that infimum of f on $[0, \frac{1}{n}]$ is $1 - \frac{1}{n}$, though it is not achieved, we get

$$L(P_n, f) = (1 - \frac{1}{n})\frac{1}{n} + (1 - \frac{2}{n})\frac{1}{n} + ... + (1 - \frac{n}{n})\frac{1}{n} \to \frac{1}{2} \text{ as } n \to \infty.$$

Hence f is not integrable.

Problem: Consider $f(x) = \dfrac{1}{x}$ on $[1, b]$. Divide the interval in geometric progression and compute $U(P_n, f)$ and $L(P_n, f)$ to show that $f \in \mathcal{R}[1, b]$.

Solution: Let $P_n = \{1, r, r^2, ..., r^n = b\}$ be a partition on $[1, b]$. Then

$$\begin{aligned} U(P_n, f) &= f(1)(r-1) + f(r)(r^2 - r) + ... + f(r^{n-1})(r^n - r^{n-1}) \\ &= (r-1) + \frac{1}{r}r(r-1) + .. + \frac{1}{r^{n-1}}r^{n-1}(r-1) \\ &= n(r-1) \\ &= n(b^{\frac{1}{n}} - 1) \end{aligned}$$

Therefore $\lim\limits_{n\to\infty} U(P_n, f) = \lim\limits_{n\to\infty} \dfrac{b^{\frac{1}{n}} - 1}{\frac{1}{n}} = \lim\limits_{n\to\infty} \dfrac{b^{\frac{1}{n}} \ln b(\frac{-1}{n^2})}{\frac{-1}{n^2}} = \ln b.$

Similarly

$$\begin{aligned}L(P_n, f) &= f(r)(r-1) + f(r^2)(r^2 - r) + ... + f(r^n)(r^n - r^{n-1})\\ &= \frac{1}{r}(r-1) + .. + \frac{1}{r^n}r^{n-1}(r-1)\\ &= \frac{n}{r}(b^{\frac{1}{n}} - 1)\\ &= n(1 - \frac{1}{b^{1/n}})\\ &= \frac{b^{1/n} - 1}{b^{1/n}\frac{1}{n}} \to \ln b \text{ as } n \to \infty.\end{aligned}$$

Theorem 4.1.12. *Suppose f is a continuous function on $[a, b]$. Then $f \in \mathcal{R}[a, b]$.*

Proof. Let $\epsilon > 0$. By Theorem 4.1.10, we need to show the existence of a partition P such that

$$U(P, f) - L(P, f) < \epsilon.$$

Since f is continuous on $[a, b]$, this implies f is uniformly continuous on $[a, b]$. Therefore there exists $\delta > 0$ such that

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\epsilon}{(b-a)}. \tag{4.1}$$

Now choose a partition P such that

$$\sup_{1\leq k\leq n} |x_k - x_{k-1}| < \delta. \tag{4.2}$$

As f is continuous on $[a, b]$ there exist $x_k', x_k'' \in (x_{k-1}, x_k)$ such that $m_k = f(x_k')$ and $M_k = f(x_k'')$. By (4.2), $|x_k' - x_k''| < \delta$ and hence by (4.1) $|f(x_k'') - f(x_k')| < \frac{\epsilon}{(b-a)}$. Thus

$$\begin{aligned}U(P, f - L(P, f) &= \sum_{k=1}^{n}(M_k - m_k)(x_k - x_{k-1})\\ &= \sum_{k=1}^{n}(f(x_k'') - f(x_k'))(x_k - x_{k-1})\\ &\leq \frac{\epsilon}{(b-a)}\sum_{k=1}^{n}(x_k - x_{k-1}) = \frac{\epsilon}{(b-a)}(b-a) = \epsilon.\end{aligned}$$

Therefore $f \in \mathcal{R}[a, b]$.

Integrability and discontinuous functions: We study the effect of discontinuity on integrability of a function $f(x)$.
Example: Consider the following function $f : [0, 1] \to \mathbb{R}$.

$$f(x) = \begin{cases} 1, & x \neq \frac{1}{2} \\ 0, & x = \frac{1}{2} \end{cases}$$

Clearly $U(P, f) = 1$ for any partition P. We notice that $L(P, f)$ will be less than 1. We can try to isolate the point $x = \frac{1}{2}$ in a subinterval of small length. Consider the partition $P_\epsilon = \{0, \frac{1}{2} - \frac{\epsilon}{2}, \frac{1}{2} + \frac{\epsilon}{2}, 1\}$. Then $L(P_\epsilon, f) = (\dfrac{1}{2} - \dfrac{\epsilon}{2}) + (1 - \dfrac{1}{2} - \dfrac{\epsilon}{2}) = 1 - \epsilon$. Therefore, for given $\epsilon > 0$ we have $U(P_\epsilon, f) - L(P_\epsilon, f) = \epsilon$. Hence f is integrable.
In fact we have the following theorem.

Theorem 4.1.13. *Suppose $f : [a, b] \to \mathbb{R}$ be a bounded function which has finitely many discontinuities. Then $f \in \mathcal{R}[a, b]$.*

Proof follows by constructing suitable partition with sub-intervals of sufficiently small length around the discontinuities as observed in the above example.

4.2 Properties of Definite Integral:

Property 1: For a constant $c \in \mathbb{R}$, $\displaystyle\int_a^b cf(x)dx = c\int_a^b f(x)dx$.
Property 2: Let $f_1, f_2 \in \mathcal{R}[a, b]$. Then

$$\int_a^b (f_1 + f_2)(x)dx = \int_a^b f_1(x)dx + \int_a^b f_2(x)dx.$$

Easy to show that for any partition P,

$$U(P, f_1 + f_2) \leq U(P, f_1) + U(P, f_2) \tag{4.3}$$
$$L(P, f_1 + f_2) \geq L(P, f_1) + L(P, f_2) \tag{4.4}$$

Since f_1, f_2 are integrable, for $\epsilon > 0$ there exists P_1, P_2 such that

$$U(P_1, f_1) - L(P_1, f_1) < \epsilon$$
$$U(P_2, f_2) - L(P_2, f_2) < \epsilon$$

Now taking $P = P_1 \cup P_2$, if necessary, we assume

$$U(P, f_1) - L(P, f_1) < \epsilon, \quad U(P, f_1) - L(P, f_2) < \epsilon \tag{4.5}$$

Therefore, using (4.3)-(4.5) we get

$$\begin{aligned} U(P, f_1 + f_2) - L(P, f_1 + f_2) &\leq U(P, f_1) + U(P, f_2) - L(P, f_2) - L(P, f_2) \\ &< \epsilon + \epsilon = 2\epsilon. \end{aligned}$$

Hence, $f_1 + f_2$ is integrable.

$$\begin{aligned} \int_a^b (f_1 + f_2)(x)dx = \lim_{n\to\infty} S(P_n, f_1 + f_2) &= \lim_{n\to\infty} \sum_{k=1}^{n} (f_1 + f_2)(\xi_k)(x_k - x_{k-1}) \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} f_1(\xi_k)(x_k - x_{k-1}) + \lim_{n\to\infty} \sum_{k=1}^{n} f_2(\xi_k)(x_k - x_{k-1}) \\ &= \int_a^b f_1(x)dx + \int_a^b f_2(x)dx \end{aligned}$$

Property 3: If $f(x) \leq g(x)$ on $[a, b]$. Then

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

First we note that $m \leq f(x) \leq M$ implies $m(b-a) \leq \int_a^b f(x) \leq M(b-a)$. Then Property 1 and $f(x) \leq g(x)$ imply $\int_a^b (g - f) \geq 0$ or $\int_a^b g(x)dx \geq \int_a^b f(x)dx$.

Property 4: If $f \in \mathcal{R}[a, b]$ then $|f| \in \mathcal{R}[a, b]$ and $|\int_a^b f(x)dx| \leq \int_a^b |f|(x)dx$. Let $m_k' = \inf_{[x_{k-1}, x_k]} |f|(x)$ and $M_k' = \sup_{[x_{k-1}, x_k]} |f|(x)$. Then we claim

Claim: $M_k - m_k \geq M_k' - m_k'$
Proof of Claim: Note that for $x, y \in [x_{i-1}, x_i]$,

$$|f|(x) - |f|(y) \leq |f(x) - f(y)| \leq M_i(f) - m_i(f).$$

Now take supremum over x and infimum over y, to conclude the claim.

