STRENGTH OF MATERIALS

PART I

Elementary Theory and Problems

BY

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SECOND EDITION—TENTH PRINTING

D. VAN NOSTRAND COMPANY, INC.

TORONTO NEW YORK LONDON

NEW YORK

D. Van Nostrand Company, Inc., 250 Fourth Avenue, New York 3

TORONTO

D. Van Nostrand Company, (Canada), Ltd., 228 Bloor Street, Toronto

LONDON

Macmillan & Company, Ltd., St. Martin's Street, London, W.C. 2

First Published, May 1930
Reprinted, March 1932, January 1936
February 1938

Second Edition, June 1940
Reprinted, October 1941, July 1942
January 1944, August 1944, May 1945
May 1946, February 1947, August 1947
August 1948

PRINTED IN THE UNITED STATES OF AMERICA
BY LANCASTER PRESS, INC., LANCASTER, PA.

PREFACE TO THE SECOND EDITION

In preparing the second edition of this volume, an effort has been made to adapt the book to the teaching requirements of our engineering schools.

With this in view, a portion of the material of a more advanced character which was contained in the previous edition of this volume has been removed and will be included in the new edition of the second volume. At the same time, some portions of the book, which were only briefly discussed in the first edition, have been expanded with the intention of making the book easier to read for the beginner. For this reason, chapter II, dealing with combined stresses, has been entirely rewritten. Also, the portion of the book dealing with shearing force and bending moment diagrams has been expanded, and a considerable amount of material has been added to the discussion of deflection curves by the integration method. A discussion of column theory and its application has been included in chapter VIII, since this subject is usually required in undergraduate courses of strength of materials. Several additions have been made to chapter X dealing with the application of strain energy methods to the solution of statically indetermined problems. In various parts of the book there are many new problems which may be useful for class and home work.

Several changes in the notations have been made to conform to the requirements of American Standard Symbols for Mechanics of Solid Bodies recently adopted by The American Society of Mechanical Engineers.

It is hoped that with the changes made the book will be found more satisfactory for teaching the undergraduate course of strength of materials and that it will furnish a better foundation for the study of the more advanced material discussed in the second volume.

S. Timoshenko

Palo Alto, California
June 13, 1940

PREFACE TO THE FIRST EDITION

At the present time, a decided change is taking place in the attitude of designers towards the application of analytical methods in the solution of engineering problems. Design is no longer based principally upon empirical formulas. The importance of analytical methods combined with laboratory experiments in the solution of technical problems is becoming generally accepted.

Types of machines and structures are changing very rapidly, especially in the new fields of industry, and usually time does not permit the accumulation of the necessary empirical data. The size and cost of structures are constantly increasing, which consequently creates a severe demand for greater reliability in structures. The economical factor in design under the present conditions of competition is becoming of growing importance. The construction must be sufficiently strong and reliable, and yet it must be designed with the greatest possible saving in material. Under such conditions, the problem of a designer becomes extremely difficult. Reduction in weight involves an increase in working stresses, which can be safely allowed only on a basis of careful analysis of stress distribution in the structure and experimental investigation of the mechanical properties of the materials employed.

It is the aim of this book to present problems such that the student's attention will be focussed on the practical applications of the subject. If this is attained, and results, in some measure, in increased correlation between the studies of strength of materials and engineering design, an important forward step will have been made.

The book is divided into two volumes. The first volume contains principally material which is usually covered in required courses of strength of materials in our engineering schools. The more advanced portions of the subject are of interest chiefly to graduate students and research engineers, and are incorporated in the second volume of the book. This contains also the new developments of practical importance in the field of strength of materials.

In writing the first volume of strength of materials, attention was given to simplifying all derivations as much as possible so that a student with the usual preparation in mathematics will be able to read it without difficulty. For example, in deriving the theory of the deflection curve, the *area moment method* was extensively used. In this manner, a considerable simplification was made in deriving the deflections of beams for various loading and supporting conditions. In discussing statically indeterminate systems, the *method of superposition* was applied, which proves very useful in treating such problems as continuous beams and frames. For explaining combined stresses and deriving principal stresses, use was made of the *Mohr's circle*, which represents a substantial simplification in the presentation of this portion of the theory.

Using these methods of simplifying the presentation, the author was able to condense the material and to discuss some problems of a more advanced character. For example, in discussing torsion, the twist of rectangular bars and of rolled sections, such as angles, channels, and I beams, is considered. The deformation and stress in helical springs are discussed in detail. In the theory of bending, the case of non-symmetrical cross sections is discussed, the *center of twist* is defined and explained, and the effect of shearing force on the deflection of beams is considered. The general theory of the bending of beams, the materials of which do not follow Hooke's law, is given and is applied in the bending of beams beyond the yielding point. The bending of reinforced concrete beams is given consideration. In discussing combinations of direct and bending stress, the effect of deflections on the bending moment is considered, and the limitation of the method of superposition is explained. In treating combined bending and torsion, the cases of rectangular and elliptical cross sections are discussed, and applications in the design of crankshafts are given. Considerable space in the book is devoted to methods for solving elasticity problems based on the consideration of the strain energy of elastic bodies. These methods are applied in discussing statically indeterminate systems. The stresses produced by impact are also discussed. All these problems of a more advanced character are printed in small type, and may be omitted during the first reading of the book.

The book is illustrated with a number of problems to which solutions are presented. In many cases, the problems are chosen so as to widen the field covered by the text and to illustrate the application of the theory in the solution of design problems. It is hoped that these problems will be of interest for teaching purposes, and also useful for designers.

The author takes this opportunity of thanking his friends who have assisted him by suggestions, reading of manuscript and proofs, particularly Messrs. W. M. Coates and L. H. Donnell, teachers of mathematics and mechanics in the Engineering College of the University of Michigan, and Mr. F. L. Everett of the Department of Engineering Research of the University of Michigan. He is indebted also to Mr. F. C. Wilharm for the preparation of drawings, to Mrs. E. D. Webster for the typing of the manuscript, and to the Van Nostrand Company for its care in the publication of the book.

S. TIMOSHENKO

ANN ARBOR, MICHIGAN
May 1, 1930

NOTATIONS

$\sigma_x, \sigma_y, \sigma_z$....Normal stresses on planes perpendicular to x, y and z axes.
σ_n.........Normal stress on plane perpendicular to direction n.
$\sigma_{Y.P.}$.......Normal stress at yield point.
σ_w.........Normal working stress
τ..........Shearing stress
$\tau_{xy}, \tau_{yz}, \tau_{zx}$..Shearing stresses parallel to x, y and z axes on the planes perpendicular to y, z and x axes.
τ_w.........Working stress in shear
δ..........Total elongation, total deflection
ϵ..........Unit elongation
$\epsilon_x, \epsilon_y, \epsilon_z$.....Unit elongations in x, y and z directions
γ..........Unit shear, weight per unit volume
E..........Modulus of elasticity in tension and compression
G..........Modulus of elasticity in shear
μ..........Poisson's ratio
Δ..........Volume expansion
K..........Modulus of elasticity of volume
M_t........Torque
M.........Bending moment in a beam
V.........Shearing force in a beam
A.........Cross sectional area
I_y, I_z......Moments of inertia of a plane figure with respect to y and z axes
k_y, k_z......Radii of gyration corresponding to I_y, I_z
I_p.........Polar moment of inertia
Z..........Section modulus
C..........Torsional rigidity
l..........Length of a bar, span of a beam
P, Q.......Concentrated forces
t..........Temperature, thickness

α.......... Coefficient of thermal expansion, numerical coefficient
U.......... Strain energy
w.......... Strain energy per unit volume
h.......... Depth of a beam, thickness of a plate
q.......... Load per unit length
ϕ, θ........ Angles
p.......... Pressure
D, d....... Diameters
R, r........ Radii
W......... Weight, load

CONTENTS

STRENGTH OF MATERIALS

PART I

CHAPTER I

TENSION AND COMPRESSION WITHIN THE ELASTIC LIMIT

1. Elasticity.—We assume that a body consists of small particles, or molecules, between which forces are acting. These molecular forces resist the change in the form of the body which external forces tend to produce. If such external forces are applied to the body, its particles are displaced and the mutual displacements continue until equilibrium is established between the external and internal forces. It is said in such a case that the body is in a *state of strain.* During deformation the external forces acting upon the body do work, and this work is transformed completely or partially into the *potential energy of strain.* An example of such an accumulation of potential energy in a strained body is the case of a watch spring. If the forces which produced the deformation of the body are now gradually diminished, the body returns wholly or partly to its initial shape and during this reversed deformation the potential energy of strain, accumulated in the body, may be recovered in the form of external work.

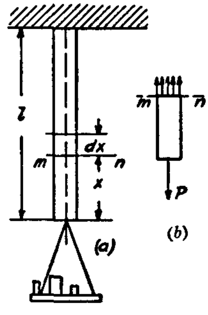

FIG. 1.

Take, for instance, a prismatical bar loaded at the end as shown in Fig. 1. Under the action of this load a certain elongation of the bar will take place. The point of application of the load will then move in a downward direction and positive work will be done by the load during this motion.

When the load is diminished, the elongation of the bar diminishes also, the loaded end of the bar moves up and the potential energy of strain will be transformed into the work of moving the load in the upward direction.

The property of bodies of returning, after unloading, to their initial form is called *elasticity*. It is said that the body is *perfectly elastic* if it recovers its original shape completely after unloading; it is *partially elastic* if the deformation, produced by the external forces, does not disappear completely after unloading. In the case of a perfectly elastic body the work done by the external forces during deformation will be completely transformed into the potential energy of strain. In the case of a partially elastic body, part of the work done by the external forces during deformation will be dissipated in the form of heat, which will be developed in the body during the non-elastic deformation. Experiments show that such structural materials as steel, wood and stone may be considered as perfectly elastic within certain limits, which depend upon the properties of the material. Assuming that the external forces acting upon the structure are known, it is a fundamental problem for the designer to establish such proportions of the members of the structure that it will approach the condition of a perfectly elastic body under all service conditions. Only under such conditions will we have continued reliable service from the structure and no *permanent set* in its members.

2. Hooke's Law.—By direct experiment with the extension of prismatical bars (Fig. 1) it has been established for many structural materials that within certain limits the elongation of the bar is proportional to the tensile force. This simple linear relationship between the force and the elongation which it produces was first formulated by the English scientist Robert Hooke [1] in 1678 and bears his name. Using the notation:

P = force producing extension of bar,
l = length of bar,

[1] Robert Hooke, De Potentia restitutiva, London, 1678.

A = cross sectional area of bar,
δ = total elongation of bar,
E = elastic constant of the material, called its *Modulus of Elasticity*,

Hooke's experimental law may be given by the following equation:

$$\delta = \frac{Pl}{AE}. \tag{1}$$

The elongation of the bar is proportional to the tensile force and to the length of the bar and inversely proportional to the cross sectional area and to the modulus of elasticity. In making tensile tests precautions are usually taken to secure central application of the tensile force. In this manner any bending of the bar will be prevented. Excluding from consideration those portions of the bar in the vicinity of the applied forces,[2] it may be assumed that during tension all longitudinal fibers of the prismatical bar have the same elongation and the cross sections of the bar originally plane and perpendicular to the axis of the bar remain so after extension.

In discussing the magnitude of internal forces let us imagine the bar cut into two parts by a cross section mn and let us consider the equilibrium of the lower portion of the bar (Fig. 1, *b*). At the lower end of this portion the tensile force P is applied. On the upper end there are acting the forces representing the action of the particles of the upper portion of the strained bar on the particles of the lower portion. These forces are continuously distributed over the cross section. A familiar example of such a continuous distribution of forces over a surface is that of a hydrostatic pressure or of a steam pressure. In handling such continuously distributed forces *the intensity of force*, i.e., the force per unit area, is of a great importance. In our case of axial tension, in which all fibers have the same elongation, the

[2] The more complicated stress distribution near the points of application of the forces will be discussed later in Part II.

distribution of forces over the cross section *mn* will be *uniform*. Taking into account that the sum of these forces, from the condition of equilibrium (Fig. 1, *b*), must be equal to P and denoting the force per unit of cross sectional area by σ, we obtain

$$\sigma = \frac{P}{A}. \tag{2}$$

This force per unit area is called *stress*. In the following, the force will be measured in pounds and the area in square inches so that the stress will be measured in pounds per square inch. The elongation of the bar per unit length is determined by the equation

$$\epsilon = \frac{\delta}{l} \tag{3}$$

and is called the *unit elongation* or the *tensile strain*. Using eqs. (2) and (3), Hooke's law may be represented in the following form:

$$\epsilon = \frac{\sigma}{E}, \tag{4}$$

and the unit elongation is easily calculated provided the stress and the modulus of elasticity of the material are known. The unit elongation ϵ is a pure number representing the ratio of two lengths (see eq. 3); therefore, from eq. (4), it may be concluded that the modulus of elasticity is to be measured in the same units as the stress σ, i.e., in pounds per square inch. In Table I, which follows, the average values of the modulus E for several materials are given in the first column.[3]

Equations (1)–(4) may be used also in the case of the compression of prismatical bars. Then δ will denote the total longitudinal contraction, ϵ the *compressive strain* and σ the *compressive stress*. The modulus of elasticity for compression is for most structural materials the same as for tension. In calculations, tensile stress and tensile strain are considered as positive, and compressive stress and strain as negative.

[3] More details on the mechanical properties of materials are given in Part II.

TABLE 1

MECHANICAL PROPERTIES OF MATERIALS

Materials	E lbs./in.^{2}	Yield Point lbs./in.^{2}	Ultimate Strength lbs./in.^{2}
Structural carbon steel 0.15 to 0.25% carbon	30×10^6	30×10^3–40×10^3	55×10^3–65×10^3
Nickel steel 3 to 3.5% nickel	29×10^6	40×10^3–50×10^3	78×10^3–100×10^3
Duraluminum	10×10^6	35×10^3–45×10^3	54×10^3–65×10^3
Copper, cold rolled	16×10^6		28×10^3–40×10^3
Glass	10×10^6		3.5×10^3
Pine, with the grain	1.5×10^6		8×10^3–20×10^3
Concrete, in compression	4×10^6		3×10^3

Problems

1. Determine the total elongation of a steel bar 25 in. long, if the tensile stress is equal to 15×10^3 lbs. per sq. in.

Answer.

$$\delta = \epsilon \times l = \frac{25}{2,000} = \frac{1}{80} \text{ in.}$$

2. Determine the tensile force on a cylindrical steel bar of one inch diameter, if the unit elongation is equal to $.7 \times 10^{-3}$.

Solution. The tensile stress in the bar, from eq. (4), is

$$\sigma = \epsilon \cdot E = 21 \times 10^3 \text{ lbs. per sq. in.}$$

The tensile force, from eq. (2), is

$$P = \sigma \cdot A = 21 \times 10^3 \times \frac{\pi}{4} = 16,500 \text{ lbs.}$$

3. What is the ratio of the moduli of elasticity of the materials of two bars of the same size if under the action of equal tensile forces the unit elongations of the bars are in the ratio 1 : 15/8. Determine these elongations if one of the bars is of steel, the other of copper and the tensile stress is 10,000 lbs. per sq. inch.

Solution. The moduli are inversely proportional to the unit elongations. For steel

$$\epsilon = \frac{10,000}{30 \times 10^6} = \frac{1}{3,000},$$

for copper

$$\epsilon = \frac{1}{1,600}.$$

4. A prismatical steel bar 25 in. long is elongated 1/40 in. under the action of a tensile force. Find the magnitude of the force if the volume of the bar is 25 in.^{3}

5. A piece of wire 100 ft. long subjected to a tensile force $P = 1{,}000$ lbs. elongates by 1 in. Find the modulus of elasticity of the material if the cross-sectional area of the wire is 0.04 sq. in.

3. The Tensile Test Diagram.—The proportionality between the tensile force and the elongation holds only up to a certain limiting value of the tensile stress, called the *limit of proportionality*, which depends upon the properties of the material. Beyond this limit, the relationship between the elongation and the tensile stress becomes more complicated. For such a material as structural steel the proportionality between the load and elongation holds within a considerable range and the limit of proportionality may be taken as high as 25×10^3–30×10^3 lbs. per sq. in. For such materials as cast iron or soft copper the limit of proportionality is very low, that is, deviations from Hooke's law may be noticed at a low tensile stress. In investigating the mechanical properties of materials beyond the limit of proportionality the relationship between the strain and the corresponding stress is usually presented graphically by the *tensile test diagram.* Figure 2 (*a*) presents a typical diagram for structural steel. Here the elongations are plotted along the horizontal axis and the corresponding stresses are given by the ordinates of the curve *OABCD*. From *O* to *A* the stress and the strain are proportional; beyond *A* the deviation from Hooke's law becomes marked; hence the stress at *A* is the *limit of proportionality*. Upon loading beyond this limit the elongation increases more quickly and the diagram becomes

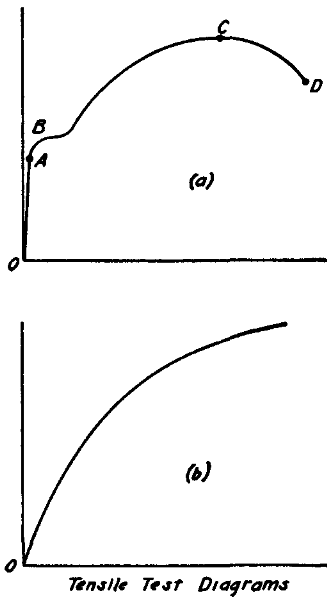

Fig. 2.

curved. At B a sudden elongation of the bar takes place without an appreciable increase in the tensile force. This phenomenon, called *yielding* of the metal, is shown in the diagram by an almost horizontal portion of the curve. The stress corresponding to the point B is called the *yield point*. Upon further stretching of the bar, the material recovers and, as is seen from the diagram, the necessary tensile force increases with the elongation up to the point C, at which this force attains its maximum value. The corresponding stress is called the *ultimate strength* of the material. Beyond the point C, elongation of the bar takes place with a diminution of the load and finally fracture occurs at a load corresponding to point D of the diagram.

It should be noted that the stretching of the bar is connected with the lateral contraction but it is an established practice in calculating the yield point and the ultimate strength to use the initial cross sectional area A. This question will be discussed later in more detail (see Part II).

Figure 2 (*b*) represents a tensile test diagram for cast iron. This material has a very low limit of proportionality [4] and has no definite yield point.

Diagrams analogous to those in tension may be obtained also for compression of various materials and such characteristic points as the limit of proportionality, the yield point, in the case of steel, and the ultimate strength for compression can be established. The mechanical properties of materials in tension and compression will be discussed later in more detail (see Part II).

4. Working Stress.—A tensile test diagram gives very valuable information on the mechanical properties of a material. Knowing the limit of proportionality, the yield point and the ultimate strength of the material, it is possible to establish for each particular engineering problem the magnitude of the stress which may be considered as a *safe stress*. This stress is usually called the *working stress*.

[4] This limit can be established only by using very sensitive extensometers in measuring elongations. See Grüneisen, Berichte d. deutsch. Phys. Gesellschaft, 1906.

In choosing the magnitude of the working stress for steel it must be taken into consideration that at stresses below the limit of proportionality this material may be considered as perfectly elastic and beyond this limit a part of the strain usually remains after unloading the bar, i.e., *permanent set* occurs. In order to have the structure in an elastic condition and to remove the possibility of a permanent set, it is usual practice to keep the working stress well below the limit of proportionality. In the experimental determination of this limit, sensitive measuring instruments (extensometers) are necessary and the position of the limit depends to some extent upon the accuracy with which the measurements are made. In order to eliminate this difficulty one takes usually the *yield point* or the *ultimate strength* of the material as a basis for determining the magnitude of the working stress. Denoting by σ_w, $\sigma_{Y.P.}$ and σ_u respectively the working stress, the yield point and the ultimate strength of the material, the magnitude of the working stress will be determined by one of the two following equations:

$$\sigma_w = \frac{\sigma_{Y.P.}}{n}, \qquad \text{or} \qquad \sigma_w = \frac{\sigma_u}{n_1}. \tag{5}$$

Here n and n_1 are factors usually called *factors of safety*, which determine the magnitude of the working stress. In the case of structural steel, it is logical to take the yield point as the basis for calculating the working stress because here a considerable permanent set may occur, which is not permissible in engineering structures. In such a case a factor of safety $n = 2$ will give a conservative value for the working stress provided that only constant loads are acting upon the structure. In the cases of suddenly applied loads, or variable loads, and these occur very often in machine parts, a larger factor of safety becomes necessary. For brittle materials such as cast iron, concrete, various kinds of stone and for such material as wood, the ultimate strength is usually taken as a basis for determining the working stresses.

The magnitude of the factor of safety depends very much upon the accuracy with which the external forces acting upon a structure are known, upon the accuracy with which the stresses in the members of a structure may be calculated and also upon the homogeneity of the materials used. This important question of working stresses will be discussed in more detail later (see Part II). Here we will include several simple examples of the determination of safe cross sectional dimensions of bars, assuming that the working stress is given.

Problems

1. Determine the diameter d of the steel bolts N of a press for a maximum compressive force $P = 100{,}000$ lbs. (Fig. 3), if the working stress for steel in this case is $\sigma_w = 10{,}000$ lbs. per sq. in. Determine the total elongation of the bolts at the maximum load, if the length between their heads is $l = 50$ in.

Solution. The necessary cross sectional area, from eq. (2),

$$A = \frac{\pi d^2}{4} = \frac{P}{2\sigma_w} = \frac{50{,}000}{10{,}000} = 5 \text{ in.}^2;$$

then

$$d = \sqrt{\frac{20}{\pi}} = 2.52 \text{ in.}$$

Total elongation, from eqs. (3) and (4),

$$\delta = \epsilon l = \frac{\sigma l}{E} = \frac{10^4 \cdot 50}{30 \cdot 10^6} = \frac{1}{60} \text{ in.}$$

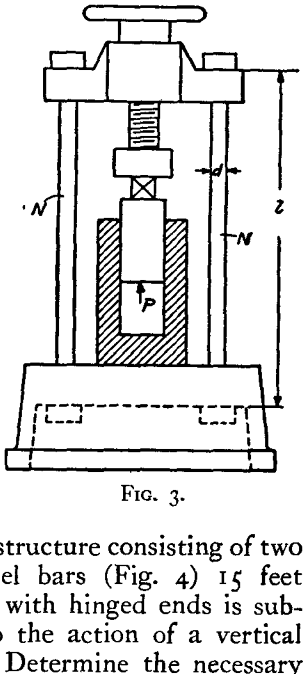

FIG. 3.

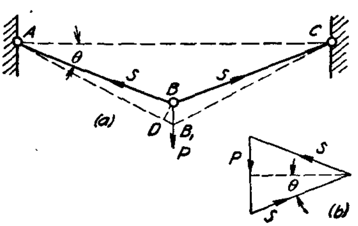

FIG. 4.

2. A structure consisting of two equal steel bars (Fig. 4) 15 feet long and with hinged ends is submitted to the action of a vertical load P. Determine the necessary cross sectional areas of the bars and the deflection of the point B when $P = 5{,}000$ lbs., $\sigma_w = 10{,}000$ lbs. per sq. in. and the initial angle of inclination of the bars $\theta = 30°$.

Solution. From Fig. 4 (b), representing the condition for equilibrium of the hinge B, the tensile force in the bars is

$$S = \frac{P}{2 \sin \theta}; \qquad \text{for} \qquad \theta = 30°; \qquad S = P = 5{,}000 \text{ lbs.}$$

The necessary cross sectional area

$$A = \frac{S}{\sigma_w} = \frac{5{,}000}{10{,}000} = \frac{1}{2} \text{ in.}^2$$

The deflection BB_1 will be found from the small right triangle DBB_1 in which the arc BD, of radius equal to the length of the bars, is considered as a perpendicular dropped upon AB_1, which is the position of the bar AB after deformation. Then the elongation of the bar AB is

$$B_1D = \epsilon \cdot l = \frac{\sigma_w l}{E} = \frac{10{,}000 \times 15 \times 12}{30 \times 10^6} = 0.06 \text{ in.}$$

and the deflection

$$BB_1 = \frac{B_1D}{\sin \theta} = 0.12 \text{ in.}$$

It is seen that the change of the angle due to the deflection BB_1 is very small and the previous calculation of S, based upon the assumption that $\theta = 30°$, is accurate enough.

3. Determine the total elongation of the steel bar AB having a cross sectional area $A = 1$ in.^{2} and submitted to the action of forces $Q = 10{,}000$ lbs. and $P = 5{,}000$ lbs. (Fig. 5).

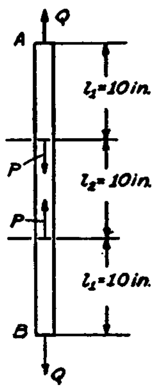

FIG. 5.

Solution. The tensile force in the upper and lower portions of the bar is equal to Q and that in the middle portion is $Q - P$. Then the total elongation will be

$$\delta = 2\frac{Ql_1}{AE} + \frac{(Q-P)l_2}{AE} = 2\frac{10{,}000 \times 10}{1 \times 30 \times 10^6}$$

$$+ \frac{5{,}000 \times 10}{1 \times 30 \times 10^6} = \frac{1}{150} + \frac{1}{600} = \frac{1}{120} = 0.00833 \text{ in.}$$

4. Determine the cross sectional dimensions of the wooden beam BC and of the steel bar AB of the structure ABC, loaded at B, when the working stress for wood is taken as $\sigma_w = 160$ lbs. per sq. in. and for steel $\sigma_w = 10{,}000$ lbs. per sq. in. The load

$P = 6{,}000$ lbs. The dimensions of the structure are shown in Fig. 6. Determine the vertical and the horizontal components of the displacement of the point B due to deformation of the bars.

Solution. From Fig. 6 (*b*) giving the condition for equilibrium of hinge B, similar to the triangle ABC of Fig. 6 (*a*), we have

$$S = \frac{P \cdot 15}{9} = 10{,}000 \text{ lbs.};$$

$$S_1 = \frac{P \cdot 12}{9} = 8{,}000 \text{ lbs.}$$

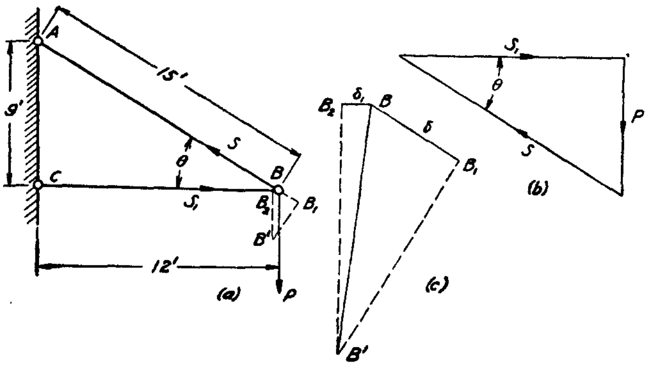

FIG. 6.

The cross sectional areas of the steel bar and of the wooden beam are

$$A = \frac{S}{\sigma_w} = \frac{10{,}000}{10{,}000} = 1 \text{ in.}^2; \qquad A_1 = \frac{S_1}{\sigma_w} = \frac{8{,}000}{160} = 50 \text{ in.}^2$$

The total elongation of the steel bar and the total compression of the wooden beam are

$$\delta = \frac{S \cdot l}{E_s A} = \frac{10{,}000 \cdot 15 \cdot 12}{30 \times 10^6} = 0.060 \text{ in.};$$

$$\delta_1 = \frac{S_1 l_1}{E_w A_1} = \frac{160 \times 12 \times 12}{1.5 \times 10^6} = 0.0154 \text{ in.}$$

To determine the displacement of the hinge B, due to deformation, arcs are drawn with centers A and C (Fig. 6, *a*) and radii equal to the lengths of the elongated bar and of the compressed beam respectively. They intersect in the new position B' of the hinge B. This is shown on a larger scale in Fig. 6 (*c*), where BB_1 is the elongation of the steel bar and BB_2 the compression of the wooden beam. The dotted perpendiculars replace the arcs mentioned above. Then BB' is the displacement of the hinge B. The components of this displacement may be easily obtained from the figure.

5. Determine in the previous problem the inclination of the bar AB to make its weight a minimum.

Solution. If θ denotes the angle between the bar and the horizontal beam and l_1 the length of the beam, then the length of the bar is $l = l_1/\cos\theta$, the tensile force in the bar is $S = P/\sin\theta$ and the necessary cross sectional area is $A = P/\sigma_w \sin\theta$. The volume of the bar will be

$$l \cdot A = \frac{l_1 P}{\sigma_w \sin\theta\cos\theta} = \frac{2l_1 P}{\sigma_w \sin 2\theta}.$$

It is seen that the volume and the weight of the bar become a minimum when $\sin 2\theta = 1$ and $\theta = 45°$.

6. The square frame $ABCD$ (Fig. 7, *a*) consisting of five steel bars of 1 in.^{2} cross sectional area is submitted to the action of two forces $P = 10{,}000$ lbs. in the direction of the diagonal. Determine the changes of the angles at A and C due to deformation of the frame. Determine the changes of the same angles if the forces are applied as shown in Fig. 7 (*b*).

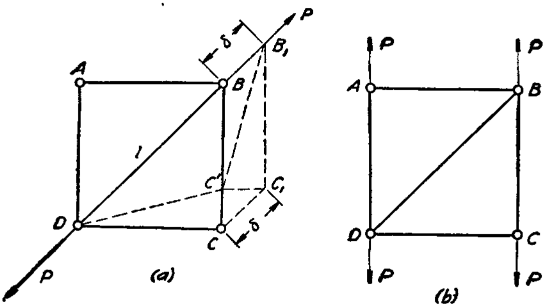

FIG. 7.

Solution. In the case shown in Fig. 7 (*a*) the diagonal will take the complete load P. Assuming that the hinge D and the direction of the diagonal are stationary, the displacement of the hinge B in the direction of the diagonal will be equal to the elongation of the diagonal $\delta = Pl/AE$. The determination of the new position C' of the hinge C is indicated in the figure by dotted lines. It is seen from the small right triangle CC_1C' that $CC' = \delta/\sqrt{2}$. Then the angle of rotation of the bar DC due to deformation of the frame is equal to

$$\frac{CC'}{DC} = \frac{\delta\sqrt{2}}{\sqrt{2}l} = \frac{\delta}{l} = \frac{P}{1 \cdot E} = \frac{1}{3,000} \text{ radian.}$$

Then the increase of the angle at C will be

$$2 \times \frac{1}{3,000} = \frac{1}{1,500} \text{ radian.}$$

The solution of the problem shown in Fig. 7 (b) is left to the student.

7. Determine the position of the load P on the beam ABD so that the force in the bar BC becomes a maximum. Determine the angle θ to make the volume of the bar BC a minimum (Fig. 8).

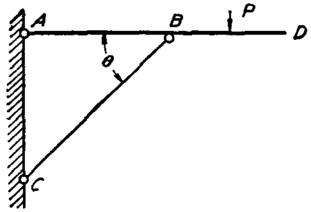

FIG. 8.

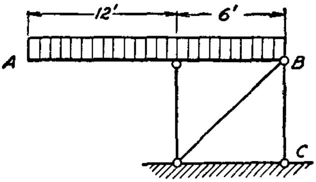

FIG. 9.

Answer. The force in the bar BC becomes maximum when the load P has its extreme position on the right at point D. The volume of the bar will be a minimum when $\theta = 45°$.

8. Determine the necessary cross sectional area of the steel bar BC (Fig. 9) if the working stress $\sigma_w = 15,000$ lbs. per sq. in. and the uniformly distributed vertical load per foot of the beam AB is $q = 1,000$ lbs.

Answer. $A = 0.6$ sq. in.

9. Determine the necessary cross sectional areas of the bars AB and BC of the structures shown in Figs. 10 (a) and (b) if $\sigma_w = 16,000$ lbs. per sq. in.

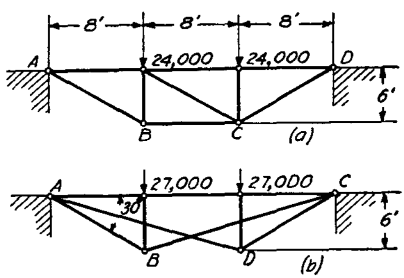

FIG. 10.

Answer. In the case of structure 10 (a) the cross sectional area of AB should be 2.5 sq. in. and of the bar BC 2.0 sq. in. In the case of Fig. 10 (b) the cross sectional area of the bar AB should be 2.25 sq. in. and of the bar BC 2.03 sq. in.

10. Solve problem 3 assuming that the material is duraluminum and that $P = Q = 10,000$ lbs. per sq. in.

11. Find the cross-sectional areas of the bars CD in Figs. 10a and 10b and the total elongation of these bars if the material is structural steel and $\sigma_w = 16{,}000$ lbs. per sq. in.

12. Solve problem 9 assuming that the load is applied at only one joint of the upper chord at a distance 8 ft. from support A.

5. Stress and Strain Produced in a Bar by its Own Weight.—In discussing the extension of a bar, Fig. 1, only the load P applied at the end was taken into consideration. If the length of the bar is large, its own weight may produce a considerable additional stress and should be taken into account. In this case the maximum stress will be at the built-in upper cross section. Denoting by γ the weight per unit volume of the bar, the complete weight will be $A\gamma l$ and the maximum stress will be given by the equation:

$$\sigma_{\max} = \frac{P + A\gamma l}{A} = \frac{P}{A} + \gamma l. \tag{6}$$

The second term on the right side of eq. (6) represents the stress produced by the weight of the bar. The weight of that portion of the bar below a cross section at distance x from the lower end (Fig. 1) is $A\gamma x$ and the stress will be given by the equation:

$$\sigma = \frac{P + A\gamma x}{A}. \tag{7}$$

Substituting the working stress σ_w for $\sigma_{\max}$ in eq. (6), the equation for calculating the safe cross sectional area will be

$$A = \frac{P}{\sigma_w - \gamma l}. \tag{8}$$

It is interesting to note that with increasing length l the bar's own weight becomes more and more important, the denominator of the right side of eq. (8) diminishes and the necessary cross sectional area A increases. When $\gamma l = \sigma_w$, i.e., the stress due to the weight of the bar alone becomes equal to the working stress, the right side of eq. (8) becomes infinite. Under such circumstances it is impossible to use a prismatical design and recourse to a bar of variable cross section is made.

In calculating the total elongation of a prismatical bar submitted to the action of a tensile force P at the end and its own weight, let us consider first the elongation of an element of length dx cut from the bar by two adjacent cross sections (see Fig. 1). It may be assumed that along the very short length dx the tensile stress is constant and is given by eq. (7). Then the elongation $d\delta$ of the element will be

$$d\delta = \frac{\sigma dx}{E} = \frac{P + A\gamma x}{AE}dx.$$

The total elongation of the bar will be obtained by summing the elongations of all the elements. Then

$$\delta = \int_0^l \frac{P + A\gamma x}{AE}dx = \frac{l}{AE}(P + \tfrac{1}{2}A\gamma l). \tag{9}$$

Comparing this with eq. (1) it is seen that the total elongation produced by the bar's own weight is equal to that produced by a load of half its weight applied at its end.

Problems

1. Determine the cross sectional area of a vertical prismatical steel bar carrying on its lower end a load $P = 70{,}000$ lbs., if the length of the bar is 720 feet, the working stress $\sigma_w = 10{,}000$ lbs. per sq. in. and the weight of a cubic foot of steel is 490 lbs. Determine the total elongation of the bar.

Solution. The cross sectional area, from eq. (8), is

$$A = \frac{70{,}000}{10{,}000 - \dfrac{490 \times 720 \times 12}{12^3}} = 9.27 \text{ in.}^2$$

The total elongation, from eq. (9), is

$$\delta = \frac{720 \times 12}{30 \times 10^6}\left(7{,}550 + \frac{1}{2}2{,}450\right) = 2.53 \text{ in.}$$

2. Determine the elongation of a conical bar under the action of its own weight (Fig. 11) if the length of the bar is l, the diameter of the base is d and the weight per unit volume of the material is γ.

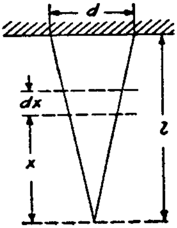

FIG. 11.

Solution. The weight of the bar will be

$$Q = \frac{\pi d^2}{4} \cdot \frac{l\gamma}{3}.$$

For any cross section at distance x from the lower end of the bar the tensile force, equal to the weight of the lower portion of the bar, is

$$\frac{Qx^3}{l^3} = \frac{\pi d^2}{4} \frac{\gamma x^3}{3l^2}.$$

Assuming that the tensile force is uniformly distributed over the cross section [5] and considering the element of length dx as a prismatical bar, the elongation of this element will be

$$d\delta = \frac{\gamma x}{3E} dx$$

and the total elongation of the bar is

$$\delta = \frac{\gamma}{3E} \int_0^l x dx = \frac{\gamma l^2}{6E}.$$

This elongation is one third that of a prismatical bar of the same length (see eq. 9).

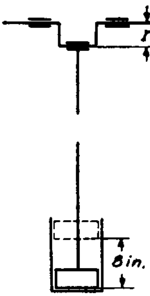

FIG. 12.

3. The vertical prismatical rod of a mine pump is moved up and down by a crank shaft (Fig. 12). Assuming that the material is steel and the working stress is $\sigma_w = 7{,}000$ lbs. per sq. in., determine the cross sectional area of the rod if the resistance of the piston during motion downward is 200 lbs. and during motion upward is 2,000 lbs. The length of the rod is 320 feet. Determine the necessary length of the radius r of the crank if the stroke of the pump is equal to 8 in.

Solution. The necessary cross sectional area of the rod will be found from eq. (8) by substituting $P = 2{,}000$ lbs. Then

$$A = \frac{2{,}000}{7{,}000 - \frac{490 \cdot 320 \cdot 12}{12^3}} = 0.338 \text{ in.}^2$$

The difference in total elongation of the rod when it moves up and when it moves down is due to the resistance of the piston and will be

[5] Such an assumption is justifiable when the angle of the cone is small.

equal to

$$\Delta\delta = \frac{(2{,}000 + 200)\cdot 320\cdot 12}{30\cdot 10^6 \times 0.338} = 0.833 \text{ in.}$$

The radius of the crank should be

$$r = \frac{8 + 0.833}{2} = 4.42 \text{ in.}$$

4. Lengths of wire of steel and aluminum are suspended vertically. Determine for each the length at which the stress due to the weight of the wire equals the ultimate strength if for steel wire $\sigma_u = 300{,}000$ lbs. per sq. in. and $\gamma = 490$ lbs. per cubic foot, and for aluminum wire $\sigma_u = 50{,}000$ lbs. per sq. in. and $\gamma = 170$ lbs. per cubic foot.

Answer. For steel $l = 88{,}200$ ft., for aluminum $l = 42{,}300$ ft.

5. In what proportion will the maximum stress produced in a prismatical bar by its own weight increase if all the dimensions of the bar are increased in the proportion $n : 1$ (Fig. 1)?

Answer. The stress will increase in the ratio $n : 1$.

6. A bridge pillar consisting of two prismatical portions of equal length (Fig. 13) is loaded at the upper end by a compressive force $P = 600{,}000$ lbs. Determine the volume of masonry if the height of the pillar is 120 ft., its weight per cubic foot is 100 lbs., and the maximum compressive stress in each portion is 150 lbs. per sq. in. Compare this volume with that of a single prismatical pillar designed for the same condition.

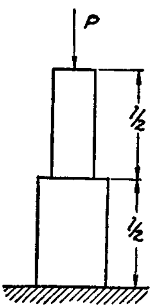

FIG. 13.

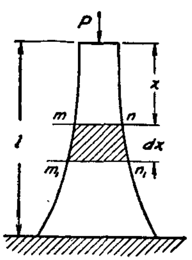

FIG. 14.

7. Solve the preceding problem assuming three prismatical portions of equal length.

8. Determine the form of the pillar in Fig. 14 such that the stress in each cross section is just equal to σ_w. The form satisfying this condition is called the *form of equal strength*.

Solution. Considering a differential element, shaded in the figure, it is evident that the compressive force on the cross section m_1n_1 is larger than that on the cross section mn by the magnitude of the weight of the element. Thus since the stress in both cross sections is to be the same and equal to σ_w, the difference dA in the cross-sectional area must be such as to compensate for the difference in the compressive force. Hence

$$dA\sigma_w = \gamma A dx \tag{a}$$

where the right side of the equation represents the weight of the element. Dividing this equation by $A\sigma_w$ and integrating we find

$$\int \frac{dA}{A} = \int \frac{\gamma dx}{\sigma_w},$$

from which

$$\log A = \frac{\gamma x}{\sigma_w} + C_1$$

and

$$A = Ce^{\gamma x/\sigma_w}, \tag{b}$$

where e is the base of natural logarithms and $C = e^{C_1}$. At $x = 0$ this equation gives for the cross-sectional area at the top of the pillar

$$(A)_{x=0} = C.$$

But the cross-sectional area at the top is equal to P/σ_w; hence $C = P/\sigma_w$ and equation (b) becomes

$$A = \frac{P}{\sigma_w} e^{\gamma x/\sigma_w}. \tag{c}$$

The cross-sectional area at the bottom of the pillar is obtained by substituting $x = l$ in equation (c), which gives

$$A_{\max} = \frac{P}{\sigma_w} e^{\gamma l/\sigma_w}. \tag{d}$$

9. Find the volume of the masonry for a pillar of equal strength designed to meet the conditions of problem 6.

Solution. By using equation (d) the difference of the cross-sectional areas at the bottom of the pillar and at its top is found to be

$$\frac{P}{\sigma_w} e^{\gamma l/\sigma_w} - \frac{P}{\sigma_w} = \frac{P}{\sigma_w} (e^{\gamma l/\sigma_w} - 1).$$

This difference multiplied by the working stress σ_w evidently gives the weight of the pillar; its volume is thus

$$V = \frac{P}{\gamma} (e^{\gamma l/\sigma_w} - 1) = 5{,}360 \text{ cubic feet.}$$

6. Statically Indeterminate Problems in Tension and Compression.—There are cases in which the axial forces acting in the bars of a structure cannot be determined from the equations of statics alone and the deformation of the structure must be taken into consideration. Such structures are called *statically indeterminate systems.*

A simple example of such a system is shown in Fig. 15. The load P produces extension in the bars OB, OC and OD, which are in the same plane. The conditions for equilibrium of the hinge O give two equations of statics which are not sufficient to determine the three unknown tensile forces in the bars, and for a third equation a consideration of the deformation of the system becomes necessary. Let us assume, for simplicity, that the system is symmetrical with respect to the vertical axis OC, that the vertical bar is of steel with A_s and E_s as the cross sectional area and the modulus of elasticity for the material, and that the inclined bars are of copper with A_c and E_c as area and modulus. The length of the vertical bar is l and that of the inclined bars is $l/\cos\alpha$. Denoting by X the tensile force in the vertical bar and by Y the forces in the inclined bars, the only equation of equilibrium for the hinge O in this case of symmetry will be

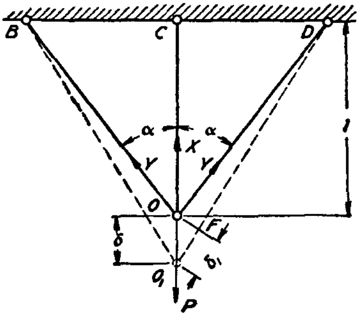

FIG. 15.

$$X + 2Y\cos\alpha = P. \qquad (a)$$

In order to derive the second equation necessary for determining the unknown quantities X and Y, the deformed configuration of the system indicated in the figure by dotted lines must be considered. Let δ be the total elongation of the vertical bar under the action of the load P; then the elongation δ_1 of the inclined bars will be found from the triangle OFO_1. Assuming that these elongations are very small, the circular arc OF from the center D may be replaced by a perpendicular line and the angle at O_1 may be taken equal to the initial angle α; then

$$\delta_1 = \delta \cos \alpha.$$

The unit elongations and the stresses for the vertical and the inclined bars will be

$$\epsilon_s = \frac{\delta}{l}; \quad \sigma_s = \frac{E_s\delta}{l} \qquad \text{and} \qquad \epsilon_c = \frac{\delta \cos^2 \alpha}{l}; \quad \sigma_c = \frac{E_c\delta \cos^2 \alpha}{l},$$

respectively. Then the forces in the bars will be obtained by multiplying the stresses by the cross sectional areas as follows:

$$X = \sigma_s A_s = \frac{A_s E_s \delta}{l}, \qquad Y = \sigma_c A_c = \frac{A_c E_c \delta \cos^2 \alpha}{l}, \qquad (b)$$

from which

$$Y = X \cos^2 \alpha \cdot \frac{A_c E_c}{A_s E_s}.$$

Substituting in eq. (a), we obtain

$$X = \frac{P}{1 + 2 \cos^3 \alpha \dfrac{A_c E_c}{A_s E_s}}. \qquad (10)$$

It is seen that the force X depends not only upon the angle of inclination α but also upon the cross sectional areas and the mechanical properties of the materials of the bars. In the particular case in which all bars have the same cross section and the same modulus we obtain, from eq. (10),

$$X = \frac{P}{1 + 2 \cos^3 \alpha}.$$

When α approaches zero, $\cos \alpha$ approaches unity, and the force in the vertical bar approaches $1/3P$. When α approaches $90°$, the inclined bars become very long and the complete load will be taken by the middle bar.

As another example of a statically indeterminate system let us consider a prismatical bar with built-in ends, loaded axially at an intermediate cross section mn (Fig. 16). The load P will be in equilibrium with the reactions R and R_1 at the ends and we have

$$P = R + R_1. \qquad (c)$$

In order to derive the second equation for determining the forces R and R_1 the deformation of the bar must be considered. The load P with the force R produces shortening of the lower portion of the bar and with the force R_1 elongation of the upper portion. The total shortening of one part is equal to the total elongation of the other. Then, by using eq. (1), we obtain

$$\frac{R_1 a}{AE} = \frac{Rb}{AE}.$$

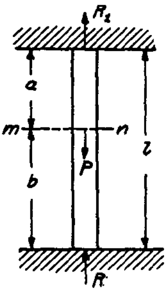

FIG. 16.

Hence

$$\frac{R}{R_1} = \frac{a}{b}, \qquad (d)$$

i.e., the forces R and R_1 are inversely proportional to the distances of their points of application from the loaded cross section mn. Now from eqs. (c) and (d) the magnitudes of these forces and the stresses in the bar may be readily calculated.

Problems

1. A steel cylinder and a copper tube are compressed between the plates of a press (Fig. 17). Determine the stresses in steel and copper and also the unit compression if $P =$ 100,000 lbs., $d = 4$ ins. and $D = 8$ ins.

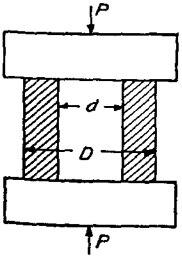

FIG. 17.

Solution. Here again static conditions are inadequate, and the deformation of cylinder and tube must be considered to get that part of the load carried by each material. The unit shortening in the steel and in the copper will be equal; therefore the stresses of each material will be in the same ratio as their moduli (eq. 4, p. 4), i.e., the compressive stress in the steel will be 15/8 the compressive stress in the copper. Then the magnitude σ_c of the stress in the copper will be found from the equation of statics,

$$P = \frac{\pi d^2}{4} \frac{15}{8} \sigma_c + \frac{\pi}{4} (D^2 - d^2)\sigma_c.$$

Substituting numerical values, we obtain

$$\sigma_c = 1{,}630 \text{ lbs. per sq. in.,} \quad \sigma_s = \frac{15}{8} \cdot \sigma_c = 3{,}060 \text{ lbs. per sq. in.;}$$

unit compression

$$\epsilon = \frac{\sigma_c}{E_c} = 102 \times 10^{-6}.$$

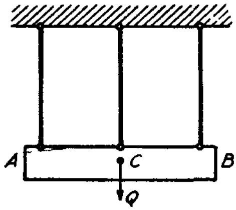

FIG. 18.

2. A column of reinforced concrete is compressed by a force $P = 60{,}000$ lbs. What part of this load will be taken by the concrete and what part by the steel if the cross sectional area of the steel is only 1/10 of the cross sectional area of the concrete?

3. A rigid body AB of weight Q hangs on three vertical wires symmetrically situated with respect to the center of gravity C of the body (Fig. 18). Determine the tensile forces in the wires if the middle wire is of steel and the two others of copper. Cross sectional areas of all wires are equal.

Suggestion. Use method of problem 1.

4. Determine the forces in four legs of a square table, Fig. 19, produced by the load P acting on one diagonal. The top of the table and the floor are assumed absolutely rigid and the legs are attached to the floor so that they can undergo tension as well as compression.

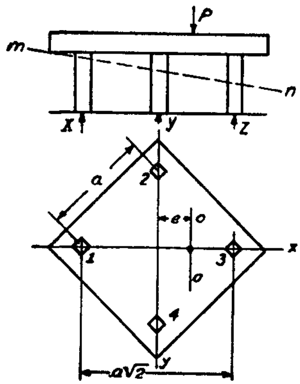

FIG. 19.

Solution. Assuming that the new position of the top of the table is that indicated by the dotted line mn, the compression of legs 2 and 4 will be the average of that of legs 1 and 3. Hence

$$2Y = X + Z$$

and since $2Y + X + Z = P$ we obtain

$$2Y = X + Z = \tfrac{1}{2}P. \qquad (a)$$

An additional equation for determining X and Z is obtained by taking the moment of all the forces with respect to the horizontal axis $O - O$ parallel to y and in the plane of the force P. Then

$$X(\tfrac{1}{2}a\sqrt{2} + e) + \tfrac{1}{2}P \cdot e = Z(\tfrac{1}{2}a\sqrt{2} - e). \qquad (b)$$

From (*a*) and (*b*) we obtain

$$X = P\left(\frac{1}{4} - \frac{e}{a\sqrt{2}}\right); \qquad Y = \frac{P}{4}; \qquad Z = P\left(\frac{1}{4} + \frac{e}{a\sqrt{2}}\right).$$

When $e > a\sqrt{2}/4$, X becomes negative. This indicates that there will be tension in leg 1.

5. Determine the forces in the legs of the above table when the load is applied at the point with the coordinates

$$x = \frac{a}{4}, \qquad y = \frac{a}{5}.$$

Hint. In solving this problem it should be noted that when the point of application of the load P is not on the diagonal of the table, this load may be replaced by two loads statically equivalent to the load P and applied at points on the two diagonals. The forces produced in the legs by each of these two loads are found as explained above. Summarizing the effects of the two component loads, the forces in the legs for any position of the load P may be found.

6. A rectangular frame with diagonals is submitted to the action of compressive forces P (Fig. 20). Determine the forces in the bars if they are all of the same material, the cross sectional area of the verticals is A, and that of the remaining bars A_1.

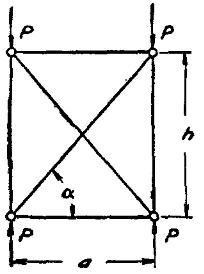

FIG. 20.

Solution. Let X be the compressive force in each vertical, Y the compressive force in each diagonal and Z the tensile force in each horizontal bar. Then from the condition of equilibrium of one of the hinges,

$$Y = 1/\sin\alpha(P - X); \qquad Z = Y\cos\alpha = (P - X)\cot\alpha. \quad (a)$$

The third equation will be obtained from the condition that the frame after deformation remains rectangular by virtue of symmetry; therefore

$$(a^2 + h^2)\left(1 - \frac{Y}{A_1E}\right)^2 = h^2\left(1 - \frac{X}{AE}\right)^2 + a^2\left(1 + \frac{Z}{A_1E}\right)^2;$$

from this, neglecting the small quantities of higher order, we get

$$\frac{(a^2 + h^2)Y}{A_1E} = \frac{h^2X}{AE} - \frac{a^2Z}{A_1E}. \quad (b)$$

Solving eqs. (*a*) and (*b*), the following value of the force in a diagonal will be obtained:

$$Y = \frac{P}{\dfrac{a^2 + h^2}{h^2} \cdot \dfrac{A}{A_1} + \dfrac{a^2}{h^2} \cdot \dfrac{A}{A_1} \cos \alpha + \sin \alpha}.$$

The forces in other bars will now be easily determined from eqs. (*a*).

7. Solve the above problem, assuming $a = h$, $A = 5A_1$ and $P = 50{,}000$ lbs.

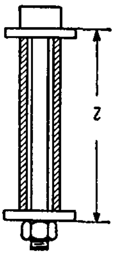

FIG. 21.

8. What stresses will be produced in a steel bolt and a copper tube (Fig. 21) by $\frac{1}{4}$ of a turn of the nut if the length of the bolt $l = 30$ ins., the pitch of the bolt thread $h = \frac{1}{8}$ in., the area of the cross section of the bolt $A_s = 1$ sq. inch, the area of the cross section of the tube $A_c = 2$ sq. inches?

Solution. Let X denote the unknown tensile force in the bolt and the compressive force in the tube. The magnitude of X will be found from the condition that the extension of the bolt plus the shortening of the tube is equal to the displacement of the nut along the bolt. In our case, assuming the length of the tube equal to the length of the bolt, we obtain

$$\frac{Xl}{A_sE_s} + \frac{Xl}{A_cE_c} = \frac{1}{4}h,$$

from which

$$X = \frac{hA_sE_s}{4l\left(1 + \dfrac{A_sE_s}{A_cE_c}\right)} = \frac{30 \times 10^6}{32 \times 30\left(1 + \dfrac{15}{16}\right)} = 16{,}100 \text{ lbs.}$$

The tensile stress in the bolt is $\sigma_s = X/A_s = 16{,}100$ lbs. per sq. in. The compressive stress in the tube is $\sigma_c = X/A_c = 8{,}050$ lbs. per sq. in.

9. What change in the stresses calculated in the above problem will be produced by tensile forces $P = 5{,}000$ lbs. applied to the ends of the bolt?

Solution. Let X denote the increase in the tensile force in the bolt and Y the decrease in the compressive force in the tube. Then from the condition of equilibrium,

$$X + Y = P. \tag{a}$$

A second equation may be written down from the consideration that the unit elongation of the bolt and tube under the application of the forces P must be equal, i.e.,

$$\frac{X}{A_sE_s} = \frac{Y}{A_cE_c}. \qquad (b)$$

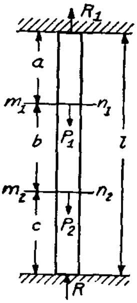

FIG. 22.

From eqs. (a) and (b) the forces X and Y and the corresponding stresses are easily calculated.

10. A prismatical bar with built-in ends is loaded axially at two intermediate cross sections (Fig. 22) by forces P_1 and P_2. Determine the reactions R and R_1.

Hint. Solution will be obtained by using eq. (d) on page 21, calculating the reactions produced by each load separately and then summarizing these reactions. Determine the reactions when

$$a = 0.3l, \quad b = 0.3l \quad \text{and} \quad P_1 = 2P_2 = 1{,}000 \text{ lbs.}$$

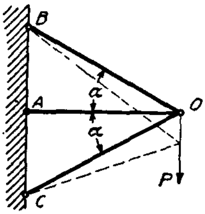

FIG. 23.

11. Determine the forces in the bars of the system, shown in Fig. 23, where OA is an axis of symmetry.

Answer. The tensile force in the bar OB is equal to the compressive force in the bar OC and is $P/2 \sin \alpha$. The force in the horizontal bar OA is equal to zero.

12. Solve problem 10 assuming that the lower portion of length c of the bar has a cross-sectional area two times larger than the cross-sectional area of the two upper parts of lengths a and b.

7. Initial and Thermal Stresses.—In a statically indeterminate system it is possible to have some initial stresses produced in assembly and due to inaccuracies in the lengths of the bars or to intentional deviations from the correct values of these lengths. These stresses will exist when external loads are absent, and depend only upon the geometrical proportions of the system, on the mechanical properties of the materials and on the magnitude of the inaccuracies. Assume, for example, that the system represented in Fig. 15 has, by mistake, $l + a$ as the length of the vertical bar instead of l. Then after assembling the bars BO and DO, the vertical bar can be put into place only after initial compression and due to this fact it will produce some tensile force in the inclined bars. Let X denote the compressive force in the vertical bar, which finally takes place after assembly. Then the corresponding tensile force in the inclined bars will be $X/2 \cos \alpha$

and the displacement of the hinge O due to the extension of these bars will be (see eq. b, p. 20)

$$\delta = \frac{Xl}{2A_cE_c \cos^3 \alpha} \cdot \qquad (a)$$

The shortening of the vertical bar will be

$$\delta_1 = \frac{Xl}{A_sE_s} \cdot \qquad (b)$$

From elementary geometrical considerations, the displacement of the hinge O, together with the shortening of the vertical bar, must be equal to the error a in the length of the vertical bar. This gives the following equation for determining X:

$$\frac{Xl}{2A_cE_c \cos^3 \alpha} + \frac{Xl}{A_sE_s} = a \cdot$$

Hence

$$X = \frac{aA_sE_s}{l\left(1 + \dfrac{A_sE_s}{2A_cE_c \cos^3 \alpha}\right)} \cdot \qquad (11)$$

Now the initial stresses in all the bars may be calculated. Expansion of the bars of a system due to changes in temperature may have also the same effect as inaccuracies in lengths. Assume a bar with built-in ends. If the temperature of the bar is raised from t_0 to t and thermal expansion is prevented by the reactions at the ends, there will be produced in the bar compressive stresses, whose magnitude may be calculated from the condition that the length remains unchanged. Let α denote the coefficient of thermal expansion and σ the compressive stress produced by the reactions. Then the equation for determining σ will be

$$\alpha(t - t_0) = \frac{\sigma}{E},$$

from which

$$\sigma = E\alpha(t - t_0). \qquad (12)$$

As a second example, let us consider the system represented in Fig. 15 and assume that the vertical bar is heated from the assembly temperature t_0 to a new temperature t. The corresponding thermal expansion will be partially prevented by the two other bars of the system, and certain compressive stresses will develop in the vertical bar and tensile stresses in the inclined bars. The magnitude of the compressive force in the vertical bar will be given by eq. (11), in which instead of the magnitude a of the inaccuracy in length we substitute the thermal expansion $\alpha l(t - t_0)$ of the vertical bar.

Problems

1. The rails of a tramway are welded together at 50° Fahrenheit. What stresses will be produced in these rails when heated by the sun to 100° if the coefficient of thermal expansion of steel is $70 \cdot 10^{-7}$?

Answer. $\sigma = 10{,}500$ lbs. per sq. in.

2. What change of stresses will be produced in the case represented in Fig. 21 by increasing the temperature from $t_0°$ to $t°$ if the coefficient of expansion of steel is α_s and that of copper α_c?

Solution. Due to the fact that $\alpha_c > \alpha_s$ the increasing temperature produces compression in the copper and tension in the steel. The unit elongations of the copper and of the steel should be equal. Denoting by X the increase in the tensile force in the bolt due to the change of temperature, we obtain

$$\alpha_s(t - t_0) + \frac{X}{A_sE_s} = \alpha_c(t - t_0) - \frac{X}{A_cE_c},$$

from which

$$X = \frac{(\alpha_c - \alpha_s)(t - t_0)A_sE_s}{1 + \dfrac{A_sE_s}{A_cE_c}}.$$

The change in the stresses in the bolt and in the tube may be calculated now in the usual way.

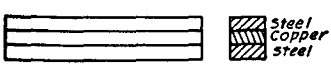

Fig. 24.

3. A strip of copper is soldered between two strips of steel (Fig. 24). What stresses will be produced in the steel and in the copper by a rise in the temperature of the strips from t_0 to t degrees?

Suggestion. The same method as in the previous problem should be used.

4. What stresses will be produced in the bars of the system

represented in Fig. 15 if the temperature of all the bars be raised from t_0 to t?

Solution. Let X denote the tensile force produced in the steel bar by an increase in temperature. Then from the condition of equilibrium of the hinge O it can be seen that in the copper bars compressive forces act, equal to $X/2 \cos \alpha$; consequently the elongation of the steel bar becomes

$$\delta = \alpha_s(t - t_0)l + \frac{Xl}{A_sE_s}$$

and the elongation of the copper bars is

$$\delta_1 = \alpha_c(t - t_0)\frac{l}{\cos \alpha} - \frac{Xl}{2 \cos^2 \alpha A_cE_c}.$$

Furthermore from previous considerations (see p. 20),

$$\delta_1 = \delta \cos \alpha;$$

therefore

$$\alpha_s(t - t_0)l + \frac{Xl}{A_sE_s} = \alpha_c(t - t_0)\frac{l}{\cos^2 \alpha} - \frac{Xl}{2 \cos^3 \alpha A_cE_c},$$

from which

$$X = \frac{(t - t_0)\left(\dfrac{\alpha_c}{\cos^2 \alpha} - \alpha_s\right)A_sE_s}{1 + \dfrac{1}{2 \cos^3 \alpha}\dfrac{A_sE_s}{A_cE_c}}.$$

The stresses in the steel and in the copper will now be obtained from the following equations:

$$\sigma_s = \frac{X}{A_s}; \qquad \sigma_c = \frac{X}{2 \cos \alpha A_c}.$$

5. Assuming that in the case shown in Fig. 17 a constant load $P = 100{,}000$ is applied at an initial temperature t_0, determine at what increase in temperature the load will be completely transmitted to the copper if $\alpha_s = 70 \times 10^{-7}$ and $\alpha_c = 92 \times 10^{-7}$.

Solution.

$$(\alpha_c - \alpha_s)(t - t_0) = \frac{4P}{\pi(D^2 - d^2)E_c},$$

from which

$$t - t_0 = 75.4 \text{ degrees Fahrenheit.}$$

6. A steel bar consisting of two portions of lengths l_1 and l_2 and cross-sectional areas A_1 and A_2 is fixed at the ends. Find the thermal stresses if the temperature rises by 100 degrees Fahrenheit. Assume $l_1 = l_2$, $A_1 = 2A_2$, and $\alpha_s = 70 \times 10^{-7}$.

7. Find the thermal stresses in the system shown in Fig. 24 if the temperature of all three strips rises by 100 degrees Fahrenheit. The thickness of each of the three strips is the same and the coefficients of thermal expansion are $\alpha_s = 70 \times 10^{-7}$ and $\alpha_c = 92 \times 10^{-7}$. Assume $E_c : E_s = \frac{8}{15}$.

8. The temperature of the system shown in Fig. 15 rises by 100 degrees Fahrenheit. Find the thermal stresses if all three bars are of steel and have equal cross-sectional areas. Take $\alpha_s = 70 \times 10^{-7}$ and $E_s = 30 \times 10^6$ lbs. per sq. in.

9. Find the stresses in the wires of the system shown in Fig. 18 if the cross-sectional area of the wires is 0.1 sq. in., the load $Q = 4,000$ lbs., and the temperature of the system rises after assembly by 10 degrees Fahrenheit.

10. Determine the stresses which will be built up in the system represented in Fig. 20 if the temperature of the upper horizontal bar rises from t_0 to t degrees.

8. Extension of a Circular Ring.—If uniformly distributed radial forces act along the circumference of a thin circular ring (Fig. 25), uniform enlargement of the ring will be pro-

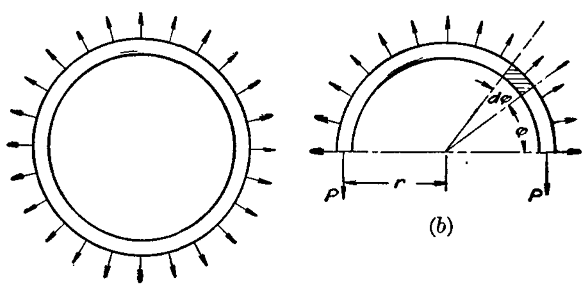

FIG. 25.

duced. In order to determine the tensile force P in the ring let us imagine that the ring is cut at the horizontal diametral section (Fig. 25, b) and consider the upper portion as a free body. If q denotes the uniform load per unit length of the center line of the ring and r is the radius of the center line, the force acting on an element of the ring cut out by two adjacent cross sections will be $qrd\varphi$, where $d\varphi$ is the central angle, corresponding to the element. Taking the sum of the vertical components of all the forces acting on the half ring, the following equation of equilibrium will be obtained:

$$2P = 2\int_0^{\pi/2} qr \sin\varphi \, d\varphi = 2qr,$$

from which

$$P = qr. \tag{13}$$

The tensile stress in the ring will now be obtained by dividing the force P by the cross sectional area of the ring.

In practical applications very often the determination of tensile stresses in a rotating ring is necessary. Then q represents the centrifugal force per unit length of the ring and is given by the equation:

$$q = \frac{w}{g}\frac{v^2}{r}, \tag{14}$$

in which w is the weight of the ring per unit length, r is the radius of the center line, v is the velocity of the ring at the radius r, and g is acceleration due to gravity. Substituting this expression for q in eq. (13), we obtain

$$P = \frac{wv^2}{g},$$

and the corresponding tensile stress will be

$$\sigma = \frac{P}{A} = \frac{wv^2}{Ag} = \frac{\gamma v^2}{g}. \tag{15}$$

It is seen that the stress is proportional to the density γ/g of the material and to the square of the peripheral velocity. For a steel ring and for the velocity $v = 100$ feet per second this stress becomes 1,060 lbs. per sq. in. Then for the same material and for any other velocity v_1 the stress will be $0.106 \times v^2$ in lbs. per sq. in., when v is in feet per sec.

Problems

1. Determine the tensile stress in the cylindrical wall of the press shown in Fig. 3 if the inner diameter is 10 ins. and the thickness of the wall is 1 in.

Solution. The maximum hydrostatic pressure p in the cylinder will be found from the equation:

$$p \cdot \frac{\pi 10^2}{4} = 100{,}000 \text{ lbs.},$$

from which $p = 1{,}270$ lbs. per sq. in. Cutting out from the cylinder an elemental ring of width 1 in. in the direction of the axis of the cylinder and using eq. (13) in which, for this case, $q = p = 1{,}270$ lbs. per in. and $r = 5$ ins., we obtain

$$\sigma = \frac{P}{A} = \frac{1{,}270 \times 5}{1 \times 1} = 6{,}350 \text{ lbs. per sq. in.}$$

2. A copper tube is fitted over a steel tube at a high temperature t (Fig. 26), the fit being such that no pressure exists between tubes at this temperature. Determine the stresses which will be produced in the copper and in the steel when cooled to room temperature t_0 if the outer diameter of the steel tube is d, the thickness of the steel tube is h_s and that of the copper tube is h_c.

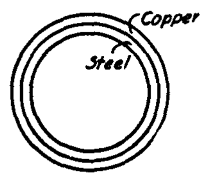

FIG. 26.

Solution. Due to the difference in the coefficients of expansion α_c and α_s there will be a pressure between the outer and the inner tubes after cooling. Let x denote the pressure per square inch; then the tensile stress in the copper tube will be

$$\sigma_c = \frac{xd}{2h_c}$$

and the compressive stress in the steel will be

$$\sigma_s = \frac{xd}{2h_s} \cdot$$

The pressure x will now be found from the condition that during cooling both tubes have the same circumferential contraction; hence

$$\alpha_c(t - t_0) - \frac{xd}{2E_c h_c} = \alpha_s(t - t_0) + \frac{xd}{2E_s h_s},$$

from which

$$\sigma_c = \frac{xd}{2h_c} = \frac{(\alpha_c - \alpha_s)(t - t_0)E_c}{1 + \frac{h_c}{h_s}\frac{E_c}{E_s}}.$$

In the same manner the stress in the steel may be calculated.

3. Referring to Fig. 26, what additional tensile stress in the tube will be produced by submitting it to a hydrostatic inner pressure $p = 100$ lbs. per sq. in. if the inner diameter $d_1 = 4$ in., $h_s = 0.1$ in. and $h_c = 15/8 \times 0.1$ in.?

Solution. Cutting out of the tube an elemental ring of width 1 in., the complete tensile force in the ring will be

$$P = \frac{pd_1}{2} = 200 \text{ lbs.}$$

Due to the fact that the unit circumferential elongations in copper and in steel are the same, the stresses will be in proportion to the moduli, i.e., the stress in the copper will be 8/15 that in the steel. At the same time the cross sectional area of the copper is 15/8 that of the steel; hence the force P will be equally distributed between two metals and the tensile stress in the copper produced by a hydrostatic pressure will be

$$\sigma_c = \frac{P}{2 \times h_c} = \frac{200}{2 \times 15/8 \times 0.1} = 533 \text{ lbs. per sq. in.}$$

The stress in the steel will be

$$\sigma_s = \frac{15}{8}\sigma_c = 1{,}000 \text{ lbs. per sq. in.}$$

4. A built-up ring consists of an inner copper ring and an outer steel ring. The inner diameter of the steel ring is smaller than the outer diameter of the copper ring by the amount δ and the structure is assembled after preliminary heating of the steel ring. When cooled the steel ring produces pressure on the copper ring (shrink fit pressure). Determine the stresses in the steel and the copper after assembly if both rings have rectangular cross sections with the dimensions h_s and h_c in radial direction and dimensions equal to unity in the direction perpendicular to the plane of the ring. The dimensions h_s and h_c may be considered small as compared with the diameter d of the surface of contact of the two rings.

Solution. Let x be the uniformly distributed pressure per square inch of the surface of contact of the rings; then the compressive stress in the copper and the tensile stress in the steel will be found from the equations:

$$\sigma_c = \frac{xd}{2h_c}; \qquad \sigma_s = \frac{xd}{2h_s}. \tag{a}$$

The decrease in the outer diameter of the copper ring will be

$$\delta_1 = \frac{\sigma_c}{E_c} \cdot d = \frac{xd^2}{2h_cE_c}.$$

The increase of the inner diameter of the steel ring will be

$$\delta_2 = \frac{\sigma_s}{E_s} \cdot d = \frac{xd^2}{2h_sE_s}.$$

The unknown pressure x will be found from the equation:

$$\delta_1 + \delta_2 = \frac{xd^2}{2}\left(\frac{1}{h_cE_c} + \frac{1}{h_sE_s}\right) = \delta,$$

from which

$$x = \frac{2\delta h_sE_s}{d^2\left(1 + \dfrac{h_sE_s}{h_cE_c}\right)}.$$

Now the stresses σ_s and σ_c, from eqs. (a), will be

$$\sigma_c = \frac{\delta}{d} \cdot \frac{h_s}{h_c} \cdot \frac{E_s}{1 + \dfrac{h_sE_s}{h_cE_c}}; \qquad \sigma_s = \frac{\delta}{d} \cdot \frac{E_s}{1 + \dfrac{h_sE_s}{h_cE_c}}.$$

5. Determine the stresses which will be produced in the built-up ring of the previous problem by rotation of the ring with a constant speed n r.p.m.

Solution. Due to the fact that copper has a greater density and a smaller modulus of elasticity than steel, the copper ring will press on the steel ring during rotation. Let x denote the pressure per square inch of the surface of contact between the two rings. Then the corresponding stresses will be given by eqs. (a) of the previous problem. In addition to these stresses the stresses produced by centrifugal forces should be taken into consideration. Denoting by γ_s and γ_c the weights per unit volume of steel and copper and using eq. (15), we obtain

$$\sigma_s = \frac{\gamma_s}{g}\left(\frac{2\pi n}{60}\right)^2\left(\frac{d + h_s}{2}\right)^2; \qquad \sigma_c = \frac{\gamma_c}{g}\left(\frac{2\pi n}{60}\right)^2\left(\frac{d - h_c}{2}\right)^2.$$

Combining these stresses with the stresses due to pressure x and noting that the unit elongation for both rings should be the same, the following equation for determining x will be obtained:

$$\frac{1}{E_s}\left[\frac{\gamma_s}{g}\left(\frac{2\pi n}{60}\right)^2\left(\frac{d+h_s}{2}\right)^2+\frac{xd}{2h_s}\right]$$
$$=\frac{1}{E_c}\left[\frac{\gamma_c}{g}\left(\frac{2\pi n}{60}\right)^2\left(\frac{d-h_c}{2}\right)^2-\frac{xd}{2h_c}\right],$$

from which x may be calculated for each particular case. Knowing x, the complete stress in the copper and the steel may be found without difficulty.

6. Determine the limiting peripheral speed of a copper ring if the working stress is $\sigma_w = 3{,}000$ lbs. per sq. in. and $\gamma_c = 550$ lbs. per cubic foot.

Answer.

$$v = 159 \text{ feet per sec.}$$

7. Referring to problem 2 and Fig. 26, determine the stress in the copper at room temperature if $t - t_0 = 100°$ Fahrenheit,

$$\alpha_c - \alpha_s = 22 \times 10^{-7}, \qquad h_s = h_c.$$

Answer.

$$\sigma_c = 2{,}300 \text{ lbs. per sq. in.}$$

8. Referring to problem 5, determine the number of revolutions n per minute at which the stress in the copper ring becomes equal to zero if the initial assembly stress in the same ring was a compression equal to σ_0, and $h_c = h_s$, and $E_s = 2E_c$.

Solution. The number of revolutions n will be determined from the equation:

$$3\sigma_0 = \left(\frac{2\pi n}{60}\right)^2\left[\frac{\gamma_c}{g}\left(\frac{d-h_c}{2}\right)^2+\frac{\gamma_s}{g}\left(\frac{d+h_s}{2}\right)^2\right].$$

9. Find the stresses in the built-up ring of problem 4 assuming $\delta = 0.001$ in., $d = 4$ in., $h_s = h_c$, and $E_s/E_c = 15/8$. Find the changes of these stresses if the temperature of the rings increases after assembly by 10 degrees Fahrenheit. Take $\alpha_c = 92 \times 10^{-7}$ and $\alpha_s = 70 \times 10^{-7}$.

10. Referring to problem 5, find the stresses in steel and in copper if $n = 3{,}000$ r.p.m., $d = 2$ ft., $h_s = h_c = \frac{1}{2}$ in., $\gamma_s = 490$ lbs. per cubic foot, and $\gamma_c = 550$ lbs. per cubic foot.

CHAPTER II

ANALYSIS OF STRESS AND STRAIN

9. Variation of the Stress with the Orientation of the Cross Section for Simple Tension and Compression.—In discussing stresses in a prismatic bar submitted to an axial tension P we have previously considered (art. 2) only the stress over cross sections perpendicular to the axis of the bar. We now take up the case in which the cross section pq (Fig. 27a), perpendicular to the plane of the figure, is inclined to the axis. Since all longitudinal fibers have the same elongation (see p. 3) the forces representing the action of the right portion of the bar on its left portion are uniformly distributed over the cross section pq. The left portion of the bar, isolated in Fig. 27b, is in equilibrium under the action of these forces and the external force P applied at the left end. Hence the resultant of the forces distributed over the cross section pq is equal to P. Denoting by A the area of the cross section normal to the axis of the bar and by φ, the angle between the axis x and the normal n to the cross section pq, the cross-sectional area of pq will be $A/\cos\varphi$ and the stress s over this cross section is

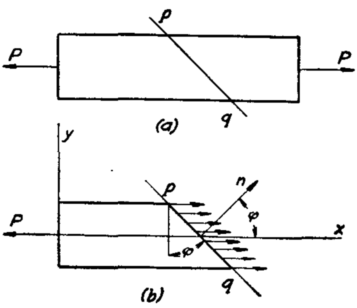

FIG. 27.

$$s = \frac{P\cos\varphi}{A} = \sigma_x \cos\varphi \tag{16}$$

where $\sigma_x = P/A$ denotes the stress on the cross section normal to the axis of the bar. It is seen that the stress s over any inclined cross section of the bar is smaller than the stress σ_x over the cross section normal to the axis of the bar and that it diminishes as the angle φ increases. For $\varphi = \pi/2$ the section pq is parallel to the axis of the bar and the stress s becomes zero, which indicates that there is no pressure between the longitudinal fibers of the bar.

The stress s, defined by equation (16), has the direction of the force P and is not perpendicular to the cross section pq. In such cases it is usual to resolve the total stress into two components, as is shown in Fig. 28. The stress component σ_n perpendicular to the cross section is called the *normal stress*. Its magnitude is

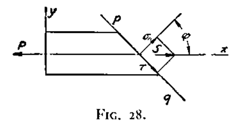

FIG. 28.

$$\sigma_n = s \cos \varphi = \sigma_x \cos^2 \varphi. \tag{17}$$

The tangential component τ is called the *shearing stress* and has the value

$$\tau = s \sin \varphi = \sigma_x \cos \varphi \sin \varphi = \frac{\sigma_x}{2} \sin 2\varphi. \tag{18}$$

To visualize the strain which each component stress produces let us consider a thin element cut out of the bar by two adjacent parallel sections pq and p_1q_1, Fig. 29a. The stresses acting on this element are shown in Fig. 29a. Figures 29b and 29c are obtained by resolving these stresses into normal and tangential components as explained above and show separately the action of each of these components. It is seen that the *normal stresses* σ_n produce extension of the element in the direction of the normal n to the cross section pq and the shearing stresses produce sliding of section pq with respect to p_1q_1.

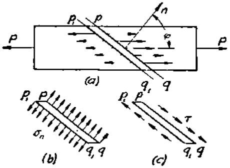

FIG. 29.

From equation (17) it is seen that the *maximum normal stress* acts over cross sections normal to the axis of the bar and we have

$$(\sigma_n)_{\max} = \sigma_x.$$

The *maximum shearing stress*, as seen from equation (18), acts over cross section s inclined at $45°$ to the axis of the bar, where $\sin 2\varphi = 1$, and has the magnitude

$$\tau_{max} = \tfrac{1}{2}\sigma_x. \tag{19}$$

Although the maximum shearing stress is one-half the maximum normal stress, this stress is sometimes the controlling factor when considering the strength of materials which are much weaker in shear than in tension. For example, in a tensile test of a bar of mild steel with a polished surface, yielding of the metal is visible to the naked eye, Fig. 30. It

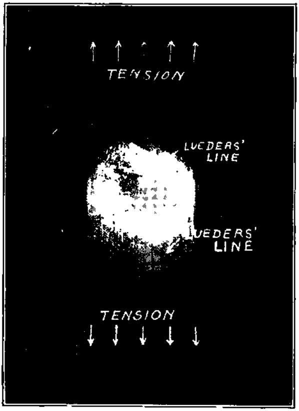

Fig. 30.

occurs along the inclined planes for which the shearing stress is a maximum and at the value of the force P which corresponds to the point B in Fig. 2a. This indicates that in the case of mild steel failure is produced by the maximum shearing stress although this stress is only equal to one-half of the maximum normal stress.

Formulas (17) and (18), derived for a bar in tension can be used also in the case of compression. Tensile stress is assumed positive and compressive negative. Hence for a bar under axial compression we have only to take σ_x with a negative sign in formulas (17) and (18). The negative sign of σ_n will then indicate that in Fig. 29*b* we obtain, instead of tension, a compressive action on the thin element between the adjacent cross sections *pq* and p_1q_1. The negative sign for τ in formula (18) will indicate that for compression of the bar the shearing action on the element has the direction opposite to that shown in Fig. 29*c*. Figure 31 illustrates the rules for signs of normal and shearing stresses which will be used. Positive sign for shearing is taken when they form a couple in clockwise direction and negative sign for opposite direction.

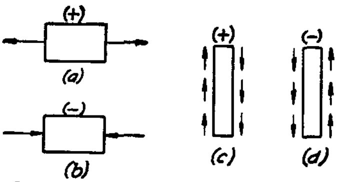

FIG. 31.

10. The Circle of Stress.—Formulas (17) and (18) can be represented graphically.[1] We take an orthogonal system of coordinates with the origin at *O* and with positive direction of axes as shown in Fig. 32. Beginning with the cross section *pq* perpendicular to the axis of the bar we have for this case $\varphi = 0$, in Fig. 28, and we find, from formulas (17) and (18) $\sigma_n = \sigma_x, \tau = 0$. Selecting a scale for stresses and measuring normal components along the horizontal axis and shearing components along the vertical axis, the stress acting on the plane with $\varphi = 0$ is represented in Fig. 32 by a point *A* having the abscissa equal to σ_x and the ordinate equal to zero. Taking now a plane parallel to the axis of the bar we have

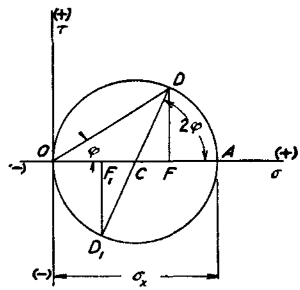

FIG. 32.

[1] This graphical representation is due to O. Mohr, Zivilingenieur, 1882, p. 113. See also his "Abhandlungen," p. 219, 1906. In this book the references to other publications on the same subject are given.

$\varphi = \pi/2$, and observing that both stress components vanish for such a plane we conclude that the origin O, in Fig. 32, corresponds to this plane. Constructing now on $\overline{OA}$ as diameter a circle it can be readily proved that the stress components for any cross section pq with an arbitrarily chosen angle φ, Fig. 28, will be represented by the coordinates of a point on that circle. To obtain the point on the circle corresponding to a definite angle φ it is only necessary to measure in the counter-clockwise direction from the point A the arc subtending an angle equal to 2φ. Let D be the point obtained in this manner; then, from the figure,

$$\overline{OF} = \overline{OC} + \overline{CF} = \frac{\sigma_x}{2} + \frac{\sigma_x}{2}\cos 2\varphi = \sigma_x \cos^2 \varphi,$$

$$\overline{DF} = \overline{CD}\sin 2\varphi = \frac{\sigma_x}{2}\sin 2\varphi.$$

Comparing these expressions for the coordinates of point D with expressions (17) and (18) it is seen that this point defines the stresses acting on the plane pq, Fig. 28. As the section pq rotates in the counter-clockwise direction about an axis perpendicular to the plane of Fig. 28, φ varying from 0 to $\pi/2$, the point D moves from A to O, so that the upper half-circle determines the stresses for all values of φ within these limits. If the angle φ is larger than $\pi/2$ we obtain a cross section as shown in Fig. 33*a* cut by a plane *mm* the external normal [2] n_1 to which makes with the x axis an angle larger than $\pi/2$. Measuring again in the counter-clockwise direction from the point A, in Fig. 32, the arc subtending an angle equal to 2φ we will obtain now a point on the lower half-circle.

Take, as an example, the case when *mm* is perpendicular to cross section pq which was previously considered. In such a case the corresponding point on the circle in Fig. 32 is point D_1 such that the angle DOD_1 is equal to π; thus DD_1 is a diameter of the circle. Using the coordinates of point D_1

[2] The portion of the bar on which the stresses act is indicated by shading. The external normal n_1 is directed outward from that portion.

we find the stress components σ_{n_1} and τ_1 for the plane mm

$$\sigma_{n_1} = \overline{OF_1} = \overline{OC} - \overline{F_1C} = \frac{\sigma_x}{2} - \frac{\sigma_x}{2}\cos 2\varphi = \sigma_x \sin^2 \varphi \qquad (20)$$

$$\tau_1 = -\overline{F_1D_1} = -\overline{CD_1}\sin 2\varphi = -\frac{\sigma_x}{2}\sin 2\varphi.^3 \qquad (21)$$

Comparing these results with expressions (17) and (18) we find

$$\sigma_n + \sigma_{n_1} = \sigma_x \cos^2 \varphi + \sigma_x \sin^2 \varphi = \sigma_x \qquad (22)$$

$$\tau_1 = -\tau. \qquad (23)$$

This indicates that the sum of the normal stresses acting on two perpendicular planes remains constant and equal to σ_x. The shearing stresses acting on two perpendicular planes are numerically equal but of opposite sign.

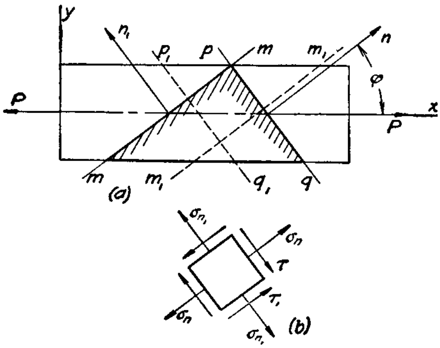

FIG. 33.

By taking the adjacent cross sections m_1m_1 and p_1q_1 parallel to mm and pq an element, such as shown in Fig. 33*b*, is isolated and the directions of stresses acting on this element are indicated. It is seen that the shearing stresses acting on the sides of the element parallel to the pq plane produce a couple in the clockwise direction, which, according to the accepted rule defined in Fig. 31*c*, must be considered positive. The shearing stresses acting on the other two sides of the element

[3] The minus sign is taken since point D_1 is on the side of negative ordinates.

produce a couple in the counter-clockwise direction which, according to the rule defined in Fig. 31*d*, is negative.

