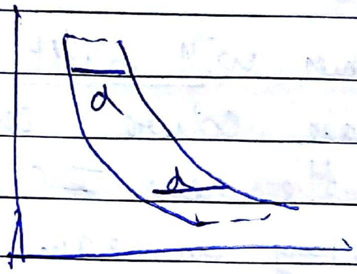
so Quasi-Linear w.r.t X_1

Take $x \sim y$

Define $x+te_1$, $y+te_1$ $\quad e_1 = \begin{pmatrix}1\\0\end{pmatrix}$

then $(x+te_1) \sim (y+te_1)$



parallel shift in $\vec{e}_1$ direction

But for representation, assume that $\parallel$ shift is rightwards.

So shift is along one axis only as Quasi-linear is w.r.t a commodity.

Utility Functions

With every bundle, a satisfaction/utility is defined
→ utility function

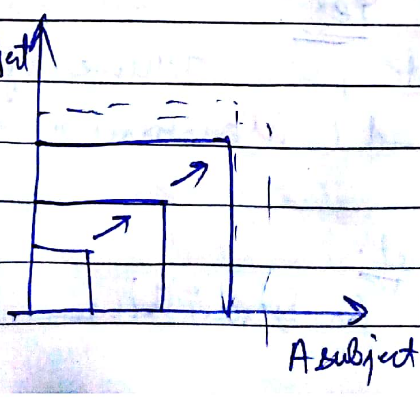
$x \succsim y$, $u : X \to \mathbb{R}$

Representation of preferences

$u, \succsim$ $\quad x \succsim y$ iff $u(x) \geq u(y)$

Q→ Prof A takes 2 exams but counts only higher of 2.

$U(x_1, x_2) = \text{Max}\{x_1, x_2\}$



(upper contour set includes the indifference curve also)

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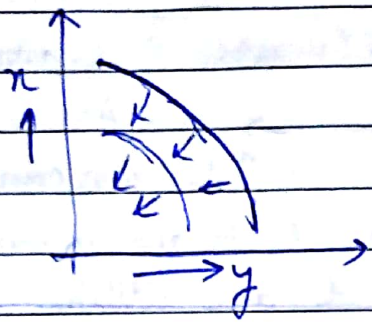
* MRS is defined as amount of dy exchanged for dx.

→ If both axis have bad goods



If $x > y \Rightarrow u(x) > u(y)$
or $\sim y \Rightarrow u(x) = u(y)$

→ Prove if $x > y$ $u(x) = u(y)$ is not true

let it be true so $u(x) = u(y) \rightarrow u(y) \geq u(x)$
(as utility def.) $\rightarrow y \succsim x$

If $x > y$ ~~also $u(y) \geq u(x)$~~
$x \succsim y$ and $y \not\succsim x$

contradiction

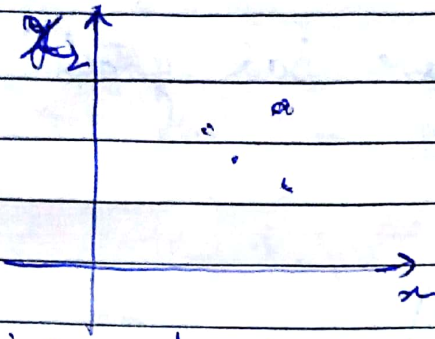
→ Preference relation must be complete and transitive ↑ & reflexive
to be represented as utility function $u(x)$
(This is a softer claim)

- Should be complete as we need to compare 2 $u(x)$ & $u(y)$ and we can always compare 2 real number $u(x)$ & $u(y)$

* Suppose that $x, y \in X$,
$x \succsim y$ if (i) $x_1 > y_1$ (ii) $x_1 = y_1$ & $x_2 \geq y_2$
This preference is called lexicographic pref. ordering.