Now since f is integrable, there exists partitions $\{P_n\}$ such that $\lim_{n\to\infty} U(P_n, f)-L(P_n, f) = 0$. i.e.,

$$\lim_{n\to\infty}\sum_{k=1}^{n}(M_k - m_k)(x_k - x_{k-1}) = 0.$$

This implies

$$\lim_{n\to\infty}\sum_{k=1}^{n}(M_k' - m_k')(x_k - x_{k-1}) = 0.$$

Hence $|f|$ is integrable. Note that $-|f| \le f \le |f|$. Thus by Property 3 we get

$$-\int_a^b |f|(x)dx \le \int_a^b f(x)dx \le \int_a^b |f|(x)dx \Longrightarrow |\int_a^b f(x)dx| \le \int_a^b |f|(x)dx.$$

Property 5: Let f be bounded on $[a, b]$ and let $c \in (a, b)$. Then f is integrable on $[a, b]$ if and only if f is integrable on $[a, c]$ and $[c, b]$. In this cases

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Let f be integrable on $[a, b]$. For $\epsilon > 0$, there exists partition P such that

$$U(P, f) - L(P, f) < \epsilon. \tag{4.6}$$

With out loss of generality we can assume that P contain c. (otherwise we can refine P by adding c and the difference will be closer than before) Let $P_1 = P \cap [a, c]$ and $P_2 = P \cap [c, b]$. Then P_1 and P_2 are partitions on $[a, c]$ and $[c, b]$ respectively. Also by (4.6), $U(P_1, f) - L(P_1, f) < \epsilon$ and $U(P_2, f) - L(P_2, f) < \epsilon$. This implies f is integrable on $[a, c]$ and $[c, b]$. Conversely, suppose f is integrable on $[a, c]$ and $[c, b]$. Then for $\epsilon > 0$, there exists partitions P_1 of $[a, c]$ and P_2 of $[c, b]$ such that $U(P_1, f) - L(P_1, f) < \frac{\epsilon}{2}$ and $U(P_2, f) - L(P_2, f) < \frac{\epsilon}{2}$. Now take $P = P_1 \cup P_2$. Then $U(P, f) - L(P, f) < \epsilon$. So by Remark 4.3, there exists $\{P_n\}$ such that

$$\begin{aligned}
\int_a^b f(x)dx &= \lim_{n\to\infty} S(P_n, f) = \lim_{n\to\infty}\sum_{P_n} f(\xi_k)(x_{k-1} - x_k) \\
&= \sum_{P_n\cap[a,c]} f(\xi_k)(x_k - x_{k-1}) + \sum_{P_n\cap[c,b]} f(\xi_k)(x_k - x_{k-1}) \\
&\to \int_a^c f(x)dx + \int_c^b f(x)dx
\end{aligned}$$

Example: Consider the following function $f : [0,1] \to \mathbb{R}$.

$$f(x) = \begin{cases} 1 & x = \frac{1}{n} \text{ for some } n \in \mathbb{N}, n \geq 2 \\ 0 & x \neq \frac{1}{n} \end{cases}$$

Then f is Riemann integrable.

Solution: Let $\epsilon > 0$. Choose N such that $\dfrac{1}{N} < \dfrac{\epsilon}{2}$. Note that $f(x)$ has only finitely many discontinuities in $[\frac{1}{N}, 1]$ say $\xi_1, \xi_2, ..., \xi_r$. Define the partition P_ϵ as

$$P_\epsilon = \{0, \frac{1}{N}, \xi_1 - \frac{\epsilon}{4r}, \xi_1 + \frac{\epsilon}{4r}, ..., \xi_r - \frac{\epsilon}{4r}, \xi_r + \frac{\epsilon}{4r}, 1\}.$$

Since ξ_r is the last discontinuity, $f = 0$ in $[\xi_r + \frac{\epsilon}{4r}, 1]$. Now $L(P_\epsilon, f) = 0$ and

$$\begin{aligned} U(P_\epsilon, f) &= 1 \cdot \frac{1}{N} + \frac{\epsilon}{2r} + \frac{\epsilon}{2r} + ... + \frac{\epsilon}{2r} + 0 \cdot (1 - \xi_r - \frac{\epsilon}{4r}) \\ &= \frac{1}{N} + \frac{\epsilon}{2} < \epsilon. \end{aligned}$$

Alternatively, using the approach of isolating the discontinuous points, we may take the Partition with length of $\frac{\epsilon}{2^k}$ at each discontinuous points $x_k, k = 1, 2, 3, 4, ...$ Then using the fact that $\sum \frac{1}{2^k}$ converges, one can prove that f is integrable.

Example: Consider the following function $f : [0,1] \to \mathbb{R}$.

$$f(x) = \begin{cases} 0 & x \in Q \\ \sin \frac{1}{x} & x \notin Q \end{cases}$$

Then f is not Riemann integrable.

Solution: We will show that f is not integrable on a sub interval of $[0,1]$. Consider the f on the subinterval $I_1 = [\frac{2}{\pi}, 1]$. Clearly $L(P, f) = 0$ for any partition P of I_1 because $f(x) \geq 0$ in the sub interval $[\frac{2}{\pi}, 1]$. Let M_k be the Supremum of f on subintervals $[x_{k-1}, x_k]$ of $[\frac{2}{\pi}, 1]$. Also the minimum of $M_k's$ is $\sin 1$. Therefore,

$$U(P, f) = \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) > (1 - \frac{2}{\pi}) \sin 1.$$

Hence $U(P,f) - L(P,f)$ can not be made less than ϵ for any $\epsilon < (1-\frac{2}{\pi})\sin 1$. ///

Mean Value Theorem

Theorem 4.2.1. *Let $f(x)$ be a continuous function on $[a,b]$. Then there exists $\xi \in [a,b]$ such that*

$$\int_a^b f(x)dx = f(\xi)(b-a).$$

Proof. Let $m = \min_{x\in[a,b]} f(x)$ and $M = \max_{x\in[a,b]} f(x)$. Then by Property 3, we have

$$m(b-a) \leq \int_a^b f \leq M(b-a),$$

i.e.

$$m \leq \frac{1}{(b-a)} \int_a^b f \leq M.$$

Now since $f(x)$ is continuous, it attains all values between it's maximum and minimum values. Therefore there exists $\xi \in [a,b]$ such that $f(\xi) = \dfrac{1}{(b-a)} \displaystyle\int_a^b f$.

Fundamental Theorem

Theorem 4.2.2. *Let $f(x)$ be a continuous function on $[a,b]$ and let $\phi(x) = \int_a^x f(s)ds$. Then ϕ is differentiable and $\phi'(x) = f(x)$.*

Proof. As $\dfrac{\phi(x+\Delta x) - \phi(x)}{\Delta x} = \dfrac{1}{\Delta x}\displaystyle\int_x^{x+\Delta x} f(s)ds$, By Mean value theorem, there exists $\xi \in [x, x+\Delta x]$ such that

$$\int_x^{x+\Delta x} f(s)ds = \Delta x f(\xi).$$

Therefore $\displaystyle\lim_{\Delta x\to 0} \frac{\phi(x+\Delta x)-\phi(x)}{\Delta x} = \lim_{\Delta x\to 0} f(\xi)$. Since f is continuous, $\displaystyle\lim_{\Delta x \to 0} f(\xi) = f(\lim_{\Delta x\to 0} \xi) = f(x)$. Thus $\phi'(x) = f(x)$. ///

Remark 4.5. *It is always not true that $\int_a^b f'(x)dx = f(b) - f(a)$.*

For example, take $f(x) = x^2 \sin\frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$. Then f is differentiable on $[0,1]$. Here the derivatives at the end points are the left/right derivatives. It is easy to check that $f'(x) = 2x\sin\frac{1}{x^2} - \frac{2}{x}\cos\frac{1}{x^2}$ for $x \in (0,1)$ and $f'(0) = 0$. Therefore f' is not bounded and so not integrable.

Definition 4.2.3. *A function $F(x)$ is called anti-derivative of $f(x)$, if $F'(x) = f(x)$.*

Second Fundamental Theorem:

Theorem 4.2.4. *Suppose $F(x)$ is an anti- derivative of continuous function $f(x)$. Then $\int_a^b f(x)dx = F(b) - F(a)$.*

Proof. By First fundamental theorem, we have

$$\frac{d}{dx}\int_a^x f(s)ds = f(x).$$

Also $F'(x) = f(x)$. Hence $\int_a^x f(s)ds = F(x) + c$ for some constant $c \in \mathbb{R}$. Taking $x = a$, we get $c = -F(a)$. Now taking $x = b$ we get $\int_a^b f(x)dx = F(b) - F(a)$. ///

Change of Variable formula

Theorem 4.2.5. *Let $u(t), u'(t)$ be continuous on $[a, b]$ and f is a continuous function on the interval $u([a, b])$. Then*

$$\int_a^b f(u(x))u'(x)dx = \int_{u(a)}^{u(b)} f(y)dy.$$

Proof. Note that $f([a, b])$ is a closed and bounded interval. Since f is continuous, it has primitive F. i.e., $F(x) = \int_a^x f(t)dt$. Then by chain rule of differentiation, $\frac{d}{dt}F(u(t)) = F'(u(t))u'(t)$. i.e., $F(u(t))$ is the primitive of $f(u(t))u'(t)$ and by Newton-Leibnitz formula, we get

$$\int_a^b f(u(t))u'(t)dt = F(u(b)) - F(u(a)).$$

On the other hand, for any two points in $u([a, b])$, we have (by Newton-Leibnitz formula)

$$\int_A^B f(y)dy = F(B) - F(A).$$

Hence $B = u(b)$ and $A = u(a)$.