The circle in Fig. 32 called the *circle of stress* is used to determine the stress components σ_n and τ for a cross section pq whose normal makes any angle φ with the x axis, Fig. 28. A similar construction can be used to solve the inverse problem, when the components σ_n and τ are given and it is required to find the tensile stress σ_x in the axial direction and the angle φ. We observe that the angle between the chord OD and the x axis is equal to φ, Fig. 32. Hence, after constructing the point D with coordinates σ_n and τ, we obtain φ by drawing the line OD. Knowing the angle φ, the radius DC making the angle 2φ with the axis OC can be drawn and the center C of the circle of stress obtained.

Problems

1. Determine σ_n and τ analytically and graphically if $\sigma_x = 15{,}000$ lbs. per sq. in. and $\varphi = 30°$ or $\varphi = 120°$. By using the angles 30° and 120° isolate an element as shown in Fig. 33*b* and show by arrows the directions of stresses acting on the element.

2. Solve the previous problem assuming that instead of tensile stress σ_x there acts compressive stress of the same amount. Observe that in this case the diameter of the circle, Fig. 32, must lie on the negative side of the abscissa.

3. On a plane pq, Fig. 28, are acting a normal stress $\sigma_n = 12{,}000$ lbs. per sq. in. and a shearing stress $\tau = 4{,}000$ lbs. per sq. in. Find the angle φ and the stress σ_x.

Answer.

$$\tan\varphi = \tfrac{1}{3}, \qquad \sigma_x = \frac{\sigma_n}{\cos^2\varphi} = 13{,}330 \text{ lbs. per sq. in.}$$

4. On the two perpendicular sides of the element in Fig. 33*b* are acting the normal stresses $\sigma_n = 12{,}000$ lbs. per sq. in. and $\sigma_{n_1} = 6{,}000$ lbs. per sq. in. Find σ_x and τ.

Answer. $\sigma_x = 18{,}000$ lbs. per sq. in., $\tau = \pm\ 8{,}485$ lbs. per sq. in.

5. Find maximum shearing stress for the case in problem 1.

6. Determine the aspect of cross sections for which the normal and the shearing stresses are numerically equal.

11. Tension or Compression in Two Perpendicular Directions.—There are cases in which the material of a structure is submitted to the action of tension or compression in two perpendicular directions. As an example of such a stress condition let us consider stresses in the cylindrical wall of a boiler submitted to internal pressure p lbs. per sq. in.[4] Let us cut out a small element from the cylindrical wall of the boiler by two adjacent axial sections and by two circumferential sections, Fig. 34*a*. Because of the internal pressure

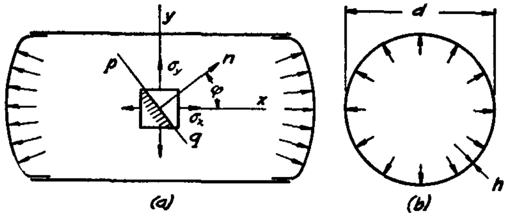

FIG. 34.

the cylinder will expand both in the circumferential and in the axial directions. The tensile stress σ_y in the circumferential direction will be determined in the same manner as in the case of a circular ring (art. 8). Denoting the inner diameter of the boiler by d and its wall thickness by h, this stress is

$$\sigma_y = \frac{pd}{2h}. \tag{24}$$

In calculating the tensile stress σ_x in the axial direction we imagine the boiler cut by a plane perpendicular to the x axis. Considering the equilibrium of one portion of the boiler it will be appreciated that the tensile force producing longitudinal extension of the boiler is equal to the resultant of the pressure on the ends of the boiler, i.e., equal to

$$P = p\left(\frac{\pi d^2}{4}\right).$$

[4] More accurately p denotes the difference between the internal pressure and the external atmospheric pressure.

The cross sectional area of the wall of the boiler is [5]

$$A = \pi dh.$$

Hence

$$\sigma_x = \frac{P}{A} = \frac{pd}{4h}. \tag{25}$$

It is seen that the element of the wall undergoes tensile stresses σ_x and σ_y in two perpendicular directions.[6] The tensile stress σ_y in the circumferential direction being twice as large as the stress σ_x in the axial direction. We consider now the stress over any cross section pq, Fig. 34*a*, perpendicular to xy plane and whose normal n makes an angle φ with the x axis. By using formulas (17) and (18) of the previous article we conclude that the tensile stresses σ_x acting in the axial direction produces on the plane pq normal and shearing stresses of magnitude

$$\sigma_n' = \sigma_x \cos^2 \varphi, \qquad \tau' = \tfrac{1}{2}\sigma_x \sin 2\varphi. \tag{a}$$

To calculate the stress components produced on the same plane pq by the tensile stress σ_y, we observe that the angle between σ_y and the normal n, Fig. 34*a*, is $\frac{\pi}{2} - \varphi$ and is measured clockwise from the y axis, while φ is measured counterclockwise from the x axis. From this we conclude that in using equations (17) and (18) we must substitute in this case σ_y for σ_x and $-\left(\frac{\pi}{2} - \varphi\right)$, instead of φ. This gives

$$\sigma_n'' = \sigma_y \sin^2 \varphi, \qquad \tau'' = -\tfrac{1}{2}\sigma_y \sin 2\varphi. \tag{b}$$

Summing up the stress components (*a*) and (*b*) produced by σ_x and σ_y stresses respectively, the resultant normal and shearing stress components for the case of tension in the two

[5] The thickness of the wall is assumed small in comparison with the diameter and the approximate formula for the cross-sectional area is used.

[6] There is also a pressure on the inner cylindrical surface of the element but this pressure is small in comparison with σ_x and σ_y and is neglected in further discussion.

perpendicular directions are obtained

$$\sigma_n = \sigma_x \cos^2 \varphi + \sigma_y \sin^2 \varphi, \tag{26}$$

$$\tau = \tfrac{1}{2}(\sigma_x - \sigma_y) \sin 2\varphi. \tag{27}$$

12. The Circle of Stress for Combined Stresses.—Proceeding as in article 10 the graphical representation of the formulas (26) and (27) can be readily obtained using the circle of stress. Assuming again that the abscissas and the ordinates represent to a certain scale the normal and the shearing stress components, we conclude that the points A and B, in Fig. 35,

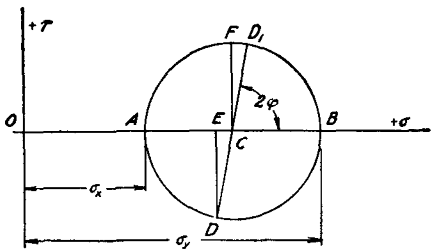

FIG. 35.

with abscissas equal to σ_x and σ_y represent the stresses acting on the sides of the element in Fig. 34a, perpendicular to the x and y axes respectively. To obtain the stress components on any inclined plane, defined by an angle φ in Fig. 34a, we have only to construct a circle on AB as a diameter and draw the radius CD making the angle ACD, measured in the counter-clockwise direction from point A, equal to 2φ. From the figure we conclude that

$$\overline{OE} = \overline{OC} - \overline{CE} = \tfrac{1}{2}(\overline{OA} + \overline{OB}) - \tfrac{1}{2}(\overline{OB} - \overline{OA}) \cos 2\varphi$$

$$= \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_y - \sigma_x}{2} \cos 2\varphi = \sigma_x \cos^2 \varphi + \sigma_y \sin^2 \varphi.$$

This indicates that the abscissa $\overline{OE}$ of the point D on the circle, if measured to the assumed scale, gives the normal stress component σ_n, (26).

The ordinate of the point D is

$$\overline{DE} = \overline{CD} \sin 2\varphi = \frac{\sigma_y - \sigma_x}{2} \sin 2\varphi.$$

Observing that this ordinate must be taken with negative sign, we conclude that the ordinate of the point D, taken with the proper sign, gives the shearing stress component (27).

When the plane pq is rotating counter-clockwise with respect to an axis perpendicular to xy plane, Fig. 34*a*, the corresponding point D is moving in the counter-clockwise direction along the circle of stress in Fig. 35 so that for each value of φ the corresponding values of the components σ_n and τ are obtained as the coordinates of the point D.

From this graphical representation of formulae (26) and (27) it follows at once that the maximum normal stress component in our case is equal to σ_y and the maximum shearing stress represented by the radius $\overline{CF}$ of the circle in Fig. 35 is

$$\tau_{\max} = \frac{\sigma_y - \sigma_x}{2} \tag{28}$$

and occurs when $\sin 2\varphi = -1$ and $\varphi = 3\pi/4$. The same magnitude of shearing stress but with negative sign is acting on the plane for which $\varphi = \pi/4$.

Taking two perpendicular planes defined by the angles φ and $\pi/2 + \varphi$, which the normals n and n_1 make with the x axis, the corresponding stress components are given by the co-ordinates of points D and D_1 in Fig. 35, and we conclude

$$\sigma_n + \sigma_{n_1} = \sigma_x + \sigma_y, \tag{29}$$

$$\tau_1 = -\tau. \tag{30}$$

This indicates that the sum of the normal stresses acting on two perpendicular planes remains constant as the angle φ varies. Shearing stresses acting on two perpendicular planes are numerically equal but of opposite sign.

The circle of stress, similar to that in Fig. 35, can be constructed also if one or both stresses σ_x and σ_y are compressive, it is only necessary to measure the compressive stresses on the negative side of the abscissa axis. Assuming, for example, that the stresses acting on an element are as shown in Fig. 36*a*, the corresponding circle is shown in Fig. 36*b*. The stress

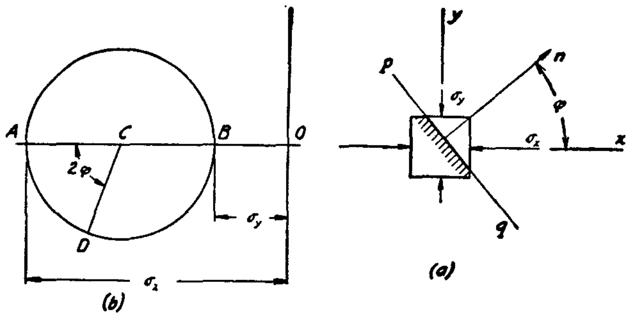

FIG. 36.

components acting on a plane pq with normal n are given by the coordinates of the point D in the diagram.

Problems

1. The boiler shown in Fig. 34 has $d = 100$ in., $h = \frac{1}{2}$ in. Determine σ_x and σ_y if $p = 100$ lbs. per sq. in. Isolate a small element by the planes for which $\varphi = 30°$ and $120°$ and show the magnitudes and the directions of the stress components acting on the lateral sides of that element.

2. Determine the stresses σ_n, σ_{n_1}, τ and τ_1 if, in Fig. 36*a*, $\sigma_x = 10{,}000$ lbs. per sq. in., $\sigma_y = -5{,}000$ lbs. per sq. in. and $\varphi = 30°$, $\varphi_1 = 120°$.

Answer. $\sigma_n = 6{,}250$ lbs. per sq. in., $\sigma_{n_1} = -1{,}250$ lbs. per sq. in., $\tau = -\tau_1 = 6{,}500$ lbs. per sq. in.

3. Determine σ_n, σ_{n_1}, τ and τ_1 in the previous problem, if the angle φ is chosen so that τ is a maximum.

Answer. $\sigma_n = \sigma_{n_1} = 2{,}500$ lbs. per sq. in., $\tau = -\tau_1 = 7{,}500$ lbs. per sq. in.

13. Principal Stresses.—It was shown in the previous article that for tension or compression in two perpendicular directions x and y one of the two stresses σ_x or σ_y is the maximum and the other the minimum normal stress. For all inclined planes, such as planes pq in Figs. 34*a* and 36*a*, the value of the normal stress σ_n lies between these limiting values.

At the same time there is acting on all inclined planes not only normal stresses σ_n, but also shearing stresses, τ. Such stresses as σ_x and σ_y, of which one is the maximum and the other the minimum normal stress, are called the *principal stresses* and the two perpendicular planes on which they act are called the *principal planes*. There are no shearing stresses acting on these planes.

In the example of the previous article, Fig. 34, the principal stresses σ_x and σ_y were found from very simple considerations and it was required to find the expressions for the normal and shearing stress components acting on any inclined plane, such as plane pq in Fig. 34*a*. In our further discussion (see p. 122) there will be cases in which it will be possible to determine the shearing and the normal stress components acting on two perpendicular planes. From the previous discussion we already know that such normal stresses do not represent the maximum stress which is the stress particularly important in design. To get the maximum value of stress, the principal stresses are required. The simplest way of solving this problem is by using the circle of stress we considered in Fig. 35. Assume that the stresses acting on an elementary rectangular parallelepiped are as shown in Fig.

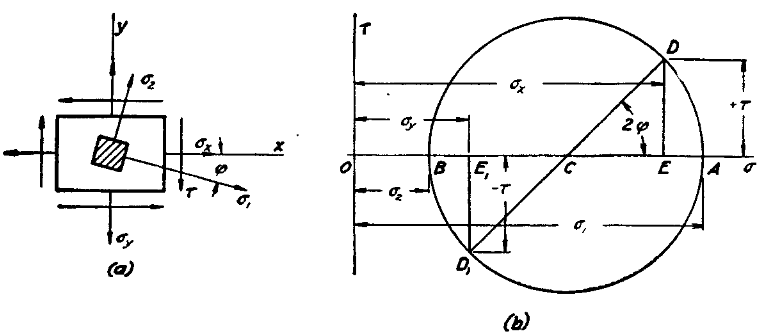

FIG. 37.

37*a*. The stresses σ_x and σ_y are not principal stresses, since not only normal but also shearing stresses are acting on the planes perpendicular to the x and y axes. To construct the circle of stress in this case, we use first the stress components σ_x, σ_y and τ and construct the points D and D_1 as shown in Fig. 37*b*. Since these two points represent the stresses acting on two perpendicular planes, the length DD_1 represents a diameter of the circle of stress. The intersection of this diameter with the x axis gives the center C of the circle, so that the circle can be readily constructed. The intersection points A and B of the circle with the x axis define the magnitudes of the maximum and the minimum normal stresses, which are the principal stresses and are denoted by σ_1 and σ_2. Using the circle, the formulas for calculating σ_1 and σ_2 can be easily obtained. From the figure we have

$$\sigma_1 = \overline{OA} = \overline{OC} + \overline{CD} = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2}, \quad (31)$$

$$\sigma_2 = \overline{OB} = \overline{OC} - \overline{CD} = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2}. \quad (32)$$

The directions of the principal stresses can also be obtained from the figure. We know that the angle DCA is the double angle between the stress σ_1 and the x axis and since 2φ is measured from D to A in the clockwise direction the direction of σ_1 must be as indicated in Fig. 37*a*. If we isolate the element shaded in the figure with the sides normal and parallel to σ_1 there will be only normal stresses σ_1 and σ_2 acting on its sides. For the calculation of the numerical value of the angle φ we have, from the figure,

$$|\tan 2\varphi| = \frac{\overline{DE}}{\overline{CE}}.$$

Regarding the sign of the angle φ, it must be taken negative in this case since it is measured from the x axis in the clockwise direction, Fig. 37*a*. Hence

$$\tan 2\varphi = -\frac{\overline{DE}}{\overline{CE}} = -\frac{2\tau}{\sigma_x - \sigma_y}. \quad (33)$$

The maximum shearing stress is given by the magnitude of the radius of the circle of stress and we have

$$\tau_{\max} = \frac{\sigma_1 - \sigma_2}{2} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2}. \tag{34}$$

The equations (31)–(34) completely solve the problem of the determination of the maximum normal and the maximum shearing stresses if the normal and shearing stresses acting on any two perpendicular planes are given since a circle is fixed by two points at the ends of a diameter.

Problems

1. An element, Fig. 37*a*, is submitted to the action of stresses $\sigma_x = 5{,}000$ lbs. per sq. in., $\sigma_y = 3{,}000$ lbs. per sq. in., $\tau = 1{,}000$ lbs. per sq. in. Determine the magnitudes and the directions of principal stresses σ_1 and σ_2.

Solution. By using formulas (31) and (32) we obtain

$$\sigma_1 = \frac{5{,}000 + 3{,}000}{2} + \sqrt{\left(\frac{5{,}000 - 3{,}000}{2}\right)^2 + 1{,}000^2}$$
$$= 4{,}000 + 1{,}414 = 5{,}414 \text{ lbs. per sq. in.,}$$

$$\sigma_2 = 4{,}000 - 1{,}414 = 2{,}586 \text{ lbs. per sq. in.}$$

From formula (33) we have

$$\tan 2\varphi = -1, \qquad 2\varphi = -45°, \qquad \varphi = -22\tfrac{1}{2}°.$$

The minus sign indicates that φ is measured from the x axis in the clockwise direction as shown in Fig. 37*a*.

2. Determine the direction of the principal stresses in the previous problem if $\sigma_x = -5{,}000$ lbs. per sq. in.

Solution. The corresponding circle of stress is shown in Fig. 38, $\tan 2\varphi = \frac{1}{4}$, $2\varphi = 14°2'$. Hence the angle which the maximum compressive stress makes with the x axis is equal to $7°1'$ and is measured counter-clockwise from the x axis.

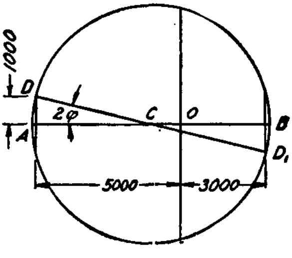

FIG. 38.

3. Find the circle of stress for the case of two equal tensions $\sigma_x = \sigma_y = \sigma$ and for two equal compressions $\sigma_x = \sigma_y = -\sigma$. $\tau = 0$ in both cases.

Answer. Circles become points on the horizontal axis with the abscissas σ and $-\sigma$ respectively.

4. On the sides of the element shown in Fig. 39*a* are acting the stresses $\sigma_x = -500$ lbs. per sq. in., $\sigma_y = 1,500$ lbs. per sq. in., $\tau = 1,000$ lbs. per sq. in. Find, by using the circle of stress, the magnitudes of the normal and shearing stresses on (*a*) the principal planes, (*b*) the planes of maximum shearing stress.

Solution. The corresponding circle of stress is shown in Fig. 39*b*. The points D and D_1 represent stresses acting on the sides of the element in Fig. 39*a* perpendicular to the x and y axes. $\overline{OB}$ and $\overline{OA}$ represent the principal stresses. Their magnitudes are $\sigma_1 = 1,914$ lbs. per sq. in. and $\sigma_2 = -914$ lbs per sq. in. respectively. The direction of the maximum compressive stress σ_2 makes the angle $= 22\frac{1}{2}°$ with the x axis, this angle being measured from the x axis in the counter-clockwise direction as shown in Fig. 39*a*. The points

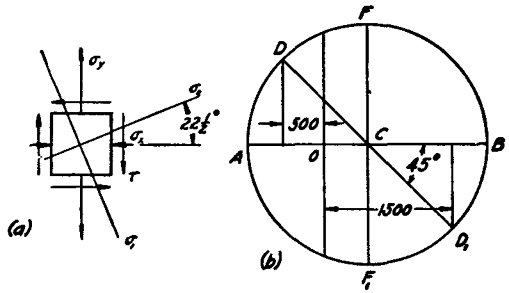

Fig. 39.

F and F_1 represent stresses acting on the planes subject to maximum shear. The magnitude of this shear is 1,414 lbs. per sq. in. $\overline{OC}$ represents the normal stresses equal to 500 lbs. per sq. in. acting on the same plane.

5. Solve the previous problem if $\sigma_x = -5,000$ lbs. per sq. in., $\sigma_y = 3,000$ lbs. per sq. in., $\tau = 1,000$ lbs. per sq. in.

14. Analysis of Strain in the Case of Simple Tension.—In article 2, the axial elongation of a bar in tension was discussed. Experiments show that such axial elongation is always accompanied by lateral contraction of the bar, and that the ratio $\dfrac{\text{unit lateral contraction}}{\text{unit axial elongation}}$ is constant for a given bar within the elastic limit. This constant will be called μ and is known as *Poisson's ratio*, after the name of the French mathematician who determined this ratio analytically by using the molecular theory of structure of the material. For materials which have the same elastic properties in all directions, so-called *isotropic materials*, Poisson found $\mu = 1/4$. Experimental investigation of the lateral contraction in structural metals[7] shows that μ is usually not very far off the value calculated by Poisson. For instance, in the case of structural steel it can be taken as $\mu = 0.30$. Knowing the Poisson ratio of a material, the change in volume of a bar in tension can be calculated. The length of the bar will increase in the ratio $(1 + \epsilon) : 1$. The lateral dimensions diminish in the ratio $(1 - \mu\epsilon) : 1$; hence the cross-sectional area diminishes in the ratio $(1 - \mu\epsilon)^2 : 1$. Then the volume of the bar changes in the ratio $(1 + \epsilon)(1 - \mu\epsilon)^2 : 1$, which becomes $(1 + \epsilon - 2\mu\epsilon) : 1$ if we recall that ϵ is a small quantity and neglect its powers. Then the *unit volume expansion* is $\epsilon(1 - 2\mu)$. It is unlikely that any material diminishes its volume when in tension; hence μ must be less than 0.50. For such materials as rubber and paraffin μ approaches the above limit and the volume of these materials during extension remains approximately constant. On the other hand such material as concrete has a small magnitude of μ ($\mu = 1/8$ to $1/12$) and for cork μ can be taken equal to zero.

The above discussion of lateral contraction during tension can be applied with suitable changes to the case of compression. Longitudinal compression will be accompanied by lateral expansion and for calculating this expansion the same value for μ as in the case of extension is used.

Problems

1. Determine the increase in unit volume of the bar in tension if $\sigma_w = 5{,}600$ lbs. per sq. in., $\mu = 0.30$, $E = 30 \cdot 10^6$ lbs. per sq. in.

Solution. Increase in unit volume is

$$\epsilon(1 - 2\mu) = \frac{\sigma_w}{E}(1 - 2\mu) = \frac{5{,}600}{30 \times 10^6}(1 - 0.6) = 74.7 \times 10^{-6}.$$

2. Determine the increase in volume of a bar the extension of

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[7] These materials can be considered as isotropic (see Part II).

which is produced by the force P at the end and the weight of the bar (see article 5), p. 14.

Answer. The increase in volume is equal to

$$\frac{Al(1 - 2\mu)}{E}\left(\frac{P}{A} + \frac{\gamma l}{2}\right).$$

15. Strain in the Case of Tension or Compression in Two Perpendicular Directions.—If a bar in the form of a rectangular parallelepiped is submitted to tensile forces acting in two perpendicular directions x and y (Fig. 34), the elongation in one of these directions will depend not only upon the tensile stress in this direction but also upon the stress in the perpendicular direction. The unit elongation in the direction of the x axis due to the tensile stress σ_x will be σ_x/E. The tensile stress σ_y will produce lateral contraction into x direction equal to $\mu\sigma_y/E$; then if both stresses σ_x and σ_y act simultaneously the unit elongation in x direction will be

$$\epsilon_x = \frac{\sigma_x}{E} - \mu\frac{\sigma_y}{E}. \tag{35}$$

Similarly, for the y direction, we obtain

$$\epsilon_y = \frac{\sigma_y}{E} - \mu\frac{\sigma_x}{E}. \tag{36}$$

In the particular case, for the two tensions equal, $\sigma_x = \sigma_y = \sigma$, we obtain

$$\epsilon_x = \epsilon_y = \frac{\sigma}{E}(1 - \mu). \tag{37}$$

From eqs. (35) and (36) the stresses σ_x and σ_y can be obtained as functions of unit strains ϵ_x and ϵ_y as follows:

$$\sigma_x = \frac{(\epsilon_x + \mu\epsilon_y)E}{1 - \mu^2}; \qquad \sigma_y = \frac{(\epsilon_y + \mu\epsilon_x)E}{1 - \mu^2}. \tag{38}$$

If in the case shown in Fig. 39*a* the elongation ϵ_x in axial direction and the elongation ϵ_y in circumferential direction are measured by an extensometer the corresponding tensile stresses σ_x and σ_y will be found from equations (38).

Problems

1. Determine the increase in the volume of the cylindrical steel boiler under internal pressure (Fig. 34), neglecting the deformation of the ends and taking $\sigma_y = 6{,}000$ lbs. per sq. in.

Solution. By using eqs. (35) and (36)

$$\epsilon_y = \frac{6{,}000}{30 \times 10^6} - 0.3\frac{3{,}000}{30 \times 10^6} = \frac{5{,}100}{30 \times 10^6} = 17 \times 10^{-5},$$

$$\epsilon_x = \frac{3{,}000}{30 \times 10^6} - 0.3\frac{6{,}000}{30 \times 10^6} = \frac{1{,}200}{30 \times 10^6} = 4 \times 10^{-5}.$$

The volume of the boiler will increase in the ratio

$$(1 + \epsilon_x)(1 + \epsilon_y)^2 : 1 = (1 + \epsilon_x + 2\epsilon_y) : 1 = 1.00038 : 1.$$

2. A cube of concrete is compressed in two perpendicular directions by the arrangement shown in Fig. 40. Determine the decrease in the volume of the cube if it is 4 inches on a side, the compressive stress is uniformly distributed over the faces, $\mu = 0.1$ and $P = 20{,}000$ lbs.

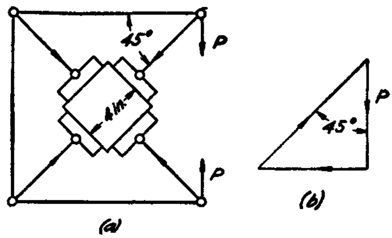

Fig. 40.

Solution. Neglecting friction in the hinges and considering the equilibrium of each hinge (Fig. 40, *b*), it can be shown that the block is submitted to equal compression in two perpendicular directions and that the compressive force is equal to $P\sqrt{2} = 28{,}300$ lbs. The corresponding strain, from eq. (37), is

$$\epsilon_x = \epsilon_y = -\frac{28{,}300}{16 \times 4 \times 10^6}(1 - 0.1) = -0.000398.$$

In the direction perpendicular to the plane of the figure a lateral expansion of the block takes place which is

$$\epsilon_z = -\mu\frac{\sigma_x}{E} - \mu\frac{\sigma_y}{E} = 0.2 \times \frac{28{,}300}{16 \times 4 \times 10^6} = 0.0000885.$$

The decrease per unit volume of the block will be

$$\epsilon_x + \epsilon_y + \epsilon_z = -2 \times 0.000398 + 0.0000885 = -0.000707.$$

3. Determine the increase in the cylindrical lateral surface of the boiler considered in problem 1 above.

Solution. Increase per unit area of lateral surface $= \epsilon_x + \epsilon_y = 21 \times 10^{-5}$.

4. Determine the unit elongation in the σ_1 direction of a bar of steel, if the stress conditions are such as indicated in problem 1, p. 49.

Solution.

$$\epsilon_1 = \frac{1}{30 \times 10^6}(5{,}414 - 0.3 \times 2{,}586) = 154.6 \times 10^{-6}.$$

16. Pure Shear. Modulus in Shear.—Let us consider the particular case of normal stresses acting in two perpendicular directions in which the tensile stress σ_x in the horizontal direction is numerically equal to the compressive stress σ_y in the vertical direction, Fig. 41*a*. The corresponding circle of

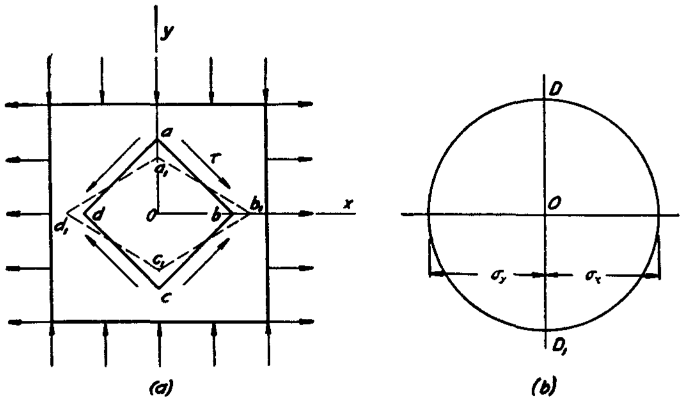

FIG. 41.

stress is shown in Fig. 41*b*. Point D on this circle represents the stresses acting on the planes *ab* and *cd* perpendicular to *xy* plane and inclined at 45° to the x axis. Point D_1 represents stresses acting on the planes *ad* and *bc* perpendicular to *ab* and *cd*. It is seen from the circle of stress that the normal stress on each of these planes is zero and that the shearing stress over the same planes, represented by the radius of the circle, is numerically equal to the normal stress σ_x, so that

$$\tau = \sigma_x = -\sigma_y. \qquad (a)$$

If we imagine the element *abcd* to be isolated it will be in equilibrium under the shearing stresses only as shown in

Fig. 41*a*. Such a state of stress is called *pure shear*. It may be concluded that pure shear is equivalent to the state of stress produced by tension in one direction and an equal compression in the perpendicular direction. If a rectangular element, similar to the element *abcd* in Fig. 41*a*, is isolated by planes which are no longer at 45° to the x axis, normal stress as well as shearing stress will act on the sides of such an element. The magnitude of these stresses may be obtained from the circle of stress, Fig. 41*b*, in the usual way.

Let us consider now the deformation of the element *abcd*. Since there are no normal stresses acting on the sides of this element the lengths *ab*, *ad*, *bc* and *cd* will not change due to the deformation, but the horizontal diagonal *bd* will be stretched and the vertical diagonal *ac* will be shrunk changing the square *abcd* into a rhombus after deformation as indicated in the figure by dotted lines. The angle at *b*, which was $\pi/2$ before deformation, now becomes less than $\pi/2$, say $(\pi/2) - \gamma$, and at the same time the angle at *a* increases and becomes equal to $(\pi/2) + \gamma$. The small angle γ determines the distortion of the element *abcd*, and is called the *shearing strain*. The shearing strain may also be visualized as follows: The element *abcd* of Fig. 41*a* is turned counter-clockwise through 45° and put into the position shown in Fig. 42. After distortion, produced by the shearing stresses τ, the same element takes the position indicated by the dotted lines. The shearing strain, represented by the magnitude of the small angle γ, may be taken equal to the ratio $\overline{aa_1}/\overline{ad}$, equal to the horizontal sliding aa_1 of the side *ab* with respect to the side *dc* divided by the distance between these two sides. If the material obeys Hooke's law this sliding is proportional to the stress τ and we can express the relation between the shearing stress and the shearing strain by the equation

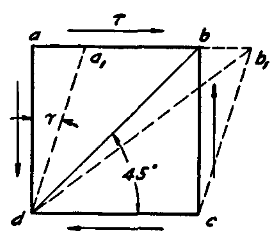

FIG. 42.

$$\gamma = \frac{\tau}{G} \tag{39}$$

in which G is a constant depending on the mechanical properties of the material. Equation (39) is similar to equation (4) which was established for simple tension and the constant G is called *modulus of elasticity in shear*, or *modulus of rigidity.*

Since the distortion of the element $abcd$, Fig. 42, is entirely defined by the elongation of the diagonal bd and the contraction of the diagonal ac, which deformations can be calculated by using the equations of the preceding article, it may be concluded that the modulus G can be expressed by the modulus in tension E and Poisson's ratio μ. To establish this relationship we consider the triangle Oab, Fig. 41*a*. The elongation of the side Ob and the shortening of the side Oa of this triangle during deformation will be found by using equations (35) and (36). In terms of ϵ_x and ϵ_y we have

$$Ob_1 = Ob(1 + \epsilon_x), \qquad Oa_1 = Oa(1 + \epsilon_y)$$

and, from the triangle Oa_1b_1,

$$\tan (Ob_1a_1) = \tan\left(\frac{\pi}{4} - \frac{\gamma}{2}\right) = \frac{Oa_1}{Ob_1} = \frac{1 + \epsilon_y}{1 + \epsilon_x}. \tag{b}$$

For a small angle γ we have also

$$\tan\left(\frac{\pi}{4} - \frac{\gamma}{2}\right) = \frac{\tan\frac{\pi}{4} - \tan\frac{\gamma}{2}}{1 + \tan\frac{\pi}{4}\tan\frac{\gamma}{2}} \approx \frac{1 - \frac{\gamma}{2}}{1 + \frac{\gamma}{2}}. \tag{c}$$

Observing that in the case of pure shear

$$\sigma_x = -\sigma_y = \tau,$$

$$\epsilon_x = -\epsilon_y = \frac{\sigma_x(1 + \mu)}{E} = \frac{\tau(1 + \mu)}{E},$$

and equating expressions (b) and (c) we obtain

$$\frac{1 - \frac{\tau(1 + \mu)}{E}}{1 + \frac{\tau(1 + \mu)}{E}} = \frac{1 - \frac{\gamma}{2}}{1 + \frac{\gamma}{2}}$$

from which

$$\frac{\gamma}{2} = \frac{\tau(1 + \mu)}{E}$$

or

$$\gamma = \frac{2\tau(1 + \mu)}{E}.$$

Comparing this result with formula (39) we conclude that

$$G = \frac{E}{2(1 + \mu)}. \tag{40}$$

We see that the modulus of elasticity in shear can be easily calculated if the modulus in tension E and Poisson's ratio μ are known. In the case of steel, for example,

$$G = \frac{30 \cdot 10^6}{2(1 + 0.30)} = 11.5 \cdot 10^6 \text{ lbs. per sq. in.}$$

It should be noted that the application of a uniform shearing stress to the sides of a block as assumed in Fig. 42 is very difficult to realize so that the condition of pure shear is usually produced by the torsion of a circular tube, Fig. 43. Due to a small rotation of one end of the tube with respect to the other the generators traced on the cylindrical surface become inclined to the axis of the cylinder and an element *abcd* formed by two generators and two adjacent circular cross sections undergoes a shearing strain similar to that shown in Fig. 42.

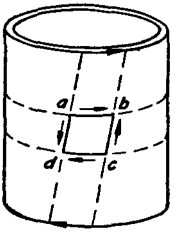

FIG. 43.

The problem of twist will be discussed later (see chapter 9) where will be shown how the shearing stress τ and the shearing strain γ of the element *abcd* can be calculated if the torque and the corresponding angle of twist of the shaft are measured. If τ and γ are found from such a torsion test, the value of the modulus G can be calculated from equation (39). With this value of G, and knowing E from a tensile test, Poisson's ratio μ can be calculated from equation (40). The direct determination of μ by measuring lateral contraction during a tensile test is more complicated since this contraction is very small and an extremely sensitive instrument is required to measure it with sufficient accuracy.

Problems

1. The block *abcd*, Fig. 42, is made of a material for which $E = 10 \cdot 10^6$ lbs. per sq. in. and $\mu = 0.25$. Find γ and the unit elongation of the diagonal *bd* if $\tau = 10{,}000$ lbs. per sq. in.

2. Find for the previous problem the sliding aa_1 of the side *ab* with respect to the side *cd* if the diagonal $bd = 2$ in.

3. Prove that the change in volume of the block *abcd* in Fig. 42 is zero if the first powers only of the strain components ϵ_x and ϵ_y are considered.

17. Working Stresses in Shear.—Submitting a material to pure shear the relation between shearing stress and shearing strain can be established experimentally. Such a relationship is usually shown by a diagram, Fig. 44, in which the abscissa represents shearing strain and the ordinate—shearing stress. The diagram is similar to that of a tensile test and we can mark on it the proportional limit A and the yield point[8] B. The experiments show that for a material such as structural steel the yield point in shear τ_{YP} is only about $0.55 - 0.60$ of σ_{YP}. Since at yield point a considerable distortion occurs without an appreciable change in stress, it is logical to take as the working stress in shear only a portion of yield point stress so that

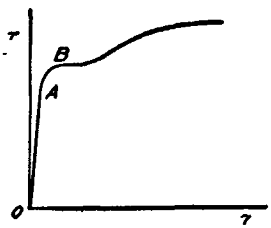

Fig. 44.

$$\tau_w = \frac{\tau_{YP}}{n} \tag{41}$$

where n is the factor of safety. Taking this factor of the same magnitude as in tension or compression we obtain

$$\tau_w = 0.55 \text{ to } 0.60 \text{ of } \sigma_w$$

which indicates that the working stress in shear should be

[8] To obtain a pronounced yield point tubular specimens are used in the torsion test.

taken much smaller than the working stress in tension. It was already indicated that in practical applications we do not encounter a uniform distribution of shearing stress over the sides of a block as was assumed in Fig. 42 and that pure shear is realized in the case of torsion. We will see later that pure shear also occurs in the bending of beams. But there are many practical problems in which a solution is obtained on the assumption that we are dealing with pure shear although this assumption is only a rough approximation. Take, for example, the case of the joint in Fig. 45. It is evident that if

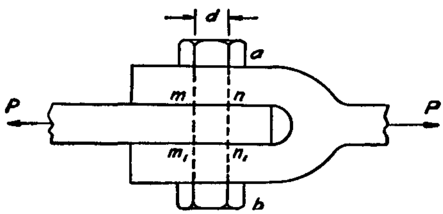

FIG. 45.

the diameter of the bolt *ab* is not large enough the joint may fail due to shear along the cross sections mn and m_1n_1. Although a more rigorous study of the problem indicates that the shearing stresses are not uniformly distributed over these cross sections and that the bolt undergoes not only shear but also bending under the action of tensile forces P, a rough approximation for the required diameter of the bolt is obtained by assuming that we have along the planes mn and m_1n_1 a uniformly distributed shearing stress τ which is obtained by dividing the force P by the sum of the cross sectional areas mn and m_1n_1. Hence

$$\tau = \frac{2P}{\pi d^2},$$

and the required diameter of the bolt is obtained from the equation

$$\tau_w = \frac{2P}{\pi d^2}. \tag{42}$$

We have another example of such a simplified treatment of shear problems in the case of riveted joints, Fig. 46. Since

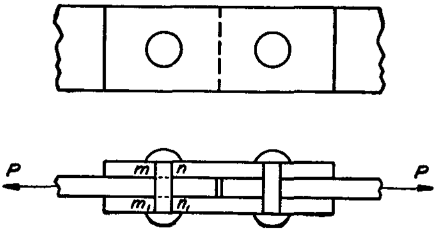

Fig. 46.

the heads of the rivets are formed at high temperature the rivets produce after cooling a great compression of the plates.[9] If tensile forces P are applied the relative motion between the plates is prevented by friction due to the above mentioned pressure between the plates. Only after friction is overcome do the rivets begin to work in shear and if the diameter of the rivets is not sufficient failure due to shear along the planes mn and m_1n_1 may occur. It is seen that the problem of stress analysis for a riveted joint is very complicated. A rough approximate solution of the problem is usually obtained by neglecting friction and assuming that the shearing stresses are uniformly distributed along the cross section mn and m_1n_1. Then the correct diameter of the rivets is obtained by using the equation (42) as in the previous example.

Problems

1. Determine the diameter of the bolt in the joint shown in Fig. 45 if $P = 10{,}000$ lbs. and $\tau_w = 6{,}000$ lbs. per sq. in.

2. Find the safe length $2l$ of the joint of two rectangular wooden bars, Fig. 47, submitted to tension, if $P = 10{,}000$ lbs., $\tau_w = 100$ lbs. per sq. in. for shear parallel to the fibers and $b = 10$ in. Determine the proper depth mn_1, if the safe limit for the local compressive stress along the fibers of wood is 800 lbs. per sq. in.

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[9] Experiments show that tensile stress in rivets is usually approaching the yield point of the material of which the rivets are made. See C. Bach, Zeitschr. d. Ver. Deutsch. Ing. 1912.

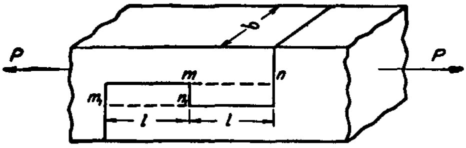


FIG. 47.

3. Find the diameter of the rivets in Fig. 46, if $\tau_w = 8{,}000$ lbs. per sq. in. and $P = 8{,}000$ lbs.

4. Determine the dimensions l and δ in the joint of two rectangular bars by steel plates, Fig. 48, if the forces, the dimensions and the working stresses are the same as in problem 2.

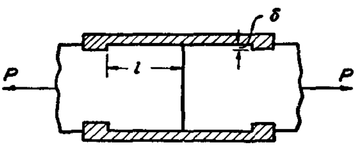


FIG. 48.

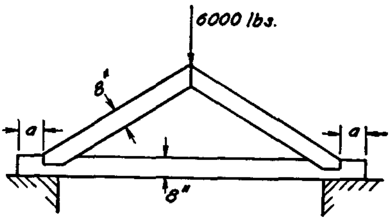


FIG. 49.

5. Determine the distance a which is required in the structure shown in Fig. 49, if the allowable shearing stress is the same as in problem 2 and the cross-sectional dimensions of all bars are 4 by 8 in. Neglect the effect of friction.

18. Tension or Compression in Three Perpendicular Directions.—If a bar in the form of a rectangular parallelepiped is submitted to the action of forces P_x, P_y and P_z (Fig. 50), the normal stresses over cross sections perpendicular to x, y and z axes are respectively

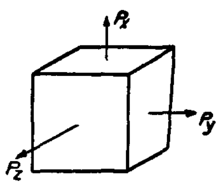


FIG. 50.

$$\sigma_x = \frac{P_x}{A_x}; \qquad \sigma_y = \frac{P_y}{A_y}; \qquad \sigma_z = \frac{P_z}{A_z}.$$

It is assumed below that $\sigma_x > \sigma_y > \sigma_z$.

Combining the effects of the forces P_x, P_y and P_z, it can be concluded that over a section through the z axis only the forces P_x and P_y produce stresses and therefore these stresses may be calculated from eqs. (26) and (27) and represented graphically by using the Mohr circle. In Fig. 51 the stress circle with a diameter AB represents these stresses. In the same manner the stresses over any section through the x axis can be represented by a circle having BC as a diameter. The circle with the diameter AC represents stresses over any section through the y axis. The three Mohr circles represent stresses over three series of sections through the x, y and z axes. For any section inclined to x, y and z axes the stress components are the coordinates of a point located in the shaded area of Fig. 51.[10] On the basis of this it can be concluded that the maximum shearing stress will be represented by the radius of the largest of the three circles and will be given by the equation $\tau_{\max} = (\sigma_x - \sigma_z)/2$. It will act on the section through the y axis bisecting the angle between the x and z axes.

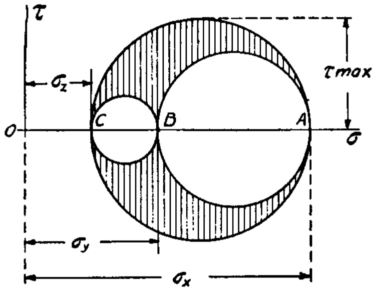


Fig. 51.

The equations for calculating the unit elongations in the directions of the x, y and z axes may be obtained by combining the effects of P_x, P_y and P_z in the same manner as in considering tension or compression in two perpendicular directions (see article 15). In this manner we obtain

$$
\begin{aligned}
\epsilon_x &= \frac{\sigma_x}{E} - \frac{\mu}{E}(\sigma_y + \sigma_z), \\
\epsilon_y &= \frac{\sigma_y}{E} - \frac{\mu}{E}(\sigma_x + \sigma_z), \qquad (43) \\
\epsilon_z &= \frac{\sigma_z}{E} - \frac{\mu}{E}(\sigma_x + \sigma_y).
\end{aligned}
$$

[10] The proof of this statement can be found in the book by A. Föppl, Technische Mechanik, Vol. 5, p. 18, 1918. See also H. M. Westergaard, Z. angew. Math. Mech., Vol. 4, p. 520, 1924.

The volume of the bar increases in the ratio

$$(1 + \epsilon_x)(1 + \epsilon_y)(1 + \epsilon_z) : 1,$$

or, neglecting small quantities of higher order,

$$(1 + \epsilon_x + \epsilon_y + \epsilon_z) : 1.$$

It is seen that the unit volume expansion is

$$\Delta = \epsilon_x + \epsilon_y + \epsilon_z. \tag{44}$$

The relation between the unit volume expansion and the stresses acting on the sides of the bar will be obtained by adding together eqs. (43). In this manner we obtain

$$\Delta = \epsilon_x + \epsilon_y + \epsilon_z = \frac{(1 - 2\mu)}{E}(\sigma_x + \sigma_y + \sigma_z). \tag{45}$$

In the particular case of uniform hydrostatic pressure we have

$$\sigma_x = \sigma_y = \sigma_z = -p.$$

Then from eqs. (43)

$$\epsilon_x = \epsilon_y = \epsilon_z = -\frac{p}{E}(1 - 2\mu), \tag{46}$$

and from eqs. (44)

$$\Delta = -\frac{3(1 - 2\mu)}{E}p, \tag{47}$$

or, using the notation

$$\frac{E}{3(1 - 2\mu)} = K, \tag{48}$$

we obtain

$$\Delta = -\frac{p}{K}. \tag{49}$$

The unit compression is proportional to the compressive stress p and inversely proportional to the quantity K, which is called the *modulus of elasticity of volume.*

Problems

1. Determine the decrease in the volume of a solid steel sphere of 10 inch diameter submitted to a uniform hydrostatic pressure $p = 10{,}000$ lbs. per sq. inch.

Solution. From eq. (49)

$$\Delta = -\frac{p}{K} = -\frac{10{,}000 \times 3(1 - 2 \times 0.3)}{30 \times 10^6} = -\frac{4}{10^4}.$$

The decrease in the volume is, therefore,

$$\frac{4}{10^4} \times \frac{\pi d^3}{6} = 0.209 \text{ cubic inch.}$$

2. Referring to Fig. 52, a rubber cylinder A is compressed in a steel cylinder B by a force P. Determine the pressure between the rubber and the steel if $P = 1{,}000$ lbs., $d = 2$ ins., Poisson's ratio for rubber $\mu = 0.45$. Friction between rubber and steel is neglected.

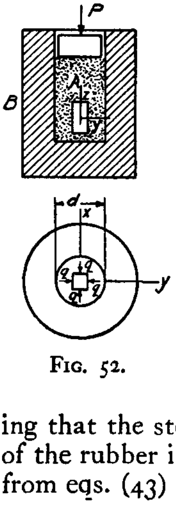


FIG. 52.

Solution. Let p denote the compressive stresses over any cross section perpendicular to the axis of the cylinder and q the pressure between the rubber and the inner surface of the steel cylinder. Compressive stress of the same magnitude will act between the lateral surfaces of the longitudinal fibers of the rubber cylinder, from which we isolate an element in the form of a rectangular parallelepiped, with sides parallel to the axis of the cylinder (see Fig. 52). This element is in equilibrium under the compressive stresses q on the lateral faces of the element and the axial compressive stress p. Assuming that the steel cylinder is absolutely rigid, the lateral expansion of the rubber in the x and y directions should be equal to zero and from eqs. (43) we obtain

$$0 = \frac{q}{E} - \frac{\mu}{E}(p + q),$$

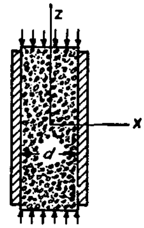
from which

$$q = \frac{\mu p}{1 - \mu} = \frac{0.45}{1 - 0.45} \cdot \frac{1{,}000 \times 4}{\pi \times 2^2} = 260 \text{ lbs. per sq. in.}$$

3. A concrete column is enclosed in a steel tube (Fig. 53). Determine the pressure between the steel and concrete and the circumfer-

ential tensile stress in the tube, assuming that there is no friction between concrete and steel and that all the dimensions and the longitudinal compressive stress in the column are known (Fig. 53).

Solution. Let p denote the longitudinal and q the lateral compressive stress, d the inner diameter and h the thickness of the tube, E_s the modulus of elasticity for steel, E_c, μ_c the modulus of elasticity and Poisson's ratio for concrete. The expansion of the concrete in a lateral direction will be, from eqs. (43),



$$\epsilon_x = -\frac{q}{E_c} + \frac{\mu_c}{E_c}(p + q). \qquad (a)$$

Fig. 53.

This expansion should be equal to the circumferential expansion of the tube (see eq. 13)

$$\epsilon = \frac{qd}{2hE_s}. \qquad (b)$$

From eqs. (a) and (b) we obtain

$$\frac{qd}{2hE_s} = -\frac{q}{E_c} + \frac{\mu_c}{E_c}(p + q),$$

from which

$$q = p\frac{\mu_c}{\frac{d}{2h}\frac{E_c}{E_s} + 1 - \mu_c}.$$

The circumferential tensile stress in the tube will now be calculated from equation

$$\sigma = \frac{qd}{2h}.$$

4. Determine the maximum shearing stress in the concrete column of the previous problem, assuming that $p = 1{,}000$ lbs. per sq. in., $\mu_c = 0.10$, $d/2h = 7.5$.

Solution.

$$\tau_{\max} = \frac{p - q}{2} = \frac{p}{2}\left(1 - \frac{0.1}{1.9}\right) = 474 \text{ lbs. per sq. in.}$$

CHAPTER III

SHEARING FORCE AND BENDING MOMENT

19. Types of Beams.—In this chapter we will discuss the simplest types of beams such as shown in Fig. 54. Figure 54*a* represents a *beam with simply supported ends*. Points of support A and B are hinged so that the ends of the beam can rotate freely during bending. It is also assumed that one of the supports is mounted on rollers and can move freely in the horizontal direction. Figure 54*b* represents a *cantilever beam*. The end A of this beam is built into the wall and cannot rotate during bending, while the end B is entirely free. Figure 54*c* represents a beam with an overhanging end. This beam is hinged to an immovable support at the end A and rests on a movable support at C.

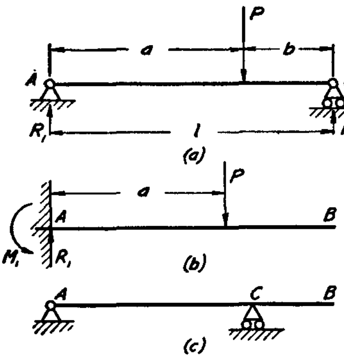


FIG. 54.

All three of the foregoing cases represent *statically determinate beams* since the reactions at the supports produced by a given load can be determined from the equations of statics. For instance, considering the simply supported beam carrying a vertical load P, Fig. 54*a*, we see that the reaction R_2 at the end B must be vertical, since this end is free to move horizontally. Then from the equation of statics, $\Sigma X = 0$, it follows that reaction R_1 is also vertical. The magnitudes of R_1 and R_2 are then determined from the equations of moments. Equating to zero the sum of the moments of all forces with respect to point B, we obtain

$$R_1 l - Pb = 0$$

from which

$$R_1 = \frac{Pb}{l}.$$

In a similar way, by considering the moments with respect to point A, we obtain

$$R_2 = \frac{Pa}{l}.$$

The reactions for the beam with an overhanging end, Fig. 54c, can be calculated in the same manner.

In the case of the cantilever beam, Fig. 54b, the load P is balanced by the reactive forces acting on the built-in end. From the equations of statics, $\Sigma X = 0$ and $\Sigma Y = 0$, we conclude at once that the resultant of the reactive forces R_1 must be vertical and equal to P. From the equation of moments, $\Sigma M = 0$, it follows that the moment M_1 of the reactive forces with respect to point A is equal to Pa and acts in the counterclockwise direction as shown in the figure.

The reactions produced by any other kind of loading on the above types of beams can be calculated by a similar procedure.

It should be noted that the special provisions permitting free rotation of the ends and free motion of the support are used in practice only in beams of large spans, such as those found in bridges. In beams of shorter span, the conditions at the support are usually as illustrated in Fig. 55. During bending of such a beam friction forces between the supporting surfaces and the beam will be produced such as to oppose rotation and horizontal movement of the ends of the beam. These forces can be of some importance in the case of flexible bars and thin metallic strips, see p. 178; but for a rigid beam the deflection of which is very small in comparison with the length l of the span these forces can be neglected, and the reactions can be calculated as though the beam were simply supported, Fig. 54a.

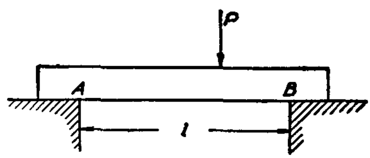


FIG. 55.

20. Bending Moment and Shearing Force.—Let us now consider a simply supported beam on which act vertical forces P_1, P_2, and P_3, Fig. 56*a*. We assume that the beam has an axial plane of symmetry and that the loads act in this plane. Then, from considerations of symmetry, we conclude that the bending also occurs in this same plane. In most practical cases this condition of symmetry is fulfilled since the usual cross-sectional shapes, such as a circle, a rectangle, an I, or a T, are symmetrical. The more general case of a non-symmetrical cross section will be discussed later (see p. 93).

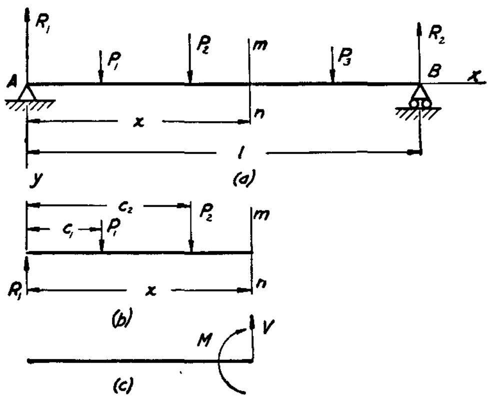


FIG. 56.

To investigate the stresses produced in a beam during bending, we proceed in the same manner as we have already used in discussing the stresses produced in a bar by a central tension, Fig. 1. We imagine that the beam AB is cut in two parts by a cross section mn taken at any distance x from the left support A, Fig. 56*a*, and that the right portion of the beam is removed. In discussing the equilibrium of the remaining left-hand portion of the beam, Fig. 56*b*, we must consider not only the external forces such as loads P_1, P_2, and reaction R_1 but also the internal forces which are distributed over the cross section mn and which represent the action of the right portion of the beam on the left portion. These internal forces must be of such a magnitude as to equilibrate the above mentioned external forces P_1, P_2, and R_1.

In the ensuing discussion it will be advantageous to reduce the actual system of external forces to a simplified equivalent system. From statics we know that a system of parallel forces can be replaced by one force equal to the algebraic sum of the given forces together with a couple. In our particular case we can replace the forces P_1, P_2, and R_1 by the vertical force V acting in the plane of the cross section mn and by the couple M. The magnitude of the force is

$$V = R_1 - P_1 - P_2, \tag{a}$$

and the magnitude of the couple is

$$M = R_1x - P_1(x - c_1) - P_2(x - c_2). \tag{b}$$

The force V, which is equal to the algebraic sum of the external forces to the left of the cross section mn, is called the *shearing force* at the cross section mn. The couple M, which is equal to the algebraic sum of the moments of the external forces to the left of the cross section mn with respect to the centroid of this cross section, is called the *bending moment* at the cross section mn. Thus the system of external forces to the left of the cross section mn can be replaced by the statically equivalent system consisting of the shearing force V acting in the plane of the cross section and the couple M, Fig. 56*c*. The stresses which are distributed over the cross section mn and which represent the action of the right portion of the beam on its left portion must then be such as to balance the bending moment M and the shearing force V.

If a distributed load rather than a number of concentrated forces acts on a beam, the same reasoning can be used as in the previous case. Take, as an example, the uniformly loaded beam shown in Fig. 57*a*. Denoting the load per unit length by q, the reactions in this case are

$$R_1 = R_2 = \frac{ql}{2}.$$

To investigate stresses distributed over a cross section mn we again consider the equilibrium of the left portion of the beam, Fig. 57*b*. The external forces acting on this portion of the beam are the reaction R_1 and the load uniformly distributed along the length x. This latter load has, of course, a resultant equal to qx. The algebraical sum of all forces to the left of the cross section mn is thus $R_1 - qx$. The algebraic sum of the moments of all forces to the left of the cross section mn with respect to the centroid of this cross section is obtained by subtracting the moment of the resultant of the distributed load from the moment R_1x of the reaction. The moment of the distributed load is evidently equal to

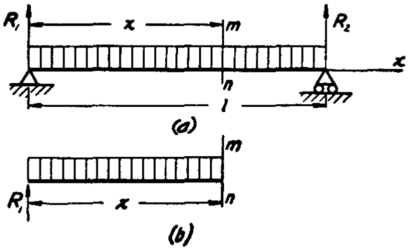


FIG. 57.

$$qx \times \frac{x}{2} = \frac{qx^2}{2}.$$

Thus we obtain for the algebraic sum of the moments the expression

$$R_1x - \frac{qx^2}{2}.$$

All the forces acting on the left portion of the beam can now be replaced by one force acting in the plane of the cross section mn and equal to

$$V = R_1 - qx = q\left(\frac{l}{2} - x\right) \qquad (c)$$

together with a couple equal to

$$M = R_1x - \frac{qx^2}{2} = \frac{qx}{2}(l - x). \qquad (d)$$

The expressions (*c*) and (*d*) represent, respectively, the shearing force and the bending moment at the cross section mn.

In the above examples the equilibrium of the left portion of the beam has been discussed. If, instead of the left portion, the right be considered, the algebraic sum of the forces to the right of a cross section and the algebraic sum of the moments of those forces have the same magnitudes V and M as have already been found but are of opposite sense. This follows from the fact that the loads acting on a beam together with the reactions R_1 and R_2 represent a system of forces in equilibrium; and the moment of all these forces with respect to any point in their plane, as well as their algebraic sum, must be equal to zero. Hence the moment of the forces acting on the left portion of the beam with respect to the centroid of a cross section mn must be equal and opposite to the moment with respect to the same point of the forces acting on the right portion of the beam. Also the algebraic sum of forces acting on the left portion of the beam must be equal and opposite to the algebraic sum of forces acting on the right portion.

In the following discussion the bending moment and the shearing force at a cross section mn are taken as positive if in considering the left portion of a beam the directions obtained are such as shown in Fig. 57*c*. To visualize this rule of sign for bending moments, let us isolate an element of the beam by two adjacent cross sections mn and m_1n_1, Fig. 58. If the

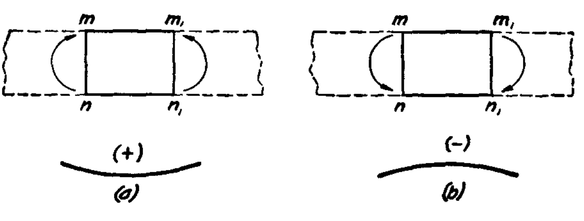


FIG. 58.

bending moments in these cross sections are positive the forces to the left of the cross section mn give a moment in the clockwise direction and the forces to the right of the cross section m_1n_1 a moment in the counter-clockwise direction as shown in Fig. 58*a*. It is thus seen that the directions of the moments are such that a bending is produced which is convex downwards. If the bending moments in the cross sections

mn and m_1n_1 are negative, a bending convex upwards is produced as shown in Fig. 58*b*. Thus in portions of a beam where the bending moment is positive, the deflection curve is convex downwards, while in portions where bending moment is negative the deflection curve is convex upwards.

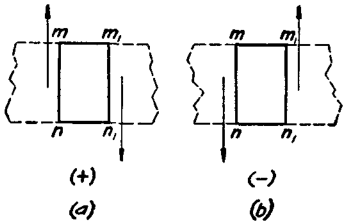


FIG. 59.

The rule of signs for shearing forces is visualized in Fig. 59.

21. Relation Between Bending Moment and Shearing Force.—Let us consider an element of a beam cut out by two adjacent cross sections mn and m_1n_1 which are a distance dx apart, Fig. 60. Assuming that there is a positive bending moment and a positive shearing force at the cross section mn, the action of the left portion of the beam on the element is represented by the force V and the couple M as indicated in Fig. 60*a*. In the same manner, assuming that at section

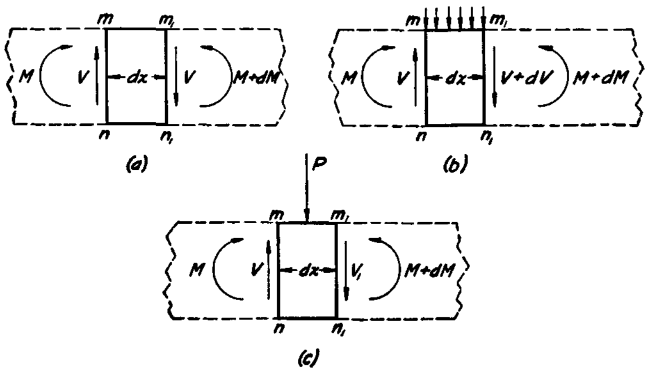


FIG. 60.

m_1n_1 the bending moment and the shearing force are positive, the action of the right portion of the beam on the element is represented by the couple and the force shown. If no forces act on the beam between cross sections mn and m_1n_1, Fig. 60*a*, the shearing forces at these two cross sections are equal.[1]

[1] The weight of the element of the beam is neglected in this discussion.

Regarding the bending moments, it can be seen from the equilibrium of the element that they are not equal at two adjacent cross sections and that the increase dM in the bending moment equals the moment of the couple represented by the two equal and opposite forces V, i.e.,

$$dM = Vdx$$

and

$$\frac{dM}{dx} = V. \tag{50}$$

Thus, on all portions of a beam between loads the shearing force is the rate of change of the bending moment with respect to x.

Let us now consider the case in which a distributed load of intensity q acts between the cross sections mn and m_1n_1, Fig. 60*b*. Then the total load acting on the element is qdx. If q is considered positive when the load acts downward, it may be concluded from the equilibrium of the element that the shearing force at the cross section m_1n_1 is different from that at mn by an amount

$$dV = -qdx,$$

from which it follows that

$$\frac{dV}{dx} = -q. \tag{51}$$

Thus the rate of change of the shearing force is equal to the intensity of the load with negative sign.

Taking the moment of all forces acting on the element we obtain

$$dM = Vdx - qdx \times \frac{dx}{2}.$$

Neglecting the second term on the right side as a small quantity of the second order, we again arrive at equation (50) and conclude that the rate of change of the bending moment is equal to the shearing force in the case of a distributed load as well.

If a concentrated load P acts between the adjacent cross sections mn and m_1n_1, Fig. 60*c*, there will be an abrupt change in the magnitude of the shearing force. Let V denote the shearing force at the cross section mn and V_1 that at the cross section m_1n_1. Then from the equilibrium of the element mm_1n_1n, we find

$$V_1 = V - P.$$

Thus the magnitude of the shearing force changes by the amount P as we pass the point of application of the load. From equation (50) it can then be concluded that at the point of application of a concentrated force there is an abrupt change in the magnitude of the derivative dM/dx.

22. Bending Moment and Shearing Force Diagrams.— It was shown in the preceding discussion that the stresses acting on a cross section mn of a beam are such as to balance the bending moment M and shearing force V at that cross section. Thus the magnitudes of M and V at any cross section entirely define the magnitudes of stresses acting on that cross section. To simplify the investigation of stresses in a beam it is advisable to use a graphical representation of the variation of the bending moment and the shearing force along the axis of the beam. In such a representation the abscissas indicate the position of the cross section and the ordinates, the values respectively of the bending moment and shearing force which act at this cross section, positive values being plotted above the horizontal axis and negative values below. Such graphical representations are called *bending moment and shearing force diagrams*, respectively.

Let us consider, as an example, a simply supported beam with a single concentrated load P, Fig. 61.[2] The reactions in this case are

$$R_1 = \frac{Pb}{l} \quad \text{and} \quad R_2 = \frac{Pa}{l}.$$

Taking a cross section mn to the left of P, it can be concluded

[2] For simplicity the rollers under the movable supports will usually be omitted in subsequent figures.

that at such a cross section

$$V = \frac{Pb}{l} \qquad \text{and} \qquad M = \frac{Pb}{l}x. \tag{a}$$

The shearing force and the bending moment have the same sense as those in Figures 58*a* and 59*a* and are therefore positive. It is seen that the shearing force remains constant along the portion of the beam to the left of the load and that the bending moment varies directly as x. For $x = 0$ the moment is zero and for $x = a$, i.e., at the cross section where the load is applied, the moment is equal to Pab/l. The corresponding portions of the shearing force and bending moment diagrams are shown in Fig. 61*b* and 61*c*, respectively, by the

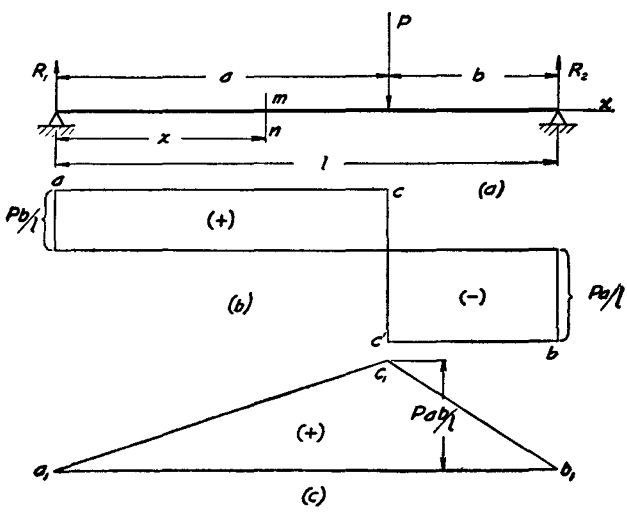


FIG. 61.

straight lines ac and a_1c_1. For a cross section to the right of the load we obtain

$$V = \frac{Pb}{l} - P = -\frac{Pa}{l} \qquad \text{and} \qquad M = \frac{Pb}{l}x - P(x - a), \tag{b}$$

x always being the distance from the left end of the beam. The shearing force for this portion of the beam remains constant and negative. In Fig. 61*b* this force is represented by the line $c'b$ parallel to the x axis. The bending moment is a linear function of x which for $x = a$ is equal to Pab/l and for $x = l$ is equal to zero. It is always positive and its variation along the right portion of the beam is represented by the straight line c_1b_1. The broken lines $acc'b$ and $a_1c_1b_1$ in Figs. 61*b* and 61*c* represent respectively the shearing force and bending moment diagrams for the whole length of the beam. At the load P there is an abrupt change in the magnitude of the shearing force from the positive value Pb/l to the negative value $-Pa/l$ and a sharp change in the slope of the bending moment diagram.

In deriving expressions (*b*) for the shearing force and bending moment, we considered the left portion of the beam, a portion which is acted upon by the two forces R_1 and P. It would have been simpler in this case to consider the right portion of the beam where only the reaction Pa/l acts. Following this procedure and using the rule of signs indicated in Figures 58 and 59, we obtain

$$V = -\frac{Pa}{l} \quad \text{and} \quad M = \frac{Pa}{l}(l - x). \tag{c}$$

Expressions (*b*) previously obtained can also be brought to this simpler form if we observe that $a = l - b$.

It is interesting to note that the shearing force diagram consists of two rectangles the areas of which are equal. Taking into consideration the opposite signs of these areas we conclude that the total area of the shearing force diagram is zero. This result is not accidental. By integrating equation (50), we have

$$\int_A^B dM = \int_A^B V dx, \tag{d}$$

where the limits A and B indicate that the integration is taken over the entire length of the beam from the end A to the end B. The right side of equation (*d*) then represents the total area of the shearing force diagram. The left side of the same equation, after integration, gives the difference $M_B - M_A$ of the bending moments at the ends B and A. In the case of a simply supported beam the moments at the ends vanish; hence the total area of the shearing force diagram vanishes.

If several loads act on a beam, Fig. 62, the beam is divided into several portions and expressions for V and M must be

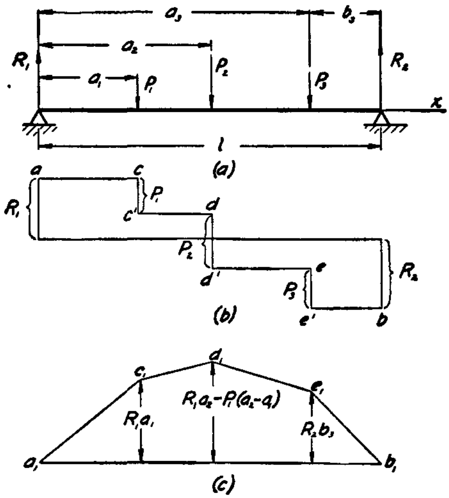


FIG. 62.

established for each portion. Measuring x from the left end of the beam and taking $x < a_1$, we obtain for the first portion of the beam

$$V = R_1 \qquad \text{and} \qquad M = R_1 x. \tag{e}$$

For the second portion of the beam, i.e., for $a_1 < x < a_2$, we obtain

$$V = R_1 - P_1 \qquad \text{and} \qquad M = R_1 x - P_1(x - a_1). \tag{f}$$

For the third portion of the beam, i.e., for $a_2 < x < a_3$, it is advantageous to consider the right portion of the beam rather than the left. In this way we obtain

$$V = -(R_2 - P_3) \qquad \text{and} \qquad M = R_2(l - x) - P_3(l - x - b_3). \tag{g}$$

Finally for the last portion of the beam we obtain

$$V = -R_2, \qquad M = R_2(l - x). \tag{h}$$

From expressions (e) $\cdots$ (h) we see that in each portion of the beam the shearing force remains constant; hence the shearing force diagram is as shown in Fig. 62b The bending moment in each portion of the beam is a linear function of x; hence in the corresponding diagram it is represented by an inclined straight line. To draw these lines we note from expressions (e) and (h) that at the ends $x = 0$ and $x = l$ the moments are zero. The moments under the loads are obtained by substituting in expressions (e), (f), and (h) $x = a_1$, $x = a_2$, and $x = a_3$, respectively. In this manner we obtain for the above mentioned moments the values

$$M = R_1a_1, \qquad M = R_1a_2 - P_1(a_2 - a_1), \qquad \text{and} \qquad M = R_2b_3.$$

By using these values the bending moment diagram, shown in Fig. 62c, is readily constructed.

In practical applications it is of importance to find the cross sections at which the bending moment has its maximum or minimum values. In the case of concentrated loads just considered, Fig. 62, the maximum bending moment occurs under the load P_2. This load corresponds in the bending moment diagram to point d_1, at which point the slope of the diagram changes sign. Further, from equation (50), we know that the slope of the bending moment diagram at any point is equal to the shearing force. Hence the bending moment has its maximum or minimum values at the cross sections in which the shearing force changes its sign. If, as we proceed along the x axis, the shearing force changes from a positive to a negative value, as under the load P_2 in Fig. 62, the slope in the bending moment diagram also changes from positive to negative. Hence we have the maximum bending moment at this cross section. A change in V from a negative to a positive value indicates a minimum bending moment. In the general case a shearing force diagram may intersect the horizontal axis in several places. To each such intersection point there then corresponds a maximum or a minimum in the bending moment diagram. The numerical values of all these maxima and minima must be investigated to find the numerically largest bending moment.

Let us next consider the case of a uniformly distributed load, Fig. 63. From our previous discussion (p. 70), we have for a cross section a distance x from the left support

$$V = q\left(\frac{l}{2} - x\right) \qquad \text{and} \qquad M = \frac{qx}{2}(l - x). \qquad (i)$$

We see that the shearing force diagram consists in this case of an inclined straight line the ordinates of which for $x = 0$ and $x = l$ are equal to $ql/2$ and $-ql/2$ respectively, as shown in Fig. 63*b*. As can be seen from expression (i) the bending

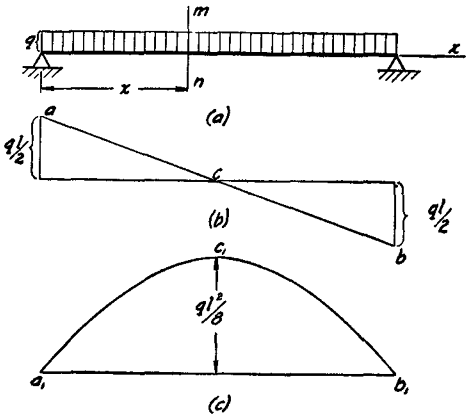


FIG. 63.

moment in this case is a parabolic curve with its vertical axis at the middle of the span of the beam, Fig. 63*c*. The moments at the ends, i.e., for $x = 0$ and $x = l$, vanish; and the maximum value of the moment occurs at the middle of the span where the shearing force changes the sign. This maximum is obtained by substituting $x = l/2$ in expression (i), which gives $M_{max} = ql^2/8$.

If a uniform load covers only a part of the span, Fig. 64, we must consider three portions of the beam of length a, b, and c. Beginning with the determination of the reactions R_1 and R_2 we replace the uniformly distributed load by its resultant qb. From the equations of statics for the moments

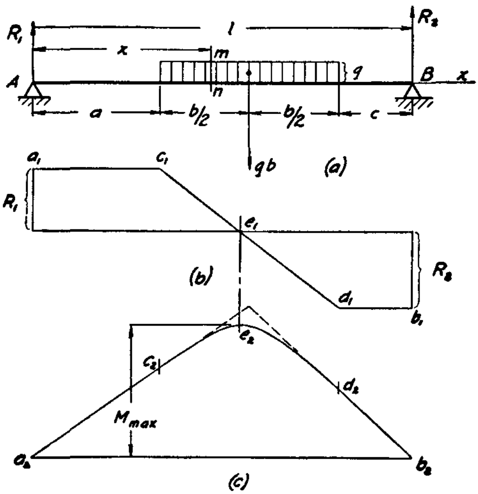


Fig. 64.

with respect to B and A, we then obtain

$$R_1 = \frac{qb}{l}\left(c + \frac{b}{2}\right) \qquad \text{and} \qquad R_2 = \frac{qb}{l}\left(a + \frac{b}{2}\right).$$

The shearing force and the bending moment for the left unloaded portion of the beam ($0 < x < a$) are

$$V = R_1 \qquad \text{and} \qquad M = R_1 x. \tag{j}$$

For a cross section mn taken in the loaded portion of the beam the shearing force is obtained by subtracting the load $q(x - a)$ to the left of the cross section from the reaction R_1. The bending moment in the same cross section is obtained by subtracting the moment of the load to the left of the cross section from the moment of the reaction R_1. In this manner we find

$$V = R_1 - q(x - a)$$

$$\text{and} \qquad M = R_1 x - q(x - a) \times \frac{x - a}{2}. \tag{k}$$

For the right unloaded portion of the beam, considering the forces to the right of a cross section, we find

$$V = -R_2 \qquad \text{and} \qquad M = R_2(l - x). \tag{l}$$

By using expressions (j), (k), and (l) the shearing force and bending moment diagrams are readily constructed. The shearing force diagram, Fig. 64b, consists of the horizontal portions a_1c_1 and d_1b_1 corresponding to the unloaded portions of the beam and the inclined line c_1d_1 corresponding to the uniformly loaded portion. The bending moment diagram, Fig. 64c, consists of the two inclined lines a_2c_2 and b_2d_2 corresponding to the unloaded portions and of the parabolic curve c_2d_2 with vertical axis corresponding to the loaded portion of the beam. The maximum bending moment is at the point e_2, which corresponds to the point e_1 where the shearing force changes sign. At points c_2 and d_2 the parabola is tangent to the inclined lines a_2c_2 and d_2b_2 respectively. This follows from the fact that at points c_1 and d_1 of the shearing force diagram there is no abrupt change in the magnitude of the shearing force; hence, by virtue of equation (50), there cannot occur an abrupt change in slope of the bending moment diagram at the corresponding points c_2 and d_2.

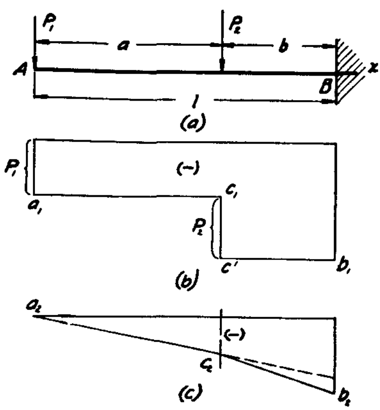


FIG. 65.

In the case of a cantilever beam, Fig. 65, the same method as before is used to construct the shearing force and bending moment diagrams. Measuring x from the left end of the beam and considering the portion to the left of the load P_2 ($0 < x < a$), we obtain

$$V = -P_1 \qquad \text{and} \qquad M = -P_1x.$$

The minus sign in these expressions follows from the rule of signs indicated in Fig. 58*b* and Fig. 59*b*. For the right portion of the beam ($a < x < l$) we obtain

$$V = -P_1 - P_2 \qquad \text{and} \qquad M = -P_1x - P_2(x - a).$$

The corresponding diagrams of shearing force and bending moment are shown in Fig. 65*b* and 65*c*. The total area of the shearing force diagram does not vanish in this case and is equal to $-P_1l - P_2b$, which is the bending moment M_B at the end B of the beam. The bending moment diagram consists of the two inclined lines a_2c_2 and c_2b_2 the slopes of which are equal to the values of the shearing force at the corresponding portions of the cantilever. The numerical maximum of the bending moment is at the built-in end B of the beam.

If a cantilever carries a uniform load, Fig. 66, the shearing force and bending moment at a distance x from the left end are

$$V = -qx \qquad \text{and} \qquad M = -qx \times \frac{x}{2} = -\frac{qx^2}{2}.$$

The shearing force is represented in the diagram by the inclined line ab and the bending moment by the parabola a_1b_1 which has a vertical axis and is tangent to the horizontal axis at a_1, where the shearing force vanishes. The numerical maximum of the bending moment and shearing force occurs at the end B of the beam.

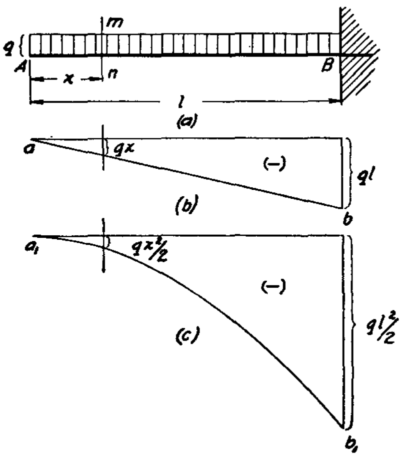


FIG. 66.

If concentrated loads and distributed loads act on the beam simultaneously, it is advantageous to draw the diagrams separately for each kind of loading and obtain the total values of V or M at any cross section by summing up the corresponding ordinates of the two partial diagrams. If, for example, we have concentrated loads P_1, P_2 and P_3, Fig. 62, acting simultaneously with a uniform load, Fig. 63, the bending moment at any cross section is obtained by summing up the corresponding ordinates of the diagrams in Fig. 62*c* and Fig. 63*c*.

Problems

1. Draw approximately to scale the shearing force and bending moment diagrams and label the values of the largest positive and negative shearing forces and bending moments for the beams shown in Fig. 67.

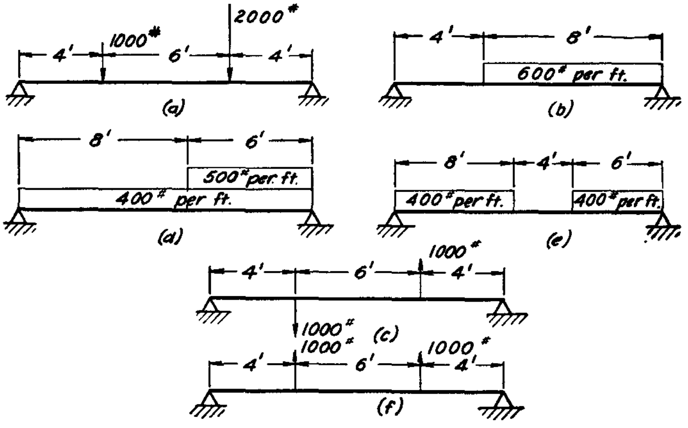


FIG. 67.

2. Draw approximately to scale the shearing force and bending moment diagrams and label the values of the largest positive and

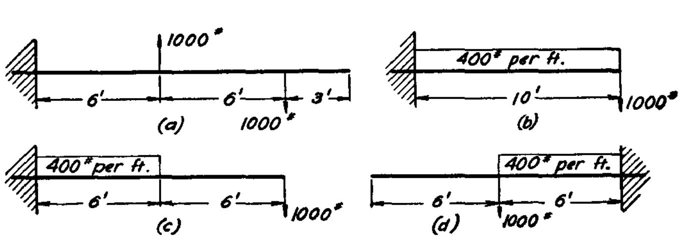


FIG. 68.

negative shearing forces and bending moments for the cantilever beams shown in Fig. 68.

3. A cantilever carrying a total load W which increases uniformly from zero at the left end as shown by the inclined line AC, Fig. 69a, is built in at the right end. Draw the diagrams of shearing force and bending moment.

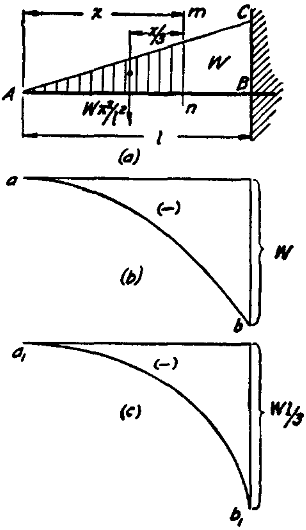


FIG. 69.

Solution. The shearing force at a cross section mn at a distance x from the left end of the cantilever is numerically equal to the shaded portion of the load. Since the total load W is represented by the triangle ACB the shaded portion is Wx^2/l^2. By using the rule of sign previously adopted, Fig. 59, we obtain

$$V = -W\frac{x^2}{l^2}.$$

The shearing force diagram is thus represented in Fig. 69b by the parabola ab which has a vertical axis at the point a. The bending moment at the cross section mn is obtained by taking the moment of the shaded portion of the load with respect to the centroid of the cross section mn. Thus

$$M = -W\frac{x^2}{l^2} \times \frac{x}{3}.$$

This moment is represented by the curve a_1b_1 in Fig. 69c.

4. A beam of length l uniformly supported along its entire length carries at the ends two equal loads P, Fig. 70. Draw the shearing force and bending moment diagrams.

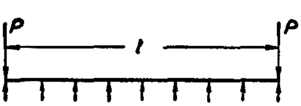


FIG. 70.

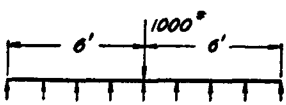


FIG. 71.

5. A beam of length l, uniformly supported along its entire length, carries at the center a concentrated load P = 1,000 lbs., Fig. 71. Find the numerical maximum of the bending moment. Draw the shearing force and bending moment diagrams.

6. A simply supported beam of length l carries a total distributed load W the intensity of which increases uniformly from zero at the

left end, as shown in Fig. 72*a*. Draw approximately to scale the shearing force and bending moment diagrams if $W = 12{,}000$ lbs. and $l = 24$ ft.

Solution. The reactions at the supports in this case are $R_1 = \frac{1}{3}W = 4{,}000$ lbs. and $R_2 = 8{,}000$ lbs. The shearing force at a cross section mn is obtained by subtracting the shaded portion of the load from the reaction R_1. Hence

$$V = R_1 - W\frac{x^2}{l^2} = W\left(\frac{1}{3} - \frac{x^2}{l^2}\right).$$

The shearing force diagram is represented by the parabolic curve acb in Fig. 72*b*. The bending moment at a cross section mn is

$$M = R_1 x - W\frac{x^2}{l^2} \times \frac{x}{3}$$

$$= \frac{1}{3}Wx\left(1 - \frac{x^2}{l^2}\right).$$

This moment is represented by the curve $a_1c_1b_1$ in Fig. 72*c*. The maximum bending moment is at c_1 where the shearing force changes its sign and where $x = l/\sqrt{3}$.

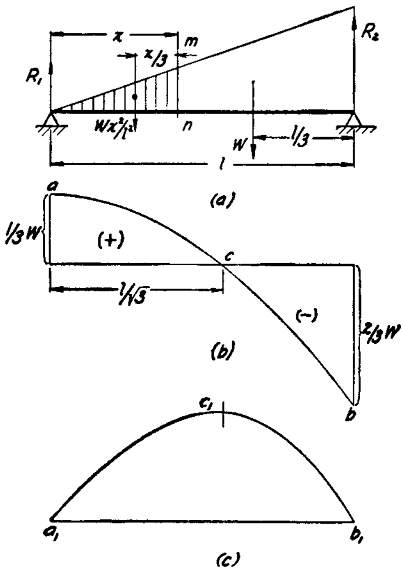


FIG. 72.

