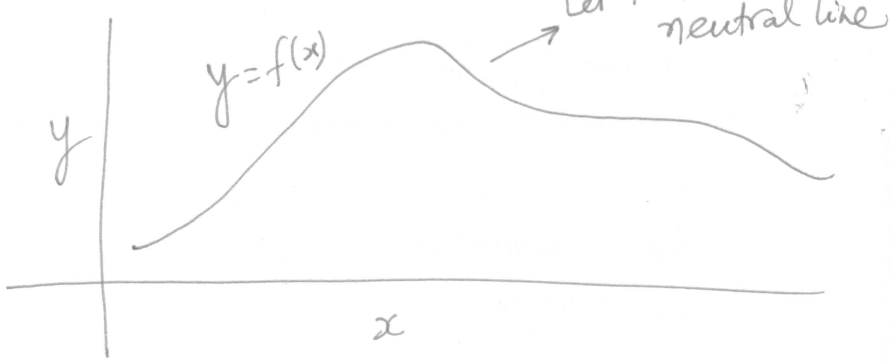
Euler bernouli beam theory

We know from bending of beams that

$$EI\frac{1}{R} = M(x)$$

, R is the radius of curvature of neutral line.

then, $\frac{1}{R} = \dfrac{d^2y/dx^2}{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}}$



If the deflection is such that $\frac{dy}{dx} \approx 0$

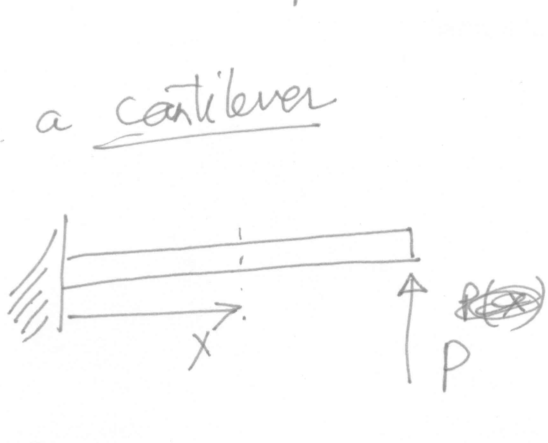
$$\Rightarrow \frac{1}{R} \approx \frac{d^2y}{dx^2},$$

Also, $\frac{dy}{dX}$ denotes rotation of cross-section

So, $$\boxed{EI\frac{d^2y}{dx^2} = M(x)}$$ → this is Euler Bernouli beam theory!

If M(x) is known, then two more boundary conditions needed to solve the above equation!

One Simple example of a cantilever



In this case, $M(x) = P(L-x)$

Hence, $$EI\frac{d^2y}{dx^2} = P(L-x)$$

Integrating twice

$$\Rightarrow EI\,y(x) = PL\frac{x^2}{2} - \frac{Px^3}{6} + cx + d$$

To obtain 'c' & 'd'

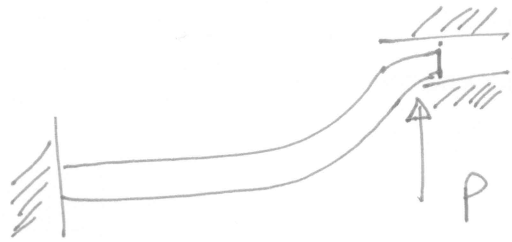
we apply, $y(0) = 0 \Rightarrow d = 0$

$\frac{dy}{dx}(0) = 0 \Rightarrow c = 0$

So, $$y(x) = \frac{PLx^2}{2EI}\left(1 - \frac{x}{3L}\right)$$

$$\boxed{y(L) = \frac{PL^3}{3EI}}$$

Another case



In this case, we are apply force 'P' at the other end but the beam is not allowed to rotate at this end

$\Rightarrow M(L) \neq 0$ is an additional unknown!!

$$M(x) = M(L) + P(L-x)$$

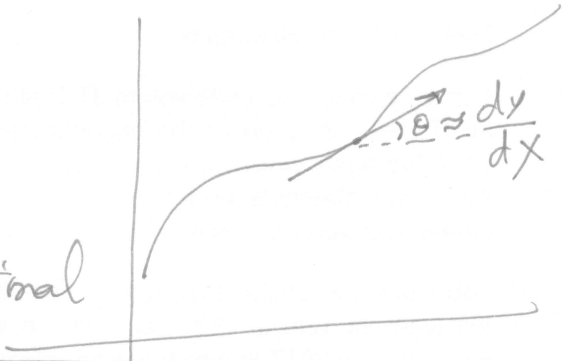
The three ~~boundary~~ conditions are:-

$$y(0) = 0 \quad , \quad \frac{dy}{dx}(0) = 0$$

$$\frac{dy}{dx}(L) = 0$$

* Euler bernouli theory holds only when $\frac{dy}{dx} \approx 0$

So, $\frac{dy}{dx} = \tan\theta \approx \theta$ (slope of neutral line)



* Furthermore, we assume that the slope of neutral axis coincides with the cross-sectional normal $\Rightarrow$ $\boxed{\text{rotation of cross-section} = \frac{dy}{dx}}$

→ this assumption is relaxed in case of Timoshenko beam theory.

→ We ~~assume that defle~~ also take into account the effect of shear force on deflection of a beam!

Hence,

$$\frac{dy}{dx} = \text{slope due to bending} + \underbrace{\text{slope due to shear}}_{\text{(shear angle)}}$$

We further assume that $\frac{dy}{dx}$ is very small!

$$\frac{dy}{dx} \simeq 0$$

$\Rightarrow \frac{dy}{dx} \simeq$ rotation due to bending + rotation due to shear force

$$\simeq \theta_b + \theta_s$$

* θ_b is such that $EI \frac{d\theta_b}{dx} = M(x)$

→ θ_b is also the rotation of a cross-section! We will simply call it "θ"

→ θ_s is shear strain $= \frac{V}{\kappa GA}$

So, the Timoshenko beam equation is

$$\boxed{\begin{aligned} EI \frac{d\theta}{dx} &= M(x) \\ \frac{dy}{dx} &= \theta + \frac{V(x)}{\kappa GA} \end{aligned}}$$

→ System of two equations in unknowns 'y' (transverse deflection) and 'θ' (cross-section rotation)

→ Unlike Euler Bernoulli theory, $\frac{dy}{dx} \neq \theta$ here!!

Cantilever problem using Timoshenko beam theory!

$$EI \frac{d\theta}{dx} = M(x)$$

$$\frac{dy}{dx} = \theta + \frac{V(x)}{kGA}$$

$$\boxed{\begin{aligned} M(x) &= P(L-x) \\ V(x) &= P \end{aligned}}$$

~~So,~~ Since, bending moment and shear force profile are known completely

$\Rightarrow$ 2 more boundary conditions needed!

They are $\boxed{\begin{aligned} y(0) &= 0 \\ \theta(0) &= 0 \end{aligned}}$ TBT.

Notice the change in boundary condition!

in Euler Bernoulli theory, we had

$\boxed{\begin{aligned} y(0) &= 0 \\ \frac{dy}{dx}(0) &= 0 \end{aligned}}$ EBT.