Monotonicity
convexity

- Indiff. curve of ~~pere~~ lexicographic ordering



Single point is the answer.
As in this preference we cannot find any indifferente bundle to another bundle

→ So, we can't find any utility function for lexicographic pref. rel. even its complete & transitive

Suppose it can be represented by U
$x \geq y \Rightarrow U(x) \geq U(y)$
$x > y \Rightarrow U(x) > U(y)$

→ Utility functions for a preference is not unique. We can always find a transformation. Absolute values of utility function doesn't matter but ranking matters.

→ Suppose u represents lexicographic preference.

Take 2 bundles $(x_1, 1)$, $(x_1, 2)$. 2nd bundle is more preferred.
$u(x,2) > u(x,1)$
Now, there will be a rational no's b/w $u(x,2)$, $u(x,1)$
Arbitrarily choose 1 such rational no r_1

Closed set is, any sequence in a set has its limit in the set itself.

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Now, suppose some x_1' such that $x_1 > x_1'$

$u(x_1', 2) > u(x_1', 1)$ $\quad x \to r_1$

$\downarrow r_1$ $\quad x_1' \to r_1'$

So $u(x_1, 2) > u(x_1, 1) > u(x_1', 2) > u(x_1', 1)$

$\uparrow r_1'$

So, each x (real no.) will be mapped to a rational no (unique) which is not possible.

→ Continuity is the ~~...~~ main thing / reason of utility functions to exist.

* Continuity of preferences $\succsim$

→ Take 2 sequences $\{x_n\}$, $\{y_n\}$ [x_1 — y_1 — — are all bundles]

S.T $x_i \succsim y_i \ \forall i$

preference is continuous if $\boxed{x \succsim y}$.

where x is the limit of x_n sequence

" " y " " " y_n " "

Alternative def:

This pref. rel. is continuous if for any x (bundle) the upper contour set & lower contour set is closed.

Q) Suppose ~~...~~ $x \succsim y$ is continuous show upper contour set is closed set.

Take ~~a sequ~~ any sequence x_1, x_2 — x_n — in upper contour set.

Then each x_1, x_2 — $\succsim x$ as it is in the upper contour set

and as Preference is continuous we know that Limit of the sequence $x_1, x_2 - x_n$ i.e x^* will be weakly preferred to x^*.
So, Limit of the sequence lies inside upper contour set itself so it's closed.

* To prove → Lexicographic sequence is not continuous.

Let $x^n = (2,1)$

$y^n = (2-\frac{1}{n}, 2)$

So $\forall n$ $x^n \succsim y^n$ a $2 > 2-\frac{1}{n}$

but in the limit $\lim_{n\to\infty} y^n = (2,2) = y^*$

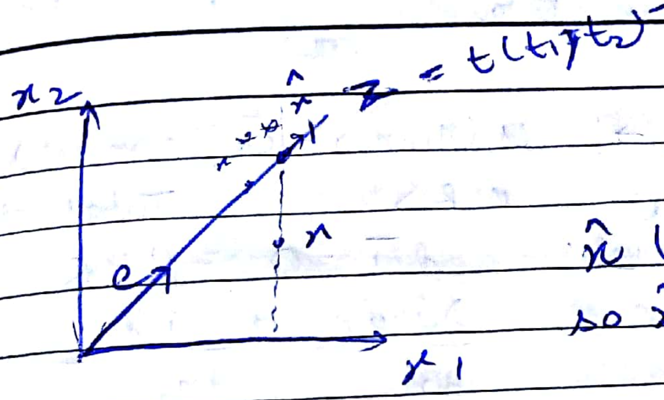
$\lim_{n\to\infty} x^n = (2,1) = x^*$

so $y^* > x^*$ so lexicographic relation is not continuous.

* Representation Theorem → (Proved by Afriat)
Suppose that $\succsim$ is well-behaved i.e it is monotonic & convex. Additionaly $\succsim$ is also continuous, then (i) $\exists$ u st u represents $\succsim$
(ii) u is continuous.
(Although convexity is not reqd. in this theorem but we assumed it in stating it).