7. A simply supported beam AB carries a distributed load the intensity of which is represented by the line ACB, Fig. 73. Find

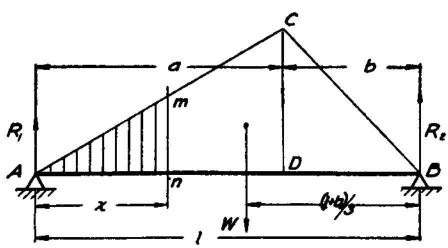


FIG. 73.

the expressions for the shearing force and the bending moment at a cross section mn.

Solution. Assuming the total load W to be applied at the centroid of the triangle ACB, the reactions at the supports are

$$R_1 = W\frac{l+b}{3l} \quad \text{and} \quad R_2 = W\frac{l+a}{3l}.$$

The total load is then divided into two parts, represented by the triangles ACD and CBD, of the amount Wa/l and Wb/l respectively. The shaded portion of the load is $W\frac{a}{l} \times \frac{x^2}{a^2} = W\frac{x^2}{al}$. For the shearing force and the bending moment at mn we then obtain

$$V = R_1 - W\frac{x^2}{al} \quad \text{and} \quad M = R_1x - W\frac{x^2}{al} \times \frac{x}{3}.$$

In a similar manner the shearing force and bending moment for a cross section in the portion DB of the beam can be obtained.

8. Find M_{max} in the previous problem if l = 12 ft., b = 3 ft., W = 12,000 lbs.

Answer. M_{max} = 22,400 ft. lbs.

9. Draw approximately to scale the shearing force and bending moment diagrams and label the values of the largest positive and negative shearing forces and bending moments for the beams with overhangs shown in Fig. 74.

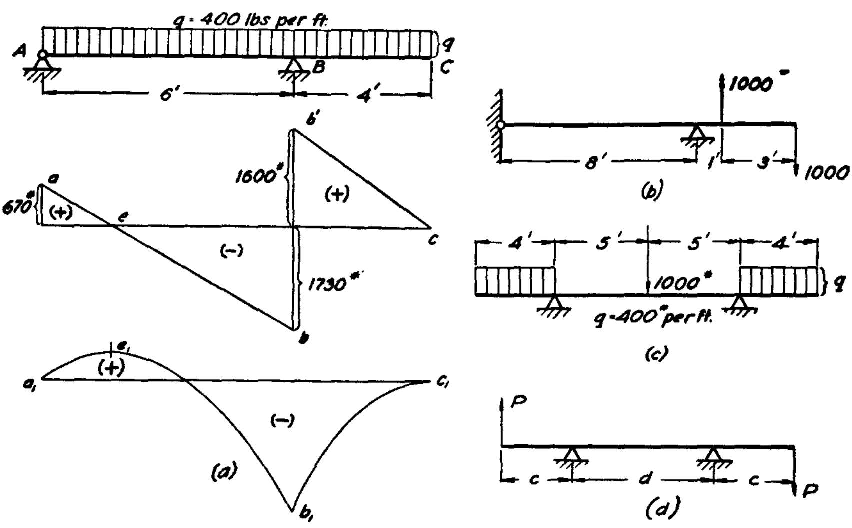


Fig. 74.

Solution. In the case shown in Fig. 74*a* the reactions are 670 lbs. and 3,330 lbs. The shearing force for the left portion of the beam is $V = 670 - 400x$. It is represented in the figure by the inclined line ab. The shearing force for the right portion of the beam is found as for a cantilever beam and is shown by the inclined line $b'c$. The bending moment for the left portion of the beam is $M = 670x - 400x^2/2$. It is represented by the parabola $a_1e_1b_1$. The maximum of the moment at e_1 corresponds to the point e, at which the shearing force changes its sign. The bending moment diagram for the right portion is the same as for a cantilever and is represented by the parabola b_1c_1 tangent at c_1.

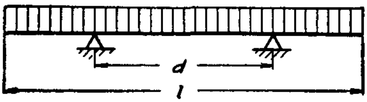


FIG. 75.

10. A beam with two equal overhangs, Fig. 75, loaded by a uniformly distributed load, has a length l. Find the distance d between the supports such that the bending moment at the middle of the beam is numerically equal to the moments at the supports. Draw the shearing force and bending moment diagrams for this case.

Answer. $d = 0.586l$.

CHAPTER IV

STRESSES IN TRANSVERSALLY LOADED BEAMS

23. Pure Bending.—It was mentioned in the previous chapter that the magnitude of the stresses at a cross section is defined by the magnitude of the shearing force and bending moment at that cross section. To calculate the stresses we shall begin with the instance in which the shearing force vanishes and only the bending moment acts. This case is called *pure bending*. An example of such bending is shown in Fig. 76. From symmetry we can conclude that the reactions in this case are equal to P. Considering the equilibrium of the portion of the beam to the left of a cross section mn, it can be concluded that the internal forces which are distributed over the cross section mn and which represent the action of the removed right portion of the beam on the left portion must be statically equivalent to a couple equal and opposite to the bending moment Pa. To find the distribution of these internal forces over the cross section, the deformation of the beam must be considered. For the simple case of a beam having a longitudinal plane of symmetry with the external bending couples acting in this plane, bending will take place in this same plane. If the beam is of rectangular cross section and two adjacent vertical lines mm and pp are drawn on its sides, direct experiment shows that these lines remain straight during bending and rotate so as to remain perpendicular to the longitudinal fibers of the beam (Fig. 77). The following theory of bending is based on the assumption that not only such lines as mm remain straight but that the entire transverse section of the beam, originally

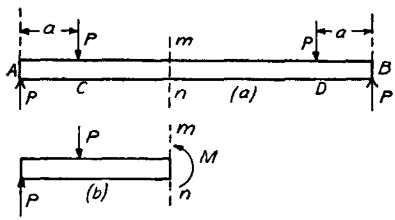


FIG. 76.

plane, remains plane and normal to the longitudinal fibers of the beam after bending. Experiment shows that the theory based on this assumption gives very accurate results for the deflection of beams and the strain of longitudinal fibers. From the above assumption it follows that during bending the cross sections mm and pp rotate with respect to each other about

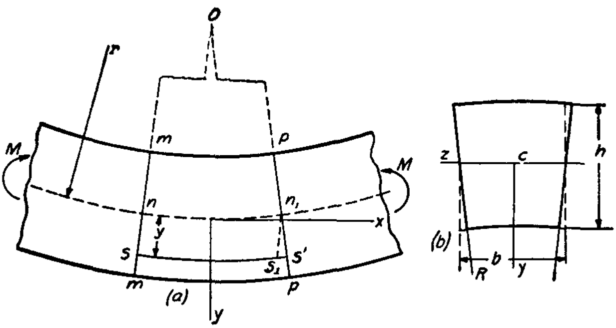


FIG. 77.

axes perpendicular to the plane of bending, so that longitudinal fibers on the convex side suffer extension and those on the concave side compression. The line nn_1 is the trace of the surface in which the fibers do not undergo strain during bending. This surface is called the *neutral surface* and its intersection with any cross section is called the *neutral axis.* The elongation $s's_1$ of any fiber, at distance y from the neutral surface, is obtained by drawing the line n_1s_1 parallel to mm (Fig. 77, a). Denoting by r the radius of curvature of the deflected axis[1] of the beam and using the similarity of the triangles non_1 and s_1n_1s', the unit elongation of the fiber ss' is

$$\epsilon_x = \frac{s's_1}{nn_1} = \frac{y}{r}. \tag{52}$$

It can be seen that the strains of the longitudinal fibers are proportional to the distance y from the neutral surface and inversely proportional to the radius of curvature.

Experiments show that longitudinal extension in the fibers

[1] The axis of the beam is the line through the centroids of its cross sections. O denotes the center of curvature.

on the convex side of the beam is accompanied by *lateral contraction* and longitudinal contraction on the concave side, by lateral expansion, as in the case of simple tension or compression (see article 14). This changes the shape of all cross sections, the vertical sides of the rectangular section becoming inclined to each other as in Fig. 77 (*b*). The unit strain in the lateral direction is

$$\epsilon_z = -\mu\epsilon_x = -\mu\frac{y}{r}, \tag{53}$$

where μ is *Poisson's ratio.* Due to this distortion all straight lines in the cross section, parallel to the z axis, curve so as to remain normal to the sides of the section. Their radius of curvature R will be larger than r in the same proportion in which ϵ_x is numerically larger than ϵ_z (see eq. 53) and we obtain

$$R = \frac{1}{\mu}r. \tag{54}$$

From the strains of the longitudinal fibers the corresponding stresses follow from Hooke's law (eq. 4, p. 4):

$$\sigma_x = \frac{Ey}{r}. \tag{55}$$

The distribution of these stresses is shown in Fig. 78. The stress in any fiber is proportional to its distance from the neutral axis nn. The position of the neutral axis and the radius of curvature r, the two unknowns in eq. (55), can now be determined from the condition that the forces distributed over any cross section of the beam must give rise to a *resisting couple* which balances the external couple M (Fig. 76).

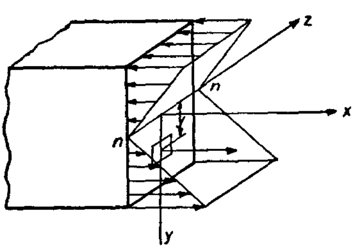


FIG. 78.

Let dA denote an elemental area of a cross section distant y from the neutral axis (Fig. 78). The force acting on this elemental area is the product of the stress (eq. 55) and its area dA, i.e., $(Ey/r)dA$. Due to the fact that all such forces distributed over the cross section represent a system equivalent to a couple, the resultant of these forces must be equal to zero and we obtain

$$\int \frac{Ey}{r} dA = \frac{E}{r}\int ydA = 0,$$

i.e., the moment of the area of the cross section with respect to the neutral axis is equal to zero; hence the *neutral axis passes through the centroid of the section.*

The moment of the force on the above element with respect to the neutral axis is $(Ey/r)\cdot dA\cdot y$. Summarizing such moments over the cross section and putting the resultant equal to the moment M of the external forces, the following equation for determining the radius of curvature r is obtained:

$$\int \frac{E}{r} y^2 dA = \frac{EI_z}{r} = M \qquad \text{or} \qquad \frac{1}{r} = \frac{M}{EI_z}, \tag{56}$$

in which

$$I_z = \int y^2 dA$$

is the *moment of inertia* of the cross section with respect to the neutral axis z (see appendix, p. 343). From eq. (56) it is seen that the curvature varies directly as the bending moment and inversely as the quantity EI_z, which is called the *flexural rigidity* of the beam. Elimination of r from eqs. (55) and (56) gives the following equation for the stresses:

$$\sigma_x = \frac{My}{I_z}. \tag{57}$$

The preceding discussion was for the case of a rectangular cross section. It will hold also for a bar of any type of cross section which has a longitudinal plane of symmetry and is bent by end couples acting in this plane, since for such cases also bending takes place in the plane of the couples and cross-sectional planes remain plane and normal to the longitudinal fibers after bending.

In eq. (57) M is positive when it produces a deflection of the bar convex down, as in Fig. 77; y is positive downwards. A negative sign for σ_x indicates compressive stress.

The maximum tensile and compressive stresses occur in the outermost fibers, and for the rectangular cross section or any other type which has its centroid at the middle of the depth h they occur for $y = \pm h/2$ and are

$$(\sigma_x)_{\max} = \frac{Mh}{2I_z} \quad \text{and} \quad (\sigma_x)_{\min} = -\frac{Mh}{2I_z}. \tag{58}$$

For simplification we will use the following notation:

$$Z = \frac{2I_z}{h}. \tag{59}$$

Then

$$(\sigma_x)_{\max} = \frac{M}{Z}; \quad (\sigma_x)_{\min} = -\frac{M}{Z}. \tag{60}$$

The quantity Z is called the *section modulus.* In the case of a rectangular cross section (Fig. 77, b) we have

$$I_z = \frac{bh^3}{12}; \quad Z = \frac{bh^2}{6}.$$

For a circular cross section of diameter d

$$I_z = \frac{\pi d^4}{64}; \quad Z = \frac{\pi d^3}{32}.$$

For the various profile sections, e.g., for I beams, channels and so on, the magnitudes of I_z and Z for the sizes manufactured are tabulated in handbooks.

When the centroid of the cross section is not at the middle of the depth, as, for instance, in the case of a T beam, let h_1 and h_2 denote the distances from the neutral axis to the outermost fibers in the downward and upward directions respectively. Then for a *positive* bending moment we obtain

$$(\sigma_x)_{\max} = \frac{Mh_1}{I_z}; \quad (\sigma_x)_{\min} = -\frac{Mh_2}{I_z}. \tag{61}$$

For a negative bending moment we obtain

$$(\sigma_x)_{\max} = -\frac{Mh_2}{I_z}; \qquad (\sigma_x)_{\min} = \frac{Mh_1}{I_z}. \tag{62}$$

The preceding derivations were made on the assumption that the beam has a longitudinal plane of symmetry in which the bending couples act. However *they can also* be used when no such plane of symmetry exists, provided the bending couples act in an axial plane which contains one of the two *principal axes* of the cross section (see appendix, p. 351). These planes are called the *principal planes of bending.*

When there is a plane of symmetry and the bending couples act in this plane, deflection occurs also in this plane. The moments of the internal forces such as shown in Fig. 78 about the horizontal axis are balanced by the external couple. The moments of these forces about the vertical axis cancel each other, because the moments of the forces on one side of this axis are just balanced by the moments of the corresponding forces on the other side.

When there is no plane of symmetry, but the bending couples act in an axial plane through one principal axis of the cross section, xy in Fig. 79, a distribution of the stresses

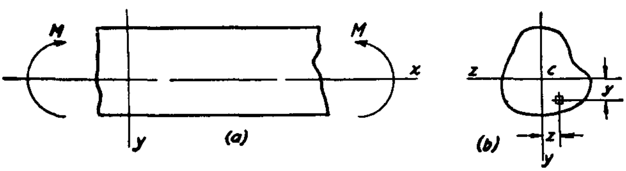


FIG. 79.

following eq. (56) will still satisfy all the conditions of equilibrium. This distribution gives a couple about the horizontal axis which balances the external couple. About the vertical axis y, it gives a resultant moment:

$$M_y = \int z \frac{Ey}{r} dA = \frac{E}{r} \int yz dA.$$

The integral on the right side is the *product of inertia* of the cross section (see appendix, p. 348) and it is zero if the y and z axes are the principal axes of the section. Therefore this couple is zero and, since the component about the y axis of the applied couple is zero, the conditions of equilibrium are satisfied.

Problems

1. Determine the maximum stress in a locomotive axle (Fig. 80) if $c = 13.5$ in., diameter d of the axle is 10 in. and the spring-borne load P per journal is 26,000 lbs.

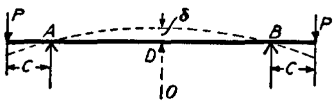


FIG. 80.

Solution. The bending moment acting in the middle portion of the axle is $M = P \times c = 26{,}000 \times 13.5$ lbs. in. The maximum stress, from eq. (60), is

$$\sigma_{\max} = \frac{M}{Z} = \frac{32 \cdot M}{\pi d^3} = \frac{32 \times 26{,}000 \times 13.5}{\pi \times 10^3} = 3{,}580 \text{ lbs. per sq. in.}$$

2. Determine the radius of curvature r and the deflection of the axle of the previous problem, if the material is steel and the distance between the centers of the journals is 59 in.

Solution. The radius of curvature r is determined from eq. (55) by substituting $y = d/2 = 5$ in., $(\sigma_x)_{\max} = 3{,}580$ lbs. per sq. in. Then

$$r = \frac{E}{\sigma} \cdot \frac{d}{2} = \frac{30 \times 10^6 \times 5}{3{,}580} = 41{,}900 \text{ in.}$$

For calculating δ (Fig. 80), the deflection curve is an arc of a circle of radius r and $\overline{DB}$ is one leg of the right triangle DOB, where O is the center of curvature. Therefore

$$\overline{DB}^2 = r^2 - (r - \delta)^2 = 2r\delta - \delta^2.$$

δ is very small in comparison with the radius r and the quantity δ^2 can be neglected in the above equation; then

$$\delta = \frac{\overline{DB}^2}{2r} = \frac{59^2}{8 \times 41{,}900} = 0.0104 \text{ in.}$$

3. A wooden beam of square cross section 10×10 inches is supported at A and B, Fig. 80, and the loads P are applied at the ends. Determine the magnitude of P and the deflection δ at the middle if $AB = 6'$; $c = 1'$; $(\sigma_x)_{\max} = 1{,}000$ lbs. per sq. in. and

$E = 1.5 \times 10^6$ lbs. per sq. in. The weight of the beam is to be neglected.

Answer. $P = 13{,}900$ lbs.; $\delta = 0.0864$ in.

4. A standard 30″ girder beam is supported as shown in Fig. 81 and loaded on the overhangs by a uniformly distributed load of 10,000 lbs. per foot. Determine the maximum stress in the middle portion of the beam and the deflection at the middle of the beam if $I_z = 9{,}150$ in.^{4}

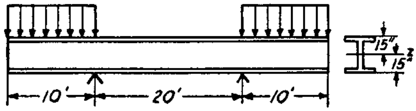


FIG. 81.

Solution. The bending moment for the middle portion of the beam is $M = 10{,}000 \times 10 \times 60 = 6 \times 10^6$ lbs. in.

$$(\sigma_x)_{\max} = \frac{M}{Z} = \frac{6 \times 10^6 \times 15}{9{,}150} = 9{,}840 \text{ lbs. per sq. in.,}$$
$$\delta = 0.157 \text{ in.}$$

5. Determine the maximum stress produced in a steel wire of diameter $d = 1/32$ in. when coiled round a pulley of diameter $D = 20$ in.

Solution. The maximum elongation due to bending, from eq. (52), is

$$\epsilon = \frac{d}{D} = \frac{1}{32 \times 20}$$

and the corresponding tensile stress is

$$(\sigma_x)_{\max} = \epsilon E = \frac{30 \times 10^6}{32 \times 20} = 46{,}900 \text{ lbs. per sq. in.}$$

6. A steel rule having a cross section $1/32 \times 1$ in. and a length $l = 10$ in. is bent by couples at the ends into a circular arc of 60°. Determine the maximum stress and deflection.

Solution. The radius of curvature r is determined from the equation $l = \frac{1}{6}2\pi r$, from which $r = 9.55$ in., and the maximum stress will be given by eq. (55),

$$(\sigma_x)_{\max} = \frac{E \times 1/64}{r} = \frac{30 \times 10^6}{64 \times 9.55} = 49{,}100 \text{ lbs. per sq. in.}$$

The deflection, calculated as for a circular arc, will become

$$\delta = r(1 - \cos 30°) = 1.28 \text{ in.}$$

7. Determine the maximum stress and the magnitude of the couples applied at the ends of the rule in the previous problem if maximum deflection at the middle is 1 in.

Answer.

$$(\sigma_x)_{\max} = 38{,}300 \text{ lbs. per sq. in.,} \qquad M = 6.23 \text{ lbs. ins.}$$

8. Determine the curvature produced in a freely supported steel beam of rectangular cross section by non-uniform heating over the depth h of the cross section. The temperature at any point at distance y from the middle plane xz of the beam (Fig. 77) is given by the equation:

$$t = \frac{t_1 + t_0}{2} + \frac{(t_1 - t_0)y}{h},$$

where t_1 is the temperature at the bottom of the beam, t_0 temperature at the top, $t_1 - t_0 = 123$ degrees Fahrenheit, and the coefficient of expansion $\alpha_s = 70 \times 10^{-7}$. What stresses will be produced if the ends of the beam are clamped?

Solution. The temperature of the middle plane xz is the constant $(t_1 + t_0)/2$, and the change in temperature of other fibers is proportional to y. The corresponding unit thermal expansions are also proportional to y, i.e., they will follow the same law as the unit elongations given by eq. (52). As a result of this non-uniform expansion of the fibers bending of the beam will occur and the radius r of curvature is found from eq. (52) with $\alpha_s(t_1 - t_0)/2$ for ϵ_x and $h/2$ for y. Then

$$r = \frac{h}{\alpha_s(t_1 - t_0)} = 1{,}160h.$$

If the ends of the beam are clamped, reactive couples at the ends will be produced of magnitude such as to remove the curvature due to non-uniform heating. Hence

$$M = \frac{E \cdot I_z}{r} = \frac{EI_z}{1{,}160h}.$$

Substituting this in eq. (57), we obtain

$$\sigma_x = \frac{Ey}{1{,}160h},$$

and the maximum stress is

$$(\sigma_x)_{\max} = \frac{E}{2 \times 1{,}160} = 12{,}900 \text{ lbs. per sq. in.}$$

9. Solve problems 6 and 7 if the arc is of 10° and material is copper.

10. Solve problem 4, assuming that the beam is of wood, has a square cross section 12″ × 12″ and the intensity of distributed load is 1,000 lbs. per foot.

24. Various Shapes of Cross Sections of Beams.[2]—From the discussion in the previous article it follows that the maximum tensile and compressive stresses in a beam in pure bending are proportional to the distances of the most remote fibers from the neutral axis of the cross section. Hence if the material has the same strength in tension and compression, it will be logical to choose those shapes of cross section in which the centroid is at the middle of the depth of the beam. In this manner the same factor of safety for fibers in tension and fibers in compression will be obtained. This is the underlying idea in the choice of sections symmetrical with respect to the neutral axis for materials such as structural steel, which have the same yield point in tension and compression. If the section is not symmetrical with respect to the above axis, e.g., a rail section, the material is so distributed between the head and the base as to have the centroid at the middle of its height.

For a material of small strength in tension and high strength in compression, e.g., cast iron or concrete, the advisable cross section for a beam is not symmetrical with respect to the neutral axis but such that the distances h_1 and h_2 from the neutral axis to the most remote fibers in tension and compression are in the same proportion as the strengths of the material in tension and in compression. In this manner equal strength in tension and compression is obtained. For example, with a T section, the centroid of the section may be put in any prescribed position along the height of the section by properly proportioning its flange and web.

For a given bending moment the maximum stress depends upon the section modulus and it is interesting to note that

[2] A very complete discussion of various shapes of cross sections of beams is given by Barré de Saint Venant in his notes to the book by Navier, Resistance des Corps Solides, 3d ed., 1864. See pp. 128–162.

there are cases in which increase in area does not give a decrease in this stress. As an example, a bar of square cross section bent by couples acting in the vertical plane through a diagonal of the cross section (Fig. 82) will have a lower maximum stress if the corners shown shaded on the figure are cut off. Letting a denote the length of the side of the square cross section, the moment of inertia with respect to the z axis is (see appendix) $I_z = a^4/12$ and the corresponding section modulus is

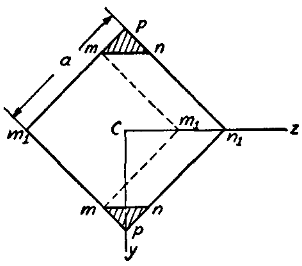


FIG. 82.

$$Z = \frac{I_z\sqrt{2}}{a} = \frac{\sqrt{2}}{12}a^3.$$

Let us now cut off the corners so that $\overline{mp} = \alpha a$, where α is a fraction to be determined later. The new cross section consists of a square mm_1mm_1 with the sides $a(1 - \alpha)$ and of two parallelograms mnn_1m_1. The moment of inertia of such a cross section with respect to the z axis is

$$I_z' = \frac{a^4(1-\alpha)^4}{12} + 2\cdot\frac{\alpha a\sqrt{2}}{3}\left[\frac{a(1-\alpha)}{\sqrt{2}}\right]^3 = \frac{a^4(1-\alpha)^3}{12}(1+3\alpha)$$

and the corresponding section modulus is

$$Z' = \frac{I_z'\sqrt{2}}{a(1-\alpha)} = \frac{\sqrt{2}}{12}\cdot a^3(1-\alpha)^2(1+3\alpha).$$

Now if we determine the value of α to make this section modulus a maximum, we find $\alpha = 1/9$. With this value of α in Z' it is found that cutting off the corners decreases the maximum bending stress by about 5 per cent. This result is easily understood once we consider that the section modulus is the quotient of the moment of inertia and half the depth of the cross section. By cutting off the corners the moment of inertia of the cross section is diminished in a smaller proportion than the depth; hence the section modulus increases and $(\sigma_x)_{\max}$ decreases. A similar effect may be obtained in other cases. For a rectangle with narrow outstanding portions (Fig. 83, *a*) the section modulus is increased, under certain conditions, by cutting off these portions. For a circular cross section (Fig. 83, *b*) the section modulus is increased by 0.7 per cent by cutting off the two shaded segments which have a depth $\delta = 0.011d$. In the case of a triangular section (Fig. 83, *c*) the section modulus can be increased by cutting off the shaded corner.

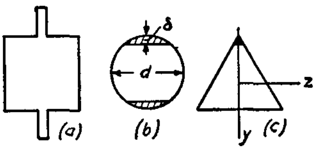


FIG. 83.

In designing a beam to undergo pure bending, not only the conditions of strength should be satisfied but also the condition of economy in the beam's own weight. Of two cross sections having the same section modulus, i.e., satisfying the condition of strength with the same factor of safety, that with the smaller area is more economical. In comparing various shapes of cross sections, we consider first the rectangle of depth h and width b. The section modulus is

$$Z = \frac{bh^2}{6} = \frac{1}{6} Ah, \tag{a}$$

where A denotes the cross-sectional area.

It is seen that the rectangular cross section becomes more and more economical with increase in its depth h. However, there is a certain limit to this increase, and the question of the stability of the beam arises as the section becomes narrower. The collapse of a beam of very narrow rectangular section may be due not to overcoming the strength of the material but to sidewise buckling (see part II).

In the case of a circular cross section we have

$$Z = \frac{\pi d^3}{32} = \frac{1}{8} A \cdot d. \tag{b}$$

Comparing circular and square cross sections of the same area, we find that the side h of the square will be $h = d\sqrt{\pi}/2$, for which eq. (*a*) gives

$$Z = 0.147 A \cdot d.$$

Comparison of this with (*b*) shows a square cross section to be more economical than a circular one.

Consideration of the stress distribution along the depth of the cross section (Fig. 78) leads to the conclusion that for economical design most of the material of the beam should be put as far as possible from the neutral axis. The most favorable case for a given cross-sectional area A and depth h would be to distribute each half of the area at a distance $h/2$ from the neutral axis. Then

$$I_z = 2 \times \frac{A}{2} \times \left(\frac{h}{2}\right)^2 = \frac{Ah^2}{4}; \qquad Z = \tfrac{1}{2}Ah. \tag{c}$$

This is a limit which may be approached in practice by use of an I section with most of the material in the flanges. Due to the necessity of putting part of the material in the web of the beam, the limiting condition (*c*) can never be realized, and for standard I profiles we have approximately

$$Z \approx 0.30Ah. \tag{d}$$

Comparison of (*d*) with (*a*) shows that an I section is more economical than a rectangular section of the same depth. At the same time, due to its wide flanges, an I beam will always be more stable with respect to sidewise buckling than a beam of rectangular section of the same depth and section modulus.

Problems

1. Determine the width x of the flange of a cast iron beam having the section shown in Fig. 84, such that the maximum

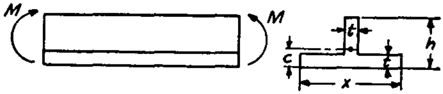


FIG. 84.

tensile stress is one third of the maximum compressive stress. The depth of the beam $h = 4$ in., the thickness of the web and of the flange $t = 1$ in.

Solution. In order to satisfy the conditions, it is necessary for the beam to have dimensions such that the distance of the centroid from the extreme bottom edge will satisfy the equation $c = \frac{1}{4}h$. Now, referring to Fig. 84, the position of the centroid is calculated from the equation:

$$c = \frac{ht \cdot \frac{h}{2} + (x - t)\frac{t^2}{2}}{ht + (x - t)t} = \frac{h}{4},$$

from which

$$x = t + \frac{h^2}{h - 2t} = 1 + \frac{16}{4 - 2} = 9 \text{ in.}$$

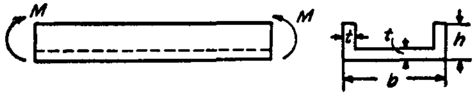


FIG. 85.

2. Determine the ratio $(\sigma_x)_{\max} : (\sigma_x)_{\min}$ for a channel, bent as shown in Fig. 85, if $t = 2$ in., $h = 10$ in., $b = 24$ in.

Answer.

$$(\sigma_x)_{\max} : (\sigma_x)_{\min} = 3 : -7.$$

3. Determine the condition at which the diminishing of the depth h_1 of the section shown in Fig. 86 is accompanied by an increase in section modulus.

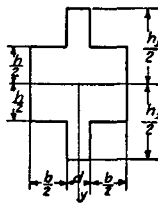


FIG. 86.

Solution.

$$Z = \frac{bh^3}{6h_1} + \frac{dh_1^2}{6},$$

$$\frac{dZ}{dh_1} = -\frac{bh^3}{6h_1^2} + \frac{dh_1}{3}.$$

The condition for increase in Z with decrease of h_1 is

$$\frac{bh^3}{6h_1^2} > \frac{dh_1}{3} \quad \text{or} \quad \frac{b}{2d} > \frac{h_1^3}{h^3}.$$

4. Determine what amount should be cut from an equilateral triangular cross section (Fig. 83, c) in order to obtain the maximum Z.

5. Determine the ratio of the weights of three beams of the same length under the same M and $(\sigma_x)_{\max}$ and having as cross sections respectively a circle, a square and a rectangle with proportions $h = 2b$.

Solution.

$$1.12 : 1 : 0.793.$$

25. General Case of Transversally Loaded Beams.—In the general case of transversally loaded beams, the stresses distributed over a cross section of a beam must balance the shearing force and the bending moment at that cross section. The calculation of the stresses is usually made in two steps by determining first the stresses produced by the bending moment, the so-called *bending stresses*, and afterwards the *shearing stresses* produced by the shearing force. In this article we shall limit ourself to the calculation of the bending stresses; the discussion of shearing stresses will be given in the next article. In calculating bending stresses we assume that these stresses are distributed in the same manner as in the case of pure bending and will use the formulas derived for the stresses in article 23. Experiments show that such a procedure gives satisfactory results if we are dealing with sections which are not very close to the point of application of a concentrated force. In the vicinity of the application of a concentrated load the stress distribution is more complicated. This problem will be discussed in Part II.

The calculation of bending stresses is usually made for the cross sections at which the bending moment has the largest positive or negative value. Having the numerical maximum of the bending moment and the magnitude of the allowable stress σ_w in bending, the required cross-sectional dimensions of a beam can be obtained from the equation

$$\sigma_w = \frac{M_{\max}}{Z}. \tag{63}$$

The application of this equation will now be shown by a number of examples.

Problems

1. Determine the necessary dimensions of a standard I beam to support a distributed load of 400 lbs. per foot, as shown in Fig. 87, when the working stress $\sigma_w = 16{,}000$ lbs. per sq. in. Only the normal stresses σ_x are to be taken into consideration and the weight of the beam is neglected.

Solution. To obtain the dangerous section the shearing force diagram should be constructed (Fig. 87, *b*). The reaction at the

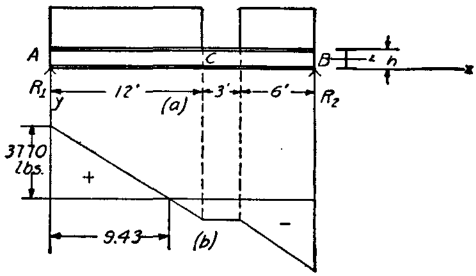


FIG. 87.

left support is

$$R_1 = \frac{12 \times 400 \times 15 + 6 \times 400 \times 3}{21} = 3,770 \text{ lbs.}$$

The shearing force for any cross section of the portion AC of the beam is

$$V = R_1 - qx = 3,770 - 400 \times x.$$

This force is zero for $x = 3,770/400 = 9.43$ feet. For this section the bending moment is a maximum,

$$M_{\max} = 3,770 \times 9.43 - 400 \times \tfrac{1}{2} \times 9.43^2$$
$$= 17,800 \times 12 \text{ lbs. in.}$$

The necessary section modulus

$$Z = \frac{17,800 \times 12}{16,000} = 13.4 \text{ in.}^3$$

This condition is satisfied by a standard I beam of depth 8 in., cross-sectional area 5.33 sq. in., and $Z = 14.2$ in.^{3}

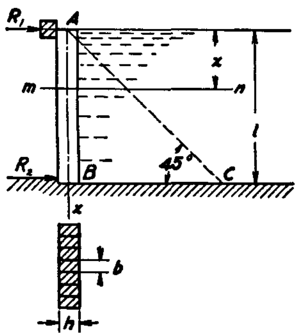


FIG. 88.

2. A wooden dam (Fig. 88) consists of vertical bars such as AB of rectangular cross section and dimension $h = 1$ foot supported at the ends. Determine $(\sigma_x)_{\max}$ if the length of the bars $l = 18$ feet and the weight of the bar is neglected.

Solution. If b is the width of one bar, the complete hydrostatic pressure on the bar, represented by the triangular prism ABC, is

$W = \frac{1}{2}bl^2 \times 62.4$ lbs. The reaction at A is $R_1 = \frac{1}{3}W = \frac{1}{6}bl^2 \times 62.4$ lbs. and the shearing force at any cross section mn is equal to the reaction R_1 minus the weight of the prism Amn of water, i.e.,

$$V = R_1 - W\frac{x^2}{l^2} = W\left(\frac{1}{3} - \frac{x^2}{l^2}\right).$$

The position of the cross section corresponding to $M_{\max}$ is found from the condition $V = 0$ or

$$\frac{1}{3} - \frac{x^2}{l^2} = 0,$$

from which

$$x = \frac{l}{\sqrt{3}} = 10.4 \text{ feet.}$$

The bending moment at any cross section mn is equal to the moment of the reaction R_1 minus the moment of the distributed load represented by the triangular prism Amn. Then

$$M = R_1x - \frac{Wx^2}{l^2}\cdot\frac{x}{3} = \frac{Wx}{3}\left(1 - \frac{x^2}{l^2}\right).$$

Substituting, from the above, $x^2/l^2 = \frac{1}{3}$ and $x = 10.4$ feet, we obtain

$$M_{\max} = \frac{1}{9}bl^2 \times 62.4 \times 10.4 \text{ lbs. feet,}$$

$$(\sigma_x)_{\max} = \frac{M_{\max}}{Z_z} = \frac{6M_{\max}}{bh^2} = \frac{2}{3}\left(\frac{l}{h}\right)^2\frac{62.4 \times 10.4}{12^2} = 973 \text{ lbs. per sq. in.}$$

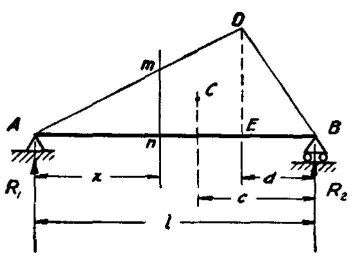


FIG. 89.

3. Determine the magnitude of $M_{\max}$ in a beam loaded by a triangular load ADB equal to $W = 12{,}000$ lbs. if $l = 12$ feet and $d = 3$ feet (Fig. 89).

Solution. The distance c to the vertical through the center of gravity C from the support B is, in the case of a triangle,

$$c = \frac{1}{3}(l + d) = 5'.$$

The reaction at the support A is then

$$R_1 = \frac{W\cdot c}{l} = \frac{12{,}000 \times 5}{12} = 5{,}000 \text{ lbs.}$$

The shearing force at any cross section mn is equal to the reaction R_1 minus the weight of the load represented by area Amn. Since the load represented by the area

$$ADE = \frac{W(l-d)}{l} = \frac{3}{4}W,$$

we obtain

$$V = R_1 - \frac{3}{4}W\frac{x^2}{(l-d)^2}.$$

The position of the section with $M_{\max}$ is found from the condition

$$R_1 - \frac{3}{4}W\frac{x^2}{(l-d)^2} = 0$$

or

$$\frac{x^2}{(l-d)^2} = \frac{4R_1}{3W} = \frac{5}{9},$$

from which

$$x = 6.71 \text{ feet.}$$

The bending moment at any cross section mn is equal to the moment of the reaction R_1 minus the moment of the load Amn. Then

$$M = R_1x - \frac{3}{4}W\frac{x^2}{(l-d)^2}\cdot\frac{x}{3}.$$

Substituting for the above value x,

$$M_{\max} = 22{,}400 \text{ lbs. ft.}$$

4. Construct the bending moment and shearing force diagrams for the case in Fig. 90 (a) and determine the necessary standard I beam if $a = c = l/4 = 6$ feet, $P = 2{,}000$ lbs., $q = 400$ lbs. per foot, $\sigma_w = 15{,}000$ lbs. per sq. in. The weight of the beam can be neglected.

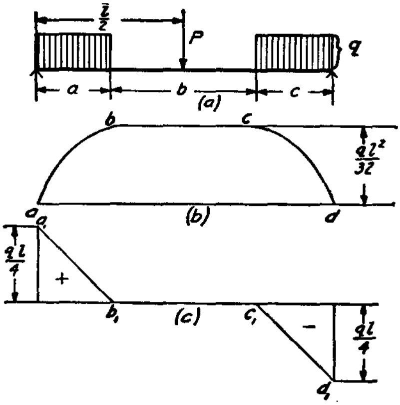


FIG. 90.

Solution. In Fig. 90 (b) and (c) the bending moment and shearing force diagrams produced by the distributed loads are shown. To this the moment and the shearing force produced by P should be added. The maximum bending moment will be at the middle of the span,

$$M_{\max} = 19{,}200 \text{ lbs. ft.}$$

The necessary

$$Z = \frac{19{,}200 \times 12}{15{,}000} = 15.4 \text{ in.}^3$$

The standard I beam of depth 8 in. and cross-sectional area 6.71 sq. in., $Z = 16.1$ in.^{3}, is the nearest cross section satisfying the strength conditions.

5. Determine the most unfavorable position of the hoisting carriage of a crane which rides on a beam as in Fig. 91. Find $M_{\max}$ if the load per wheel is $P = 10{,}000$ lbs., $l = 24$ feet, $d = 6$ feet. The weight of the beam is neglected.

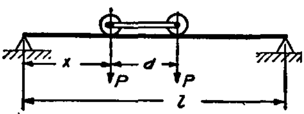


FIG. 91.

Solution. If x is the distance of the left wheel from the left support of the beam, the bending moment under this wheel is

$$\frac{2P(l - x - \frac{1}{2}d)x}{l}.$$

This moment becomes a maximum when

$$x = \frac{l}{2} - \frac{d}{4};$$

hence in order to obtain the maximum bending moment under the left wheel the carriage must be displaced from the middle position by a distance $d/4$ towards the right support. The same magnitude of bending moment can be obtained also under the right wheel by displacing the carriage by $d/4$ from the middle position towards the left support.

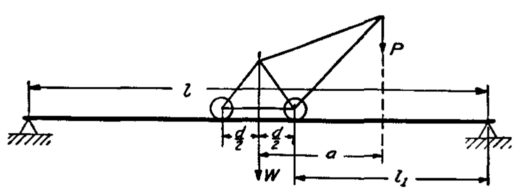


FIG. 92.

6. The rails of a crane (Fig. 92) are supported by two standard I beams. Determine the most unfavorable position of the crane, the

corresponding M_{max} and the dimensions of the I beams if $\sigma_w = 15{,}000$ lbs. per sq. in., $l = 30$ feet, $a = 12$ feet, $d = 6$ feet, the weight of the crane $W = 10{,}000$ lbs., the load lifted by the crane $P = 2{,}000$ lbs. The loads are acting in the middle plane between the two I beams and are equally distributed between them.

Solution. The maximum bending moment will be under the right wheel when the distance of this wheel from the right support is equal to $l_1 = \frac{1}{2}(l - \frac{1}{6}d)$; $M_{max} = 1{,}009{,}000$ lbs. in. Dividing the moment equally between the two beams, we find the necessary

$$Z = \frac{M_{max}}{2\sigma_w} = 33.6 \text{ in.}^3$$

The necessary I beam has a depth 12 in., cross-sectional area 9.26 sq. in.; $Z = 36.0$ in.³ The weight of the beam is neglected.

7. A circular wooden beam supported at C and attached to the foundation at A (Fig. 93) carries a load $q = 300$ lbs. per foot uni-

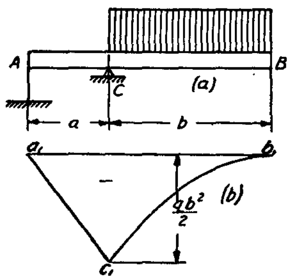


FIG. 93.

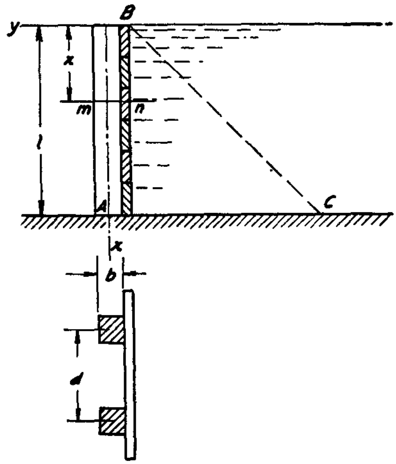


FIG. 94.

formly distributed along the portion BC. Construct the bending moment diagram and determine the necessary diameter d if $\sigma_w = 1{,}200$ lbs. per sq. in., $a = 3$ feet, $b = 6$ feet.

Solution. The bending moment diagram is shown in Fig. 93 (*b*). Numerically the largest moment will be at C and is equal to 64,800 lbs. in.:

$$d = \sqrt[3]{\frac{32}{\pi} \cdot \frac{M}{\sigma_w}} = 8.2 \text{ in.}$$

8. The wooden dam backed by vertical pillars built in at the lower ends (Fig. 94) consists of horizontal boards. Determine the dimension of the square cross section of the pillars if $l = 6$ feet, $d = 3$ feet and $\sigma_w = 500$ lbs. per sq. in. Construct the bending moment and shearing force diagrams.

Solution. The total lateral load on one pillar is represented by the weight W of the triangular prism ABC of water. At any cross section mn, the shearing force and the bending moment are

$$V = -\frac{W \cdot x^2}{l^2}; \qquad M = -\frac{Wx^2}{l^2} \cdot \frac{x}{3}.$$

In determining the signs of V and M it is assumed that Fig. 94 is rotated 90° in the counter-clockwise direction so as to bring the axes x and y into coincidence with those of Fig. 56. The necessary dimension b is found from equation

$$Z = \frac{b^3}{6} = \frac{M_{\max}}{\sigma_w} = \frac{3 \times 6^2 \times 62.4 \times 12}{500},$$

from which

$$b = 9.90 \text{ in.}$$

The construction of diagrams is left to the reader.

9. Determine the necessary dimensions of a cantilever beam of a standard I section which carries a uniform load $q = 200$ lbs. per foot and a concentrated load $P = 500$ lbs. at the end if the length $l = 5$ feet and $\sigma_w = 15{,}000$ lbs. per sq. in.

Solution.

$$Z = \frac{(500 \times 5 + 1{,}000 \times 2.5)12}{15{,}000} = 4 \text{ in.}^3$$

The necessary standard I beam is 5 in. deep and of 2.87 in.² cross-sectional area.

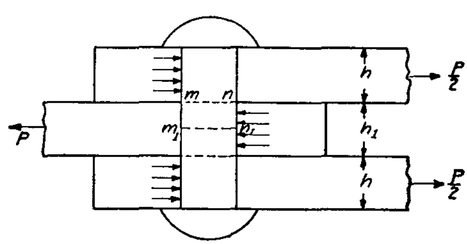


FIG. 95.

10. Determine the bending stresses in a rivet by assuming that the loads acting on the rivet are distributed as shown in Fig. 95.

The diameter of the rivet $d = 3/4$ in., $h = 1/4$ in., $h_1 = 3/8$ in., $P = 10{,}000$ lbs. per sq. in.

Solution. The bending moment at the cross section mn is $P/2 \times h/2$. The bending moment at the middle cross section m_1n_1 is

$$\frac{P}{2}\left(\frac{h}{2} + \frac{h_1}{4}\right).$$

This latter moment is the one to be taken into account in calculating the stresses. Then

$$(\sigma_x)_{\max} = \frac{P}{2}\left(\frac{h}{2} + \frac{h_1}{4}\right) : \frac{\pi d^3}{32} = \frac{4P}{\pi d^2} \cdot \frac{2h + h_1}{d} = 26{,}400 \text{ lbs. per sq. in.}$$

11. Determine the necessary standard I beams for the cases in Figures 67*a*, 67*d*, and 68*b*, assuming working stress of 16,000 lbs. per sq. in.

12. Determine the necessary dimensions of a simply supported beam of standard I-section such as to carry a uniform load of 400 lbs. per ft. and a load of $P = 4000$ lbs. placed at the middle. The length of the beam is 15 ft. and working stress $\sigma_w = 16{,}000$ lbs. per sq. in.

13. A channel the cross section of which is shown in Fig. 85 is simply supported at the ends and carries a concentrated load at the middle. Calculate the maximum value of the load which the beam will carry if the working stress is 1,000 lbs. per sq. in. for tension and 2,000 lbs. per sq. in. for compression.

26. Shearing Stresses in Bending.—In the preceding article it was shown that when a beam is bent by transverse loads not only normal stresses σ_x but also shearing stresses τ are produced in any cross section mn of the beam, Fig. 96. Considering the action on the right portion of the beam, Fig. 96, it can be concluded from the conditions of equilibrium that the magnitude of these shearing stresses is such that their summation is equal to the shearing force V. In investigating the law of their distribution over the area of the cross section we begin with the simple case of a rectangular cross section, Fig. 97. In such a case it is natural to assume that the shear-

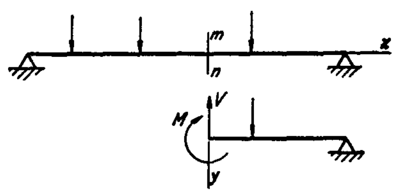


FIG. 96.

ing stress at each point of the cross section is parallel to the shearing force V, i.e., parallel to the sides mn of the cross section. We denote the stress in such a case by τ_{yx}. The subscript y in τ_{yx} indicates that the shearing stress is parallel to the y axis and the subscript x that the stress acts in a plane perpendicular to the x axis. As a second assumption we take

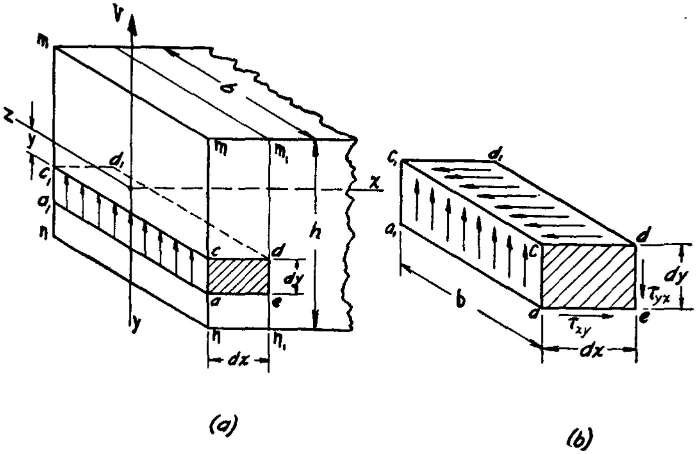


FIG. 97.

the distribution of the shearing stresses to be uniform across the width of the beam cc_1. These two assumptions will enable us to determine completely the distribution of the shearing stresses. A more elaborate investigation of the problem shows that the approximate solution thus obtained is usually sufficiently accurate and that for a narrow rectangle (h large in comparison with b, Fig. 97) it practically coincides with the exact solution.[3]

If an element be cut from the beam by adjacent cross sections and by adjacent planes parallel to the neutral plane,

[3] The exact solution of this problem is due to de Saint Venant, Journal de Math. (Liouville), 1856. An account of de Saint Venant's famous work is given in Todhunter and Pearson's "History of the Theory of Elasticity." The approximate solution given below is by Jouravski. For the French translation of his work see Annales des Ponts et Chaussées, 1856. The exact theory shows that when the depth of the beam is small in comparison with the width the discrepancy between the exact and the approximate theories becomes considerable.

as in Fig. 97 (*b*), in accordance with our assumption there is a uniform distribution of the shearing stresses τ_{yx} over the vertical face acc_1a_1. These stresses have a moment $(\tau_{yx}bdy)dx$ about the lower rear edge ee of the element which must be balanced by the moment $(\tau_{xy}bdx)dy$ due to shearing stresses distributed over the horizontal face of the element, cdd_1c_1. Then

$$\tau_{yx}bdydx = \tau_{xy}bdxdy \qquad \text{and} \qquad \tau_{yx} = \tau_{xy},$$

i.e., the shearing stresses acting on the two perpendicular faces of the element are equal. The same conclusion was met before in simple tension (see p. 40) and also in tension or compression in two perpendicular directions (see p. 45). The existence of these shearing stresses in the planes parallel to the neutral plane can be demonstrated by simple experiments. Take two equal rectangular bars put together on simple supports as shown in Fig. 98 and bent by a concentrated load P. If there is no friction between the bars, the bending of each bar will be independent of that of the other; each will have compression of the upper and tension of the lower longitudinal fibers and the condition will be that indicated in Fig. 98 (*b*). The lower longitudinal fibers of the upper bar slide with respect to the upper fibers of the lower bar. In a solid bar of depth $2h$ (Fig. 98, *a*) there will be shearing stresses along the neutral plane nn of such magnitude as to prevent this sliding of the upper portion of the bar with respect to the lower, shown in Fig. 98 (*b*). Due to this prevention of sliding the single bar of depth $2h$ is much stiffer and stronger than two bars each of depth h. In practice keys such as a, b, c, . . . are sometimes used with built-up wooden beams in order to prevent sliding (Fig. 99, *a*). Observation of the clearances around a key, Fig. 99 (*b*), enables us to determine the direction of sliding in the case of a built-up beam and therefore

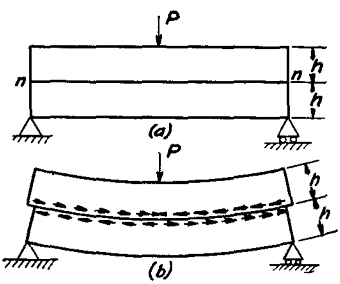


FIG. 98.

the direction of the shearing stresses over the neutral plane in the case of a solid beam.

The above discussion shows that the shearing stress τ_{yx} at any point of the vertical cross section is vertical in direction and numerically equal to the horizontal shearing stress τ_{xy} in the horizontal plane through the same point. This latter stress can easily be calculated from the condition of equilibrium of the element pp_1nn_1 cut out from the beam by two adjacent cross sections mn and m_1n_1 and by the horizontal plane pp_1, Fig. 100 (*a*) and (*b*). The only forces on

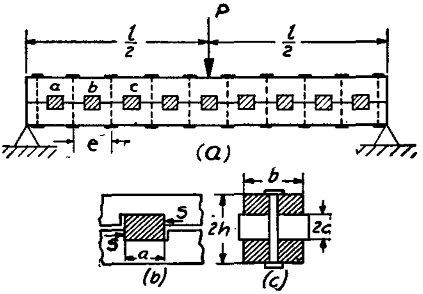


FIG. 99.

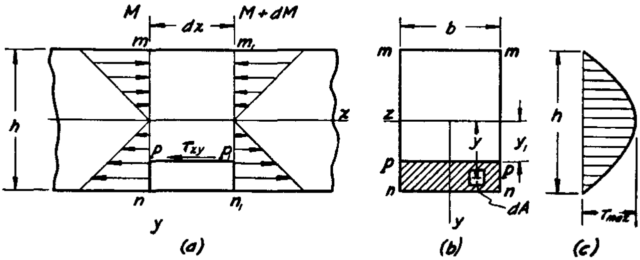


FIG. 100.

this element in the direction of the x axis are the shearing stresses τ_{xy} over the side pp_1 and the normal stress σ_x over the sides pn and p_1n_1. If the bending moments at cross sections mn and m_1n_1 are equal, i.e., in the case of pure bending, the normal stresses σ_x over the sides np and n_1p_1 will be also equal and will be in balance between themselves. Then the shearing stress τ_{xy} must be equal to zero.

Let us consider now the more general case of a varying bending moment, denoting by M and $M + dM$ the moments in the cross sections mn and m_1n_1 respectively. Then the normal force acting on an elemental area dA of the side $nppn$

will be (eq. 57)

$$\sigma_x dA = \frac{My}{I_z} dA.$$

The sum of all these forces distributed over the side *nppn* of the element will be

$$\int_{y_1}^{h/2} \frac{My}{I_z} dA. \tag{a}$$

In the same manner the sum of the normal forces acting on the side $n_1p_1p_1n_1$ is

$$\int_{y_1}^{h/2} \frac{(M + dM)y}{I_z} dA. \tag{b}$$

The force due to the shearing stresses τ_{xy} acting on the top side pp_1 of the element is

$$\tau_{xy} b dx. \tag{c}$$

The forces given in (*a*), (*b*) and (*c*) must satisfy eq. $\Sigma X = 0$, hence

$$\tau_{xy} b dx = \int_{y_1}^{h/2} \frac{(M + dM)y}{I_z} dA - \int_{y_1}^{h/2} \frac{My}{I_z} dA,$$

from which

$$\tau_{xy} = \frac{dM}{dx} \frac{1}{b \cdot I_z} \int_{y_1}^{h/2} y dA,$$

or, by using eq. (50),

$$\tau_{xy} = \tau_{yx} = \frac{V}{bI_z} \int_{y_1}^{h/2} y dA. \tag{64}$$

The integral in this equation has a very simple meaning. It represents the moment of the shaded portion of the cross section, Fig. 100 (*b*), with respect to the neutral axis *z*. For the rectangular section discussed

$$dA = bdy$$

and the integral becomes

$$\int_{y_1}^{h/2} bydy = \left| \frac{by^2}{2} \right|_{y_1}^{h/2} = \frac{b}{2}\left(\frac{h^2}{4} - y_1^2\right). \tag{d}$$

The same result can be obtained by multiplying the area

$b[(h/2) - y_1]$ of the shaded portion by the distance $\frac{1}{2}[(h/2) + y_1]$ of its centroid from the neutral axis.

Substituting (d) in eq. (64), we obtain for the rectangular section

$$\tau_{xy} = \tau_{yx} = \frac{V}{2I_z}\left(\frac{h^2}{4} - y_1^2\right). \tag{65}$$

It is seen that the shearing stresses τ_{yx} are not uniformly distributed from top to bottom of the beam. The maximum value of τ_{yx} occurs for $y_1 = 0$, i.e., for points on the neutral axis; and is, from equation (65),

$$(\tau_{yx})_{\max} = \frac{Vh^2}{8I_z}$$

or, since $I_z = bh^3/12$,

$$(\tau_{yx})_{\max} = \frac{3}{2} \cdot \frac{V}{bh}. \tag{66}$$

Thus the maximum shearing stress in the case of a rectangular cross section is 50 per cent greater than the average shearing stress, obtained by dividing the shearing force by the area of the cross section.

For the bottom and for the top of the cross section, $y_1 = \pm h/2$ and equation (65) gives $\tau_{yx} = 0$. The graph of equation (65) (Fig. 100, c) shows that the distribution of the shearing stresses along the depth of the beam follows a parabolic law. The shaded area bounded by the parabola multiplied with the width b of the beam gives $\frac{2}{3}(\tau_{yx})_{\max}hb = V$, as it should.

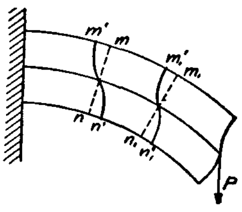


FIG. 101.

A natural consequence of these shearing stresses is shearing strain, which causes cross sections, initially plane, to become warped. This warping can be easily demonstrated by bending with a force on the end a rectangular piece of rubber (Fig. 101), on whose sides vertical lines have been drawn. The lines will not remain straight as indicated by the dotted lines, but become curved, so that the maximum shear strain occurs at the neutral surface. At the points m', m_1', n', n_1'

the shearing strain is zero, so that the curves $m'n'$ and $m_1'n_1'$ are normal to the upper and lower surfaces of the bar after bending. At the neutral surface the angles between the tangents to the curves $m'n'$ and $m_1'n_1'$ and the normal sections mn and m_1n_1 are equal to $\gamma = 1/G \cdot (\tau_{yx})_{\max}$. As long as the shearing force remains constant along the beam, the warping of all cross sections is the same, so that $mm' = m_1m_1'$, $nn' = n_1n_1'$ and the stretching or the shrinking produced by the bending moment in the longitudinal fibers is unaffected. This fact explains the validity here of eq. (57), which was developed for pure bending and based on the assumption that cross sections of a bar remain plane during bending.

A more elaborate investigation of the problem [4] shows that the warping of cross sections also does not substantially affect the strain in longitudinal fibers if a distributed load acts on the beam and the shearing force varies continuously along the beam. In the case of concentrated loads the stress distribution near the loads is more complicated, but this deviation from the straight line law is of a local type (see Part II).

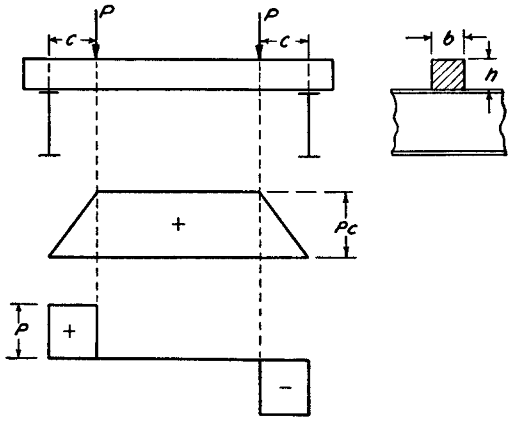


FIG. 102.

Problems

1. Determine the limiting values of the loads P acting on the wooden rectangular beam, Fig. 102, if $b = 8$ in., $h = 10$ in., $\sigma_w = 800$ lbs. per sq. in., $\tau_w = 200$ lbs. per sq. in., $c = 1.5$ feet.

Solution. The bending moment and shearing force diagrams are given in Fig. 102.

[4] See W. Voigt, Göttingen Abhandlungen, Bd. 34 (1887); J. H. Michell, Quart. J. of Math., Vol. 32 (1901); and L. N. G. Filon, Phil. Trans. Roy. Soc. (Ser. A), Vol. 201 (1903), and London Roy. Soc. Proc., Vol. 72 (1904).

$$V_{\max} = P; \qquad M_{\max} = P \cdot c.$$

From equations

$$\frac{Pc}{Z} = \sigma_w \qquad \text{and} \qquad \frac{3}{2}\frac{P}{bh} = \tau_w,$$

we obtain

$$P = 5{,}930 \text{ lbs.} \qquad \text{and} \qquad P = 10{,}700 \text{ lbs.};$$

therefore $P = 5{,}930$ lbs. is the limiting value of the load P.

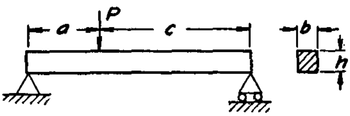


FIG. 103.

2. Determine the maximum normal stress σ_x and the maximum shearing stress τ_{yx} in the neutral plane of the beam represented in Fig. 103 if $a = 2$ feet, $c = 4$ feet, $b = 8$ in., $h = 10$ in. and $P = 6{,}000$ lbs.

Answer.

$$(\sigma_x)_{\max} = 720 \text{ lbs. per sq. in.}; \qquad (\tau_{yx})_{\max} = 75 \text{ lbs. per sq. in.}$$

3. Determine the maximum shearing stress in the neutral plane of a uniformly loaded rectangular beam if the length of the beam $l = 6$ feet, the load per foot $q = 1{,}000$ lbs., the depth of the cross section $h = 10$ in. and the width $b = 8$ in.

Answer.

$$\tau_{\max} = 56.3 \text{ lbs. per sq. in.}$$

4. Determine the maximum shearing stresses in problem 2 of article 25.

27. Distribution of Shearing Stresses in the Case of a Circular Cross Section.—In considering the distribution over a circular cross section (Fig. 104) there is no foundation for the assumption that the shearing stresses are all parallel to the shearing force V. In fact we can readily show that at points p (Fig. 104, b) of the cross section along the boundary the shearing stress is tangent to the boundary.

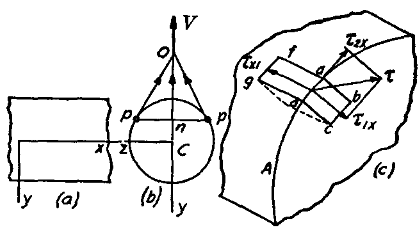


FIG. 104.

Let us consider an infinitesimal element $abcd$ (Fig. 104, c) in the form of a rectangular parallelepiped with the face $adfg$ in the surface of the beam and the face $abcd$ in the plane yz of the cross section. If the shearing stress acting over the side $abcd$ of the element has a direction such as τ, it can always be resolved into two components τ_{1x} in a radial direction and τ_{2x} in the direction of the tangent to the boundary. Now it has been proved before (see p. 111), by using the equation of equilibrium of an element, that if a shearing stress τ acts over an elemental area, an equal shearing stress will act also over an elemental area perpendicular to τ. Applying this in our case it must be concluded that if a shearing stress τ_{1x} is acting on the element *abcd* in a radial direction there must be an equal shearing stress τ_{x1} on the side *adfg* of the element lying in the surface of the beam. If the lateral surface of the beam is free from shearing stresses, the radial component τ_{1x} of the shearing stress τ must be equal to zero, i.e., τ must be in the direction of the tangent to the boundary of the cross section of the beam. At the midpoint *n* of the chord *pp* symmetry requires that the shearing stress have the direction of the shearing force V. Then the directions of the shearing stresses at the points *p* and *n* will intersect at some point *O* on the *y* axis (Fig. 104, *b*). Assuming now that the shearing stress at any other point of the line *pp* is also directed toward the point *O*, we have a complete determination of the directions of the shearing stresses. As a second assumption we take the vertical components of the shearing stresses equal for all points of the line *pp*.[5] As this assumption coincides completely with that made in the case of a rectangular cross section, we can use eq. (64) for calculating this component. Knowing the actual direction of the shearing stress and its vertical component, its magnitude may be easily calculated for any point of the cross section.

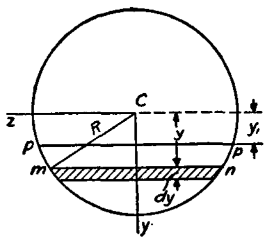


FIG. 105.

Let us calculate now the shearing stresses along the line *pp* of the cross section (Fig. 105). In applying eq. (64) to the calculation of the vertical component τ_{yx} of these stresses we must find the moment of the segment of the circle below the line *pp* with respect to the *z* axis. The elemental area *mn* has the length $2\sqrt{R^2 - y^2}$ and the width dy. The area is $dA = 2\sqrt{R^2 - y^2}dy$. The moment of this strip about Cz is ydA and the total moment for the entire segment is

[5] The approximate theory based on the above two assumptions gives satisfactory accuracy and comparison with the exact theory shows that the error in the magnitude of the maximum shearing stress is about 5 per cent, which is not high for practical application. See Saint Venant, loc. cit., p. 110. See also the book by A. E. H. Love, "Mathematical Theory of Elasticity," 4th ed., 1927, p. 346.

$$\int_{y_1}^{R} 2\sqrt{R^2 - y^2}\cdot ydy = \tfrac{2}{3}(R^2 - y_1^2)^{3/2}.$$

Substituting this in eq. (64) and taking $2\sqrt{R^2 - y_1^2}$ for b, we obtain

$$\tau_{yx} = \frac{V(R^2 - y_1^2)}{3I_z}, \tag{67}$$

and the total shearing stress at points p (Fig. 105) is

$$\tau = \frac{\tau_{yx}\cdot R}{\sqrt{R^2 - y_1^2}} = \frac{VR\sqrt{R^2 - y_1^2}}{3I_z}.$$

It is seen that the maximum τ is obtained for $y_1 = 0$, i.e., for the neutral axis of the cross section. Then, substituting $I_z = \pi R^4/4$,

$$\tau_{\max} = \frac{4}{3}\frac{V}{\pi R^2} = \frac{4}{3}\cdot\frac{V}{A}. \tag{68}$$

In the case of a circular cross section, therefore, the maximum shearing stress is 33 per cent larger than the average value obtained by dividing the shearing force by the cross-sectional area.

28. Distribution of Shearing Stresses in I Beams.—In considering the distribution of the shearing stresses in I beams (Fig. 106) for the section of the web, the same assumptions are made as for a rectangular cross section; these were that the shearing stresses are parallel to the shearing force V and are uniformly distributed over the thickness b_1 of the web. Then eq. (64) will be used for calculating the stresses τ_{yx}. For points on the line pp at a distance y_1 from the neutral axis, where the width of the cross section is b_1, the moment of the shaded portion with respect to the neutral axis z is

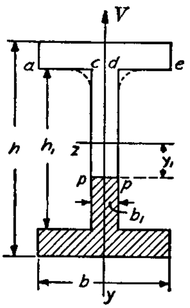


FIG. 106.

$$\int_{y_1}^{h/2} ydA = \frac{b}{2}\left(\frac{h^2}{4} - \frac{h_1^2}{4}\right) + \frac{b_1}{2}\left(\frac{h_1^2}{4} - y_1^2\right).$$

Substituting in eq. (64), we obtain

$$\tau_{yx} = \frac{V}{b_1 I_z}\left[\frac{b}{2}\left(\frac{h^2}{4} - \frac{h_1^2}{4}\right) + \frac{b_1}{2}\left(\frac{h_1^2}{4} - y_1^2\right)\right]. \tag{69}$$

The stress then varies along the depth of the beam following a parabolic law. The maximum and minimum values of τ_{yx} in the web of the beam are obtained by putting $y_1 = 0$ and $y_1 = h_1/2$:

$$(\tau_{yx})_{\max} = \frac{V}{b_1 I_z}\left[\frac{bh^2}{8} - \frac{h_1^2}{8}(b - b_1)\right]; \qquad (70)$$

$$(\tau_{yx})_{\min} = \frac{V}{b_1 I_z}\left(\frac{bh^2}{8} - \frac{bh_1^2}{8}\right). \qquad (71)$$

When b_1 is very small in comparison with b there is no great difference between $(\tau_{yx})_{\max}$ and $(\tau_{yx})_{\min}$ and the distribution of the shearing stresses over the cross section of the web is practically uniform.

A good approximation for $(\tau_{yx})_{\max}$ is obtained by dividing the complete shearing force V by the cross-sectional area of the web alone. This follows from the fact that the shearing stresses distributed over the cross section of the web yield a force which is nearly equal to V, which means that the web takes nearly all the shearing force, and the flanges have only a secondary part in its transmission. To prove this the summation of the stresses τ_{yx} over the web which we will call V_1 is $\int_{-h_1/2}^{h_1/2} \tau_{yx} b_1 dy$, from eq. (69):

$$V_1 = \frac{V}{b_1 I_z}\int_{-h_1/2}^{h_1/2}\left[\frac{b}{2}\left(\frac{h^2}{4} - \frac{h_1^2}{4}\right) + \frac{b_1}{2}\left(\frac{h_1^2}{4} - y_1^2\right)\right] b_1 dy$$

and, after integration,

$$V_1 = \frac{V}{I_z}\left[\frac{b(h - h_1)}{2} \cdot \frac{h + h_1}{2} \cdot \frac{h_1}{2} + \frac{b_1 h_1^3}{12}\right]. \qquad (a)$$

For small thickness of flanges, i.e., when h_1 approaches h, the moment of inertia I_z is represented with sufficient accuracy by the equation:

$$I_z = \frac{b(h - h_1)}{2} \cdot \frac{(h + h_1)^2}{8} + \frac{b_1 h_1^3}{12}, \qquad (b)$$

in which the first term represents the cross-sectional area of the flanges multiplied by the square of the distance $(h + h_1)/4$ of their centers from the z axis, which is approximately the moment of inertia of the cross section of the flanges. The second term is the moment of inertia of the cross section of the web. Comparing (a) and (b), we see that as h_1 approaches h the force V_1 approaches V and the shearing force will be taken by the web alone.

In considering the distribution of the shearing stresses over the cross sections of the flanges the assumption of no variation along the width of the section cannot be made. For example, at the level ae (Fig. 106), along the lower boundary of the flange, ac and de, the shearing stress τ_{yx} must be zero since the corresponding equal stress τ_{xy} in the free bottom surface of the flange is zero (see p. 116 and also Fig. 104, c). In the part cd, however, the shearing stresses are not zero, but have the magnitudes calculated above for $(\tau_{yx})_{\min}$ in the web. This indicates that at the junction cd of the web and the flange the distribution of shearing stresses follows a more complicated law than can be investigated by our elementary analysis. In order to dissipate a stress concentration at the points c and d, the sharp corners are usually replaced by fillets as indicated in the figure by dotted lines. A more detailed discussion of the distribution of shearing stresses in flanges will be given later (see Part II).

Problems

1. Determine $(\tau_{yx})_{\max}$ and $(\tau_{yx})_{\min}$ in the cross section of the web of an I beam, Fig. 106, if $b = 5$ in., $b_1 = \frac{1}{2}$ in., $h = 12$ in., $h_1 = 10\frac{1}{2}$ in., $V = 30{,}000$ lbs. Determine the shearing force transmitted by the web V_1.

Answer. $(\tau_{yx})_{\max} = 5{,}870$ lbs. per sq. in., $(\tau_{yx})_{\min} = 4{,}430$ lbs. per sq. in., $V_1 = 0.945V$.

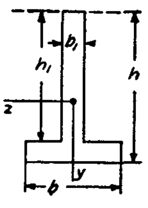


FIG. 107.

2. Determine the maximum shearing stress in the web of a T beam (Fig. 107) if $h = 8$ in., $h_1 = 7$ in., $b = 4$ in., $b_1 = 1$ in. and $V = 1{,}000$ lbs.

Answer. Using the same method as in the case of an I beam, we find $(\tau_{yx})_{\max} = 176$ lbs. per sq. in.

3. Determine the maximum shearing stress in problems 1 and 6 of article 25.

29. Principal Stresses in Bending.—By using eqs. (57) and (64) the normal stress σ_x and the shearing stress τ_{yx} can easily be calculated for any point of a cross section provided the bending moment M and the shearing force V are known for this cross section. The maximum numerical value of σ_x will be in the fiber most remote from the neutral axis, and the maximum value of τ_{yx} usually at the neutral axis. In the majority of cases only the maximum values of σ_x and τ_{yx} obtained in this manner are used in design and the cross-sectional dimensions of beams are taken so as to satisfy the conditions

$$(\sigma_x)_{\max} \lesseqgtr \sigma_w \quad \text{and} \quad (\tau_{yx})_{\max} \lesseqgtr \tau_w.$$

It is assumed here that the material is equally strong in tension and compression and σ_w is the same for both. Other-

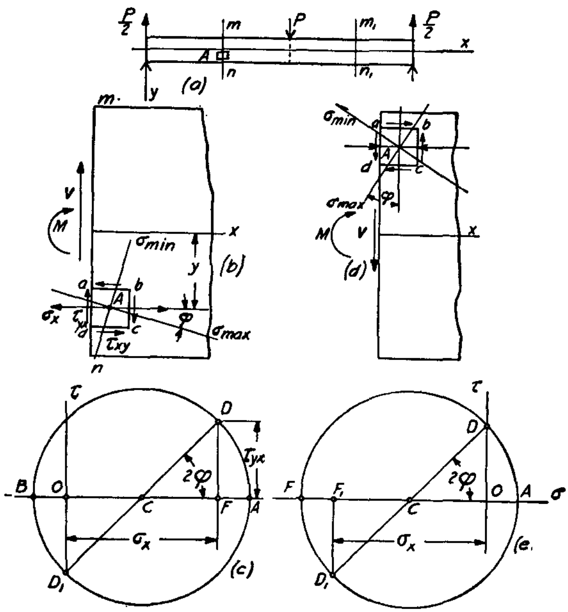


FIG. 108.

wise the conditions of strength in tension and in compression must be satisfied separately and we obtain

$$(\sigma_x)_{\max} \lesseqgtr \sigma_w \text{ in tension; } (\sigma_x)_{\min} \lesseqgtr \sigma_w \text{ in compression.}$$

There are cases, however, which require a more detailed analysis of stress conditions. We shall now demonstrate the method of analysis necessary for such cases with a beam simply supported and loaded at the middle (Fig. 108). For a point A below the neutral axis in the cross section mn, the magnitudes of the stresses σ_x and $\tau_{yx} = \tau_{xy}$ are given by eqs. (57) and (64). In Fig. 108 (*b*) those stresses are shown acting on an infinitesimal element cut out of the beam at the point A, their senses being easily determined from those of M and V. For such an infinitesimal element the changes in stresses σ_x and τ_{yx} for various points of the element can be neglected and it can be assumed that the element is in a *homogeneous state of stress*, i.e., the quantities σ_x and τ_{yx} may be regarded as the same throughout the element. Such a state of stress is illustrated by the element of finite dimensions in Fig. 37a.

From our previous investigation (see p. 46) we know that the stresses on the sides of an element cut out from a stressed body vary with the directions of these sides and that it is possible to so rotate the element that only normal stresses are present (see p. 47). The directions of the sides are then called *principal directions* and the corresponding stresses *principal stresses*. The magnitudes of these stresses can be found from eqs. (31) and (32) by substituting in these equations $\sigma_y = 0$. Then we obtain

$$\sigma_{\max} = \frac{\sigma_x}{2} + \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{yx}^2}, \tag{72}$$

$$\sigma_{\min} = \frac{\sigma_x}{2} - \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{yx}^2}. \tag{73}$$

It should be noted that $\sigma_{\max}$ is always tension and $\sigma_{\min}$ always compression. Knowing principal stresses, the maximum shearing stress at any point will be obtained from eq. (34) (see p. 49):

$$\tau_{\max} = \frac{\sigma_{\max} - \sigma_{\min}}{2} = \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{yx}^2}. \tag{74}$$

For determining the directions of principal stresses Mohr's circle will be used. For an element such as at point A (Fig.

108, *b*), the corresponding Mohr's circle is shown in Fig. 108 (*c*). By taking the distance $\overline{OF} = \sigma_x$ and $\overline{DF} = \tau_{yx}$, the point D, representing stresses over the sides *bc* and *ad* of the element, is obtained. The distance $\overline{OF}$ is taken in the direction of positive σ and $\overline{DF}$ in the upward direction because σ_x is tensile stress and shearing stresses τ_{yx} over sides *bc* and *ad* give a clockwise couple (see p. 38). Point D_1 represents the stresses over the sides *ab* and *dc* of the element on which the normal stresses are zero and the shearing stresses are negative. The circle constructed on the diameter DD_1 determines $\sigma_{max} = \overline{OA}$ and $\sigma_{min} = -\overline{OB}$. From the same construction the angle 2φ is determined and the direction of σ_{max} in Fig. 108 (*b*) is obtained by measuring φ from the x axis in the clockwise direction. Of course σ_{min} is perpendicular to σ_{max}.

By taking a section m_1n_1 to the right of the load P (Fig. 108, *a*) and considering a point A above the neutral axis, the direction of the stresses acting on an element *abcd* at A will be that indicated in Fig. 108 (*d*). The corresponding Mohr's circle is shown in Fig. 108 (*e*). Point D represents the stresses for the sides *ab* and *dc* of the element *abcd* and point D_1 the stresses over the sides *ad* and *bc*. The angle φ determining the direction σ_{max} must be measured in the clockwise direction from the outer normal to the side *ab* or *cd* as shown in Fig. 108 (*d*).

If we take a point at the neutral surface, then σ_x becomes zero. An element at this point will be in the condition of pure shear. The directions of the principal stresses will be at 45° to the x and y axes.

It is possible to construct two systems of orthogonal curves whose tangents at each point are in the directions of the principal stresses at this point. Such curves are called the *trajectories of the stresses*. Figure 109 shows the stress trajectories for a rectangular cantilever beam, loaded at the end. All these curves intersect the neutral surface at 45° and have horizontal or vertical tangents at points where the shearing stress τ_{yx} is zero, i.e., at the top and at the bottom surfaces of the beam. The trajectories giving the directions of σ_{max}

(tension) are represented by full lines and the other system of trajectories by dotted lines. Figure 110 gives the trajectories and the stress distribution diagrams for σ_x and τ_{yx} over several cross sections of a simply supported rectangular

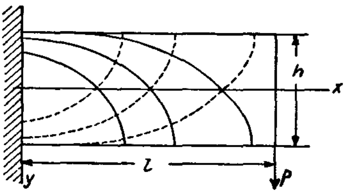


FIG. 109.

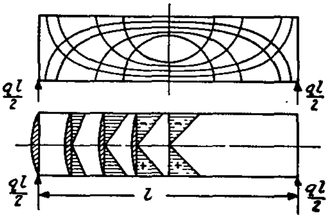


FIG. 110.

beam under uniform load. It is clearly seen that σ_x has a maximum value at the middle, where the bending moment M is a maximum, and τ_{yx} is maximum at the supports, where the maximum shearing force acts.[6] In the design of beams the concern is for the numerically maximum values of σ. From eq. (72) it can be seen that for the most remote fibers in tension, where the shear is zero, the longitudinal normal stress σ_x becomes the principal stress, i.e., $\sigma_{\max} = (\sigma_x)_{\max}$. For fibers nearer to the neutral axis the longitudinal fiber stress σ_x is less than at the extreme fiber; however we now have a shear stress τ_{yx} also and the stresses σ_x and τ_{yx} acting together at this point may produce a principal stress, given by eq. (72), which will be numerically larger even than that at the extreme fiber. In the case of beams of rectangular or circular cross section, in which the shearing stress τ_{yx} varies continuously down the depth of the beam, this is not usually the case, that is, the stress $(\sigma_x)_{\max}$ calculated for the most remote fiber at the section of maximum bending moment is the maximum stress acting in the beam. However, in such a case as an I beam, where a sudden change occurs in the magnitude of shearing stress at the junction of flange and web (see p. 120), the maximum stress calculated at this joint

[6] Several examples of construction of the trajectories of stresses are discussed by I. Wagner, Zeitschr. d. Österr. Ing. u. Archit. Ver., 1911, p. 645.

from eq. (72) may be larger than the tensile stress $(\sigma_x)_{\max}$ in the most remote fiber and should be taken into account in design. To illustrate, consider the case represented in Fig. 108 (*a*) with a beam of I section and the same dimensions as in problem 1, page 120, the length l = 2 feet and P = 60,000 lbs. Then $M_{\max}$ = 30,000 lbs. feet; $V_{\max}$ = 30,000 lbs. From eq. (57) the tensile stress in the most remote fiber is

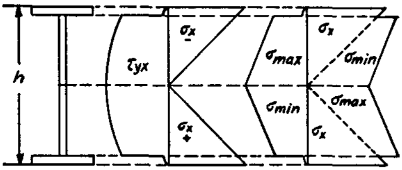


Fig. 111.

$$(\sigma_x)_{\max} = \frac{30{,}000 \times 12 \times 6}{286} = 7{,}550 \text{ lbs. per sq. in.}$$

Now for a point at the junction of flange and web we obtain the following values of normal and shearing stresses:

$$\sigma_x = \frac{7{,}550 \times 10\frac{1}{2}}{12} = 6{,}610 \text{ lbs. per sq. in.};$$

$$\tau_{yx} = 4{,}430 \text{ lbs. per sq. in.}$$

Then, from eq. (72), the principal stress is

$$\sigma_{\max} = 8{,}830 \text{ lbs. per sq. in.}$$

It can be seen that $\sigma_{\max}$ at the joint between the flange and the web is larger than the tensile stress at the most remote fiber and therefore it must be considered in design. The variations of σ_x, τ_{yx}, $\sigma_{\max}$ and $\sigma_{\min}$ along the depth of the beam are shown in Fig. 111.

Problems

1. Determine $\sigma_{\max}$ and $\sigma_{\min}$ at a point 2 in. below the neutral axis in the section 3 feet from the loaded end of the cantilever (Fig. 109) if the depth h = 8 in., the width b = 4 in. and P = 2,000 lbs. Determine the angle between $\sigma_{\max}$ at this point and the x axis.

Solution. $(\sigma_x) = -844$ lbs. per sq. in.; $(\tau_{yx}) = 70.3$ lbs. per sq. in.; $\sigma_{\max} = 5.7$ lbs. per sq. in.; $\sigma_{\min} = -849.7$ lbs. per sq. in. The angle between $\sigma_{\max}$ and the x axis is 85° 16′ measured clockwise.

2. Determine $\sigma_{\max}$ and $\sigma_{\min}$ at the neutral axis and in the cross section 1 foot from the left support for the uniformly loaded

rectangular beam supported at the ends (Fig. 110). The cross-sectional dimensions are the same as in the previous problem, and $q = 1{,}000$ lbs. per foot; $l = 10$ feet.

Answer.

$$\sigma_{\max} = -\sigma_{\min} = 187.5 \text{ lbs. per sq. in.}$$

3. Determine the length of the I beam considered on p. 125 if $(\sigma_x)_{\max}$ is equal to $\sigma_{\max}$ at the junction of flange and web.

Answer. $l = 39.8$ in.

30. Stresses in Built-up Beams.—In engineering practice built-up beams are frequently used and the stresses in them are usually calculated on the assumption that their parts are rigidly connected. The computation will then involve (*a*) the designing of the beam as a solid beam and (*b*) the designing and spacing of the elements which unite the parts of the beam. In the first case the formulas for solid beams are used, making an allowance for the effect of rivet holes, bolts, slots, etc., by the use of reduced sections. The computations necessary for the uniting of the elements will be indicated by illustrations.

Let us discuss first a wooden beam built up as shown in Fig. 99. It is assumed that the keys used between the two parts of the beam are strong enough to resist the shearing forces S (Fig. 99, *b*). Then eq. (57) can be used for calculating σ_x. In order to take into account the weakening of the section by the keyways and the bolt holes, only the shaded portion of the section, indicated in Fig. 99 (*c*), should be taken into consideration. Then

$$I_z = \frac{(b-d)}{12}\left[(2h)^3 - (2c)^3\right].$$

In calculating the shearing force S acting on each key we assume that this force is equal to the shearing force distributed in a solid beam over the area eb of the neutral surface where b is the width of the beam and e is the distance between the middle points of the keys (see Fig. 99, *a*). Then by using eq. (66) and considering that the depth of the beam is equal to $2h$ in this case, we obtain

$$S = eb \cdot \frac{3}{2} \frac{V}{b2h} = \frac{3}{2} \frac{Ve}{2h}. \tag{75}$$

The dimensions of the keys and the distance e between them should be chosen so as to insure sufficient strength against shear of the key and against crushing of the wood on the lateral sides of the key and the keyway. In such calculations the rough assumption is usually made that the shearing stresses are uniformly distributed over the middle section $a \times b$ of the key and that the pressure on the lateral sides of the keys is uniformly distributed over the areas $c \times b$. Then denoting by τ_w the working shearing stress for the keys, and by σ_w' the working stress in lateral compression of the wood of the keys or the keyways, the following equations for designing the keys are obtained:

$$\frac{S}{ab} \mathrel{\substack{=\\<}} \tau_w ; \qquad \frac{S}{bc} \mathrel{\substack{=\\<}} \sigma_w'.$$

It is necessary to insure also sufficient strength against shearing of the wood of the beam along the fibers between two keys. The shearing force will be again equal to S and the resisting area is $b \times (e - a)$. Denoting with τ_w' the working stress in shear of the material of the beam along the fibers, the condition of strength becomes

$$\frac{S}{b(e - a)} \mathrel{\substack{=\\<}} \tau_w'.$$

In addition to keys there are bolts (Fig. 99) uniting the parts of the beam. By tightening them friction between the parts of the beam is produced. This friction is usually neglected in calculations and it is assumed that the total shearing force is taken by keys. Experiments show that such built-up wooden beams are weaker than solid beams of the same dimensions.[7]

[7] The experiments made by Prof. E. Kidwell at the Michigan College of Mines show that built-up wooden beams have about 75 per cent of the strength of the solid beam of the same dimensions.

In calculating the σ_x stresses in built-up I beams the weakening effect of rivet holes is usually taken into account by assuming that all the holes are in the same cross section (Fig. 112, *a*) of the beam [8] and subtracting their diametral sections in calculating I_z in eq. (57).