Example: Evaluate $\int_0^1 x\sqrt{1 + x^2}dx$.

Taking $u = 1 + x^2$, we get $u' = 2x$ and $u(0) = 1, u(1) = 2$. Then

$$\int_0^1 x\sqrt{1+x^2}dx = \frac{1}{2}\int_1^2 \sqrt{u}du = \frac{1}{3}u^{\frac{2}{3}}\Big|_{u=1}^{2} = \frac{1}{3}(2^{\frac{2}{3}} - 1).$$

4.3 Improper Integrals

In the previous section, we defined Riemann integral for functions defined on closed and bounded interval $[a, b]$. In this section our aim is to extend the concept of integration to the the following cases:

1. The function $f(x)$ defined on unbounded interval $[a, \infty)$ and $f \in \mathcal{R}[a, b]$ for all $b > a$.

2. The function is not defined at some points on the interval $[a, b]$.

We first consider
Improper integral of first kind: Suppose f is a bounded function defined on $[a, \infty)$ and $f \in \mathcal{R}[a, b]$ for all $b > a$.

Definition 4.3.1. *The improper integral of f on $[a, \infty)$ is defined as*

$$\int_a^\infty f(x)dx := \lim_{b\to\infty} \int_a^b f(x)dx.$$

If the limit exists and is finite, we say that the improper integral converges. If the limit goes to infinity or does not exist, then we say that the improper integral diverges.

Examples:

1. (i) $\int_1^\infty \frac{1}{x^2}dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}dx = \lim_{b\to\infty} 1 - \frac{1}{b} = 1.$

2. (ii) $\int_0^\infty \frac{dx}{1+x^2} = \lim_{b\to\infty} \int_0^b \frac{dx}{1+x^2} = \lim_{b\to\infty} \arctan x\Big|_0^b = \frac{\pi}{2}.$

Theorem 4.3.2. ***Comparison test:*** *Suppose $0 \le f(x) \le \phi(x)$ for all $x \ge a$, then*

1. *$\int_a^\infty f(x)dx$ converges if $\int_a^\infty \phi(x)dx$ converges.*

2. *$\int_a^\infty \phi(x)dx$ diverges if $\int_a^\infty f(x)dx$ diverges.*

Proof. Define $F(x) = \int_a^x f(t)dt$ and $G(x) = \int_a^x g(t)dt$. Then by properties of Riemann integral, $0 \le F(x) \le G(x)$ and we are given that $\lim\limits_{x\to\infty} G(x)$ exists. So $G(x)$ is bounded. F is monotonically increasing and bounded above. Therefore, $\lim\limits_{x\to\infty} F(x)$ exists.

Examples:

1. $\int_1^\infty \frac{dx}{x^2(1+e^x)}$. Note that $\frac{1}{x^2(1+e^x)} < \frac{1}{x^2}$ and $\int_1^\infty \frac{dx}{x^2}$ converges.

2. $\int_1^\infty \frac{x^3}{x+1}dx$. Note that $\frac{x^3}{x+1} > \frac{x^2}{2}$ on $[1,\infty)$ and $\int_1^\infty x^2dx$ diverges.

Definition 4.3.3. *Let $f \in \mathcal{R}[a,b]$ for all $b > a$. Then we say $\int_a^\infty f(x)dx$ converges absolutely if $\int_a^\infty |f(x)|dx$ converges.*

In the following we show that absolutely convergence implies convergence of improper integral.

Theorem 4.3.4. *If the integral $\int_a^\infty |f(x)|dx$ converges, then the integral $\int_a^\infty f(x)dx$ converges.*

Proof. Note that $0 \leq f(x) + |f(x)| \leq 2|f(x)|$. So the improper integral $\int_a^\infty f(x) + |f(x)|dx$ converges by comparison theorem above. Also $\int_a^\infty |f(x)|$ converges. Therefore, $\int_a^\infty f(x)dx = \int_a^\infty f(x) + |f(x)|dx - \int_a^\infty |f(x)|dx$ also converges. ///

The converse of the above theorem is not true. For example take the integral $\int_\pi^\infty \frac{\sin x}{x}dx$. This integral does not converge absolutely. Indeed,

$$\begin{aligned}
\int_\pi^\infty \frac{|\sin x|}{x}dx &= \sum_{n=1}^\infty \int_{n\pi}^{(n+1)\pi} \frac{|\sin x|}{x}dx \\
&\geq \sum_{n=1}^\infty \frac{1}{n\pi} \int_{n\pi}^{(n+1)\pi} |\sin x| \\
&= \sum_{n=0}^\infty \frac{1}{n\pi} \int_0^\pi \sin x = \frac{2}{\pi}\sum_{n=1}^\infty \frac{1}{n}.
\end{aligned}$$

On the other hand, by integration by parts,

$$\begin{aligned}
\lim_{b\to\infty} \int_1^b \frac{\sin x}{x}dx &= \lim_{b\to\infty} \int_1^b \frac{1}{x}d(1-\cos x) \\
&= \lim_{b\to\infty} \left(\frac{(1-\cos b}{b} + \int_1^b \frac{1-\cos x}{x^2}dx \right)
\end{aligned}$$

It is not difficult to show that the limits on the right exist.

Examples:

1. $\int_1^\infty \frac{\sin x}{x^3}dx$. Easy to see that $|\frac{\sin x}{x^3}| \leq |\frac{1}{x^3}|$ and $\int_1^\infty \frac{dx}{x^3}$ converges.

2. $\int_0^\infty \frac{e^{-x^2}\sin x}{\log(1+x)}$. Here first note that $\lim_{x\to 0} \frac{e^{-x^2}\sin x}{\log(1+x)} = 1$. Therefore the integral is proper

at $x = 0$. For $x > 10$(say):

$$|f(x)| \leq \frac{e^{-x^2}}{\log(1+x)} < e^{-x^2} \leq e^{-x}$$

Hence the integral $\int_{10}^{\infty} \frac{e^{-x^2}\sin x}{\log(1+x)}$ converges by comparison theorem.

Theorem 4.3.5. ***Limit comparison test:*** *Let $f(x), g(x)$ are defined and positive for all $x \geq a$ and $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = L$.*

1. *If $L \in (0, \infty)$, then the improper integrals $\int_a^{\infty} f(x)dx$ and $\int_a^{\infty} g(x)dx$ are either both convergent or both divergent. i.e., $\int_a^{\infty} f(x)dx$ converges $\iff$ $\int_a^{\infty} g(x)$ converges.*

2. *If $L = 0$, then $\int_a^{\infty} f(x)dx$ converges if $\int_a^{\infty} g(x)dx$ converges. i.e, $\int_a^{\infty} g(x)dx$ converges $\implies$ $\int_a^{\infty} f(x)dx$ converges.*

3. *If $L = \infty$, then $\int_a^{\infty} f(x)dx$ diverges if $\int_a^{\infty} g(x)dx$ diverges. i.e., $\int_a^{\infty} g(x)dx$ diverges $\implies$ $\int_a^{\infty} f(x)dx$ diverges.*

Proof. From the definition of limits, for any $\epsilon > 0$, there exists $M > 0$ such that

$$x \geq M \implies L - \epsilon < \frac{f(x)}{g(x)} < L + \epsilon.$$

Thus for $x \geq M$, we have $(L - \epsilon)g(x) < f(x) < (L + \epsilon)g(x)$.
Now in case (1), since $L > 0$, we can find $\epsilon > 0$ such that $L - \epsilon > 0$. Using the property 3, it is enough to prove the convergence/divergence for x large, say $x \geq M$. In this interval, we have the comparison $(L - \epsilon)g(x) < f(x) < (L + \epsilon)g(x)$. Now integrating this, we get the result.
In case (2), we have $f(x) < (L + \epsilon)g(x)$. Again, integrate on both sides.
In case (3) by the definition, for every $M > 0$, there exists, R such that $f(x) > Mg(x)$ for all $x > R$. Now the result follows similar to (1) and (2).

Examples:

1. $\int_1^{\infty} \frac{dx}{\sqrt{x+1}}$. Take $f(x) = \frac{1}{\sqrt{x+1}}$ and $g(x) = \frac{1}{\sqrt{x}}$. Then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$ and $\int_1^{\infty} g(x)dx$ diverges. So by above theorem, $\int_1^{\infty} f(x)dx$ diverges.

2. $\int_1^{\infty} \frac{dx}{1+x^2}$. Take $f(x) = \frac{1}{1+x^2}$ and $g(x) = \frac{1}{x^2}$. Then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$ and $\int_1^{\infty} g(x)dx$ converges. So by above theorem, $\int_1^{\infty} f(x)dx$ converges.

3. $\int_0^\infty \frac{x}{\cosh x}dx$. Let $f(x) = \frac{x}{\cosh x} = \frac{2xe^x}{e^{2x}+1} \sim xe^{-x}$. So choose $g(x) = xe^{-x}$. Then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 2$ and $\int_0^\infty g(x)$ converges.