Microeconomic theory

Mas Colell · Whinston & Green Book to Look for Proof representation theorem. Bunk Pages



$e = (1,1)$ vector

$\hat{x}$ lies to right of x vertical line. so $\hat{x} \gtrsim x$

~~αe~~ represents a pt. on Z (where α is a constant) then if x^{**} is a bundle on Z s.T it is indifferent to x, then $\alpha e = x^{**}$ → this α is just the utility function for x^{**}.

→ Step 1) → existence of x^{**}
→ step 2) → continuity of α

upper contour set of $x \cap Z = \{\alpha \mid \alpha e \gtrsim x\}$
upper " " U lower contour set of $x \cap Z$ is whole Z: $(\alpha \mid \alpha e \gtrsim x)$ $(\alpha \mid \alpha e \lesssim x)$

→ Can't have multiple α's indifferent as by weakly preferred (coordinates are different so not indifferent)

⇒ Now, if ~~x~~ $x \gtrsim y$ prove $\alpha(x) \gtrsim \alpha(y)$

$\alpha(x) e \gtrsim x \gtrsim y$, $x \gtrsim \alpha(x) e$

→ Perfect substitutes → $ax_1 + bx_2$
→ " complement → ~~$\min\{ax_1, bx_2\}$~~ $\min\{ax_1, bx_2\}$
→ Homothetic → $u(tx) = t\,u(x)$
→ Quasilinear → $x_2 = K - f(x_1)$ → $u(x) = f(x_1) + x_2$

→ Equilibrium is attained in max. guaranteed by convexity.

→ max. utility pt. lies on budget line guaranteed by monotonicity

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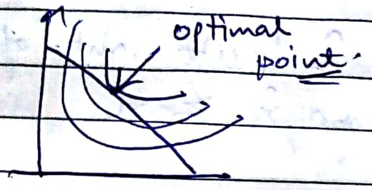
$\ast$ Utility Function → $u(x_1, x_2) = \bar{u}$ (constant)



Budget line

MB.

MRS

Total differential

$d\bar{u} = 0 = RHS$

$\frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2$

$\Rightarrow \frac{dx_2}{dx_1} = -\frac{(\partial u / \partial x_1)}{(\partial u / \partial x_2)}$

↓

Slope of indifference curve

$\ast$ $\frac{\partial u}{\partial x_i}$ ⇒ Change in utility level with changing consumption of ith commodity while other commodity consumption is constant

↳ Marginal utility of x_i

↳ Change in utility with marginal change in x_i keeping other x_j's constant

→ Utility Max. Problem (UMP)

Max $(u(x_1, x_2))$ s.T

$\vec{p}\vec{x} \leq M$ $\vec{p}$ = price vector

$\vec{x}$ = quantity vector

$P > 0$, $M > 0$, $x \geq 0$

Existence of solution → Budget set is closed and bounded.

And max. exists as any closed, bounded set has max. & min.

To solve $Px \leq M$?

$\ast$ The Lagrangean Multiplier technique →

Introduce a variable λ into system $(\lambda \geq 0)$

So, we converted constrained optimisation into unconstrained optimisation problem.

Here the only constrain is $x, \lambda \geq 0$.

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$$\mathcal{L}(x_1, x_2, \lambda) = U(x) + \lambda[M - P \cdot x]$$

Now, we maximize this

Constraint Qualification → Constraints for Lagrangian Multiplier technique

→ Kuhn-Tucker.

→ Utility max → At x' : $MRS \neq \frac{P_1}{P_2}$

Case 1) Slope of the BL is flatter than slope of I-C

→ Langrange $L(x, \lambda) = U(x) + \lambda[M - Px]$ $(x, \lambda \geq 0)$

Kuhn Tucker condition

$$\frac{\partial L}{\partial x_i} \leq 0 \; ; i = 1, 2 \quad = \frac{\partial U}{\partial x_i} - \lambda P_i \leq 0$$

$$\frac{\partial L}{\partial \lambda} \geq 0 \qquad M \geq Px$$

These condition are satisfied as some optimal point x^*, λ^* only.