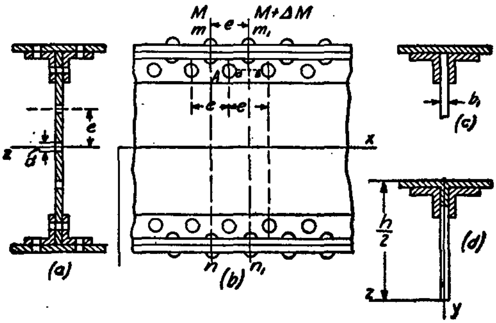


FIG. 112.

In calculating the maximum shearing stress τ_{yx} it is also the practice to take into account the weakening effect of the rivet holes. It can be seen that the cross-sectional area of the web is diminished, by holes, in the ratio $(e - d)/e$, where e is the distance between the centers of the holes and d the diameter of the holes. Hence the factor $e/(e - d)$ is usually included in the right side of eq. (64) for calculating τ_{yx} in the web of built-up I beams. It should be noted that this manner of calculating the weakening effect of rivet holes is only a rough approximation. The actual distribution of stresses near the holes is very complicated. Some discussion of stress concentration near the edge of a hole will be given later (see Part II).

In calculating the shearing force acting on one rivet, such as rivet A (Fig. 112, *b*), let us consider the two cross sections mn and m_1n_1. Due to the difference of bending moments in these two cross sections the normal stresses σ_x on sections mn and m_1n_1 will be different and there is a tendency for the

[8] The holes in vertical web are present in sections where vertical stiffeners are riveted to the girder.

flange of the beam shaded in Fig. 112 (*c*) to slide along the web. This sliding is prevented by friction forces and by the rivet *A*. Neglecting friction, the force acting on the rivet becomes equal to the difference of forces acting in sections *mn* and m_1n_1 of the flange. The force in the flange in the cross section *mn* is (see eq. (*a*), p. 113)

$$\frac{M}{I_z}\int ydA,$$

where the integration should be extended over the shaded cross sectional area of the flange. In the same manner for the cross section m_1n_1 we obtain

$$\frac{(M + \Delta M)}{I_z}\int ydA.$$

Then the force transmitted by the rivet *A* from the flange to the web will be

$$S = \frac{\Delta M}{I_z}\int ydA. \qquad (a)$$

By using eq. (50) and substituting the distance *e* between the rivets instead of *dx*, we obtain

$$\Delta M = Ve,$$

where *V* is the shearing force in the cross section of the beam through the rivet *A*. Substituting in eq. (*a*), we obtain

$$S = \frac{Ve}{I_z}\int ydA. \qquad (76)$$

The integral entering in this equation represents the moment of the shaded cross section (Fig. 112, *c*) of the flange with respect to the neutral axis *z*.

It is easy to see that in order to get sliding of the flange along the web the rivet must be sheared through two cross sections. Assuming that the force *S* is uniformly distributed over these two cross sections, the shearing stress in the rivet will be

$$\tau = \frac{S}{2 \cdot \dfrac{\pi d^2}{4}} = \frac{2Ve}{\pi d^2 I_z} \int y dA. \tag{77}$$

The force S sometimes produces considerable shearing stress in the web of the beam along the plane ab (see Fig. 112, b) which must be taken into consideration. Assuming that these stresses are uniformly distributed and dividing S by the area $b_1(e - d)$, we obtain

$$\tau' = \frac{V}{b_1 I_z} \cdot \frac{e}{e - d} \int y dA. \tag{b}$$

In addition to this stress produced by forces S transmitted from the flanges there will act along the same plane ab shearing stresses τ'' due to bending of the web. The magnitude of these stresses will be obtained by using the above eq. (b) and substituting for the integral $\int y dA$ the statical moment with respect to the neutral axis z of the portion of the rectangular cross section of the web above the plane ab. In this manner we arrive at the following equation for the shearing stress τ_{xy} in the web along the plane ab:

$$\tau_{xy} = \tau' + \tau'' = \frac{V}{b_1 I_z} \cdot \frac{e}{e - d} \int y dA, \tag{78}$$

in which the integral is extended over the shaded area of the cross section shown in Fig. 112 (d). Knowing σ_x and τ_{xy}, the σ_{max} and σ_{min} for the points in the plane ab can be calculated from eqs. (72) and (74), as was explained in the previous article, and the directions of principal stresses can be determined.

From the above discussion it is seen that in calculating stresses in built-up I beams several assumptions are made for simplifying the calculations. This to a certain extent reduces the accuracy of the calculated stresses, which fact should be considered in choosing the working stresses for built-up I beams.[9]

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[9] Experiments show that the failure of I beams usually occurs due to buckling of the compressed flanges or of the web (see H. F. Moore,

Problems

1. A built-up wooden beam (Fig. 99) consists of two bars of rectangular cross section connected by keys. Determine the shearing force acting on the keys, the shearing stress in the key and pressure per unit area on its lateral sides if the load $P = 5{,}000$ lbs., the width of the beam $b = 5$ in., the depth $2h = 16$ in., the width of the key $a = 3$ in., the depth of the key $2c = 2\frac{1}{2}$ in., and the distance between centers of the keys $e = 11$ in.

Answer.

$$S = \frac{3}{2} \cdot \frac{2{,}500 \times 11}{16} = 2{,}580 \text{ lbs.}$$

Shearing stress in the key is

$$\tau = \frac{2{,}580}{5 \times 3} = 172 \text{ lbs. per sq. in.}$$

The pressure per unit area on the lateral side is

$$p = \frac{S}{bc} = \frac{2{,}580 \times 2}{2\frac{1}{2} \times 5} = 413 \text{ lbs. per sq. in.}$$

2. Determine the shearing stress at the neutral axis of a girder, the web of which is $\frac{3}{4}$ in. thick and 50 in. high, the flanges consisting of two pairs of angles 6 in. $\times$ 6 in. $\times$ $\frac{1}{2}$ in., when the total shearing force on the section is 150,000 lbs. Determine also the shearing stresses in the rivets attaching the flanges to the web if the diameter of these rivets is 1 in. and the pitch $e = 4$ in. (Fig. 112).

Solution. For the dimensions given we have

$$I_z = \frac{3}{4} \times \frac{50^3}{12} + 4(19.9 + 5.75 \times 23.3^2) = 20{,}400 \text{ in.}^4$$

The moment of half of the cross section with respect to the neutral axis is

$$\int_0^{h/2} y dA = \frac{3}{4} \frac{25 \times 25}{2} + 2 \times 5.75 \times 23.3 = 502 \text{ in.}^3$$

In this calculation 5.75 in.^{2} is the cross sectional area of an angle, 19.9 in.^{4} is the moment of inertia of the cross section of an angle with

University of Illinois, Bulletin 68, 1913). This question of buckling will be considered later. The effect of bending of rivets on the distribution of stresses in I beams has been discussed by I. Arnovlevic, Zeitschr. f. Architekt. u. Ingenieurwesen, 1910, p. 57. He found that due to this bending stresses for usual proportions increase about 6 per cent.

respect to the axis through its centroid parallel to the neutral axis of the beam, 23.3 in. is the distance of the centroid of each angle from the neutral axis z of the beam. All such numerical data can be taken directly from a handbook. Now we obtain, from eq. (64),

$$(\tau_{xy})_{\max} = \frac{150{,}000 \times 502}{\frac{3}{4} \times 20{,}400} = 4{,}920 \text{ lbs. per sq. in.}$$

If we consider weakening of the web by the rivet holes, then

$$(\tau_{xy})_{\max} = \frac{e}{e-d} \cdot 4{,}920 = \frac{4}{3} 4{,}920 = 6{,}560 \text{ lbs. per sq. in.}$$

The force S transmitted by one rivet, from eq. (76),

$$S = \frac{150{,}000 \times 4 \times 268}{20{,}400} = 7{,}880 \text{ lbs.}$$

The shearing stress in the rivet, from eq. (77),

$$\tau = \frac{7{,}880 \times 2}{3.14} = 5{,}020 \text{ lbs. per sq. in.}$$

3. Determine $\sigma_{\max}$ in points of the plane ab (Fig. 112) a distance of 21.5 in. from the neutral axis if the dimensions of the beam are the same as in the previous problem, $V = 150{,}000$ and the bending moment $M = 3 \times 10^6$ lbs. in.

Solution. From eq. (78),

$$\tau_{xy} = \frac{150{,}000}{\frac{3}{4} \times 20{,}400} \cdot \frac{4}{3}(268 + 61) = 4{,}300 \text{ lbs. per sq. in.,}$$

$$\sigma_x = \frac{3 \times 10^6 \times 21.5}{20{,}400} = 3{,}160 \text{ lbs. per sq. in.,}$$

$$\sigma_{\max} = \frac{\sigma_x}{2} + \sqrt{\frac{\sigma_x^2}{4} + \tau_{xy}^2} = 6{,}160 \text{ lbs. per sq. in.}$$

4. Determine the shearing force in the rivets connecting the two rails of the beam shown in Fig. 113 if the cross-sectional area of a rail is $A = 10$ sq. in., the distance from the bottom of the rail to the centroid of its cross section $c = 3$ in., the moment of inertia of the cross section of the rail with respect to the axis through its centroid c and parallel to the z axis is 40 in.^{4}, the distance between the rivets $e = 6$ in., and the shearing force $V = 5{,}000$ lbs.

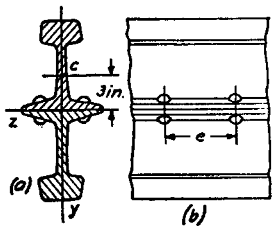

FIG. 113.

Solution.

$$S = \frac{1}{2} \cdot \frac{5{,}000 \times 6 \times 30}{2(40 + 10 \times 9)} = 1{,}730 \text{ lbs.}$$

CHAPTER V

DEFLECTION OF TRANSVERSALLY LOADED BEAMS

31. Differential Equation of the Deflection Curve.—In the design of a beam the engineer is usually interested not only in the stresses produced by the loads acting but also in the deflections produced by these loads. In many cases, furthermore, it is specified that the maximum deflection shall not exceed a certain small portion of the span.

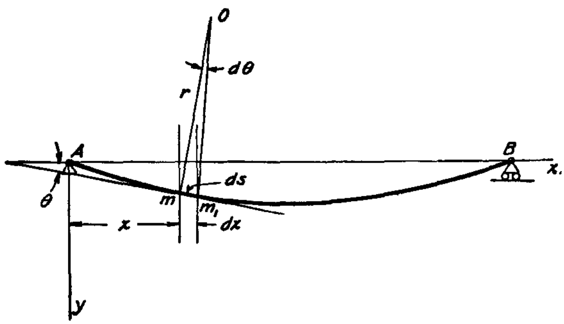

FIG. 114.

Let the curve *AmB* in Fig. 114 represent the shape of the axis of the beam after bending. This curve is called the *deflection curve*. To derive the differential equation of this curve we take the coordinate axes as shown in the figure and assume that the curvature of the deflection curve at any point depends only on the magnitude of the bending moment M at that point.[1] In such a case the relation between the curvature and the moment is the same as in the case of pure bending (see equation (56)), and we obtain

$$\frac{1}{r} = \frac{M}{EI_z}. \tag{a}$$

[1] The effect of shearing force on the curvature will be discussed later (see art. 39). It will be shown that this effect is usually small and can be neglected.

To derive an expression for the relation between the curvature and the shape of the curve, we shall consider two adjacent points m and m_1, ds apart on the deflection curve. If the angle which the tangent at m makes with the x axis is denoted by θ, the angle between the normals to the curve at m and m_1 is $d\theta$. The intersection point O of these normals gives the center of curvature and defines the length r of the radius of the curvature. Then

$$ds = rd\theta \qquad \text{and} \qquad \frac{1}{r} = \left|\frac{d\theta}{ds}\right|, \tag{b}$$

the bars indicating that we consider here only the numerical value of the curvature. Regarding the sign, it should be noted that the bending moment is taken positive in equation (a) if it produces upward concavity (see p. 71). Hence the curvature is positive when the center of curvature is above the curve as in Fig. 114. However, it is easy to see that for such a curvature the angle θ decreases as the point m moves along the curve from A to B. Hence, to a positive increment ds corresponds a negative $d\theta$. Thus to have the proper sign equation (b) must be written in the form

$$\frac{1}{r} = -\frac{d\theta}{ds}. \tag{c}$$

In practical applications only very small deflections of beams are allowable, and the deflection curves are very flat. In such cases we can assume with sufficient accuracy that

$$ds \approx dx \qquad \text{and} \qquad \theta \approx \tan\theta = dy/dx. \tag{d}$$

Substituting these approximate values for ds and θ in equation (c) we obtain

$$\frac{1}{r} = -\frac{d^2y}{dx^2}. \tag{e}$$

Equation (a) thus becomes

$$EI_z\frac{d^2y}{dx^2} = -M. \tag{79}$$

This is the differential equation of the deflection curve which must be integrated in each particular case to find deflections of beams.

It should be noted that the sign in equation (79) depends upon the direction of the coordinate axes. For example, if we take y positive upwards, it is necessary to put

$$\theta \approx -\, dy/dx$$

in place of equation (d); and we obtain plus instead of minus on the right side of equation (79).

In the case of very slender bars, in which the deflection may be large, it is not permissible to use the simplifications (d); and we must have recourse to the exact expression

$$\theta = \arc\tan\left(\frac{dy}{dx}\right).$$

Then

$$\frac{1}{r} = -\frac{d\theta}{ds} = -\frac{d\,\arc\tan\left(\dfrac{dy}{dx}\right)}{dx}\frac{dx}{ds}$$

$$= -\frac{\dfrac{d^2y}{dx^2}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}. \qquad (f)$$

Comparing this result with equation (e), it can be concluded that the simplifications shown in equation (d) are equivalent to assuming that the quantity $(dy/dx)^2$ in the denominator of the exact formula (f) is small in comparison with unity and can therefore be neglected.[2]

By differentiating equation (79) with respect to x and using equations (50) and (51), we obtain

$$EI_z\frac{d^3y}{dx^3} = -V$$

[2] The exact expression (f) for the curvature was used by the first investigators of the deflection curves. It was used, for example, by L. Euler in his famous work on "Elastic Curves," an English translation of which was published in "Isis," No. 58 (vol. XX, 1) November, 1933.

and

$$EI_z \frac{d^4y}{dx^4} = q. \tag{80}$$

The last equation is sometimes used in considering the deflection of beams under a distributed load.

32. Bending of a Uniformly Loaded Beam.—In the case of a *simply supported beam* which is uniformly loaded, Fig. 63, the bending moment at a cross section mn, a distance x from the left support, is

$$M = \frac{qlx}{2} - \frac{qx^2}{2}$$

and the differential equation (79) becomes

$$EI_z \frac{d^2y}{dx^2} = -\frac{qlx}{2} + \frac{qx^2}{2}.$$

Multiplying both sides by dx and integrating, we obtain

$$EI_z \frac{dy}{dx} = -\frac{qlx^2}{4} + \frac{qx^3}{6} + C \tag{a}$$

where C is the constant of integration which is to be adjusted to satisfy the conditions of this particular problem. To this end, we note that as a result of symmetry the slope at the middle of the span is zero. Setting $dy/dx = 0$ when $x = l/2$, we thus obtain

$$C = \frac{ql^3}{24}$$

and equation (a) becomes

$$EI_z \frac{dy}{dx} = -\frac{qlx^2}{4} + \frac{qx^3}{6} + \frac{ql^3}{24}. \tag{b}$$

A second integration gives

$$EI_z y = -\frac{qlx^3}{12} + \frac{qx^4}{24} + \frac{ql^3x}{24} + C_1. \tag{c}$$

The new constant of integration C_1 is determined from the condition that the deflection at the supports is zero. Substituting $y = 0$ and $x = 0$ into equation (c) we find $C_1 = 0$. Equation (c) then becomes

$$y = \frac{q}{24EI_z}(l^3x - 2lx^3 + x^4). \tag{81}$$

This is the deflection curve of a simply supported and uniformly loaded beam. The maximum deflection of this beam is evidently at the middle of the span. Substituting $x = l/2$ in equation (81) we thus find

$$y_{\max} = \frac{5}{384}\frac{ql^4}{EI_z}. \tag{82}$$

The maximum slope occurs at the left end of the beam where, by substituting $x = 0$ in equation (b), we obtain

$$\left(\frac{dy}{dx}\right)_{\max} = \frac{ql^3}{24EI_z}. \tag{83}$$

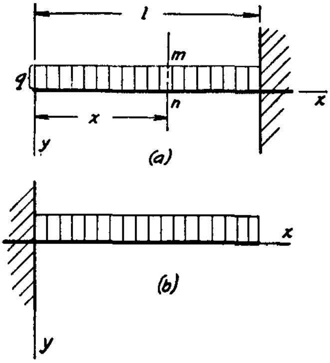

FIG. 115.

In the case of a *uniformly loaded cantilever beam*, Fig. 115a, the bending moment at a cross section mn a distance x from the left end is

$$M = -\frac{qx^2}{2},$$

and equation (79) becomes

$$EI_z\frac{d^2y}{dx^2} = \frac{qx^2}{2}.$$

The first integration gives

$$EI_z\frac{dy}{dx} = \frac{qx^3}{6} + C. \tag{a}$$

The constant of integration is found from the condition that the slope at the built-in end is zero, that is $dy/dx = 0$ for $x = l$. Substituting these values in equation (a) we find

$$C = -\frac{ql^3}{6}.$$

The second integration gives

$$EI_z y = \frac{qx^4}{24} - \frac{ql^3x}{6} + C_1. \qquad (b)$$

The constant C_1 is found from the condition that the deflection vanishes at the built-in end. Thus, by substituting $x = l$, $y = 0$ in equation (b), we obtain

$$C_1 = \frac{ql^4}{8}.$$

Substituting this value in equation (b), we find

$$y = \frac{q}{24EI_z}(x^4 - 4l^3x + 3l^4). \qquad (84)$$

This equation defines the deflection curve of the uniformly loaded cantilever.

If the left end, instead of the right end, is built in, as in Fig. 115b, the deflection curve is evidently obtained by substituting $l - x$ instead of x in equation (84). In this way we find

$$y = \frac{q}{24EI_z}(x^4 - 4lx^3 + 6l^2x^2). \qquad (85)$$

Problems

1. A simply supported and uniformly loaded wooden beam of square cross section has a span $l = 10$ ft. Find the maximum deflection if $(\sigma_x)_{\max} = 1{,}000$ lbs. per sq. in., $E = 1.5 \times 10^6$ lbs. per sq. in. and $q = 400$ lbs. per ft.

2. Find the depth of a uniformly loaded and simply supported steel I beam having a span of 10 ft., if the maximum bending stress is 16,000 lbs. per sq. in. and the maximum deflection $\delta = 0.1$ in.

Answer: $h = 16$ in.

3. A uniformly loaded cantilever beam of a span $l = 10$ ft. has a deflection at the end equal to $0.01l$. What is the slope of the deflection curve at the end?

4. What is the length of a uniformly loaded cantilever beam if its deflection at the free end is 1 in. and the slope of the deflection curve at the same point is 0.01?

5. A uniformly loaded steel I beam supported at the ends has a deflection at the middle of $\delta = 5/16$ in. while the slope of the deflection curve at the end $\theta = 0.01$. Find the depth h of the beam if the maximum bending stress is $\sigma = 18{,}000$ lbs. per sq. in.

Solution:

We use the known formulas

$$\delta = \frac{5}{384}\frac{ql^4}{EI}, \qquad \theta = \frac{ql^3}{24EI}, \qquad \sigma_{\max} = \frac{ql^2}{8} \times \frac{h}{2I}.$$

From the first two formulas we find

$$\frac{5}{16}l = \frac{\delta}{\theta} = \frac{5}{16} \times 100 \text{ in.} \qquad \text{and} \qquad l = 100 \text{ in.}$$

The second formula then gives

$$\frac{ql^2}{8I} = \frac{3E\theta}{l} = \frac{3 \times 30 \times 10^6 \times 0.01}{100}.$$

Substituting this in the third formula, we obtain

$$h = \frac{2 \times 18{,}000 \times 100}{3 \times 30 \times 10^6 \times 0.01} = 4 \text{ in.}$$

33. Deflection of a Simply Supported Beam Loaded with a Concentrated Load.—In this case there are two different expressions for the bending moment (see p. 75) corresponding to the two portions of the beam, Fig. 116. Equation (79) for the deflection curve must therefore be written for each portion. In this way we obtain

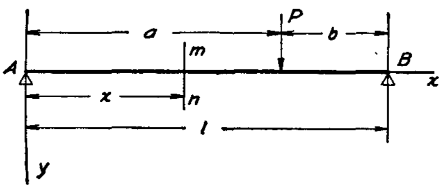

Fig. 116.

$$EI_z \frac{d^2y}{dx^2} = -\frac{Pb}{l}x \qquad \text{for} \qquad x \leqq a$$

and

$$EI_z \frac{d^2y}{dx^2} = -\frac{Pb}{l}x + P(x-a) \qquad \text{for} \qquad x \geqq a.$$

By integrating these equations we obtain

$$\left.\begin{aligned} EI_z\frac{dy}{dx} &= -\frac{Pbx^2}{2l} + C \qquad \text{for} \qquad x \leqq a \\ \text{and} \\ EI_z\frac{dy}{dx} &= -\frac{Pbx^2}{2l} + \frac{P(x-a)^2}{2} + C_1 \qquad \text{for} \qquad x \geqq a. \end{aligned}\right\} \quad (a)$$

Since the two branches of the deflection curve must have a common tangent at the point of application of the load P, the above expressions (a) for the slope must be equal for $x = a$. From this we conclude that the constants of integration are equal, i.e., $C = C_1$. Performing the second integration and substituting C for C_1, we obtain

$$\left.\begin{aligned} EI_zy &= -\frac{Pbx^3}{6l} + Cx + C_2 \qquad \text{for} \qquad x \leqq a \\ \text{and} \\ EI_zy &= -\frac{Pbx^3}{6l} + \frac{P(x-a)^3}{6} + Cx + C_3 \qquad \text{for} \qquad x \geqq a. \end{aligned}\right\} \quad (b)$$

Since the two branches of the deflection curve have a common deflection at the point of application of the load, the two expressions (b) must be identical for $x = a$. From this it follows that $C_2 = C_3$. Finally we need to determine only two constants C and C_2, for which determination we have two conditions, namely that the deflection at each of the two ends of the beam is zero. Substituting $x = 0$ and $y = 0$ in the first of expressions (b), we find

$$C_2 = C_3 = 0. \qquad (c)$$

Substituting $y = 0$ and $x = l$ in the second of expressions (b) we obtain

$$C = \frac{Pbl}{6} - \frac{Pb^3}{6l} = \frac{Pb(l^2 - b^2)}{6l}. \qquad (d)$$

Substituting the values (c) and (d) of the constants into equations (b) for the deflection curve, we obtain

$$EI_z y = \frac{Pbx}{6l}(l^2 - b^2 - x^2) \qquad \text{for} \qquad x \leqq a \tag{86}$$

and

$$EI_z y = \frac{Pbx}{6l}(l^2 - b^2 - x^2) + \frac{P(x-a)^3}{6} \qquad \text{for} \qquad x \geqq a. \tag{87}$$

The first of these equations gives the deflections for the left portion of the beam and the second gives the deflections for the right portion.

Substituting the value (d) into equations (a) we obtain

$$\left.\begin{aligned} EI_z \frac{dy}{dx} &= \frac{Pb}{6l}(l^2 - b^2 - 3x^2) \qquad \text{for} \qquad x \leqq a \\ &\text{and} \\ EI_z \frac{dy}{dx} &= \frac{Pb}{6l}(l^2 - b^2 - 3x^2) + \frac{P(x-a)^2}{2} \qquad \text{for} \qquad x \geqq a. \end{aligned}\right\} \tag{e}$$

From these equations the slope at any point of the deflection curve can readily be calculated. Often we need the values of the slopes at the ends of the beam. Substituting $x = 0$ in the first of equations (e), $x = l$ in the second, and denoting the slopes at the corresponding ends by θ_1 and θ_2 we obtain [3]

$$\theta_1 = \left(\frac{dy}{dx}\right)_{x=0} = \frac{Pb(l^2 - b^2)}{6lEI_z}, \tag{88}$$

$$\theta_2 = \left(\frac{dy}{dx}\right)_{x=l} = -\frac{Pab(l + a)}{6lEI_z}. \tag{89}$$

The maximum deflection occurs at the point where the tangent to the deflection curve is horizontal. If $a > b$ as in Fig. 116, the maximum deflection is evidently in the left portion of the beam. We can find the position of this point by equating the first of the expressions (e) to zero to obtain

$$l^2 - b^2 - 3x^2 = 0,$$

[3] For flat curves, which we have in most cases, the slopes θ_1 and θ_2 can be taken numerically equal to the angles of rotation of the ends of the beam during bending, the slopes being taken positive when the rotation is clockwise.

from which

$$x = \frac{\sqrt{l^2 - b^2}}{\sqrt{3}}. \qquad (f)$$

This is the distance from the left support to the point of maximum deflection. To find the maximum deflection itself we substitute expression (f) in equation (86), which gives

$$y_{\max} = \frac{Pb(l^2 - b^2)^{3/2}}{9\sqrt{3}lEI_z}. \qquad (g)$$

If the load P is applied at the middle of the span the maximum deflection is evidently at the middle also. Its magnitude is obtained by substituting $b = l/2$ in equation (g), which gives

$$(y)_{\substack{x=l/2 \\ a=b}} = \frac{Pl^3}{48EI_z}. \qquad (90)$$

From equation (f) it can be concluded that in the case of one concentrated force the maximum deflection is always near the middle of the beam. When $b = l/2$ it is at the middle; in the limiting case, when b is very small and P is at the support, the distance x as given by equation (f) is $l/\sqrt{3}$, and the point of maximum deflection is only a distance

$$\frac{l}{\sqrt{3}} - \frac{l}{2} = 0.077l$$

from the middle. Due to this fact the deflection at the middle is a close approximation to the maximum deflection. To obtain the deflection at the middle we substitute $x = l/2$ in equation (86), which gives

$$(y)_{\substack{x=l/2 \\ a>b}} = \frac{Pb}{48EI_z}(3l^2 - 4b^2). \qquad (91)$$

The difference of the deflections (g) and (91) in the most unfavorable case, that is when b approaches zero, is only about 2.5 per cent of the maximum deflection.

Problems

1. Find the position of the load P, Fig. 116, if the ratio of the numerical values of the slopes at the ends of the beam is $|\theta_1/\theta_2| = \frac{3}{4}$.

2. Find the difference between the maximum deflection and the deflection at the middle of the beam in Fig. 116 if $b = 2a$.

3. Find the maximum deflection of the beam shown in Fig. 116 if AB is an American Standard I beam, 8 in. in depth and 5.34 sq. in. in cross-sectional area, and $a = 12$ ft., $b = 8$ ft., and $P = 2000$ lbs.

4. What will be the maximum deflection if the I beam of the previous problem is replaced by a wooden beam having a cross section 10 in. by 10 in. The modulus of elasticity for wood can be taken as $E = 1.5 \times 10^6$ lbs. per sq. in.

34. Determination of Deflections by the use of the Bending Moment Diagram; Method of Superposition.—In the preceding articles it was shown how the deflection curve of a beam can be determined by integration of the differential equation (79). In many cases, however, especially if we need the deflection in a prescribed point rather than the general equation of the deflection curve, the calculation can be considerably simplified by the use of the bending moment diagram as will be described in the following discussion.[4]

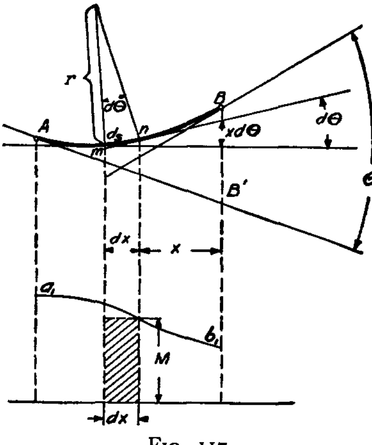

FIG. 117.

In Fig. 117 AB represents a portion of a deflection curve and a_1b_1 the corresponding portion of the bending moment

[4] The use of the bending moment diagram in calculating deflections of beams was developed by O. Mohr, see Zeitschr. d. Architekten und Ingenieur-Vereins zu Hannover, p. 10, 1868. See also O. Mohr, Abhandlungen aus dem Gebiete der Technischen Mechanik, p. 294, Berlin, 1906. A similar method was developed independently of O. Mohr by Prof. C. E. Green, University of Michigan, 1874.

diagram. Two adjacent cross sections of the beam at distance ds apart will intersect after bending, at an angle $d\theta$, and, from eq. (56),

$$d\theta = \frac{1}{r} ds = \frac{M}{EI_z} ds.$$

For beams used in structures, the curvature is very small, and we may use dx for ds. Then

$$d\theta = \frac{1}{EI_z} (Mdx). \tag{a}$$

Graphically interpreted, this means that the elemental angle $d\theta$ between two consecutive radii or two consecutive tangents to the deflection curve equals the shaded elemental area Mdx of the bending moment diagram, divided by the flexural rigidity.[5] This being so for each element, the angle θ between the tangents at A and B will be obtained by summarizing such elements as given by eq. (a). Then

$$\theta = \int_A^B \frac{1}{EI_z} Mdx, \tag{92}$$

that is, the angle between the tangents at two points A and B of the deflection curve equals the area of the bending moment diagram between the corresponding verticals, divided by the flexural rigidity of the beam.

Let us consider now the distance of the point B from the tangent AB' at point A. Recalling that a deflection curve is a flat curve, the above distance can be measured along the vertical BB'. The contribution made to this distance by the bending of an element mn of the beam and included between the two consecutive tangents at m and n is equal to

$$xd\theta = x \frac{Mdx}{EI_z} \cdot$$

Interpreted graphically this is $1/EI_z$ (moment of shaded area Mdx with respect to the vertical through B). Integration

[5] By way of dimensional check: $d\theta$ is in radians, i.e., a pure number, Mdx in inch lbs. $\times$ inches, EI_z in lbs. per sq. in. $\times$ (inches)^4.

gives the total deflection BB':

$$\overline{BB'} = \delta = \int_A^B \frac{1}{EI_z} xMdx, \tag{93}$$

that is, the distance of B from the tangent at A is equal to the moment with respect to the vertical through B of the area of the bending moment diagram between A and B, divided by the flexural rigidity EI_z. By using eqs. (92) and (93) the slope of the deflection curve and the magnitude of deflection at any cross section of the beam can easily be calculated. This method of calculating deflections is called *Area-Moment Method.*

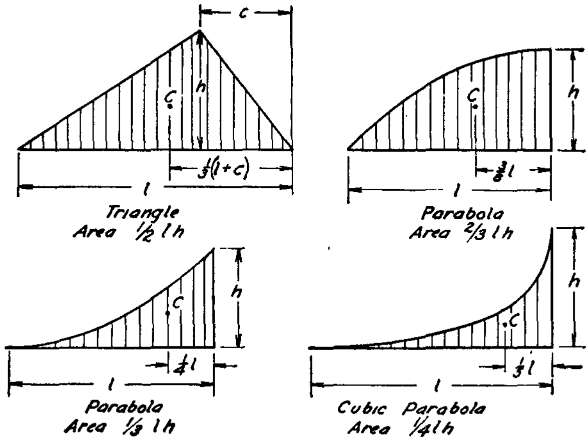

FIG. 118.

It should be noted that the deflection of a beam of a given flexural rigidity (see equation (93)) is entirely defined by the bending moment diagram. From this fact a very important conclusion can be drawn. It will be appreciated from the definition of the bending moment (art. 19) that the bending moment produced at a cross section mn of a beam by several simultaneously acting loads is equal to the sum of the moments produced at the same cross section by the individual loads acting separately. On the basis of this conclusion, together with equation (93), it can be stated that the deflection produced at a point of a beam by a system of simultaneously acting loads can be obtained by summing up the deflections produced at that point by the individual loads. For example, if the deflection curve produced by a single load (equations (86) and (87)) is known, the deflection produced by several loads is obtained by simple summation. This method of calculating deflections will be called *the method of superposition* in the subsequent discussion.

The calculation of the integrals in equations (92) and (93) can often be simplified by the use of known formulas concerning areas and centroids. Several formulas which are often encountered in applications are given in Fig. 118.

35. Deflection of a Cantilever Beam by the Area-Moment Method.—For the case of a cantilever beam with a concentrated load at the end (Fig. 119, *a*) the bending moment diagram is shown in Fig. 119 (*b*). Since a tangent at the built-in end A remains fixed, the distances of points of the deflection curve from this tangent are actual deflections. The angle θ_b which the tangent to the deflection curve at B makes with the tangent at A is called the *angular deflection* of B with respect to A. Then from eq. (92) [6]

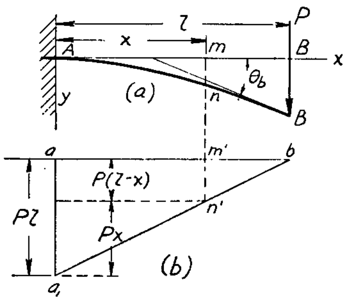

FIG. 119.

$$\theta_b = Pl \times \frac{l}{2} \times \frac{1}{EI_z} = \frac{Pl^2}{2EI_z}. \tag{94}$$

The deflection δ is calculated from eq. (93) as the moment of the area aba_1 about the vertical through b divided by EI_z. Then

$$\delta = Pl \times \frac{l}{2} \times \frac{2}{3} l \times \frac{1}{EI_z} = \frac{Pl^3}{3EI_z}. \tag{95}$$

For any cross section such as mn, the angular deflection from

[6] The numerical value of the angular deflection is calculated. The direction of the deflection is readily seen from the loading conditions.

A is the area $m'n'aa_1$ of Fig. 119 (*b*), divided by EI_z. In the case of flat curves such as deflection curves of beams, angular deflection can be taken equal to the slope of the curve and we obtain

$$\theta = \frac{dy}{dx} = \frac{Pl^2}{2EI_z}\left[1 - \frac{(l-x)^2}{l^2}\right]. \tag{96}$$

The deflection y at the same cross section is the moment of the area $m'n'aa_1$ about $m'n'$ divided by EI_z (see eq. 93). Separating this area into the rectangle and the triangle indicated in the figure, this is

$$y = \frac{1}{EI_z}\left[P(l-x)\frac{x^2}{2} + \frac{Px^2}{2}\frac{2x}{3}\right] = \frac{P}{EI_z}\left(\frac{lx^2}{2} - \frac{x^3}{6}\right). \tag{97}$$

For a cantilever with a concentrated load P at a cross section a distance c from the support (Fig. 120, *a*) the bending moment diagram is shown in Fig. 120 (*b*). The slope and the deflection for any section to the left of the point of application of the load are determined from eqs. (96) and (97) with c in place of l. For any cross section to the right of the load the bending moment and the curvature are zero; hence this portion of the beam remains straight. The slope is constant and equal to the slope at D, i.e., from eq. (94), $Pc^2/2EI_z$. The deflection at any cross section mn is the moment of the area of the triangle aa_1d about the vertical $m'n'$ divided by EI_z, which gives

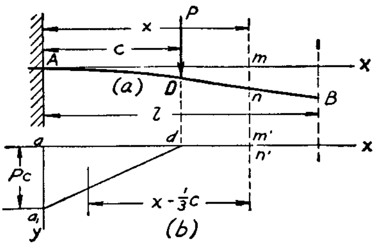

FIG. 120.

$$y = \frac{1}{EI_z}\frac{Pc^2}{2}\left(x - \frac{1}{3}c\right). \tag{98}$$

In the case of a cantilever with a uniform load of intensity q (Fig. 121, *a*) the bending moment at any cross section

mn distant x_1 from the built-in end is

$$M = -\frac{q(l - x_1)^2}{2}.$$

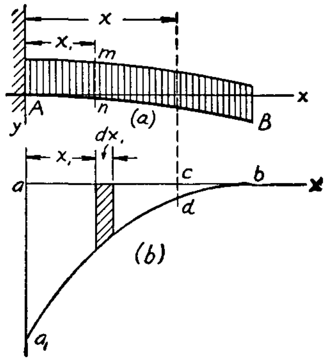

FIG. 121.

The slope at any cross section a distance x from the support is, from eq. (92),

$$\theta = \frac{dy}{dx} = \frac{1}{EI_z}\int_0^x \frac{q(l - x_1)^2}{2}\,dx_1$$

$$= \frac{q}{2EI_z}\left(l^2x - lx^2 + \frac{x^3}{3}\right). \quad (99)$$

The slope at the end is obtained by substituting l for x in the above equation, giving

$$\left(\frac{dy}{dx}\right)_{x=l} = \frac{ql^3}{6EI_z}. \quad (100)$$

The deflection at any section a distance x from the built-in end is the moment of the area aa_1cd about the vertical cd divided by EI_z (Fig. 121, b). The moment of the element of this area, shown shaded, is

$$(x - x_1)\,\frac{q(l - x_1)^2}{2}\,dx_1$$

and the total moment is the integral of this with respect to x_1 from $x_1 = 0$ to $x_1 = x$. Hence

$$y = \frac{1}{EI_z}\frac{q}{2}\int_0^x (x - x_1)(l - x_1)^2 dx_1.$$

The deflection at any point a distance x from the support is then, after integration,

$$y = \frac{q}{2EI_z}\left(\frac{l^2x^2}{2} - \frac{lx^3}{3} + \frac{x^4}{12}\right). \quad (101)$$

For the deflection at the end $x = l$,

$$\delta = (y)_{x=l} = \frac{ql^4}{8EI_z}. \tag{102}$$

The same problem can be solved by using the method of superposition. The uniform load can be considered as a system of infinitesimal loads qdc as indicated by the shaded area in Fig. 122. The deflection produced at the cross section mn by each elemental load qdc to its left can be found from eq. (98) by substituting qdc for P. The deflection y_1 produced by the total load to the left of mn is the summation of the deflections produced by all such elemental loads with c varying from $c = 0$ to $c = x$:

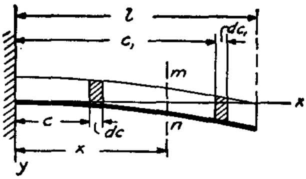

FIG. 122

$$y_1 = \frac{1}{EI_z}\int_0^x \frac{qc^2}{2}(x - \tfrac{1}{3}c)dc = \frac{q}{2EI_z}\frac{x^4}{4}.$$

The deflection produced at the cross section mn by an elemental load qdc_1 to its right is found from eq. (97) by substituting qdc_1 for P and c_1 for l. The deflection y_2 produced at mn by the total load to the right is the summation of the deflections due to all such elemental load, with c_1 varying from $c_1 = x$ to $c_1 = l$:

$$y_2 = \frac{1}{EI_z}\int_x^l q\left(\frac{c_1x^2}{2} - \frac{x^3}{6}\right)dc_1 = \frac{q}{2EI_z}\left(-\frac{x^4}{6} + \frac{x^2l^2}{2} - \frac{lx^2}{3}\right).$$

Then the total deflection at the section mn is

$$y = y_1 + y_2 = \frac{q}{2EI_z}\left(\frac{l^2x^2}{2} - \frac{lx^3}{3} + \frac{x^4}{12}\right),$$

which agrees with eq. (101) found above.

Problems

1. Determine the deflection and the slope of the cantilever beam in problem 9, p. 108.

Solution.

$$\delta = \frac{Pl^3}{3EI_z} + \frac{ql^4}{8EI_z}.$$

2. Determine the deflection of the top of the pillar represented in Fig. 94.

Solution. The bending moment at any cross section mn, a distance x from the top, is

$$M = -\frac{Wx^3}{3l^2},$$

where $W = \frac{1}{2}dl^2 \times 62.4$ lbs. is the total hydrostatic pressure transmitted to one pillar. Using eq. (93), the deflection of the top of the pillar is

$$\delta = \frac{W}{EI_z}\int_0^l \frac{x^4dx}{3l^2} = \frac{Wl^3}{15EI_z} = \frac{3\times6^2\times62.4\times6^3\times12^3\times12}{2\times15\times1.5\times10^6\times9.9^4} = 0.070 \text{ in.}$$

3. Determine the deflection and the slope at the end of a cantilever bent by a couple M (Fig. 123).

Answer.

$$(y)_{x=l} = -\frac{Ml^2}{2EI_z}; \qquad \left(\frac{dy}{dx}\right)_{x=l} = -\frac{Ml}{EI_z}.$$

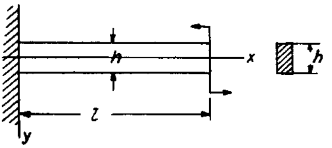

FIG. 123.

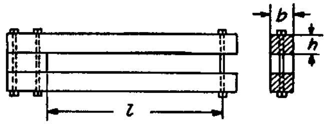

FIG. 124.

4. Two wooden rectangular beams clamped at the left end (Fig. 124) are bent by tightening the bolt at the right end. Determine the diameter d of the bolt to make the factors of safety for the wooden beams and for the steel bolt the same. The length of the beams $l = 3$ feet, the depth $h = 8$ in., the width $b = 6$ in., working stress for steel $\sigma_w = 12{,}000$ lbs. per sq. in., for wood $\sigma_w = 1{,}200$ lbs. per sq. in. Determine the deflection of the beams when the tensile stress in the bolt is 12,000 lbs. per sq. in.

Solution. If P is the force in the bolt, the equation for determining the diameter d will be

$$\frac{4P}{\pi d^2} : \frac{6Pl}{bh^2} = \frac{12{,}000}{1{,}200} = 10,$$

from which

$$d = 0.476 \text{ in.} \qquad \text{and} \qquad P = 12{,}000 \times \frac{\pi d^2}{4} = 2{,}130 \text{ lbs.}$$

Then from eq. (95), by taking $E = 1.5 \times 10^6$ lbs. per sq. in. the deflection $\delta = 0.0864$ in.

5. What is the ratio of the deflections at the ends of the cantilevers shown in Fig. 125 if the intensity of uniform load is the same in both cases?

Answer. 7 : 41.

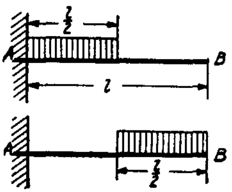

FIG. 125.

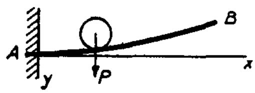

FIG. 126.

6. What must be the equation of the axis of the curved bar AB before it is bent if the load P, moving along the bar, remains always on the same level (Fig. 126)?

Answer.

$$y = -\frac{Px^3}{3EI_z}.$$

7. Determine the safe deflection of the beam shown in Fig. 123 when the working stress σ_w is given. Determine this also for a cantilever loaded at the end (Fig. 119).

Answer.

$$(1)\ \delta = \frac{\sigma_w l^2}{Eh}, \qquad (2)\ \delta = \frac{2}{3}\frac{\sigma_w l^2}{Eh}.$$

FIG. 127.

8. A circular disc N of radius R (Fig. 127) produces on a thin steel strip of thickness h an attraction of q lbs. per sq. in. uniformly distributed. Determine the length l of the unsupported part AC of the strip and the maximum stress in it if $h = 0.01$ inch, $R = 3$ in., and $q = 15$ lbs. per sq. in.

Solution. The length of the unsupported part of the strip can be determined from the condition that at the point C the curvature produced by the uniformly distributed load q must be equal to

$1/R$. Therefore

$$\frac{ql^2}{2} = \frac{EI_z}{R},$$

from which

$$l = \sqrt{\frac{2EI_z}{qR}} = 1/3 \text{ in.}$$

The maximum stress is determined from the equation $\sigma_{max} = Eh/2R = 50{,}000$ lbs. per sq. in.

9. Determine the deflections of the cantilever beams shown in Fig. 68, assuming that the material is steel, the depth of each beam is 10 in., and the maximum bending stress is 16,000 lbs. per sq. in.

36. Deflection of a Simply Supported Beam by the Area-Moment Method.—Let us consider the case of a simply supported beam with a load P applied at point F, Fig. 128. The bending moment diagram is the triangle $a_1b_1f_1$.

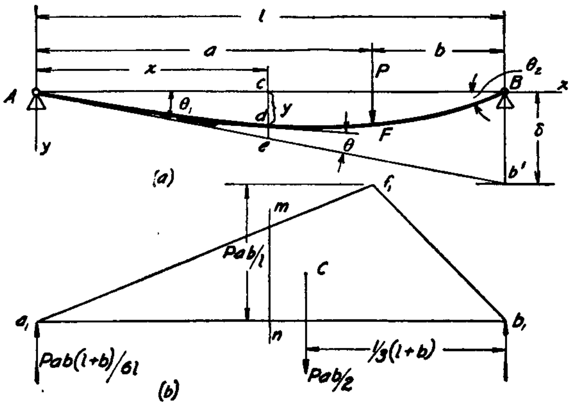

FIG. 128.

Its area is $Pab/2$, and its centroid C is at distance $(l + b)/3$ from the vertical Bb_1. The distance δ from the end B to the line Ab' which is tangent to the deflection curve at A is obtained from equation (93) and is

$$\delta = \frac{1}{EI_z}\frac{Pab}{2} \times \frac{l + b}{3} = \frac{Pab(l + b)}{6EI_z}.$$

By using this value the slope θ_1 at the left end of the beam is found to be

$$\theta_1 = \frac{\delta}{l} = \frac{Pab(l + b)}{6lEI_z}, \tag{a}$$

which coincides with previously obtained formula (88).[7] A simple interpretation of the formula (*a*) is obtained if we consider a_1b_1 as a simply supported beam, carrying the triangular load represented by the triangle $a_1f_1b_1$. The reaction at the left support a_1 of this imaginary beam is evidently

$$R = \frac{Pab}{2} \times \frac{l + b}{3} \times \frac{1}{l} = \frac{Pab(l + b)}{6l}.$$

By comparing this with formula (*a*), it can be concluded that the slope θ_1 is equal to the reaction at the corresponding support of the imaginary beam divided by the flexural rigidity of the real beam. The slope θ_2 at the right end of the beam can be obtained in a similar way; to get the correct sign for θ_2 the reaction at the right end must be taken with minus sign, which represents the shearing force at the right end of the imaginary beam. The imaginary beam a_1b_1 which carries the fictitious load represented by the area of the bending moment diagram is called a *conjugate beam*. It can thus be concluded that the numerical values of the slopes at the ends of a simply supported beam can be obtained by dividing the reactions at the ends of the conjugate beam by the flexural rigidity EI_z. This conclusion, which was derived for the case of a single load, holds for any transverse loading, since, as has been shown (p. 146) the moments and the deflections in the case of several loads can be obtained by the superposition of the moments and deflections due to single loads.

To calculate the slope at any point *d* of the deflection curve, Fig. 128, it is necessary to subtract the angle θ between the tangents at *A* and at *d* from the angle θ_1 at the support. Using equation (92) for the calculation of the angle θ, we obtain

[7] Note that $a = l - b$.

$$\frac{dy}{dx} = \theta_1 - \theta = \frac{1}{EI_z}(R - \Delta a_1mn).$$

The first term in the parenthesis is the reaction at the left support of the conjugate beam a_1b_1 and the second is the load on the conjugate beam to the left of the cross section mn. The expression in the parenthesis therefore represents the shearing force at the cross section mn of the conjugate beam. Consequently the slope of the actual beam at a point d can be obtained by dividing the shearing force at the corresponding cross section of the conjugate beam by the flexural rigidity EI_z.

Considering next the deflection y at a point d, it may be seen from Fig. 128 that

$$y = \overline{ce} - \overline{de}. \tag{b}$$

From the triangle Ace we obtain the relation

$$\overline{ce} = \theta_1 x = \frac{Rx}{EI_z} \tag{c}$$

where R is the reaction at the left support of the conjugate beam. The second term on the right side of equation (b) represents the distance of the point d of the deflection curve from the tangent Ae and is obtained from equation (93) as

$$\overline{de} = \frac{1}{EI_z}\text{area } \Delta a_1mn \times \frac{x}{3}. \tag{d}$$

Substituting expressions (c) and (d) in equation (b), we obtain

$$y = \frac{1}{EI_z}\left(Rx - \Delta a_1mn \times \frac{x}{3}\right). \tag{e}$$

The expression in parenthesis is seen to be the bending moment at the cross section mn of the conjugate beam. Thus the deflection at any point of a simply supported beam is obtained by dividing the bending moment at the corresponding cross section of the conjugate beam by the flexural rigidity EI_z. Substituting the actual value of R in equation (e) and noting that

$$\text{area } \Delta a_1mn = \frac{Pbx^2}{2l},$$

we obtain

$$y = \frac{1}{EI_z}\left[\frac{Pabx(l+b)}{6l} - \frac{Pbx^3}{6l}\right] = \frac{Pbx}{6lEI_z}(l^2 - b^2 - x^2).$$

This checks with equation (86), which was previously obtained by integration of the differential equation of the deflection curve. The deflection for a point to the right of the load P can be calculated in a similar manner. The result will, of course, be the same as equation (87).

Having the deflection curve produced by a single load P, the deflection curve produced by any system of transverse concentrated loads can readily be obtained by employing the method of superposition. It is simply necessary to use equations (86) and (87) for each individual load.

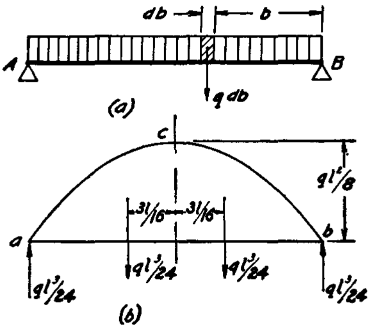

FIG. 129.

The same method is also applicable to the case of a distributed load. As an example we shall take the case of a simply supported beam under a uniformly distributed load, Fig. 129, and calculate the slopes at the ends and the deflection at the middle. From equation (a) the increment of slope $d\theta_1$ produced at the left end of the beam by the element of load qdb shown in the Fig. 129 is

$$d\theta_1 = \frac{qabdb(l+b)}{6lEI_z} = \frac{qb(l^2 - b^2)db}{6lEI_z}.$$

The slope θ_1 produced by the total load is then the summation of the increments of slope produced by all the elements qdb from $b = 0$ to $b = l$. Thus

$$\theta_1 = \int_0^l \frac{qb(l^2 - b^2)db}{6lEI_z} = \frac{ql^3}{24EI_z}. \qquad (f)$$

The deflection at the middle is obtained from equation (91), which was derived on the assumption that the load is to the right of the middle. Any element of load qdb to the right of the middle produces at the middle a deflection

$$(dy)_{x=l/2} = \frac{qbdb}{48EI_z}(3l^2 - 4b^2).$$

Summing up the deflections produced by all such elements of load to the right of the middle, and noting that the load on the left half of the beam produces the same deflection at the middle as the load on the right half, we obtain for the total deflection

$$\delta = (y)_{x=l/2} = 2\int_0^{l/2} \frac{qbdb}{48EI_z}(3l^2 - 4b^2) = \frac{5}{384}\frac{ql^4}{EI_z}. \quad (g)$$

The results (f) and (g) coincide with formulas (83) and (82) previously obtained by integration of the differential equation of the deflection curve.

The same results are readily obtained by considering the conjugate beam ab, Fig. 129b, loaded by the parabolic segment acb, which is the bending moment diagram in this case. The total fictitious load on the conjugate beam is

$$\frac{2}{3} \times \frac{ql^2}{8} \times l,$$

and each reaction is equal to $ql^3/24$. The slope (f) is then obtained by dividing this reaction by EI_z. To calculate the deflection at the middle we find the bending moment at the middle $\bar{d}$ of the conjugate beam, which is

$$\frac{ql^3}{24}\left(\frac{l}{2} - \frac{3l}{16}\right) = \frac{5ql^4}{384}.$$

The deflection (g) is then obtained by dividing this moment by EI_z.

The method of superposition is especially useful if the distributed load covers only a part of the span as in Fig. 130. Using the expression developed above for

FIG. 130.

$(dy)_{x=l/2}$, the deflection produced at the middle by the load to the right of the middle is

$$\delta_1 = \int_d^{l/2} \frac{qbdb}{48EI_z}(3l^2 - 4b^2).$$

The load to the left of the middle produces the deflection

$$\delta_2 = \int_c^{l/2} \frac{qbdb}{48EI_z}(3l^2 - 4b^2).$$

The total deflection at the middle is therefore

$$\delta = \delta_1 + \delta_2 = \int_d^{l/2} \frac{qbdb}{48EI_z}(3l^2 - 4b^2) + \int_c^{l/2} \frac{qbdb}{48EI_z}(3l^2 - 4b^2).$$

In the case of a simply supported beam AB with a couple M acting at the end, Fig. 131, the bending moment diagram is a triangle abd, as shown in Fig. 131b. Considering ab as the conjugate beam, the total fictitious load is $Ml/2$. The reactions at the ends of the conjugate beam are thus $Ml/6$ and $Ml/3$. Hence the numerical values of the slopes at the ends of the actual beam are

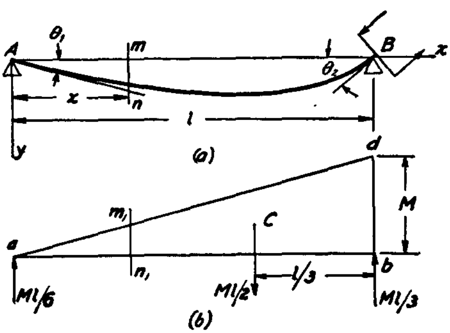

FIG. 131.

$$\theta_1 = \frac{Ml}{6EI_z} \tag{103}$$

and

$$\theta_2 = \frac{Ml}{3EI_z}. \tag{104}$$

The sign of the slope at the right end is of course negative. The deflection at a cross section mn of the beam is obtained by dividing the bending moment at the corresponding cross section m_1n_1 of the conjugate beam by EI_z, which gives

$$y = \frac{1}{EI_z}\left(\frac{Ml}{6}x - \frac{Ml}{2}\cdot\frac{x^2}{l^2}\cdot\frac{x}{3}\right) = \frac{Mlx}{6EI_z}\left(1 - \frac{x^2}{l^2}\right). \quad (105)$$

Problems

1. Determine the angles at the ends and the deflection under the loads and at the middle of the beam shown in Fig. 132.

Solution. The conjugate beam will be loaded by the trapezoid $adeb$, the area of which is $Pc(l - c)$. The angles at the ends are

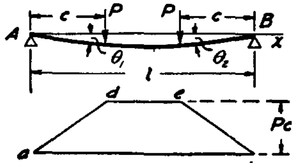

FIG. 132.

$$\theta_1 = \theta_2 = \frac{1}{EI_z}\frac{Pc(l-c)}{2}$$

The deflection under the loads is

$$(y)_{x=c} = \frac{1}{EI_z}\left[\frac{Pc^2(l-c)}{2} - \frac{Pc^2}{2}\cdot\frac{c}{3}\right] = \frac{Pc^2}{EI_z}\left(\frac{l}{2} - \frac{2}{3}c\right).$$

The deflection at the middle, from eq. (91), is

$$(y)_{x=l/2} = \frac{Pc}{24EI_z}(3l^2 - 4c^2).$$

2. Determine the slope at the ends of the beam shown in Fig. 88.

Answer.

$$\left(\frac{dy}{dx}\right)_{x=0} = \frac{7}{180}\frac{Wl^2}{EI_z}; \qquad \left(\frac{dy}{dx}\right)_{x=l} = -\frac{8}{180}\frac{Wl^2}{EI_z}.$$

3. Determine the deflection at the middle of the beam AB, shown in Fig. 133, when $I_z = 91.4$ in.^{4}, $q = 500$ lbs. per foot, $l = 24$ feet, $a = 12$ feet, $b = 8$ feet. $E = 30 \times 10^6$ lbs. per sq. in.

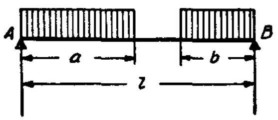

FIG. 133.

Solution. Due to the fact that $a = l/2$ the deflection produced at the middle by the load acting on the left half of the beam, from eq. (82), is

$$(y_1)_{x=l/2} = \frac{1}{2}\frac{5}{384}\frac{ql^4}{EI_z}.$$

The deflection produced at the middle by the load on the right half of the beam is

$$(y_2)_{x=l/2} = \int_0^b \frac{qcdc}{48EI_z}(3l^2 - 4c^2) = \frac{25}{48 \times 162} \times \frac{ql^4}{EI_z}.$$

The total deflection is

$$(y)_{x=l/2} = (y_1)_{x=l/2} + (y_2)_{x=l/2} = \left(\frac{1}{2}\frac{5}{384} + \frac{25}{48 \times 162}\right)\frac{ql^4}{EI_z} = 1.02 \text{ in.}$$

4. Determine the deflection at the middle of the beam shown in Fig. 91 when the load is in a position to produce the maximum bending moment.

Suggestion. The deflection can be obtained by using eq. (91) together with the method of superposition and substituting $b = l/2 - d/4$ in this equation for one load and $b = l/2 - \frac{3}{4}d$ for the other.

5. Determine the deflections at the middle and the angles of rotation of the ends of the beams shown in Figs. 67 (*b*) and 67 (*e*). Assume in these calculations a standard I beam of 8 in. depth and 5.97 sq. in. area, $E = 30 \times 10^6$ lbs. per sq. in.

6. Determine the angles θ_1 and θ_2 and the deflection at any cross section mn of a beam simply supported at the ends and bent by a couple Pc (Fig. 134).

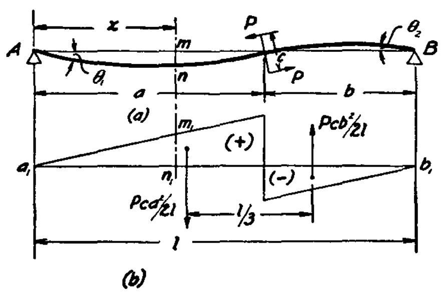

FIG. 134.

Solution. The loading of the conjugate beam is indicated in Fig. 134 (*b*). The reactions at a_1 and b_1 are

$$R_a = \frac{1}{l}\left[\frac{Pca^2}{2l}\left(b + \frac{a}{3}\right) - \frac{Pcb^2}{2l}\cdot\frac{2}{3}b\right];$$

$$R_b = \frac{1}{l}\left[\frac{Pca^2}{2l}\frac{2a}{3} - \frac{Pcb^2}{2l}\left(a + \frac{b}{3}\right)\right].$$

Therefore

$$\theta_1 = \frac{Pc}{2l^2EI_z}\left[a^2\left(b + \frac{a}{3}\right) - \frac{2}{3}b^3\right] = \frac{Pc}{2lEI_z}\left(\frac{l^2}{3} - b^2\right);$$

$$\theta_2 = -\frac{Pc}{2l^2EI_z}\left[\frac{2}{3}a^3 - b^2\left(a + \frac{b}{3}\right)\right] = \frac{Pc}{2lEI_z}\left(\frac{l^2}{3} - a^2\right).$$

If $a = b = l/2$, we obtain

$$\theta_1 = \theta_2 = \frac{Pcl}{24EI_z}.$$

If $a > l/\sqrt{3}$ the angle θ_2 changes its sign. The bending moment at the cross section m_1n_1 of the conjugate beam is

$$A_1x - \frac{Pca^2}{2l}\frac{x^2}{a^2}\frac{x}{3} = \frac{Pcx}{2l^2}\left[a^2\left(b + \frac{a}{3}\right) - \frac{2}{3}b^3\right] - \frac{Pcx^3}{6l}.$$

Therefore the deflection curve for the left part of the actual beam is

$$y = \frac{Pcx}{2l^2EI_z}\left[a^2\left(b + \frac{a}{3}\right) - \frac{2}{3}b^3\right] - \frac{Pcx^3}{6lEI_z}.$$

7. A beam with supported ends is bent by two couples M_1 and M_2, applied at the ends (Fig. 135). Determine the angles of rotation of the ends and the position of the cross section in which the deflection is a maximum.

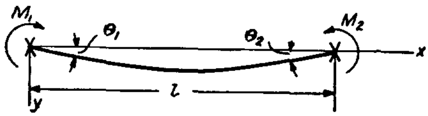

Fig. 135.

Solution. The absolute values of the angles are

$$\theta_1 = \frac{M_1l}{3EI_z} + \frac{M_2l}{6EI_z}; \qquad \theta_2 = \frac{M_2l}{3EI_z} + \frac{M_1l}{6EI_z}.$$

The deflection curve, by using eq. (105), is

$$y = \frac{M_1l(l - x)}{6EI_z}\left[1 - \left(\frac{l - x}{l}\right)^2\right] + \frac{M_2lx}{6EI_z}\left(1 - \frac{x^2}{l^2}\right).$$

The position of maximum deflection can be found from this equation by equating the first derivative to zero.

8. A beam is bent by two couples as shown in Fig. 136. Determine the ratio $M_1 : M_2$ if the point of inflection is at a distance $l/3$ from the left support.

Answer. $M_2 = 2M_1$.

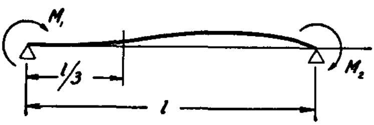

Fig. 136.

9. Two planks of different thicknesses h_1 and h_2, resting one on the other, support a uniformly distributed load as shown in Fig. 137. Determine the ratio of the maximum stresses occurring in each.

Solution. Both planks have the same deflection curves and curvature; hence their bending moments are in the same ratio as the moments of inertia of their cross sections, i.e., in the ratio $h_1^3 : h_2^3$. The section moduli are in the ratio $h_1^2 : h_2^2$; hence the maximum stresses are in the ratio $h_1 : h_2$.

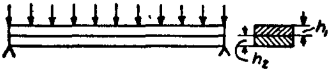

FIG. 137.

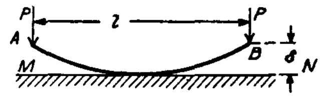

FIG. 138.

10. A steel bar AB has such an initial curvature that after being straightened by the forces P (Fig. 138) it produces a uniformly distributed pressure along the length of the rigid plane surface MN. Determine the forces P necessary to straighten the bar and the maximum stress produced in it if $l = 20$ in., $\delta = 0.1$ in., and the cross section of the bar is a square having 1 in. sides.

Solution. To obtain a uniformly distributed pressure, the initial curvature of the bar must be the same as the deflection curve of a simply supported beam carrying a uniformly distributed load of intensity $2P/l$. Then we obtain

$$M_{\max} = \frac{2P}{l}\frac{l^2}{8} = \frac{Pl}{4}, \tag{a}$$

$$\delta = \frac{5}{384} \times \frac{2P}{l} \times \frac{l^4}{EI_z}. \tag{b}$$

The maximum stress will be

$$\sigma_{\max} = \frac{M_{\max}}{Z} = \frac{Plh}{8I_z}. \tag{c}$$

Now from (b) and (c)

$$\sigma_{\max} = \frac{24E\delta h}{5l^2} = \frac{24 \times 30 \times 10^6 \times 0.1 \times 1}{5 \times 20^2} = 36{,}000 \text{ lbs. per sq. in.}$$

and from (c)

$$P = 1{,}200 \text{ lbs.}$$

37. Deflection of Beams with Overhangs.—A beam with an overhang can be divided into two parts: the one between the supports which is to be treated as a beam with supported ends and the overhang which is to be treated as a cantilever. As an illustration, we consider the bending of a beam with an overhang under the action of a uniformly distributed load q (Fig. 139). The beam is divided into the parts AB and BC and the action of the overhang on the portion of the beam between the supports is replaced by a shearing force qa and a couple $M = qa^2/2$. We find that the shearing force is directly transmitted to the support and that only the couple $qa^2/2$ need be considered. Then the deflection at any cross section between the supports is obtained by subtracting the deflection produced by the couple $qa^2/2$ from the deflections produced by the uniform load q (Fig. 139, *b*). Using eqs. (81) and (105), we obtain

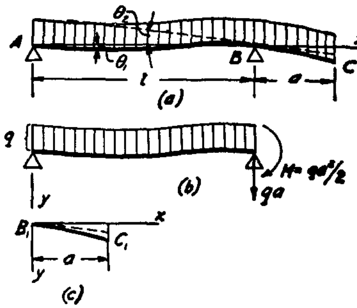

FIG. 139.

$$y = \frac{q}{24EI_z}(l^3x - 2lx^3 + x^4) - \frac{qa^2lx}{12EI_z}\left(1 - \frac{x^2}{l^2}\right).$$

The angle of rotation of the cross section at B is obtained by using eqs. (83) and (104), from which, by considering rotation positive when in the clockwise direction, we have

$$\theta_2 = \frac{qa^2l}{6EI_z} - \frac{ql^3}{24EI_z}.$$

The deflection at any cross section of the overhang (Fig. 139, *c*) is now obtained by superposing the deflection of a cantilever (eq. 101) on the deflection

$$\theta_2 x = \left(\frac{qa^2l}{6EI_z} - \frac{ql^3}{24EI_z}\right)x$$

due to the rotation of the cross section B.

Problems

1. Determine the deflection and the slope at the end C of the beam shown in Fig. 141*a*.

Answer. Deflection $= \dfrac{Pa^2(l+a)}{3EI_z}$; Slope $= \dfrac{Pa(2l+3a)}{6EI_z}$.

2. For the beam shown in Fig. 140 determine the deflection at the end C and also that at the midpoint between the supports.

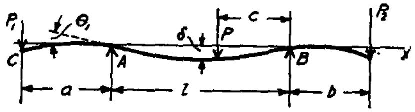

FIG. 140.

Solution. That part of the beam between the supports will be in the condition of a beam loaded by the force P and by the couples P_1a and P_2b at the supports. By using eqs. (91) and (105), pp. 143, 159, and the method of superposition, the deflection at the middle is

$$\delta = \frac{Pc}{48EI_z}(3l^2 - 4c^2) - \frac{P_1al^2}{16EI_z} - \frac{P_2bl^2}{16EI_z}.$$

The angle θ_1 at the support A is obtained from eqs. (88), (103) and (104) on pp. 142, 158,

$$\theta_1 = \frac{Pc(l^2 - c^2)}{6lEI_z} - \frac{P_1al}{3EI_z} - \frac{P_2bl}{6EI_z}.$$

From eq. (95) the deflection at the end C is

$$\frac{P_1a^3}{3EI_z} - a\theta_1.$$

3. A beam with an overhang is bent in one case by the force P at the end (Fig. 141, a), and in another case by the same force applied at the middle of the span (Fig. 141, b). Prove that the deflection at the point D in the first case is equal to the deflection at the end C in the second case.

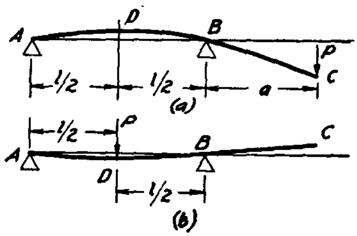

FIG. 141.

Answer. In each case the deflection is

$$\frac{Pl^2a}{16EI_z}.$$

4. A beam of length l with two equal overhangs is loaded by two equal forces P at the ends (Fig. 142). Determine the ratio x/l

at which (1) the deflection at the middle is equal to the deflection at either end, (2) the deflection at the middle has its maximum value.

Answer. (1) $x = 0.152l$; (2) $x = l/6$.

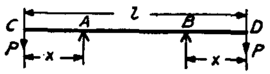

FIG. 142.

FIG. 143.

5. A wooden beam of circular cross section supported at C, with the end attached at A, carries a uniformly distributed load q on the overhang CD (Fig. 143). Determine the diameter of the cross section and the deflection at D if $l = 3$ feet, $a = 6$ feet, $q = 300$ lbs. per foot, $\sigma_w = 1{,}200$ lbs. per sq. in.

Solution. The diameter d is found from equation

$$\frac{qa^2}{2} : \frac{\pi d^3}{32} = \sigma_w.$$

Then the deflection at the end D is

$$\delta = \frac{qa^4}{8EI_z} + \frac{qa^3 l}{6EI_z}.$$

6. A beam of length l carries a uniformly distributed load of intensity q (Fig. 144). Determine the length of overhangs to make the numerical maximum value of the bending moment as small as possible. Determine the deflection at the middle for this condition.

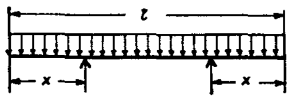

FIG. 144.

Solution. Making the numerical values of the bending moments at the middle and at the supports equal we obtain

$$x = 0.207l.$$

The deflection at the middle is determined from the equation

$$\delta = \frac{5}{384} \cdot \frac{q(l - 2x)^4}{EI_z} - \frac{qx^2(l - 2x)^2}{16EI_z},$$

in which the first term on the right side represents the deflection produced by the load between the support (eq. 82) and the second, the deflection produced by the load on the overhangs (eq. 105).

7. Determine the deflections at the ends of the overhangs for the beams represented in Fig. 74 *a*, *b*, *c*. Assume a standard I beam of 8 in. depth and 5.97 sq. in. area. $E = 30 \times 10^6$ lbs. per sq. in.

38. The Deflection of Beams When the Loads Are Not Parallel to One of the Two Principal Planes of Bending.—Let us consider first a simple example of a cantilever, whose cross section has two axes of symmetry (Fig. 145). The load P at the end is perpendicular to the axis of the beam and makes an angle α with the principal axis y of the cross section. In calculating the stresses and deflections of the beam the method of superposition will be used. The load P will be resolved into two components $P\cos\alpha$ and $P\sin\alpha$ in the directions of the two principal axes of the cross section. The deflection produced by each of these components can easily be obtained by using the theory of bending in the plane of symmetry. Then the resultant deflection is obtained by superposition. The absolute values of the two components of the bending moment at any cross section mn of the cantilever are $M_z = P\cos\alpha(l - x)$ about the z axis, and $M_y = P\sin\alpha(l - x)$ about the y axis. From the directions of the two components and of the axes y and z, it will be seen that the moment M_z produces compression in points with a positive y and M_y produces compression in points with a positive z. Then the normal stress σ_x at any point of the cross section (y, z) is obtained by adding together the stresses produced by M_z and M_y separately. We thus arrive at the equation:

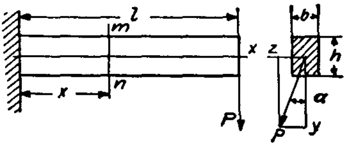

FIG. 145.

$$\sigma_x = -\frac{P\cos\alpha(l - x)y}{I_z} - \frac{P\sin\alpha(l - x)z}{I_y}$$

$$= -P(l - x)\left[\frac{y\cos\alpha}{I_z} + \frac{z\sin\alpha}{I_y}\right]. \qquad (a)$$

The neutral axis is found by taking points with such coordinates that the expression in brackets in eq. (a) equals zero. The equation of the neutral axis is therefore

$$\frac{y\cos\alpha}{I_z} + \frac{z\sin\alpha}{I_y} = 0. \qquad (b)$$

This is an axis through the centroid of the cross section making an angle β with the z axis (see Fig. 146) such that, from eq. (*b*),

$$\tan \beta = -\frac{y}{z} = \tan \alpha \frac{I_z}{I_y}. \qquad (c)$$

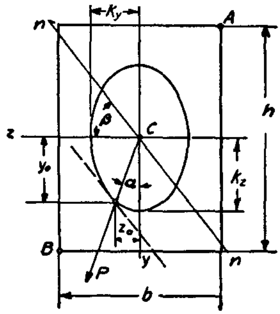

Fig. 146.

It will be seen that, in general, $\tan \beta$ is not equal to $\tan \alpha$; hence the neutral axis *nn* is not perpendicular to the plane of the bending forces and the plane of the deflection curve, which is perpendicular to *nn*, does not coincide with the plane of the bending forces. These two planes coincide only when $\tan \alpha = 0$ or ∞ or $I_z = I_y$. In the first two cases the plane of the bending forces coincides with one of the principal planes of bending. In the last case the ellipse of inertia becomes a circle since the two principal moments of inertia are equal and any two perpendicular directions may be taken as the two principal axes of the cross section. When I_z/I_y is a large number, i.e., when the rigidity of the beam in the *xy* plane is much larger than that in the *xz* plane, $\tan\beta$ becomes large in comparison with $\tan\alpha$ and when angle α is small, angle β will approach 90° and the neutral axis will approach the vertical axis. The deflection will be principally in the *xz* plane, i.e., there is a tendency to deflect in the plane of greatest flexibility. This can be demonstrated in a very simple manner on a thin rule. The slightest deviation of the bending force from the plane of greatest rigidity results in a bending in the perpendicular direction. This can be shown also by resolving the force P (Fig. 145) into two components and calculating deflections produced by each component. If the flexural rigidity of the cantilever in the horizontal plane is very small in comparison with the rigidity in vertical plane, a small horizontal component may produce a much greater horizontal deflection than the deflection in the vertical plane; hence the resultant deflection will be principally in the plane of greatest flexibility. It is interesting to note that the neutral axis nn is parallel to the tangent drawn to the ellipse of inertia at the point of intersection of this ellipse with the direction of the force P. This can be proved as follows. The equation of the ellipse is

$$\frac{y^2}{k_z^2} + \frac{z^2}{k_y^2} = 1$$

and the equation of the tangent at the point with the coordinates y_0 and z_0 (Fig. 146) is

$$\frac{yy_0}{k_z^2} + \frac{zz_0}{k_y^2} = 1.$$

The tangent of the angle between the z axis and this tangent will be

$$\frac{z_0}{y_0} \cdot \frac{k_z^2}{k_y^2} = \tan \alpha \frac{I_z}{I_y} = \tan \beta.$$

When the direction of the neutral axis is determined the points of maximum normal stress will be those most distant from it. In our case the maximum tension will be at point A and maximum compression at point B. Substituting in eq. (a) $x = 0$; $y = -(h/2)$; $z = -(b/2)$, we obtain

$$(\sigma_x)_{\max} = Pl\left(\frac{h \cos \alpha}{2I_z} + \frac{b \sin \alpha}{2I_y}\right). \qquad (d)$$

The compressive stresses at the point B will have the same magnitude. The method developed above for the case of a cantilever with two planes of symmetry and loaded at the end can be applied also to beams supported at the ends and loaded by several loads. Resolving each force into two components parallel to the two axes of symmetry of the cross section, the problem is reduced to two simple problems of bending of a beam in the two principal planes. The resultant deflections will be obtained by superposing the two deflections in the principal planes.

Problems

1. A cantilever beam of Z section is loaded at the end by a vertical load $P = 400$ lbs. (Fig. 147). Determine the maximum normal stress σ_x and the vertical and horizontal components of the deflection at the end. The dimensions are as indicated in the figure. $\alpha = 17^\circ\ 20'$, principal moments of inertia

$$I_{z_1} = 60.3 \text{ in.}^4; \qquad I_{y_1} = 3.54 \text{ in.}^4$$

Answer. $(\sigma_x)_{\max} = 6{,}420$ lbs. per sq. in. at B; $\delta_{\text{vert.}} = 0.178$ in.; $\delta_{\text{horiz.}} = 0.336$ in.

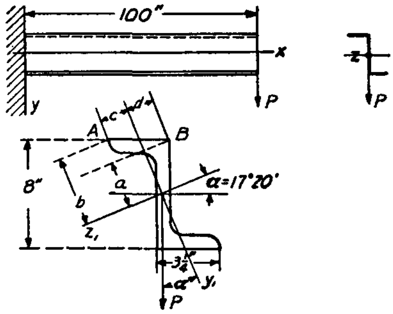

FIG. 147.

2. A cantilever of rectangular cross section is bent by a force P at the end. What curve will be described by the loaded end, when the angle α (Fig. 145) varies from 0 to 2π?

Answer. The curve will be an ellipse with the semi axes

$$\frac{Pl^3}{3EI_z} \quad \text{and} \quad \frac{Pl^3}{3EI_y}.$$

3. A wooden beam of rectangular cross section carries a uniformly distributed load of intensity q and is supported at the ends in the position shown in Fig. 148. Determine the maximum normal stress and the vertical deflection at the middle if the length of the beam $l = 10$ feet, $q = 200$ lbs. per foot, $h = 8$ in., $b = 6$ in., $\tan \alpha = 1/3$.

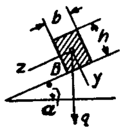

FIG. 148.

Solution. The maximum bending moment will be at the middle

$$M_{\max} = \frac{ql^2}{8} = \frac{200 \times 10^2}{8} = 2{,}500 \text{ lbs. feet} = 30{,}000 \text{ lbs. ins.}$$

The components of the bending moment in the principal planes are $M_z = M_{\max} \cos \alpha = 30{,}000 \times 0.949 = 28{,}500$ lbs. ins. and $M_y = M_{\max} \sin \alpha = 30{,}000 \times 0.316 = 9{,}480$ lbs. in. The maximum stress at the point B is

$$(\sigma_x)_{\max} = \frac{6 \times 28{,}500}{bh^2} + \frac{6 \times 9{,}480}{hb^2} = 643 \text{ lbs. per sq. in.}$$

The deflections at the middle in the two principal planes are

$$\delta_y = \frac{5}{384} \frac{ql^4 \cos \alpha}{EI_z} \quad \text{and} \quad \delta_z = \frac{5}{384} \frac{ql^4 \sin \alpha}{EI_y}.$$

The vertical deflection at the middle is

$$\delta = \delta_y \cos\alpha + \delta_z \sin\alpha = \frac{5}{384}\frac{ql^4}{EI_z}\left(\cos^2\alpha + \frac{I_z}{I_y}\sin^2\alpha\right)$$

$$= 0.117 \text{ in.} \times 1.08 = .126 \text{ in.}$$

4. Solve the above problem if the distance between the supports is 6 feet and the beam has two equal overhangs each 2 feet long.

39. Effect of Shearing Force on the Deflection of Beams.—In the previous discussion (see p. 134) only the action of the bending moment in causing deflection was considered. An additional deflection will be produced by the shearing force, in the form of a mutual sliding of adjacent cross sections along each other. As a result of the non-uniform distribution of the shearing stresses, the cross sections, previously plane, become curved as in Fig. 149, which shows the bending due to shear alone.[8] The elements of the cross sections at the centroids remain vertical and slide along one another; therefore the slope of the deflection curve, due to the shear alone, is equal at each cross section to the shearing strain at the centroid of this cross section. Denoting by y_1 the deflections due to shear, we obtain for any cross section the following expression for the slope:

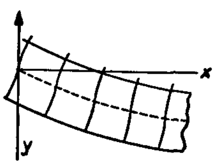

FIG. 149.

$$\frac{dy_1}{dx} = \frac{(\tau_{yx})_{y=0}}{G} = \frac{\alpha V}{AG}, \qquad (a)$$

in which V/A is the average shearing stress τ_{yx}, G is the modulus in shear and α is a numerical factor with which the average shearing stress must be multiplied in order to obtain the shearing stress at the centroid of the cross sections. For a rectangular cross-section $\alpha = 3/2$ (see eq. 66, p. 114); for a circular cross-section $\alpha = 4/3$ (see eq. 68, p. 118). With a continuous load on the beam, the shearing force V is a continuous function which may be differentiated with respect to x. The curvature caused by the shear alone is then

$$\frac{d^2y_1}{dx^2} = \frac{\alpha}{AG}\frac{dV}{dx} = -\frac{\alpha}{AG}q.$$

[8] The deformation produced by the bending moment and consisting of a mutual rotation of adjacent cross sections is removed.

The sum of this and the curvature produced by the bending moment (see eq. 79) gives

$$\frac{d^2y}{dx^2} = -\frac{1}{EI_z}\left(M + \frac{\alpha EI_z}{AG}q\right). \tag{106}$$

This equation must be used instead of eq. (79) to determine deflections in all cases in which the effect of the shearing force should be taken into consideration.[9] Knowing M and q as functions of x, eq. (106) can be easily integrated in the same manner as has been shown in article 32.

The conjugate beam method (see p. 154) may also be applied to good advantage in this case by taking as ordinates of the imaginary load diagram

$$M + \alpha\frac{EI_z}{AG}q, \tag{b}$$

instead of only M.

Let us consider, for example, the case of a simply supported beam with a uniform load (Fig. 150). The bending moment at any section x is

$$M = \frac{ql}{2}x - \frac{qx^2}{2}. \tag{c}$$

The load on the conjugate beam consists of two parts: (1) that represented by the first term of (b) and given by the parabolic bending moment diagram (Fig. 150, b) and (2) that represented by the second term of (b), which is $\alpha(EI_z/AG)q$. Since q is constant, the second term is a uniformly distributed load shown in Fig. 150 (c).

The additional deflection at any section due to the shearing force is the bending moment produced at this section of the conjugate beam by such a load, divided by EI_z. At the middle of the beam the additional deflection is consequently

$$\delta_1 = \frac{1}{EI_z}\left(\alpha\frac{EI_z}{AG}q\right)\frac{l^2}{8} = \frac{\alpha l^2 q}{8AG}.$$

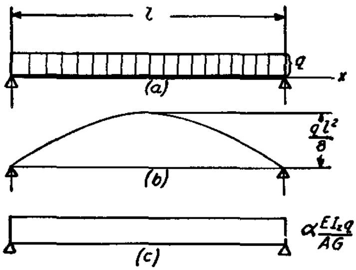

FIG. 150.

Adding this to the deflection due to the bending moment (see eq.

[9] Another way of determining additional deflection due to shear is discussed on page 298.

82, p. 138), we obtain the total deflection

$$\delta = \frac{5}{384}\frac{ql^4}{EI_z} + \frac{\alpha l^2 q}{8AG} = \frac{5}{384}\frac{ql^4}{EI_z}\left(1 + \frac{48\alpha}{5}\frac{k_z^2}{l^2}\frac{E}{G}\right), \qquad (d)$$

in which $k_z = \sqrt{I_z/A}$ is the radius of gyration of the cross section with respect to the z axis.

For a rectangular cross-section of depth h, $k_z^2 = \frac{1}{12}h^2$, $\alpha = 3/2$. Putting $E/G = 2(1 + \mu) = 2.6$, we obtain from (d)

$$\delta = \frac{5}{384}\frac{ql^4}{EI_z}\left(1 + 3.12\frac{h^2}{l^2}\right).$$

It may be seen that for $l/h = 10$ the effect of the shearing force on the deflection is about 3 per cent. As the ratio l/h decreases this effect increases.

The factor α is usually larger than 2 for I beams and when they are short the effect of the shearing force may be comparatively great. Using eq. (70) and Fig. 106, we have

$$\frac{\alpha V}{A} = \frac{V}{b_1 I_z}\left[\frac{bh^2}{8} - \frac{h_1^2}{8}(b - b_1)\right],$$

from which

$$\alpha = \frac{A}{b_1 I_z}\left[\frac{bh^2}{8} - \frac{h_1^2}{8}(b - b_1)\right]. \qquad (e)$$

For example, suppose $h = 24$ in., $A = 31.0$ sq. in., $I_z = 2{,}810$ in.^{4}, the thickness of the web $b_1 = 5/8$ in., $l = 6h$. Then eq. (e) gives $\alpha = 2.42$. Substituting in eq. (d), we find

$$\delta = \frac{5}{384}\frac{ql^4}{EI_z}\left(1 + \frac{48}{5} \times 2.42 \times \frac{2{,}810}{31 \times 144^2} \times 2.6\right) = 1.265\frac{5ql^4}{384EI_z}.$$

The additional deflection due to shear in this case is equal to 26.5 per cent of the deflection produced by the bending moment and must therefore be considered.

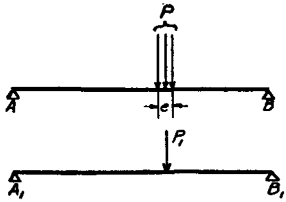

FIG. 151.

In the case of a concentrated load P (Fig. 151) such a load can be considered as the limiting case of a load distributed over a very short portion e of the beam. The amount of the imaginary loading P_1 on the conjugate beam A_1B_1, corresponding to the second term in expression (b), will be

$$P_1 = \alpha \frac{EI_z}{AG} P. \tag{f}$$

The additional deflection due to shearing forces is obtained by dividing by EI_z the bending moment produced in the conjugate beam by the imaginary concentrated load given by eq. (*f*). For instance, for central loading of a beam the bending moment at the middle of the conjugate beam produced by the load (*f*) will be $\alpha(EI_z/AG)Pl/4$ and the additional deflection at the middle due to shearing forces is

$$\delta_1 = \frac{\alpha}{AG}\frac{Pl}{4}. \tag{g}$$

Adding this to the deflection produced by the bending moment alone (see eq. 90, p. 143), the following expression for the complete deflection is obtained:

$$\delta = \frac{Pl^3}{48EI_z} + \frac{\alpha}{AG}\cdot\frac{Pl}{4} = \frac{Pl^3}{48EI_z}\left(1 + \frac{12\alpha k_z^2}{l^2}\frac{E}{G}\right).$$

For a beam of rectangular cross-section of depth h we have

$$\frac{k_z^2}{l^2} = \frac{h^2}{12l^2}; \qquad \alpha = \frac{3}{2},$$

and we obtain

$$\delta = \frac{Pl^3}{48EI_z}\left(1 + 3.90\frac{h^2}{l^2}\right). \tag{h}$$

For $h/l = 1/10$ the additional effect of the shearing force is about 4 per cent.