Improper integrals of second kind

Definition 4.3.6. *Let $f(x)$ be defined on $[a, c)$ and $f \in \mathcal{R}[a, c-\epsilon]$ for all $\epsilon > 0$. Then we define*

$$\int_a^c f(x)dx = \lim_{\epsilon\to 0} \int_a^{c-\epsilon} f(x)dx.$$

Then $\int_a^b f(x)dx$ is said to converge if the limit exists and is finite. Otherwise, we say improper integral $\int_a^b f(x)dx$ diverges.

Example: $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{\epsilon\to 0} \int_\epsilon^1 \frac{dx}{\sqrt{x}} = \lim_{\epsilon\to 0} 2(1-\sqrt{\epsilon}) = 2.$
Suppose $a_1, a_2, ...a_n$ are finitely many discontinuities of $f(x)$ in $[a, b]$. Then

$$\int_a^b f(x)dx = \int_a^{a_1} f(x)dx + \int_{a_1}^{a_2} f(x)dx + \int_{a_2}^{a_3} f(x)dx + ... + \int_{a_n}^b f(x)dx$$

If all the improper integrals on the right hand side converge, then we say the improper integral of f over $[a, b]$ converges. Otherwise, we say it diverges.

The following comparison and Limit comparison tests can be proved following similar arguments:

Theorem 4.3.7. *(Comparison Theorem:) Suppose $0 \le \phi(x) \le f(x)$ for all $x \in [a, c)$ and are discontinuous at c.*

1. *If $\int_a^c f(x)dx$ converges then $\int_a^c \phi(x)dx$ converges.*
2. *If $\int_a^c \phi(x)dx$ diverges then $\int_a^c f(x)dx$ diverges.*

Theorem 4.3.8. *(Limit comparison theorem:) Suppose $0 < f(x), g(x)$ be continuous in $[a, c)$ and $\lim_{x\to c} \frac{f(x)}{g(x)} = L$. Then*

1. *If $L \in (0, \infty)$. Then $\int_a^c f(x)dx$ and $\int_a^c g(x)dx$ both converge or diverge together.*
2. *If $L = 0$ and $\int_a^c g(x)dx$ converges then $\int_a^c f(x)dx$ converges.*
3. *If $L = \infty$ and $\int_a^c g(x)dx$ diverges then $\int_a^c f(x)dx$ diverges.*

Proof. From the definition of limit, for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$x \in (c-\delta, c) \implies (L-\epsilon)g(x) < f(x) < (L+\epsilon)g(x)$$

Rest of the proof follows in the similar lines theorems on first kind, by choosing $\epsilon < L$.

Transforming improper integrals:

Sometimes improper integrals may be transformed into proper integrals. for example consider the improper integral $I = \int_1^3 \frac{dx}{\sqrt{x}\sqrt{3-x}}$. Taking the transformation $y = \frac{1}{3-x}$, we get $I = \int_{1/2}^{\infty} \frac{dy}{y\sqrt{3y-1}}$. This is an improper integral of first kind. Instead, if we choose the transformation $3 - x = u^2$ then $I = \int_0^{\sqrt{2}} \frac{2udu}{u\sqrt{3-u^2}}$, which is a proper integral.

Remark 4.6. *It is important to note that the "symmetric" limit could be convergent but the limit may not exist. For example,*

$$\begin{aligned}\int_{-1}^{1} \frac{dx}{x^3} &= \int_{-1}^{0} \frac{dx}{x^3} + \int_0^1 \frac{dx}{x^3} \\ &= \lim_{\epsilon_1 \to 0} \int_{-1}^{-\epsilon_1} \frac{dx}{x^3} + \lim_{\epsilon_2 \to 0} \int_{\epsilon_2}^{1} \frac{dx}{x^3} \\ &= \frac{1}{2} \lim_{\epsilon_1, \epsilon_2 \to 0} \left(\frac{1}{\epsilon_1^2} - \frac{1}{\epsilon_2^2}\right), \end{aligned} \tag{4.7}$$

where $\epsilon_1, \epsilon_2 \to 0$ as $n \to \infty$. It is easy to see that if one takes $\epsilon_1 = \epsilon_2$, then the limit exits and is equal to 0*. But if one takes $\epsilon_1 = \frac{1}{(n+1)^2}, \epsilon_2 = \frac{1}{n^2}$, then the above limit in* (4.7) *does not exist. So through different sequences, we are getting different limits. By now, by our familiarity with existence of limits, we say integral diverges.*

Gamma and Beta functions:

Consider the *Gamma function* defined as improper integral for $p > 0$,

$$\Gamma(p) = \int_0^{\infty} x^{p-1} e^{-x} dx$$

This integral is improper of second kind in the neighbourhood of 0 as x^{p-1} goes to infinity as $x \to 0$ (when $p < 1$). Since the domain of integration is $(0, \infty)$, the integral is improper of first kind. To prove the convergence, we divide the integral into

$$\begin{aligned}\Gamma(p) =& \int_0^1 x^{p-1}e^{-x}dx + \int_1^\infty x^{p-1}e^{-x}dx \\ =& I_1 + I_2\end{aligned}$$

To see the convergence of I_1 we take $f(x) = x^{p-1}e^{-x}$ and $g(x) = x^{p-1}$, then $\lim_{x\to 0} \frac{f(x)}{g(x)} = 1$ and $\int_0^1 x^{p-1}dx$ converges. To see the convergence of I_2, take $f(x) = x^{p-1}e^{-x}$ and $g(x) = \frac{1}{x^2}$. Then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} x^{2+p-1}e^{-x} = 0$ and $\int_1^\infty \frac{1}{x^2}dx$ converges. Hence by (2) of limit comparison theorem, the integral converges.

Next we consider the *Beta function* defined as improper integral for $p > 0, q > 0$,

$$\beta(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx$$

If $p > 1$ and $q > 1$, then the integral is definite integral. When $p < 1$ and/or $q < 1$, this integral is improper of second kind at 0 and/or 1. To prove the convergence, we divide as before

$$\begin{aligned}\int_0^1 x^{p-1}(1-x)^{q-1}dx =& \int_0^{1/2} x^{p-1}(1-x)^{q-1}dx + \int_{1/2}^1 x^{p-1}(1-x)^{q-1}dx \\ =& I_1 + I_2.\end{aligned}$$

To see the convergence of I_1, take $f(x) = x^{p-1}(1-x)^{q-1}$ and $g(x) = x^{p-1}$. Then $\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0}(1-x)^{q-1} = 1$ and $\int_0^{1/2} x^{p-1}dx$ converges. Similarly, for convergence of I_2, we take $f(x) = x^{p-1}(1-x)^{q-1}$ and $g(x) = (1-x)^{q-1}$.

Some identities of beta and gamma functions:

1. $\Gamma(1) = \int_0^\infty e^{-x}dx = 1$.
2. $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$.
   Integration by parts formula implies,

$$\Gamma(\alpha + 1) = \int_0^\infty x^\alpha e^{-x} = -(x^\alpha e^{-x})|_0^\infty + \alpha \int_0^\infty x^{\alpha-1}e^{-x}dx = \alpha\Gamma(\alpha).$$

Therefore, $\Gamma(m+1) = m! \ \forall m \in I\!N$.

3. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

$$\begin{aligned}\left(\Gamma(\frac{1}{2})\right)^2 =&4\int_0^\infty \int_0^\infty e^{-u^2}e^{-v^2}dudv, \quad \text{take } \ u = r\cos\theta, v = r\sin\theta,\\ =&4\int_0^\infty \int_0^{\pi/2} e^{-r^2}rdrd\theta = \pi\end{aligned}$$

4. $\beta(m,n) = \beta(n,m)$. Substituting $t = 1-x$ in the definition of $\beta(m,n)$, we get

$$\beta(m,n) = \int_0^1 t^{n-1}(1-t)^{m-1}dt = \beta(n,m)$$

5. $\beta(m,n) = 2\displaystyle\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta d\theta$
   Taking $x = \sin^2\theta$ in $\beta(m,n)$, we get

$$\beta(m,n) = \int_0^\pi \cos^{2m-2}\theta\sin^{2n-2}\theta\cos\theta\sin\theta d\theta = 2\int_0^{\pi/2}\cos^{2m-1}\theta\sin^{2n-1}\theta d\theta.$$

6. $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

Problem: Evaluate $(i)\ \displaystyle\int_0^\infty x^{2/3}e^{-\sqrt{x}}dx \quad (ii)\ \int_0^1 x^{\frac{3}{2}}(1-\sqrt{x})^{\frac{1}{2}}dx$
For (i), take $t = \sqrt{x}$, then the given integral becomes

$$\int_0^\infty t^{4/3}e^{-t}2tdt = 2\int_0^\infty e^{-t}t^{7/3}dt = 2\Gamma(\frac{10}{3}) = \frac{56}{27}\Gamma(1/3).$$