Interior solution → $\frac{\partial U}{\partial x_1} = \lambda P_i$ (sort of max. lies within interior)

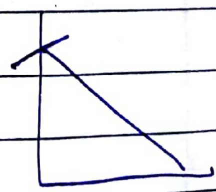
↳ .

theory

Function that we need to maximize is

Level curve → Curve when all values remain constn

Quasi-concave, which comes from convexity.



ruled out in this way

Convexity → for any $x \in X$, the upper contour set is convex.

Mathematical Analysis by Walter Rudin

$u(x_1, x_2)$ = Maximand

Bunk Pages

Quasiconcave ($u(\cdot)$ is quasiconcave function)

↳ Take 2 values $x, x' \in X$, take a convex combination of x & x', $x'' = \alpha x + (1-\alpha)x'$

$$u(x'') \geq \min\{u(x), u(x')\}$$

At x^* → $\frac{\partial u}{\partial x_1} \leq \lambda p_1$, $\frac{\partial u}{\partial x_2} \leq \lambda p_2$

$$M \geq p_1 x_1 + p_2 x_2$$

Assume it's a Interior Solution → $\frac{\partial u}{\partial x_1} = \lambda p_1$, $\frac{\partial u}{\partial x_2} = \lambda p_2$

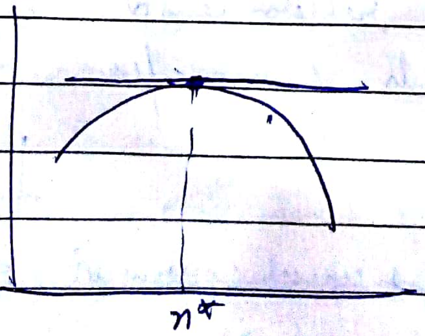
$$M = px$$

$$\underbrace{\frac{\partial u}{\partial x_1} \Big/ \frac{\partial u}{\partial x_2} = \frac{p_1}{p_2}}_{\text{MRS}} \quad \text{and} \quad \underbrace{M = px}_{\text{Budget slope}}$$

Gradient at a point on level curve $= \left(\frac{\partial u}{\partial x_1}\hat{i} + \frac{\partial u}{\partial x_2}\hat{j}\right)$
(Perp. to tangent at that pt.)

Sufficient condition for optimization function is convexity of preferences or quasiconcavity of utility function.

$\boxed{u(\cdot) \text{ is negative semidefinite}}$ → We get this by a matrix of double partials
(Bordered Hessian)



Find taylor expansion about x^*

$$f(x^*) + (x - x^*) f'(x^*) + \frac{(x - x^*)^2}{2!} f''(x^*)$$

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We see that as we increase no. of term we further dip in value so each f', f'' etc are -ve this is ~~semi~~ negative semidefinite

→ **Cobb - Douglas utility function**

$u(x_1, x_2) = x_1^{\alpha} x_2^{\beta}$

$\alpha + \beta = 1$

$\alpha + \beta < 1$

$\alpha + \beta > 1$

x_1^*

x_2^*

$x_1^{\alpha} x_2^{\beta} + \lambda(M - P_x) = \mathcal{L}(x)$

$\frac{\partial \mathcal{L}(x)}{\partial x_1} = \alpha x_1^{\alpha-1} x_2^{\beta} - \lambda P_1 \leq 0$

$\beta x_1^{\alpha} x_2^{\beta-1} - \lambda P_2 \leq 0$

$M \geq P_1 x_1 + P_2 x_2$

(Marginal utility) $\frac{MU_1}{MU_2} = \frac{P_1}{P_2} \geq \frac{x_2 \alpha}{x_1 \beta} \rightarrow x_2 \leq \frac{P_1 x_1 \beta}{P_2 \alpha}$

