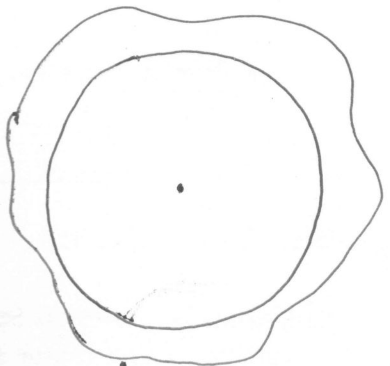
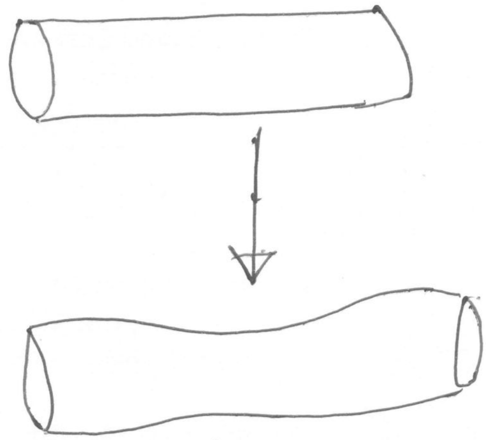
Minor - 2 Solution

1. (a) $u_r(r)$: If u_r were a function of 'θ', a circular line in cross-section will not remain circular !

2 If u_r were a function of 'z', the radius of tube would change along the tube's axis.



We don't want this !



We don't want this !

(b) $u_z \equiv 0$, u_z is not a function of 'r' or 'θ' otherwise

1 planar cross-section will become curved !.

If, u_z were a function of 'z', it will lead to $\underline{u_z' \neq 0}$

but we know $\underline{u_z' \text{ has to be constant}}$, ~~Furthermore, no elon~~ from $\underline{z}$ equation.

Furthermore, no axial elongation allowed $\Rightarrow u_z' = 0$

$\Rightarrow \underline{u_z = 0}$

(b) $$\underline{\underline{\epsilon}} = \begin{bmatrix} u_r' & 0 & 0 \\ 0 & \frac{u_r}{r} & \frac{r\theta_0}{2L} \\ 0 & \frac{r\theta_0}{2L} & 0 \end{bmatrix}$$ (2)

(2)

(3)

$$\sigma_{rr} = \lambda\left(u_r' + \frac{u_r}{r}\right) + 2\mu u_r', \quad \sigma_{\theta\theta} = \lambda\left(u_r' + \frac{u_r}{r}\right) + 2\mu \frac{u_r}{r}$$

$$\sigma_{zz} = \lambda\left(u_r' + \frac{u_r}{r}\right), \quad \sigma_{r\theta} = \sigma_{rz} = 0, \quad \sigma_{\theta z} = \frac{G r \theta_0}{L}$$

(c) 'θ' and 'z' equation get ~~satisfied~~ automatically.

radial eq. $\Rightarrow$ $$\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$$ (2)

$$\Rightarrow (\lambda + 2\mu)\left(u_r'' + \frac{u_r'}{r} - \frac{u_r}{r^2}\right) = 0$$ (3)

$$\cancel{\Rightarrow \left(u_r' + \frac{u_r}{r}\right)' = c}$$

or, $$\boxed{u_r'' + \frac{u_r'}{r} - \frac{u_r}{r^2} = 0}$$

(d) From (c) $u_r' + \frac{u_r}{r}$ = Constant $= C$ (2)

$$\Rightarrow \sigma_{rr} = \lambda C + 2\mu u_r', \quad \sigma_{\theta\theta} = \lambda C + 2\mu \frac{u_r}{r}$$

or $$\boxed{\sigma_{rr} + \sigma_{\theta\theta} = 2(\lambda + \mu) C}$$

(1)