For (ii), again take $t = \sqrt{x}$, then the integral becomes

$$2\int_0^1 t^3(1-t)^{1/2}tdt = 2\int t^4(1-t)^{1/2}dt = 2\beta(5,3/2) = 2\frac{\Gamma(5)\Gamma(3/2)}{\Gamma(13/2)} = \frac{512}{3465}$$

Cauchy Principal Value:

Consider the improper integral $I = \int_0^\infty \sin xdx$. It is easy to see from the definition that $I = \lim_{a\to\infty}(1-\cos a)$ which does not exist.Similarily, $\int_{-\infty}^0 \sin xdx$ does not exist. But

$$\lim_{c\to\infty}\int_{-c}^{c}\sin xdx$$

exists and is equal to 0. Though the improper integral does not exist, this symmetric limit exists. This is called *Cauchy Principal value* of improper integral

Definition 4.3.9. *The Cauchy Principal value of improper integral of first kind is defined as*

$$CPV \int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx$$

For the improper integral of second kind, with $c \in (a,b)$ as point of discontinuity of $f(x)$ as

$$CPV \int_{a}^{b} f(x)dx = \lim_{\delta\to 0} \int_{a}^{c-\delta} f(x)dx + \int_{c+\delta}^{b} f(x)dx$$

Examples:

First kind: $\int_{-\infty}^{\infty} x^{2n+1}dx$ for all $n = 1, 2, 3, ...$ In this case it is easy to check that $\lim_{a\to\infty} \int_{-a}^{a} x^{2n+1} = 0$. But the the improper integrals $\int_0^\infty x^{2n+1}$ and $\int_{-\infty}^{0} x^{2n+1}$ does not converge.

Second kind: $\int_{-1}^{1} x^{-(2n+1)}dx$, for $n = 1, 2, 3, ...$ Simply evaluate

$$\int_{-1}^{-\epsilon} x^{-(2n+1)} + \int_{\epsilon}^{1} x^{-(2n+1)}$$

to see that the the limit is 0.

Integrals dependent on a Parameter

Cosider an integral

$$I(\alpha) = \int_{a}^{b} f(x,\alpha)dx$$

where the integrand is depend on the parameter α. At times we can differentiate under the integral sign to evaluate the integral. It is sometimes not possible and leads to wrong assertions. For example, we know that $I = \int_0^\infty \frac{\sin x}{x} = \frac{\pi}{2}$. It is easy to notice with change of variable formula, taking $tx = y$, that $I = I(t) = \int_0^\infty \frac{\sin(tx)}{x} = \frac{\pi}{2}$. Now differentiating this, taking derivative inside integral, we get $I'(t) = \int_0^\infty \cos(tx)dx = 0$, which doesn't make sense.

Here we have a theorem, which explains under which conditions we can do the differentiation under integral sign.

Theorem 4.3.10. *Suppose,*

1. *Suppose $f, \frac{d}{d\alpha}f(x,\alpha)$ are continuous functions for $x \in [a,b]$ and α in an interval of containing α_0.*

2. *$|f(x,\alpha)| \le a(x), |\frac{d}{d\alpha}f(x,\alpha)| \le b(x)$ such that a, b are integrable on $[a,b]$.*

Then I is differentiable, and

$$I'(\alpha) = \int_a^b \frac{d}{d\alpha} f(x,\alpha)dx.$$

Proof. .

$$\begin{aligned}\frac{d}{d\alpha}I(\alpha) &= \lim_{\Delta\alpha \to 0} \frac{I(\alpha+\Delta\alpha) - I(\alpha)}{\Delta\alpha} \\ &= \lim \frac{1}{\Delta\alpha}\left[\int_a^b (f(x,\alpha+\Delta\alpha) - f(x,\alpha))dx\right]\end{aligned}$$

Now by Taylor's theorem, $f(x,\alpha+\Delta\alpha) - f(x,\alpha) = \Delta\alpha\frac{d}{d\alpha}f(x,\alpha+\theta\Delta\alpha)$. Since $\frac{d}{d\alpha}f(x,\alpha)$ is continuous, we have $\frac{d}{d\alpha}f(x,\alpha+\theta\Delta\alpha) = \frac{d}{d\alpha}f(x,\alpha) + \epsilon$, where $0 < \theta < 1$ and $\epsilon \to 0$ as $\Delta\alpha \to 0$. Thus

$$\frac{d}{d\alpha}I(\alpha) = \lim_{\Delta\alpha \to 0}\int_a^b \frac{d}{d\alpha}f(x,\alpha) + \epsilon = \int_a^b \frac{d}{d\alpha}f(x,\alpha)dx$$

///

In fact the following holds.

Newton-Leibnitz Formula:

Let $h(x) = \displaystyle\int_{a(x)}^{b(x)} f(x,t)dt$. Then $h'(x) = \displaystyle\int_{a(x)}^{b(x)} \frac{df}{dx}(x,t)dt + f(x,b(x))b'(x) - f(x,a(x))a'(x)$

Examples:

1. Evaluate $I(\alpha) = \displaystyle\int_0^\infty e^{-x}\frac{\sin\alpha x}{x}dx$.
   By the above formula, $I'(\alpha) = \int_0^\infty e^{-x}\cos\alpha x dx = \frac{1}{1+\alpha^2}$. Therefore, $I(\alpha) = \arctan\alpha + C$. Also $I(0) = \int_0^\infty e^{-x}\sin 0x = 0$. Hence $C = 0$.

2. Test the convergence and evaluate the integral $\displaystyle\int_0^\infty e^{\frac{1}{2}(t^2-x^2)}\cos(tx)dx$.

$$|I| \le e^{t^2/2}\int_0^\infty |e^{-x^2}\cos(tx)|dx \le C\int_0^\infty e^{-x^2}dx = \frac{1}{2}\Gamma(\frac{1}{2}).$$

Hence the integral converges. By Newton Leibnitz formula,

$$
\begin{aligned}
I_a'(t) &= \int_0^a e^{\frac{1}{2}(t^2-x^2)}(t\cos tx - x\sin tx)dx \\
&= \int_0^a \frac{\partial}{\partial x} e^{\frac{1}{2}(t^2-x^2)} \sin tx dx \\
&= e^{\frac{1}{2}(t^2-a^2)} \sin at
\end{aligned}
$$

Therefore, $I'(t) = \lim_{a\to\infty} I_a'(t) = 0$. Now note that $I(0) = \int_0^\infty e^{-x^2/2}dx = \frac{1}{\sqrt{2}}\Gamma(1/2)$. Hence $I(t) = \sqrt{\frac{\pi}{2}}$.

4.4 Approximation of definite integrals

Suppose we are given a function $f : [a,b] \to \mathbb{R}$ that is integrable on $[a,b]$ and has its Taylor series in an interval containing $[a,b]$. Then we can approximate the definite integral using the Taylor series. Let $f(x) = \sum_{k=0}^\infty a_n t^n$ for $|t| < R$. Then we know from the term-by-term integration theorem,

$$
\int_0^t f(x)dx = \sum_{k=0}^\infty \int_0^t a_n x^n dx = \sum_{k=0}^\infty a_n \frac{t^{n+1}}{n+1} \quad |t| < R
$$

Example: Find the approximate value of the definite integral $\int_0^1 x^2\sin(x^2)dx$ with error less than 10^{-4}.
We note that if the n^th term has k zeros in the first k decimals, then the approximation is with error 10^{k-1}.

$$
\begin{aligned}
\int_0^1 x^2\sin(x^2)dx &= \int_0^1 \sum_{n=0}^\infty (-1)^n \frac{x^{4n+4}}{(2n+1)!} \\
&= \sum_{n=0}^\infty \frac{(-1)^n x^{4n+5}}{(4n+5)(2n+1)!}\Bigg|_0^1 \\
&= \sum_{n=0}^\infty \frac{(-1)^n}{(4n+5)(2n+1)!}
\end{aligned}
$$

Hence

$$
\int_0^1 x^2\sin(x^2) = \frac{1}{5} - \frac{1}{54} + \frac{1}{13\times 5!} - \frac{1}{17\times 7!}
$$

Since the 4th term has 4 zeros in the first 4 decimal places. So the approximation has error less than 10^{-4}.

Application: A pendulum of mass m suspended by a inextensible string of length L is released from initial position of angle α (from equilibrium position). Let θ be the angular coordinate. Then the quarter-period time $\tau/4$ is given by

$$\frac{\tau}{4} = \sqrt{\frac{L}{4g}} \int_0^{\alpha} \left(\sin^2(\frac{\alpha}{2}) - \sin^2(\frac{\theta}{2}) \right)^{-\frac{1}{2}} d\theta.$$

By taking the transformation $\theta \mapsto \phi$ defined as $\sin\phi = \frac{\sin(\theta/2)}{\sin(\alpha/2)}$, this integral is transformed to

$$\int_0^{\pi/2} (1 - k^2 \sin^2\phi)^{-\frac{1}{2}} d\phi, \quad k = \sin^2(\alpha/2).$$

This integral is an improper integral which is difficult to evaluate. We can use Taylor series to find approximate value as described in the previous example.

