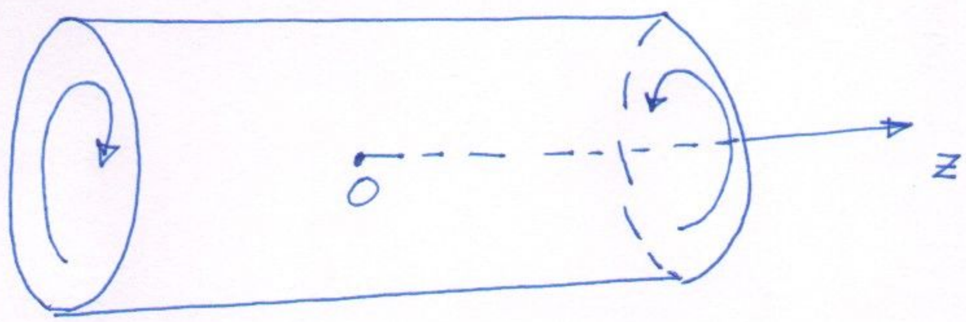
Combined extension-torsion of Circular beams



By physical considerations:-

$$u_r(r)\ ,\quad u_\theta(r,z) = r\frac{\theta_0}{L}z\ ,\quad u_z(z)$$

$$\Rightarrow \underset{=}{\epsilon} = \begin{bmatrix} u_r' & 0 & 0 \\ 0 & u_r/r & \frac{r\theta_0}{2L} \\ 0 & \frac{r\theta_0}{2L} & u_z' \end{bmatrix} \quad \text{in cylinderical } \underline{\text{coordinate}}.$$

So, $$\sigma_{rr} = \lambda\left(u_r' + \frac{u_r}{r} + u_z'\right) + 2\mu u_r'$$

$$\sigma_{\theta\theta} : \lambda\left(\quad '' \quad\right) + 2\mu\frac{u_r}{r}$$

$$\sigma_{zz} = \lambda\left(\quad '' \quad\right) + 2\mu u_z'$$

$$\sigma_{r\theta} = \sigma_{rz} = 0, \qquad \sigma_{\theta z} = \frac{G\theta_0}{L}r$$

Let us look at θ- equation

$$\Rightarrow \cancel{\frac{\partial\sigma_{r\theta}}{\partial r}} + \frac{1}{r}\cancel{\frac{\partial\sigma_{\theta\theta}}{\partial\theta}} + \cancel{\frac{\partial\sigma_{\theta z}}{\partial z}} + 2\cancel{\frac{\sigma_{r\theta}}{r}} = 0$$

Hence, it gets trivially satisfied

Z- equation.

$$\Rightarrow \cancel{\frac{\partial \sigma_{rz}}{\partial r}} + \frac{1}{r}\cancel{\frac{\partial \sigma_{\theta z}}{\partial \theta}} + \frac{\partial \sigma_{zz}}{\partial z} + \cancel{\frac{\sigma_{rz}}{r}} = 0$$

$$\Rightarrow \frac{\partial \sigma_{zz}}{\partial z} = 0 \quad \text{or, } \sigma_{zz} \text{ is independent of } \underline{\underline{z}}$$

$$\Rightarrow \lambda\left(u_r' + \frac{u_r}{r} + u_z'\right) + 2\mu\, u_z' = \text{indep. of } z$$

$$\Rightarrow \lambda\left(u_r' + \frac{u_r}{r}\right) + (\lambda + 2\mu) u_z' = \text{''}$$

$$\Rightarrow \boxed{u_z' = \text{constant} = \epsilon}$$

Now, let us look at radial eq.

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\cancel{\frac{\partial \sigma_{r\theta}}{\partial \theta}} + \cancel{\frac{\partial \sigma_{rz}}{\partial z}} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$$

$$\Rightarrow \frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$$

$$\Rightarrow \lambda\left(u_r' + \frac{u_r}{r}\right)' + 2\mu\, u_r'' + 2\mu\left(\frac{u_r'}{r} - \frac{u_r}{r^2}\right) = 0$$

$$\Rightarrow (\lambda + 2\mu) u_r'' + (\lambda + 2\mu)\frac{u_r'}{r} - (\lambda + 2\mu)\frac{u_r}{r^2} = 0$$

$$\Rightarrow u_r'' + \frac{u_r'}{r} - \frac{u_r}{r^2} = 0$$

$$\Rightarrow u_r'' + \left(\frac{u_r}{r}\right)' = 0 \quad \Rightarrow \boxed{u_r' + \frac{u_r}{r} = C}$$

$$\text{or, } \boxed{\epsilon_{rr} + \epsilon_{\theta\theta} = C}$$

As, $u_r' + \frac{u_r}{r} = C \Rightarrow \sigma_{zz} = \lambda C + (\lambda + 2\mu)\epsilon$

$$\boxed{\sigma_{zz} = \lambda (C + \epsilon) + 2\mu\epsilon} \rightarrow \text{a constant}$$