Again, looking at radial eq:-

$$\frac{d\sigma_{rr}}{dr} + 2\frac{\sigma_{rr}}{r} = \frac{\sigma_{rr} + \sigma_{\theta\theta}}{r} = \frac{2(\lambda+\mu) C}{r}$$

$$\Rightarrow \frac{1}{r^2}\frac{d}{dr}\left(\sigma_{rr} r^2\right) = \frac{2(\lambda+\mu)C}{r}$$

$$\Rightarrow \boxed{\sigma_{rr} = (\lambda+\mu)C + \frac{D}{r^2}} \quad 1$$

Let us use boundary condition $\sigma_{rr}(r_1) = -p$, $\sigma_{rr}(r_2) = 0$

$$\Downarrow$$

$$D = -(\lambda+\mu)C r_2^2$$

$$\Downarrow$$

$$\sigma_{rr} = (\lambda+\mu)C\left[1 - \frac{r_2^2}{r^2}\right]$$

$$1 \quad -p = (\lambda+\mu)C\left[1 - \frac{r_2^2}{r_1^2}\right]$$

$$\Downarrow$$

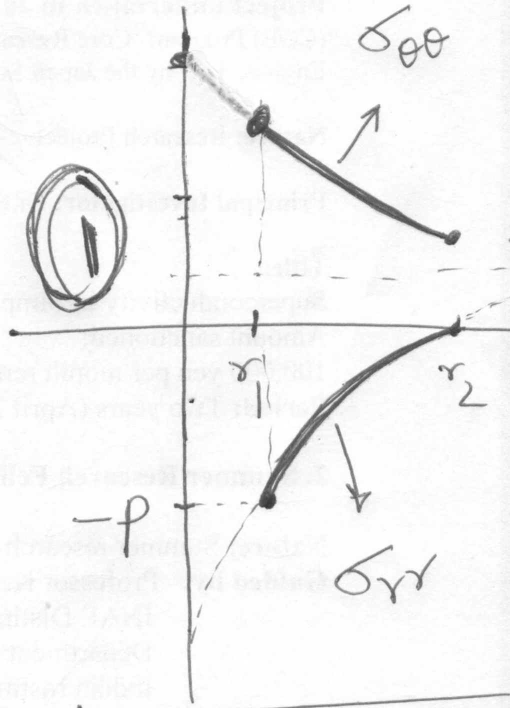
$$(\lambda+\mu)C = \frac{p r_1^2}{r_2^2 - r_1^2}$$

So,

$$\boxed{\sigma_{rr} = \frac{p r_1^2}{(r_2^2 - r_1^2)}\left(1 - \frac{r_2^2}{r^2}\right)}$$

$$\boxed{\sigma_{\theta\theta} = \frac{p r_1^2}{(r_2^2 - r_1^2)}\left(1 + \frac{r_2^2}{r^2}\right)}$$

1



(e) Axial force: $\sigma_{zz} \cdot$ Area

$$1 \; = \lambda\left(u_r' + \frac{u_r}{r}\right)\pi\left(r_2^2 - r^2\right)$$

$$= \lambda C \; \pi\left(r_2^2 - r_1^2\right)$$

$$2 \; = \frac{\lambda}{(\lambda+\mu)}\frac{p r_1^2}{(r_2^2 - r_1^2)}\pi\cancel{\left(r_2^2 - r_1^2\right)} = \frac{\lambda}{\lambda+\mu}\cdot \pi r_1^2 p$$

$$\boxed{\begin{aligned} &\equiv -\nu(\sigma_{rr} + \sigma_{\theta\theta})\pi(r_2^2 - r_1^2) \\ &\equiv -2\nu p \pi r_1^2 \end{aligned}}$$

(f) To get displacement.

$$u_r' + \frac{u_r}{r} = C$$

$$\Rightarrow \frac{1}{r}\frac{d}{dr}(r u_r) = C$$

$$\Rightarrow \boxed{u_r = \frac{C}{2} r + \frac{B}{r}}$$

1

To obtain B,

We see, $\sigma_{\theta\theta} = \lambda C + 2\mu \frac{u_r}{r} = \frac{p r_1^2}{(r_2^2 - r_1^2)}\left(1 + \frac{r_2^2}{r^2}\right)$