Now suppose the function is not continuous but only integrable. Then we can still approximate the definite integral using Riemann sums. Recall from section 3.1, that if the function is integrable then definite integral can be computed as

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(\xi_k)\Delta x_k$$

where $\xi_k \in [x_{k-1}, x_k]$ and $\Delta x_k = x_k - x_{k-1}$. When we take equal partitions, we get $\Delta x_k = \frac{b-a}{n}$. We denote $y_0 = f(x_0), y_1 = f(x_1), ...y_n = f(x_n)$. When we take $\xi_k = x_{k-1}$ and $\xi_k = x_k$ we get the sums

$$((y_0 + y_1 + ... + y_{n-1})\,\Delta x$$

$$(y_1 + y_2 + ... + y_n)\,\Delta x$$

Each of these sums is a Riemann sum and converges to definite integral $\int_a^b f(x)dx$ when $n \to \infty$. So we write *Rectangular formula*

$$\int_a^b f(x)dx \approx (y_0 + y_1 + ... + y_{n-1})\,\frac{b-a}{n}$$

$$\int_a^b f(x)dx \approx (y_1 + y_2 + ... + y_n)\,\frac{b-a}{n}.$$

These approximations involve the area of rectangles with sides $x_{k-1}x_k$ and height y_{k-1} or y_k . Instead of taking the end points, we could take mid-points of the interval for "better approximation" to obtain *Trapezoidal Rule*:

$$\int_a^b f(x)dx \approx \left(\frac{y_0 + y_1}{2}\Delta x + \frac{y_1 + y_2}{2}\Delta x + ... + \frac{y_{n-1} + y_n}{2}\Delta x\right)$$
$$\approx \frac{b-a}{n}\left(\frac{y_0 + y_n}{2} + y_1 + y_2 + ... + y_{n-1}\right)$$

As the name suggests, the first two formulas involve the sum of areas of rectangles and trapezoidal rule consists of areas of trapezoids. This is also seen as piecewise linear approximation of the function $y = f(x)$. (Explain Geometrically)!

We can also take quadratic approximation or with exact values at three points. Approximations like this and estimation of errors is part of Numerical integration theory which is beyond the scope of this course.

4.5 Applications to Area, Arc length, Volume and Surface area

Suppose $f(x) \geq 0$ on $[a, b]$. Then it is clear from the definition of Definite integral that the area under the curve $y = f(x)$ can be approximated by Riemann sums. i.e.,

$$A \cong \sum_{k=1}^{n} f(\xi_i)(x_i - x_{i-1}) \to \int_a^b f(x)dx \quad \text{as } n \to \infty.$$

Similarly, the area bounded by the curves $y = f(x)$ and $y = g(x)$ where $f(x) \geq g(x)$ on $[a, b]$ is

$$A = \int_a^b (f(x) - g(x))dx.$$

Example: Find the area bounded by $y = x^2$ and $y^2 = x$

The curves interest at $x = 0, 1$. The upper curve is $y^2 = x$ and lower curve is $y^2 = x$. So by the above formula

$$A = \int_0^1 (\sqrt{x} - x^2)dx = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

One can also find by integrating along y: $A = \displaystyle\int_0^1 (\sqrt{y} - y^2)dy = \frac{1}{3}$.

Polar coordinates

A point (x, y) on the xy-plane is assigned polar coordinates (r, θ) if the point is at a distance $r = \sqrt{x^2 + y^2}$ from the origin on the ray at an angle θ with positive x-axis. We allow r negative with convention: $(-r, \theta) = (r, \theta + \pi)$. Each point on the plane has infinitely many representations in polar form, for example $(1, 0)$ is at a distance of 1 units from the origin on the x-axis. So it can be represented in polar form also as $(r, \theta) = (1, 0)$. Also it is same as $(1, 2n\pi), n \in \mathbb{N}$ and $(-1, \pi)$. Each point (r, θ) is same as $(r, \theta + 2n\pi)$ for all $n \in \mathbb{N}$.

Example: The point $(2, \pi/6)$ can also be represented by $(-2, \frac{5\pi}{6})$ and $(-2, \frac{7\pi}{6})$

Relation with cartesian coordinates:

We often use the following relations:

1. Given the polar coordinates $(r\theta)$, we can write the cartesian coordinates using $x = r\cos\theta$, and $y = r\sin\theta$.

2. Given the cartesian coordinates (x, y), we can write polar coordinates using $r = \sqrt{x^2 + y^2}$, and $\theta = \tan^{-1}(\frac{y}{x})$

Circles and Straight lines:

1. The circle $x^2+y^2 = a^2$ in cartesian coordinates, using (1) above, $r^2 \cos^2\theta + r^2 \sin^2\theta = a^2$ which is $r = a$.

2. The circle $(x-a)^2 + y^2 = a^2 \implies x^2 + y^2 - 2ax = 0$, again by (1) above we get $r^2 - 2ar\cos\theta = 0 \implies r = 2a\cos\theta$.

3. The circle $x^2 + (y-a)^2 = a^2 \implies x^2 + y^2 - 2ay = 0$, again by (1) above we get $r^2 - 2ar\sin\theta = 0 \implies r = 2a\sin\theta$.

4. The straight line $y = mx$ is $\theta = \tan^{-1} m$

5. The straight line $x = a$ is $r = a\sec\theta$ and $y = b$ is $r = b\csc\theta$.

Symmetry in polar coordinates: The symmetry of the graph of the function in polar coordinates helps one to plot/trace the graph. There are three types of symmetry principles.

1. For (r, θ) on the graph, suppose $(r, -\theta)$ is also on the graph. Then the graph is symmetric about $x-$ axis.

2. For (r, θ) on the graph, suppose $(r, \pi - \theta)$ is also on the graph. Then the graph is symmetric about y- axis.

3. For (r, θ) on the graph, suppose $(r, \pi + \theta)$ is also on the graph. Then the graph is symmetric about the origin.

Examples:

1. (leminiscate): Consider the function $r^2 = \cos 2\theta$. If (r, θ) is on the graph, then $r^2 = \cos 2(-\theta) = \cos 2\theta$ implies $(r, -\theta)$ is also on the graph. So the graph is symmetric about $x-$ axis.
   Again, $r^2 = \cos 2(\pi - \theta) = \cos(2\pi - 2\theta) = \cos 2\theta$ implies $(r, \pi + \theta)$ is also on the graph. Therefore, graph is symmetric about y-axis.
   We can also see that $(r, \pi + \theta)$ is also on the graph. So the graph is also symmetric about the origin.
   Hence it is enough to trace the curve in the first quadrant. Now since $r^2 \geq 0$, the domain of θ in the first quadrant is $[-\frac{\pi}{4}, \frac{\pi}{4}]$. Also one can see by the derivative test that $\theta = 0$ is a point of local maxima(see figure 1).

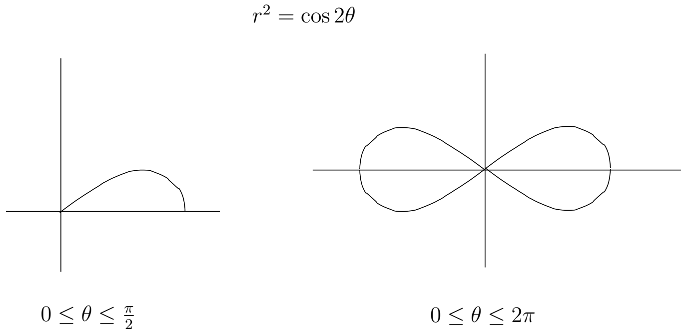

Figure 1: leminiscate

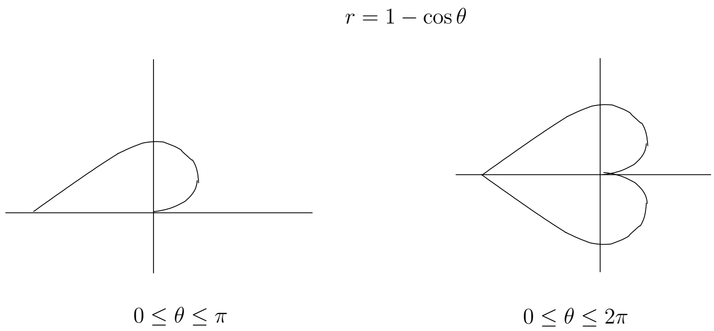

Figure 2: Cardioid

2. (Cardioid): Consider the function $r = 1 - \cos\theta$. Then if $(r, \theta) \in$ graph $\implies$ $(r, -\theta) \in$ graph. So the graph is symmetric with respect to $x-$ axis. So it is enough to trace the curve for $0 \le \theta \le \pi$. Again by derivative test we see that $\theta = 0$ is a point of minimum and $\theta = \pi$ is point of maximum(see figure 2).