Also, note that

$$\sigma_{rr} + \sigma_{\theta\theta} = 2(\lambda + \cancel{2}\mu)\left(u_r' + \frac{u_r}{r}\right) + 2\lambda u_z'$$

or, $$\boxed{\sigma_{rr} + \sigma_{\theta\theta} = 2(\lambda + \mu) C + 2\lambda\epsilon} \rightarrow \text{another constant}$$

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Let us look at radial equation again

$$\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$$

$$\Rightarrow \frac{d\sigma_{rr}}{dr} + 2\frac{\sigma_{rr}}{r} = \frac{\sigma_{rr} + \sigma_{\theta\theta}}{r} = \frac{A}{r}$$

$$\Rightarrow \frac{1}{r^2}\frac{d}{dr}\left(\sigma_{rr} r^2\right) = \frac{A}{r}$$

$$\Rightarrow \boxed{\sigma_{rr} = \frac{A}{2} + \frac{B}{r^2}}$$

Solid tube

σ_{rr} = finite at $r = 0$

$\Rightarrow B = 0$

$\sigma_{rr} = 0$ at $r = r_2$

$\Rightarrow A = 0$

$\Rightarrow \sigma_{rr} = 0$

Hollow tube

$\sigma_{rr} = 0$ at $r = r_1, r_2$

$\Rightarrow A = B = 0$

Hence, $\sigma_{rr} = 0$ for both solid and hollow tube

Plugging $\sigma_{rr} = 0$ in radial eq

$$\Rightarrow \sigma_{\theta\theta} = 0$$

Hence, $\underline{\underline{\sigma_{rr} = \sigma_{\theta\theta} = 0}}$ for both solid and hollow beams.

Again $u_r' + \frac{u_r}{r} = C$

$$\Rightarrow \frac{1}{r}\frac{d}{dr}(r u_r) = C \quad \Rightarrow \boxed{u_r = \frac{C}{2} r + \frac{D}{r}}$$

plugging it into expression for σ_{rr}

$$\Rightarrow \lambda(C + \epsilon) + 2\mu\left(\frac{C}{2} - \frac{D}{r^2}\right) = 0$$

$$\Rightarrow (\lambda+\mu)C + \lambda\epsilon - \frac{2\mu D}{r^2} = 0$$

$\Rightarrow D = 0$ (no dependence on 'r')

$$\Rightarrow (\lambda+\mu)C + \lambda\epsilon = 0$$

$$\boxed{\begin{gathered} u_r = \frac{C}{2} r \\ \Rightarrow \epsilon_{rr} = \epsilon_{\theta\theta} = \frac{C}{2} \end{gathered}}$$

$$\Rightarrow C = -\frac{\lambda}{\lambda+\mu}\epsilon$$

$$\Rightarrow \epsilon_{rr} = \epsilon_{\theta\theta} = \frac{C}{2} = -\frac{\lambda}{2(\lambda+\mu)}\epsilon$$

or, $\boxed{\epsilon_{rr} = \epsilon_{\theta\theta} = -\nu\epsilon}$ and $\boxed{u_r = -\nu\epsilon r}$

Now, we have to understand how to generate $\underline{\underline{\epsilon}}$ and $\underline{\theta_0}$

for this note that

$$\epsilon = \frac{1}{E}\left(\sigma_{zz} - \nu\cancel{(\sigma_{rr} + \sigma_{\theta\theta})}\right)$$

$$\Rightarrow \sigma_{zz} = E\epsilon$$

or, Axial force: $\int \sigma_{zz}\, dA = EA\epsilon$

So, Axial force of $\underline{\underline{EA\epsilon}}$ is needed to generate strain "ϵ" ~~and~~

This does not change even if torsion is present

Similarly,

$$\sigma_{\theta z} = \frac{G\theta_0}{L} r$$

$$\Rightarrow \text{Torque } (T) = \int_{r_1}^{r_2} r\,(2\pi r\,dr)\,\frac{G\theta_0}{L} r$$

$$= \frac{G\theta_0}{L}\int r^2\, dA = \frac{G\theta_0}{L} J$$

or, $\boxed{T = \frac{G\theta_0}{L} J}$

* Basically torsion and axial extension become independent of each other

* Also, $\underline{u_r = -\nu\epsilon r}$ does not arise if only torsion is present