Comparing coefficient of $\frac{1}{r^2}$

$$\Rightarrow B = \frac{p r_1^2 r_2^2}{2\mu(r_2^2 - r_1^2)}$$

2

So, $u_r = \frac{p r_1^2}{(r_2^2 - r_1^2)}\left[\frac{1}{2(\lambda+\mu)} r + \frac{r_2^2}{2\mu r}\right]$

So, $u_r(r_1) = \frac{p r_1}{2(r_2^2 - r_1^2)}\left[\frac{r_1^2}{\lambda+\mu} + \frac{r_2^2}{\mu}\right]$

Now, $r_1 + u_r(r_1) = \alpha r_1$, ($\alpha$ = inflation)

1

$$\Rightarrow \alpha = 1 + \frac{p}{2(r_2^2 - r_1^2)}\left[\frac{r_1^2}{\lambda+\mu} + \frac{r_2^2}{\mu}\right]$$

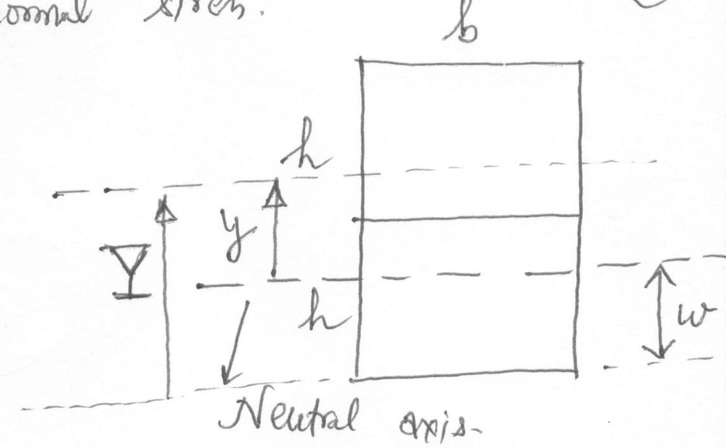
$$\Rightarrow \text{inflation strain } (\alpha - 1) = \frac{p}{2(r_2^2 - r_1^2)}\left[\frac{r_1^2}{\lambda+\mu} + \frac{r_2^2}{\mu}\right]$$

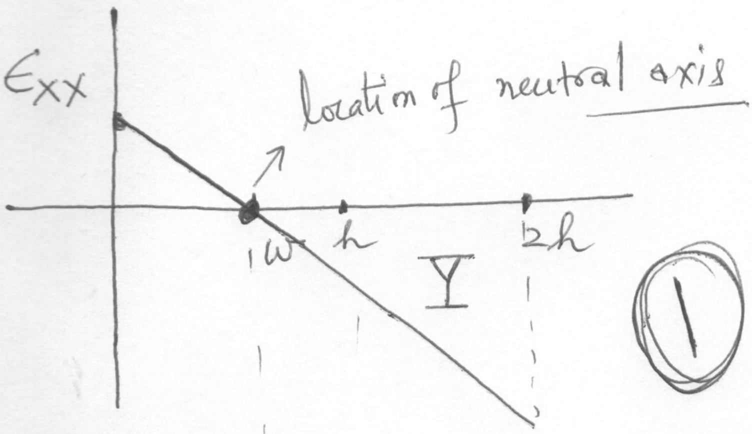
So, $\boxed{p = \frac{2(r_2^2 - r_1^2)(\alpha - 1)}{\left(r_1^2/_{\lambda+\mu} + r_2^2/_{\mu}\right)}}$

Problem 2

(a) Variation in normal strain and normal stress.