$M \geq P_1 x_1 + \frac{P_1 x_1 \beta}{\alpha}$

$\boxed{\frac{M}{P_1\left(1+\frac{\beta}{\alpha}\right)} \geq x_1^*}$, $\boxed{\left[\frac{M - \frac{M}{(1+\frac{\beta}{\alpha})}}{P_2}\right] \geq x_2}$

When it's mentioned assume interior sol^n. so $\geq$ will be replaced by $=$

$\boxed{\frac{M(\beta/\alpha)}{P_2(1+\beta/\alpha)} \geq x_2^*}$

x_1^*, x_2^* are called ordinary demand function (function of money & Price)

maybe it's also called marshally or Marshian ~~or~~ demand functions

Kuhn Tucker sol^n occurs at boundaries → chk

Sticky variable → Preference don't change over time
Loose " " → " " change "

Bunk Pages

x^* is a function of P_1, P_2 & M

Homogeneous function of deg. k → $f(\lambda x) = \lambda^k f(x)$

Then function is Homogeneous of deg (0) in P_1, P_2, M
as $f: U(\lambda P_1, \lambda P_2, \lambda M) = \lambda^0 U(P_1, P_2, M)$

* We assume weak monotonicity & convexity unless mentioned to consider strict one

* Interior sol.^{n} we also get by $\frac{MU_1}{MU_2} = \frac{\partial u/\partial x_1}{\partial u/\partial x_2} = \frac{P_1}{P_2}$
& then Budget constraint

* $x_i^* = x_i^*(P_1, P_2, M)$, $i = 1, 2$
↳ Ordinary demand function

→ Maximised level of utility,
When the prices/income are given by P_1, P_2, M ;
~~$U(x_1^*, x_2^*$~~ $U(x_1^*(P,M), x_2^*(P,M)) = v(P_1, P_2, M)$
v is a function which takes in P_1, P_2 & M, keeping U ~~constant~~ function same and outputs max utility.
→ also called Indirect utility function.
↳ also " " value function.

→ Finding economic significance of λ.

$$v(P, M) = U(x_1^*(P,M), x_2^*(P,M))$$

$$\frac{\partial v}{\partial M} = \left.\frac{\partial u}{\partial x_1}\right|_{x_1^*} \frac{\partial x_1}{\partial M} + \left.\frac{\partial u}{\partial x_2}\right|_{x_2^*} \frac{\partial x_2}{\partial M} = \lambda$$

↳ Marginal change in maximised utility with change in income keeping other P_1, P_2 constant
↳ Marginal utility of Money

$$\Rightarrow \lambda P_1 \frac{\partial x_1}{\partial M} + \lambda P_2 \frac{\partial x_2}{\partial M} = \lambda \qquad \left[\frac{\partial x_1}{\partial M} = \frac{1}{P_1} \text{ by budget constraint} \right]$$

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→ So, λ computes change in maximised utility with money

* Properties of ordinary demand function

1) $x_i(P, M)$ is homogenous of degree 0 in prices & income

2) $Px^* = M$ → Walras's law (Market clear)

3) $x^*(P,M)$ is a convex set.

Proof → Suppose $x, x' \in x^*(P,m)$, $x'' = \alpha x + (1-\alpha)x'$, $\alpha \in [0,1]$

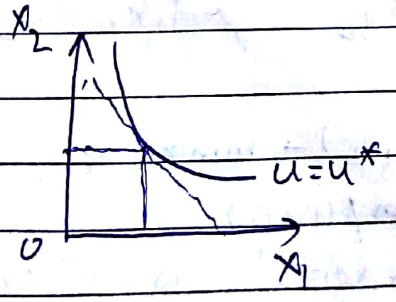
To prove → $x'' \in x^*(P,M)$

- x'' lies in the upper contour set of x' & x where x' & x lie on same indifference line, by def. of convexity of preference.
- Also x & x' both lie on budget line (clear the market) so its convex combination x'' also lies on same budget line so under same budget constraint.
- As x & x' are maximized in this constraint & x'' is of higher preference so x'' is also maximised.