Area in polar coordinates: Let a region be bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and the curve $r = f(\theta)$. We approximate the region with n non-overlapping circular sectors based on the partition P of angle $\theta \in [\alpha, \beta]$. The typical sector has radius $r_k = f(\theta_k)$ and central angle of radian measure $\Delta\theta_k$. Its area is $\frac{\Delta\theta_k}{2\pi}$ times the area of a circle r_k. i.e.,

$$A_k = \frac{1}{2} r_k^2 \Delta\theta_k = \frac{1}{2} f(\theta_k)^2 \Delta\theta_k$$

The area of the region is approximately $\sum_{k=1}^{n} A_k$. Taking $n \to \infty$ so that $\|P\| \to 0$, we get

$$A = \int_\alpha^\beta \frac{1}{2} f(\theta)^2 d\theta = \frac{1}{2}\int_\alpha^\beta r^2 d\theta.$$

Example: Find the area of the region enclosed by the cardioid $r = 2(1-\cos\theta)$.
Solution: From the graph discussed above, the range of θ is from 0 to 2π. Therefore, the area is

$$A = \frac{1}{2}\int_0^{2\pi} r^2 d\theta = \int_0^{2\pi} (3 + \cos 2\theta - 4\cos\theta) d\theta = 6\pi.$$

Arc length

Consider a curve defined by $y = f(x)$ between $x = a$ and $x = b$. For example $y = \sin x$ between $x = 0$ and π. The length of this curve can be approximated by sum of lengths of straight lines connecting $(0,0) \to (\pi/4, \sin(\pi/4)) \to (\pi/2, \sin(\pi/2)) \to (\pi, 0)$. The arc length s is approximately

$$\sqrt{(\frac{\pi}{4})^2 + (\frac{1}{\sqrt{2}})^2} + \sqrt{(\frac{\pi}{4})^2 + (1 - \frac{1}{\sqrt{2}})^2} + \sqrt{(\frac{\pi}{2})^2 + 1}.$$

This approximation becomes better and better as we refine the partition $\mathcal{P} = \{0, \pi/4, \pi/2, \pi\}$. For a given curve defined by function $y = f(x)$ between $x = a, b$, we consider the partition $\mathcal{P} = \{a = x_0, x_1, x_2, ...x_{k-1}, x_k, ...x_n = b\}$. Then the length of this curve may be approximated by the formula

$$\begin{aligned} s &\sim \sum_{i=1}^n \sqrt{(x_i - x_{i-1})^2 + (f(x_i) + f(x_{i-1}))^2} \\ &= \sum_{i=1}^n \sqrt{1 + \left(\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}\right)^2} \ (x_i - x_{i-1}) \\ &\to \int_a^b \sqrt{1 + (f'(x))^2} dx \ \text{ as } n \to \infty \end{aligned}$$

The following two formulas are used for finding the *Arc length* or *length of curve*:

1. For a function $y = f(x)$ between $x = a$ and $x = b$

$$s = \int_a^b \sqrt{1 + \left(\frac{df}{dx}\right)^2} \, dx.$$

2. For a function $x = f(y)$ between $y = c$ and $y = d$

$$s = \int_c^d \sqrt{1 + \left(\frac{df}{dy}\right)^2}\, dy.$$

Parametric form: Suppose if an arc is defined in the parametric form $x = x(t), y = y(t)$ between $t = T_1$ and $t = T_2$. Then we note from above approximation, that s may be approximated by taking the partition $\mathcal{P} = \{T_1 = t_0, t_1, ..., t_n = T_2\}$ and

$$\begin{aligned} s \sim & \sum_{i=1}^{n} \sqrt{\left(\frac{x_i - x_{i-1}}{t_i - t_{i-1}}\right)^2 + \left(\frac{y_i - y_{i-1}}{t_i - t_{i-1}}\right)^2}(t_i - t_{i-1}) \\ \to & \int_{T_1}^{T_2} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt \text{ as } n \to \infty. \end{aligned}$$

Example: Find the arc length of the curve defined by $x = 2\cos^2\theta,\ y = 2\cos\theta\sin\theta$, $0 \leq \theta \leq \pi$.

Solution: This curve is a circle with radius 1 at $(1, 0)$. So the answer should be 2π. Applying formula

$$\begin{aligned} s = & \int_0^\pi \sqrt{x'(\theta)^2 + y'(\theta)^2}\, d\theta = 2\int_0^\pi \sqrt{(2\cos\theta\sin\theta)^2 + (\cos^2\theta - \sin^2\theta)^2}\, d\theta \\ =& 2\int_0^\pi \sqrt{\cos^4\theta + 2\cos^2\theta\sin^2\theta + \sin^4\theta}\, d\theta = 2\pi \end{aligned}$$

Example 2: Find the arc length of an arc of $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$.

Solution: Take the parametrization: $x = \cos^3 t, y = \sin^3 t, 0 \leq t \leq \frac{\pi}{2}$. Then

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9\cos^2 t\sin^2 t.$$

Hence the arc length is

$$l = 3\int_0^{\frac{\pi}{2}} \cos t\ \sin t\ dt = \frac{3}{2}$$

Problem: Find the approximate value of the length of ellipse $x = a\cos t, y = b\sin t, 0 \leq t \leq 2\pi$ when $a = 1, e = 1/2$.

solution: By the arc length formula,

$$l = 4a\int_0^{\pi/2} \sqrt{1 - e^2\cos^2 t} dt$$

where e is ellipse's eccentricity. This integral is non-elementary except when $e = 0$ or 1. The integrals in this form are called *elliptic integrals.* We can use Trapezoidal rule to evaluate the value when $a = 1$ and $e = 1/2$. The answer with $n = 10$ is $l = 5.870$.

Suppose the curve is given in polar form $r = f(\theta),\ \alpha \leq \theta \leq \beta$. Then by taking the parametrization $x = r\cos\theta = f(\theta)\cos\theta$ and $y = r\sin\theta = f(\theta)\sin\theta$ with $\theta \in [\alpha, \beta]$, we get

$$\frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta,\ \ \frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta.$$

Hence the arc length is

$$l = \int_\alpha^\beta \sqrt{f^2(\theta) + (f')^2(\theta)}d\theta.$$

Application to Work done

Suppose the force $f(x)$ depends on position x is along a straight line from $x = a$ to $x = b$. Let $n \in \mathbb{N},\ \Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$ for $i = 1, 2, ..., n$. Then the work done in moving a particle under the force $f(x)$ from x_{k-1} to x_k is approximately $W_k = f(x_k)\Delta x$. The total work done (in moving from a to b) approximately is $W \sim \sum_{i=1}^n f(x_k^*)\Delta x, x_k^* \in [x_{k-1}, x_k]$. Taking $n \to \infty$ we get the total work done as $W = \int_a^b f(x)dx$.

Example: Find the work required to compress a spring from its natural length of 1 foot to a length of 0.75 foot if the force constant is $k = 16$ kg/foot.

Solution: Hooks law ways that the force it takes to stretch or compress a spring x length nits from its natural length is proportional to x. i.e.,$F = kx$, k is constant measured in force units per unit length.

Suppose the given springs is placed on the $x-$ axis. It is fixed at $x = 1$ and movable end at the origin. From the above formula, the force required to compress the spring from 0 to x with the formula $F = 16x$. To compress the spring from 0 to 0.25 ft, the force must increase from $F(0) = 0$ to $F(0.25) = 16 \times 0.25 = 4$ foot-kg. Therefore, the work done by F over this interval is

$$W = \int_0^{0.25} 16xdx = 0.5\ ft - kg.$$

Volume of "symmetrical" objects

Method of Slicing:

Consider a solid lying alongside some interval $[a, b]$ of the x-axis. For each x let $A(x)$ be the area of the cross section (of the solid) obtained by cutting it with a plane perpendicular to the x-axis at x. We divide the interval into n subintervals $[x_{i-1}, x_i]$. The planes that are perpendicular to the x-axis at the points $x_0, x_1, x_2, ..., x_n$ divide the solid into n slices. If the cross section between $[x_{i-1}, x_i]$ changes "little bit" along the that subinterval, then it can be approximated by a cylinder of height $x_i - x_{i-1}$ with base $A(x_i^*), x_i^* \in [x_{i-1}, x_i]$. So the volume of the slice is $V_i = A(x_i^*)(x_i - x_{i-1})$. Then the volume of the solid can be approximated as

$$V \approx \sum_{i=1}^{n} V_i = \sum_{i=1}^{n} A(x_i^*)(x_i - x_{i-1}) \to \int_a^b A(x)dx$$

as $n \to \infty$. Now this can be done along any axis, say y-axis. In this case we get the formula:

$$V = \int_a^b A(y)dy.$$

Solid of Revolution: Consider the area between the function $y = f(x), x \in [a, b]$ and x-axis. By revolving this area along $x-$ axis, we obtain a solid which is called "solid of revolution". It is easy to see that for this solid, the cross section is disc of radius $f(x)$ and the area of cross section $A(x)$ is equal to $\pi[f(x)]^2$. Hence the volume is

$$V = \int_a^b A(x)dx = \pi \int_a^b f^2(x)dx.$$

For example, take a cone of radius r and height h. Then this cone can be obtained by revolving $y = \frac{rx}{h}$ about $x-$ axis between $x = 0, h$. Then the volume is

$$V = \pi \int_0^h \frac{r^2x^2}{h^2}dx = \frac{\pi r^2 h}{3}.$$

Suppose the area revolved is bounded by two curves $y = f(x) \geq 0$ and $y = g(x) \geq 0$, with $f(x) \geq g(x)$. Then each cross section looks like washer with outer radius $r_1(x) = f(x)$ and inner radius $r_2(x) = g(x)$. The area of the cross section is $\pi(f^2(x) - g^2(x))$. The volume of the solid is

$$V = \pi \int_a^b (r_1^2 - r_2^2)(x)dx = \pi \int_a^b (f^2(x) - g^2(x))dx.$$

If the revolution is performed about y axis. Then

$$V = \pi \int_a^b (f^2(y) - g^2(y))dy$$

Example: The volume of the solid obtained by revolving the area bounded by $y = x^2$ and $y = \sqrt{x}$ about the $x-$ axis.