It has been assumed throughout the above discussion that the cross sections of the beam can warp freely as shown in Fig. 149. The uniformly loaded beam is one case in which this condition is approximately satisfied. The shearing force at the middle of such a beam is zero and there will be no warping here. The warping increases gradually with the shearing force as we proceed along the beam to the left or to the right of the middle. The condition of symmetry of deformation with respect to the middle section is therefore satisfied. Consider now bending by a concentrated load at the middle. From the condition of symmetry the middle cross section of the beam must remain plane. At the same time, adjacent cross sections to the right and to the left of the load carry a shearing force equal to $P/2$, and warping of cross sections caused by these shearing forces should take place. From the condition of continuity of deformation, however, there can be no abrupt change from a plane middle section to warped adjacent sections. There must be a continuous increase in warping as we proceed along the beam in either direction from the middle, and only at some distance from the load can the warping be such as a shearing force $P/2$ produces under conditions of freedom in warping. From this discussion it must be concluded that in the neighborhood of the middle cross section the stress distribution will not be that predicted by the elementary theory of bending (see p. 113). Warping will be partially prevented and the additional deflection due to shearing forces will be somewhat less than that found above (see eq. *g*). A more detailed investigation[10] shows that in the case of a concentrated load at the middle the deflection at the middle is

$$\delta = \frac{Pl^3}{48EI}\left[1 + 2.85\frac{h^2}{l^2} - 0.84\left(\frac{h}{l}\right)^3\right]. \tag{k}$$

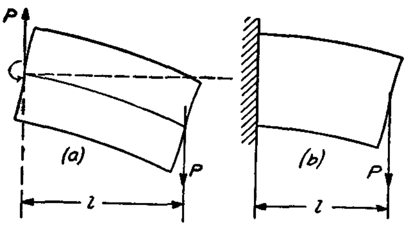

FIG. 152.

We have an analogous condition also in the case of a cantilever beam. If the built-in cross section can warp freely as shown in Fig. 152 (*a*), the conditions will be as assumed in the derivation of eq. (*h*). The deflection of a cantilever of rectangular cross section will be obtained by substituting l for $l/2$ and P for $P/2$ in this equation, giving

$$\delta = \frac{Pl^3}{3EI}\left(1 + 0.98\frac{h^2}{l^2}\right). \tag{l}$$

When the built-in cross section is completely prevented from warping (Fig. 152, *b*), the conditions will be the same as assumed in the derivation of eq. (*k*) and the deflection will be

$$\delta = \frac{Pl^3}{3EI}\left[1 + 0.71\frac{h^2}{l^2} - 0.10\left(\frac{h}{l}\right)^3\right], \tag{m}$$

which is less than the deflection given by (*l*).

[10] See L. N. G. Filon, Phil. Trans. Roy. Soc. (A), Vol. 201, p. 63, 1903, and S. Timoshenko, Phil. Mag., Vol. 47, p. 1095, 1924. See also Th. v. Kármán, Scripta Universitatis atque Bibliothecae Hierosolmitanarum, 1923, and writer's "Theory of Elasticity," p. 95, 1934.

CHAPTER VI

STATICALLY INDETERMINATE PROBLEMS IN BENDING

40. Redundant Constraints.—In our previous discussion three types of beams have been considered: (*a*) a cantilever beam, (*b*) a beam supported at the ends and (*c*) a beam with overhangs. In all three cases the reactions at the supports can be determined from the fundamental equations of statics; hence the problems are *statically determinate*. We will now consider problems on the bending of beams in which the equations of statics are not sufficient to determine all the reactive forces at the supports, so that additional equations, based on a consideration of the deflection of the beams, must be derived. Such problems are called *statically indeterminate*.

Let us consider the various types of supports which a beam may have. The support represented in Fig. 153 (*a*) is

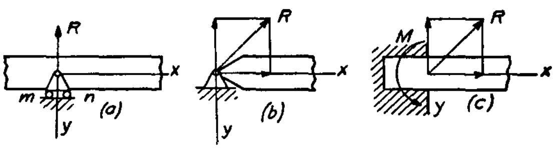

FIG. 153.

called a *hinged movable support*. Neglecting the friction in the hinge and in the rollers, it is evident that in this type of support the reaction must act through the center of the hinge and must be perpendicular to the plane *mn* on which the rollers are moving. Hence we know the point of application of the reaction and its direction. There remains only one unknown element, the magnitude of the reaction.

In Fig. 153 (*b*) a *hinged immovable* support is shown. In this case the reaction must go through the center of the hinge but it may have any direction in the plane of the figure. We have two unknowns to determine from the equations of statics, the direction of the reaction and its magnitude, or, if we like, the vertical and horizontal components of the reaction.

In Fig. 153 (*c*) a *built-in end* is represented. In this case not only the direction and the magnitude of the reaction are unknown, but also the point of application. The reactive forces distributed over the built-in section can be replaced by a force R applied at the centroid of the cross section and a couple M. We then have three unknowns to determine from the equations of statics, the two components of the reactive force R and the magnitude of the couple M.

For beams loaded by transverse loads in one plane we have, for determining the reactions at the supports, the three equations of statics, namely,

$$\Sigma X = 0; \qquad \Sigma Y = 0; \qquad \Sigma M = 0. \tag{a}$$

If the beam is supported so that there are only three unknown *reactive elements*, they can be determined from eqs. (*a*); hence the problem is statically determinate. These three elements are just sufficient to assure the immovability of the beam. When the number of reactive elements is larger than three, we say there are *redundant constraints* and the problem is statically indeterminate.

A cantilever is supported at the built-in end. In this case, as was explained above, the number of unknown reactive elements is three and they can be determined from the equations of statics. For beams supported at both ends and beams with overhangs it is usually assumed that one of the supports is an immovable and the other a movable hinge. In such a case we have again three unknown reactive elements, which can be determined from the equations of statics.

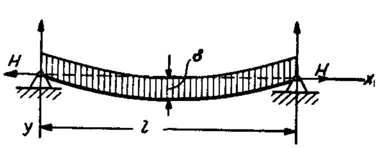

FIG. 154.

If the beam has immovable hinges at both ends (Fig. 154), the problem becomes statically indeterminate. At each end we have two unknown elements, the two components of the corresponding reaction, and for determining these four unknowns we have only the three equations (*a*). Hence we have one redundant constraint and a consideration of the deformation of the beam becomes necessary to determine the reactions. The vertical components of the reactions can be calculated from the equations of statics. In the case of vertical loads it can be concluded also from statics that the horizontal components H are equal and opposite in direction. To find the magnitude of H let us consider the elongation of the axis of the beam during bending. A good approximation to this elongation can be obtained by assuming that the deflection curve of the beam is a parabola,[1] the equation of which is

$$y = \frac{4\delta x(l - x)}{l^2} \tag{b}$$

where δ is the deflection at the middle. The length of the curve is

$$s = 2\int_0^{l/2} \sqrt{dx^2 + dy^2} = 2\int_0^{l/2} dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \tag{c}$$

In the case of a flat curve the quantity $(dy/dx)^2$ is small in comparison with unity and neglecting small quantities of order higher than the second we obtain approximately

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \approx 1 + \frac{1}{2}\left(\frac{dy}{dx}\right)^2.$$

Substituting this expression in equation (*c*) and using equation (*b*) we find the length of the curve to be

$$s = l\left(1 + \frac{8}{3}\frac{\delta^2}{l^2}\right).$$

Then the difference between the length of the curve and the distance l between the supports represents the total axial elongation of the beam and is $(8/3)(\delta^2/l)$; the unit elongation

[1] The exact expression for the deflection curve will be given later (see Part II).

is then $(8/3)(\delta^2/l^2)$. Knowing this and denoting by E the modulus of elasticity of the material of the beam and by A its cross-sectional area, we obtain the horizontal reaction from equation:

$$H = \frac{8}{3}\frac{\delta^2}{l^2}EA. \tag{b}$$

It is interesting to note that for most beams in practice the deflection δ is very small in comparison with the length and the tensile stress $(8/3)(\delta^2/l^2)E$ produced by the forces H is usually small in comparison with bending stresses and can be neglected. This justifies the usual practice of calculating beams with supported ends by assuming that one of the two supports is a movable hinge, although the special provisions for permitting free motion of the hinge are actually used only in cases of large spans such as bridges.

In the case of the bending of flexible bars and thin metallic strips, where the deflection δ is no longer very small in comparison with l, the tensile stresses produced by the longitudinal forces H can not be neglected. Such problems will be discussed later (see Part II). In the following discussion of statically indeterminate problems of bending the method of superposition will be used and the solutions will be obtained by combining statically determinate cases in such a manner as to satisfy the conditions at the supports.

41. Beam Built-in at One End and Supported at the Other.—Consider first the case where there is a single concentrated load P (Fig. 155).[2] In this case we have three unknown reactive elements at the left end and one at the right end. Hence the problem is statically indeterminate with one redundant constraint. In the solution of this problem let us consider as redundant the constraint which prevents the left end A of the beam from rotating during bending. Removing this constraint, we obtain the statically determinate problem shown in Fig. 155 (*b*). The bending produced by the statically indeterminate couple M_a will be studied separately as shown in Fig. 155 (c).[2] It is evident that the bending of the beam represented in Fig. 155 (a) can be obtained by a combination of cases (b) and (c). It is only necessary to adjust the magnitude of the couple M_a at the support in such a manner as to satisfy the condition

$$\theta_1 = -\theta_1'. \qquad (a)$$

Thus, the rotation of the left end of the beam, due to the force P, will be annihilated by M_a and the condition of a built-in end with zero slope will be satisfied. To obtain the statically indeterminate couple M_a it is necessary only to substitute in eq. (a) the known values for the angles θ_1 and θ_1' from equations (88), p. 142, and (104), p. 158. Then

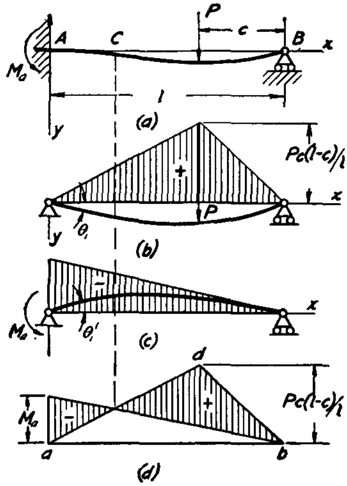

Fig. 155.

$$\frac{Pc(l^2 - c^2)}{6lEI_z} = -\frac{M_a l}{3EI_z},$$

from which

$$M_a = -\frac{Pc(l^2 - c^2)}{2l^2}. \qquad (107)$$

The bending moment diagram can now be obtained by combining the diagrams for cases (b) and (c) as shown by the shaded area in Fig. 155 (d). The maximum bending moment will be either that at a or that at d.

The deflection at any point can easily be obtained by subtracting from the deflection at this point produced by the load P the deflection produced by the couple M_a. The equations of the deflection curves for both these cases have already been given in (86) and (87), p. 142, and in (105), p. 159. Let us take, for instance, the case $c < \frac{1}{2}l$ and calcu-

[2] Deflection curves and bending moment diagrams are shown together.

late the deflection at the middle of the span. From eqs. (91) and (105)

$$\delta = \frac{Pc}{48EI_z}(3l^2 - 4c^2) + \frac{M_a l^2}{16EI_z}$$

or, by using eq. (107),

$$\delta = \frac{Pc}{96EI_z}(3l^2 - 5c^2).$$

At the point C where the bending moment becomes zero, the curvature of the deflection curve is also zero and we have a point of inflection, i.e., a point where the curvature changes sign.

It may be seen from eq. (107) that the bending moment at the built-in end depends on the position of the load P. If we equate to zero the derivative of eq. (107) with respect to c, we find that the moment M_a has its numerical maximum value when $c = l/\sqrt{3}$. Then

$$|M_a|_{\max} = \frac{Pl}{3\sqrt{3}} = 0.192Pl. \tag{108}$$

The bending moment under the load, from Fig. 155 (d), is

$$M_d = \frac{Pc(l-c)}{l} - \frac{c}{l}\frac{Pc(l^2-c^2)}{2l^2} = \frac{Pc}{2l^3}(l-c)^2(2l+c). \tag{b}$$

If we take the derivative of (b) with respect to c and equate it to zero, we find that M_d becomes a maximum when

$$c = \frac{l}{2}(\sqrt{3} - 1) = 0.366l.$$

Substituting this in eq. (b), we obtain

$$(M_d)_{\max} = 0.174Pl.$$

Comparing this with eq. (108), we find that in the case of a moving load the maximum normal stresses σ_x are at the built-in section.

Having the solution for the single concentrated load and using the method of superposition, the problem can be solved for other types of transverse loading by simple extension of the above theory. Take, for instance, the case represented in Fig. 156. The moment at the support A, produced by any element qdc of the load, is obtained from eq. (107) by substituting qdc for P. The total moment M_a at the support will be

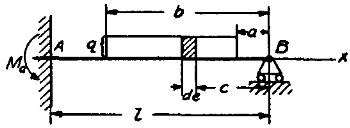

FIG. 156.

$$M_a = -\int_a^b \frac{qcdc(l^2 - c^2)}{2l^2} = -\frac{q}{2l^2}\left[\frac{l^2(b^2 - a^2)}{2} - \frac{b^4 - a^4}{4}\right]. \qquad (c)$$

If the load be distributed along the entire length of the beam, then substituting in eq. (c) $a = 0$, $b = l$, we obtain

$$M_a = -\frac{ql^2}{8}. \qquad (109)$$

The bending moment diagram is obtained by subtracting the triangular diagram due to the couple M_a (Fig. 157) from the parabolic diagram, due to uniform loading. It can be seen that the maximum bending stresses will be at the built-in section. The deflection at any point is obtained by subtracting the deflection at this point produced by the couple M_a (see eq. 105, p. 159) from the deflection at the same point produced by the uniform load (see eq. 81, p. 138).

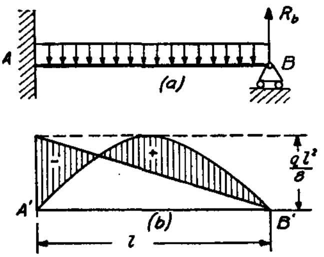

FIG. 157.

For the middle of the span we will obtain

$$\delta = \frac{5}{384}\frac{ql^4}{EI_z} + \frac{M_a l^2}{16EI_z} = \frac{ql^4}{192EI_z}. \qquad (110)$$

Problems

1. Draw shearing force diagrams for the cases shown in Figures 155 and 157.

2. Determine the maximum deflection for the case of a uniformly distributed load shown in Fig. 157.

Solution. Combining eqs. (81) and (105), the following equation for the deflection curve is obtained:

$$y = \frac{q}{48EI_z}(3l^2x^2 - 5lx^3 + 2x^4). \tag{d}$$

Setting the derivative dy/dx equal to zero, we find the point of maximum deflection at $x = (l/16)(15 - \sqrt{33}) = 0.579l$. Substituting in ($d$), we obtain

$$\delta_{\max} = \frac{ql^4}{185EI_z}.$$

3. Determine the reaction at the right support of the beam shown in Fig. 157, considering this reaction as the redundant constraint.

Solution. Removing support B, the deflection of this end of the beam, considered as a cantilever, will be $ql^4/8EI_z$ from eq. (84). Reaction R_b at B (Fig. 157, a) must be such as to eliminate the above deflection. Then by using eq. (95) we obtain the equation:

$$\frac{ql^4}{8EI_z} = \frac{R_b l^3}{3EI_z},$$

from which

$$R_b = \frac{3}{8}ql.$$

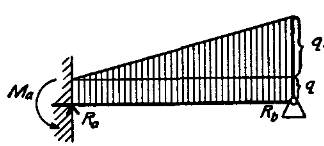

FIG. 158.

4. A beam is loaded as shown in Fig. 158. Determine the moment M_a and the reactions R_a and R_b at the supports.

Answer.

$$-M_a = \frac{ql^2}{8} + \frac{7}{120}q_1l^2; \quad R_a = \frac{5}{8}ql + \frac{9}{40}q_1l; \quad R_b = \frac{3}{8}ql + \frac{11}{40}q_1l.$$

5. Determine the reaction R_b at the support B of a uniformly loaded beam such as shown in Fig. 157 if the support B is elastic, so that a downward force of the magnitude k lowers the support a unit distance.

Solution. Using the same method as in problem 3 above, the equation for determining R_b will be

$$\frac{ql^4}{8EI_z} - \frac{R_b l^3}{3EI_z} = \frac{R_b}{k},$$

from which

$$R_b = \frac{3}{8}ql \frac{1}{1 + \frac{3EI_z}{kl^3}}.$$

6. Construct the bending moment diagram for a uniformly loaded beam supported at three equidistant points.

Suggestion. From the condition of symmetry the middle cross section does not rotate during bending and each half of the beam will be in the condition of a beam built in at one end and supported at the other.

7. Determine the deflection of the end C of the beam shown in Fig. 159.

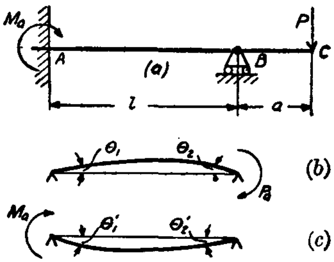

FIG. 159.

Solution. Replacing the action of the overhang by a couple Pa, the bending of the beam between the supports will be obtained by superposing cases (b) and (c). The statically indeterminate couple M_a will be found from the equation $\theta_1 = -\theta_1'$ or

$$\frac{Pal}{6EI_z} = \frac{M_a l}{3EI_z},$$

from which $M_a = Pa/2$. The deflection at C will be

$$\delta = \frac{Pa^3}{3EI_z} + a(\theta_2 - \theta_2') = \frac{Pa^3}{3EI_z} + \frac{Pa^2 l}{4EI_z}.$$

The first term on the right side represents the deflection of a cantilever and the second represents the deflection due to rotation of the cross section at B.

8. Determine the additional pressure of the beam AB on the support B (Fig. 155) due to non-uniform heating of the beam, provided that the temperature varies from t_0 at the bottom to t at the top of the beam according to a linear law.

Solution. If the support at B is removed, the non-uniform heating will cause the deflection curve of the beam to become an arc of a circle. The radius of this circle can be determined from the equation $1/r = \alpha(t - t_0)/h$, in which h = the depth of the beam and α = the coefficient of thermal expansion. The corresponding deflection at B can be found as in problem 2, p. 94, and is

$$\delta = \frac{l^2}{2r} = \frac{l^2\alpha(t - t_0)}{2h}.$$

This deflection is eliminated by the reaction of the support B. Letting R_b denote this reaction,

$$\frac{R_b l^3}{3EI_z} = \frac{l^2\alpha(t - t_0)}{2h},$$

from which

$$R_b = \frac{3EI_z}{2hl} \cdot \alpha(t - t_0).$$

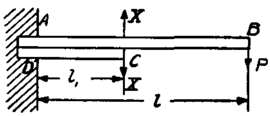

Fig. 160.

9. A cantilever AB, loaded at the end, is supported by a shorter cantilever CD of the same cross section as cantilever AB. Determine the pressure X between the two beams at C.

Solution. Pressure X will be found from the condition that at C both the cantilevers have the same deflection. Using eq. (95) for the lower cantilever and eq. (97) together with eq. (95) for the upper we obtain

$$\frac{Xl_1^3}{3EI_z} = \frac{P}{EI_z}\left(\frac{ll_1^2}{2} - \frac{l_1^3}{6}\right) - \frac{Xl_1^3}{3EI_z},$$

from which

$$X = \frac{3P}{4}\left(\frac{l}{l_1} - \frac{1}{3}\right).$$

From a consideration of the bending moment diagrams for the upper and lower cantilevers it can be concluded that at C the upper cantilever has a larger angular deflection than the lower has. This indicates that there will be contact between the two cantilevers only at points D and C.

10. Solve problem 7 assuming, instead of a concentrated load P, a uniformly distributed load of intensity q to be distributed (1) along the length a of the overhand, (2) along the entire length of the beam.

11. Draw the bending moment and shearing force diagrams for the case shown in Fig. 156 if a = 4 ft., b = 12 ft., l = 15 ft. and q = 400 lbs. per ft.

42. Beam with Both Ends Built in.—In this case we have six reactive elements (three at each end), i.e., the problem has three statically indeterminate elements. However, for ordinary beams, the horizontal components of the reactions can be neglected (see p. 178), which reduces the number of statically indeterminate quantities to two. Let us take the moments M_a and M_b at the supports for the statically indeterminate quantities. Then for the case of a single concentrated load P (Fig. 161, *a*) the solution can be obtained by combining the two statically determinate problems shown in Fig. 161 (*b*) and (*c*). It is evident that the conditions at the built-in ends of the beam AB will be satisfied if the couples M_a and M_b are adjusted so as to make

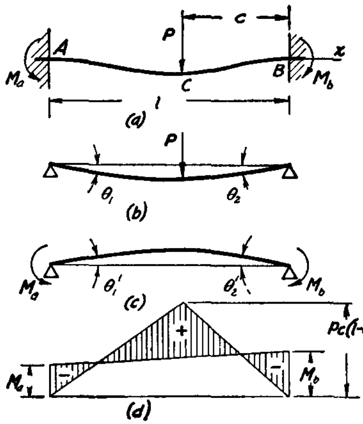

FIG. 161.

$$\theta_1 = -\theta_1'; \qquad \theta_2 = -\theta_2'. \tag{a}$$

From these two equations the two statically indeterminate couples are obtained. Using eqs. (88) and (89) for a concentrated load and eqs. (103) and (104) for the couples, eqs. (*a*) become

$$-\frac{Pc(l^2 - c^2)}{6lEI_z} = \frac{M_a l}{3EI_z} + \frac{M_b l}{6EI_z},$$

$$-\frac{Pc(l - c)(2l - c)}{6lEI_z} = \frac{M_a l}{6EI_z} + \frac{M_b l}{3EI_z},$$

from which

$$M_a = -\frac{Pc^2(l - c)}{l^2}; \qquad M_b = -\frac{Pc(l - c)^2}{l^2}. \tag{111}$$

Combining the bending moment diagrams for cases (*b*) and (*c*), the diagram shown in Fig. 161 (*d*) is obtained. The maximum positive bending moment is under the load at the point C. Its magnitude can be found from Fig. 161 (d) and is given by the following:

$$M_c = \frac{Pc(l-c)}{l} + \frac{M_a c}{l} + \frac{M_b(l-c)}{l} = \frac{2Pc^2(l-c)^2}{l^3}. \quad (112)$$

From Fig. 161 (d) it may be seen that the numerically maximum bending moment is either that at C or that at the nearest support. For a moving load, i.e., when c varies, assuming $c < l/2$, the maximum numerical value of M_b is obtained by putting $c = 1/3l$ in eq. (111). This maximum is equal to $4/27Pl$. The bending moment under the load is a maximum when $c = l/2$ and this maximum, from eq. (112), is equal to $1/8Pl$. Hence for a moving load the greatest moment is at the end.

By using the method of superposition the deflection at any point can also be obtained by combining the deflection produced by load P with that produced by couples M_a and M_b.

Having the solution for a single concentrated load P, any other type of transverse loading can easily be studied by using the method of superposition.

Problems

1. Draw the shearing force diagram for the case in Fig. 161 (a) if $P = 1{,}000$ lbs., $l = 12$ ft., and $c = 4$ ft.

2. Construct the bending moment diagram for a uniformly loaded beam with built-in ends (Fig. 162).

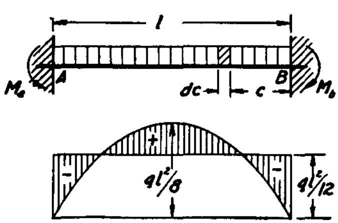

FIG. 162.

Solution. The moment at A produced by one element qdc of the load (Fig. 162, a) is, from eq. (111),

$$dM_a = -\frac{qdcc^2(l-c)}{l^2}.$$

The moment produced by the load over the entire span is then

$$M_a = -\int_0^l \frac{qdcc^2(l-c)}{l^2} = -\frac{ql^2}{12};$$

the moment at the support B will have the same magnitude. Combining the parabolic bending moment diagram produced by the uniform load with the rectangular diagram given by two equal couples applied at the ends, we obtain the diagram shown in Fig. 162 (*b*) by the shaded area.

3. Determine the moments at the ends of a beam with built-in ends and loaded by the triangular load shown in Fig. 163.

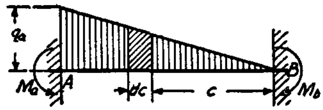

FIG. 163.

Solution. The intensity of the load at distance c from the support B is $q_a c/l$ and the load represented by the shaded element is $q_a c dc/l$. The couples acting at the ends, produced by this elementary load, as given by eqs. (111), are

$$dM_a = -\frac{q_a c^3(l-c)dc}{l^3}; \qquad dM_b = -\frac{q_a c^2(l-c)^2 dc}{l^3};$$

therefore

$$M_a = -\int_0^l \frac{q_a c^3(l-c)dc}{l^3} = -\frac{q_a l^2}{20};$$

$$M_b = -\int_0^l \frac{q_a c^2(l-c)^2 dc}{l^3} = -\frac{q_a l^2}{30}.$$

4. Determine the reactive couples M_a and M_b in a beam with built-in ends and bent by a couple Pc (Fig. 164).

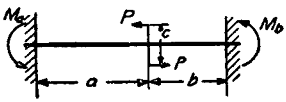

FIG. 164.

Solution. By using the solution of problem 5, p. 160, and eqs. (104) and (105), the following equations are obtained:

$$2M_a + M_b = -\frac{3Pc}{l^3}\left[a^2\left(b + \frac{a}{3}\right) - \frac{2}{3}b^3\right],$$

$$2M_b + M_a = \frac{3Pc}{l^3}\left[\frac{2}{3}a^3 - b^2\left(a + \frac{b}{3}\right)\right],$$

from which M_a and M_b can easily be calculated.

5. Determine the bending moments at the ends of a built-in beam due to non-uniform heating of the beam if the temperature varies from t_0 at the bottom to t at the top of the beam according to a linear law.

Answer.

$$M_a = M_b = \frac{\alpha EI_z(t - t_0)}{h},$$

where α is the coefficient of thermal expansion and h is the depth of the beam.

6. Determine the effect on the reactive force and reactive couple at A of a small vertical displacement δ of the built-in end A of the beam AB (Fig. 161).

Solution. Remove the support A; then the deflection δ_1 at A and the slope θ_1 at this point will be found as for a cantilever built in at B and loaded by P, i.e.,

$$\delta_1 = \frac{Pc^3}{3EI_z} + \frac{Pc^2}{2EI_z}(l - c); \qquad \theta_1 = \frac{Pc^2}{2EI_z}.$$

Applying at A an upward reactive force X and a reactive couple Y in the same direction as M_a, of such magnitude as to annihilate the slope θ_1 and to make the deflection equal to δ, the equations for determining the unknown quantities X and Y become

$$\frac{Xl^2}{2EI_z} - \frac{Yl}{EI_z} = \frac{Pc^2}{2EI_z},$$

$$\frac{Xl^3}{3EI_z} - \frac{Yl^2}{2EI_z} = \delta_1 - \delta.$$

7. Draw the shearing force and bending moment diagrams for the beam shown in Fig. 163 if $q_a = 400$ lbs. per ft. and $l = 15$ ft.

8. Draw the shearing force and bending moment diagrams for a beam with built-in ends if the left half of the beam is uniformly loaded with a load $q = 400$ lbs. per ft. The length of the beam is $l = 16$ ft.

43. Frames.—The method used above for the cases of statically indeterminate beams can be applied also to the study of frames. Take, as a simple example, the symmetrical and symmetrically loaded frame, Fig. 165, hinged at C and D. The shape of the frame after deformation is shown by the dotted lines. Neglecting the change in the length of the bars and the effect of axial forces on the bending of bars,[3] the frame can be considered to be made up of three beams as shown in

[3] Simultaneous action of bending and thrust will be discussed later (see Part II).

Fig. 165 (*b*). It is evident that there will be couples M at the ends of the horizontal beam AB which oppose the free rotation of these ends and represent the action of the vertical bars on the horizontal beam. This couple M can be considered as the only statically indeterminate quantity. Knowing M, the

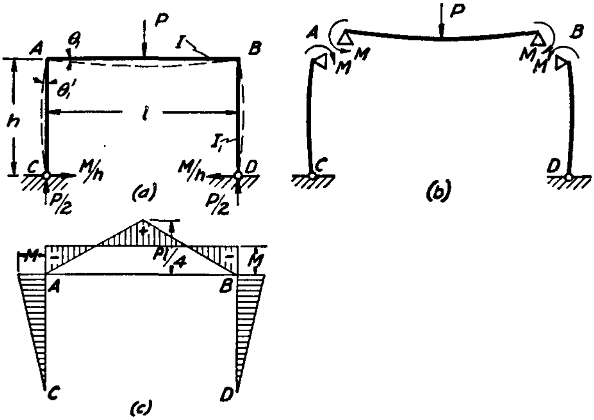

FIG. 165.

bending of all three bars can be investigated without any difficulty. For determining M we have the condition that at A and at B there are rigid joints between the bars so that the rotation of the top end of the vertical bar AC must be equal to the rotation of the left end of the horizontal bar. Hence the equation for determining M is

$$\theta_1 = \theta_1'. \tag{a}$$

θ_1 must be determined from the bending of the horizontal beam AB. Denoting by l the length of this beam and by EI its flexural rigidity, the rotation of the end A due to the load P, by eq. (88) ($b = l/2$), is $Pl^2/16EI$. The couples at the ends resist this bending and give a rotation in the opposite direction, which, from eqs. (103) and (104), equals $Ml/2EI$. The final value of the angle of rotation will be

$$\theta_1 = \frac{Pl^2}{16EI} - \frac{Ml}{2EI}.$$

Considering now the vertical bar AC as a beam with supported ends, bent by a couple M, and denoting by h its length and by EI_1 its flexural rigidity, the angle at the top, from eq. (104), will be

$$\theta_1' = \frac{Mh}{3EI_1}.$$

Substituting in eq. (a), we obtain

$$\frac{Pl^2}{16EI} - \frac{Ml}{2EI} = \frac{Mh}{3EI_1},$$

from which

$$M = \frac{Pl}{8} \frac{1}{1 + \frac{2}{3}\frac{h}{l}\frac{I}{I_1}}. \tag{113}$$

This is the absolute value of M. Its direction is shown in Fig. 165 (b). Knowing M, the bending moment diagram can be constructed as shown in Fig. 165 (c). The reactive forces at the hinges C and D are also shown (Fig. 165, a). The vertical components of these forces, from considerations of symmetry, are each equal to $P/2$. As regards the horizontal components, their magnitude M/h is obtained by considering the vertical bars as simply supported beams loaded at the top by the couples M.

The same problem can be solved in another way by taking the horizontal reaction H at the hinges C and D as the statically indeterminate quantity, instead of M (Fig. 166). The statically indeterminate problem is solved by superposing the two statically determinate problems shown in Fig. 166 (b) and (c). In case (b) the redundant constraint prevent-

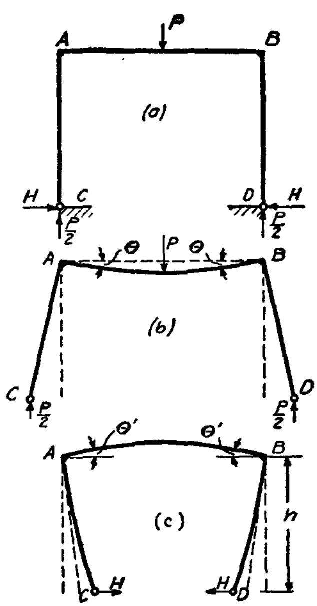

Fig. 166.

ing the horizontal motion of the hinges C and D is removed. The vertical bars no longer have any bending. The horizontal bar AB is in the condition of a bar with simply supported ends whose angles of rotation are equal to $Pl^2/16EI$, and the horizontal motion of each hinge C and D is therefore $h(Pl^2/16EI)$. In case (c) the effect of the forces H is studied. These forces produce bending couples on the ends of the horizontal bar AB equal to $H \cdot h$, so that the angles of rotation of its ends θ' will be $Hh \cdot l/2EI$. The deflection of each hinge C and D consists of two parts, the deflection $\theta' h = Hh^2 l/2EI$ due to rotation of the upper end and the deflection $Hh^3/3EI_1$ of the vertical bars as cantilevers. In the actual case (Fig. 166, a) the hinges C and D do not move; hence the horizontal displacements produced by the force P (Fig. 166, b) must be counteracted by the forces H (Fig. 166, c), i.e.,

$$\frac{Pl^2}{16EI} h = \frac{Hh^2 l}{2EI} + \frac{Hh^3}{3EI_1},$$

from which

$$H = \frac{1}{h}\frac{Pl}{8}\frac{1}{1 + \frac{2}{3}\frac{h}{l} \cdot \frac{I}{I_1}}.$$

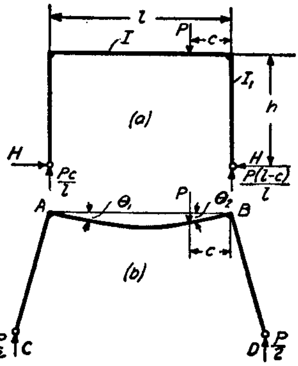

FIG. 167.

Remembering that $Hh = M$, this result agrees with the equation (113) above.

This latter method of analysis is especially useful for nonsymmetrical loading such as shown in Fig. 167. Removing the constraint preventing the hinges C and D from horizontal motion, we have the condition represented in Fig. 167 (b). It is evident that the increase in distance between C and D may be obtained by multiplying by h the sum of the angles θ_1 and θ_2. Using eqs. (88) and (89), this increase in distance becomes

$$h\left[\frac{Pc(l^2 - c^2)}{6lEI} + \frac{Pc(l - c)(2l - c)}{6lEI}\right] = \frac{Pc(l - c)h}{2EI}.$$

It must be eliminated by the horizontal reactions H (Fig. 166, c). Then, using the results obtained in the previous problem, we get the following equation for determining H:

$$2\left(\frac{Hh^2l}{2EI} + \frac{Hh^3}{3EI_1}\right) = \frac{Pc(l - c)h}{2EI},$$

from which

$$H = \frac{Pc(l - c)}{2hl} \frac{1}{1 + \frac{2}{3}\frac{I}{I_1}\frac{h}{l}}. \tag{114}$$

Having the solution for one concentrated load, any other case of loading of the beam AB of the frame can easily be studied by the method of superposition.

Let us consider now a frame with built-in supports and an unsymmetrical loading as shown in Fig. 168. In this case we have

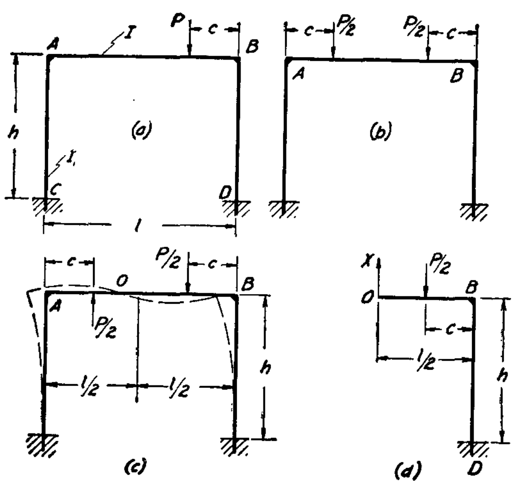

FIG. 168.

three reactive elements at each support and the system has three statically indeterminate elements. In the solution of this problem we will use a method based on the method of superposition in which the given system of loading is split into parts such that for each partial loading a simple solution can be found.[4] The problem shown in Fig. 168 (*a*) can be solved by superposing the solutions of the two problems shown in Fig. 168 (*b*) and (*c*). The case shown in (*b*) is a symmetrical one and can be considered in the same manner as the first example shown in Fig. 165. A study of the case shown in (*c*) will show that the point of inflection O of the horizontal bar AB is situated at the middle of the bar. This follows from the condition that the loads $P/2$ are equally distant from the vertical axis of symmetry of the frame and are opposite in sense. The moment, the deflection, and the axial force produced at the mid point O of the horizontal beam AB by one of the loads $P/2$ will be removed by the action of the other load $P/2$. Hence there will be no bending moment, no vertical deflection and no axial force at O. The magnitude of the shearing force at the same point X can be found from the condition that the vertical deflection of O is equal to zero (Fig. 168, *d*). This deflection consists of two parts, a deflection δ_1 due to the bending of the cantilever OB and a deflection δ_2 due to rotation of the end B of the vertical bar BD. Using the known equations for a cantilever (eq. 98), and using the notations given in the figure, the following equations are obtained:

$$\delta_1 = \frac{P}{2}\frac{c^3}{3EI} + \frac{P}{2}\frac{c^2}{2EI}\left(\frac{l}{2} - c\right) - \frac{X\left(\frac{l}{2}\right)^3}{3EI},$$

$$\delta_2 = \left(\frac{Pc}{2} - X\frac{l}{2}\right)\frac{h}{EI_1}\frac{l}{2}.$$

Substituting this in the equation $\delta_1 + \delta_2 = 0$, the magnitude X of the shearing force can be found. Having determined X, the bending moment at every cross section of the frame for case (*c*) can be calculated. Combining this with the bending moments for the symmetrical case (*b*), the solution of the problem (*a*) is obtained.[5]

Problems

1. Determine the bending moments at the corners of the frame shown in Fig. 169.

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[4] Such a method was extensively used by W. L. Andrée; see his book "Das *B-U* Verfahren," München and Berlin, 1919.

[5] Solutions of many important problems on frames can be found in the book by Kleinlogel, "Mehrstielige Rahmen," Berlin, 1927.

Solution. Considering the bar AB as a beam supported at the

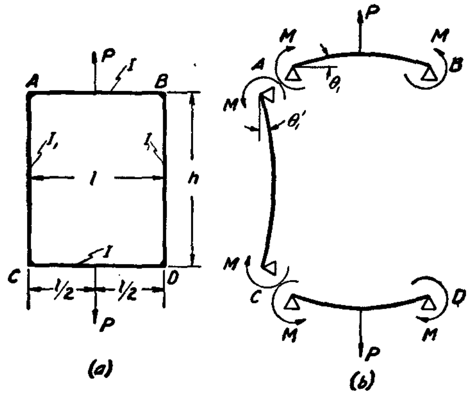

FIG. 169.

ends (Fig. 169, *b*) and denoting by M the moments at the corners, the angle θ_1 will be

$$\frac{Pl^2}{16EI} - \frac{Ml}{2EI}.$$

Putting this equal to the angle θ_1' at the ends of the vertical bars which are bent by the couples M only, the following equation for M is obtained:

$$\frac{Pl^2}{16EI} - \frac{Ml}{2EI} = \frac{Mh}{2EI_1},$$

from which

$$M = \frac{Pl}{8} \frac{1}{1 + \frac{h}{l}\frac{I}{I_1}}.$$

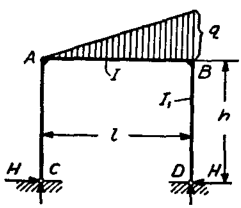

FIG. 170.

2. Determine the horizontal reactions H for the case shown in Fig. 170.

Suggestion. By using eq. (114) and applying the method of superposition, we get

$$H = \frac{ql^2}{24h} \frac{1}{1 + \frac{2}{3}\frac{I}{I_1}\frac{h}{l}}.$$

3. Draw the bending moment diagram for the three bars of the preceding problem assuming $h = l$ and $I = I_1$.

4. Determine the bending moments at the joints of the frame shown in Fig. 171.

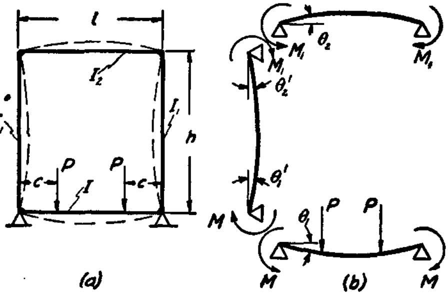

FIG. 171.

Solution. Disjointing the frame as shown in Fig. 171 (*b*), the equations for determining the couples M and M_1 are

$$\theta_1 = \theta_1' \qquad \text{and} \qquad \theta_2 = \theta_2'.$$

Substituting in these equations

$$\theta_1 = \frac{Pc(l-c)}{2EI} - \frac{Ml}{2EI};$$

$$\theta_1' = \frac{Mh}{3EI_1} - \frac{M_1 h}{6EI_1};$$

$$\theta_2 = \frac{M_1 l}{2EI_2};$$

$$\theta_2' = \frac{Mh}{6EI_1} - \frac{M_1 h}{3EI_1},$$

we obtain two equations for determining M and M_1.

5. A symmetrical rectangular frame is submitted to the action of a horizontal force H as shown in Fig. 172. Determine the bending moments M and M_1 at the joints.

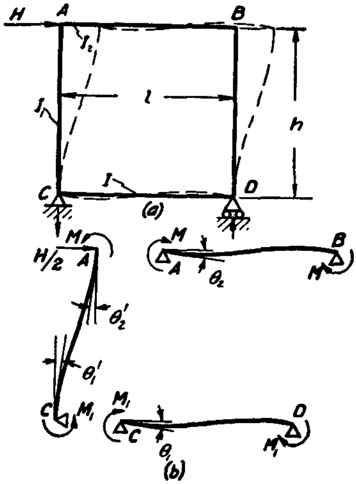

FIG. 172.

Solution. The deformed shape of the frame is shown in Fig. 172 (*a*). Disjointing the frame as shown in Fig. 172 (*b*), and applying moments the directions of which are chosen to comply with the distorted shape of the frame, Fig. 172 (*a*), we have for the bar CD

$$\theta_1 = \frac{M_1 l}{6EI} = \left(\frac{Hh}{2} - M\right)\frac{l}{6EI}. \tag{a}$$

Considering now the vertical bar AC as a cantilever built in at the end C at an angle θ_1, the slope at the end A will be

$$\theta_2' = \theta_1 + \frac{H}{2}\frac{h^2}{2EI_1} - \frac{Mh}{EI_1}. \tag{b}$$

Finally, due to bending of the bar AB,

$$\theta_2 = \theta_2' = \frac{Ml}{6EI_2}. \tag{c}$$

Then, from eqs. (a), (b) and (c),

$$M = \frac{Hh}{2}\left(1 + \frac{3h}{l}\frac{I}{I_1}\right)\frac{1}{1 + \frac{I}{I_2} + 6\frac{h}{l}\frac{I}{I_1}}. \tag{d}$$

Substituting in eq. (a), the bending moment M_1 can be found. When the horizontal bar CD has very great rigidity, we approach the condition of the frame shown in Fig. 168, submitted to a lateral load H. Substituting in (d) $I = \infty$, we obtain for this case

$$M = \frac{Hh}{4}\frac{1}{1 + \frac{1}{6}\frac{l}{h}\frac{I_1}{I_2}}. \tag{e}$$

The case of a frame such as shown in Fig. 165 with hinged supports and submitted to the action of a lateral load applied at A can also be obtained by substituting $I = 0$ in eq. (d).

6. Determine the horizontal reactions H, and the moments M_c and M_b at the joints A and B for the frame shown in Fig. 173.

Answer.

$$H = \frac{qh}{20}\frac{11m + 20}{2m + 3},$$

$$M_a = M_b = \frac{qh^2}{60}\frac{7m}{2m + 3},$$

where

$$m = \frac{I}{I_1}\cdot\frac{h}{l}.$$

7. A frame consists of two bars joined rigidly at B and built in at A and C (Fig. 174). Determine the bending moment M at

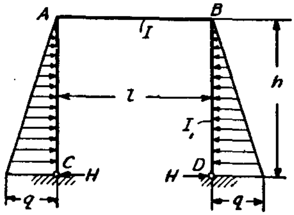

FIG. 173.

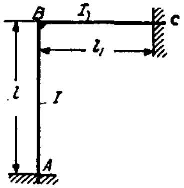

FIG. 174.

B and the compressive force P in AB when, due to a rise in temperature, the bar AB increases in length by $\Delta = \alpha l(t - t_0)$.

Answer. P and M can be found from equations:

$$\frac{Pl_1^3}{3EI} - \frac{Ml_1^2}{2EI} = \Delta,$$

$$\frac{Pl_1^2}{2EI} - \frac{Ml_1}{EI} = \frac{Ml}{4EI}.$$

44. Beams on Three Supports.—In the case of a beam on three supports (Fig. 175, *a*) there is one statically indeterminate reactive element. Let the reaction of the intermediate support be this element. Then by using the method of superposition the solution of case (*a*) may be obtained by combining the cases represented in (*b*) and (*c*), Fig. 175. The intermediate reaction X is found by using the condition that the deflection δ produced at C by the load P must be eliminated by the reaction X. Using eq. (86), we get the following equation for determining X:

$$\frac{Pcl_1[(l_1 + l_2)^2 - c^2 - l_1^2]}{6(l_1 + l_2)EI_z} = \frac{Xl_1^2l_2^2}{3(l_1 + l_2)EI_z},$$

from which

$$X = \frac{Pc[(l_1 + l_2)^2 - c^2 - l_1^2]}{2l_1l_2^2}. \qquad (115)$$

If P is acting on the left span of the beam, the same equation can be used, but the distance c must be measured from the support A and l_1 and l_2 must be interchanged. For $l_1 = l_2 = l$, from (115),

$$X = \frac{Pc(3l^2 - c^2)}{2l^3}. \tag{116}$$

Having the solution for a single load P, any other loading can easily be studied by using the method of superposition.

The same problem can be solved in another manner. Imagine the beam cut into two parts at C (Fig. 175, d) and let

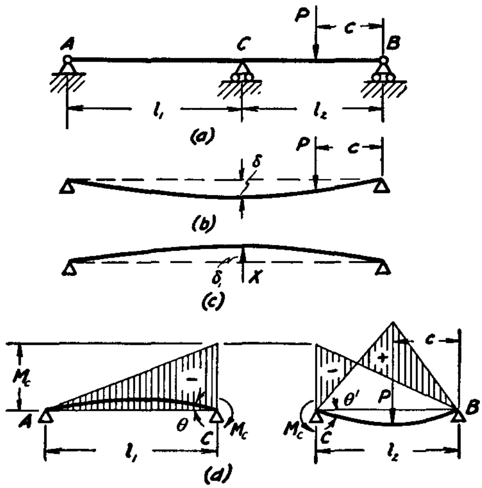

FIG. 175.

M_c denote the magnitude of the bending moment of the original beam at this cross section. In this manner the problem is reduced to the consideration of the two simply supported beams shown in (d) which are statically determinate. The magnitude of M_c is determined from the condition of continuity of the deflection curve at the support C. From this $\theta = \theta'$, whence, using the eqs. (88), p. 142, and (104), p. 158, we obtain

$$-\frac{M_c l_1}{3EI_z} = \frac{Pc(l_2^2 - c^2)}{6l_2 EI_z} + \frac{M_c l_2}{3EI_z},$$

from which

$$M_c = -\frac{Pc(l_2^2 - c^2)}{2l_2(l_1 + l_2)}. \tag{117}$$

The direction of M_c as determined from the condition of bending is shown by arrows in Fig. 175 (*d*). The bending moment diagram is shown by the shaded area in Fig. 175 (*d*).

Problems

1. For the example in Fig. 175 prove that the magnitude of the bending moment M_c given by eq. (117) is the same as that given for the cross section C by eq. (115).

2. Draw the shearing force diagram for the beam of the preceding problem if $l_1 = l_2$, $c = l_2/2$, and $P = 1{,}000$ lbs.

3. A beam on three supports (Fig. 175, *a*) is carrying a uniformly distributed load of intensity q. Determine the bending moment at the support C.

Solution. By the method of superposition, substituting qdc for P in eq. (117) and integrating along both spans, we obtain

$$M_c = -\int_0^{l_2} \frac{qc(l_2^2 - c^2)dc}{2l_2(l_1 + l_2)} - \int_0^{l_1} \frac{qc(l_1^2 - c^2)dc}{2l_1(l_1 + l_2)} = -\frac{q}{8}\frac{l_2^3 + l_1^3}{l_1 + l_2},$$

when

$$l_1 = l_2 = l, \qquad M_c = -\frac{ql^2}{8}.$$

The direction of this moment is as shown in Fig. 175 (*d*).

4. Draw the shearing force diagram for the preceding problem assuming $l_1 = l_2$ and $q = 500$ lbs. per ft.

5. Determine the numerical maximum bending moment in the beam ACB (Fig. 175) if $P = 10{,}000$ lbs., $l_1 = 9$ ft., $l_2 = 12$ ft., $c = 6$ ft.

Answer. $M_{max} = 23{,}600$ lbs. ft.

6. A beam on three equidistant supports is carrying a uniformly distributed load of intensity q. What effect will it have on the middle reaction if the middle support is lowered a distance δ?

Solution. Using the method shown in Fig. 175 (*b*) and (*c*), the middle reaction X is found from the equation:

$$\frac{5}{384}\frac{q(2l)^4}{EI} = \frac{X(2l)^3}{48EI} + \delta,$$

from which

$$X = \frac{5}{8}2ql - \frac{6\delta EI}{l^3}.$$

7. Determine the additional pressure of the beam AB on the support C (Fig. 175, *a*) due to non-uniform heating of the beam

if the temperature varies from t at the bottom to t_1 at the top of the beam according to a linear law, $t > t_1$ and $l_1 = l_2 = l$.

Solution. If the support at C were removed, then, due to the non-uniform heating, the deflection curve of the beam would become the arc of a circle. The radius of this circle is determined by the equation:

$$\frac{1}{r} = \frac{\alpha(t - t_1)}{h},$$

in which $h =$ the depth of the beam and $\alpha =$ the coefficient of thermal expansion. The corresponding deflection at the middle is $\delta = l^2/2r$ and the reaction X at C can be found from the equation:

$$\frac{X(2l)^3}{48EI} = \delta.$$

8. Determine the bending moment diagram for the beam ABC supported by three pontoons (Fig. 176) if the horizontal cross-sectional area of each pontoon is A and the weight of unit volume of water is γ.

Solution. Removing the support at C, the deflection δ produced at this point by the load P consists of two parts: (a) the deflection due to bending of the beam and (b) the deflection due to sinking of pontoons A and B. From eq. (91) we obtain

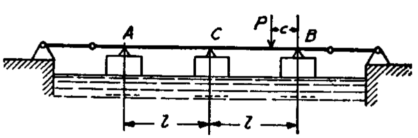

Fig. 176.

$$\delta = \frac{Pc}{48EI_z}[3(2l)^2 - 4c^2] + \frac{P}{2A\gamma}. \tag{a}$$

The reaction X of the middle support diminishes the above deflection by

$$\frac{X(2l)^3}{48EI_z} + \frac{X}{2A\gamma}. \tag{b}$$

The difference between (a) and (b) represents the distance the middle pontoon sinks, from which we get the following equation for determining X:

$$\frac{Pc}{48EI_z}[3(2l)^2 - 4c^2] + \frac{P}{2A\gamma} - \frac{X(2l)^3}{48EI_z} - \frac{X}{2A\gamma} = \frac{X}{A\gamma}.$$

Knowing X, the bending moment diagram can readily be obtained.

45. Continuous Beams.—In the case of a continuous beam on many supports (Fig. 177) one support is usually considered as an immovable hinge while the other supports are hinges on rollers. In this arrangement every intermediate support has only one unknown reactive element, the magnitude of the vertical reaction; hence the number of statically indeterminate

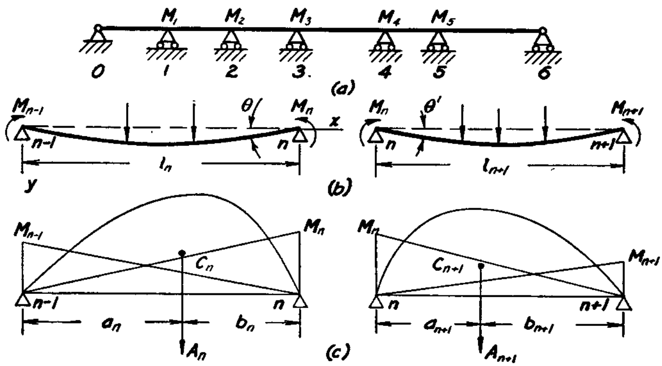

FIG. 177.

elements is equal to the number of intermediate supports. For instance, in the case shown in Fig. 177 (*a*) the number of statically indeterminate elements is five. Both methods shown in the previous article can be used here also. But if the number of supports is large, the second method, in which the bending moments at the supports are taken as the statically indeterminate elements, is by far the simpler method. Let Fig. 177 (*b*) represent two adjacent spans n and $n + 1$ of a continuous beam cut at supports $n - 1$, n and $n + 1$. Let M_{n-1}, M_n and M_{n+1} denote the bending moments at these supports. The directions of these moments depend on the loads on the beam. We will assume the directions shown in the figure.[6] It is evident that if the bending moments at the supports are known the problem of the continuous beam will be reduced to that of calculating as many simply supported

[6] If finally we would obtain negative signs for some moments, this will indicate that the directions of these moments are opposite to that shown in the figure.

beams as there are spans in the continuous beam. For calculating the bending moments M_{n-1}, M_n, M_{n+1} the condition of continuity of the deflection curve at the supports will be used. For any support n this condition of continuity is satisfied if the deflection curves of the two adjacent spans have a common tangent at the support n, i.e., if the slope at the right end of span n is equal to the slope at the left end of span $n + 1$. To calculate these slopes the area moment method will be used. Let A_n denote the area of the bending moment diagram for the span n, considered as a simply supported beam, due to the actual load on this span; let a_n and b_n represent the horizontal distances of the centroid C_n of the moment area from the supports $n - 1$ and n. Then the slope at the right end for this condition of loading is (see Art. 35)

$$-\frac{A_n a_n}{l_n EI_z}.$$

In addition to the deflection caused by the load on the span itself, the span n is also bent by the couples M_{n-1} and M_n. From equations (103) and (104) the slope produced at the support n by these couples is

$$-\left(\frac{M_n l_n}{3EI_z} + \frac{M_{n-1} l_n}{6EI_z}\right).$$

The total angle of rotation is then *

$$\theta = -\left(\frac{M_n l_n}{3EI_z} + \frac{M_{n-1} l_n}{6EI_z} + \frac{A_n a_n}{l_n EI_z}\right). \tag{a}$$

In the same manner, for the left end of the span $n + 1$, we obtain

$$\theta' = \frac{A_{n+1} b_{n+1}}{l_{n+1} EI_z} + \frac{M_n l_{n+1}}{3EI_z} + \frac{M_{n+1} l_{n+1}}{6EI_z}. \tag{b}$$

From the condition of continuity it follows that

$$\theta = \theta'. \tag{c}$$

* The angle is taken positive if rotation is in clockwise direction.

Substituting expressions (a) and (b) in this equation we obtain

$$M_{n-1}l_n + 2M_n(l_n + l_{n+1}) + M_{n+1}l_{n+1} = -\frac{6A_na_n}{l_n} - \frac{6A_{n+1}b_{n+1}}{l_{n+1}}. \quad (118)$$

This is the well-known *equation of three moments.*[7] It is evident that the number of these equations is equal to the number of intermediate supports and the bending moments at the supports can be calculated without difficulty.

In the beginning it was assumed that the ends of the continuous beam were supported. If one or both ends are built in, then the number of statically indeterminate quantities will be larger than the number of intermediate supports and derivation of additional equations will be necessary to express the condition that no rotation occurs at the built-in ends (see problem 5 below).

Knowing the moments at the supports, there is no difficulty in calculating the reactions at the supports of a continuous beam. Taking, for instance, the two adjacent spans n and $n + 1$ (Fig. 177, b), and considering them as two simply supported beams, the reaction R_n' at the support n, due to the loads on these two spans, can easily be calculated. In addition to this there will be a reaction due to the moments M_{n-1}, M_n and M_{n+1}. Taking the directions of these moments as indicated in Fig. 177 (b), the additional pressure on the support n will be

$$\frac{M_{n-1} - M_n}{l_n} + \frac{-M_n + M_{n+1}}{l_{n+1}}.$$

Adding this to the above reaction R_n', the total reaction will be

$$R_n = R_n' + \frac{M_{n-1} - M_n}{l_n} + \frac{-M_n + M_{n+1}}{l_{n+1}}. \quad (119)$$

If concentrated forces are applied at the supports they will

[7] This equation was established by Bertot; see Comptes rendus de la Société des Ingenieurs civils, p. 278, 1855; see also Clapeyron, Paris, C. R., t. 45 (1857).

be transmitted directly to the corresponding supports and must be added to the right side of equation (119).

The general equation of continuity (c) can also be used for those cases where, by mis-alignment, the supports are not situated on the same level (Fig. 178). Let β_n and β_{n+1} denote the angles of inclination to the horizontal of the straight lines connecting the points of supports in the nth and $(n + 1)$th spans. The angle of rotation given by eqs. (a) and (b) was measured from the line connecting the centers of the hinges; hence the angle θ between the tangent at n and the horizontal line will be, for the span n,

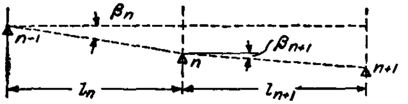

FIG. 178.

$$\theta = -\left(\frac{M_n l_n}{3EI_z} + \frac{M_{n-1} l_n}{6EI_z} + \frac{A_n a_n}{l_n EI_z} - \beta_n\right).$$

In the same manner for the span $n + 1$

$$\theta' = \frac{A_{n+1} b_{n+1}}{l_{n+1} EI_z} + \frac{M_n l_{n+1}}{3EI_z} + \frac{M_{n+1} l_{n+1}}{6EI_z} + \beta_{n+1}.$$

Equating these angles we obtain

$$M_{n-1} l_n + 2M_n(l_n + l_{n+1}) + M_{n+1} l_{n+1}$$
$$= -\frac{6A_n a_n}{l_n} - \frac{6A_{n+1} b_{n+1}}{l_{n+1}} - 6EI_z(\beta_{n+1} - \beta_n). \quad (120)$$

If h_{n-1}, h_n, h_{n+1} denote the vertical heights of the supports $n - 1$, n and $n + 1$ above a horizontal reference line, we have

$$\beta_n = \frac{h_{n-1} - h_n}{l_n}; \qquad \beta_{n+1} = \frac{h_n - h_{n+1}}{l_{n+1}}.$$

Substituting in eq. (120), the bending moments at the supports due to mis-alignment can easily be calculated.

Problems

1. Determine the bending moment and shearing force diagrams for a continuous beam with three equal spans carrying a uniformly distributed load of intensity q (Fig. 179).

Solution. For a simply supported beam and a uniformly distributed load the bending moment diagram is a parabola with maximum ordinate $ql_n^2/8$. The area of the parabolic segment is

$$A_n = \frac{2}{3} l_n \frac{ql_n^2}{8} = \frac{ql_n^3}{12}.$$

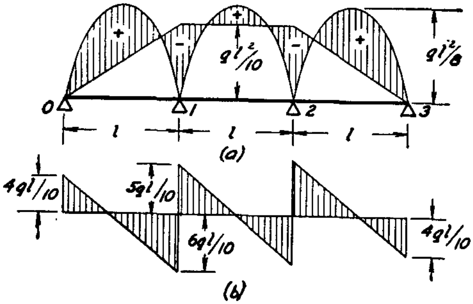

FIG. 179.

The centroid is at the middle of the span, so $a_n = b_n = l_n/2$. Substituting in eq. (118), we obtain

$$M_{n-1}l_n + 2M_n(l_n + l_{n+1}) + M_{n+1}l_{n+1} = -\frac{ql_n^3}{4} - \frac{q(l_{n+1})^3}{4}. \quad (118')$$

Applying this equation to our case (Fig. 179) for the first and the second span and noting that at the support 0 the bending moment is zero, we obtain

$$4M_1l + M_2l = -\frac{ql^3}{2}. \quad (d)$$

From the condition of symmetry it is evident that $M_1 = M_2$. Then, from (d), $M_1 = -(ql^2/10)$. The bending moment diagram is shown in Fig. 179 (a) by the shaded area. The reaction at support 0 is

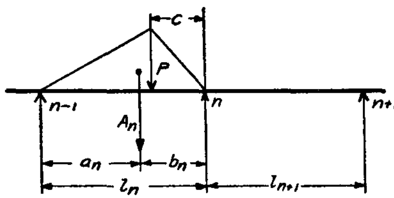

FIG. 180.

$$R_0 = \frac{ql}{2} - \frac{ql^2}{10}\frac{1}{l} = \frac{4}{10}ql.$$

The reaction at support 1 is

$$R_1 = ql + \frac{ql^2}{10}\frac{1}{l} = \frac{11}{10}ql.$$

The shearing force diagram is shown in Fig. 179 (b). The maximum moment will evidently be at a distance $4l/10$ from the ends of the beam, where the shearing force is zero. The numerical maximum bending moment is at the intermediate supports.

2. Set up the expression for the right side of eq. (118) when there is a concentrated force in the span n and no load in the span $n + 1$ (Fig. 180).

Solution. In this case A_n is the area of the triangle of height $Pc(l_n - c)/l_n$ and with the base l_n; hence $A_n = Pc(l_n - c)/2$ and $a_n = l_n - b_n = l_n - (l_n + c)/3$. Substituting in (118), we get

$$M_{n-1}l_n + 2M_n(l_n + l_{n+1}) + M_{n+1}l_{n+1} = -\frac{Pc(l_n - c)(2l_n - c)}{l_n}.$$

3. Determine the bending moments at the supports and the reactions for the continuous beam shown in Fig. 181.

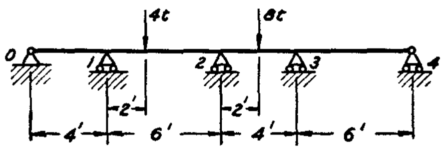

FIG. 181.

Answer. $M_1 = -1.54$ ton ft.; $M_2 = -3.74$ ton ft.; $M_3 = -1.65$ ton ft. The reactions are $R_0 = -0.386$ ton; $R_1 = 2.69$ tons; $R_2 = 6.22$ tons; $R_3 = 3.75$ tons; $R_4 = -0.275$ ton. The moments at the supports are negative and produce bending convex up.

4. Construct the bending moment and shearing force diagrams for the continuous beam shown in Fig. 182 (*a*) if $P = ql$, $c = l/4$.

Solution. In this case the imaginary loading for the first span is $A_1 = ql^3/12$, for the second span $A_2 = 0$, and for the third span

$$A_3 = \frac{Pc(l - c)}{2}; \qquad a_3 = \frac{2l - c}{3}; \qquad b_3 = \frac{l + c}{3}.$$

Substituting in eq. (118) gives the following equations for determining the bending moments M_1 and M_2:

$$4M_1l + M_2l = -\frac{ql^3}{4},$$

$$M_1l + 4M_2l = -\frac{Pc(l^2 - c^2)}{l},$$

from which

$$M_1 = -\frac{49}{960}ql^2; \qquad M_2 = -\frac{44}{960}ql^2.$$

Both these moments are negative, so that the bending moment diagram will be as shown in Fig. 182 (*b*). To obtain the shearing

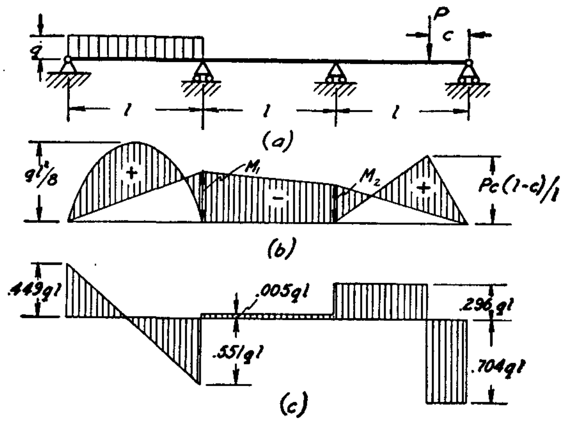

FIG. 182.

force diagram it is necessary to find the reactions at the supports of the separate spans (eq. 119). The pressures on supports 0 and 1 for the first span of the beam are

$$\frac{ql}{2} + \frac{M_1}{l} = 0.449ql \qquad \text{and} \qquad \frac{ql}{2} - \frac{M_1}{l} = 0.551ql.$$

The pressure on supports 1 and 2 of the second span of the beam are

$$\frac{-M_1 + M_2}{l} = 0.005ql \qquad \text{and} \qquad \frac{-M_2 + M_1}{l} = -0.005ql$$

and on supports 2 and 3 of the third span

$$\frac{Pc}{l} - \frac{M_2}{l} = 0.296ql \qquad \text{and} \qquad \frac{P(l-c)}{l} + \frac{M_2}{l} = 0.704ql.$$

From this, the shearing force diagram is obtained as shown in Fig. 182 (*c*).

5. Determine the bending moment diagram for the case shown in Fig. 183 (*a*).

Solution. Equation (118) for this case becomes

$$M_0 l + 4M_1 l + M_2 l = 0.$$

Now, $M_2 = -Pc$, while the condition at the built-in end (support 0) gives (from eqs. 103, 104)

$$\frac{M_0 l}{3EI} + \frac{M_1 l}{6EI} = 0.$$

From the above equations we obtain $M_0 = -\frac{1}{7} Pc$; $M_1 = +\frac{2}{7} Pc$; $M_2 = -Pc$. The bending moment diagram is as shown in Fig. 183 (*b*).

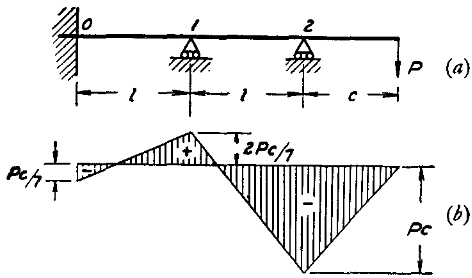

FIG. 183.

6. Determine the bending moments at the supports of a continuous beam with seven equal spans when the middle span alone is loaded by a uniformly distributed load q.

Answer.

$$M_3 = M_4 = -\frac{ql^2}{18.9}; \quad M_2 = M_5 = -\frac{1}{3.75} M_3; \quad M_1 = M_6 = \frac{1}{15} M_3.$$

7. A continuous beam having four equal spans of length 16 ft. each is uniformly loaded over the last span. Draw the shearing force and bending moment diagrams if $q = 400$ lbs. per ft.

8. Solve problem 5 assuming that a uniform load of intensity q is distributed along the entire length of the beam and that $c = l/2$. Draw the shearing force diagram for this loading condition.

CHAPTER VII

BEAMS OF VARIABLE CROSS SECTION. BEAMS OF TWO MATERIALS

46. Beams of Variable Cross Section.—In the preceding discussion all of the beams considered were of prismatical form. More elaborate investigation shows that equations (56) and (57), which were derived for prismatical bars, can also be used with sufficient accuracy for bars of variable cross section provided the variation is not too extreme. Cases of abrupt changes in cross section, in which considerable stress concentration takes place, will be discussed in Part II.

As a first example of a beam of a variable cross section let us consider the deflection of a cantilever beam of *uniform strength*, i.e., a beam in which the section modulus varies along the beam in the same proportion as the bending moment. Then, as is seen from eqs. (60), $(\sigma_x)_{\max}$ remains constant along the beam and can be taken equal to σ_w. Such a condition is favorable as regards the amount of material used, because each cross section will have only the area necessary to satisfy the conditions of strength.

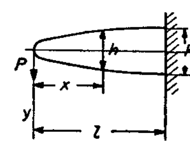

FIG. 184.

For a cantilever with an end load (Fig. 184), the bending moment at a cross section a distance x from the load is numerically equal to Px. In order to have a beam of uniform strength the section modulus also must be proportional to x. This condition can be fulfilled in various ways.

Let us take as a first example the case of a rectangular cross section of constant width b and variable depth h. From the definition of the beam of equal strength it follows that

$$\frac{M}{Z} = \frac{6Px}{bh^2} = \frac{6Pl}{bh_0^2} = \text{const.},$$

in which h_0 is the depth of the beam at the built-in end. Then

$$h^2 = \frac{h_0^2 x}{l}.$$

It may be seen that in this case the depth of the beam varies following a parabolic law. At the loaded end the cross-sectional area is zero. This result is obtained because the shearing stress was neglected in the derivation of the form of the beam of uniform strength. In practical applications this stress must be taken into account by making certain changes in the above form at the loaded end to have a cross-sectional area sufficient to transmit the shearing force. The deflection of the beam at the end is found from eq. (93):

$$\delta = \int_0^l \frac{12Px^2dx}{Ebh^3} = \frac{12Pl^{3/2}}{Ebh_0^3}\int_0^l \sqrt{x}dx = \frac{2}{3}\frac{Pl^3}{EI_0}, \qquad (121)$$

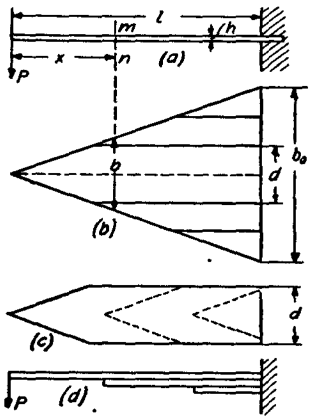

Fig. 185.

where $I_0 = bh_0^3/12$ represents the moment of inertia of the cross section at the built-in end. Comparison with eq. (95) shows that this deflection is twice that of a prismatical bar having the flexural rigidity EI_0 and subjected to the same load. That is, the bar has the same strength, but not the same stiffness as the prismatical bar.

As a second example we consider a cantilever of rectangular cross section of constant depth h and variable width b (Fig. 185a, b). As the section modulus and moment of inertia I_z of a beam of triangular shape increases with x in the same proportion as the bending moment, the maximum stress $(\sigma_x)_{\max}$ and the curvature (see eq. 56) remain constant along the beam and the magnitude of the radius of curvature can be determined from the equation (see eq. 55):

$$(\sigma_x)_{\max} = \frac{hE}{2r}. \qquad (c)$$

The deflection at the end of a circular arc can be taken, for small deflections, equal to

$$\delta = \frac{l^2}{2r} = \frac{Pl^3}{2EI_0} \tag{122}$$

or, by using (*c*),

$$\delta = (\sigma_x)_{\max} \frac{l^2}{hE} \cdot \tag{123}$$

It is seen from this equation that for this type of cantilever of uniform strength the deflection at the end varies as the square of the length and inversely as the depth.

These results may be used to compute the approximate stresses and deflections in a *spring of leaf type*. The triangular plate considered above is thought of as divided into strips, arranged as shown in Fig. 185 (*b*, *c*, *d*). The initial curvature and the friction between the strips are neglected for a first approximation and eq. (123) can then be considered as sufficiently accurate.[1]

The conjugate beam theory can also be used in calculating the deflection of beams of variable cross section. In this connection it is only necessary to bear in mind that the curvature of the deflection line at any cross section is equal to the ratio M/EI_z (eq. 56, p. 91). Therefore an increase in the flexural rigidity at a given section will have the same effect on the deflection as a decrease of the bending moment there in the same ratio. Consequently the problem of the deflection of beams of variable cross section can be reduced to that of beams of constant cross section, by using the *modified*

[1] This solution was obtained by E. Phillips, Annales des Mines, Vol. 1, pp. 195–336, 1852. See also Todhunter and Pearson, History of Elasticity, Vol. 2, part 1, p. 330, and Theorie der Biegungs- und Torsions-Federn v. A. Castigliano, Wien, 1888. The effect of friction between the leaves was discussed by G. Marié, Annales des Mines, Vols. 7–9, 1905 and 1906. D. Landau and P. H. Parr investigated the distribution of load between the individual leaves of the spring, Journ. of the Franklin Inst., Vols. 185, 186, 187. A complete bibliography on mechanical springs was published by the Amer. Soc. Mech. Eng., New York, 1927. See also the book by S. Gross and E. Lehr, "Die Federn," V. D. I. Verlag, 1938.

bending moment diagram in which each ordinate is multiplied by I_0/I, where I is the moment of inertia at that cross section and I_0 is the constant moment of a uniform bar to the deflection of which we will reduce our bar of variable cross section.

For example, the problem of the deflection of a circular shaft (Fig. 186) which has sections of two different diameters, with moments of inertia I_0 and I_1, and loaded by P, can be reduced to that for a circular shaft having a constant moment of inertia of the cross section I_0 as follows. In considering the conjugate beam A_1B_1 instead of a triangular loading $A_1C_1B_1$ representing the bending moment diagram, we use the loading represented by the shaded area. This area is obtained by reducing ordinates of the diagram along the middle portion of the shaft in the ratio I_0/I_1. Determination of deflections and slopes can now be made as in the case of prismatical bars, the magnitude of the deflection and slope at any cross section of the beam being equal to the bending moment and shearing force of the conjugate beam divided by EI_0. It should be noted that in the case represented in Fig. 186 an abrupt change in the diameter of the shaft takes place at a distance $l/4$ from the supports, producing local stresses at these points. These have no substantial effect upon the deflection of the shaft provided the difference in diameter of the two portions is small in comparison with the lengths of these portions.

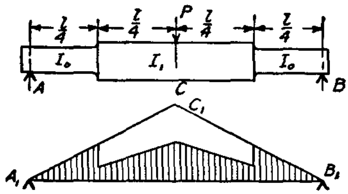

FIG. 186.

The method used for a shaft of variable cross section can be applied also to built-up I beams of variable cross section. An example of an I beam supported at the ends and uniformly loaded is shown in Fig. 187. The bending moment decreases from the middle towards the ends of the beam and the weight of the beam can be reduced by diminishing the number of plates in the flanges as shown schematically in the figure. The deflection of such a beam may be calculated on the basis of the moment of inertia of the middle cross section. The load on the conjugate beam, instead of being a single parabola indicated by the dotted line, is taken as represented by the shaded

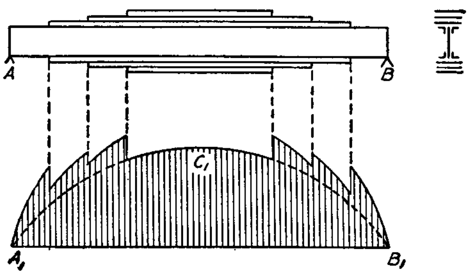

FIG. 187.

diagram in Fig. 187, each diminution in the cross section being compensated for by an increase of the ordinates of the moment diagram in the ratio I_{middle}/I.

Problems

1. A steel plate of the form shown in Fig. 188 is built in at one end and loaded by a force P at the other. Determine the deflection at the end if the length is $2l$, a is the width, h the thickness of the plate, and P the load at the end.

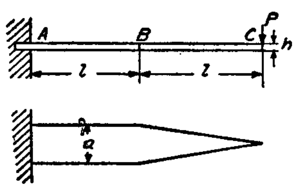

FIG. 188.

Solution. The deflection will consist of three parts:

(1) $\delta_1 = \dfrac{Pl^3}{3EI_z} + \dfrac{Pl^3}{2EI_z}$, the deflection at B,

(2) $\delta_2 = \dfrac{3Pl^3}{2EI_z}$, the deflection at C due to the slope at B,

and

(3) $\delta_3 = \dfrac{Pl^3}{2EI_z}$, the deflection due to bending of part BC of the plate.

The complete deflection is given by $\delta = \delta_1 + \delta_2 + \delta_3$.

2. Solve the previous problem assuming $l = 10$ in., $a = 3$ in., $P = 1{,}000$ lbs. and $\sigma_{\max} = 70{,}000$ lbs. per sq. in.

3. Determine the width d of a carriage leaf spring (Fig. 185) and its deflection if $P = 6{,}000$ lbs., $h = \frac{1}{2}$ in., $l = 24$ in., $\sigma_w = 70{,}000$ lbs. per sq. in., and the number of leaves $n = 10$.

Solution. Considering the leaves of the spring as cut out of a triangular plate (Fig. 185, *b*), the maximum stress will be

$$\sigma_{\max} = \frac{6Pl}{ndh^2},$$

from which

$$d = \frac{6Pl}{n\sigma_w h^2} = \frac{6 \times 6{,}000 \times 24 \times 4}{10 \times 70{,}000} = 4.94 \text{ in.}$$

The deflection is determined from eq. (123),

$$\delta = \frac{70{,}000 \times 24^2}{\frac{1}{2} \times 30 \times 10^6} = 2.69 \text{ in.}$$

4. Compare the deflection at the middle and the slope at the ends of the shaft shown in Fig. 186 with those of a shaft of the same length but of constant cross section whose moment of inertia is equal to I_0. Take $I_1 : I_0 = 2$.

Solution. Due to the greater flexural rigidity at the middle, the slopes at the ends of the shaft shown in Fig. 186 will be less than those at the ends of the cylindrical shaft in the ratio of the shaded area to the total area of the triangle $A_1C_1B_1$. The total area is the loading for the case of the cylindrical shaft. For the values given, this ratio is 5/8 : 1.

The deflections at the middle for the two types of shaft are in the ratio given by the bending moment produced by the shaded area, divided by that produced by the area of the triangle $A_1C_1B_1$. This will be 9/16 : 1.

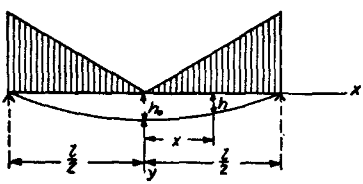

FIG. 189.

5. A beam supported at the ends is loaded as shown in Fig. 189. How should the depth h of the beam vary in order to have a form of equal strength if the width b of the rectangular cross section remains constant along the beam?

Answer.

$$h^2 = h_0^2\left(1 - 8\,\frac{x^3}{l^3}\right).$$

6. Determine the deflection of a steel plate $\frac{1}{2}$ in. thick shown in Fig. 190 under the action of the load $P = 20$ lbs. at the middle.

Solution. Reducing the problem to that of the deflection of a plate of constant width $= 4$ in., the transformed moment area for this case will be represented by the trapezoid $adeb$, and we obtain

$$\delta = \frac{11}{8} \cdot \frac{Pl^3}{48EI_z},$$

where I_z is the moment of inertia at the middle of the span. The numerical value of the deflection can now be easily calculated.

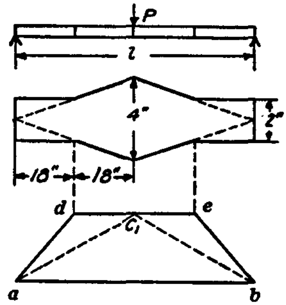

FIG. 190.

7. Determine the maximum deflection of a leaf spring (Fig. 185) if $l = 36$ in., $h = \frac{1}{2}$ in., $E = 30 \times 10^6$ lbs. per sq. in., $\sigma_w =$ 60,000 lbs. per sq. in.

Answer. $\delta = 5.18$ in.

8. A simply supported rectangular beam carries a load P which moves along the span. How should the depth h of the beam vary in order to have a form of equal strength if the width b of the rectangular cross section remains constant along the beam?

Solution. For any given position of the load, the maximum moment occurs under the load. Denoting the distance of the load from the middle of the span by x, the bending moment under the load is

$$M = \frac{P\left(\frac{l}{2} - x\right)\left(\frac{l}{2} + x\right)}{l}.$$

The required depth h of the beam under the load is obtained from the equation

$$\sigma_w = \frac{6M}{bh^2},$$

from which

$$h^2 = \frac{6M}{b\sigma_w} = \frac{6P}{lb\sigma_w}\left(\frac{l^2}{4} - x^2\right)$$

and

$$\frac{h^2}{6Pl/4b\sigma_w} + \frac{x^2}{l^2/4} = 1.$$

It may be seen that in this case the depth of the beam varies following an elliptical law, the semi-axes of the ellipse being

$$l/2 \qquad \text{and} \qquad \sqrt{6Pl/4b\sigma_w}.$$

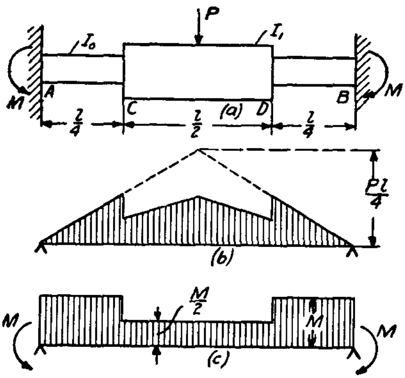

FIG. 191.

9. Determine the bending moments at the ends of the beam AB shown in Fig. 191, with built-in ends and centrally loaded. Take $I_1/I_0 = 2$.

Solution. A solution is obtained by combining the two simple cases shown in (*b*) and (*c*). It is clear that the condition at the built-in ends will be satisfied if the slopes at the ends are equal to zero, i.e., if the reactions due to the imaginary loading (see p. 154) represented by the shaded areas in (*b*) and (*c*) are equal. Therefore the equation for calculating the numerical value of M is

$$\frac{Pl}{4}\frac{l}{2} - \frac{3}{8}\frac{Pl}{4}\frac{l}{2} = Ml - \frac{Ml}{4},$$

from which

$$M = 5/48Pl.$$

10. Solve the above problem on the assumption that two equal loads P are applied at C and D.

Answer. $M = Pl/6$.

47. Beams of Two Different Materials.—There are cases when beams of two or more different materials are used. Figure 192 (*a*) represents a simple case, a wooden beam reinforced by a steel plate bolted to the beam at the bottom. Assuming that there is no sliding between the steel and wood during bending, the theory of solid beams can also be used here. According to this theory elongations and contractions of longitudinal fibers are proportional to the distance from the neutral axis. Due to the fact that the modulus of elasticity of wood is much smaller than that of steel, the wooden part of the beam in bending will be equivalent to a much narrower web of steel as shown in Fig. 192 (*b*). To have the moment

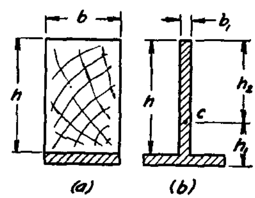

FIG. 192.

of the internal forces unchanged, for a given curvature, i.e., for a given elongation and contraction, the thickness b_1 of this web must be as follows:

$$b_1 = \frac{bE_w}{E_s}. \qquad (a)$$

In this manner the problem is reduced to that of the bending of a steel beam of T section, which can be solved on the basis of the previous theory. Consider, for instance, a simply supported beam 10 feet long loaded at the middle by 1,000 lbs. The cross-sectional dimensions of the wooden part of the beam are $b = 4$ in. and $h = 6$ in. and at the convex side it is reinforced by a steel plate 1 inch wide and $\frac{1}{2}$ inch thick. Assuming $E_w/E_s = 1/20$ and using eq. (a), the equivalent section will have a web 6×0.20 and the flange 1×0.50. The distances of the outermost fibers from the neutral axis (Fig. 192, b) are $h_1 = 2.54$ in. and $h_2 = 3.96$ in. The moment of inertia with respect to the neutral axis is $I_z = 7.37$ in.^{4}, whence the stresses in the outermost fibers are (from eqs. 61, p. 92)

$$\sigma_{\max} = \frac{M_{\max}h_1}{I_z} = \frac{30{,}000 \times 2.54}{7.37} = 10{,}300 \text{ lbs. per sq. in.},$$

$$\sigma_{\min} = -\frac{M_{\max}h_2}{I_z} = -\frac{30{,}000 \times 3.96}{7.37} = -16{,}000 \text{ lbs. per sq. in.}$$

To obtain the maximum compressive stress in the wood of the actual beam the stress $\sigma_{\min}$ obtained above for steel must be multiplied by $E_w/E_s = 1/20$.