$$\epsilon_{xx} = -\frac{y}{R} = -\frac{(Y-w)}{R}$$





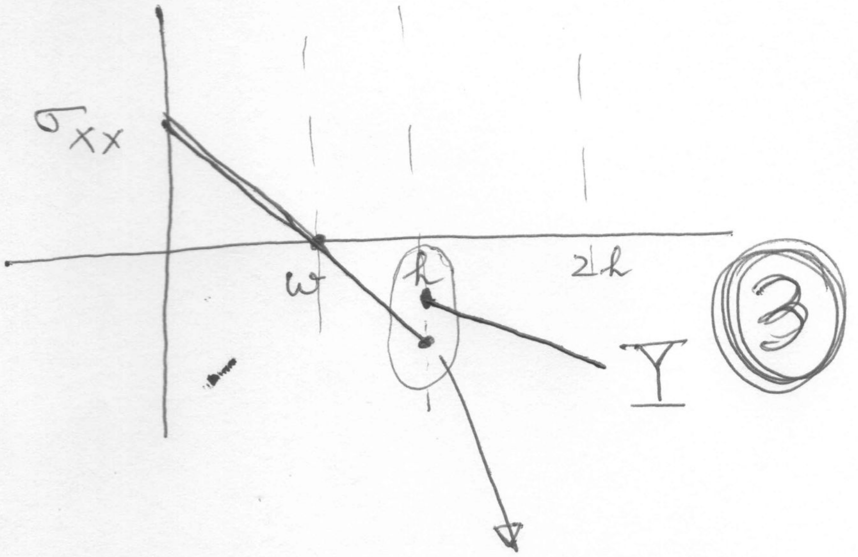
1

$Y = \underline{\underline{w + y}}$

$\downarrow$

distance from bottom of cross-section

$\underline{\underline{E_A < E_s}}$



3

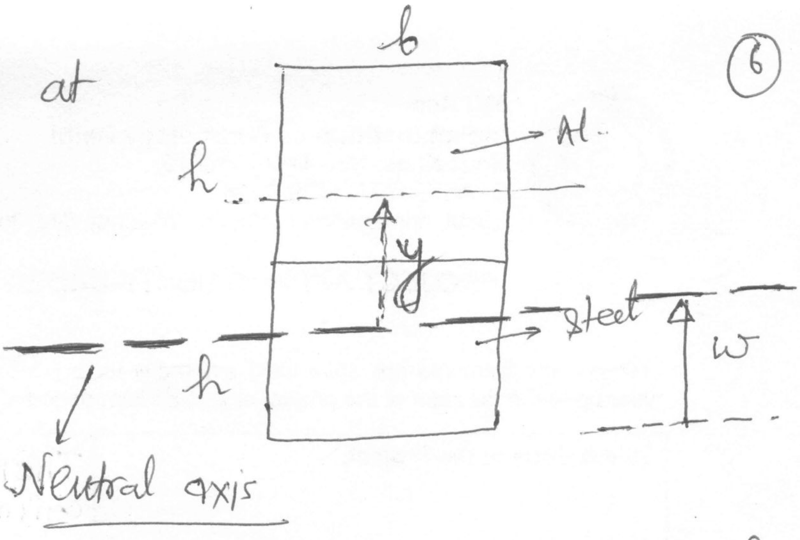
$$\sigma_{xx} = -\frac{E_s y}{R} \quad (-w < y < h-w)$$

$$= -\frac{E_A y}{R} \quad (h-w < y < 2h-w)$$

* Notice the jump in σ_{xx}

* Also the slope of the curve varies in Aluminium and steel!

(b) Let the neutral axis lies at a distance 'w' from the bottom (see figure)



then, $\epsilon_{xx} = -y/R$ (y is the distance from neutral axis)

$$\sigma_{xx} = E\,\epsilon_{xx} = -\frac{E_s y}{R} \quad (-w < y < h-w)$$

$$= -E_A \frac{y}{R} \quad (y > h-w)$$

2

To obtain 'w', we use the fact that the total axial force must vanish

$$\Rightarrow -\frac{E_s}{R}\int_{-w}^{h-w} y\,dy - \frac{E_A}{R}\int_{h-w}^{2h-w} y\,dy = 0$$