Solution: First we solve these two equation to find the interval of integration. Easy to see that (real) solution of $y = x^2, y = \sqrt{x}$ is $x = 0, 1$. Next we can see that $y = \sqrt{x}$ is above $y = x^2$ in this interval. Hence by above formula, the required Volume is

$$V = \pi \int_0^1 (x - x^4)dx = \frac{3\pi}{10}.$$

Volume by cylindrical shells:

A cylindrical shell is the region between two concentric cylinders of same height h. It is something like top portion of "Well" above earth surface. Let r_1 be the radius of outer cylinder and r_2 be that of inner cylinder. Then the volume of this shell is

$$V = \pi(r_1^2 - r_2^2)h = 2\pi r_a th,$$

where r_a is the average radius $(r_1 + r_2)/2$ and t is thickness of shell.

Consider the solid generated by revolving $y = f(x), a \leq x \leq b$ around the y-axis. We divide the interval $[a, b]$ into n subintervals $[x_{i-1}, x_i]$. The volume V of the solid may be approximated by the sum of the volumes V_i of the shells between $[x_{i-1}, x_i]$. Each shell is approximately cylindrical. Its height is $f(x_i^*)$, where $x_i^* = (x_{i-1} + x_i)/2$, the mid point. Its thickness is $(x_i - x_{i-1})/2$. Its average radius is x_i^* . Hence its volume is

$$V \approx \sum_{i=1}^{n} V_i = \sum_{i=1}^{n} 2\pi x_i^* f(x_i^*)(x_i - x_{i-1}) \to 2\pi \int_a^b xf(x)dx \quad \text{as } n \to \infty.$$

If the region is revolved about the x axis, then

$$V = \int_a^b 2\pi y f(y)dy.$$

Suppose the solid is obtained by revolving(about y axis) the area between two curves $y = f(x)$ and $y = g(x)$ with $f(x) \geq g(x)$. Then the shell height will be $f(x_i^*) - g(x_i)^*)$. Hence the volume will be given by

$$V = 2\pi \int_a^b x(f(x) - g(x))dx.$$

Example: Find the volume obtained by revolving the area bounded by $y = 2x^2 - x^3$ and

$y = 0$ about y axis.

Solution: The points of intersection of $y = 0$ and $y = 2x - x^3$ are $x = 0, 2$. Height of the shell is $f(x) = 2x^2 - x^3$. So the volume is

$$V = \int_0^2 2\pi x f(x) = 2\pi \int_0^2 (2x^3 - x^4) = \frac{16\pi}{5}.$$

If we revolve the area about **Arbitrary line** parallel to axis , say $x = c$. Then the radius of the shell be $x - c$ (or $c - x$ whichever is positive) instead of x. So the volume in this case is

$$V = \int_a^b 2\pi(x - c)(f(x) - g(x))dx.$$

Similarly, if the region is revolved about $x = d$, then

$$V = \int_a^b 2\pi(y - d)(f(y) - g(y))dy.$$

Example: Find the volume of the solid obtained by rotating the region bounded by $y = 0$ and $y = x - x^2$ about $x = 2$.

Solution: The points of intersection are 0 and 1. So the radius is $2 - x$ and height is $x - x^2$. Hence the volume is

$$V = \int_0^1 2\pi(2 - x)(x - x^2)dx = 2\pi \int_0^1 (x^3 - 3x^2 + 2x)dx = \frac{\pi}{2}.$$

Surface area of solids of revolution

Consider an object obtained by revolving a curve $y = f(x), a \leq x \leq b$ about x-axis. We assume that f is differentiable and f' is integrable. We find the surface area of this by approximating the surface by cylinders having the radius r_1 on one end and r_2 at the other end with lateral height l. The surface area of such cylinder is $2\pi \frac{r_1+r_2}{2} l$. Now we divided the interval into sub-intervals $[x_{i-1}, x_i]$. Let L be the line segment connecting $f(x_{i-1})$ and $f(x_i)$. Consider the small cylinders with radii $r_1 = f(x_i)$ and $r_2 = f(x_{i-1})$. Then the surface area of this cylinder is $S_i = 2\pi \frac{f(x_i)+f(x_{i-1})}{2} |L|$, where $|L|$ is the length of line segment touching $f(x_{i-1})$ and $f(x_i)$. We note as in arc length, $|L|$ is

$\sqrt{1+\left(\frac{f(x_{i-1})-f(x_i)}{x_i-x_{i-1}}\right)^2}(x_i-x_{i-1})$. Hence applying mean value theorem,

$$\begin{aligned}
S &\approx \sum_{i=1}^{n} 2\pi \frac{f(x_i)+f(x_{i-1})}{2}|L| \\
&= 2\pi \sum_{i=1}^{n} \frac{f(x_i)+f(x_{i-1})}{2}\sqrt{1+\left(\frac{f(x_{i-1})-f(x_i)}{x_i-x_{i-1}}\right)^2}(x_i-x_{i-1}) \\
&= 2\pi \sum_{i=1}^{n} \frac{f(x_i)+f(x_{i-1})}{2}\sqrt{1+[f'(x_i^*)]^2}(x_i-x_{i-1}),\ x_i^* \in [x_{i-1},x_i] \\
&\to 2\pi \int_a^b f(x)\sqrt{1+(f'(x))^2}dx.
\end{aligned}$$

Example: Find the surface area of the solid obtained by revolving the curve $y = \sqrt{4-x^2}, -1 \leq x \leq 1$ about x-axis.
Solution: This is the portion of the circle $x^2+y^2=4$ between $[-1,1]$.

$$\begin{aligned}
S &= 2\pi \int_a^b f(x)\sqrt{1+(f'(x))^2}dx \\
&= 2\pi \int_{-1}^{1} \sqrt{4-x^2}\sqrt{1+\frac{x^2}{4-x^2}}dx \\
&= 2\pi \int_{-1}^{1} dx = 4\pi
\end{aligned}$$

If the curve is described as $x = g(y)$, then we have:

$$S = 2\pi \int_a^b y\sqrt{1+g'(y)^2}dy$$

Also, if the rotation is about the y-axis, the formula becomes,

$$S = 2\pi \int_a^b y\sqrt{1+g'(y)^2}dy.$$

Problem: Find the surface area of the solid obtained by rotating $y = \sqrt[3]{x}, 1 \leq y \leq 2$ about y-axis.

Solution: Given curve is $x = y^3, 1 \leq y \leq 2$. By the given formula,

$$S = 2\pi \int_{y=1}^{2} y^3 \sqrt{1 + (\frac{dx}{dy})^2} dy = 2\pi \int_1^2 y^3 \sqrt{1 + 9y^4} dy$$
$$= \frac{2\pi}{36} \int_1^2 36y^3 \sqrt{1 + 9y^4} dy = \frac{\pi}{27}(1 + 9y^4)^{3/2}\Big|_{y=1}^{2} = \frac{\pi}{27}(145^{3/2} - 10^{3/2})$$

Problem: Find the surface area and volume of the solid generated by infinite curve $y = \frac{1}{x}, x \geq 1$. Interpret the result.
Solution: The surface area and volume are given by

$$S = 2\pi \int_1^{\infty} \frac{1}{x} \sqrt{1 + \frac{1}{x^4}} dx, \quad V = \pi \int_1^{\infty} \frac{1}{x^2} dx$$

It is easy to see that the integral in S diverges. Indeed,

$$\int_1^b \frac{1}{x} \sqrt{1 + \frac{1}{x^4}} dx > \int_1^b \frac{1}{x} dx.$$

However, the integral for V converges. This is sometimes described as a can that does not hold enough paint to cover its own interior. Of course, a finite amount of paint cannot cover infinite surface. But if we fill the can with finite amount of paint we will have covered an infinite surface. This is known as **Painter's paradox.**

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