As another example of the bending of a beam of two different materials let us consider the case of a bi-metallic strip built up of nickel steel and monel metal (Fig. 193). The bending of such a strip by external forces can be discussed in the same manner as in the above problem of wood and steel, it being necessary only to know the ratio E_m/E_s, in which E_m and E_s are respectively the moduli of elasticity of monel metal and steel. Let us consider now the bending of such a strip due to a change in temperature. The coefficient of thermal expansion of monel metal is larger than that of nickel steel and when the temperature rises bending will occur concave on the side of the steel. This phenomenon of bending of bi-metallic strips under varying temperatures is used in various automatic instruments for regulating temperature[3] (thermostats). Let $h/2$ be the thickness of each metal, b the width of the strip, t the increase in temperature, r the radius of curvature, α_s and α_m

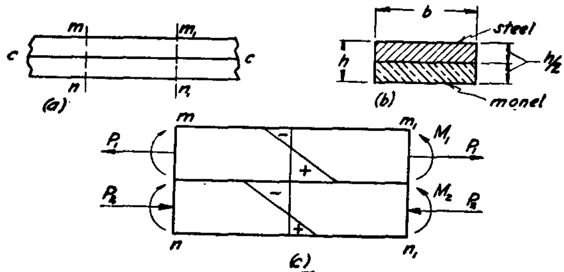

FIG. 193.

the coefficients of thermal expansion of steel and monel respectively, E_sI_s = the flexural rigidity of the steel, E_mI_m = the flexural rigidity of monel metal. When the temperature rises, the strip of monel metal, having a greater coefficient of expansion, will be subjected to both bending and compression and the steel will be subjected to bending and tension. Considering an element of the strip cut out by two adjacent cross sections mn and m_1n_1 (Fig. 193, c), the internal forces over the cross section of the steel can be reduced to a tensile force P_1 and a couple M_1. In the same manner for the monel metal, the internal forces can be reduced to a compressive force P_2 and a couple M_2. The internal forces over any cross section of the beam must be in equilibrium; therefore

$$P_1 = P_2 = P$$

and

$$\frac{Ph}{2} = M_1 + M_2. \qquad (a)$$

Using equations:

$$M_1 = \frac{E_sI_s}{r}; \qquad M_2 = \frac{E_mI_m}{r}.$$

Then, from eq. (a),

$$\frac{Ph}{2} = \frac{E_sI_s}{r} + \frac{E_mI_m}{r}, \qquad (b)$$

[3] See author's paper in Journal of the Optical Soc. of Amer., Vol. 11, 1925, p. 233.

another equation for determining P and r can be derived from the condition that at the joining surface, c–c, the unit elongation of monel metal and steel must be the same; therefore

$$\alpha_s t + \frac{2P_1}{E_s hb} + \frac{h}{4r} = \alpha_m t - \frac{2P_2}{E_m hb} - \frac{h}{4r}$$

or

$$\frac{2P}{hb}\left(\frac{1}{E_s} + \frac{1}{E_m}\right) = (\alpha_m - \alpha_s)t - \frac{h}{2r}. \tag{c}$$

From eqs. (b) and (c) we obtain

$$\frac{4}{bh^2 r}(E_s I_s + E_m I_m)\left(\frac{1}{E_s} + \frac{1}{E_m}\right) = (\alpha_m - \alpha_s)t - \frac{h}{2r}. \tag{d}$$

Substituting in this equation

$$I_s = I_m = \frac{bh^3}{96} \qquad \text{and} \qquad E_s = 1.15 E_m,$$

the following approximate equation is obtained:

$$\frac{1}{r} = \frac{3}{2}\frac{(\alpha_m - \alpha_s)t}{h}. \tag{e}$$

Now, from eq. (b),

$$P = \frac{3}{h^2}(\alpha_m - \alpha_s)t(E_s I_s + E_m I_m) = \frac{bh}{32}(\alpha_m - \alpha_s)t(E_s + E_m) \tag{f}$$

and

$$M_1 = \frac{3}{2}\frac{(\alpha_m - \alpha_s)t}{h}E_s I_s\,; \qquad M_2 = \frac{3}{2}\frac{(\alpha_m - \alpha_s)t}{h}E_m I_m. \tag{g}$$

From eqs. (f) and (g) P, M_1 and M_2 can be determined. The maximum stress in the steel is obtained by adding to the tensile stress produced by the force P the tensile stress due to the curvature $1/r$:

$$\sigma_{\max} = \frac{2P}{bh} + \frac{h}{4}\frac{E_s}{r} = \frac{4}{bh^2 r}\left(E_s I_s + E_m I_m + \frac{bh^3}{16}E_s\right).$$

Assuming, for example, that both metals have the same modulus E, we obtain

$$\sigma_{\max} = \frac{hE}{3r},$$

or, by using eq. (e),[4]

$$\sigma_{\max} = \tfrac{1}{2}Et(\alpha_m - \alpha_s).$$

[4] This equation holds also for $E_s = E_m$.

For $E = 27 \times 10^6$ lbs. per sq. in., $t = 200$ degrees Centigrade and $\alpha_m - \alpha_s = 4 \times 10^{-6}$ we find

$$\sigma_{\max} = 10{,}800 \text{ lbs. per sq. in.}$$

The distribution of stresses due to heating is shown in Fig. 193 (c).

Problems

1. Find the safe bending moment for the wooden beam reinforced by a steel plate, Fig. 192, if $b = 6$ in., $h = 8$ in., and the thickness of the steel plate is $\frac{1}{2}$ in. Assume $E_w = 1.5 \times 10^6$ lbs. per sq. in., $E_s = 30 \times 10^6$ lbs. per sq. in., $\sigma_w = 1{,}200$ lbs. per sq. in. for wood and $\sigma_w = 16{,}000$ lbs. per sq. in. for steel.

2. Assume that the wooden beam of the previous problem is reinforced at the top with a steel plate 2 in. wide and 1 in. thick and at the bottom with a steel plate 6 in. wide and $\frac{1}{2}$ in. thick. Calculate the safe bending moment if E and σ_w are the same as in the previous problem.

Answer. $M = 308{,}000$ in. lbs.

3. A bimetallic strip has a length $l = 1$ in. Find the deflection at the middle produced by a temperature increase equal to 200 degrees Centigrade if $E_s = 1.15E_m$ and $\alpha_m - \alpha_s = 4 \times 10^{-6}$.

48. Reinforced-Concrete Beams.—It is well known that the strength of concrete is much greater in compression than in tension. Hence a rectangular beam of concrete will fail from the tensile stresses on the convex side. The beam can be greatly strengthened by the addition of steel bars on the convex side as shown in Fig. 194. As concrete grips the steel strongly there will be no sliding of the steel bars with respect to the concrete during bending and the methods developed in the previous article can also be used here for calculating bending stresses.

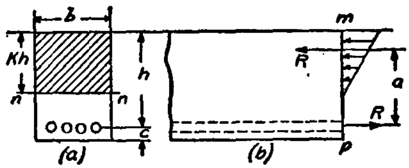

FIG. 194.

In practice the cross-sectional area of the steel bars is usually such that the tensile strength of the concrete on the convex side is overcome before yielding of the steel begins and at larger loads the steel alone takes practically all the tension. Hence it is the established practice in calculating bending stresses in reinforced-concrete beams to assume that all the tension is taken by the steel and all the compression by the concrete. Replacing the tensile forces in the steel bars by their resultant R, the distribution of internal forces over any cross section mp will be as shown in Fig. 194 (b). Assuming as before that cross sections remain plane during bending and denoting by kh the distance of the neutral axis nn from the top,[5] the maximum longitudinal unit contraction in the concrete ϵ_c, and the unit elongation of the axes of the steel bars ϵ_s, are given by the following:

$$\epsilon_c = -\frac{kh}{r}; \qquad \epsilon_s = \frac{(1 - k)h}{r}. \tag{a}$$

Concrete does not follow Hooke's law and a compression test diagram for this material has a shape similar to that for cast iron in Fig. 2 (b). As the compression stress increases, the slope of the tangent to the diagram decreases, i.e., the modulus of concrete decreases with an increase in stress. In calculating stresses in reinforced-concrete beams it is the usual practice to assume that Hooke's law holds for concrete, and to compensate for the variable modulus by taking a lower value for this modulus than that obtained from compression tests when the stresses are small. In specifications for reinforced-concrete it is usually assumed that $E_s/E_c = 15$. Then, from eqs. (a), the maximum compressive stress in the concrete and the tensile stress in the steel[6] are, respectively,

$$\sigma_c = -\frac{kh}{r}E_c; \qquad \sigma_s = \frac{(1 - k)h}{r}E_s. \tag{b}$$

We will now calculate the position of the neutral axis from the condition that the normal forces over the cross section mp reduce to a couple equal to the bending moment. The sum of the compressive forces in the concrete must equal the tensile force R in the steel bars, or

$$-\frac{bkh\sigma_c}{2} = \sigma_s A_s, \tag{c}$$

---

[5] k is a numerical factor less than unity.

[6] The cross-sectional dimensions of the steel bars are usually small and the average tensile stress is used instead of the maximum stress.

where A_s is the total cross sectional area of steel. Using the notation $A_s/bh = n_1$ and $E_s/E_c = n$, we get from (*c*) and (*b*)

$$k^2 = 2(1 - k)nn_1, \tag{d}$$

from which

$$k = -nn_1 + \sqrt{(nn_1)^2 + 2nn_1}. \tag{124}$$

After determining the position of the neutral axis from eq. (124), the ratio between the maximum stress in the concrete and the stress in the steel becomes, from eqs. (*b*),

$$-\frac{\sigma_c}{\sigma_s} = \frac{k}{(1 - k)n}. \tag{125}$$

The distance a between the resultants R of the compressive and tensile forces acting over the cross section of the beam (Fig. 194, *b*) is

$$a = \frac{2}{3}kh + (1 - k)h = \left(1 - \frac{k}{3}\right)h \tag{126}$$

and the moment of the internal forces equal to the bending moment M is

$$aR = aA_s\sigma_s = -\frac{akbh}{2}\sigma_c = M,$$

from which

$$\sigma_s = \frac{M}{aA_s}, \tag{127}$$

$$\sigma_c = -\frac{2M}{akbh}. \tag{128}$$

By using eqs. (124) to (128) the bending stresses in reinforced-concrete beams are easily calculated.

Problems

1. If $E_s/E_c = 15$ and $A_s = 0.008bh$, determine the distance of the neutral axis from the top of the beam (Fig. 194).

Solution. Substituting in eq. (124) $n = 15$, $n_1 = 0.008$, we obtain $k = 0.384$ and the distance from the top of the beam is $kh = 0.384h$.

2. Determine the ratio $n_1 = A_s/bh$ if the maximum tensile stress

in the steel is 12,000 lbs. per sq. in., the maximum compressive stress in the concrete is 645 lbs. per sq. in., and $E_s/E_c = n = 15$.

Solution. From eq. (125) $k = 0.446$. Then, from eq. (*d*),

$$n_1 = \frac{k^2}{2(1 - k)n} = 0.012.$$

3. Determine the ratio n_1 if the maximum compressive stress in the concrete is one-twentieth of the tensile stress in the steel.

Answer. $n_1 = 0.0107$.

4. If $n = 15$ and the working compressive stress for concrete is 650 lbs. per sq. in., determine the safe load at the middle of a reinforced-concrete beam 10 feet long supported at the ends and having $b = 10$ in., $h = 12$ in., $A_s = 1.17$ sq. in.

Answer. $P = 5{,}570$ lbs.

5. Calculate the maximum moment which a concrete beam will safely carry if $b = 8$ in., $h = 12$ in., $A_s = 2$ sq. in., $E_s/E_c = 12$, and the working stress for steel is 15,000 lbs. per sq. in. and for concrete 800 lbs. per sq. in.

Answer. $M = 16{,}000$ ft. lbs.

6. Determine the value of k for which the maximum permissible stresses in the concrete and the steel are realized simultaneously.

Solution. Let σ_c and σ_s be the allowable stresses for the concrete and the steel. Then taking the ratio of these stresses, as given by formulas (*b*), and considering only the absolute value of this ratio, we obtain

$$\frac{\sigma_c}{\sigma_s} = \frac{kE_c}{(1 - k)E_s},$$

from which

$$k = \frac{\sigma_c}{\sigma_c + \sigma_s \dfrac{E_c}{E_s}}.$$

If this condition is satisfied the beam is said to have *balanced reinforcement*. Having k and using equation (126) the depth is obtained from equation (128) and the area A_s from equation (127).

7. Determine the steel ratio $n_1 = A_s/bh$ if $\sigma_s = 12{,}000$ lbs. per sq. in., $\sigma_c = 645$ lbs. per sq. in., and $n = E_s/E_c = 15$.

Solution. From the formula of the preceding problem we find

$$k = 0.446.$$

Then, substituting in equation (*d*), we obtain

$$n_1 = 0.012.$$

8. Design a beam 10 in. wide to withstand safely a moment of 22,500 ft. lbs. if $\sigma_c = 750$ lbs. per sq. in., $\sigma_s = 12{,}000$ lbs. per sq. in., and $E_s/E_c = 12$. Find the depth h and the steel area A_s. Assume balanced reinforcement as in problem 6.

49. Shearing Stresses in Reinforced-Concrete Beams.—Using the same method as in article 26, by considering an element mnm_1n_1 between the two adjacent cross sections mp and m_1p_1 (Fig. 195) it can be concluded that the maximum shearing stress τ_{xy} will act over the neutral surface nn_1.

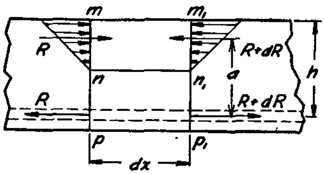

FIG. 195.

Denoting by dR the difference between the compressive forces on the concrete on cross sections mp and m_1p_1, the shearing stress τ_{xy} over the neutral surface is found from the following:

$$(\tau_{xy})_{\max} b dx = dR,$$

from which

$$(\tau_{xy})_{\max} = \frac{1}{b}\frac{dR}{dx}. \qquad (a)$$

Since the bending moment is

$$M = Ra,$$

eq. (a) becomes

$$(\tau_{xy})_{\max} = \frac{1}{ab}\frac{dM}{dx} = \frac{V}{ab}, \qquad (b)$$

in which V is the shearing force at the cross section considered. Using eq. (126), the above equation for shearing stresses becomes

$$(\tau_{xy})_{\max} = \frac{3V}{bh(3 - k)}. \qquad (129)$$

In practical calculations not only the shearing stresses over the neutral surface but also the shearing stresses over the surface of contact between the steel and concrete are of importance. Considering again the two adjacent cross sections (Fig. 195), the difference between the tensile forces in the steel bars at these two sections is

$$dR = \frac{Vdx}{a}.$$

This difference is balanced by the shearing stresses distributed over the surface of the bars. Denoting by A the total lateral surface of all the steel bars per unit length of the beam, the shearing stress over the surface of the bars is

$$\frac{dR}{Adx} = \frac{V}{Aa} = \frac{3V}{(3 - k)hA}. \tag{130}$$

This stress becomes larger than the stress on the neutral surface (eq. 129) if A is less than b. To increase A and at the same time keep the cross sectional area of steel constant, it is only necessary to increase the number of bars and decrease their diameter.

CHAPTER VIII

COMBINED BENDING AND TENSION OR COMPRESSION; THEORY OF COLUMNS

50. Bending Accompanied by Compression or Tension.—It is assumed here that a prismatical bar is loaded by forces in one of its planes of symmetry, but, whereas in the previous discussion these forces were all transverse, here they may have components along the axis of the bar. A simple case of this kind is shown in Fig. 196, which represents a column loaded by an inclined force P. This force is resolved into a transverse component N and a longitudinal T and it is assumed that the column is comparatively stiff with a deflection so small that it can be neglected in discussing the stresses produced by the force T. Then the resultant stress at any point is obtained by superposing the compressive stress due to the force T on the bending stress produced by the transverse load N. The case of a flexible column, in which the thrust, due to deflection of the column (Fig. 196, b), has a considerable effect on the bending, will be discussed later (see Art. 53). The stress due to force T is constant for all cross sections of the column and equal to T/A where A is the cross-sectional area. The bending stress depends upon the moment, which increases from zero at the top to a maximum Nl at the bottom. Hence the dangerous section is at the built-in end, and the stress there, for a point at distance y from the z axis, is

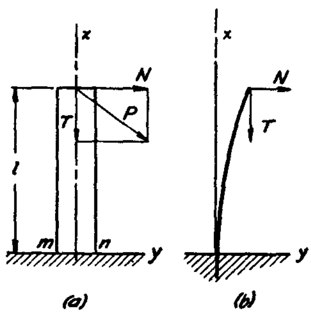

FIG. 196.

$$\sigma_x = -\frac{T}{A} - \frac{Nly}{I_z}. \qquad (a)$$

Assuming, for instance, that the cross section of the column is a rectangle $b \times h$ with the side h parallel to the plane of bending, we have $A = bh$ and $I_z = bh^3/12$. Maximum compressive stress will be at point n, at which

$$(\sigma_x)_{\min} = -\frac{6Nl}{bh^2} - \frac{T}{bh}. \qquad (b)$$

This stress is numerically the largest.

At point m we obtain

$$(\sigma_x)_{\max} = \frac{6Nl}{bh^2} - \frac{T}{bh}.$$

When the force P is not parallel to one of the two principal planes of bending, the bending stresses, produced by its transverse component N, are found by resolving N into components parallel to those planes (see the discussion in art. 38). The resultant stress at any point is obtained by superposing these bending stresses with the compressive stress produced by the longitudinal force.

Problems

1. Determine the maximum compressive stress in the circular wooden poles 20 feet high and 8 inches in diameter shown in Fig. 197 if the load P on the wire ABC is 60 lbs. The tensile force in

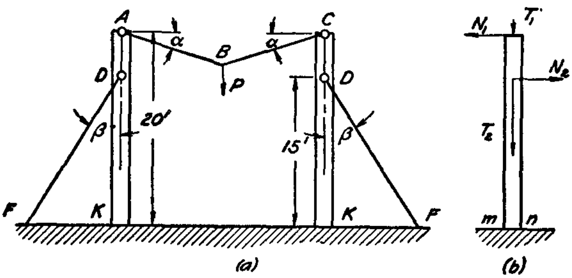

Fig. 197.

each cable DF is $S = 1{,}000$ lbs.; $\tan\alpha = 1/10$; $\sin\beta = 1/5$, and $\overline{Dk} = 15$ feet.

Solution. The components of the force in the wire BC (Fig. 197, b) are $N_1 = 300$ lbs.; $T_1 = 30$ lbs. The components of the force in the cable DF are $N_2 = 200$ lbs.; $T_2 = 980$ lbs. The maximum bending moment is found to be at the built-in end, where $M_{max} = 36{,}000$ lbs. in. The thrust at the same cross section is $T_1 + T_2 = 1{,}010$ lbs. The maximum compressive stress at the point m is

$$\sigma = \frac{4 \times 1{,}010}{\pi d^2} + \frac{32 \times 36{,}000}{\pi d^3} = 21 + 715 = 736 \text{ lbs. per sq. in.}$$

2. Determine the maximum tensile stress in the rectangular wooden beam shown in Fig. 198 if $S = 4{,}000$ lbs.; $b = 8$ in.; $h = 10$ in.

Answer.

$$(\sigma_x)_{max} = \frac{6 \times 72 \times 1{,}000}{8 \times 100} + \frac{4{,}000}{80} = 590 \text{ lbs. per sq. in.}$$

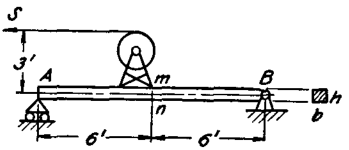

FIG. 198.

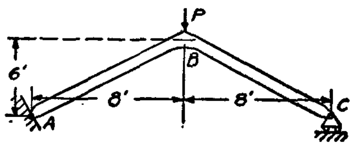

FIG. 199.

3. Determine the maximum compressive stress in the structure ABC, which supports a load $P = 2{,}000$ lbs. (Fig. 199), has a rigid connection between the bars at B, an immovable hinge at A, and a movable support at C. The cross section of the bars AB and BC is a square 10×10 in.

Answer.

$$\frac{6 \times 1{,}000 \times 8 \times 12}{10^3} + \frac{600}{10^2} = 582 \text{ lbs. per sq. in.}$$

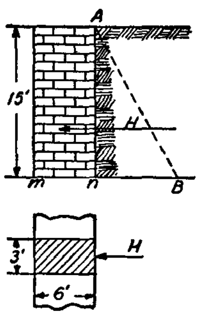

FIG. 200.

4. A brick wall 6 feet thick and 15 feet high supports sand pressure (Fig. 200). Determine the maximum tensile compressive stresses at the bottom of the wall if its weight is $\gamma = 150$ lbs. per cubic foot and the lateral pressure of the sand is 10,000 lbs. per yard of the wall. The distribution of the sand pressure along the height of the wall follows a linear law, given by the line AB.

Answer. The stress at $m = -\frac{150 \times 15}{144} - \frac{10{,}000 \times 60 \times 6}{36 \times 72^2}$ $= -15.6 - 19.3 = -34.9$ lbs. per sq. in. The stress at

$$n = -\frac{150 \times 15}{144} + \frac{10{,}000 \times 60 \times 6}{36 \times 72^2} = 3.7 \text{ lbs. per sq. in.}$$

5. Determine the thickness of the wall in the previous problem which will give zero stress at n.

Answer. 80 in.

6. A circular column, 6 ft. high, Fig. 196, is acted upon by a force P the components N and T of which are equal to 1,000 lbs. Find the diameter of the column if the maximum compressive stress is 1,000 lbs. per sq. in.

7. Find σ_{max} and σ_{min} at the cross section at the middle of the bar BC, Fig. 199, if, instead of the concentrated load P, a uniform vertical load $q = 400$ lbs. per ft. is distributed along the axis ABC.

8. A circular bar AB hinged at B and supported by a smooth vertical surface (no friction) at A is submitted to the action of its own weight. Determine the position of the cross section mn (Fig. 201) at which the compressive stress is maximum.

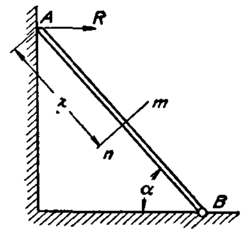

FIG. 201.

Solution. Denote by l the length of the bar, by q its weight per unit length and by α its angle of inclination to the horizon. The horizontal reaction at A is $R = (ql/2) \cot \alpha$. The compressive force at any cross section mn, distant x from A, is $qx \sin \alpha + (ql/2) \times (\cos^2 \alpha/\sin \alpha)$; the bending moment at the same cross section is $M = (ql/2) \cos \alpha \cdot x - (q \cos \alpha/2)x^2$. The maximum compressive stress at the cross section mn is

$$\frac{4}{\pi d^2}\left(qx \sin \alpha + \frac{ql}{2}\frac{\cos^2 \alpha}{\sin \alpha}\right) + \frac{32}{\pi d^3}\left(\frac{ql}{2}\cos \alpha \cdot x - \frac{q \cos \alpha}{2} x^2\right),$$

where d is the diameter of the bar.

Equating the derivative of this stress with respect to x to zero, we obtain the required distance

$$x = \frac{l}{2} + \frac{d}{8} \tan \alpha.$$

9. The bar shown in Fig. 196 is 6′ long and has a 12″ diameter. Determine the magnitude of the force P if its components N and T are equal and the maximum compressive stress at n is 1,000 lbs. per sq. in.

Answer. $P = 3{,}260$ lbs.

10. A force P produces bending of the bar ABC built in at A (Fig. 202). Determine the angle of rotation of the end C, during

bending, if the bending moments at A and at B are numerically equal.

Solution. From the equality of the bending moments at A and B, it follows that the force P passes through the midpoint D of the bar AB. Then $P_x = P_y l/2a$, and the components P_x and P_y may now be calculated. The rotation of the cross section B due to bending of the portion AB by the component P_y is $P_y l^2/2EI$ in a clockwise direction. The rotation of the same cross section due to the component P_x is $P_x al/EI$ in a counter-clockwise direction. The rotation of the cross section C with respect to the cross section B, due to bending of the portion BC of the bar, is $P_x a^2/2EI$ in a counter-clockwise direction. The total angle of rotation of the end C in a clockwise direction is

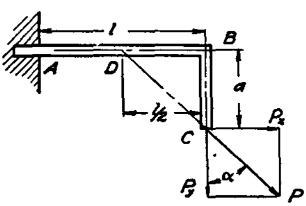

FIG. 202.

$$\frac{P_y l^2}{2EI} - \frac{P_x al}{EI} - \frac{P_x a^2}{2EI} = -\frac{P_x a^2}{2EI}.$$

51. Eccentric Loading of a Short Strut.—Eccentric loading is a particular case of the combination of direct and bending stresses. When the length of the bar is not very large in comparison with its lateral dimensions, its deflection is so small that it can be neglected in comparison with the initial eccentricity e; hence the method of superposition may be used.[1] Take, for example, the case of compression by a longitudinal force P applied with an eccentricity e (Fig. 203) on one of the two principal axes of the cross section. Then, if we put two equal and opposite forces P at the centroid O of the cross section, the problem is not changed, as they are equivalent to zero, and we obtain an axial compression by the force P producing direct compressive stresses, $-(P/A)$, as shown in

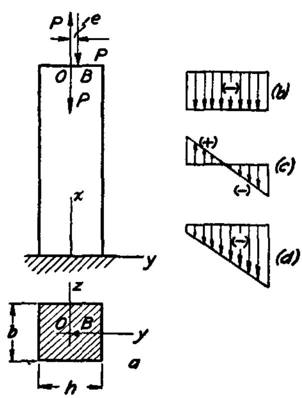

FIG. 203.

[1] For the case of eccentric loading of long bars see art. 53.

Fig. 203 (*b*), and bending in one of the principal planes by the couple Pe producing bending stresses, $-(Pey/I_z)$, as shown in Fig. 203 (*c*). The total stress is then

$$\sigma_x = -\frac{P}{A} - \frac{Pey}{I_z}. \tag{a}$$

The distribution diagram of this total stress is shown in Fig. 203 (*d*). It is assumed that the maximum bending stress is less than the direct stress; then there will be compressive stresses all over the cross section of the bar. If the maximum bending stress is larger than the direct compressive stress, there will be a line of zero stress parallel to the z axis, dividing the cross section into two zones with tensile stresses on the left and compressive stresses on the right. For a rectangular cross section with sides h and b (Fig. 203, *a*) eq. (*a*) becomes

$$\sigma_x = -\frac{P}{bh} - \frac{12Pey}{bh^3} \tag{a'}$$

and we obtain, by putting $y = -(h/2)$,

$$(\sigma_x)_{\max} = -\frac{P}{bh} + \frac{6Pe}{bh^2} = \frac{P}{bh}\left(-1 + \frac{6e}{h}\right), \tag{b}$$

and, by putting $y = h/2$,

$$(\sigma_x)_{\min} = -\frac{P}{bh} - \frac{6Pe}{bh^2} = -\frac{P}{bh}\left(1 + \frac{6e}{h}\right). \tag{c}$$

It may be seen that when $e < h/6$ there is no reversal of sign of the stresses over the cross section; when $e = h/6$, the maximum compressive stress, from eq. (*c*), is $2P/bh$, and the stress on the opposite side of the rectangular cross section is zero; when $e > h/6$, there is a reversal of sign of the stress and the position of the line of zero stress is obtained by equating to zero the general expression (*a'*) for σ_x, giving

$$y = -\frac{h^2}{12e} \tag{d}$$

or, using the notation k_z for the *radius of gyration* with respect to the z axis (see appendix),

$$y = -\frac{k_z^2}{e}. \tag{131}$$

It will be seen that the distance of the line of zero stress from the centroid O diminishes as the eccentricity e increases. The same discussion applies as well to the case of eccentric loading in tension. Equation (131) may also be used for other shapes of cross sections if the point of application of the load is on one of the principal axes of inertia.

Let us consider now the case in which B, the point of application of the eccentric compressive force P, is not on one of the two principal axes of the cross section, taken as the y and the z axes, in Fig. 204. Using m and n as the coordinates of this point, the moments of P with respect to the y and z axes are Pn and Pm respectively. By superposition, the stress at any point F of the cross section is

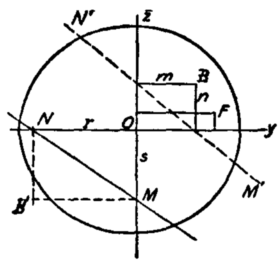

FIG. 204.

$$\sigma_x = -\frac{P}{A} - \frac{Pmy}{I_z} - \frac{Pnz}{I_y}, \tag{e}$$

in which the first term on the right side represents the direct stress and the two other terms are the bending stresses produced by the moments Pm and Pn respectively. It may be seen that the stress distribution follows a linear law. The equation of the line of zero stress is obtained by equating the right side of eq. (e) to zero. Using the notation $I_z/A = k_z^2$ and $I_y/A = k_y^2$, where k_z and k_y are the radii of gyration with respect to the z and y axes respectively, this gives

$$\frac{my}{k_z^2} + \frac{nz}{k_y^2} + 1 = 0.$$

By substituting in this equation first $y = 0$ and then $z = 0$ we obtain the points M and N of intersection of the line of zero stress with the axes of coordinates z and y (Fig. 204). The coordinates of these points, s and r, are

$$s = -\frac{k_y^2}{r}. \qquad r = -\frac{k_z^2}{m}. \tag{g}$$

From these equations we obtain

$$n = -\frac{k_y^2}{s}; \qquad m = -\frac{k_z^2}{r}.$$

These equations have the same form as eqs. (g) and it can be concluded that when the load is put at the point B' with the coordinates s and r, the corresponding line of zero stress will be the line $N'M'$, indicated in the figure by the dotted line, and cutting off from the y and z axes the lengths m and n.

There is another important relation between the point of application B of the load and the position of the corresponding line of zero stress, namely, as B moves along a line B_1B_2 (Fig. 205), the corresponding line of zero stress turns about a certain constant point B'. This is proven as follows: Resolve the load at B into two parallel components, one at B_1 and the other at B_2. The component at B_1 acts in the principal plane xz; hence the corresponding line of zero stress is parallel to the y axis and its intercept on OZ, as found from an equation analogous to eq. (131), is

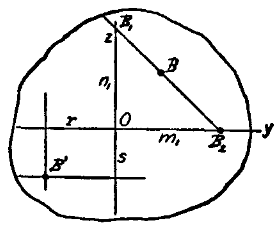

FIG. 205.

$$s = -\frac{k_y^2}{n_1}. \tag{h}$$

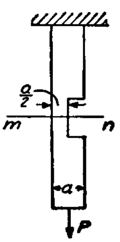

FIG. 206.

Similarly the line of zero stress for the component B_2 is parallel to the z axis and its distance from this axis is

$$r = -\frac{k_z^2}{m_1}. \tag{k}$$

For any position of the load on the line B_1B_2 there will be zero stress at B'; hence as the point of application of the load moves along the straight line B_1B_2, the corresponding line of zero stress turns about the point B', the coordinates of which are determined by eqs. (h) and (k).

Problems

1. The cross-sectional area of a square bar is reduced one half at mn (Fig. 206). Determine the maximum tensile stress at this cross section produced by an axial load P.

Answer.

$$(\sigma_x)_{\max} = \frac{2P}{a^2} + \frac{Pa}{4}\frac{24}{a^3} = \frac{8P}{a^2}.$$

2. Solve the above problem, assuming the bar to have a circular cross section.

3. A bar of ⊥ section is excentrically loaded by the forces P (Fig. 207). Determine the maximum tensile and compressive stresses in this bar if $d = 1''$, $h = 5''$, the width of the flange $b = 5''$, $P = 4{,}000$ lbs.

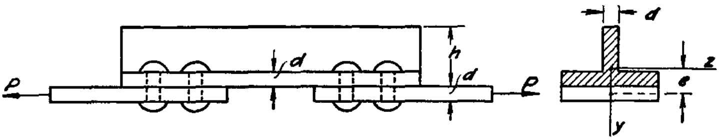

FIG. 207.

Solution. The distances of the centroid of the ⊥ section from the bottom and the top are respectively $h_1 = \frac{29}{18}$ in. and $h_2 = \frac{61}{18}$ in. The eccentricity of the force P is $e = \frac{1}{2} + \frac{29}{18} = 2\frac{1}{9}$ in. The moment of inertia $I_z = 19.64$ in.[4] The bending stresses are

$$(\sigma_x)_{\max} = \frac{P \cdot e \cdot h_1}{I_z} = \frac{4{,}000 \times 2\frac{1}{9} \times 29}{19.64 \times 18} = 693 \text{ lbs. per sq. in.},$$

$$(\sigma_x)_{\min} = -\frac{Peh_2}{I_z} = \frac{4{,}000 \times 2\frac{1}{9} \times 61}{19.64 \times 18} = 1{,}458 \text{ lbs. per sq. in.}$$

Combining with the direct stress $P/A = 4{,}000/9 = 444$ lbs. per sq. in., we obtain the maximum tensile stress $693 + 444 = 1{,}137$ lbs. per sq. in., maximum compressive stresses $1{,}458 - 444 = 1{,}014$ lbs. per sq. in.

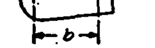

FIG. 208.

4. Determine the maximum tensile stress at the section mn of the clamp shown in Fig. 208 if $P = 300$ lbs., $b = 3$ in., and the cross section is a rectangle with the dimensions 1 in. $\times \frac{1}{4}$ in.

Answer. $\sigma_{\max} = 22{,}800$ lbs. per sq. in.

5. Determine the width of the cross section mn in the previous problem to make $\sigma_{\max} = 20{,}000$ lbs. per sq. in.

6. Find the maximum and the minimum stress at the built-in cross section of the rectangular column shown in Fig. 203, if $b = 10$ in., $h = 12$ in., $P = 5{,}000$ lbs., and the coordinates of the point B of application of the load, Fig. 204, are $m = n = 2$ in. Find the position of the neutral axis.

52. The Core of a Section.—In the previous article it was shown that for a small eccentricity e the normal stresses have the same sign over all of the cross section of an eccentrically loaded bar. For larger values of e the line of zero stress cuts the cross section and there is a reversal of sign of the stress. In the case of a material very weak in tension, such as brick work, the question arises to determine the region in which the compressive load may be applied without producing any tensile stress on the cross section. This region is called the *core of the cross section.* The method of determining the core is illustrated in the following simple examples.

In the case of a *circular* cross section of radius R we can conclude from symmetry that the core is a circle. The radius a of this circle is found from the condition that when the point of application of the load is on the boundary of the core the corresponding line of zero stress must be tangent to the boundary of the cross section. Remembering that the moment of inertia of a circle about a diameter is $\pi R^4/4$ (see appendix), and hence the radius of gyration is $k = \sqrt{I/A} = R/2$, we find from eq. (131) (p. 232), by substituting a for e and R for $-y$, that

$$a = \frac{k^2}{R} = \frac{R}{4}, \tag{132}$$

i.e., the radius of the core is one quarter of the radius of the cross section.

For the case of a *circular ring section* with outer radius R_0 and inner radius R_i we have

$$I = \frac{\pi}{4}(R_0^4 - R_i^4); \qquad k^2 = \frac{I}{A} = \frac{R_0^2 + R_i^2}{4},$$

and the radius of the core, from eq. (131), becomes

$$a = \frac{k^2}{R_0} = \frac{R_0^2 + R_i^2}{4R_0}. \tag{133}$$

For $R_i = 0$, eq. (133) coincides with eq. (132). For a very narrow ring, when R_i approaches R_0 the radius a of the core approaches the value $R_0/2$.

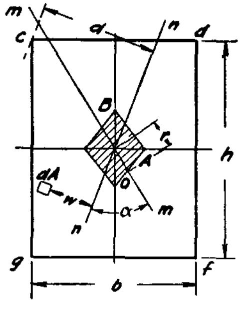

FIG. 209.

In the case of a *rectangular cross section* (Fig. 209) the line of zero stress coincides with the side cg when the load is applied at point A, a distance $b/6$ from the centroid (see p. 231). In the same manner the line of zero stress coincides with the side gf when the load is at the point B, a distance $h/6$ from the centroid. As the load moves along the line AB the neutral axis rotates about the point g (see p. 233) without cutting the cross section. Hence AB is one of the sides of the core. The other sides follow from symmetry. The core is therefore a rhombus with diagonals equal to $h/3$ and $b/3$. As long as the point of application of the load remains within this rhombus the line of zero stress does not cut the cross section and there will be no reversal in the sign of the stress.

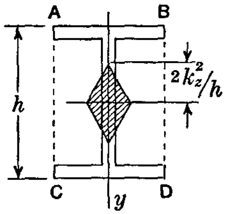

FIG. 210.

For an I section (Fig. 210) the extreme positions of the lines of zero stress, in which they do not cut the cross section, are given by the sides AB and CD and by the dotted lines AC and BD. The corresponding positions of the point of application of the load may be determined from eq. (131). From symmetry it may be concluded that these points will be the corners of a rhombus, shaded in Fig. 210.

If the point of application of the eccentric load is outside the core of a cross section, the corresponding line of zero stress crosses the section and the load produces not only compressive but also tensile stresses. If the material does not resist tensile stresses at all, part of the cross section will be inactive and the rest will carry compressive stresses only. Take, for example, a rectangular cross section (Fig. 211) with the point of application A of the load on the principal axis y and at a distance c from the edge of the section. If c is less than $h/3$, part of the cross section will not work. The working portion may be found from the condition that the distri-

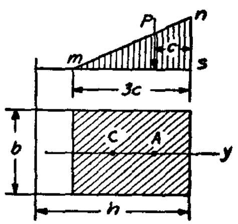

FIG. 211.

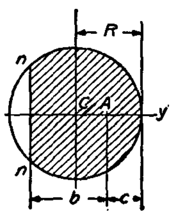

FIG. 212.

bution of the compressive forces over the cross section follows a linear law, represented in the figure by the line mn, and that the resultant of these forces is P. Since this resultant must pass through the centroid of the triangle mns, the dimension ms of the working portion of the cross section must be equal to $3c$.

In the case of a circular cross section (Fig. 212), if the eccentricity $\overline{CA}$ of the load is larger than $R/4$ and the material does not resist tensile stresses, only a portion of the cross section will work. Let the line nn, perpendicular to AC, be the limit of this portion. Its distance b from the point A may be found from the conditions that (1) the compressive stresses are proportional to the distance y from nn, (2) the sum of the compressive forces over the working portion of the cross section is equal to the load P, and (3) the moment of these forces with respect to nn is equal to the moment Pb of the load P with respect to the same axis. Denoting the maximum compressive stress by σ_{max}, the compressive stress at any distance y from nn is

$$\sigma = \frac{y\sigma_{max}}{b + c}$$

and the equations for determining b become

$$\int \frac{y\sigma_{max}}{b + c} dA = P; \qquad \int \frac{y^2\sigma_{max}}{b + c} dA = Pb,$$

from which

$$b = \frac{I_{nn}}{Q_{nn}}, \tag{a}$$

in which $I_{nn} = \int y^2 dA$ is the moment of inertia of the working portion of the cross section with respect to the nn axis and $Q_{nn} = \int y dA$ is the moment of the working portion of the cross section with respect to the same axis. By using eq. (*a*) the position of A for any given position of nn may easily be found. The same equation may also be used for other shapes of cross sections, provided A is on one of the principal axes.[2] If the load is not on a principal axis, the problem of determining the working portion of the cross section becomes more complicated.[3]

By using the notion of the core, the calculation of maximum bending stresses when the bending is not in a principal plane may be greatly simplified. For example, in Fig. 209 let mm be the axial

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[2] For the cases of circular cross sections and circular ring sections, which are of importance in calculating stresses in chimneys, tables have been published which simplify these calculations. See Keck, Z. Hannover. Arch. u. Ing. Ver., 1882, p. 627; see also V. D. I., 1902, p. 1321, and the paper by G. Dreyer in "Die Bautechnik," 1925.

[3] Some calculations for a rectangular cross section will be found in the following papers: Engesser, Zentralblatt d. Bauv., 1919, p. 429; K. Pohl, Der Eisenbau, 1918, p. 211; O. Henkel, Zentralbl. d. Bauv., 1918, p. 447; F. K. Esling, Proc. of the Institute of Civ. Eng., 1905–1906, part 3.

plane of the beam in which a bending moment M acts and nn the corresponding neutral axis, which makes an angle α with the plane mm (see p. 166). Denoting by $\sigma_{\max}$ the maximum stress in the most remote point c and by d its distance from the neutral axis nn, the stress at any other point, distant w from nn, is $\sigma = \sigma_{\max}w/d$, and the moment of all forces distributed over the cross section with respect to the axis nn is

$$\int \frac{\sigma_{\max} w^2}{d}\, dA = \frac{\sigma_{\max}}{d} I_{nn}, \tag{b}$$

in which I_{nn} is the moment of inertia of the cross section with respect to the n–n axis. The moment of the external forces with respect to the same axis is $M \sin \alpha$. Equating this to (b), we have

$$\sigma_{\max} = \frac{Md \sin \alpha}{I_{nn}}. \tag{c}$$

This equation may be greatly simplified by using the property of the core of the cross section.[4] Let O be the point of intersection of the plane mm with the core and r its distance from the centroid of the cross section. From the property of the core it follows that a compressive force P at O produces zero stress at the corner c; hence the tensile stress produced at c by the bending moment Pr, acting in the plane mm, is numerically equal to the direct compressive stress P/A, or, substituting Pr for M in eq. (c), we obtain

$$\frac{P}{A} = \frac{Prd \sin \alpha}{I_{nn}},$$

from which

$$\frac{d \sin \alpha}{I_{nn}} = \frac{1}{Ar}. \tag{d}$$

Substituting this into eq. (c), we obtain

$$\sigma_{\max} = \frac{M}{Ar}. \tag{134}$$

The product Ar is called the *section modulus* of the cross section in the plane mm. This definition agrees with the definition which we had previously (see p. 92), and for bending in a principal plane Ar becomes equal to Z.

Problems

1. Determine the core of a standard I beam, 24″ depth, for

[4] See R. Land, Zeitschr. f. Architektur und Ingenieurwesen, 1897, p. 291.

which $A = 23.33$ sq. in., $I_z = 2{,}087$ in.^{4}, $k_z = 9.46$ in., $I_y = 42.9$ in.^{4}, $k_y = 1.36$ in. The width of the flanges $b = 7$ in.

Answer. The core is a rhombus with diagonals equal to 14.9 in. and 1.06 in.

2. Determine the radius of the core of a circular ring section if $R_0 = 10$ in. and $R_i = 8$ in.

Answer. The radius of the core $a = 4.10$ in.

3. Determine the core of a cross section in the form of an equilateral triangle.

4. Determine the core of the cross section of a square thin tube.

Solution. If h is the thickness of the tube and b the side of the square cross section, we have

$$I_z = I_y \approx \frac{2}{3} hb^3; \qquad k_z^2 = k_y^2 = \frac{b^2}{6}.$$

The core is a square with diagonal

$$d = 2\frac{k^2}{\frac{1}{2}b} = \frac{2b}{3}.$$

53. Eccentric Compression of a Slender Column.—In discussing the bending of a slender column under the action of an eccentric load, Fig. 213, the deflection δ can no longer be neglected as being small in comparison with the eccentricity e. Assuming that the eccentricity is in the direction of one of the principal axes of the cross section of the column, the deflection occurs in the same axial plane xy in which the load P acts, and the bending moment at any cross section mn is

$$M = -P(\delta + e - y). \qquad (a)$$

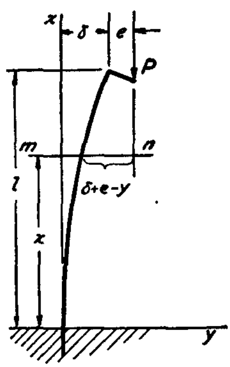

FIG. 213.

In determining the sign of the moment it should be noted that by rotating Fig. 213 in the clockwise direction by an angle $\pi/2$ the same directions of the coordinate axes are obtained as those used in deriving equation (79). Hence, to follow the rule shown in Fig. 58 (*b*), the moment (*a*) is taken with a minus sign since the deflection curve is concave in the positive direction of the y

axis. The differential equation of the deflection curve obtained by substituting (*a*) in equation (79) is

$$EI_z \frac{d^2y}{dx^2} = P(\delta + e - y). \tag{b}$$

Using the notation

$$\frac{P}{EI_z} = p^2 \tag{135}$$

we obtain from equation (*b*)

$$\frac{d^2y}{dx^2} + p^2y = p^2(\delta + e). \tag{c}$$

By substitution it can be readily proved that

$$y = C_1 \sin px + C_2 \cos px + \delta + e \tag{d}$$

is the solution of equation (*c*). This solution contains two constants of integration C_1 and C_2 whose magnitudes must be adjusted so as to satisfy the conditions at the ends of the column if we are to obtain the true deflection curve of the column. At the lower end, which is built-in, the conditions are

$$(y)_{x=0} = 0, \qquad \left(\frac{dy}{dx}\right)_{x=0} = 0. \tag{e}$$

Using these conditions together with expression (*d*) and its first derivative, we obtain

$$C_1 = 0, \qquad C_2 = -(\delta + e).$$

The equation of the deflection curve (*d*) thus becomes

$$y = (\delta + e)(1 - \cos px). \tag{f}$$

To obtain the magnitude of the deflection δ at the upper end of the column, we substitute $x = l$ in the right side of equation (*f*). The deflection y on the left side must then be equal to δ and we obtain the equation

$$\delta = (\delta + e)(1 - \cos pl)$$

from which

$$\delta = \frac{e(1 - \cos pl)}{\cos pl}. \tag{136}$$

Substituting this into equation (f) we obtain the deflection curve

$$y = \frac{e(1 - \cos px)}{\cos pl}. \tag{137}$$

By using this equation the deflection at any cross section of the column can readily be calculated.

In the case of short columns, which were considered in Art. 51, the quantity pl is small in comparison with unity and it is sufficiently accurate to take

$$\cos pl \approx 1 - \tfrac{1}{2}p^2l^2. \tag{g}$$

Using this value of $\cos pl$ and neglecting the quantity $p^2l^2/2$ in the denominator of expression (136), as being small in comparison with unity, we obtain

$$\delta = \frac{ep^2l^2}{2} = \frac{ePl^2}{2EI_z}. \tag{h}$$

This represents the magnitude of the deflection at the end of a cantilever bent by a couple Pe applied at the end. Hence the use of the approximate expression (g) is equivalent to neglecting the effect of the deflections upon the magnitude of the bending moment and taking instead a constant moment equal to Pe.

If pl is not small, as is usually the case when the column is slender, we must use expression (136) in calculating δ. In this way we find that the deflection is no longer proportional to the load P. Instead, it increases more rapidly than P, as is seen from the values of this deflection as given in the second line of the table below.

Deflections Produced by an Eccentric Longitudinal Load

pl	0.1	0.5	1.0	1.5	$\pi/2$
δ	$0.005e$	$0.139e$	$0.851e$	$13.1e$	∞
Approximate δ	$0.005e$	$0.139e$	$0.840e$	$12.8e$	∞
$\sec pl$	1.005	1.140	1.867	13.2	∞
P/P_{cr}	0.004	0.101	0.405	0.911	1

The maximum bending moment occurs at the built-in end of the column and has a magnitude

$$M_{\max} = P(e + \delta) = Pe \sec pl. \qquad (138)$$

A series of values of $\sec pl$ is given in the fourth line of the above table. These values show how rapidly the moment increases as pl approaches the value $\pi/2$. This phenomenon will be discussed in the next article.

Here, however, we should like to repeat that in the case under discussion there is no proportionality between the magnitude of the compressive force and the deflection δ which it produces. Hence the method of superposition (p. 147) cannot be used here. An axially applied force P produces only compression of the bar; but when the same force acts in conjunction with a bending couple Pe, it produce not only compression but also additional bending, so that the resulting deformation cannot be obtained by simple superposition of an axial compression due to the force P and a bending due to the couple Pe. The reason why in this case the method of superposition is not applicable can readily be seen if we compare this problem with the bending of a beam by transverse loads. In the latter case, it can be assumed that small deflections of the beam do not change the distances between the forces, and the bending moments can be calculated without considering the deflection of the beam. In the case of eccentric compression of a column the deflections produced by the couple Pe entirely change the character of the action of the axial load by causing it to have a bending action as well as a compressive action. In each case in which the deformation produced by one load changes the action of the other load it will be found that the final deformation cannot be obtained by the method of superposition.

In the previous discussion bending in one of the principal planes of the column was considered. If the eccentricity e is not in the direction of one of the principal axes of the cross section, it is necessary to resolve the bending couple Pe into two component couples each acting in a principal plane of the column. The deflection in each of the two principal planes can then be investigated in the same manner as discussed above.

The preceding discussion of bending of a column built-in at one end can also be applied to the case of a strut eccentrically compressed by two equal and opposite forces P, Fig. 214. From symmetry it can be appreciated that the cross section A at the middle does not rotate during bending and each half of the strut in Fig. 214 is in exactly the same condition as the strut in Fig. 213. Hence the deflection and the maximum bending moment are obtained by substituting $l/2$ for l in equations (136) and (138). In this way we obtain

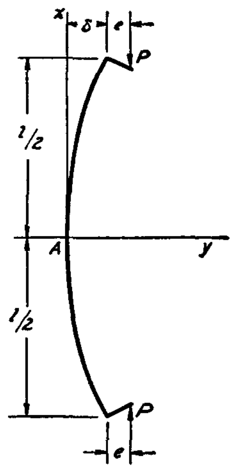

FIG. 214.

$$\delta = \frac{e\left(1 - \cos\frac{pl}{2}\right)}{\cos\frac{pl}{2}} \tag{139}$$

and

$$M_{max} = Pe \sec\frac{pl}{2}. \tag{140}$$

Problems

1. Find the deflection at the middle and the maximum tensile and compressive stresses in an eccentrically compressed strut 10 ft. long with hinged ends if the cross section is a channel of 8 in. depth with $I_z = 1.3$ in.^{4}, $I_y = 32.3$ in.^{4}, and $A = 3.36$ sq. in. The distance between the centroid and the back of the channel is 0.58 in., and the compressive forces $P = 4{,}000$ lbs. act in the plane of the back of the channel and in the symmetry plane of the channel.

2. A square steel bar 2 by 2 in. and 6 ft. long is eccentrically compressed by forces $P = 1{,}000$ lbs. The eccentricity e is directed along a diagonal of the square and is equal to 1 in. Find the maximum compressive stress, assuming that the ends of the bar are hinged.

3. A steel bar 4 ft. long and having a rectangular cross section 1 by 2 in. is compressed by two forces $P = 1{,}000$ lbs. applied at the corners of the end cross sections so that the eccentricity is in the direction of a diagonal of the cross section and is equal to half the length of the diagonal. Considering the ends to be hinged, find the maximum compressive stress.

54. Critical Load.—It was indicated in the previous article that the deflection of an eccentrically compressed column increases very rapidly as the quantity pl in equation (136) approaches the value $\pi/2$. When pl becomes equal to $\pi/2$, the formula (136) for the deflections and (138) for the maximum bending moment both give infinite values. To find the corresponding value of the load we use formula (135). Substituting $p = \pi/2l$ in this expression we find that the value of the load at which the expressions (136) and (138) become infinitely large is

$$P_{cr} = \frac{\pi^2 EI_z}{4l^2}. \tag{141}$$

This value depending, as we see, only on the dimensions of the column and on the modulus of the material is called the *critical load* or Euler's load since Euler was the first to derive its value in his famous study of elastic curves.[5] To see more clearly the physical significance of this load let us plot curves representing the relation between the load P and the deflection δ as given by equation (136). Several curves of this kind, made for various values of the eccentricity ratio e/k_z, are shown in Fig. 215. The abscissas of these curves are the values of the ratio δ/k_z while the ordinates are the ratio P/P_{cr}, that is the values of the ratio of the acting load to its critical value defined by equation (141).

It is seen from the curves that the deflections δ become smaller and smaller and the curves approach closer and closer to the vertical axis as the eccentricity e decreases. At the same time the deflections increase rapidly as the load P ap-

[5] An English translation of this work is given in "Isis" No. 58 (vol. XX, I), 1933.

proaches its critical value (141), and all the curves have as their assymptote the horizontal line $P/P_{cr} = 1$.

The differential equation of the deflection curve (79), which was used in the discussion of the preceding article, was derived on the assumption that the deflections are small in comparison with the length l of the column. Hence formula (136) for the deflection δ cannot give us an accurate result when P is very close to P_{cr}. However, the curves in Fig. 215

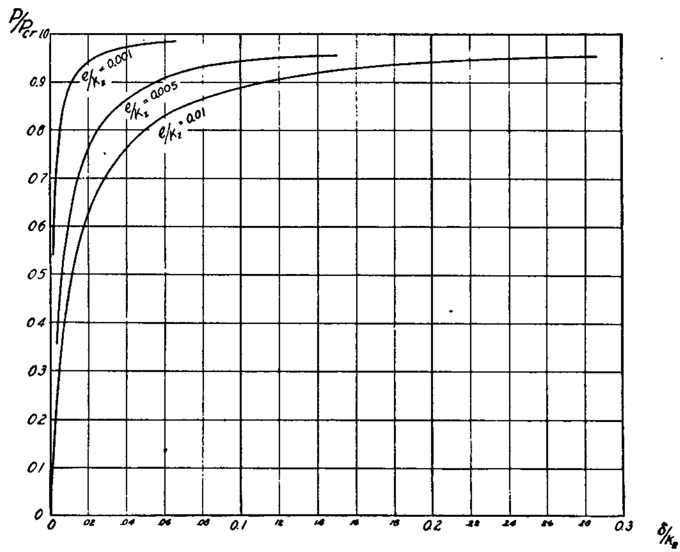

FIG. 215.

indicate that, irrespective of how small the eccentricity e may be, very large deflections are produced if the load P is sufficiently close to its critical value. If the deflection becomes large, the bending moment at the built-in end and the stresses are also large.

Experiments dealing with the compression of columns show that even when all practicable precautions are taken to apply the load centrally there always exists some unavoidable small eccentricities. Consequently in such experiments the load P produces not only compression but also bending. The curves in Fig. 216 show the results of such experiments as obtained by several experimenters. It may be seen that with increasing accuracy in the application of the load the curves become closer and closer to the vertical axis and the rapid increase in the deflection as the load approaches its critical value becomes more and more pronounced. The loads P which are close to their critical values always produce large deformations which usually go beyond the elastic limit of the material, so that after such a loading the column loses its

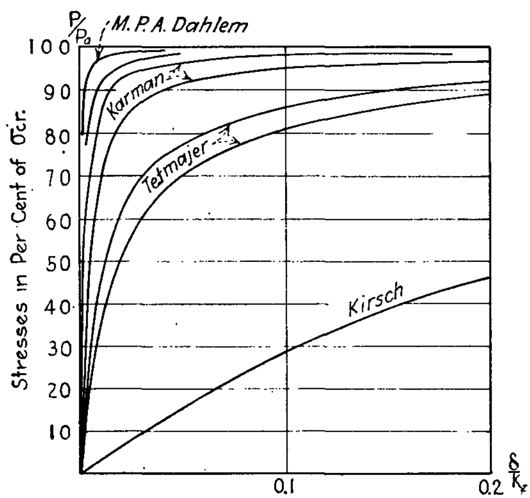

FIG. 216.

practical usefulness. This indicates that the critical value of the load, as given by equation (141), must be considered as an *ultimate load* which will produce complete failure of the column. In practical applications the allowable load should be smaller than the critical load and is obtained by dividing the critical value of the load by a certain factor of safety. Further discussion of this question is given in the next two articles.

In the preceding discussion a column with one end built-in and the other end free was considered. Similar conclusions can be made in the case of a strut with hinged ends, Fig. 214.

Equations (139) and (140) give infinite values when

$$\frac{pl}{2} = \frac{\pi}{2}.$$

Substituting for p its value from formula (135) we obtain

$$P_{cr} = \frac{\pi^2 EI_z}{l^2}. \tag{142}$$

This is the *critical value* of the compressive force for a strut with hinged ends.

In the case of compression of columns with built-in ends the deflection has the form shown in Fig. 217. The deflection curve can be considered as consisting of four portions each similar to the curve previously obtained for a column with one end built-in and the other free. The critical value of the load is found in such a case by substituting $l/4$ instead of l into equation (141), which gives

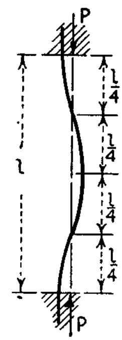

FIG. 217.

$$P_{cr} = \frac{4\pi^2 EI_z}{l^2}. \tag{143}$$

This is the *critical load* for a column with built-in ends.

It should be noted that in the derivation of equation (136) it was assumed that the eccentricity is in the direction of the y axis and that the bending occurs in the xy plane. Similar formulas will be obtained if the initial eccentricity is in the direction of the z axis. The bending then occurs in the xz plane and, to calculate the deflections, I_y must be substituted in place of I_z in equation (136). If an attempt is made to apply the load centrally and bending occurs as a result of small unavoidable eccentricities, we must consider deflections in both principal planes xy and xz; and, in calculating the critical value of the load, we must use the smaller of the two principal moments of inertia in equations (141), (142) and (143). In the following discussion it is assumed that I_z is the smaller principal moment of inertia and k_z is the corresponding radius of gyration.

In calculating deflections it is sometimes advantageous to use approximate formulas instead of the accurate formulas (136) and (138). It was shown in the preceding article that for small loads, that is when pl is a small fraction, say less than 1/10, the deflection is given with sufficient accuracy by the equation

$$\delta = \frac{Pel^2}{2EI_z} \tag{a}$$

in which the influence of the longitudinal force on the bending is neglected and a constant bending moment Pe is assumed. For larger loads the equation (a) is not accurate enough, and the influence of the compressive force on bending should be considered. This influence depends principally on the ratio P/P_{cr} and the deflection can be obtained with very satisfactory accuracy from the approximate formula

$$\delta = \frac{Pel^2}{2EI_z} \cdot \frac{1}{1 - \dfrac{P}{P_{cr}}}. \tag{b}$$

The deflections calculated from this formula are given in the third line of the table on p. 241. Comparison of these figures with those of the second line of the same table shows that the formula (b) is sufficiently accurate almost up to the critical value of the load.

A similar approximate formula for the deflection of a strut with hinged ends is [6]

$$\delta = \frac{Pel^2}{8EI_z} \cdot \frac{1}{1 - \dfrac{P}{P_{cr}}}. \tag{c}$$

The first factor on the right side is the deflection produced by the two couples Pe applied at the ends. The second factor

[6] This approximate solution was given by Prof. J. Perry. See Engineer, December 10 and 24, 1886.

represents the effect on the deflection of the longitudinal compressive force P.

Equation (c) is very useful for determining the critical load from an experiment with a compressed strut. If the results of such experiment are represented in the form of a curve, such as the one shown in Fig. 216, the horizontal asymptote to that curve must be drawn to determine P_{cr}. This operation cannot be done with sufficient accuracy, especially if the unavoidable eccentricities are not very small and the curve does not turn very sharply in approaching the horizontal asymptote. A more satisfactory determination of P_{cr} is obtained by using equation (c). Dividing this equation by P/P_{cr} we obtain

$$\frac{\delta}{P} \cdot P_{cr} = \frac{e\pi^2}{8} \frac{1}{1 - \dfrac{P}{P_{cr}}}$$

and

$$\frac{\delta}{P} \cdot P_{cr} - \delta = \frac{e\pi^2}{8}.$$

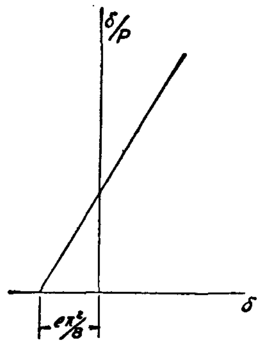

Fig. 218.

This equation shows that, if we plot the ratio δ/P against the deflection δ measured during experiment, the points will fall on a straight line, Fig. 218. This line will cut the horizontal axis ($\delta/P = 0$) at the distance $e\pi^2/8$ from the origin, and the inverse slope of the line gives the critical load.[7]

55. Critical Stress; Design of Columns.—Considering the case of a strut with hinged ends, the critical stress is obtained by dividing the critical load given by equation (142) by the cross-sectional area A. In this way we find

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 E}{(l/k_z)^2}. \tag{144}$$

[7] This method, suggested by R. V. Southwell, Proc. Roy. Soc. London, series A, vol. 135, p. 601, 1932, has proved a very useful one and is widely used in column tests.

It is seen that for a given material the value of the critical stress depends on the magnitude of the ratio l/k_z, which is called the *slenderness ratio.* In Fig. 219 the curve ACB represents [8] the relation between σ_{cr} and l/k_z for the case of steel having $E = 30 \times 10^6$ lbs. per sq. in. It will be appreciated that the curve is entirely defined by the magnitude of the modulus of the material and is independent of its ultimate strength. For large values of the slenderness ratio l/k_z the critical stress becomes small, which indicates that a very

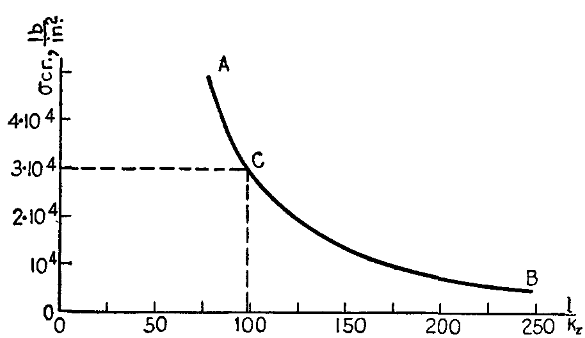

Fig. 219.

slender strut buckles sidewise and loses its strength at a very small compressive stress. This condition cannot be improved by taking a steel of higher strength, since the modulus of steel does not vary much with alloy and heat treatment and remains *practically constant.* The strut can be made stronger by increasing the moment of inertia I_z and the radius of gyration k_z, which can very often be accomplished without any increase in the cross-sectional area by placing the material of the strut as far as possible from the axis. Thus tubular sections are more economical as columns than are solid sections. As the slenderness ratio diminishes the critical stress increases and the curve ACB approaches the vertical axis asymptotically. However, there must be a certain limitation to the use of the Euler curve for shorter struts. The derivation of the ex-

[8] This curve is sometimes called the Euler curve, since it is derived from Euler's formula for the critical load.

pression for the critical load is based on the use of the differential equation (79) for the deflection curve, which equation assumes that the material is perfectly elastic and follows Hooke's law (see art. 31). Hence the curve ACB in Fig. 219 gives satisfactory results only for comparatively slender bars for which σ_{cr} remains within the elastic region of the material. For shorter struts, for which σ_{cr} as obtained from equation (144) is higher than the proportional limit of the material, the Euler curve does not give a satisfactory result and recourse

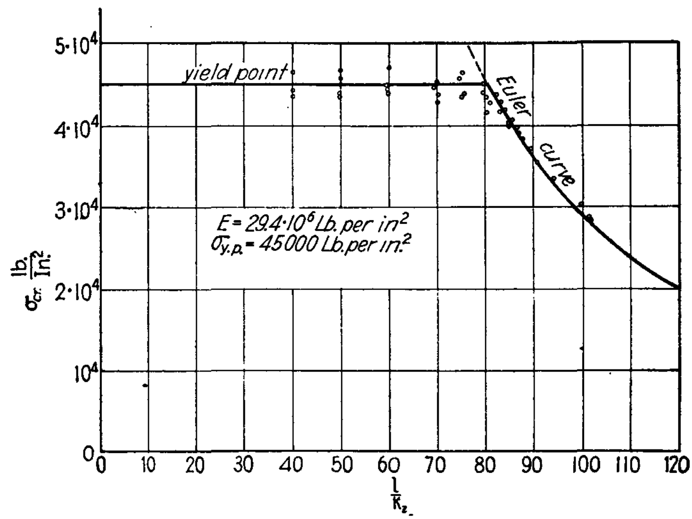

FIG. 220.

should be had to experiments with the buckling of struts compressed beyond the proportional limit. These experiments show that struts of materials such as structural steel, which have a pronounced yield point, lose all their stability and buckle sidewise as soon as the compressive stress becomes equal to the yield point stress. Some experimental results are shown in Fig. 220. The material is a structural steel having a very pronounced yield point at $\sigma_{Y.P.} = 45{,}000$ lbs. per sq. in. It is seen that for struts of relatively large slenderness ($l/k_z > 80$) the experimental values of the critical stresses coincide satisfactorily with the Euler curve, while for shorter struts the critical stress remains practically independent of the slenderness ratio l/k_z and is equal to the yield point stress.

In the case of an ordinary low carbon structural steel the yield point is not as pronounced as in the preceding example and occurs at a much lower stress. For such steel we may take $\sigma_{Y.P.} = 34{,}000$ lbs. per sq. in. The proportional limit is also much lower, so that the Euler curve is satisfactory only for slenderness ratios beginning with $l/k_z = 100$, to which value corresponds the compressive stress $\sigma_{cr} = 30{,}000$ lbs. per sq. in. For higher stresses, i.e., for $l/k_z < 100$, the material does not follow Hooke's law and the Euler curve cannot be used. It is usually replaced in the inelastic region by the two straight lines AB and BC as shown in Fig. 221. The

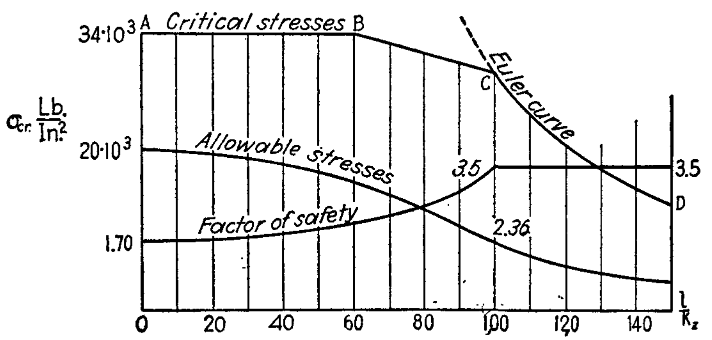

FIG. 221.

horizontal line AB corresponds to the yield point stress and the inclined line BC is taken for the stresses between the proportional limit and the yield point of the material.

Having such a diagram as the line $ABCD$ in Fig. 221, constructed for ordinary structural steel, the critical stress for a steel strut of any dimensions can readily be obtained. It is only necessary to calculate in each particular case the value of the slenderness ratio l/k_z and take the corresponding ordinate from the curve. To obtain the safe stress on the strut the critical stress must then be divided by a proper factor of safety. In selecting this factor it must be considered that as the slenderness ratio increases various imperfections, such as an initial crookedness of the column, are likely to increase. It appears logical therefore to introduce a variable factor of safety which increases with the slenderness ratio. In some specifications the factor of safety increases from 1.7 for $l/k_z = 0$ to 3.5 for $l/k_z = 100$. It varies in such a way that the allowable stress in the inelastic range follows a parabolic law. For $l/k_z > 100$, the factor of safety is taken as constant at 3.5, and the allowable stresses are calculated from the Euler curve. In Fig. 221 curves are given which represent the allowable stress and the factor of safety as functions of the slenderness ratio for ordinary structural steel.

In the preceding discussion a strut with hinged ends was considered. This case is sometimes called the *fundamental case* of buckling of struts, since it is encountered very often in the design of compressed members of trusses with hinged joints. The allowable stresses established by the diagram in Fig. 221 for the fundamental case can also be used in other cases provided we take, instead of the actual length of the column, a *reduced length* the magnitude of which depends on the conditions at the ends of the column. Considering, for example, the case of a column with one end built-in and the other end free and also the column with both ends built-in, the corresponding formulas for the critical loads can be put respectively in the form

$$P_{cr} = \frac{\pi^2 EI_z}{(2l)^2} \qquad \text{and} \qquad P_{cr} = \frac{\pi^2 EI_z}{(\frac{1}{2}l)^2}.$$

Comparing these formulas with formula (142) for the fundamental case it can be concluded that in the design of a column with one end built-in and the other free we must take a length two times larger than the actual length of the column when using the diagram of Fig. 221. In the case of a column with both ends built-in the reduced length is equal to half of the actual length.

The selection of proper cross-sectional dimensions of a column is usually made by trial and error. Having the load P which acts on the column, we assume certain cross-sectional dimensions and calculate k_z and l/k_z for these dimensions. Then the safe value of the compressive stress is obtained from the diagram of Fig. 221. Multiplying this value by the area of the assumed cross section, the safe load on the column is obtained. If this load is neither smaller nor overly larger than P, the assumed cross section is satisfactory. Otherwise the calculations must be repeated. In the case of built-up columns the gross cross section is used in calculating k_z, since the rivet holes do not appreciably affect the magnitude of the critical load. However, in calculating the safe load on the column the safe stress is multiplied by the net cross-sectional area in order to insure against excessive stresses in the column.

Problems

1. A steel bar of rectangular cross section 1 by 2 in. and having hinged ends is compressed axially. Determine the minimum length at which equation (144) for the critical stress can be applied if the limit of proportionality of the material is 30,000 lbs. per sq. in. and $E = 30 \times 10^6$ lbs. per sq. in. Determine the magnitude of the critical stress if the length of the bar is 5 ft.

Answer. Minimum length = 28.9 in. The critical stress for $l = 5$ ft. is 6,850 lbs. per sq. in.

2. Solve the preceding problem assuming a bar with a circular cross section 1 in. in diameter and built-in ends.

3. Determine the critical compressive load for a standard I section 6 ft. long and 6 in. deep assuming hinged ends. $I_z = 1.8$ in.^{4}, $I_y = 21.8$ in.^{4} and $A = 3.61$ sq. in. Determine the safe load from the curve of Fig. 121.

4. Solve the preceding problem assuming that the ends of the column are built-in.

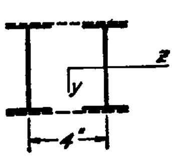

Fig. 222.

5. Calculate, by the use of Fig. 221, the safe load on a member built-up of two I beams of the same cross section as those in problem 3, Fig. 222. The length of the member is 10 ft. and the ends are hinged. Assume that the connecting details are so rigid that both I beams work together as a single bar.

6. Solve the preceding problem assuming that the ends of the member are built-in.

7. A column 10 ft. long with hinged ends is made of two channels 8 in. deep having $I_z = 1.3$ in.^{4}, $I_y = 32.3$ in.^{4}, $A = 3.36$ in.^{2} and a distance of $c = 0.58$ in. between the centroid and the back of channel. Find the safe load on the column if the back to back distance between the channels is 4 in.

8. Determine the cross-sectional dimension of a square steel strut 6 ft. long if the load $P = 40{,}000$ lbs. and the ends are hinged. Use Fig. 121.

9. Solve the preceding problem assuming that the ends of the strut are built-in. Use Fig. 121.

56. Design of Columns on Basis of Assumed Inaccuracies. —In the preceding article the safe load on a column was obtained by dividing the critical load for the column by a proper factor of safety. The weakness of that method lies in a certain arbitrariness in the selection of the factor of safety which, as we have seen, varies with the slenderness ratio. To make the procedure of column design more rational, another method based on assumed inaccuracies has been developed.[9] On the basis of existing experimental data we can assume certain values for the magnitude of the unavoidable eccentricity e in the application of the compressive force P. Then, by using these values in the formulas of Article 53, we can calculate the magnitude $P_{\text{Y.P.}}$ of the load at which the maximum stress in the compressed strut becomes equal to the yield point stress of the material. The safe load is then obtained by dividing the load $P_{\text{Y.P.}}$ by a proper factor of safety. Thus instead of using the critical load, which is equivalent to the ultimate load, we use the load at which yielding begins as a basis for calculating the safe load.

This method of column design can be simplified by the use of diagrams the calculations of which will now be explained. Taking the case of a strut with hinged ends, Fig. 214, the maximum bending moment is obtained from equation (140) and the maximum compressive stress is

$$\sigma_{\max} = \frac{P}{A} + \frac{Pe}{Z} \sec \sqrt{\frac{P}{EI_z}}\,\frac{l}{2}. \qquad (a)$$

[9] See paper by D. H. Young, Proceedings Am. Soc. Civil Eng., December, 1934.

The first term on the right side is the direct stress and the second is the maximum compressive bending stress. The load at which yielding begins is obtained by substituting $\sigma_{Y.P.}$ for σ_{max} in this equation, which gives

$$\sigma_{Y.P.} = \frac{P_{Y.P.}}{A}\left(1 + \frac{e}{r}\sec\frac{l}{2k_z}\sqrt{\frac{P_{Y.P.}}{EA}}\right). \qquad (b)$$

We now introduce the notations $r = Z/A$ for the radius of the *core* of the cross section (see p. 238) and $k_z = \sqrt{I_z/A}$ for the smaller principal radius of gyration. The quantity $P_{Y.P.}/A$ is the average compressive stress or *centroidal compressive stress* at which yielding begins. Denoting this stress by σ_c we obtain

$$\sigma_{Y.P.} = \sigma_c\left(1 + \frac{e}{r}\sec\frac{l}{2k_z}\sqrt{\frac{\sigma_c}{E}}\right). \qquad (c)$$

From this equation, for a given value of the eccentricity ratio e/r, the value of σ_c can readily be obtained for any value of the slenderness ratio l/k_z. The results of such calculations for a structural steel having $\sigma_{Y.P.} = 36{,}000$ lbs. per sq. in. are repre-

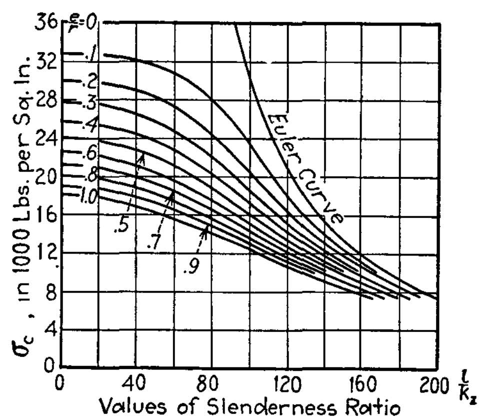

FIG. 223.

sented by curves in Fig. 223. By the use of these curves the average compressive stress σ_c and the compressive load $P_{Y.P.} = A\sigma_c$ at which yielding begins can readily be calculated if e/r and l/k_z are given. The safe load is then obtained by dividing $P_{Y.P.}$ by the factor of safety.

We assumed in the foregoing discussion that the unavoidable inaccuracies in the column could be represented by an eccentricity of the load. In a similar manner we can also consider the inaccuracies to be equivalent to an initial crookedness of the column. Denoting the maximum initial deviation of the axis of the column from a straight line[10] by a, curves similar to those shown in Fig. 223 and representing σ_c as a function of the ratio a/r and the slenderness ratio l/k_z can be obtained.

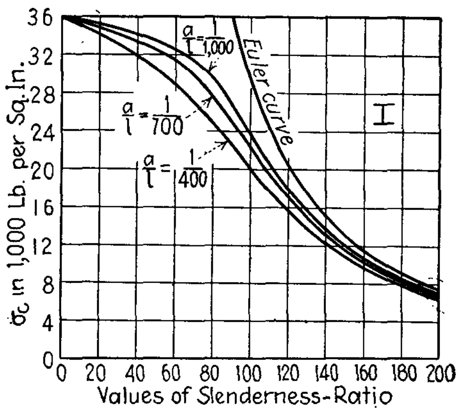

FIG. 224.

In practical design it is usually assumed that the initial deflection a is in a certain ratio to the length l of the column. Taking a certain magnitude for that ratio,[11] the magnitude of a is calculated and the value of σ_c is then obtained from the above-mentioned curves. The results obtained in this way for three different values of the ratio a/l and for $\sigma_{Y.P.} = 36{,}000$ lbs. per sq. in. are shown for an I section in Fig. 224. For very short columns all three curves give $\sigma_c = 36{,}000$ lbs.

[10] A half wave of a sinusoidal form is usually taken as representing the initial crookedness of a column.

[11] It is usually taken within the limits $\frac{l}{400} \geqq a \geqq \frac{l}{1000}$.

per sq. in. For very slender columns the values given by the curves approach those obtained from the Euler curve. Using one of the curves and dividing the value σ_c from the curve by a proper factor of safety, say by 2, the safe value of the average compressive stress is obtained. The advantage of this method is that it employs a constant factor of safety, since the increase of inaccuracies with the length l of the column has already been taken into consideration by assuming that the eccentricity is proportional to the span. However, the magnitude of the inaccuracies which should be taken remains to a certain extent indefinite and dependent upon existing experimental data.

57. Empirical Formulas for Column Design.—In both of the methods of column design developed in the last two articles on the basis of theoretical considerations there occur some uncertainties such as a variable factor of safety in the design procedure illustrated by Fig. 221 or the assumed inaccuracies as used in making the curves in Fig. 224. These quantities can be properly selected only on the basis of experiments with actual columns. Under such circumstances it is natural that many practical engineers prefer to use directly the results of experiments as represented by empirical formulas. Such a procedure is entirely legitimate so long as the application of these formulas remains within the limits for which they were established and for which there is sufficient experimental information. However, as soon as it is necessary to go beyond those limits, the formulas must be modified to conform with the new conditions. In this work the theoretical considerations become of primary importance.

One of the oldest empirical formulas was originated by Tredgold.[12] It was adapted by Gordon to represent the results of Hodgkinson's experiments and was given in final form by Rankine. The allowable average compressive stress as

[12] Regarding the history of the formula see E. H. Salmon, "Columns," London, 1921. See also Todhunter and Pearson, "History of the Theory of Elasticity," vol. 1, p. 105, Cambridge, 1886.

given by the Rankine formula is

$$\sigma_w = \frac{a}{1 + b\left(\frac{l}{k_z}\right)^2}, \tag{a}$$

in which a is a stress and b is a numerical factor, both of which are constant for a given material. By a proper selection of these constants the formula can be made to agree satisfactorily with the results of experiments within certain limits.

The American Institute of Steel Construction in its specifications of 1928 takes for the safe stress on the cross section of a column

$$\sigma_w = \frac{18{,}000}{1 + \frac{l^2}{18{,}000 k_z^2}} \tag{b}$$

for $l/k_z > 60$. For shorter columns σ_w is taken as 15,000 lbs. per sq. in.

The straight-line formula used by the *American Railway Engineering Association* and incorporated also in the building codes of the cities of Chicago and New York gives the working stress in the form

$$\sigma_w = 16{,}000 - 70 l/r \tag{c}$$

to be used for $30 < l/k_z < 120$ for main members and as high as $l/k_z = 150$ for secondary members. For values of $l/k_z < 30$, $\sigma_w = 14{,}000$ lbs. per sq. in. is used.

The *parabolic formula* proposed by A. Ostenfeld [13] is also sometimes used. It gives for the critical compressive stress

$$\sigma_{cr} = a - b\left(\frac{l}{k_z}\right)^2 \tag{d}$$

in which the constants a and b depend upon the mechanical properties of the material. For structural steel equation (d)

[13] Zeitschr. Ver. Deutsch Ing. vol. 42, p. 1462, 1898. See also C. E. Fuller and W. A. Johnston, "Applied Mechanics," vol. 2, p. 359, 1919.

is sometimes taken in the form

$$\sigma_{cr} = 40{,}000 - 1.33\left(\frac{l}{k_z}\right)^2. \qquad (e)$$

This gives a parabola tangent to the Euler curve at $l/k_z = 122.5$ and makes $\sigma_{cr} = 40{,}000$ lbs. per sq. in. for short columns. A suitable factor of safety varying from 2-1/2 to 3 should be used with this formula to obtain the working stress.

Problems

1. A 6 in. × 6 in. 22-1/2 lb. H-beam, $I_z = 12.2$ in.^{4}, $A = 6.61$ sq. in., is to be used as a column with hinged ends. Three lengths are to be considered: $l = 5$ ft., 10 ft., and 13 ft.-4 in. What are the safe loads using the formulas (*b*), (*c*), and (*e*), the latter with a factor of safety of 2-1/2.

Answers.

Formulas	$l = 60$ in.	$l = 120$ in.	$l = 160$ in.
(*b*)	99,000	83,000	67,300
(*c*)	85,300	65,000	51,000
(*e*)	98,700	78,300	57,000

2. Select a Carnegie Beam section to serve as a column 25 ft. long with fixed ends to carry a load of 200,000 lbs. Use formula (*b*).

Solution. Taking the reduced length $l = \frac{1}{2}\cdot 25$ ft. $= 150$ in., equation (*b*) gives

$$\frac{200{,}000}{A} \approx \frac{18{,}000}{1 + \frac{1.25}{k_z^2}}. \qquad (f)$$

The minimum area may be found by taking $\sigma_w = 15{,}000$ lbs. per sq. in. as for a short column. This gives $A = 200{,}000/15{,}000 = 13.3$ sq. in. We therefore need not try any section which has an area less than 13.3 sq. in. We try a 12 in. × 10 in. 53 lb. section, for which $A = 15.57$ sq. in., $k_z = 2.48$, and $l/k_z = 60.5$. The safe stress as given by the right side of equation (*f*) is 14,970 lbs. per sq. in. The actual stress given by the left side is 12,850 lbs. per sq. in. It may be seen that a smaller section should be tried. Taking a 12 in. × 8 in. 50 lb. section, for which $A = 14.69$ sq. in., $k_z = 1.96$ in., and $l/k_z = 76.5$, the permissible stress is found to be 13,600 lbs. per sq. in. This value is also given by the left side of equation (*f*). The above section is therefore satisfactory.

CHAPTER IX

TORSION AND COMBINED BENDING AND TORSION

58. Torsion of a Circular Shaft.—Let us consider a circular shaft built in at the upper end and twisted by a couple applied to the lower end (Fig. 225). It can be shown by measurements at the surface that circular sections of the shaft remain circular during twist, and that their diameters and the distances between them do not change provided the angle of twist is small.

A disc, isolated as in Fig. 225 (*b*), will be in the following state of strain. There will be a rotation of its bottom cross section with reference to its top through an angle $d\varphi$ where φ measures the rotation of the section *mn* with reference to the built-in end. An element *abcd* of the surface of the disc whose sides were vertical before strain takes the form shown

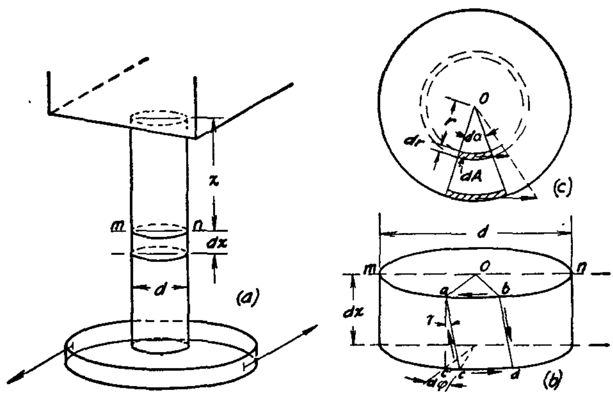

FIG. 225.

in Fig. 225 (*b*). The lengths of the sides remain essentially the same, and only the angles at the corners change. The element is in a state of *pure shear* (see article 16) and the magnitude of the shearing strain is found from the small triangle cac':

$$\gamma = \frac{c'c}{ac'}.$$

Since $c'c$ is the small arc of radius $d/2$ corresponding to the difference $d\varphi$ in the angle of rotation of the two adjacent cross sections, $c'c = (d/2)d\varphi$, we obtain

$$\gamma = \frac{1}{2}\frac{d\varphi}{dx}d. \tag{a}$$

For a shaft twisted by a torque at the end the angle of twist is proportional to the length and the quantity $d\varphi/dx$ is constant. It represents *the angle of twist per unit length of the shaft* and will be called θ. Then, from (*a*),

$$\gamma = \tfrac{1}{2}\theta d. \tag{145}$$

The shearing stresses which act on the sides of the element and produce the above shear have the directions shown. The magnitude of each, from eq. (39), is

$$\tau = \tfrac{1}{2}G\theta d. \tag{146}$$

So much for the state of stress of an element at the surface of the shaft. As for that within the shaft the assumption will now be made that not only the circular boundaries of the cross sections of the shaft remain undistorted but also the cross sections themselves remain plane and rotate as if absolutely rigid, that is, every diameter of the cross section remains straight and rotates through the same angle. The tests of circular shafts show that the theory developed on this assumption is in very good agreement with the experimental results. Such being the case, the discussion for the element *abcd* at the surface of the shaft (Fig. 225, *b*) will hold also for a similar element of the surface of an inner cylinder, whose radius r replaces $d/2$ (Fig. 225, *c*). The thickness of the element in radial direction is considered as very small.