2

$$\Rightarrow \left[(h-w)^2 - w^2\right] + \frac{E_A}{E_s}\left[(2h-w)^2 - (h-w)^2\right] = 0$$

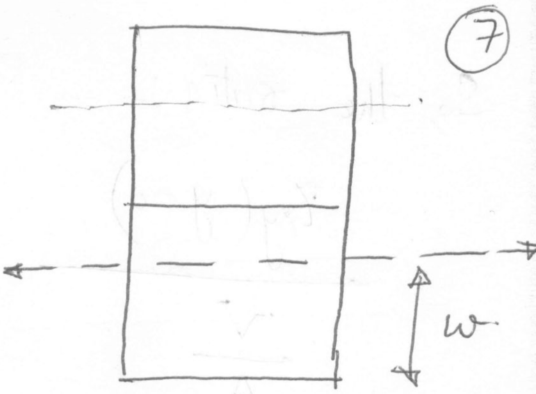
$$\Rightarrow (h^2 - 2hw) + \frac{E_A}{E_s}(3h^2 - 2hw) = 0$$

$$\Rightarrow h^2\left(1 + 3\frac{E_A}{E_s}\right) - 2hw\left(1 + \frac{E_A}{E_s}\right) = 0$$

1

$$\Rightarrow \boxed{w = \frac{h}{2}\left(\frac{1 + 3\,E_A/E_s}{1 + E_A/E_s}\right)}$$

(c) I_{zz} about neutral axis..



$$I_{zz} = I_{zz}(Al) + I_{zz}(steel)$$

11 $$= \frac{bh^3}{12} + bh\left(\frac{3h}{2} - w\right)^2 + \frac{bh^3}{12} + \left(\frac{h}{2} - w\right)^2$$ 1

$$= \frac{bh^3}{6} + bh\left(\frac{9h^2}{4} + \frac{h^2}{4} + 2w^2 - 4hw\right)$$

$$= \frac{8}{3}bh^3 + 2bhw(w - 2h)$$ 1

(d) Let us first obtain the bending moment due to above distribution!!

$$M_z = -\iint \sigma_{xx}\, y\, dA$$ 1

(y from neutral axis).

$$= \cancel{-b\int \sigma_{xx}\, y\, dy}$$

$$= \frac{E_s}{R}\iint y^2\, dA + \frac{E_{Al}}{R}\iint y^2\, dA$$ 1

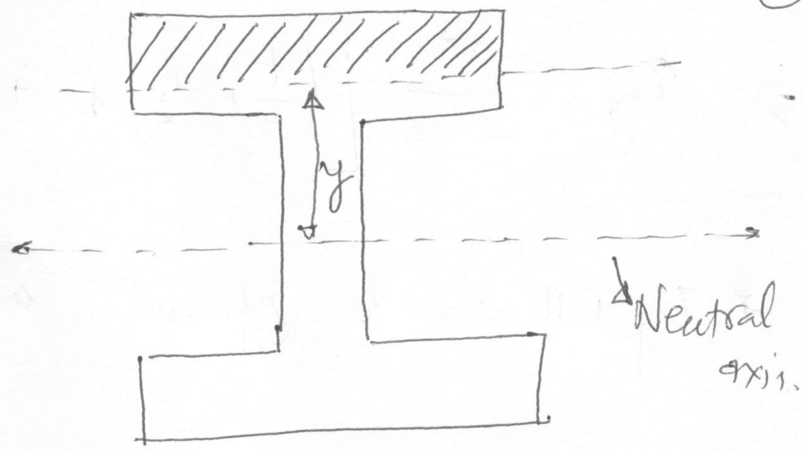
$$= \frac{E_s}{R} I_{zz}(\text{steel about neutral axis}) + \frac{E_{Al}}{R} I_{zz}(\text{Al about neutral axis})$$

So, $$M_z = \frac{E_s I_{zz}^{s} + E_{Al} I_{zz}^{Al}}{R}.$$ Effective rigidity!! 1