Q) Change Cobb-douglas utility fn to have a set of x^* not singleton.

Q) If strictly convex preference then prove x^* is singleton.

→ To minimise expenditure when your utility level doesn't fall below a utility level u^*.



$(P_1, P_2 = \text{fixed})$ (Expenditure minimisation Problem (EMP))

$h_1(P_1, P_2, u^*)$
$h_2(P_1, P_2, u^*)$ compensated demand function

Call this line as Expenditure line

It is also called Hicksian

→ $\text{Min}(P.x)$ s.t. $u(x) \geq u^*$

Langragian → $Z(x_1, x_2, \mu) = P.x + \mu(u(x) - u^*)$

$\frac{\partial u}{\partial h_1} = \mu P_1$, $\frac{\partial u}{\partial h_2} = \mu P_2$ (Interior sol^n)

$u(h) = u^*$

→ Connection in UMP & EMP

$(v(P, M) = u^*)$ in a special case.

→ $P_1 h_1^* + P_2 h_2^* = e(P_1, P_2, u^*)$ → Expenditure function.

↳ minimized level of expenditure when prices are P_1, P_2 & utility is u^*

→ Given indirect utility function, can we find out ordinary demand function.

→ From e can you derive compensated demand function.

Test → $x_1^* = \begin{cases} m/P_1, & \alpha/\beta > P_1/P_2 \\ 0, & \frac{\alpha}{\beta} < P_1/P_2 \\ \left[0, \frac{m}{P}\right], & \frac{\alpha}{\beta} = \frac{P_1}{P_2} \end{cases}$

Perfect Substitutes ?

where $u(x_1, x_2) = \alpha x_1 + \beta x_2$, $P_1 x_1 + P_2 x_2 \leq M$

→ Perfect complements → $u(x_1, x_2) = \min\{x_1, x_2\}$, 1:1

$x_1^* = x_2^* = m/(P_1 + P_2)$

$V(p, m)$ is also homogenous in deg '0' as x^* is homogen in deg (0)

$t^0 x(p, m) = x(tp, tm)$

$u(x_1, x_2) = v(tp, tm) = t^0 v(p, m)$

..., strict inequality holds both in UMP & EMP.

$\{P : V(P,m) \le k\}$ is convex.

Take P, P' s.t.

$V(P,m) \le k$ $\quad$ $V(P',m) \le k$

$P'' = (1-t)P' + tP$ $\quad$ $P''x \le M$

$B = \{x : Px \le M\}$

$B' = \{x : P'x \le M\}$

→ Assume $Px'' > m$, $P'x'' > m$

then $tPx'' > mt$, $(1-t)P'x'' > m(1-t)$

then sum also lies outside but

$P''x = tPx + (1-t)P'x \le M$.

UMP	EMP
$\max u(x)$ s.t. $P \cdot x \le M$	$\min P \cdot x$ s.t. $u(x) \ge u^*$
i) $\frac{MU_1}{MU_2} = MRS = \frac{P_1}{P_2}$	i) $\frac{MU_1}{MU_2} = MRS = \frac{P_1}{P_2}$
ii) $P \cdot X = M$	ii) $u(x) = u^*$
• $x_i^* = x_i^*(P,M)$, $i=1,2$	• $h_i^* = h_i^*(P,u^*)$ $i=1,2$
• Ordinary / Marshian / Walrasian Demand function	• Hicksian / Compensated Demand function.
• $\underbrace{u(x^*) = V(P,M)}$ ↳ Indirect utility function / value function	• $e(P,u^*) = P \cdot h^*$
• $\lambda = \frac{\partial V}{\partial M}$	• This is homogeneous with degree 1 in P_1, P_2 only.
• $V(P_1,P_2,M)$ is homogeneous with deg 0 in P_1, P_2, M.	
• V is quasi-convex in P_1, P_2, M (Lower contour set is convex)	

Lower contour

→ v is quasiconvex → ↑ Set $\{(P,M) \mid v(P,M) \le v^*\}$ is convex.