Such elements are then also in pure shear, and the shearing stress on their sides is

$$\tau = Gr\theta. \tag{b}$$

This states that the shearing stress varies directly as the distance r from the axis of the shaft. Figure 226 pictures this stress distribution. The maximum stress occurs in the outer surface of the shaft. For a ductile material, plastic flow begins first in this surface. For a material which is weaker in shear longitudinally than transversely, e.g., a wooden shaft with the fibers parallel to the axis, the first cracks will be produced by shearing stresses acting in the axial sections and they will appear on the surface of the shaft in the longitudinal direction. In the case of a material which is weaker in tension than in shear, e.g., a circular shaft of cast iron or a cylindrical piece of chalk, a crack along a helix inclined at 45° to the axis of the shaft often occurs (Fig. 227). The explanation is simple. We recall that the state of pure shear is equivalent to one of tension in one direction and equal compression in the perpendicular direction (see Fig. 41). A rectangular element cut from the outer layer of a twisted shaft with sides at 45° to the axis will be submitted to such stresses as shown in Fig. 227. The tensile stresses shown produce the helical crack mentioned.

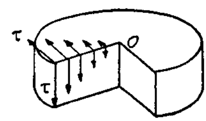

FIG. 226.

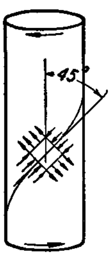

FIG. 227.

We seek now the relationship between the applied twisting couple M_t and the stresses which it produces. From the equilibrium of that portion of the shaft between the bottom and the imaginary section mn, we conclude that the shearing stresses distributed over the cross section are statically equivalent to a couple equal and opposite to the torque M_t. For each element of area dA (Fig. 225, c), shearing force $= \tau dA$. The moment of this force about the axis of the shaft $= (\tau dA)r = G\theta r^2 dA$, from eq. ($b$). The total moment M_t about the axis of the shaft is the summation, taken over the entire cross-sectional area, of these moments on the individual elements, i.e.,

$$M_t = \int_{r=0}^{r=d/2} G\theta r^2 dA = G\theta \int_{r=0}^{r=d/2} r^2 dA = G\theta I_p, \qquad (c)$$

where I_p is the polar moment of inertia of the circular cross section. From the appendix, for a circle of diameter d, $I_p = \pi d^4/32$;

$$\therefore \quad M_t = G\theta \frac{\pi d^4}{32}$$

and

$$\theta = \frac{M_t}{G}\frac{32}{\pi d^4} = \frac{M_t}{GI_p}. \qquad (147)$$

We see that θ, the angle of twist per unit length of the shaft, varies directly as the applied torque and inversely as the modulus of shear G and the fourth power of the diameter. If the shaft is of length l, the total angle of twist will be

$$\varphi = \theta l = \frac{M_t l}{GI_p}. \qquad (148)$$

This equation is useful in the physical verification of the theory, and is checked by numerous experiments which prove the assumptions made in deriving the theory. It should be noted that experiments in twist are commonly used for determining the modulus of materials in shear. If the angle of twist produced in a given shaft by a given torque be measured, the magnitude of G can be easily calculated from eq. (148).

Substituting θ from eq. (147) in eq. (146), we obtain an equation for calculating the maximum shearing stress in twist of a circular shaft:

$$\tau_{\max} = \frac{M_t d}{2I_p} = \frac{16M_t}{\pi d^3}. \qquad (149)$$

That is, this stress is proportional to the torque M_t and inversely proportional to the cube of the diameter of the shaft. In practical applications the diameter of the shaft must usually be calculated from the horse-power H which it transmits. Given H, the torque is obtained as lbs. ins. from the well-known equation:

$$M_t \cdot \frac{2\pi n}{60} = 550 \times 12 \times H \tag{150}$$

in which n denotes the number of revolutions of the shaft per minute. The quantity $2\pi n/60$ is then the angle of rotation per second and the left side of the equation (150) represents the work done during one second by the torque M_t measured in in. lbs. The right side of the same equation represents the work done in in. lbs. per second as calculated from the horse-power H. Taking M_t from eq. (150) and substituting it into eq. (149), we obtain

$$d = 68.5 \sqrt[3]{\frac{H}{n\tau_{\max}}}. \tag{151}$$

Taking, for instance, the working stress for shear as $\tau_w = 9{,}000$ lbs. per sq. in., we have

$$d = 3.29 \sqrt[3]{\frac{H}{n}}.$$

Problems

1. Determine the shaft diameter d of a machine of 200 h.p. of speed $n = 120$ r.p.m., for the working stress $\tau_w = 3{,}000$ lbs. per sq. in.

Answer.

$$d = 5.63 \text{ in.}$$

2. Determine the horse-power transmitted by a shaft if $d = 6$ ins., $n = 120$ r.p.m., $G = 12 \times 10^6$ lbs. per sq. inch, and the angle of twist, as measured between two cross sections 25 ft. apart, is 1/15 of a radian.

Solution. From eq. (148)

$$M_t = \frac{\pi d^4}{32} \cdot \frac{\varphi \cdot G}{l} = \frac{\pi \times 6^4}{32} \cdot \frac{12 \times 10^6}{15 \times 25 \times 12}.$$

The power transmitted is, from eq. (150),

$$H = \frac{M_t \cdot 2\pi n}{60 \times 550 \times 12} = \frac{\pi \times 6^4 \times 12 \times 10^6 \times 2\pi \times 120}{32 \times 15 \times 25 \times 12 \times 60 \times 550 \times 12} = 646.$$

3. A shaft of diameter $d = 3.5$ in. makes 45 r.p.m. Determine the power transmitted if the maximum shearing stress is 4,500 lbs. per sq. in.

4. A steel shaft ($G = 12 \times 10^6$ lbs. per sq. in.) is to have such proportions that the maximum shearing stress is 13,500 lbs. per sq. in. for an angle of twist of 90°. Determine the ratio l/d.

Answer.

$$\frac{l}{d} = 698.$$

5. A steel shaft with built-in ends (Fig. 228) is submitted to the action of a torque M_t, applied at an intermediate cross section mn. Determine the angle of twist if the working stress τ_w is known.

Solution. For both parts of the shaft the angles of twist are equal; therefore, from eq. (148), the twisting moments are inversely proportional to the lengths of these parts. If $a > b$, the greater twisting moment is in the right part of the shaft and its magnitude is

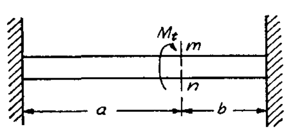

FIG. 228.

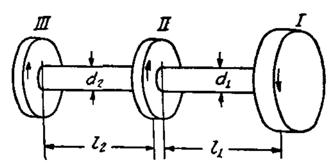

FIG. 229.

$M_t \cdot a/(a + b)$. Substituting this for the torque, and τ_w for τ_{max} in eq. (149), the following equation for d is obtained:

$$d = \sqrt[3]{\frac{16aM_t}{(a + b)\pi\tau_w}}.$$

Now the angle of twist can be obtained by using eq. (148).

6. 500 h.p. is transmitted from pulley I, 200 h.p. to pulley II and 300 h.p. to pulley III (Fig. 229). Find the ratio of the diameters d_1 and d_2 to give the same maximum stress in both parts of the shaft. Find the ratio of the angles of twist for these two parts.

Solution. The torques in the two parts of the shaft are in the ratio 5 : 3. In order to have the same maximum stress from eq.

(149),

$$\frac{d_1}{d_2} = \sqrt[3]{\frac{5}{3}}.$$

The angles of twist, from eqs. (148) and (149), must be in the ratio

$$\varphi_1 : \varphi_2 = \frac{l_1}{l_2}\sqrt[3]{\frac{3}{5}}.$$

7. Assuming that the shaft of the preceding problem has a constant diameter and turns at 200 r.p.m., find the magnitude of the diameter if $\tau_w = 6{,}000$ lbs. per sq. in. Find the angle of twist for each portion of the shaft if $G = 12 \times 10^6$ lbs. per sq. in. and $l_1 = l_2 = 4$ ft.

8. Determine the length of the steel shaft of 2 in. diameter ($G = 12 \times 10^6$ lbs. per sq. in.) if the maximum stress is equal to 13,500 lbs. per sq. in. when the angle of twist is 6°.

Answer.

$$l = 93 \text{ in.}$$

9. Determine the diameter beginning from which the angle of twist of the shaft, and not the maximum stress, is the controlling factor in design, if $G = 12 \times 10^6$ lbs. per sq. in., $\tau_w = 3{,}000$ lbs. per sq. in. and the maximum allowable twist is $\frac{1}{4}°$ per yard.

Solution. Eliminating M_t from the equations

$$\frac{16M_t}{\pi d^3} = 3{,}000; \qquad \frac{32M_t}{G\cdot\pi d^4} = \frac{\pi}{180 \times 4 \times 36},$$

we obtain $d = 4.12$ in.; for $d < 4.12$ in., the angle of twist is the controlling factor in design.

10. Determine the torque in each portion of a shaft with built-in ends which is twisted by the moments M_t' and M_t'' applied in two intermediate sections (Fig. 230).

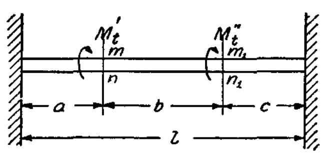

FIG. 230.

Solution. By finding the torques produced in each portion of the shaft by each of the moments M_t' and M_t'' (see problem 5 above) and adding these moments for each portion we obtain

$$\frac{M_t'(b+c) + M_t''c}{l}, \quad \frac{M_t'a - M_t''c}{l}, \quad \frac{M_t'a + M_t''(a+b)}{l}.$$

11. Determine the diameters and the angles of twist for the shaft of problem 6 if $n = 120$ r.p.m., $\tau_{max} = 3{,}000$ lbs. per sq. in., $l_1 = 6$ feet, $l_2 = 4$ feet.

59. Torsion of a Hollow Shaft.—From the previous discussion of the twist of a solid shaft, it is seen (see Fig. 226) that only the material at the outer surface of the shaft can be stressed to the limit assigned as the working stress. The material within will work at a lower stress, and in the cases in which reduction in weight is of great importance, e.g., propeller shafts of aeroplanes, it is advisable to use hollow shafts. In discussing the torsion of hollow shafts the same assumptions are made as in the case of solid shafts. The general expression for shearing stresses will then be the same as is given by eq. (*b*) of the previous article. In calculating the moment of the shearing stresses, however, the radius r varies from the radius of the inner hole, which we will denote by $\frac{1}{2}d_1$, to the outer radius of the shaft, which, as before, will be $\frac{1}{2}d$. Then eq. (*c*) of the previous article must be replaced by the following equation:

$$G\theta \int_{\frac{1}{2}d_1}^{\frac{1}{2}d} r^2 dA = M_t = G\theta I_p,$$

where $I_p = (\pi/32)(d^4 - d_1^4)$ is the polar moment of inertia of the ring section. Then

$$\theta = \frac{32M_t}{\pi(d^4 - d_1^4)G} = \frac{M_t}{GI_p} \qquad (152)$$

and the angle of twist will be

$$\varphi = \theta l = \frac{M_t l}{GI_p}. \qquad (153)$$

Substituting eq. (152) in eq. (146), we obtain

$$\tau_{\max} = \frac{16M_t}{\pi d^3\left(1 - \frac{d_1^4}{d^4}\right)} = \frac{M_t d}{2I_p}. \qquad (154)$$

We see from eqs. (153) and (154) that by taking, for instance, $d_1 = \frac{1}{2}d$ the angle of twist and the maximum stress, as compared with the same quantities for a solid shaft of diameter d, will increase about 6 per cent while the reduction in the weight of the shaft will be 25 per cent.

Problems

1. A hollow cylindrical steel shaft, 10 in. outside diameter and 6 in. inside diameter, turns at 1,000 r.p.m. What horse-power is being transmitted if $\tau_{max} = 8{,}000$ lbs. per sq. in.

Answer. $H = 21{,}700$ h.p.

2. Find the maximum torque that may be applied to a hollow circular shaft if $d = 6$ in., $d_1 = 4$ in., and $\tau_w = 8{,}000$ lbs. per sq. in.

3. A hollow propeller shaft of a ship transmits 8,000 h.p. at 100 r.p.m. with a working stress of 4,500 lbs. per sq. in. If $d/d_1 = 2$, find d.

Solution.

$$M_t = \frac{8{,}000 \times 12 \times 33{,}000}{2\pi \times 100}.$$

Eq. (154) becomes

$$\tau_{max} = \frac{16}{15} \cdot \frac{16M_t}{\pi d^3},$$

from which

$$d = \sqrt[3]{\frac{16 \times 16 \times 8{,}000 \times 12 \times 33{,}000}{15 \times 2\pi \times 100 \times \pi \times 4{,}500}} = 18.2 \text{ in.}$$

Then $d_1 = 9.1$ in.

60. The Shaft of Rectangular Cross Section.—The problem of the twist of a shaft of rectangular cross section is complicated, due to the warping of the cross section during twist. This warping can be shown experimentally with a rectangular bar of rubber on whose faces a system of small squares has been traced. It is seen from photograph 231 that during twist the lines originally perpendicular to the axis of the bar become curved. This indicates that the distortion of the small squares, mentioned above, varies along the sides of this cross section, reaches a maximum value at the middle and disappears at the corners. We therefore expect that the shearing stress varies as this distortion, namely, is maximum at the middle of the sides and zero at the corners of the cross section. Investigation

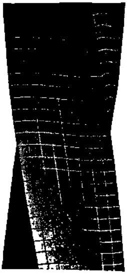

FIG. 231.

of the problem [1] indicates that the maximum shearing stress occurs at the middle of the longer sides of the rectangular cross section and is given by the equation:

$$\tau_{\max} = \frac{M_t}{\alpha b c^2}, \tag{155}$$

in which b is the longer and c the shorter side of the rectangular cross section and α is a numerical factor depending upon the ratio b/c. Several values of α are given in Table 3 below. It is interesting to note that the magnitude of the maximum stress can be calculated with satisfactory accuracy from the following approximate equation:

$$\tau_{\max} = \frac{M_t}{bc^2}\left(3 + 1.8\frac{c}{b}\right).$$

TABLE 3

DATA FOR THE TWIST OF A SHAFT OF RECTANGULAR CROSS SECTION

$\frac{b}{c}$ = 1.00	1.50	1.75	2.00	2.50	3.00	4.00	6	8	10	∞
α = 0.208	0.231	0.239	0.246	0.258	0.267	0.282	0.299	0.307	0.313	0.333
β = 0.141	0.196	0.214	0.229	0.249	0.263	0.281	0.299	0.307	0.313	0.333

The angle of twist per unit length in the case of a rectangular cross section is given by the equation:

$$\theta = \frac{M_t}{\beta b c^3 G}. \tag{156}$$

The values of the numerical factor β are given in the third line of the above table.

In all cases considered the angle of twist per unit length is proportional to the torque and can be represented by the equation

$$\theta = \frac{M_t}{C}, \tag{a}$$

where C is a constant called the *torsional rigidity* of the shaft.

In the case of a circular shaft (eq. 147), $C = GI_p$.

For a rectangular shaft (eq. 156), $C = \beta b c^3 G$.

[1] The complete solution is due to de Saint Venant, Mém. des Savants étrangers, t. 14 (1855). An account of this work will be found in Todhunter and Pearson's "History of the Theory of Elasticity," Vol. II, p. 312.

61. Helical Spring, Close Coiled.—Assume that a helical spring of circular cross section is submitted to the action of axial forces P (Fig. 232), and that any one coil lies nearly in a plane perpendicular to the axis of the helix. Considering the equilibrium of the upper portion of the spring bounded by an axial section such as mn (Fig. 232, b), it can be concluded from the equations of statics that the stresses over the cross section mn of the coil reduce to a shearing force P through the center of the cross section and a couple acting in a counter clockwise direction in the plane of the cross section of mag-

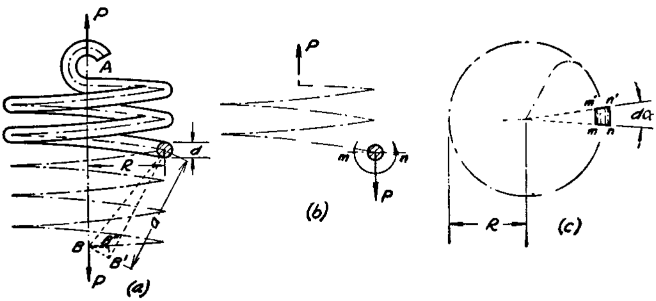

FIG. 232.

nitude PR, where R is the radius of the cylindrical surface containing the center line of the spring. The couple PR twists the coil, and causes a maximum shearing stress given by eq. (149), which becomes here

$$\tau_1 = \frac{16PR}{\pi d^3}, \tag{a}$$

where d is the diameter of the cross section mn of the coil. Upon this stress due to twist that due to the shearing force P is superposed. For a rough approximation, this shearing force is assumed to be uniformly distributed over the cross section; the corresponding shearing stress will be

$$\tau_2 = \frac{4P}{\pi d^2}. \tag{b}$$

At the point m the directions of τ_1 and τ_2 coincide so that the maximum shearing stress occurs here and has the magnitude

$$\tau_{max} = \tau_1 + \tau_2 = \frac{16PR}{\pi d^3}\left(1 + \frac{d}{4R}\right). \qquad (157)$$

It can be seen that the second term in the parenthesis, which represents the effect of the shearing force, increases with the ratio d/R. It becomes of practical importance in heavy helical springs, such as are used on railway cars. Due to this term points such as m on the inner side of a coil are in a less favorable condition than points such as n. Experience with heavy springs shows that cracks usually start on the inner side of the coil.

There is another reason to expect higher stresses at the inner side of the coil. In calculating the stresses due to twist, we used eq. (a), which was derived for cylindrical bars. In reality each element of the spring will be in the condition shown in Fig. 233. It is seen that if the cross section bf rotates with respect to ac, due to twist, the displacement of the point b with respect to a will be the same as that of the point f with respect to c. Due to the fact that the distance ab is smaller than the distance cf, the *shearing strain* at the inner side ab will be larger than that at the outer side cf, and therefore the shearing stresses produced by the couple PR will be larger at b than at f. Taking this into consideration, together with the effect of the shearing force,[2] we replace eq. (157) by the following equation for calculating the maximum shearing stress:

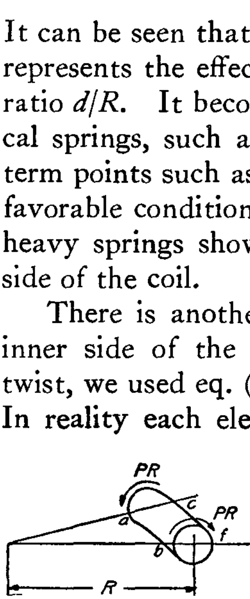

Fig. 233.

$$\tau_{max} = \frac{16PR}{\pi d^3}\left(\frac{4m - 1}{4m - 4} + \frac{0.615}{m}\right), \qquad (158)$$

[2] Such investigations were made by V. Roever, V. D. I., Vol. 57, p. 1906, 1913; also A. M. Wahl, Trans. Am. Soc. Mech. Eng., 1928. The latter also determined the stresses experimentally by making measurements at the surface of the coil. Recent literature on helical springs is given in the paper by J. R. Finniecome. See Trans. Am. Soc. Mech. Eng., Vol. 6 A, p. 188, 1939.

in which

$$m = \frac{2R}{d}.$$

It can be seen that the *correction factor* in the parenthesis increases with a decrease of m; for instance, in case $m = 4$ this factor is about 1.40 and for $m = 10$ it is equal to 1.14. In calculating the deflection of the spring, usually only the effect of the twist of the coils is taken into consideration. For the angle of twist of one element between the two adjacent cross sections mn and $m'n'$ (Fig. 232, *c*), using eq. (148), in which $Rd\alpha$ is used instead of l, we obtain

$$d\varphi = \frac{P \cdot R \cdot Rd\alpha}{I_pG}.$$

Due to this twist the lower portion of the spring rotates with respect to the center of mn (Fig. 232, *a*), and the point of application B of the force P describes the small arc BB' equal to $ad\varphi$. The vertical component of this displacement is

$$B'B'' = BB'\frac{R}{a} = Rd\varphi = \frac{PR^3d\alpha}{I_pG}. \qquad (c)$$

The complete deflection of the spring is obtained by summation of the deflections $B'B''$ due to each element $mnm'n'$, over the length of the spring. Then

$$\delta = \int_0^{2\pi n} \frac{PR^3}{I_pG}d\alpha = \frac{64nPR^3}{d^4G}, \qquad (159)$$

in which n denotes the number of coils.

For a spring of other than circular cross section, the method above can be used to calculate stresses and deflections if, instead of eqs. (148) and (149), we take the corresponding equations for this shape of cross section. For example, in the case of a rectangular cross section eqs. (155) and (156) should be used.

Problems

1. Determine the maximum stress and the extension of the helical spring (Fig. 232) if $P = 250$ lbs., $R = 4$ in., $d = 0.8$ in., the

number of coils is 20 and $G = 12 \times 10^6$ lbs. per sq. in.

Answer.

$$\tau_{\max} = 11{,}300 \text{ lbs. per sq. in.}, \qquad \delta = 4.17 \text{ in.}$$

2. Solve the previous problem, assuming that the coil has a square cross section 0.8 in. on a side.

Solution. Assuming that the correction factor for the shearing force and the curvature of the coils (see eq. 158) in this case is the same as for a circular cross section, we obtain from eq. (155)

$$\tau_{\max} = \frac{PR}{0.208 \times b^3} 1.14 = \frac{250 \times 4 \times 1.14}{0.208 \times 0.8^3} = 10{,}700 \text{ lbs. per sq. in.}$$

In calculating extension $0.141d^4$ (see eq. 156) instead of $\pi d^4/32$ must be used in eq. (159); then

$$\delta = \frac{4.17\pi}{32 \times 0.141} = 2.90 \text{ in.}$$

3. Compare the weights of two helical springs, one of circular, the other of square cross section, designed for the conditions stated in problem 1 and having the same maximum stress. Take the correction factor in both cases as 1.14. Compare the deflections of these two springs.

Solution. The length of the side of the square cross section is found from the equation $\pi d^3/16 = 0.208b^3$, from which $b = \sqrt[3]{0.944} \cdot d = 0.981d$. The weights of the springs are in the same ratio as the cross-sectional areas, i.e., in the ratio

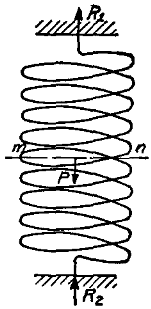

FIG. 234.

$$\frac{\pi d^2}{d} : 0.981^2 d^2 = 0.816.$$

The deflections of the two springs are in the ratio

$$0.141b^4 : \frac{\pi d^4}{32} = 0.141 \times 0.926 : \frac{\pi}{32} = 1.33.$$

4. How will the load P be distributed between the two ends of the helical spring shown in Fig. 234 if the number of coils above the point of application of the load is 6 and that below this point is 5?

Answer. $R_1 : R_2 = 5 : 6.$

5. Two helical springs of the same material and of equal circular cross sections and lengths, assembled as shown in Fig. 235, are

compressed between two parallel planes. Determine the maximum stress in each spring if $d = 0.5$ and $P = 100$ lbs.

Solution. From eq. (159) it follows that the load P is distributed between two springs in inverse proportion to the cubes of the radii of the coils, i.e., the forces compressing the outer and the inner springs will be in the ratio 27 : 64. The maximum stresses in these springs are then (from eq. 158) 2,860 lbs. per sq. in. and 5,380 lbs. per sq. in. respectively.

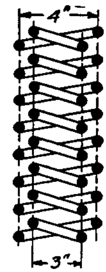

FIG. 235.

6. What will be the limiting load for the spring of problem 1 if the working stress is $\tau_w = 20{,}000$ lbs. per sq. in.? What will be the deflection of the spring at this limiting load?

Answer. 442 lbs. $\delta = 7.38$ in.

7. A conical spring (Fig. 236) is submitted to the action of axial forces P. Determine the safe magnitude of P for a working stress $\tau_w = 45{,}000$ lbs. per sq. in.; diameter of the cross section $d = 1$ in.; radius of the cone at the top of the spring $R_1 = 2$ in.; and at the bottom, $R_2 = 8$ in. Determine the extension of the spring if the number of coils is n, and the horizontal projection of the center line of the spring is a spiral given by the equation

$$R = R_1 + \frac{(R_2 - R_1)\alpha}{2\pi n}.$$

Solution. For any point A of the spring, determined by the magnitude of the angle α, the distance from the axis of the spring is

$$R = R_1 + \frac{(R_2 - R_1)\alpha}{2\pi n} \tag{a}$$

and the corresponding torque is

$$M_t = P\left(R_1 + \frac{(R_2 - R_1)\alpha}{2\pi n}\right).$$

The maximum torque, at $\alpha = 2\pi n$, is $P \cdot R_2$. The safe limit for P, from eq. (158), will be

$$P = \frac{45{,}000 \times \pi}{16 \times 8 \times 1.09} = 1{,}010 \text{ lbs.}$$

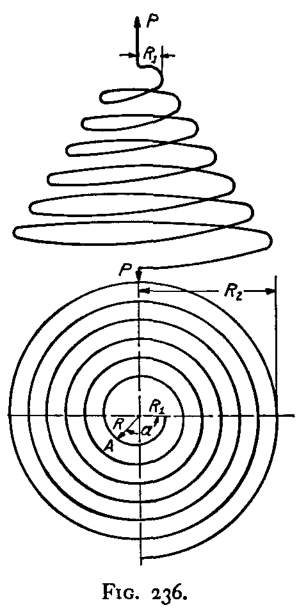

FIG. 236.

The deflection of the spring will be obtained, from eq. (*c*) (see p. 273), as follows:

$$\delta = \frac{32P}{\pi d^4 G}\int_0^{2\pi n}\left[R_1 + \frac{(R_2 - R_1)\alpha}{2\pi n}\right]^3 d\alpha$$

$$= \frac{16Pn}{d^4 G}(R_1^2 + R_2^2)(R_1 + R_2).$$

8. Determine the necessary cross sectional area of coils of a conical spring, designed for the same conditions as in the previous problem, but of a square cross section. Take 1.09 as the correction factor (see previous problem).

Answer. $b^2 = 0.960$ sq. in.

62. Combined Bending and Twist in Circular Shafts.—In the previous discussion of twist (see p. 261) it was assumed that the circular shaft was in simple torsion. In practical applications we often have cases where torque and bending moment are acting simultaneously. The forces transmitted to a shaft by a pulley, a gear or a flywheel can usually be reduced to a torque and a bending force. A simple case of this kind is shown in Fig. 237. A circular shaft is built in at one end and loaded at the other by a vertical force P at a distance R from the axis. This case reduces to one of loading by a torque $M_t = PR$ and by a transverse force P at the free end.[3] The torque is constant along the axis and the bending moment due to P, at any cross section, is

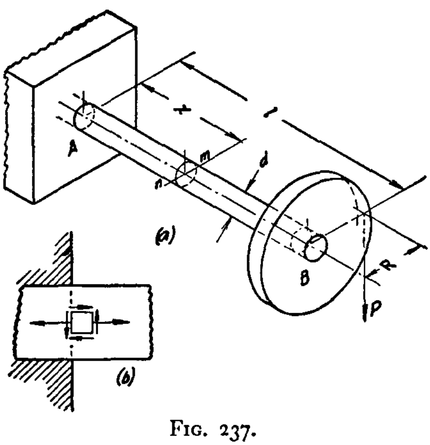

FIG. 237.

$$M = -P(l - x). \qquad (a)$$

In discussing the maximum stress produced in the shaft it is necessary to consider (1) shearing stresses due to the torque

[3] The weight of the shaft and of the pulley is neglected in this problem.

M_t, (2) normal stresses due to the bending moment (a), and (3) shearing stresses due to the shearing force P. The maximum torsional stress occurs at the circumference of the shaft and has the value

$$\tau_{\max} = \frac{16M_t}{\pi d^3}. \qquad (b)$$

The maximum normal stress σ_x due to bending occurs in the fibers most remote from the neutral axis at the built-in end; where the bending moment is numerically a maximum, it has the value

$$(\sigma_x)_{\max} = \frac{M}{Z} = \frac{32M}{\pi d^3}. \qquad (c)$$

The stress due to the shearing force is usually of only secondary importance. Its maximum value occurs at the neutral axis where the normal stress due to bending is zero; hence the maximum combined stress usually occurs at the point where stresses (1) and (2) are a maximum, in this case at the top and bottom surface elements at the built-in end.

Figure 237 (b) is a top view of the portion of the shaft at the built-in end, showing an element and the stresses acting on it. The principal stresses on this element are found from eqs. (72) and (73) (p. 122):

$$\sigma_{\max} = \frac{\sigma_x}{2} + \frac{1}{2}\sqrt{\sigma_x^2 + 4\tau^2},$$

or, using eqs. (b) and (c),

$$\sigma_{\max} = \frac{1}{2Z}(M + \sqrt{M^2 + M_t^2})$$
$$= \frac{16}{\pi d^3}(M + \sqrt{M^2 + M_t^2}). \qquad (160)$$

In the same manner using eq. (73)

$$\sigma_{\min} = \frac{1}{2Z}(M - \sqrt{M^2 + M_t^2})$$
$$= \frac{16}{\pi d^3}(M - \sqrt{M^2 + M_t^2}). \qquad (160')$$

It will be noted that σ_{max} would have the same value for a case of simple bending in which the *equivalent bending moment* is

$$M_{\text{equivalent}} = \tfrac{1}{2}(M + \sqrt{M^2 + M_t^2}).$$

The maximum shearing stress at the same element (Fig. 237, *b*), from eq. (34) (p. 49), is

$$\tau_{max} = \frac{\sigma_{max} - \sigma_{min}}{2} = \frac{16}{\pi d^3}\sqrt{M^2 + M_t^2}. \qquad (161)$$

For ductile metals such as are used in shafting it is now common practice to use the maximum shearing stress to determine the safe diameter of the shaft. Calling the working stress in shear τ_w, and substituting it into eq. (161) for τ_{max}, the diameter must then be

$$d = \sqrt[3]{\frac{16}{\pi \tau_w}\sqrt{M^2 + M_t^2}}. \qquad (162)$$

The above discussion can be used also in the case of a hollow shaft of outer diameter d and inner diameter d_1. Then

$$Z = \frac{\pi(d^4 - d_1^4)}{32d} = \frac{\pi d^3}{32}\left[1 - \left(\frac{d_1}{d}\right)^4\right],$$

and setting $d_1/d = n$, eqs. (160) and (160′) for a hollow shaft become

$$\sigma_{max} = \frac{16}{\pi d^3(1 - n^4)}(M + \sqrt{M^2 + M_t^2}), \qquad (163)$$

$$\sigma_{min} = \frac{16}{\pi d^3(1 - n^4)}(M - \sqrt{M^2 + M_t^2}). \qquad (164)$$

The maximum shearing stress is

$$\tau_{max} = \frac{16}{\pi d^3(1 - n^4)}\sqrt{M^2 + M_t^2}, \qquad (165)$$

and d becomes

$$d = \sqrt[3]{\frac{16}{\pi \tau_w(1 - n^4)}\sqrt{M^2 + M_t^2}}. \qquad (166)$$

If several parallel transverse forces act on the shaft, the total bending moment M and the total torque M_t at each cross section must be taken in calculating the necessary diameter at that point, from eq. (162) or (166). If the transverse forces acting on the shaft are not parallel, the bending moments due to them must be added vectorially to get the resultant bending moment M. An example of such a calculation is discussed in problem 3 below.

Problems

1. A $2\frac{1}{2}$-in. circular shaft carries a 30-inch diameter pulley weighing 500 lbs. (Fig. 238). Determine the maximum shearing

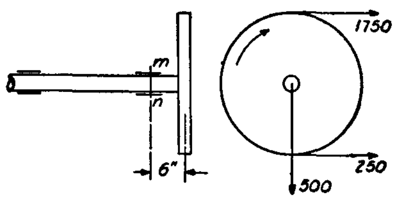

FIG. 238.

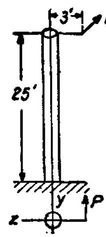

FIG. 239.

stress at cross section mn if the horizontal pulls in the upper and lower portions of the belt are 1,750 lbs. and 250 lbs. respectively.

Solution. At cross section mn,

$$M_t = (1{,}750 - 250)15 = 22{,}500 \text{ lbs. ins.},$$

$$M = 6\sqrt{500^2 + 2{,}000^2} = 12{,}370 \text{ lbs. ins.}$$

Then, from eq. (161),

$$\tau_{\max} = 8{,}370 \text{ lbs. per sq. in.}$$

2. A vertical tube, shown in Fig. 239, is submitted to the action of a horizontal force $P = 250$ lbs. acting 3 feet from the axis of the tube. Determine $\sigma_{\max}$ and $\tau_{\max}$ if the length of the tube is $l = 25'$ and the section modulus $Z = 10$ in.^{3}

Answer.

$$\sigma_{\max} = 7{,}530 \text{ lbs. per sq. in.} \quad \tau_{\max} = 3{,}780 \text{ lbs. per sq. in.}$$

3. Determine the necessary diameter for a uniform shaft (Fig. 240) carrying two equal pulleys, 30 in. in diameter, weighing 500 lbs. each. The horizontal forces in the belt for one pulley and the vertical forces for another are shown in the figure. $\tau_w = 6{,}000$ lbs. per sq. in.

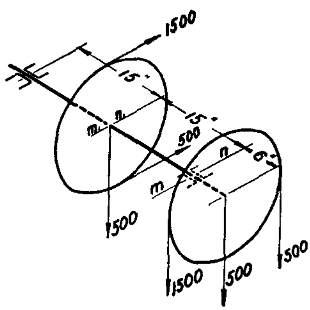

FIG. 240.

Solution. The worst sections are mn and m_1n_1, which carry the full torque and the highest bending moments. The torque at both points is $M_t = (1{,}500 - 500)15 = 15{,}000$ lbs. ins. The bending moment at mn is $(1{,}500 + 500 + 500)6 = 15{,}000$ lbs. ins. The bending moment at m_1n_1 in the horizontal plane is

$$\tfrac{1}{4}(1{,}500 + 500) \times 30 = 15{,}000 \text{ lbs. ins.}$$

The bending moment at the same cross section in the vertical plane is

$$\frac{500 \times 30}{4} - \frac{2{,}500 \times 6 \times 15}{30} = -\ 3{,}750 \text{ lbs. ins.}$$

The combined bending moment at cross section m_1n_1 is

$$M = \sqrt{15{,}000^2 + 3{,}750^2} = 15{,}460 \text{ lbs. ins.}$$

This is larger than the moment at cross section mn and should therefore be used together with the above calculated M_t in eq. (162), from which

$$d = 2.63 \text{ in.}$$

4. Determine the diameter of the shaft shown in Fig. 238 if the working stress in shear is $\tau_w = 6{,}000$ lbs. per sq. in.

5. Determine the outer diameter of a hollow shaft if $\tau_w = 6{,}000$ lbs. per sq. in., $d_1/d = 1/3$, and the other dimensions and forces are as in Fig. 240.

6. Solve problem 3 assuming that the same torque is produced by a horizontal force tangent to the periphery of the pulley instead of by vertical tensions of 1,500 lbs. and 500 lbs. in the belt acting on the right-hand pulley.

CHAPTER X

ENERGY OF STRAIN

63. Elastic Strain Energy in Tension.—In the discussion of a bar in simple tension (see Fig. 1), we saw that, during elongation under a gradually increasing load, work was done on the bar, and that this work was transformed, either partially or completely, into potential energy of strain. If the strain remains within the elastic limit, the work done will be completely transformed into potential energy and can be recovered during a gradual unloading of the strained bar.

If the final magnitude of the load is P and the corresponding elongation is δ, the tensile test diagram will be as shown in Fig. 241, in which the abscissas are the elongations and the ordinates are the corresponding loads. P_1 represents an intermediate value of the load, and δ_1 the elongation due to it. An increase dP_1 in the load causes an increase $d\delta_1$ in the elongation. The work done by P_1 during this elongation is $P_1 d\delta_1$, represented in the figure by the shaded area. If allowance is made for the increase of P_1 during the elongation, the work done will be represented by the area of the trapezoid *abcd*. The total work done in increasing the load from O to P is the summation of such elemental areas, and is given by the area of the triangle OAB. This represents the total energy U stored up in the bar during strain. Then

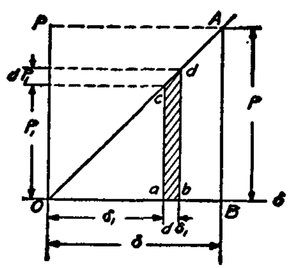

FIG. 241.

$$U = \frac{P\delta}{2}\,. \tag{167}$$

By use of eq. (1), we obtain the following two expressions for the strain energy in a prismatical bar:

$$U = \frac{P^2 l}{2AE}, \tag{168}$$

$$U = \frac{AE\delta^2}{2l}. \tag{169}$$

The first of these gives the strain energy as a function of the load P and the second the same energy as a function of the elongation δ. For a bar of given dimensions and a given modulus of elasticity the strain energy is completely determined by the value of the force P or the value of the elongation δ.

In practical applications the strain energy per unit volume is often of importance; this is, from eqs. (168) and (169):

$$w = \frac{U}{Al} = \frac{\sigma^2}{2E}, \quad (170) \qquad \text{or} \qquad w = \frac{E\epsilon^2}{2}, \quad (171)$$

in which $\sigma = P/A$ is the tensile stress and $\epsilon = \delta/l$ is the unit elongation.

The greatest amount of strain energy per unit volume which can be stored in a bar without permanent set [1] is found by substituting the elastic limit of the material in place of σ in eq. (170). Steel, with an elastic limit of 30,000 lbs. per sq. in. and $E = 30 \times 10^6$ lbs. per sq. in., gives $w = 15$ inch lbs. per cubic inch; rubber, with a modulus of elasticity $E = 150$ lbs. per sq. in. and an elastic limit of 300 lbs. per sq. in., gives $w = 300$ inch lbs. per cubic inch. It is sometimes of interest to know the greatest amount of strain energy per unit weight of a material w_1 which can be stored without producing permanent set. This quantity is calculated from eq. (170) by substituting the elastic limit for σ and dividing w by the weight of one cubic inch of the material. Several numerical results calculated in this manner are given in the table on the following page.

[1] This quantity sometimes is called the *modulus of resilience.*

Material	Density	E lbs. per sq. inch	Elastic limit lbs. per sq. inch	w per cu. inch	w_1 per pound
Structural steel.....	7.8	30×10^6	28,000	13.1 inch lbs.	46 inch lbs.
Tool steel...	7.8	30×10^6	120,000	240 " "	850 " "
Copper.....	8.5	16×10^6	4,000	.5 " "	1.6 " "
Oak.......	1.0	1.5×10^6	4,000	5.3 " "	146 " "
Rubber....	.93	150	300	300 " "	8,900 " "

This indicates that the quantity of energy which can be stored in a given weight of rubber is about 10 times larger than for tool steel and about 200 larger than for structural steel.

Problems

1. A prismatical steel bar 10 inches long and 4 sq. ins. in cross sectional area is compressed by a force $P = 4{,}000$ lbs. Determine the amount of strain energy.

Answer.

$$U = \tfrac{2}{3} \text{ inch lb.}$$

2. Determine the amount of strain energy in the previous problem if the cross-sectional area is 2 sq. in. instead of 4 sq. in.

Answer.

$$U = 1\tfrac{1}{3} \text{ inch lbs.}$$

3. Determine the amount of strain energy in a vertical uniform steel bar strained by its own weight if the length of the bar is 100 feet and its cross-sectional area 1 sq. in., the weight of steel being 490 lbs. per cubic foot.

Answer.

$$U = 0.772 \text{ inch lb.}$$

4. Determine the amount of strain energy in the previous problem if in addition to its own weight the bar carries an axial load $P = 1{,}000$ lbs. applied at the end.

Answer.

$$U = 27.58 \text{ inch lbs.}$$

5. Check the solution of the problem shown in Fig. 15 for the case in which all bars have the same cross section and the same

modulus by equating the strain energy of the system to the work done by the load P.

Solution. If X is the force in the vertical bar, its elongation is Xl/AE and the work done by P is $\frac{1}{2}P(Xl/AE)$. Equating this to the energy of strain, we obtain

$$\frac{1}{2}P\frac{Xl}{AE} = \frac{X^2l}{2AE} + 2\frac{(X\cos^2\alpha)^2l}{2AE\cos\alpha},$$

from which

$$X = \frac{P}{1 + 2\cos^3\alpha},$$

which checks the previous solution.

6. Check problem 2, p. 9, by showing that the work done by the load is equal to the strain energy of the two bars.

7. A steel bar 30 inches long and of 1 sq. in. cross-sectional area is stretched 0.02 in. Find the amount of strain energy.

Answer. From eq. (169),

$$U = \frac{(0.02)^2 \times 30 \times 10^6}{2 \times 30} = 200 \text{ inch lbs.}$$

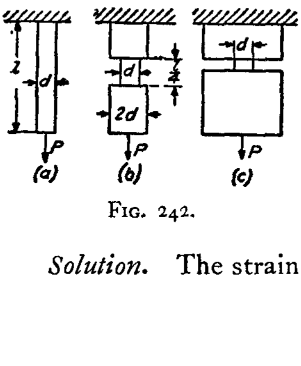

FIG. 242.

8. Compare the amounts of strain energy in the two circular bars shown in Fig. 242 (*a*) and (*b*) assuming a uniform distribution of stresses over cross sections of the bars.

Solution. The strain energy of the prismatical bar is

$$U = \frac{P^2l}{2AE}.$$

The strain energy of the grooved bar is

$$U_1 = \frac{P^2\frac{1}{4}l}{2AE} + \frac{P^2\frac{3}{4}l}{8AE} = \frac{7}{16}\frac{P^2l}{2AE}.$$

Hence

$$U_1 : U = \frac{7}{16}.$$

For a given maximum stress the quantity of energy stored in a grooved bar is less than that in a bar of uniform thickness. It takes only a very small amount of work to bring the tensile stress to a dangerous limit in a bar such as shown in Fig. 242 (*c*), having a very narrow groove and a large outer diameter, although its diameter at the weakest place is equal to that of the cylindrical bar.

64. Tension Produced by Impact.—A simple arrangement for producing tension by impact is shown in Fig. 243. A weight W falls from a height h onto the flange mn and during the impact produces an extension of the vertical bar AB, which is fixed at the upper end. If the masses of the bar and flange are small in comparison with the mass of the falling body, a satisfactory approximate solution is obtained by neglecting the mass of the bar and assuming that there are no losses of energy during impact. After striking the flange mn the body W continues to move downward, causing an extension of the bar. Due to the resistance of the bar the velocity of the moving body diminishes until it becomes zero. At this moment the elongation of the bar and the corresponding tensile stresses are a maximum and their magnitudes are calculated on the assumption that the total work done by the weight W is transformed into strain energy of the bar.[2] If δ denotes the maximum elongation, the work done by W is $W(h + \delta)$. The strain energy of the bar is given by eq. (169). Then the equation for calculating δ is

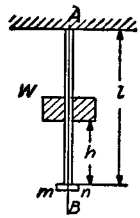

FIG. 243.

$$W(h + \delta) = \frac{AE}{2l}\delta^2, \tag{a}$$

from which

$$\delta = \delta_{st} + \sqrt{\delta_{st}^2 + \frac{1}{g}\delta_{st}v^2}, \tag{172}$$

where

$$\delta_{st} = \frac{Wl}{AE}$$

is the static elongation of the bar by the load W and $v = \sqrt{2gh}$ is the velocity of the falling body at the moment of striking the flange mn. If the height h is large in comparison with

[2] In actual cases part of the energy will be dissipated and the actual elongation will always be less than that calculated on the above assumption.

δ_{st}, this reduces to approximately

$$\delta = \sqrt{\frac{1}{g}\,\delta_{st}v^2}.$$

The corresponding tensile stress in the bar is

$$\sigma = \frac{\delta E}{l} = \frac{E}{l}\sqrt{\frac{1}{g}\,\delta_{st}v^2} = \sqrt{\frac{2E}{Al}\cdot\frac{Wv^2}{2g}}\,. \tag{173}$$

The expression under the radical is directly proportional to the kinetic energy of the falling body, to the modulus of elasticity of the material of the bar and inversely proportional to the volume Al of the bar. Hence the stress can be diminished not only by an increase in the cross-sectional area but also by an increase in the length of the bar or by a decrease in the modulus E. This is quite different from static tension of a bar where the stress is independent of the length l and the modulus E.

By substituting the working stress for σ in eq. (173) we obtain the following equation for proportioning a bar submitted to an axial impact:

$$Al = \frac{2E}{\sigma_w^{\,2}}\cdot\frac{Wv^2}{2g}, \tag{174}$$

i.e., for a given material the *volume* of the bar must be proportional to the kinetic energy of the falling body in order to keep the maximum stress constant.

Let us consider now another extreme case in which h is equal to zero, i.e., the body W is suddenly put on the support mn (Fig. 243) without an initial velocity. Although in this case we have no kinetic energy at the beginning of extension of the bar, the problem is quite different from that of a static loading of the bar. In the case of a static tension we assume a gradual application of the load and consequently there is always equilibrium between the acting load and the resisting forces of elasticity in the bar. The question of the kinetic energy of the load does not enter into the problem at all under such conditions. In the case of a sudden application of the load, the elongation of the bar and the stress in the bar are zero at the beginning and the suddenly applied load begins to fall under the action of its own weight. During this motion the resisting force of the bar gradually increases until it just equals W when the vertical displacement of the weight is δ_{st}. But at this moment the load has a certain kinetic energy, acquired during the displacement δ_{st}; hence it continues to move downward until its velocity is brought to zero by the resisting force in the bar. The maximum elongation for this condition is obtained from eq. (172) by setting $v = 0$. Then

$$\delta = 2\delta_{st}, \qquad (175)$$

i.e., a suddenly applied load, due to dynamic conditions, produces a deflection which is twice as great as that which is obtained when the load is applied gradually.

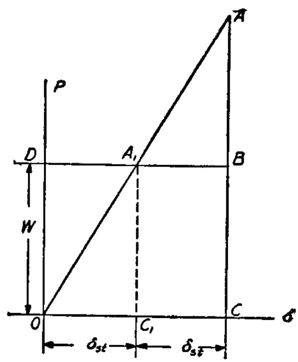

FIG. 244.

This may also be shown graphically as in Fig. 244. The inclined line OA is the tensile test diagram for the bar shown in Fig. 243. Then for any elongation such as OC the area AOC gives the corresponding strain energy in the bar. The horizontal line DB is at distance W from the δ axis and the area $ODBC$ gives the work done by the load W during the displacement OC. When δ is equal to δ_{st}, the work done by W is represented in the figure by the area of the rectangle ODA_1C_1. At the same time the energy stored in the bar is given by the area of the triangle OA_1C_1, which is only half the area of the above rectangle. The other half of the work done is transformed into the kinetic energy of the moving body. Due to its acquired velocity the body continues to move and comes to rest only at the distance $\delta = 2\delta_{st}$ from the origin. At this moment the total work done by the load W, represented by the rectangle $ODBC$, equals the amount of energy stored in the bar and represented by the triangle OAC.

The above discussion of impact is based on the assumption that the stress in the bar remains within the elastic limit. Beyond this limit the problem becomes more involved, because the elongation of the bar is no longer proportional to the tensile force. Assuming that the tensile test diagram does not depend upon the speed of straining the bar,[3] elongation beyond the elastic limit during impact can be determined from an ordinary tensile test diagram such as shown in Fig. 245. For any assumed maximum elongation δ the corresponding area $OADF$ gives the work necessary to produce such an elongation; this must equal the work $W(h + \delta)$ produced by the weight W. When $W(h + \delta)$ is equal or larger than the total area $OABC$ of the tensile test diagram, the falling body will fracture the bar.

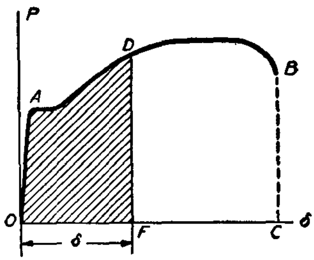

FIG. 245.

From this it follows that any change in the form of the bar which results in diminishing the total area $OABC$ of the diagram diminishes also the resisting power of the bar to impact. In the grooved specimens shown in Fig. 242 (*b*) and (*c*), for instance, the plastic flow of metal will be concentrated at the groove and the total elongation and the work necessary to produce fracture will be much smaller than in the case of the cylindrical bar shown in the same figure. Such grooved specimens are very weak in impact; a slight shock may produce fracture, although the material itself is ductile. Members having rivet holes or any sharp variation in cross section are similarly weak against impact.[4]

In the previous discussion we neglected the mass of the bar

[3] Experiments with ductile steel show that with a high velocity of straining the yield point is higher and the amount of work necessary to produce fracture is greater than in a static test. See N. N. Davidenkoff, Bulletin Polyt. Institute, St. Petersburg, 1913; also Welter, Ztschr. f. Metallkunde, 1924.

[4] See Hackstroh, Baumaterialienkunde, 1905, p. 321, and H. Zimmermann, Zentralbl. d. Bauverw., 1899, p. 265.

in comparison with the mass of the falling body W. Only then may we assume that the total energy of the falling body is transformed into strain energy of the bar. The actual conditions of impact are more complicated and when the bar has an appreciable mass a part of the energy will be lost during impact. It is well known that when a mass W/g moving with a velocity v strikes centrally a stationary mass W_1/g and the deformation at the point of contact is plastic the final common velocity v_a, of the two bodies, is

$$v_a = \frac{W}{W + W_1} v. \tag{b}$$

In the case of the bar shown in Fig. 243 the conditions are more complicated. During impact the upper end A is at rest while the lower end B acquires the velocity of the moving body W. Hence, to calculate the final velocity v_a from eq. (b) we use a *reduced mass* in place of the actual mass of the bar. Assuming that the velocity of the bar varies linearly along its length, it can be shown that the reduced mass in such a case is equal to one third of the mass of the bar.[5] For a bar of weight q per unit length, eq. (b) becomes

$$v_a = \frac{W}{W + \frac{ql}{3}} v.$$

This is the common velocity of the load W and the lower end of the bar which is established at the first moment of impact. Assuming plastic deformation at the surface of contact between the falling load and the support mn (Fig. 243) so that there will be no question of rebounding, the corresponding kinetic energy is

$$\frac{v_a^2}{2g}(W + ql/3) = \frac{Wv^2}{2g} \cdot \frac{1}{1 + \frac{ql}{3W}}.$$

This quantity must be substituted for

$$\frac{Wv^2}{2g} = Wh$$

in eq. (a) in order to take into account the loss of energy at the

[5] This solution was obtained by H. Cox, Cambridge Phil. Soc. Trans., 1849, p. 73. See also Todhunter and Pearson, History, Vol. 1, p. 895.

first moment of impact. Then, instead of eq. (172), we obtain

$$\delta = \delta_{st} + \sqrt{\delta_{st}^2 + \frac{1}{g}\delta_{st}v^2 \frac{1}{1 + \frac{ql}{3W}}}. \tag{176}$$

The method described gives satisfactory results as long as the mass of the bar is small in comparison with the mass of the falling body. Otherwise a consideration of longitudinal vibrations of the bar becomes necessary.[6] The local deformation at the point of contact during impact has been discussed by J. E. Sears[7] and J. E. P. Wagstaff.[8]

Problems

1. A weight of 10 lbs. attached to a steel wire $\frac{1}{8}$ in. in diameter (Fig. 246) falls from A with the acceleration g. Determine the stress produced in the wire when its upper end A is suddenly stopped. Neglect the mass of the wire.

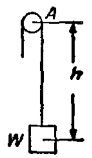

FIG. 246.

Solution. If the acceleration of the weight W is equal to g, there is no tensile stress in the wire. The stress after stopping the wire at A is obtained from eq. (173), in which δ_{st} is neglected. Substituting $v^2 = 2gh$ and $l = h$, we obtain

$$\sigma = \sqrt{\frac{2EW}{A}} = \sqrt{\frac{2 \times 30 \times 10^6 \times 10 \times 8^2}{0.785}} = 221 \times 10^3 \text{ lbs. per sq. in.}$$

It may be seen that the stress does not depend upon the height h through which the load falls, because the kinetic energy of the body increases in the same proportion as the volume of the wire.

2. A weight $W = 1{,}000$ lbs. falls from a height $h = 3$ ft. upon a vertical wooden pole 20 feet long and 12 in. in diameter, fixed at the lower end. Determine the maximum compressive stress in the pole, assuming that for wood $E = 1.5 \times 10^6$ lbs. per sq. in. and neglecting the mass of the pole and the quantity δ_{st}.

---

[6] The longitudinal vibrations of a prismatical bar during impact were considered by Navier. A more comprehensive solution was developed by St. Venant; see his translation of Clebsch, "Theorie der Elasticität fester Körper," note on par. 61. See also I. Boussinesq, "Application des Potentiels," p. 508, and C. Ramsauer, Ann. d. Phys., Vol. 30, 1909.

[7] J. E. Sears, Trans. Cambridge Phil. Soc., Vol. 21 (1908), p. 49.

[8] J. E. P. Wagstaff, London Royal Soc. Proc. (Ser. A), Vol. 105, 1924, p. 544.

Answer. $\sigma = 2{,}000$ lbs. per sq. in.

3. A weight $W = 10{,}000$ lbs. attached to the end of a steel wire rope (Fig. 246) moves downwards with a constant velocity $v = 3$ feet per sec. What stresses are produced in the rope when its upper end is suddenly stopped? The free length of the rope at the moment of impact is $l = 60$ feet, its net cross-sectional area is $A = 2.5$ sq. in. and $E = 15 \times 10^6$ lbs. per sq. in.

Solution. Neglecting the mass of the rope and assuming that the kinetic energy of the moving body is completely transformed into the potential energy of strain of the rope, the equation for determining the maximum elongation δ of the rope is

$$\frac{AE\delta^2}{2l} - \frac{AE\delta_{st}^2}{2l} = \frac{W}{2g}v^2 + W(\delta - \delta_{st}), \tag{d}$$

in which δ_{st} denotes the statical elongation of the rope. Noting that $W = AE\delta_{st}/l$ we obtain, from eq. (d),

$$\frac{AE}{2l}(\delta - \delta_{st})^2 = \frac{Wv^2}{2g},$$

from which

$$\delta = \delta_{st} + \sqrt{\frac{Wv^2l}{AEg}}.$$

Hence, upon sudden stopping of the motion, the tensile stress in the rope increases in the ratio

$$\frac{\delta}{\delta_{st}} = 1 + \frac{v}{\delta_{st}}\sqrt{\frac{Wl}{AEg}} = 1 + \frac{v}{\sqrt{g\delta_{st}}}. \tag{e}$$

For the above numerical data

$$\delta_{st} = \frac{Wl}{AE} = \frac{10{,}000 \times 60 \times 12}{2.5 \times 15 \times 10^6} = .192 \text{ in.},$$

$$\frac{\delta}{\delta_{st}} = 1 + \frac{3 \times 12}{\sqrt{386 \times .192}} = 5.18.$$

Hence

$$\sigma = 5.18\frac{W}{A} = 20{,}700 \text{ lbs. per sq. in.}$$

4. Solve the previous problem if a spring, which elongates .5 in. per thousand pounds load, is put between the rope and the load.

Solution. $\delta_{st} = 0.192 + 0.5 \times 10 = 5.192$ in. Substituting into eq. (*e*),

$$\frac{\delta}{\delta_{st}} = 1 + 0.80 = 1.80; \quad \sigma = 1.80\frac{W}{A} = 7{,}200 \text{ lbs. per sq. in.}$$

5. For the case shown in Fig. 243 determine the height h for which the maximum stress in the bar during impact is 30,000 lbs. per sq. in. Assume $W = 25$ lbs., $l = 6$ feet, $A = \frac{1}{2}$ sq. in., $E = 30 \times 10^6$ lbs. per sq. in. Neglect the mass of the bar.

Answer. $h = 21.6$ in.

65. Elastic Strain Energy in Shear and Twist.—The strain energy stored in an element submitted to pure shearing stress (Fig. 247) may be calculated by the method used in the case of simple tension. If the bottom side *ad* of the element is taken as fixed, only the work done during strain by the force P at the upper side *bc* need be considered. Assuming that the material follows Hooke's law, the shearing strain is proportional to the shearing stress and the diagram showing this relationship is analogous to that shown in Fig. 241. The work done by the force P and stored in the form of elastic strain energy is then (see eq. 167)

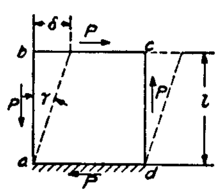

Fig. 247.

$$U = \frac{P\delta}{2}. \tag{167'}$$

Remembering that

$$\frac{\delta}{l} = \gamma = \frac{\tau}{G} = \frac{P}{AG},$$

we obtain the following two equations from (167'):

$$U = \frac{P^2 l}{2AG}, \quad (177) \qquad\qquad U = \frac{AG\delta^2}{2l}. \quad (178)$$

We obtain two expressions for the shearing strain energy per unit volume by dividing these equations by the volume Al of the block:

$$w = \frac{\tau^2}{2G}, \quad (179) \qquad\qquad w = \frac{\gamma^2 G}{2}, \quad (180)$$

in which $\tau = P/A$ is the shearing stress and $\gamma = \delta/l$ is the shearing strain. The amount of shear energy per unit volume, which can be stored in the block without permanent set, is obtained by substituting the elastic limit for τ in eq. (179).

The energy stored in a twisted circular shaft is easily calculated by use of eq. (179). If τ_{max} is the maximum shearing stress at the surface of the shaft, then $\tau_{max}(2r/d)$ is the shearing stress at a point a distance r from the axis, where d is the diameter of the shaft. The energy per unit volume at this point is, from eq. (179),

$$w = \frac{2\tau_{max}^2 r^2}{Gd^2}. \tag{a}$$

The energy stored in the material included between two cylindrical surfaces of radii r and $r + dr$ is

$$\frac{2\tau_{max}^2 r^2}{Gd^2} 2\pi l r dr,$$

where l is the length of the shaft. Then the total energy stored in the shaft is

$$U = \int_0^{d/2} \frac{2\tau_{max}^2 r^2}{Gd^2} 2\pi l r dr = \frac{1}{2}\frac{\pi d^2 l}{4}\frac{\tau_{max}^2}{2G}. \tag{181}$$

This shows that the total energy is only half what it would be if all elements of the shaft were stressed to the maximum shearing stress τ_{max}.

The energy of twist may be calculated from a diagram of twist (Fig. 248) in which the torque is represented by the ordinates and the angle of twist by the abscissas. Within the elastic limit, the angle of twist is proportional to the twisting moment, as represented by an inclined line OA. A small area shaded in the figure represents the work done by the torque during an increase $d\varphi$ in the angle of twist φ. The area $OAB = M_t\varphi/2$

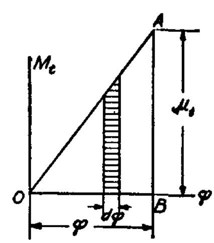

FIG. 248.

represents the total energy stored in the shaft during twist Recalling that $\varphi = M_t l/GI_p$, we obtain

$$U = \frac{M_t^2 l}{2GI_p} \qquad \text{or} \qquad U = \frac{\varphi^2 GI_p}{2l}. \tag{182}$$

In the first of these two equations the energy is represented as a function of the torque, in the second as a function of the angle of twist.

In the general case of any shape of cross section and a torque varying along the length of the shaft, the angle of twist between the two adjacent cross sections is given by the equation (see p. 270)

$$\frac{d\varphi}{dx}dx = \frac{M_t}{C}dx.$$

The strain energy of one element of the shaft is

$$\frac{1}{2}M_t\frac{d\varphi}{dx}dx = \frac{C}{2}\left(\frac{d\varphi}{dx}\right)^2 dx$$

and the total energy of twist is

$$U = \frac{C}{2}\int_0^l \left(\frac{d\varphi}{dx}\right)^2 dx. \tag{183}$$

Problems

1. Determine the ratio between the elastic limit in shear and the elastic limit in tension if the amount of strain energy per cubic inch, which can be stored without permanent set, is the same in tension and in shear.

Solution. From eqs. (170) and (179),

$$\frac{\sigma^2}{2E} = \frac{\tau^2}{2G},$$

from which

$$\frac{\tau}{\sigma} = \sqrt{\frac{G}{E}}.$$

For steel

$$\tau = \sigma\sqrt{\frac{1}{2.6}} = 0.62\sigma.$$

2. Determine the deflection of a helical spring (Fig. 232) by using the expression for the strain energy of twist.

Solution. Denote by P the force acting in the direction of the axis of the helix (Fig. 232), by R the radius of the coils and by n the number of coils. The energy of twist stored in the spring, from eq. (182), is

$$U = \frac{(PR)^2 2\pi Rn}{2GI_p}.$$

Equating this to the work done, $P\delta/2$, we obtain

$$\delta = \frac{2\pi nPR^3}{GI_p} = \frac{64nPR^3}{Gd^4}.$$

3. The weight of a steel helical spring is 10 lbs. Determine the amount of energy which can be stored in this spring without producing permanent set if the elastic limit in shear is 74,300 lbs. per sq. in.

Solution. The amount of energy per cubic inch, from eq. (179), is

$$w = \frac{(74{,}300)^2}{2 \times 11.5 \times 10^6} = 240 \text{ lbs. ins.}$$

The energy per pound of material (see p. 282) is 850 lbs. ins. Then the total energy of twist [9] which can be stored in the spring is

$$\tfrac{1}{2} \times 10 \times 850 = 4{,}250 \text{ lbs. ins.}$$

4. A solid circular shaft and a thin tube of the same material and the same weight are submitted to twist. In what ratio are the amounts of energy in shaft and tube if the maximum stresses in both are equal?

Answer. $\tfrac{1}{2} : 1$.

5. A circular steel shaft with a flywheel at one end rotates at 120 r.p.m. It is suddenly stopped at the other end. Determine the maximum stress in the shaft during impact if the length of the shaft $l = 5$ feet, the diameter $d = 2$ inches, the weight of the flywheel $W = 100$ lbs., its radius of gyration $r = 10$ inches.

Solution. Maximum stress in the shaft is produced when the

[9] The stress distribution is assumed to be the same as that in a twisted circular bar.

total kinetic energy of the flywheel is transformed into strain energy of the twisted shaft. The kinetic energy of the flywheel is

$$\frac{Wr^2\omega^2}{2g} = \frac{100 \times 10^2 \times (4\pi)^2}{2 \times 386} = 2{,}050 \text{ lbs. ins.}$$

Substituting this for U in eq. (181),

$$\tau_{\max} = \sqrt{\frac{16 \times 11.5 \times 10^6 \times 2{,}050}{\pi \times 4 \times 60}} = 22{,}400 \text{ lbs. per sq. in.}$$

6. Two circular bars of the same material, the same length but different cross sections A and A_1, are twisted by the same torque. In what ratio are the amounts of energy of strain stored in these two bars?

Answer. Inversely proportional to the squares of the cross sectional areas.

66. Elastic Strain Energy in Bending.—Let us begin with pure bending. For a prismatic bar built in at one end and bent by a couple M applied at the other end (Fig. 249) the angular displacement at the free end is

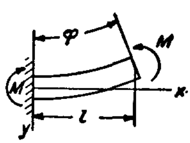

FIG. 249.

$$\varphi = \frac{Ml}{EI_z}. \tag{a}$$

This displacement is proportional to the bending moment M and by using a diagram similar to that in Fig. 248 we may conclude from similar reasoning that the work done during deflection by the bending moment M, or also the energy stored in the bar, is

$$U = \frac{M\varphi}{2}. \tag{b}$$

By use of eq. (a) this energy may be expressed in either of these forms:

$$U = \frac{M^2 l}{2EI_z}; \quad (184) \qquad\qquad U = \frac{\varphi^2 EI_z}{2l}. \quad (185)$$

It is sometimes useful to have the potential energy expressed as a function of the maximum normal stress $\sigma_{max} = M_{max}/Z$; thus, for a rectangular bar $\sigma_{max} = 6M/bh^2$ or $M = bh^2\sigma_{max}/6$, and eq. (184) becomes

$$U = \frac{1}{3} bhl \frac{\sigma_{max}^2}{2E}. \tag{186}$$

In this case the total energy is evidently only one third as much as it would be if all fibers carried the stress σ_{max}.

In the discussion of bending by transverse forces, the strain energy of shear will be neglected, at first. The energy stored in an element of the beam of length dx is, from eqs. (184) and (185),

$$dU = \frac{M^2 dx}{2EI_z} \quad \text{or} \quad dU = \frac{EI_z(d\varphi)^2}{2dx}.$$

Here the bending moment M is variable with respect to x, and

$$d\varphi = \frac{dx}{r} = \left|\frac{d^2y}{dx^2}\right| dx$$

(see p. 135). The total energy stored in the beam is consequently

$$U = \int_0^l \frac{M^2 dx}{2EI_z}, \quad (187) \quad \text{or} \quad U = \int_0^l \frac{EI_z}{2}\left(\frac{d^2y}{dx^2}\right)^2 dx. \quad (188)$$

Take, for instance, the cantilever AB (Fig. 250). The bending moment at any cross section mn is $M = -Px$. Substitution into eq. (187) gives

$$U = \int_0^l \frac{P^2x^2 dx}{2EI_z} = \frac{P^2l^3}{6EI_z}. \tag{c}$$

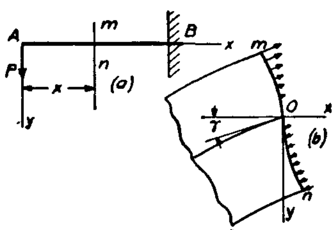

FIG. 250.

For a rectangular bar, $\sigma_{max} = 6Pl/bh^2$, and eq. (c) may be put in the form

$$U = \frac{1}{9} bhl \frac{\sigma_{max}^2}{2E}. \tag{c'}$$

This shows that the quantity of energy which can be stored in a rectangular cantilever beam, loaded at the end, without producing permanent set, is one third of that for pure bending of the same bar and one ninth of that for the same bar in simple tension. This consideration is of importance in designing springs, which must absorb a given amount of energy without damage and yet have as small a weight as possible. The capacity of a cantilever to absorb energy may be increased by giving it a variable cross section. For example, a cantilever of *uniform strength* with a rectangular cross section of constant depth h (Fig. 185), and with the same values for P, h, and σ_{max}, has a deflection and hence an amount of stored energy 50 per cent greater than for the prismatical bar. At the same time the bar of uniform strength has half the weight of the prismatical bar, so it can store three times as much energy per pound of material.

Returning to eq. (c) and equating the strain energy to the work done by the load P during deflection, we obtain

$$\frac{P\delta}{2} = \frac{P^2 l^3}{6EI_z}, \tag{d}$$

from which the deflection at the end is

$$\delta = \frac{Pl^3}{3EI_z},$$

which coincides with eq. (95).

The additional deflection due to shear may also be determined from the potential energy of strain. For the cantilever shown in Fig. 250, with a rectangular cross section, the shearing stress at a distance y from the neutral axis is (see eq. 65)

$$\frac{P}{2I_z}\left(\frac{h^2}{4} - y^2\right).$$

The energy of shear in an elemental volume $bdxdy$ is, therefore, from eq. (179),

$$\frac{P^2}{8GI_z^2}\left(\frac{h^2}{4} - y^2\right)^2 bdxdy,$$

$$U = \int_0^l \int_{-h/2}^{+h/2} \frac{P^2}{8GI_z^2}\left(\frac{h^2}{4} - y^2\right)^2 bdxdy = \frac{P^2lh^2}{20GI_z}. \qquad (e)$$

This must be added to the right side of eq. (d) above [10] to obtain the equation for determining the total deflection:

$$\frac{P\delta}{2} = \frac{P^2l^3}{6EI_z} + \frac{P^2lh^2}{20GI_z}; \qquad (f)$$

consequently

$$\delta = \frac{Pl^3}{3EI_z}\left(1 + \frac{3}{10}\frac{h^2}{l^2}\frac{E}{G}\right). \qquad (g)$$

The second term in the parentheses represents the effect of the shearing stresses on the deflection of the beam. By use of the method developed in article 39 under the assumption that the element of the cross section at the centroid of the built-in end remains vertical (Fig. 250, b), the additional slope due to shear is

$$\gamma = \frac{\tau_{\max}}{G} = \frac{3}{2}\frac{P}{bhG},$$

and the additional deflection is

$$\frac{3}{2}\frac{Pl}{bhG};$$

hence

$$\delta = \frac{Pl^3}{3EI_z} + \frac{3}{2}\frac{Pl}{bhG} = \frac{Pl^3}{3EI_z}\left(1 + \frac{3}{8}\frac{h^2}{l^2}\frac{E}{G}\right). \qquad (g')$$

It will be seen that eqs. (g) and (g') do not coincide. The discrepancy is explained as follows: The derivation of article 39 was based on the assumption that the cross sections of the beam can warp freely under the action of shearing stresses. In such a case the built-in cross section will be distorted to a curved surface *mon* (Fig. 250, b) and in calculating the total work done on the cantilever we must consider not only the work done by the force P, Fig. 250 (a), but also the work done by the stresses acting on the built-in cross section, Fig. 250 (b). If this later work is taken into account, the

[10] Such an addition of the energy of shear to the energy due to normal stresses is justified, because the shearing stresses acting on an element (Fig. 247) do not change the lengths of the sides of the element and if normal forces act on these sides, they do no work during shearing strain. Hence shearing stresses do not change the amount of energy due to tension or compression and the two kinds of energy may be simply added together.

deflection calculated from the consideration of the strain energy coincides with that obtained in article 39 and given in equation (g') above.[11]

In the case of a simply supported beam loaded at the middle, the middle cross section does not warp, as can be concluded from considerations of symmetry. In such a case equation (g), if applied to each half of the beam, will give a better approximation for the deflection than will equation (g'). This can be seen by comparing the approximate equations (g) and (g') with the more rigorous solution given in article 39.

Problems

1. A wooden cantilever beam, 6 feet long, of rectangular cross section 8″ × 5″ carries a uniform load q = 200 lbs. per foot. Determine the amount of strain energy stored if $E = 1.5 \times 10^6$ lbs. per sq. in.

Answer.

$$U = \frac{q^2 l^5}{40EI_z} = \frac{1200^2 \times 72^3 \times 12}{40 \times 1.5 \times 10^6 \times 5 \times 8^3} = 42 \text{ lbs. ins.}$$

2. In what ratio does the amount of strain energy calculated in the previous problem increase if the depth of the beam is 5″ and the width 8″?

Answer. The strain energy increases in the ratio $8^2/5^2$.

3. Two identical bars, one simply supported, the other with built-in ends, are bent by equal loads applied at the middle. In what ratio are the amounts of strain energy stored?

Answer. 4 : 1.

4. Solve the above problem for a uniformly distributed load of the same intensity q for both bars.

5. Find the ratio of the amounts of strain energy stored in beams of rectangular section equally loaded, having the same length and the same width of cross sections but whose depths are in the ratio 2/1.

Solution. For a given load the strain energy is proportional to the deflection and this is inversely proportional to the moment of inertia of the cross section. By halving the depth the deflection is therefore increased 8 times and the amount of strain energy increases in the same proportion.

67. Bending Produced by Impact.—The dynamic deflection of a beam which is struck by a falling body W may be

[11] See "Theory of Elasticity," p. 150, 1934.

determined by the method used in the case of impact causing tension (art. 64). Take, as an example, a simply supported beam struck at the middle (Fig. 251), and assume that the mass of the beam may be neglected in comparison with the mass of the falling body, and that the beam is not stressed beyond the yield point. Then there will be no loss of energy during impact and the work done by the weight W during its fall is completely transformed into strain energy of bending of the beam.[12] Let δ denote the maximum deflection of the beam during impact. If we assume that the deflection curve during impact has the same shape as that during static deflection, the force which would cause such a deflection is, from eq. (90),

$$P = \delta \cdot \frac{48EI_z}{l^3}. \qquad (a)$$

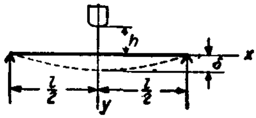

FIG. 251.

The total energy stored in the beam is equal to the work done by the force P:

$$U = \frac{P\delta}{2} = \delta^2 \frac{24EI_z}{l^3}.$$

If h denotes, as before, the distance fallen, the equation for determining δ is

$$W(h + \delta) = \delta^2 \frac{24EI_z}{l^3}, \qquad (b)$$

from which

$$\delta = \delta_{st} + \sqrt{\delta_{st}^2 + \frac{1}{g}\delta_{st}v^2}, \qquad (189)$$

where

$$\delta_{st} = \frac{Wl^3}{48EI_z} \qquad \text{and} \qquad v = \sqrt{2gh}.$$

Equation (189) is exactly the same as that for impact causing tension (eq. 172).

It should be noted that the form of the equation remains the same for any other case of impact, provided the deflection

[12] Local deformation at the surface of contact of the load and the beam is neglected in this calculation.

at the point of impact is proportional to the force P, exerted at this point. If we represent by α the factor of proportionality which depends upon the structure, we have

$$\alpha P = \delta \qquad \text{and} \qquad U = \frac{P\delta}{2} = \frac{\delta^2}{2\alpha}.$$

Then

$$W(h + \delta) = \frac{\delta^2}{2\alpha},$$

and since $\delta_{st} = W\alpha$, this reduces to eq. (189) above.

It should be noted also that the deflection δ calculated from (189) represents the upper limit, which the maximum dynamic deflection approaches when there are no losses of energy during impact. Any such loss will reduce the dynamic deflection. When the dynamic deflection is found from eq. (189), the corresponding stresses can be found by multiplying by δ/δ_{st} the stresses obtained for a statical application of the load W.

When h is large in comparison with δ_{st}, eq. (189) may take the simpler form

$$\delta = \sqrt{\frac{1}{g}\,\delta_{st}v^2}. \tag{c}$$

For the case of a beam supported at the ends and struck at the middle this equation gives

$$\delta = \sqrt{\frac{Wv^2}{2g}\frac{l^3}{24EI_z}}. \tag{d}$$

The maximum bending moment in this case is

$$M_{\max} = \frac{Pl}{4} = \frac{\delta \cdot 48EI_z}{l^3}\frac{l}{4}$$

and

$$\sigma_{\max} = \frac{M_{\max}}{Z} = \frac{\delta \cdot 48EI_z}{l^3}\frac{l}{4Z}.$$

For a rectangular cross section, using eq. (d),

$$\sigma_{\max} = \sqrt{\frac{Wv^2}{2g}\frac{18E}{lA}}. \tag{e}$$

This indicates that the maximum stress depends upon the kinetic energy of the falling body and the volume lA of the beam.

In determining the effect of the mass of the beam on the maximum deflection we will assume that the deflection curve during impact has the same shape as during static deflection. Then it can be shown that the reduced mass of the beam [13] supported at the ends is $17/35\ (ql/g)$ and the common velocity which will be established at the first moment of impact is

$$v_a = \frac{W}{W + (17/35)\,ql}v.$$

The total kinetic energy after the establishment of common velocity v_a is

$$\frac{v_a^2}{2g}(W + (17/35)\,ql) = \frac{Wv^2}{2g}\,\frac{1}{1 + \frac{17}{35}\frac{ql}{W}};$$

using this instead of

$$\frac{Wv^2}{2g} = Wh$$

in eq. (b), we obtain

$$\delta = \delta_{st} + \sqrt{\delta_{st}^2 + \frac{\delta_{st}v^2}{g}\,\frac{1}{1 + \frac{17}{35}\frac{ql}{W}}}, \tag{190}$$

which takes account of the effect of the mass of the beam on the deflection δ.[14]

[13] See paper by Homersham Cox mentioned before (see p. 289).

[14] Several examples of the application of this equation will be found in the paper by Prof. Tschetsche, Zeitschr. d. Ver. d. Ing., 1894, p. 134. A more accurate theory of transverse impact on the beam is based on the investigation of its lateral vibration together with the local deformations at the point of impact. See St. Venant, loc. cit., p. 537. Note finale du par. 61; C. R., Vol. 45, 1857, p. 204. See also writer's paper in Ztschr. f. Math. u. Phys., Vol. 62, 1913, p. 198. Experiments with beams subjected to impact have been made in Switzerland and are in satisfactory agreement with the above approximate theory; see "Tech. Komm. d. Verband Schweiz. Brückenbau- u. Eisenhochbaufabriken," Bericht von M. Roš, March, 1922. See also the recent articles by Tuzi, Z., and Nisida, M., Phil. Mag. (7), Vol. 21, p. 448; and R. N. Arnold, Proc. of the Institution of Mechanical Engineers, Vol. 137, 1937, p. 217.

In the case of a cantilever, if the weight W strikes the beam at the end, the magnitude of the reduced mass of the beam is 33/140 (ql/g). When a beam simply supported at ends is struck at a point whose distances from the supports are respectively a and b, the reduced mass is

$$\frac{1}{105}\left[1 + 2\left(1 + \frac{l^2}{ab}\right)^2\right]\frac{ql}{g}.$$

Problems

1. A simply supported rectangular wooden beam 9 feet long is struck at the middle by a 40 lb. weight falling from a height $h = 12$ in. Determine the necessary cross-sectional area if the working stress is $\sigma_w = 1{,}000$ lbs. per sq. in., $E = 1.5 \times 10^6$ lbs. per sq. in.

Solution. Using eq. (e), p. 302,[15]

$$A = \frac{Wv^2}{2g}\frac{18E}{l\sigma_w^2} = 40 \times 12\frac{18 \times 1.5 \times 10^6}{9 \times 12 \times 1{,}000^2} = 120 \text{ sq. in.}$$

2. In what proportion does the area in the previous problem change (1) if the span of the beam increases from 9 to 12 feet; (2) if the weight W increases by 50 per cent?

Answer. (1) The area diminishes in the ratio 3 : 4. (2) The area increases by 50 per cent.

3. A weight $W = 100$ lbs. drops 12 inches upon the middle of a simply supported I beam, 10 ft. long. Find safe dimensions if $\sigma_w = 30 \times 10^3$ lbs. per sq. in.

Solution. Neglecting δ_{st} in comparison with h (see eq. c), the ratio between the dynamic and the static deflections is

$$\frac{\delta}{\delta_{st}} = \sqrt{\frac{v^2}{g\delta_{st}}} = \sqrt{\frac{2h}{\delta_{st}}}.$$

If the deflection curve during impact is of the same shape as for static deflection, the maximum bending stresses will be in the same ratio as the deflections; hence

$$\sqrt{\frac{2h}{\delta_{st}}}\frac{Wl}{4Z} = \sigma_w, \text{ from which } \frac{Z}{c} = \frac{6EW}{\sigma_w^2}\frac{h}{l},$$

in which Z is the section modulus and c is the distance from the neutral axis of the most remote fiber, which is half the depth of

[15] Local deformation at the surface of contact of the load and the beam is neglected in this calculation.

the beam in our case. Substituting the numerical data,

$$\frac{Z}{c} = \frac{6 \times 30 \times 10^6 \times 100 \times 12}{30{,}000^2 \times 120} = 2 \text{ in.}^2$$

The necessary I beam is of 5″ depth, wt. per foot 12.25 lbs.

4. What stress is produced in the beam of the previous problem by a 200 lb. weight falling onto the middle of the beam from a height of 6 in.?

Answer. $\sigma_{\max} = 28{,}900$ lbs. per sq. in.

5. A wooden cantilever beam 6 feet long and of square cross section 12″ × 12″ is struck at the end by a weight $W = 100$ lbs. falling from a height $h = 12$ in. Determine the maximum deflection, taking into account the loss in energy due to the mass of the beam.

Solution. Neglecting δ_{st} in comparison with h, the equation analogous to eq. (190) becomes

$$\delta = \sqrt{\frac{\delta_{st}v^2}{g} \frac{1}{1 + \frac{33}{140}\frac{ql}{W}}}.$$

For $ql = 40 \times 6 = 240$ lbs.,

$$\delta = \sqrt{\delta_{st} \cdot \frac{24}{1 + \frac{33 \times 240}{140 \times 100}}} = \sqrt{\frac{15.3 \times 100 \times 72^3}{3 \times 1.5 \times 10^6 \times 12^3}} = 0.271 \text{ in.}$$

6. A beam simply supported at the ends is struck at the middle by a weight W falling down from the height h. Neglecting δ_{st} in comparison with h, find the magnitude of the ratio ql/W at which the effect of the mass of the beam reduces the dynamical deflection by 10 per cent.

Answer.

$$\frac{ql}{W} = 0.483.$$

68. The General Expression for Strain Energy.—In the discussion of problems in tension, compression, twist and bending it has been shown that the energy of strain can be represented in each case by a function of the second degree in the external forces (eqs. 168, 177 and 184) or by a function of the second degree in the displacements (eqs. 169, 178 and

185). This is also true for the most general deformation of an elastic body, with the following provisions: the material follows Hooke's law; the conditions are such that the small displacements, due to strain, do not affect the action of the external forces and are negligible in calculating the stresses.[16] With these two provisions, the displacements of an elastic system are linear functions of the external loads; if these loads increase in a certain proportion, all the displacements increase in the same proportion. Consider a body submitted to the action of the external forces P_1, P_2, P_3, $\cdots$ (Fig. 252) and supported in such a manner that movement as a rigid body is impossible and displacements are due to elastic deformations only. Let δ_1, δ_2, δ_3, $\cdots$ denote the displacements of the points of application of the forces each measured in the direction of the corresponding force.[17] If the external forces increase gradually so that they are always in equilibrium with the resisting internal elastic forces, the work which they do during deformation will be equal to the strain energy stored in the deformed body. The amount of this energy does not depend upon the order in which the forces are applied and is completely determined by their final magnitudes. Let us assume that all external forces P_1, P_2, P_3, $\cdots$ increase simultaneously in the same ratio; then the relation between each force and its corresponding displacement can be represented by a diagram analogous to that shown in Fig. 241, and the work done by all the forces P_1, P_2, P_3, $\cdots$, equal to the strain energy stored in the body, is

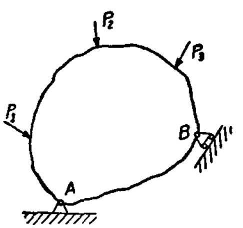

FIG. 252.

$$U = \frac{P_1\delta_1}{2} + \frac{P_2\delta_2}{2} + \frac{P_3\delta_3}{2} + \cdots, \tag{191}$$

[16] Such problems as the bending of bars by lateral forces with simultaneous axial tension or compression do not satisfy the above condition and are excluded from this discussion. Regarding these exceptional cases see article 72.

[17] The displacements of the same points in the directions perpendicular to the corresponding forces are not considered in the following discussion.

i.e., the total energy of strain is equal to half the sum of the products of each external force and its corresponding displacement.[18] On the assumptions made above, the displacements $\delta_1, \delta_2, \delta_3, \cdots$ are linear functions of the forces $P_1, P_2, P_3, \cdots$. The substitution of these functions into eq. (191) gives a general expression for the strain energy in the form of a homogeneous function of the second degree in the external forces. If the forces be represented as linear functions of displacements and these functions be substituted into eq. (191), an expression for the strain energy in the form of a homogeneous function of the second degree in displacements is obtained.

In the above discussion the reactions at the supports were not taken into consideration. The work done by these reactions during the deformation is equal to zero since the displacement of an immovable support, such as A (Fig. 252), is zero and the displacement of a movable support, such as B, is perpendicular to the reaction, friction at the supports being neglected. Consequently, the reactions add nothing to the expression for the potential energy (191).

As an example of the application of eq. (191) let us consider the energy stored in a cubic element submitted to uniform tension in three perpendicular directions (Fig. 50). If the edge of the cube is of unit length, the tensile forces on its faces are numerically σ_x, σ_y, σ_z and the corresponding elongations, ϵ_x, ϵ_y, ϵ_z. Then the strain energy stored in one cubic inch, from eq. (191), is

$$w = \frac{\sigma_x \epsilon_x}{2} + \frac{\sigma_y \epsilon_y}{2} + \frac{\sigma_z \epsilon_z}{2}.$$

Substituting, for the elongations, the values given by (43),[19]

$$w = \frac{1}{2E}(\sigma_x^2 + \sigma_y^2 + \sigma_z^2) - \frac{\mu}{E}(\sigma_x \sigma_y + \sigma_y \sigma_z + \sigma_z \sigma_x). \quad (192)$$

[18] This conclusion was obtained first by Clapeyron; see Lamé, Leçons sur la théorie mathématique de l'élasticité, 2 ed., 1866, p. 79.