3 a.

$$\tau_{xy}(y,z) = \tau_{xy}(y)$$

does not vary with "z"!



$$\tau_{xy}(y) = \frac{V \cdot Q_y}{I_{zz}\, t_y}$$

for $y > \frac{h}{2}$, $t_y = b$ 1

$$Q_y = \bar{Y} A$$

$$= \left(\frac{y}{2} + \frac{1}{4}(t+h)\right)\left(\frac{t}{2} + \frac{h}{2} - y\right) b$$

2

$$= \frac{1}{2}\left(\frac{t+h}{2} + y\right)\left(\frac{t+h}{2} - y\right) b$$

$$= \frac{1}{2}\left(\frac{(t+h)^2}{4} - y^2\right) b$$

$$\tau_{xy} = \frac{V}{2I_{zz}}\left[\frac{(t+h)^2}{4} - y^2\right], \quad y > h/2$$

for $y < h/2$, $t_y = t$ 1

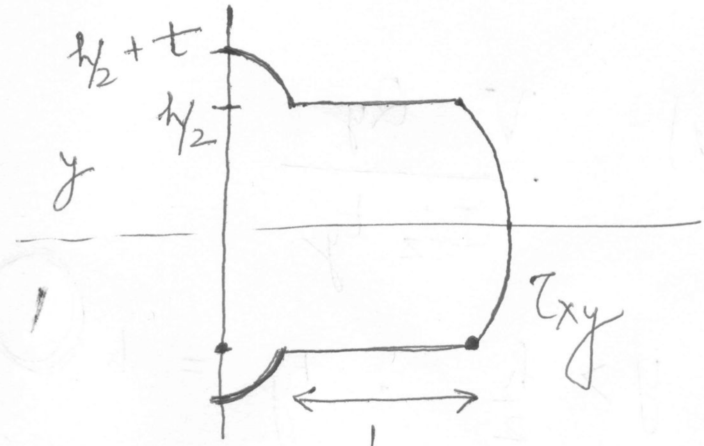
$$Q_y = \frac{1}{2}\left(\frac{t^2}{4} + \frac{th}{2}\right) b + \frac{t}{2}\left(\frac{h^2}{4} - y^2\right)$$

2

$$\Rightarrow \tau_{xy} = \frac{V}{2I_{zz}}\left[\left(\frac{t}{4}+\frac{h}{2}\right)b + \left(\frac{h^2}{4}-y^2\right)\right], \quad y < h/2$$

* τ_{xy} will also be symmetric in "y"!

(b)



(1) → showing jump

(1) → zero at $y = \pm\left(\frac{h}{2}+t\right)$

(1) → max at $y=0$

Notice the jump in τ_{xy} due to abrupt change in the flange width!

(c) shear stress is maximum at centre! $(y=0)$ (1)

$$\tau_{xy}(y=0) = \frac{V}{2I_{zz}}\left[\frac{tb}{4}+\frac{hb}{2}+\frac{h^2}{4}\right]$$

Average shear stress $= \frac{V}{A} = \frac{V}{ht+2bt} = \frac{V}{t(h+2b)}$ → (2)

$$I_{zz} = \frac{th^3}{12} + 2\left(\frac{bt^3}{12}+\frac{bt}{4}(h+t)^2\right)$$

$$= t\left[\frac{h^3}{12}+\frac{bt^2}{6}+\frac{b}{2}(h+t)^2\right]$$

So, the ratio :-

$$\frac{\tau_{xy}(y=0)}{\frac{V}{A}} = \frac{3\left(tb + 2hb + h^2\right)(h+2b)}{2\left(h^3 + 2bt^2 + 6b(h+t)^2\right)}$$

this is independent of $\underline{\underline{V}}$

Also, note that to compute average shear stress, you don't need to integrate $\underline{\underline{\tau_{xy}}}$ since, ~~int~~ the integration of τ_{xy} will give us $\underline{\underline{V}}$ itself!