To show: (P_1, P_2, M) , (P_1', P_2', M')

$$(P_1'', P_2'', M'') = (\alpha P_1 + (1-\alpha)P_1'), (\alpha P_2 + (1-\alpha)P_2'), (\alpha M + (1-\alpha)M')$$

$\alpha \in [0,1]$

$v(P'', M'') \le v^*$ (To show)

→ For any

$x \in B_{P'', M''}$

We show $u(x) \le v^*$

$P''x \le M''$

$\alpha P x + (1-\alpha)P'x \le \alpha M + (1-\alpha)M'$

so either $P \cdot x \le M$

$P'x \le M'$ as Both

Now if x is in then maximised utility is v^* so this x is finally less than v^*

H.W Prove this diagramatically.

→ $e(P, u^*)$ is Concave in P

$$e(\alpha P + (1-\alpha)P', u^*) \ge \alpha e(P, u^*) + (1-\alpha)e(P', u^*)$$

$u(x) \ge u^*$

$\alpha e(P, u^*) \le \alpha P \cdot x$

$(1-\alpha)e(P', u^*) \le (1-\alpha)P' \cdot x$

$\alpha e(P, u^*) + (1-\alpha)e(P', u^*) \le \alpha P x + (1-\alpha)P'x$

* **Recoverability** → UMP ⇒ $\vartheta(P, M)$ → Indirect utility fn

→ We need to find 'x_i' from $\vartheta(P, M)$
demand function.

$$\frac{\partial \vartheta(P_1, P_2, M)}{\partial P_1} = \frac{\partial U(x_1^*(P_1,P_2,M), x_2^*(P_1,P_2,M))}{\partial P_1}$$

$$= \underbrace{\left.\frac{\partial U}{\partial x_1}\right|_{x_1^*}}_{\lambda P_1} \frac{\partial x_1}{\partial P_1} + \underbrace{\left.\frac{\partial U}{\partial x_2}\right|_{x_2^*}}_{\lambda P_2} \frac{\partial x_2}{\partial P_1}$$

$$= \lambda\left[P_1 \frac{\partial x_1}{\partial P_1} + P_2 \frac{\partial x_2}{\partial P_1}\right] \qquad \left[\begin{array}{l} P_1x_1 + P_2x_2 = M \\ x_1 + P_1\frac{\partial x_1}{\partial P_1} + P_2\frac{\partial x_2}{\partial P_1} = \frac{\partial M}{\partial P_1} \\ \qquad\qquad\qquad\qquad \downarrow \\ \qquad\qquad\qquad\qquad 0 \end{array}\right.$$

$$= -\lambda x_1$$

* so $\dfrac{\partial \vartheta / \partial P_i}{\partial \vartheta / \partial M} = -x_i$ (Demand function)

This is an identity
Roy's identity (as true for all prices & income)

→ $e(P_1, P_2, u^*) \rightarrow h_1(P_1, P_2, u^*)$

$$e(P, u^*) = ~~P_1Q_1 + P_2Q_2~~ \; P_1 h_1(P, u^*) + P_2 h_2(P, u^*)$$

$$\frac{\partial e(P, u^*)}{\partial P_1} = h_1(P, u^*) + P_1 \frac{\partial h_1}{\partial P_1}(P, u^*) + P_2 \frac{\partial h_2}{\partial P_1}(P, u^*)$$

$$= h_1(P, u^*) \; \cancel{+ \frac{\partial}{\partial P_1}} \qquad \underset{0}{\downarrow}$$

$$\boxed{\frac{\partial e}{\partial P_1}(P, u^*) = h_1(P, u^*)} \leftarrow \text{Shephard's lemma}$$