[19] Here the changes in temperature due to strain are considered of no practical importance. For further discussion see the book by T. Weyrauch, "Theorie elastischer Körper," Leipzig, 1884, p. 163. See also Z. f. Architektur- und Ingenieurwesen, Vol. 54, 1908, p. 91 and p. 277.

This expression can also be used when some of the *normal* stresses are compressive, in which case they must be given a negative sign.

If in addition to normal stresses there are shearing stresses acting on the faces of the element, the energy of shear can be added to the energy of tension or compression (see p. 299), and using eq. (179) the total energy stored in one cubic inch is

$$w = \frac{1}{2E}(\sigma_x^2 + \sigma_y^2 + \sigma_z^2) - \frac{\mu}{E}(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x) + \frac{1}{2G}(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2). \quad (193)$$

As a second example let us consider a beam supported at the ends, loaded at the middle by a force P and bent by a couple M applied at the end A. The deflection at the middle is, from eqs. (90) and (105),

$$\delta = \frac{Pl^3}{48EI} + \frac{Ml^2}{16EI}. \quad (a)$$

The slope at the end A is, from eqs. (88) and (104),

$$\theta = \frac{Pl^2}{16EI} + \frac{Ml}{3EI}. \quad (b)$$

Then the strain energy of the beam, equal to the work done by the force P and by the couple M, is

$$U = \frac{P\delta}{2} + \frac{M\theta}{2} = \frac{1}{EI}\left(\frac{P^2l^3}{96} + \frac{M^2l}{6} + \frac{MPl^2}{16}\right). \quad (c)$$

This expression is a homogeneous function of the second degree in the external force and the external couple. Solving eqs. (*a*) and (*b*) for M and P and substituting in eq. (*c*), an expression for the strain energy in the form of a homogeneous function of the second degree in displacements may be obtained. It must be noted that when external couples are acting on the body the corresponding displacements are the angular displacements of surface elements on which these couples are acting.

69. The Theorem of Castigliano.—From the expressions for the energy of strain in various cases a very simple method for calculating the displacements of points of an elastic body during deformation may be established. For example, in the case of simple tension (Fig. 1), the strain energy as given by eq. (168) is

$$U = \frac{P^2 l}{2AE}.$$

By taking the derivative of this expression with respect to P we obtain

$$\frac{dU}{dP} = \frac{Pl}{AE} = \delta,$$

i.e., the derivative of the strain energy with respect to the load gives the displacement *corresponding* to the load, i.e., at the point of application of the load in the direction of the load. In the case of a cantilever loaded at the end, the strain energy is (eq. c, p. 297)

$$U = \frac{P^2 l^3}{6EI}.$$

The derivative of this expression with respect to the load P gives the known deflection at the free end $Pl^3/3EI$.

In the twist of a circular shaft the strain energy is (eq. 182)

$$\frac{M_t^2 l}{2GI_p}.$$

The derivative of this expression with respect to the torque gives

$$\frac{dU}{dM_t} = \frac{M_t l}{GI_p} = \varphi,$$

which is the angle of twist of the shaft, and represents the displacement *corresponding* to the torque.

When several loads act on an elastic body, the same method of calculation of displacements may be used. For example, expression (c) of the previous article gives the strain energy of a beam bent by a load P at the middle and by a couple M at the end. The partial derivative of this expression with respect to P gives the deflection under the load and the partial derivative with respect to M gives the angle of rotation of the end of the beam on which the couple M acts.

The theorem of Castigliano is a general statement of these results.[20] If the material of the system follows Hooke's law and the conditions are such that the small displacements due to deformation can be neglected in discussing the action of forces, the strain energy of such a system may be given by a homogeneous function of the second degree in the acting forces (see art. 68). Then the partial derivative of strain energy with respect to any such force gives the displacement corresponding to this force (exceptional cases see art. 72). The terms "force" and "displacement" here may have their generalized meanings, that is, they include "couple" and "angular displacement" respectively.

Let us consider a general case such as shown in Fig. 252. Assume that the strain energy is represented as a function of the forces P_1, P_2, P_3, $\cdots$, so that

$$U = f(P_1, P_2, P_3, \cdots). \tag{a}$$

If a small increase dP_n is given to any external load P_n, the strain energy will increase also and its new amount will be

$$U + \frac{\partial U}{\partial P_n} dP_n. \tag{b}$$

But the magnitude of the strain energy does not depend upon the order in which the loads are applied to the body—it depends only upon their final values. It can be assumed, for instance, that the infinitesimal load dP_n was applied first, and afterwards the loads P_1, P_2, P_3, $\cdots$. The final amount of strain energy remains the same, as given by eq. (*b*). The load dP_n, applied first, produces only an infinitesimal displacement, so that the corresponding work done is a small quantity of the second order and can be neglected. Applying now the loads P_1, P_2, P_3, $\cdots$, it must be noticed that their effect will not be

[20] See the paper by Castigliano, "Nuova Teoria Intorno dell' Equilibrio dei Sistemi Elastici," Atti della Academia delle scienze, Torino, 1875. See also his "Théorie de l'équilibre des systèmes élastiques," Turin, 1879. For an English translation of Castigliano's work see E. S. Andrews, London, 1919.

modified by the load dP_n previously applied [21] and the work done by these loads will be equal to U (eq. a), as before. But during the application of these forces, however, dP_n is given some displacement δ_n in the direction of P_n, and does the work $(dP_n)\delta_n$. The two expressions for the work must be equal; therefore

$$U + \frac{\partial U}{\partial P_n}(dP_n) = U + (dP_n)\delta_n,$$

$$\delta_n = \frac{\partial U}{\partial P_n}\,. \tag{194}$$

As an application of the theorem let us consider a cantilever beam carrying a load P and a couple M_a at the end, Fig. 253. The bending moment at a cross section mn is $M = -Px - M_a$ and the strain energy, from equation (184), is

$$U = \int_0^l \frac{M^2dx}{2EI}\,.$$

To obtain the deflection δ at the end of the cantilever we have only to take the partial derivative of U with respect to P, which gives

$$\delta = \frac{\partial U}{\partial P} = \frac{1}{EI}\int_0^l M\frac{\partial M}{\partial P}\,dx.$$

Substituting for M its equivalent expression, in terms of P and M_a, we obtain

$$\delta = \frac{1}{EI}\int_0^l (Px + M_a)x dx = \frac{Pl^3}{3EI} + \frac{M_a l^2}{2EI}\,.$$

The same expression would have been obtained by applying one of the previously described methods, such as the area moment method.

To obtain the slope at the end we calculate the partial

[21] This follows from the provisions made on page 306 on the basis of which the strain energy was obtained as a homogeneous function of the second degree.

derivative of the strain energy with respect to the couple M_a. Then

$$\theta = \frac{\partial U}{\partial M_a} = \frac{1}{EI}\int_0^l M\frac{\partial M}{\partial M_a}dx$$

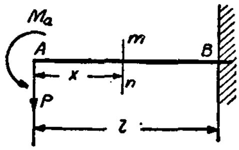

Fig. 253.

$$= \frac{1}{EI}\int_0^l (Px + M_a)dx = \frac{Pl^2}{2EI} + \frac{M_a l}{EI}.$$

The positive signs obtained for δ and θ indicate that the deflection and rotation of the end have the same directions respectively as the force and the couple in Fig. 253.

It should be noted that the partial derivative $\partial M/\partial P$ is the rate of increase of the moment M with respect to the increase of the load P and can be visualized by the bending moment diagram for a load equal to unity, as shown in Fig. 254 (a). The partial derivative $\partial M/\partial M_a$ can be visualized

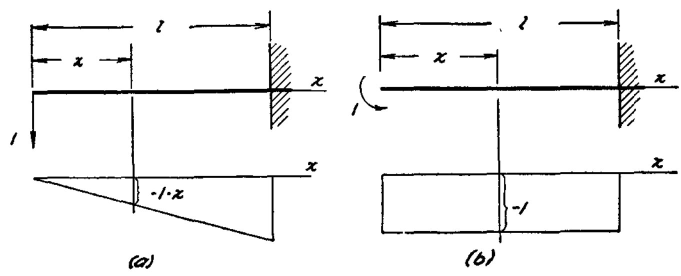

Fig. 254.

in the same manner by the bending moment diagram in Fig. 254 (b). Using the notations

$$\frac{\partial M}{\partial P} = M_p' \qquad \text{and} \qquad \frac{\partial M}{\partial M_a} = M_m'$$

we can represent our previous results in the following form:

$$\delta = \frac{1}{EI}\int_0^l MM_p'dx; \qquad \theta = \frac{1}{EI}\int_0^l MM_m'dx. \qquad (195)$$

These equations, derived for the particular case shown in Fig. 253, also hold for the general case of a beam with any kind of loading and any kind of support. They can also be used in the case of distributed loads.

Let us consider, for example, the case of a uniformly loaded and simply supported beam, Fig. 255, and calculate the deflection at the middle of this beam by using the Castigliano theorem. In the preceding cases concentrated forces and couples acted, and partial derivatives with respect to these forces and couples gave the corresponding displacements and

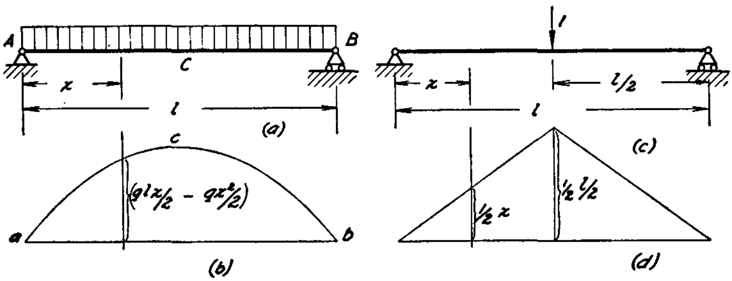

FIG. 255.

rotations. In the case of a uniform load there is no vertical force acting at the middle of the beam which would *correspond* to the deflection at the middle. Thus we cannot proceed as in the previous problem. This difficulty can, however, be readily removed by assuming that there is a fictitious load P of infinitely small magnitude at the middle. Such a force evidently will not affect the deflection and the bending moment diagram shown in Fig. 255 (*b*). At the same time, the rate of increase of the bending moment due to the increase of P, represented by the partial derivative $\partial M/\partial P$, is as represented by Fig. 255 (*c*) and 255 (*d*). With these values of M and $\partial M/\partial P$ the value of the deflection is

$$\delta = \frac{\partial U}{\partial P} = \frac{1}{EI_z}\int_0^l M\frac{\partial M}{\partial P}\,dx.$$

Observing that M and $\partial M/\partial P$ are both symmetrical with respect to the middle of the span, we obtain

$$\delta = \frac{2}{EI_z}\int_0^{l/2} M\frac{\partial M}{\partial P}\,dx$$
$$= \frac{2}{EI_z}\int_0^{l/2}\left(\frac{qlx}{2} - \frac{qx^2}{2}\right)\frac{x}{2}\,dx = \frac{5}{384}\frac{ql^4}{EI_z}.$$

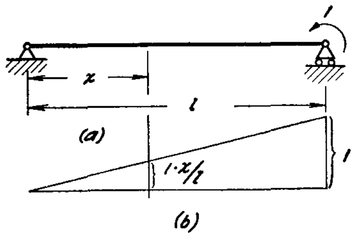

FIG. 256.

If it is required to calculate the slope at the end B of the beam in Fig. 255 (a) by using the Castigliano theorem, we have only to assume an infinitely small couple M_b applied at B. Such a couple does not change the bending moment diagram in Fig. 255 (b). The partial derivative $\partial M/\partial M_b$ is then as represented in Fig. 256 (a) and 256 (b). The required rotation of the end B of the beam is then

$$\theta = \frac{\partial U}{\partial M_b} = \frac{1}{EI_z}\int_0^l M\frac{\partial M}{\partial M_b}\,dx$$
$$= \frac{1}{EI_z}\int_0^l\left(\frac{qlx}{2} - \frac{qx^2}{2}\right)\frac{x}{l}\,dx$$
$$= \frac{ql^3}{24EI_z}.$$

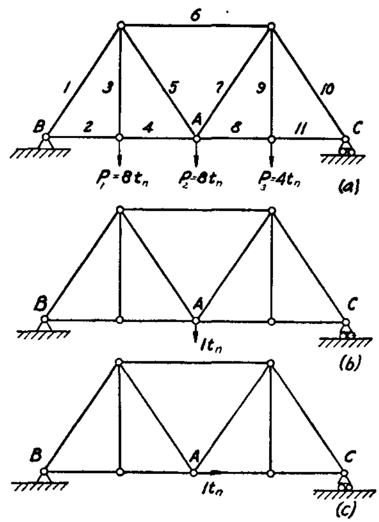

FIG. 257.

We see that the results obtained by the use of Castigliano's theorem coincide with those previously obtained (p. 138).

The Castigliano theorem is especially useful in the calculation of deflections in trusses. As an example let us consider the case shown in Fig. 257. All members of the system are numbered and their lengths and cross-sectional areas given in the table below. The force S_i produced in any bar i of the system by the loads P_1, P_2, P_3

may be calculated from simple equations of statics. These forces are given in column 4 of the table. The strain energy

TABLE OF DATA FOR THE TRUSS IN FIGURE 257

1	2	3	4	5	6	7
i	l_i in.	A_i in.^{2}	S_i tn.	S_i'	$\frac{S_iS_i'l_i}{A_i}$	S_i''
1	250	6	−13.75	−0.625	358	0
2	150	3	8.25	0.375	155	1
3	200	2	8.00	0	0	0
4	150	3	8.25	0.375	155	1
5	250	2	3.75	0.625	293	0
6	300	4	−10.50	−0.750	59	0
7	250	2	6.25	0.625	488	0
8	150	3	6.75	0.375	127	0
9	200	2	4.00	0	0	0
10	250	6	−11.25	−0.625	293	0
11	150	3	6.75	0.375	127	0

$$\sum_{i=1}^{i=m} \frac{S_iS_i'l_i}{A_i} = 2{,}055 \text{ tns. per inch.}$$

of any bar i, from eq. (168), is $S_i^2l_i/2A_iE$. The amount of strain energy in the whole system is

$$U = \sum_{i=1}^{i=m} \frac{S_i^2l_i}{2A_iE}, \tag{196}$$

in which the summation is extended over all the members of the system, which in our case is $m = 11$. The forces S_i are functions of the loads P, and the deflection δ_n under any load P_n is, therefore, from eq. (194),

$$\delta_n = \frac{\partial U}{\partial P_n} = \sum_{i=1}^{i=m} \frac{S_il_i}{A_iE} \cdot \frac{\partial S_i}{\partial P_n}. \tag{197}$$

The derivative $\partial S_i/\partial P_n$ is the rate of increase of the force S_i with increase of the load P_n. Numerically it is equal to the force produced in the bar i by a unit load applied in the position of P_n, and we will use this fact in finding the above derivative. These derivatives will hereafter be denoted by

S_i'. The equation for calculating the deflections then becomes

$$\delta_n = \sum_{i=1}^{i=m} \frac{S_i S_i' l_i}{A_i E}. \tag{198}$$

Consider for instance the deflection δ_2 corresponding to P_2 at A in Fig. 257 (*a*). The magnitudes S_i' tabulated in column 5 above are obtained by the simple principles of statics from the loading conditions shown in Fig. 257 (*b*), in which all actual loads are removed and a vertical load of one ton is applied at the hinge A. The values tabulated in column 6 are calculated from those entered in columns 2 through 5. Summation and division by the modulus $E = (30/2,000) \times 10^6$ tns. per sq. in. gives the deflection at A, eq. (198),

$$\delta_2 = \frac{2,055 \times 2,000}{30 \times 10^6} = 0.137 \text{ in.}$$

The above discussion was concerned with the computation of displacements $\delta_1, \delta_2, \cdots$ corresponding to the given external forces $P_1, P_2, \cdots$. In investigating the deformation of an elastic system, it may be necessary to calculate the displacement of a point at which there is no load at all, or the displacement of a loaded point in a direction different from that of the load. The method of Castigliano may also be used here. We merely apply at that point an additional infinitely small *imaginary load* Q in the direction in which the displacement is wanted, and calculate the derivative $\partial U/\partial Q$. In this derivative the added load Q is put equal to zero, and the desired displacement obtained. For example, in the truss shown in Fig. 257 (*a*), let us calculate the horizontal displacement of the point A. A horizontal force Q is applied at this point, and the corresponding horizontal displacement is

$$\delta_h = \left(\frac{\partial U}{\partial Q}\right)_{Q=0} = \sum_{i=1}^{i=m} \frac{S_i l_i}{A_i E} \cdot \frac{\partial S_i}{\partial Q}, \tag{d}$$

in which the summation is extended over all the members of the system. The forces S_i in eq. (*d*) have the same meaning as before, because the added load Q is zero, and the derivatives $\partial S_i/\partial Q = S_i''$ are obtained as the forces in the bars of the truss produced by the loading shown in Fig. 257 (*c*). These are tabulated in column 7. Substituting these forces into eq. (*d*), we find that the horizontal displacement of A is equal to the sum of the elongations of the bars 2 and 4, namely,

$$\delta_h = \frac{1}{E}\left(\frac{S_2 l_2}{A_2} + \frac{S_4 l_4}{A_4}\right) = \frac{150 \times 2{,}000}{3 \times 30 \times 10^6}(8.25 + 8.25) = 0.055 \text{ in.}$$

In investigating the deformation of trusses it is sometimes necessary to know the change in distance between two points of the system. This can also be done by the Castigliano method. Let us determine, for instance, what decrease δ, in the distance between the joints A and B (Fig. 258, *a*), is produced by the loads P_1, P_2, P_3. At these joints, two equal and opposite imaginary forces Q are applied as indicated in the figure by the dotted lines. It follows from the Castigliano theorem that the partial derivative $(\partial U/\partial Q)_{Q=0}$ gives the sum of the displacements of A and B, in the direction AB, produced by the loads P_1, P_2, P_3. Using eq. (194), this displacement is[22]

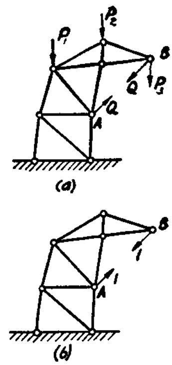

FIG. 258.

$$\delta = \left(\frac{\partial U}{\partial Q}\right)_{Q=0} = \sum_{i=1}^{i=m} \frac{S_i l_i}{A_i E}\frac{\partial S_i}{\partial Q_i} = \sum_{i=1}^{i=m} \frac{S_i l_i}{A_i E} \cdot S_i', \qquad (199)$$

in which S_i are the forces produced in the bars of the system by the actual loads P_1, P_2, P_3; S_i' are the quantities to be determined from the loading shown in Fig. 258 (*b*), in which all actual loads are removed and two opposite unit forces are applied at A and B; and m is the number of members.

Problems

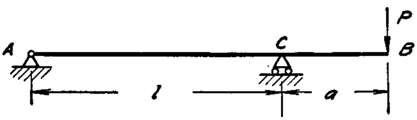

FIG. 259.

1. Determine by the use of Castigliano's theorem the deflection and the slope at the end of a uniformly loaded cantilever beam.

2. Determine the deflection at the end B of the overhang of the beam shown in Fig. 259.

[22] This problem was first solved by J. C. Maxwell, "On the Calculation of the Equilibrium and Stiffness of Frames," Phil. Mag. (4), Vol. 27, 1864, p. 294. Scientific Papers, Vol. 1, Cambridge, 1890, p. 598.

3. A system consisting of two prismatical bars of equal length and equal cross section (Fig. 260) carries a vertical load P. Determine the vertical displacement of the hinge A.

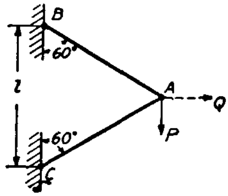

FIG. 260.

Solution. The tensile force in the bar AB and compressive force in the bar AC are equal to P. Hence the strain energy of the system is

$$U = 2\frac{P^2 l}{2AE}.$$

The vertical displacement of A is

$$\delta = \frac{dU}{dP} = \frac{2Pl}{AE}.$$

4. Determine the horizontal displacement of the hinge A in the previous problem.

Solution. Apply a horizontal imaginary load Q as shown in Fig. 260 by the dotted line. The potential energy of the system is

$$U = \frac{(P + 1/\sqrt{3}Q)^2 l}{2AE} + \frac{(P - 1/\sqrt{3}Q)^2 l}{2AE}.$$

The derivative of this expression with respect to Q for $Q = 0$ gives the horizontal displacement

$$\delta_h = \left(\frac{\partial U}{\partial Q}\right)_{Q=0} = \left(\frac{2Ql}{3AE}\right)_{Q=0} = 0.$$

5. Determine the angular displacement of the bar AB produced by the load P in Fig. 261.

Solution. An imaginary couple M is applied to the system as shown in the figure by dotted lines. The displacement corresponding to this couple is the angular displacement φ of the bar AB due to the load P. The forces S_i in this case are: $P + 1/\sqrt{3}(M/l)$ in the bar AB and $-P - 2/\sqrt{3}(M/l)$ in the bar AC. The strain energy is

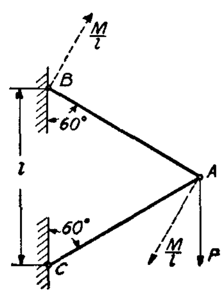

FIG. 261.

$$U = \frac{l}{2AE}\left[\left(P + 1/\sqrt{3}\,\frac{M}{l}\right)^2 + \left(-P - 2/\sqrt{3}\,\frac{M}{l}\right)^2\right],$$

from which

$$\varphi = \left(\frac{\partial U}{\partial M}\right)_{M=0} = \left(\frac{P\sqrt{3}}{AE} + \frac{5M}{3lAE}\right)_{M=0} = \frac{P\sqrt{3}}{AE}.$$

6. What horizontal displacement of the support B of the frame shown in Fig. 262 is produced by horizontal force H?

Answer.

$$\delta_h = \frac{2}{3}\frac{Hh^3}{EI_1} + \frac{Hh^2l}{EI}.$$

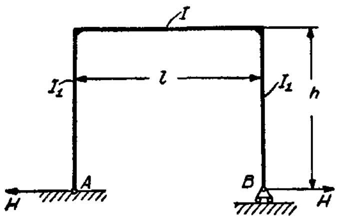

FIG. 262.

7. Determine the vertical displacement of the point A and horizontal displacement of the point C of the steel truss shown in Fig. 263 if $P = 2{,}000$ lbs., the cross-sectional areas of the compressed bars are 5 sq. in., and of the other bars 2 sq. in.

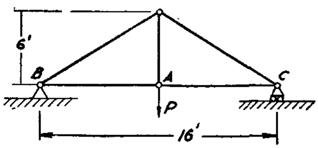

FIG. 263.

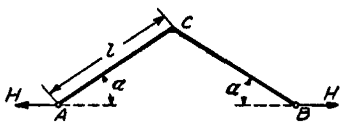

FIG. 264.

8. Determine the increase in the distance AB produced by forces H (Fig. 264) if the bars AC and BC are of the same dimensions and only the bending of the bars need be taken into account. It is assumed that α is not small, so that the effect of deflections on the magnitude of bending moment can be neglected.

Answer.

$$\delta = \frac{2}{3}\frac{H\sin^2\alpha\, l^3}{EI}.$$

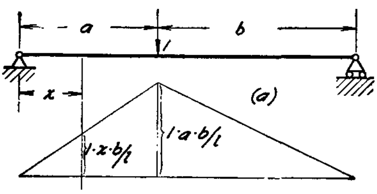

FIG. 265.

9. Determine the deflection at a distance a from the left end of the uniformly loaded beam shown in Fig. 255 (a).

Solution. Applying an infinitely small load P at a distance a from the left end, the partial derivative $\partial M/\partial P$ is as visualized in Fig. 265 (a) and 265 (b). Using for M the parabolic diagram in Fig. 255 (b), the desired deflection is

$$\delta = \frac{\partial U}{\partial P} = \frac{1}{EI}\int_0^l M\frac{\partial M}{\partial P}\,dx = \frac{1}{EI}\int_0^a \left(\frac{qlx}{2} - \frac{qx^2}{2}\right)\frac{xb}{l}\,dx$$
$$+ \frac{1}{EI}\int_b^l \left(\frac{qlx}{2} - \frac{qx^2}{2}\right)\frac{a(l-x)}{l}\,dx = \frac{qab}{24EI}(a^2 + b^2 + 3ab).$$

Substituting x for a and $l - x$ for b this result can be brought into agreement with the equation for the deflection curve previously obtained (p. 138).

70. Application of Castigliano Theorem in Solution of Statically Indeterminate Problems.—The Castigliano theorem is very useful also in the solution of statically indeterminate problems. Let us begin with problems in which the reactions at the supports are considered as the statically indeterminate quantities. Denoting by X, Y, Z, $\cdots$ the statically indeterminate reactive forces, the strain energy of the system is a function of these forces. For the immovable supports and for the supports whose motion is perpendicular to the direction of the reactions the partial derivatives of the strain energy with respect to the unknown reactive forces must be equal to zero by the Castigliano theorem. Hence

$$\frac{\partial U}{\partial X} = 0; \qquad \frac{\partial U}{\partial Y} = 0; \qquad \frac{\partial U}{\partial Z} = 0; \qquad \cdots. \tag{200}$$

In this manner we obtain as many equations as there are statically indeterminate reactions.

It can be shown that eqs. (200) represent the conditions for a minimum of function U, from which it follows that the magnitudes of statically indeterminate reactive forces are such as to make the strain energy of the system a minimum. This is the *principle of least work* as applied to the determination of redundant reactions.[23]

[23] The principle of least work was stated first by F. Menabrea in his article, "Nouveau principe sur la distribution des tensions dans les systèmes élastiques," Paris, C. R., Vol. 46 (1858), p. 1056. See also C.R., Vol. 98 (1884), p. 714. The complete proof of the principle was given by Castigliano, who made of this principle the fundamental method of solution of statically indeterminate systems. The application of

As an example of application of the above principle let us consider a uniformly loaded beam built in at one end and supported at the other (Fig. 266). This is the problem with one statically indeterminate reaction. Taking the vertical reaction X at the right support as the statically indeterminate quantity, this unknown force is found from the equation:

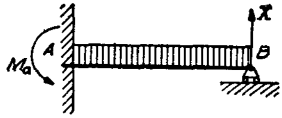

FIG. 266.

$$\frac{dU}{dX} = 0. \qquad (a)$$

The strain energy of the beam, from eq. (187), is

$$U = \int_0^l \frac{M^2 dx}{2EI}, \qquad (b)$$

in which

$$M = Xx - \frac{qx^2}{2}.$$

Substituting in (*a*), we obtain

$$\frac{dU}{dX} = \frac{1}{EI}\int_0^l M\frac{dM}{dX}dx = \frac{1}{EI}\int_0^l \left(Xx - \frac{qx^2}{2}\right)xdx$$
$$= \frac{1}{EI}\left(X\frac{l^3}{3} - q\frac{l^4}{8}\right) = 0,$$

from which

$$X = \tfrac{3}{8}ql.$$

Instead of the reactive force X the reactive couple M_a at the left end of the beam could have been taken as the statically indeterminate quantity. The strain energy will now be a function of M_a. Equation (*b*) still holds, where now the

strain energy methods in engineering was developed by O. Mohr (see his "Abhandlungen aus dem Gebiete d. technischen Mechanik"), by H. Müller-Breslau in his book, "Die neueren Methoden der Festigkeitslehre," and F. Engesser, "Über die Berechnung statisch unbestimmter Systeme," Zentralbl. d. Bauverwalt. 1907, p. 606. A very complete bibliography of this subject is given in the art. by M. Grüning, Encyklopädie d. Math. Wiss., Vol. IV, 2, II, p. 419.

bending moment at any cross section is

$$M = \left(\frac{ql}{2} - \frac{M_a}{l}\right)x - \frac{qx^2}{2}.$$

From the condition that the left end of the actual beam does not rotate when the beam is bent the derivative of the strain energy with respect to M_a must be equal to zero. From this we obtain

$$\frac{dU}{dM_a} = \frac{1}{EI}\int_0^l M\frac{dM}{dM_a}dx = -\frac{1}{EI}\int_0^l \left[\left(\frac{ql}{2} - \frac{M_a}{l}\right)x - \frac{qx^2}{2}\right]\frac{x}{l}dx = -\frac{1}{EI}\left(\frac{ql^3}{24} - \frac{M_a l}{3}\right) = 0,$$

from which the absolute value of the moment is

$$M_a = \tfrac{1}{8}ql^2.$$

Problems in which we consider the forces acting in redundant members of the system as the statically indeterminate quantities can also be solved by using the Castigliano theorem. Take, as an example, the system represented in Fig. 15 which was already discussed (see p. 19). Considering the force X in the vertical bar OC as the statically indetermined quantity, the forces in the inclined bars OB and OD are $(P - X)/2\cos\alpha$. Denoting by U_1 the strain energy of the inclined bars (Fig. 267, a) and by U_2 the strain energy of the vertical bar (Fig. 267, b), the total strain energy of the system is,[24]

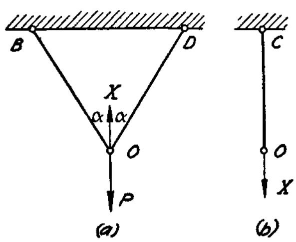

FIG. 267.

$$U = U_1 + U_2 = \left(\frac{P - X}{2\cos\alpha}\right)^2 \frac{l}{AE\cos\alpha} + \frac{X^2 l}{2AE}. \qquad (c)$$

[24] It is assumed that all bars have the same cross-sectional area A and the same modulus of elasticity E.

If δ is the actual displacement downwards of the joint O in Fig. 15 the derivative with respect to X of the energy U_1 of the system in Fig. 267 (a) should be equal to $-\delta$, since the force X of the system has the direction opposite to that of the displacement δ. In the same time the derivative $\partial U_2/\partial X$ will be equal to δ. Hence

$$\frac{\partial U}{\partial X} = \frac{\partial U_1}{\partial X} + \frac{\partial U_2}{\partial X} = -\delta + \delta = 0. \qquad (d)$$

It is seen that the true value of the force X in the redundant member is such as to make the total strain energy of the system a minimum. Substituting for U its expression (c) in equation (d) we obtain

$$-\frac{(P - X)}{2\cos^2\alpha}\,\frac{l}{AE\cos\alpha} + \frac{Xl}{AE} = 0$$

from which

$$X = \frac{P}{1 + 2\cos^3\alpha}.$$

A similar reasoning can be applied to any statically indetermined system with one redundant member, and we can state that the force in that member is such as to make the strain energy of the system a minimum. To illustrate the procedure of calculating stresses in such systems let us consider the frame

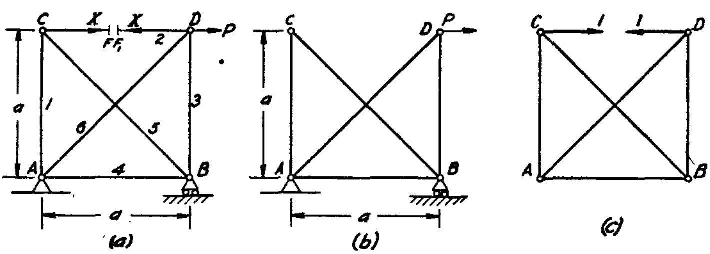

FIG. 268.

shown in Fig. 268 (a). The reactions here are statically determinate, but when we try to compute the forces in the bars, we find that there is one redundant member. Let us consider the bar CD as this redundant member. Cut this bar at any point and apply to each end F and F_1 a force X, equal to that in the bar. We thus arrive at a statically determinate system acted upon by the known force P, and, in addition, unknown forces X. The forces in the bars of this system are found in two groups: first, those produced by the external loading P assuming $X = 0$, Fig. 268, *b*, and denoted by S_i^0, where i indicates the number of the bar; second, those produced when the external force P is removed and unit forces replace the X forces (Fig. 268, *c*). The latter forces are denoted by S_i'. Then the total force in any bar, when the force P and the forces X are all acting, is

$$S_i = S_i^0 + S_i'X. \tag{e}$$

The total strain energy of the system, from eq. (196), is

$$U = \sum_{i=1}^{i=m} \frac{S_i^2 l_i}{2A_iE} = \sum_{i=1}^{i=m} \frac{(S_i^0 + S_i'X)^2 l_i}{2A_iE}, \tag{f}$$

in which the summation is extended over all the bars of the system including the bar CD, which is cut.[25] The Castigliano theorem is now applied and the derivative of U with respect to X gives the displacement of the ends F and F_1 towards each other. In the actual case the bar is continuous and this displacement is equal to zero. Hence

$$\frac{dU}{dX} = 0, \tag{g}$$

i.e., the force X in the redundant bar is such as to make the strain energy of the system a minimum. From eqs. (*f*) and (*g*)

$$\frac{d}{dX} \sum_{i=1}^{i=m} \frac{(S_i^0 + S_i'X)^2 l_i}{2A_iE} = \sum_{i=1}^{i=m} \frac{(S_i^0 + S_i'X) l_i S_i'}{A_iE} = 0,$$

from which

$$X = -\frac{\displaystyle\sum_{i=1}^{i=m} \frac{S_i^0 S_i' l_i}{A_iE}}{\displaystyle\sum_{i=1}^{i=m} \frac{S_i'^2 l_i}{A_iE}}. \tag{201}$$

[25] For this bar $S_i^0 = 0$ and $S_i' = 1$.

This process may be extended to a system in which there are several redundant bars.

The principle of least work can be applied also when the statically unknown quantities are couples. Take, as an example, a uniformly loaded beam on three supports (Fig. 269). If the bending moment at the middle support be considered the statically indeterminate quantity, the beam is cut at B and we obtain two simply supported beams (Fig.

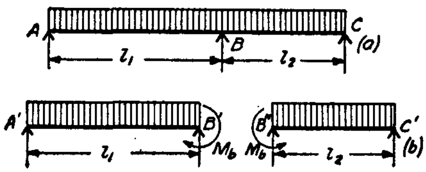

FIG. 269.

269, b) carrying the unknown couples M_b in addition to the known uniform load q. There is no rotation of the end B' with respect to the end B'' because in the actual case (Fig. 269, a) there is a continuous deflection curve. Hence

$$\frac{dU}{dM_b} = 0. \tag{202}$$

Again the magnitude of the statically indeterminate quantity is such as to make the strain energy of the system a minimum.

Problems

1. The vertical load P is supported by a vertical bar DB of length l and cross-sectional area A and by two equal inclined bars of length l and cross sectional area A_1 (Fig. 270). Determine the forces in the bars and also the ratio A_1/A which will make the forces in all bars numerically equal.

Solution. The system is statically indeterminate. Let X be the tensile force in the vertical bar. The compressive forces in the inclined bars are $1/\sqrt{2}(P - X)$ and the strain energy of the system is

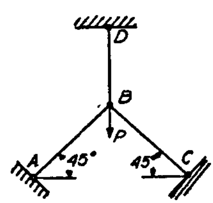

FIG. 270.

$$U = \frac{X^2 l}{2AE} + \frac{(P - X)^2 l}{2A_1E} \cdot$$

The principle of least work gives

$$\frac{dU}{dX} = \frac{Xl}{AE} - \frac{(P - X)l}{A_1E} = 0,$$

from which

$$X = \frac{P}{1 + \frac{A_1}{A}}.$$

Substituting this into equation

$$X = \frac{1}{\sqrt{2}}(P - X),$$

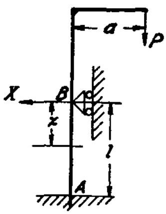

FIG. 271.

we obtain

$$A_1 = \sqrt{2}A.$$

2. Determine the horizontal reaction X in the system shown in Fig. 271.

Solution. The unknown force X will enter only into the expression for the potential energy of bending of the portion AB of the bar. For this portion, $M = Pa - Xx$, and the equation of least work gives

$$\frac{dU}{dX} = \frac{d}{dX}\int_0^l \frac{M^2dx}{2EI} = \frac{1}{EI}\int_0^l M\frac{dM}{dX}dx = -\frac{1}{EI}\int_0^l (Pa - Xx)xdx$$
$$= \frac{1}{EI}\left(\frac{Xl^3}{3} - \frac{Pal^2}{2}\right) = 0,$$

from which

$$X = \frac{3}{2}P\frac{a}{l}.$$

3. Determine the horizontal reactions X of the system shown in Fig. 272. All dimensions are given in the table below.

Solution. From the principle of least work we have

$$\frac{dU}{dX} = \frac{d}{dX}\sum \frac{S_i^2l_i}{2A_iE}$$
$$= \sum \frac{S_il_i}{A_iE}\frac{dS_i}{dX} = 0.$$

Let S_i^0 be the force in bar i produced by the known load P assum-

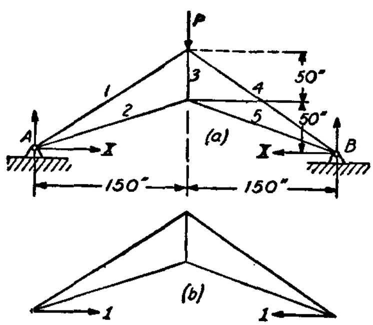

FIG. 272.

ing $X = 0$, S_i' the force produced in the same bar by unit forces which replace the X forces (Fig. 272, b). The values of S_i^0 and S_i' are determined from statics. They are given in columns 4 and 5 of the table below. Then the total force in any bar is

$$S_i = S_i^0 + S_i'X.$$

i	l_i in.	A_i in.^{2}	S_i^0	S_i'	$\dfrac{S_i^0 S_i' l_i}{A_i}$	$\dfrac{S_i'^2 l_i}{A_i}$
1	180.3	5	$-1.803P$	1.202	$-78.1P$	52.0
2	158.1	3	$1.581P$	-2.108	$-175.7P$	234
3	50.0	2	$1.000P$	-1.333	$-33.3P$	44.5
4	180.3	5	$-1.803P$	1.202	$-78.1P$	52.0
5	158.1	3	$1.581P$	-2.108	$-175.7P$	234

$\Sigma = -540.9P$; $\Sigma = 616.5$.

Substituting into the equation of least work (200),

$$\sum_1^5 \frac{(S_i^0 + S_i'X)l_i}{A_iE} S_i' = 0,$$

from which

$$X = -\frac{\displaystyle\sum_{i=1}^{i=5} \frac{S_i^0 S_i' l_i}{A_i}}{\displaystyle\sum_{i=1}^{i=5} \frac{S_i'^2 l_i}{A_i}}. \qquad (f)$$

The necessary figures for calculating X are given in columns 6 and 7. Substituting this data into eq. (f), we obtain

$$X = 0.877P.$$

4. Determine the force in the redundant horizontal bar of the system shown in Fig. 273, assuming that the length of this bar is $l_0 = 300''$ and the cross-sectional area is A_0. The other bars have the same dimensions as in problem 3.

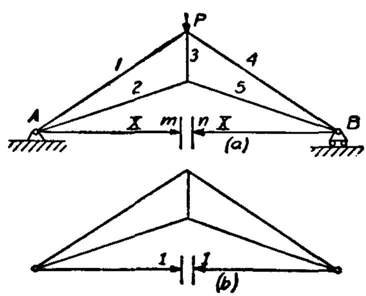

FIG. 273.

Solution. The force in the horizontal bar is calculated from eq. (201). This equation is of the same kind as eq. (f) in problem 3 but in the system of Fig. 273 there is the additional horizontal bar. The force produced in this bar by the force P alone ($X = 0$) is zero, i.e., $S_0^0 = 0$. The force produced by two forces equal to unity (Fig. 273, b) is $S_0' = 1$. The additional term in the numerator of eq. (f) is

$$\frac{S_0^0 S_0' l_0}{A_0} = 0.$$

The additional term in the denominator is

$$\frac{S_0'^2 l_0}{A_0} = \frac{1 \cdot l_0}{A_0} = \frac{300}{A_0}.$$

Then, by using the data of problem 3,

$$X = \frac{540.9P}{\frac{300}{A_0} + 616.5}.$$

Taking, for instance, $A_0 = 10$ sq. in.,

$$X = \frac{540.9P}{30 + 616.5} = 0.836P.$$

That is only 4.7 per cent less than the value obtained in problem 3 for immovable supports.[26]

Taking the cross-sectional area $A_0 = 1$ sq. in.,

$$X = \frac{540.9}{300 + 616.5} = 0.590P.$$

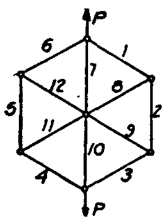

FIG. 274.

It can be seen that in statically indeterminate systems the forces in the bars depend also on their cross-sectional areas.

5. Determine the forces in the bars of the systems shown in Fig. 20 by using the principle of least work.

6. Determine the forces in the bars of the system shown in Fig. 274, assuming that all bars are of the same dimensions and material.

Solution. If one bar be removed, the forces in the remaining bars can be determined from statics; hence the system has one redundant bar. Let 1 be this bar and X the force acting in it. Then all the

[26] Taking $A_0 = \infty$, we obtain the same condition as for immovable supports.

bars on the sides of the hexagon will have tensile forces X, bars 8, 9, 11 and 12 have compressive forces X, and bars 7 and 10 have the force $P - X$. The strain energy of the system is

$$U = 10\frac{X^2 l}{2AE} + 2\frac{(P - X)^2 l}{2AE}.$$

From the equation $dU/dX = 0$ we obtain

$$X = \frac{P}{6}.$$

7. Determine the forces in the system shown in Fig. 268, assuming the cross-sectional areas of all bars equal and taking the force X in the diagonal AD as the statically indeterminate quantity.

Solution. Substituting the data, given in the table below, in eq. (201),

$$X = \frac{3 + 2\sqrt{2}}{4 + 2\sqrt{2}}P.$$

i	l_i	S_i^0	S_i'	$S_i^0 S_i' l_i$	$S_i'^2 l_i$
1	a	P	$-1/\sqrt{2}$	$-aP/\sqrt{2}$	$a/2$
2	a	P	$-1/\sqrt{2}$	$-aP/\sqrt{2}$	$a/2$
3	a	0	$-1/\sqrt{2}$	0	$a/2$
4	a	P	$-1/\sqrt{2}$	$-aP/\sqrt{2}$	$a/2$
5	$a\sqrt{2}$	$-P\sqrt{2}$	$+1$	$-2aP$	$a\sqrt{2}$
6	$a\sqrt{2}$	0	$+1$	0	$a\sqrt{2}$

$$\Sigma = \frac{-(3 + 2\sqrt{2})aP}{\sqrt{2}}; \quad \Sigma = 2a(1 + \sqrt{2}).$$

8. A rectangular frame of uniform cross section (Fig. 275) is submitted to a uniformly distributed load of intensity q as shown. Determine the bending moment M at the corner.

Answer.

$$M = \frac{(a^3 + b^3)q}{12(a + b)}.$$

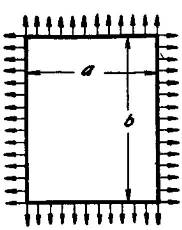

FIG. 275.

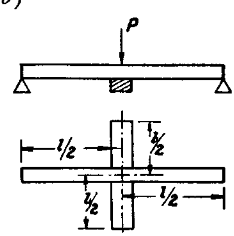

FIG. 276.

9. A load P is supported by two beams of equal cross section, crossing each other as shown in Fig. 276. Determine the pressure X between the beams.

Answer.

$$X = \frac{Pl^3}{l^3 + l_1^3}.$$

10. Find the statically indeterminate quantity in the frame shown in Fig. 167 by using the principle of least work.

Solution. The strain energy of bending of the frame is

$$U = 2\int_0^h \frac{H^2x^2dx}{2EI_1} + \int_0^l \frac{(M_0 - Hh)^2dx}{2EI}, \tag{g}$$

in which M_0 denotes the bending moment for the horizontal bar calculated as for a beam simply supported at the ends. Substituting in equation:

$$\frac{dU}{dH} = 0, \tag{h}$$

we find

$$\frac{2H}{EI_1}\frac{h^3}{3} + \frac{Hh^2l}{EI} = \frac{h}{EI}\int_0^l M_0dx. \tag{k}$$

The integral on the right side is the area of the triangular moment diagram for a beam carrying the load P. Hence

$$\int_0^l M_0dx = \frac{1}{2}Pc(l - c).$$

Substituting in (k), we obtain for H the same expression as in (114). (See p. 192.)

11. Find the statically indeterminate quantities in the frames shown in Figs. 166, 169 and 171 by using the principle of least work.

12. Find the bending moment in Fig. 269 assuming that $l_1 = 2l_2$.

71. The Reciprocal Theorem.—Let us begin with a problem of a simply supported beam shown in Fig. 277 (a) and calculate the deflection at a point D when the load P is acting at C. This deflection is obtained by substituting $x = d$ into equation (86) which gives

$$(y)_{x=d} = \frac{Pbd}{6l}(l^2 - b^2 - d^2). \tag{a}$$

It is seen that the deflection (*a*) does not change if we put d for b and b for d, which indicates that for the case shown in Fig. 277 (*b*) the deflection at D_1 is the same as the deflection at D in Fig. 277 (*a*). From Fig. 277 (*b*) we obtain Fig. 277 (*c*) by simply rotating the beam through 180 degrees which brings point C_1 in coincidence with point D and point D_1 with point C. Hence the deflection at C in Fig. 277 (*c*) is equal to the deflection at D in Fig. 277 (*a*). This means that if the load P is moved from point C to point D, the deflection which in the first case of loading was measured at D will be obtained in the second case at point C. This is a particular case of the *reciprocal theorem*.

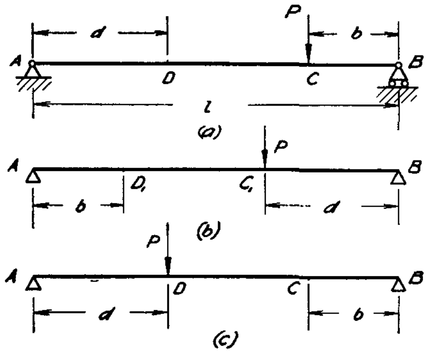

FIG. 277.

To establish the theorem in general form [27] we consider an elastic body, shown in Fig. 278, loaded in two different manners and supported in such a way that displacement as a rigid body is impossible. In the first state of stress the applied forces are P_1 and P_2, and in the second state P_3 and P_4. The displacements of the points of application in the directions of the forces are $\delta_1, \delta_2, \delta_3, \delta_4$ in the first state and $\delta_1', \delta_2', \delta_3', \delta_4'$ in the second state. The reciprocal theorem states: The work done by the forces of the first state on the corresponding displacements of the second state is equal to the work done by the forces of the second state on the corresponding dis-

[27] A particular case of this theorem was obtained by J. C. Maxwell, loc. cit., p. 317. The theorem is due to E. Betti, Il nuovo Cimento (Ser. 2), V. 7 and 8 (1872). In a more general form, the theorem was given by Lord Rayleigh, London Math. Soc. Proc., Vol. 4 (1873), or Scientific Papers, Vol. 1, p. 179. Various applications of this theorem to the solution of engineering problems were made by O. Mohr, loc. cit., p. 327, and H. Müller-Breslau, loc. cit., p. 327.

placements of the first. In symbols this means

$$P_1\delta_1' + P_2\delta_2' = P_3\delta_3 + P_4\delta_4. \tag{203}$$

To prove this theorem let us consider the strain energy of the body when all forces $P_1, \cdots P_4$ are acting together and let us

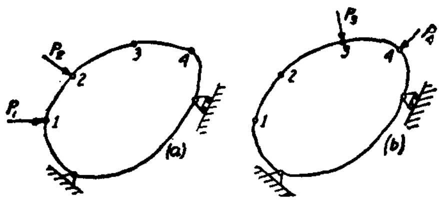

FIG. 278.

use the fact that the amount of the strain energy does not depend upon the order in which the forces are applied but only upon the final values of the forces. In the first manner of loading assume that forces P_1 and P_2 are applied first and later forces P_3 and P_4. The strain energy stored during the application of P_1 and P_2 is

$$\frac{P_1\delta_1}{2} + \frac{P_2\delta_2}{2}. \tag{a}$$

Applying now P_3 and P_4, the work done by these forces is

$$\frac{P_3\delta_3'}{2} + \frac{P_4\delta_4'}{2}. \tag{b}$$

It must be noted, however, that during the application of P_3 and P_4 the points of application of the previously applied forces P_1 and P_2 will be displaced by δ_1' and δ_2'. Then P_1 and P_2 do the work

$$P_1\delta_1' + P_2\delta_2'.^{28} \tag{c}$$

Hence the total strain energy stored in the body, by summing

[28] These expressions are not divided by 2 because forces P_1 and P_2 remain constant during the time in which their points of application undergo the displacements δ_1' and δ_2'.

(a), (b) and (c), is

$$U = \frac{P_1\delta_1}{2} + \frac{P_2\delta_2}{2} + \frac{P_3\delta_3'}{2} + \frac{P_4\delta_4'}{2} + P_1\delta_1' + P_2\delta_2'. \quad (d)$$

In the second manner of loading, let us apply the forces P_3 and P_4 first and afterwards P_1 and P_2. Then, repeating the same reasoning as above, we obtain

$$U = \frac{P_3\delta_3'}{2} + \frac{P_4\delta_4'}{2} + \frac{P_1\delta_1}{2} + \frac{P_2\delta_2}{2} + P_3\delta_3 + P_4\delta_4. \quad (e)$$

Putting (d) and (e) equal, eq. (203) is obtained. This theorem can be proven for any number of forces, and also for couples, or for forces and couples. In the case of a couple the corresponding angle of rotation is considered as the displacement.

For the particular case in which a single force P_1 acts in the first state of stress, and a single force P_2 in the second state, eq. (203) becomes [29]

$$P_1\delta_1' = P_2\delta_2. \quad (204)$$

If $P_1 = P_2$, it follows that $\delta_1' = \delta_2$, i.e., the displacement of the point of application of the force P_2 in the direction of this force, produced by the force P_1, is equal to the displacement of the point of application of the force P_1 in the direction of P_1, produced by the force P_2. A verification of this conclusion for a particular case was given in considering the beam shown in Fig. 277.

As another example let us again consider the bending of a simply supported beam. In the first state let it be bent by a load P at the middle, and in the second state by a bending couple M at the end. The load P produces the slope $\theta = Pl^2/16EI$ at the end. The couple M, applied at the end, produces the deflection $Ml^2/16EI$ at the middle. Equation (204) becomes

$$P\frac{Ml^2}{16EI} = M\frac{Pl^2}{16EI}.$$

[29] This was proved first by J. C. Maxwell, and is frequently called Maxwell's theorem.

The reciprocal theorem is very useful in the problem of finding the most unfavorable position of moving loads on a statically indeterminate system. An example is shown in Fig. 279, which represents a beam built in at one end and simply supported at the other and carrying a concentrated load P. The problem is to find the variation in the magnitude of the reaction X at the left support as the distance x of the load from this support changes.

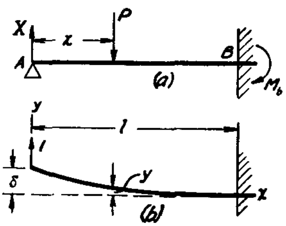

FIG. 279.

Let us consider the *actual condition* of the beam (Fig. 279, *a*) as the first state of stress. The second, or *fictitious*, state is shown in Fig. 279 (*b*). The external load and the redundant support are there removed and a unit force upward replaces the unknown reaction X. This second state of stress is statically determinate and the corresponding deflection curve is known (see eq. 97, p. 148). If the coordinate axes are taken as shown in Fig. 279 (*b*),

$$y = \frac{1}{6EI}(l - x)^2(2l + x). \tag{f}$$

Let δ denote the deflection at the end and y the deflection at distance x from the left support. Then, applying the reciprocal theorem, the work done by the forces of the first state on the corresponding displacements of the second state is

$$X\delta - Py.$$

In calculating now the work done by the forces of the second state, there is only the unit force on the end,[30] and the corresponding displacement of the point A in the first state is equal to zero. Consequently this work is zero and the reciprocal theorem gives

$$X\delta - Py = 0,$$

from which

$$X = P\frac{y}{\delta}. \tag{g}$$

[30] The reactions at the built-in end are not considered in either case because the corresponding displacement is zero.

It is seen that, as the load P changes position, the reaction X is proportional to the corresponding values of y in Fig. 279 (b). Hence the deflection curve of the second state (eq. f) gives a complete picture of the manner in which X varies with x. Such a curve is called the *influence line* for the reaction X.[31]

If several loads act simultaneously, the use of eq. (g) together with the method of superposition gives

$$X = \frac{1}{\delta} \sum P_n y_n,$$

where y_n is the deflection corresponding to the load P_n and the summation is extended over all the loads.

Problems

1. Construct the influence lines for the reactions at the supports of the beam on three supports (Fig. 280).

Solution. To get the influence line for the middle support the actual state shown in Fig. 280 (a) is taken as the first state of stress. The second state is indicated in Fig. 280 (b), in which the load P is removed and the reaction X is replaced by a unit force upward. This second state of stress is statically determinate and the deflection curve is known (eqs. 86 and 87, p. 142); hence the deflections δ and y can be calculated. Then the work done by the forces of the first state on the corresponding displacements of the second state is

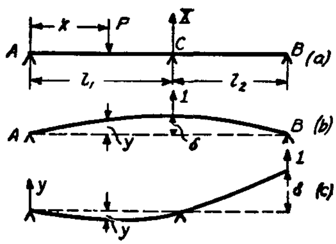

Fig. 280.

$$X\delta - Py.$$

The work of the forces of the second state (force-unity) on the corresponding displacements of the first state (zero deflection at C) is zero; hence

$$X\delta - Py = 0; \qquad X = P\frac{y}{\delta}.$$

Hence the deflection curve of the second state is the influence line for the reaction X. In order to get the influence line for the reac-

[31] The use of models in determining the influence lines was developed by G. E. Beggs, Journal of Franklin Institute 1927.

tion at B, the second state of stress should be taken as shown in Fig. 280 (c).

2. By using the influence line of the previous problem, determine the reaction at B if the load P is at the middle of the first span ($x = l_1/2$) (Fig. 280, a).

Answer. Reaction is downward and equal to

$$\frac{3P}{16}\frac{l_1^2}{l_2^2 + l_2 l_1}.$$

3. Find the influence line for the bending moment at the middle support C of the beam on three supports (Fig. 281). By using this line calculate the bending moment M_c when the load P is at the middle of the second span.

Solution. The first state of stress is the actual state (Fig. 281, a) with a bending moment M_c acting at the cross section C. For the second state of stress the load P is removed, the beam is cut at C and two equal and opposite unit couples replace M_c (Fig. 281, b). This case is statically determinate. The angles θ_1 and θ_2 are given by eq. (104) and the deflection y by eq. (105). The sum of the angles $\theta_1 + \theta_2$ represents the displacement in the second state corresponding to the bending moment M_c acting in the first state. The work done by the forces of the first state on the corresponding displacement in the second state is [32]

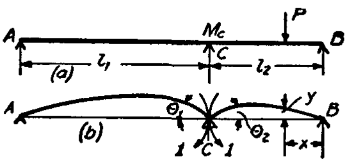

FIG. 281.

$$M_c(\theta_1 + \theta_2) - Py.$$

The work done by the forces of the second state on the displacements of the first state is zero because there is no cut at the support C in the actual case and the displacement corresponding to the two unit couples of the second state is zero. Hence

$$M_c(\theta_1 + \theta_2) - Py = 0$$

and

$$M_c = P\frac{y}{\theta_1 + \theta_2}. \tag{h}$$

It will be seen that as the load P changes its position, the bending moment M_c changes in the same ratio as the deflection y. Hence

[32] It is assumed that the bending moment M_c produces a deflection curve concave downward.

the deflection curves of the second state represent the influence line for M_c. Noting that

$$\theta_1 + \theta_2 = \frac{l_1 + l_2}{3EI}$$

and that the deflection at the middle of the second span is

$$(y)_{x=l_2/2} = \frac{1 \cdot l_2^2}{16EI},$$

the bending moment when the load P is at the middle of the second span is, from eq. (h),

$$M_c = \frac{3}{16} \cdot \frac{Pl_2^2}{l_1 + l_2}.$$

The positive sign obtained for M_c indicates that the moment has the direction indicated in Fig. 281 (b). Following our general rule for the sign of moments (Fig. 58) we then consider M_c as a negative bending moment.

4. Find the influence line for the bending moment at the built-in end B of the beam AB shown in Fig. 279, and calculate this moment when the load is at the distance $x = l/3$ from the left support.

Answer.

$$M_b = (4/27)lP.$$

5. Construct the influence line for the horizontal reactions H of the frame shown in Fig. 167 (a) as the load P moves along the bar AB.

Answer. The influence line has the same shape as the deflection curve of the bar AB for the loading condition shown in Fig. 166 (c).

6. Construct the influence line for the force X in the horizontal bar CD (Fig. 282, a) as the load P moves along the beam AB. Calculate X when the load is at the middle. The displacements due to elongation and contraction of the bars are to be neglected and only the displacement due to the bending of the beam AB is to be taken into account.

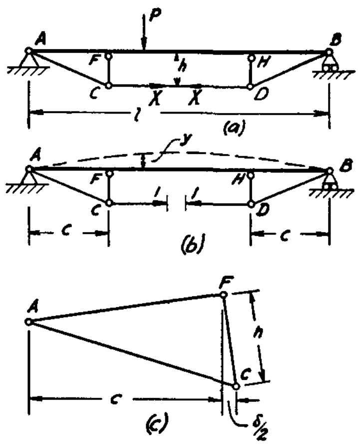

FIG. 282.

Solution. The actual condition (Fig. 282, *a*) is taken as the first state of stress. In the second state the load P is removed and the forces X are replaced by unit forces (Fig. 282, *b*). Due to these forces, upward vertical pressures equal to $(1 \cdot h)/c$ are transmitted to the beam AB at the points F and H and the beam deflects as indicated by the dotted line. If y is the deflection of the beam at the point corresponding to the load P, and δ is the displacement of the points C and D towards one another in the second state of stress, the reciprocal theorem gives

$$X\delta - Py = 0 \qquad \text{and} \qquad X = P\frac{y}{\delta}. \tag{i}$$

Hence the deflection curve of the beam AB in the second state is the required influence line. The bending of the beam by the two symmetrically situated loads is discussed in problem 1, p. 159. Substituting $(1 \cdot h)/c$ for P in the formulas obtained there, the deflection of the beam at F and that at the middle are

$$(y)_{x=c} = \frac{ch}{6EI}(3l - 4c) \qquad \text{and} \qquad (y)_{x=l/2} = \frac{h}{24EI}(3l^2 - 4c^2),$$

respectively.

Considering the rotation of the triangle AFC (Fig. 282, *c*) as a rigid body, the horizontal displacement of the point C is equal to the vertical displacement of the point F multiplied by h/c; hence

$$\delta = 2\frac{h}{c}(y)_{x=c} = \frac{h^2}{3EI}(3l - 4c).$$

Substituting this and the deflection at the middle into eq. (i) gives

$$X = \frac{P}{8h}\frac{3l^2 - 4c^2}{3l - 4c}.$$

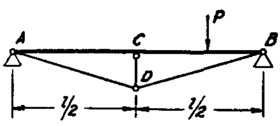

FIG. 283.

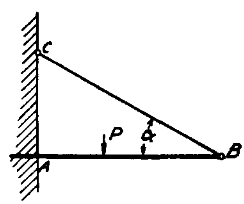

FIG. 284.

7. Find the influence line for the force in the bar CD of the system shown in Fig. 283, neglecting displacements due to con-

tractions and elongations and considering only the bending of the beam AB.

Answer. The line will be the same as that for the middle reaction of the beam on three supports (see problem 1, p. 335).

8. Construct the influence line for the bar BC which supports the beam AB. Find the force in BC when P is at the middle (Fig. 284).

Answer. Neglecting displacements due to elongation of the bar BC and contraction of the beam AB, the force in BC is $\frac{5}{16}$ $(P/\sin\alpha)$.

72. Exceptional Cases.—In the derivation of both the Castigliano theorem and the reciprocal theorem it was assumed that the displacements due to strain are proportional to the loads acting on the elastic system. There are cases in which the displacements are not proportional to the loads, although the material of the body may follow Hooke's law. This always occurs when the displacements due to deformations must be considered in discussing the action of external loads. In such cases, the strain energy is no longer a second degree function and the theorem of Castigliano does not hold. In order to explain this limitation let us consider a simple case in which only one force P acts on the elastic system. Assume first that the displacement δ is proportional to the corresponding force P as represented by the straight line OA in Fig. 285

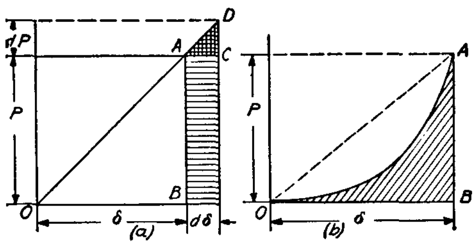

FIG. 285.

(*a*). Then the area of the triangle OAB represents the strain energy stored in the system during the application of the load P. For an infinitesimal increase $d\delta$ in the displacement the strain energy increases by an amount shown in the figure by the shaded area and we obtain

$$dU = Pd\delta. \tag{a}$$

With a linear relationship the infinitesimal triangle ADC is similar to the triangle OAB; therefore

$$\frac{d\delta}{dP} = \frac{\delta}{P} \quad \text{or} \quad d\delta = \frac{dP\delta}{P}. \tag{b}$$

Substituting this into eq. (*a*),

$$dU = P\frac{dP\delta}{P},$$

from which the Castigliano statement is obtained:

$$\frac{dU}{dP} = \delta. \tag{c}$$

An example to which the Castigliano theorem cannot be applied is shown in Fig. 286. Two equal horizontal bars AC and BC hinged at A, B and C are subjected to the action of the vertical force P at C. Let C_1 be the position of C after deformation and α the angle of inclination of either bar in its deformed condition. The unit elongation of the bars, from Fig. 286 (*a*), is

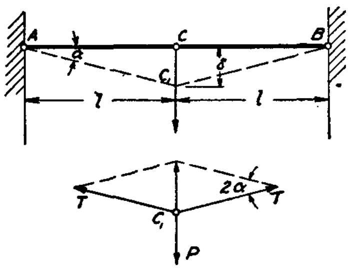

FIG. 286.

$$\epsilon = \left(\frac{l}{\cos\alpha} - l\right) : l. \tag{d}$$

If only small displacements are considered, α is small and $1/\cos\alpha = 1 + (\alpha^2/2)$ approximately. Then, from (*d*),

$$\epsilon = \frac{\alpha^2}{2}.$$

The corresponding forces in the bars are

$$T = AE\epsilon = \frac{AE\alpha^2}{2}. \tag{e}$$

From the condition of equilibrium of the point C_1 (Fig. 286, *b*),

$$P = 2\alpha T, \tag{f}$$

and for T, as given in eq. (*e*),

$$P = AE\alpha^3,$$

from which

$$\alpha = \sqrt[3]{\frac{P}{AE}} \tag{g}$$

and

$$\delta = l\alpha = l\sqrt[3]{\frac{P}{AE}}. \tag{205}$$

In this case the displacement is not proportional to the load P, although the material of the bars follows Hooke's law. The relation between δ and P is represented in Fig. 285 (*b*) by the curve OA. The shaded area OAB in this figure represents the strain energy stored in the system. The amount of strain energy is

$$U = \int_0^\delta P d\delta. \tag{h}$$

Substituting, from (205),

$$P = AE\frac{\delta^3}{l^3}, \tag{i}$$

we obtain

$$U = \frac{AE}{l^3}\int_0^\delta \delta^3 d\delta = \frac{AE\delta^4}{4l^3} = \frac{P\delta}{4} = \frac{Pl}{4}\sqrt[3]{\frac{P}{AE}}. \tag{l}$$

This shows that the strain energy is no longer a function of the second degree in the force P. Also it is not one half but only one quarter of the product $P\delta$ (see art. 68). The Castigliano theorem of course does not hold here:

$$\frac{dU}{dP} = \frac{d}{dP}\left(\frac{Pl}{4}\sqrt[3]{\frac{P}{AE}}\right) = \frac{1}{3}l\sqrt[3]{\frac{P}{AE}} = \frac{1}{3}\delta.$$

Analogous results are obtained in all cases in which the displacements are not proportional to the loads.

APPENDIX

MOMENTS OF INERTIA OF PLANE FIGURES

I. The Moment of Inertia of a Plane Area with Respect to an Axis in Its Plane

In discussing the bending of beams, we encounter integrals of this type:

$$I_z = \int_A y^2 dA, \tag{1}$$

in which each element of area dA is multiplied by the square of its distance from the z-axis and integration is extended over the cross sectional area A of the beam (Fig. 1). Such an integral is called the *moment of inertia* of the area A with

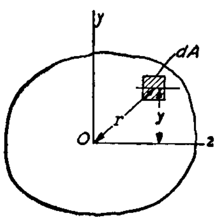

FIG. 1.

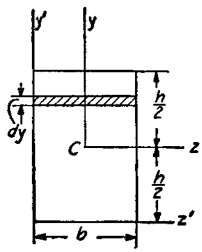

FIG. 2.

respect to the z-axis. In simple cases, moments of inertia can readily be calculated analytically. Take, for instance, a rectangle (Fig. 2). In calculating the moment of inertia of this rectangle with respect to the horizontal axis of symmetry z we can divide the rectangle into infinitesimal elements such as shown in the figure by the shaded area. Then

$$I_z = 2\int_0^{h/2} y^2 b dy = bh^3/12. \tag{2}$$

In the same manner, the moment of inertia of the rectangle with respect to the y-axis is

$$I_y = 2\int_0^{b/2} z^2 h dz = hb^3/12.$$

Equation (2) can also be used for calculating I_z, for the parallelogram shown in Fig. 3, because this parallelogram can be obtained from the rectangle shown by dotted lines by a displacement parallel to the axis z of elements such as the one shown. The areas of the elements and their distances from the z-axis remain unchanged during such displacement so that I_z is the same as for the rectangle.

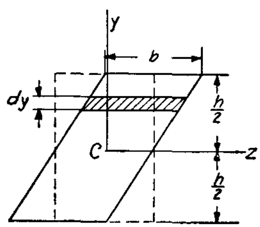

FIG. 3.

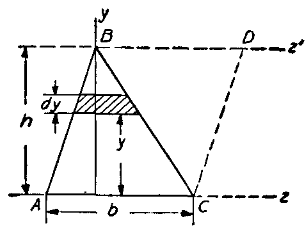

FIG. 4.

In calculating the moment of inertia of a triangle with respect to the base (Fig. 4), the area of an element such as shown in the figure is

$$dA = b\frac{h-y}{h}dy$$

and eq. (1) gives

$$I_z = \int_0^h b\frac{h-y}{h}y^2 dy = bh^3/12.$$

The method of calculation illustrated by the above examples can be used in the most general case. The moment of inertia is obtained by dividing the figure into infinitesimal strips parallel to the axis and then integrating as in eq. (1).

The calculation can often be simplified if the figure can be divided into portions whose moments of inertia about the axis are known. In such case, the total moment of inertia is the sum of the moments of inertia of all the parts.

From its definition, eq. (1), it follows that the moment of inertia of an area with respect to an axis has the dimensions of a length raised to the fourth power; hence, by dividing the moment of inertia with respect to a certain axis by the cross sectional area of the figure, the square of a certain length is obtained. This length is called the *radius of gyration* with respect to that axis. For the y and z axes, the radii of gyration are

$$k_y = \sqrt{I_y/A}; \qquad k_z = \sqrt{I_z/A}. \tag{3}$$

Problems

1. Find the moment of inertia of the rectangle in Fig. 2 with respect to the base. (Ans. $I_z = bh^3/3$.)

2. Find the moment of inertia of the triangle ABC with respect to the axis z' (Fig. 4).

Solution. This moment is the difference between the moment of inertia of the parallelogram $ABDC$ and the triangle BDC. Hence,

$$I_{z'} = bh^3/3 - bh^3/12 = bh^3/4.$$

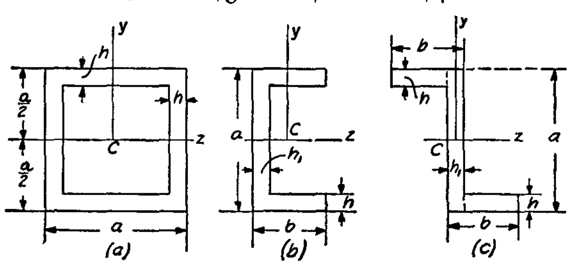

FIG. 5.

3. Find I_z for the cross sections shown in Fig. 5. (Ans. $I_z = a^4/12 - (a - 2h)^4/12$; $I_z = ba^3/12 - \dfrac{(b - h_1)(a - 2h)^3}{12}$.)

4. Find the moment of inertia of a square with sides a with respect to a diagonal. (Ans. $I = a^4/12$.)

5. Find k_y and k_z for the rectangle shown in Fig. 2. (Ans. $k_y = b/2\sqrt{3}$; $k_z = h/2\sqrt{3}$.)

6. Find k_z for Figs. 5a and 5b.

II. Polar Moment of Inertia of a Plane Area

The moment of inertia of a plane area with respect to an axis perpendicular to the plane of the figure is called the *polar* *moment of inertia* with respect to the point, where the axis intersects the plane (point O in Fig. 1). It is defined as the integral

$$I_p = \int_A r^2 dA, \tag{4}$$

in which each element of area dA is multiplied by the square of its distance to the axis and integration is extended over the entire area of the figure.

Referring back to Fig. 1, $r^2 = y^2 + z^2$, and from eq. (4)

$$I_p = \int_A (y^2 + z^2) dA = I_z + I_y. \tag{5}$$

That is, the polar moment of inertia with respect to any point O is equal to the sum of the moments of inertia with respect to two perpendicular axes y and z through the same point.

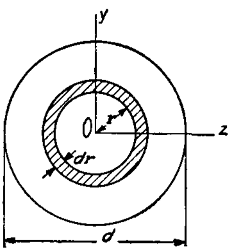

FIG. 6.

Let us consider a *circular cross section.* We encounter the polar moment of inertia of a circle with respect to its center in discussing the twist of a circular shaft (see article 58). If we divide the area of the circle into thin elemental rings, as shown in Fig. 6, we have $dA = 2\pi r dr$, and from eq. (4),

$$I_p = 2\pi \int_0^{d/2} r^3 dr = \pi d^4/32. \tag{6}$$

We know from symmetry that in this case $I_y = I_z$; hence, from eqs. (5) and (6),

$$I_y = I_z = \tfrac{1}{2} I_p = \pi d^4/64. \tag{7}$$

The moment of inertia of an ellipse with respect to a principal axis z (Fig. 7) can be obtained by comparing the ellipse with the circle shown in the figure by dotted line.

The height y of any element of the ellipse such as the one shown shaded can be obtained by reducing the height y_1 of the corresponding element of the circle in the ratio b/a. From eq. (2), the moments of inertia of these two elements with respect to the z-axis are in the ratio b^3/a^3. The moments of inertia of the ellipse and of the circle are evidently in the same ratio; hence, the moment of inertia of the ellipse is

$$I_z = \pi(2a)^4/64 \cdot b^3/a^3 = \pi ab^3/4. \tag{8}$$

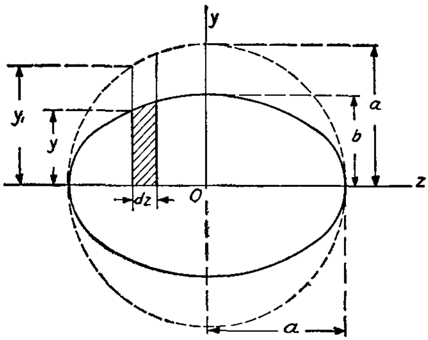

FIG. 7.

In the same manner, for the vertical axis

$$I_y = \pi ba^3/4;$$

the polar moment of inertia of an ellipse is then, from eq. (5),

$$I_p = I_y + I_z = \pi ab^3/4 + \pi ba^3/4. \tag{9}$$

Problems

1. Find the polar moment of a rectangle with respect to the centroid (Fig. 2). (Ans. $I_p = bh^3/12 + hb^3/12$.)

2 Find the polar moments of inertia with respect to their centroids of the areas shown in Fig. 5.

III. Transfer of Axis

If the moment of inertia of an area with respect to an axis z through the centroid (Fig. 8) is known, the moment of inertia with respect to any parallel axis z' can be calculated from the equation:

$$I_{z'} = I_z + Ad^2, \tag{10}$$

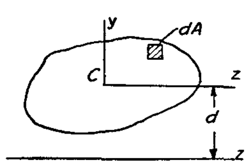

FIG. 8.

in which A is the area of the figure and d is the distance between the axes. This can be proved as follows: from eq. (1)

$$I_{z'} = \int_A (y + d)^2 dA = \int_A y^2 dA + 2\int_A yd dA + \int_A d^2 dA.$$

The first integral on the right side is equal to I_z, the third integral is equal to Ad^2 and the second integral vanishes due to the fact that z passes through the centroid; hence, this equation reduces to (10). Equation (10) is especially useful in calculating moments of inertia of cross sections of built-up beams (Fig. 9). The positions of the centroids of standard angles and the moments of inertia of their cross sections with respect to an axis through their centroid are given in hand books. By transfer of axis, the moment of inertia of such a built-up section with respect to the z-axis can readily be calculated.

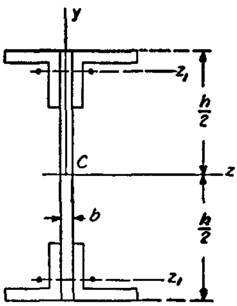

FIG. 9.

Problems

1. By transfer of axis, find the moment of inertia of a triangle (Fig. 4) with respect to the axis through the centroid and parallel to the base. (Ans. $I = bh^3/36$.)

2. Find the moment of inertia I_z of the section shown in Fig. 9 if $h = 20''$, $b = \frac{1}{2}''$, and the angles have the dimensions $4'' \times 4'' \times \frac{1}{2}''$.

Solution. $I_z = 20^3/(2 \times 12) + 4[5.56 + 3.75(10 - 1.18)^2] = 1{,}522$ in.^{4}.

3. Find the moment of inertia with respect to the neutral axis of the cross section of the channel shown in Fig. 85.

IV. Product of Inertia, Principal Axes

The integral

$$I_{yz} = \int_A yz dA, \tag{11}$$

in which each element of area dA is multiplied by the product of its coordinates and integration is extended over the entire area A of a plane figure, is called the *product of inertia* of the figure. If a figure has an axis of symmetry which is taken for the y or z axis (Fig. 10), the product of inertia is equal to zero. This follows from the fact that in this case for any element such as dA with a positive z there exists an equal and symmetrically situated element dA' with a negative z. The corresponding elementary products $yzdA$ cancel each other; hence integral (11) vanishes.

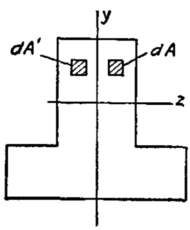

FIG. 10.

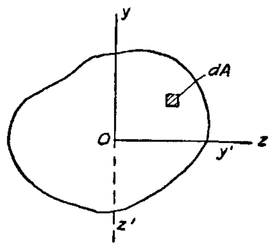

FIG. 11.

In the general case, for any point of any plane figure, we can always find two perpendicular axes such that the product of inertia for these axes vanishes. Take, for instance, the axes y and z, Fig. 11. If the axes are rotated about O 90° in the clock-wise direction, the new positions of the axes are y' and z' as shown in the figure. There is then the following relation between the old coordinates of an element dA and its new coordinates:

$$y' = z; \qquad z' = -y.$$

Hence, the product of inertia for the new coordinates is

$$I_{y'z'} = \int_A y'z'dA = -\int_A yzdA = -I_{yz};$$

thus, during this rotation, the product of inertia changes its sign. As the product of inertia changes continuously with the angle of rotation, there must be certain directions for which this quantity becomes zero. The axes in these directions are called the *principal axes*. Usually the centroid is taken as the origin of coordinates and the corresponding principal axes are then called the *centroidal principal axes.* If a figure has an axis of symmetry, this axis and an axis perpendicular to it are principal axes of the figure, because the product of inertia with respect to these axes is equal to zero, as explained above.

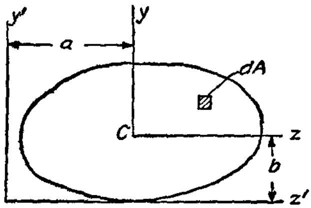

FIG. 12.

If the product of inertia of a figure is known for axes y and z (Fig. 12) through the centroid, the product of inertia for parallel axes y' and z' can be found from the equation:

$$I_{y'z'} = I_{yz} + Aab. \tag{12}$$

The coordinates of an element dA for the new axes are

$$y' = y + b; \qquad z' = z + a.$$

Hence,

$$I_{y'z'} = \int_A y'z'dA = \int_A (y + b)(z + a)dA$$

$$= \int_A yzdA + \int_A abdA + \int_A yadA + \int_A bzdA.$$

The last two integrals vanish because C is the centroid so that the equation reduces to (12).

Problems

1. Find $I_{y'z'}$ for the rectangle in Fig. 2. (Ans. $I_{y'z'} = (b^2h^2/4)$.)

2. Find the product of inertia of the angle section with respect to the y and z axes. Do the same for the y_1 and z_1 axes (Fig. 13).

Solution. Dividing the figure into two rectangles and using eq. (12) for each of these rectangles, we find

$$I_{yz} = a^2h^2/4 + h^2(a^2 - h^2)/4.$$

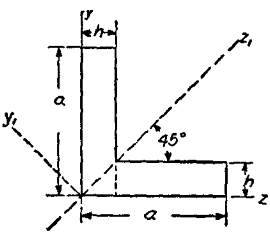

FIG. 13.

From the symmetry condition $I_{y_1z_1} = 0$.

3. Determine the products of inertia I_{yz} of the sections shown in Fig. 5 if C is the centroid.

Solution. For Figs. 5*a* and 5*b*, $I_{yz} = 0$ because of symmetry. In the case of Fig. 5*c*, dividing the section into three rectangles, and using eq. (12), we find

$$I_{yz} = -2(b - h_1)h\frac{a-h}{2}\frac{b}{2}.$$

V. Change of Direction of Axis. Determination of the Principal Axes

Suppose that the moments of inertia

$$I_z = \int_A y^2 dA; \qquad I_y = \int_A z^2 dA \tag{a}$$

and the product of inertia

$$I_{yz} = \int_A yz dA \tag{b}$$

are known, and it is required to find the same quantities for

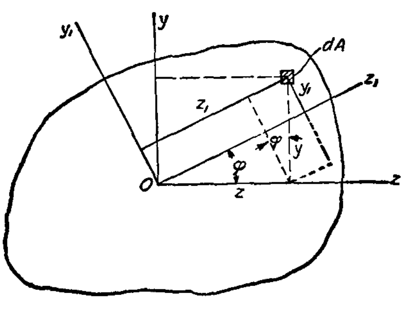

FIG. 14.

the new axes y_1 and z_1 (Fig. 14). Considering an elementary area dA, the new coordinates from the figure are

$$z_1 = z\cos\varphi + y\sin\varphi; \qquad y_1 = y\cos\varphi - z\sin\varphi, \tag{c}$$

in which φ is the angle between z and z_1. Then,

$$I_{z_1} = \int_A y_1^2 dA = \int_A (y\cos\varphi - z\sin\varphi)^2 dA = \int_A y^2\cos^2\varphi dA$$
$$+ \int_A z^2\sin^2\varphi dA - \int_A 2yz\sin\varphi\cos\varphi dA,$$

or by using (*a*) and (*b*)

$$I_{z_1} = I_z \cos^2 \varphi + I_y \sin^2 \varphi - I_{yz} \sin 2\varphi. \tag{13}$$

In the same manner

$$I_{y_1} = I_z \sin^2 \varphi + I_y \cos^2 \varphi + I_{yz} \sin 2\varphi. \tag{13'}$$

By taking the sum and the difference of eqs. (13) and (13′) we find

$$I_{z_1} + I_{y_1} = I_z + I_y, \tag{14}$$

$$I_{z_1} - I_{y_1} = (I_z - I_y) \cos 2\varphi - 2I_{yz} \sin 2\varphi. \tag{15}$$

These equations are very useful for calculating I_{z_1} and I_{y_1}. For calculating $I_{y_1z_1}$, we find

$$I_{y_1z_1} = \int_A y_1 z_1 dA = \int_A (y \cos \varphi - z \sin \varphi)(z \cos \varphi + y \sin \varphi) dA = \int_A y^2 \sin \varphi \cos \varphi dA - \int_A z^2 \sin \varphi \cos \varphi dA + \int_A yz(\cos^2 \varphi - \sin^2 \varphi) dA,$$

or by using (*a*) and (*b*)

$$I_{y_1z_1} = (I_z - I_y)\tfrac{1}{2} \sin 2\varphi + I_{yz} \cos 2\varphi. \tag{16}$$

The principal axes of inertia are those two perpendicular axes for which the product of inertia vanishes. The axes y_1 and z_1 in Fig. 14 are principal axes if the right side of eq. (16) vanishes.

$$(I_z - I_y)\tfrac{1}{2} \sin 2\varphi + I_{yz} \cos 2\varphi = 0;$$

this gives

$$\tan 2\varphi = 2I_{yz}/(I_y - I_z). \tag{17}$$

Let us determine, as an example, the directions of the principal axes of a rectangle through a corner of the rectangle (Fig. 2). In this case,

$$I_{z'} = bh^3/3; \qquad I_{y'} = hb^3/3; \qquad I_{y'z'} = b^2h^2/4;$$

hence,

$$\tan 2\varphi = \frac{b^2h^2}{2(hb^3/3 - bh^3/3)} = 3bh/2(b^2 - h^2). \qquad (d)$$

In the derivation of eq. (17), the angle φ was taken as positive in the counter-clockwise direction (Fig. 14), so φ must be taken in this direction if it comes out positive. Equation (d) gives two different values for φ differing by 90°. These are the two perpendicular directions of the principal axes. Knowing the directions of the principal axes, the corresponding moments of inertia can be found from eqs. (14) and (15).

The radii of gyration corresponding to the principal axes are called *principal radii of gyration.*

If y_1 and z_1 are the principal axes of inertia (Fig. 15) and k_{y_1} and k_{z_1} the principal radii of gyration, the ellipse with k_{y_1} and

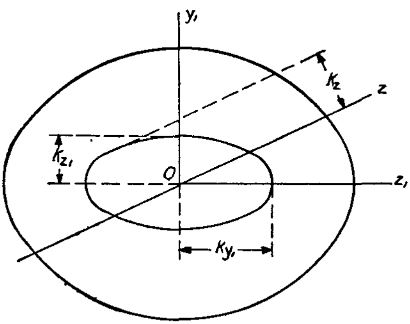

FIG. 15.

k_{z_1} as semi-axes, as shown in the figure, is called the *ellipse of inertia.* Having this ellipse, the radius of gyration k_z for any axis z can be obtained graphically by drawing a tangent to the ellipse parallel to z. The distance of the origin O from this tangent is the length of k_z. The ellipse of inertia gives a picture of how the moment of inertia changes as the axis z rotates in the plane of the figure about the point O, and shows that the maximum and minimum of the moment of inertia are the principal moments of inertia.

Problems

1. Determine the directions of the centroidal principal axes of the Z section (Fig. 5c) if $h = h_1 = 1''$; $b = 5''$; $a = 10''$.

2. Find the directions of the centroidal principal axes and the corresponding principal moments of inertia for an angle section $5'' \times 2\frac{1}{2}'' \times \frac{1}{2}''$. (Ans. $\tan 2\varphi = 0.547$; $I_{max} = 9.36$ in.^{4}; $I_{min} = 0.99$ in.^{4}.)

3. Determine the semi-axes of the ellipse of inertia for an elliptical cross section (Fig. 7). (Ans. $k_z = b/2$; $k_y = a/2$.)

4. Under what conditions does the ellipse of inertia become a circle? (Ans. When the moments of inertia with respect to the principal axes are equal.)

AUTHOR INDEX

SUBJECT INDEX

Numbers refer to pages

R

S

T

U

V

W

Y