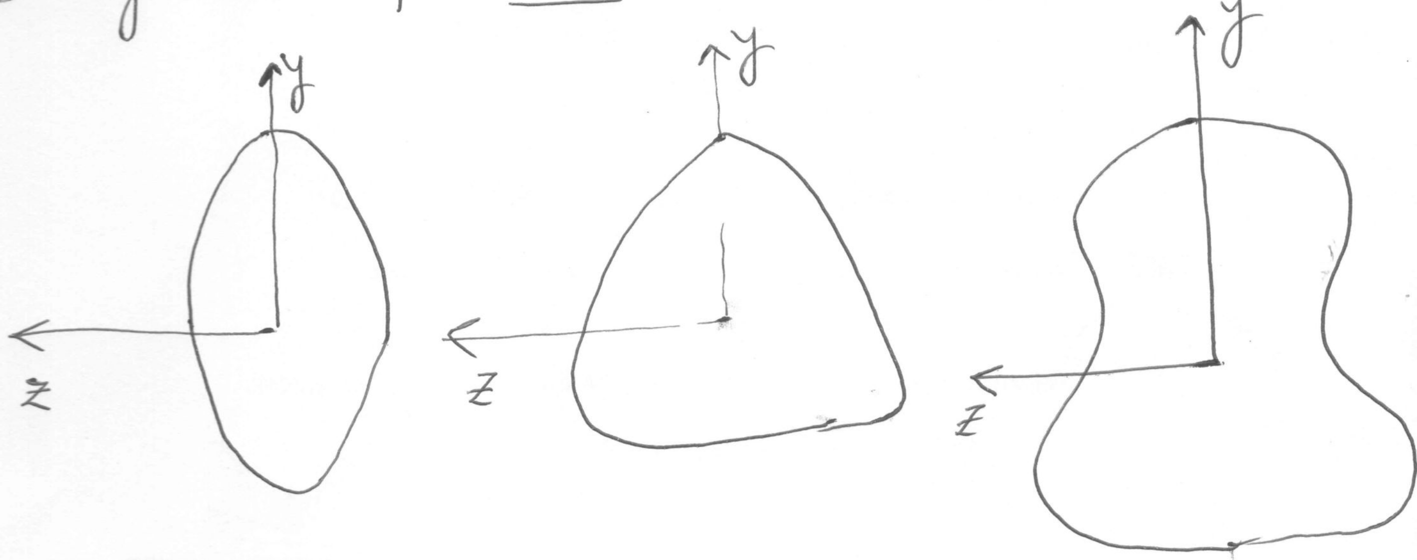
Bending of Symmetrical beams.

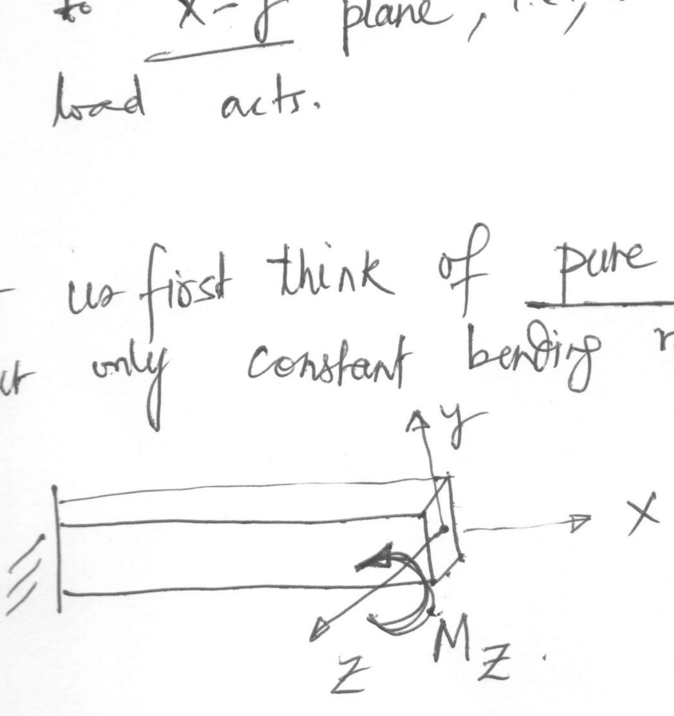
Think of beams such that its cross-section has a line of symmetry about 'Y' axis.



When such beams are acted upon by a transverse load along its line of symmetry (in 'y' direction), it creates bending moment in the beam along 'z' direction.

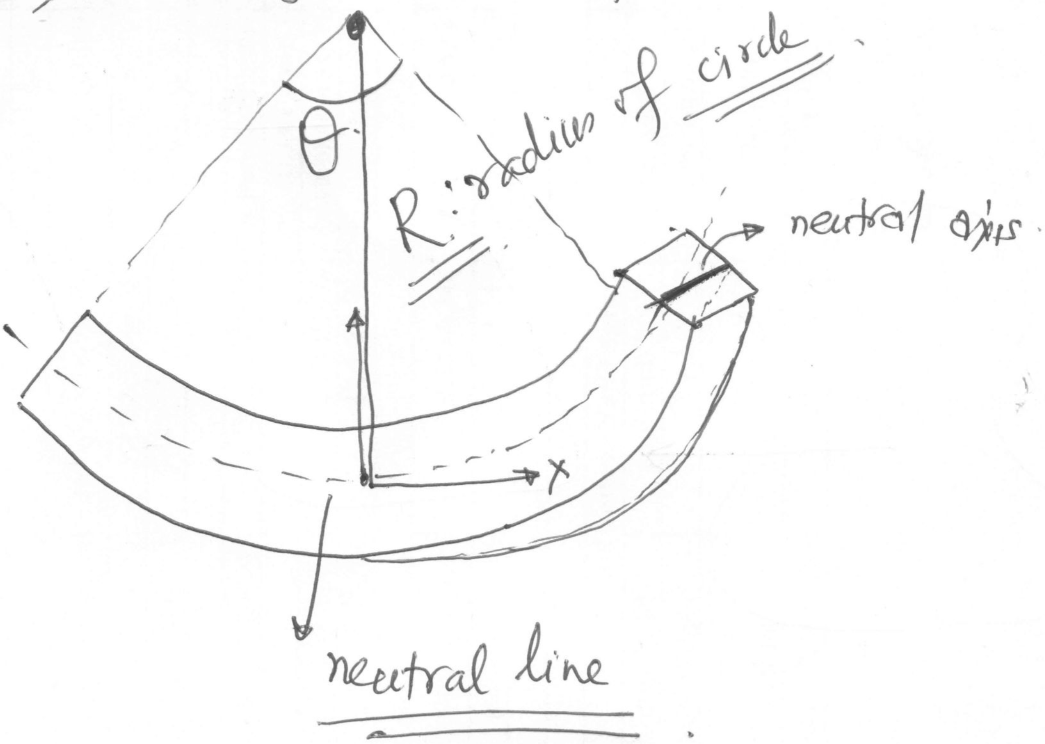
→ In such cases, bending of beams is restricted to X-Y plane, i.e., the plane in which transverse load acts.

* Let us first think of pure bending, i.e., no load acts but only constant bending moment



Pure bending of beams

- In this case, a straight beam deforms into an arc of a circle



→ There will be a neutral plane such that on the concave side, beam fibers will compress but on convex side, beam fibers will elongate. No compression or extension takes place of the neutral line.

Since, neutral line does not compress or stretch

$$\Rightarrow R\theta = L \text{ (Length of beam)}$$

At a distance 'y' above the neutral line,

deformed length of fiber $= (R-y)\theta$

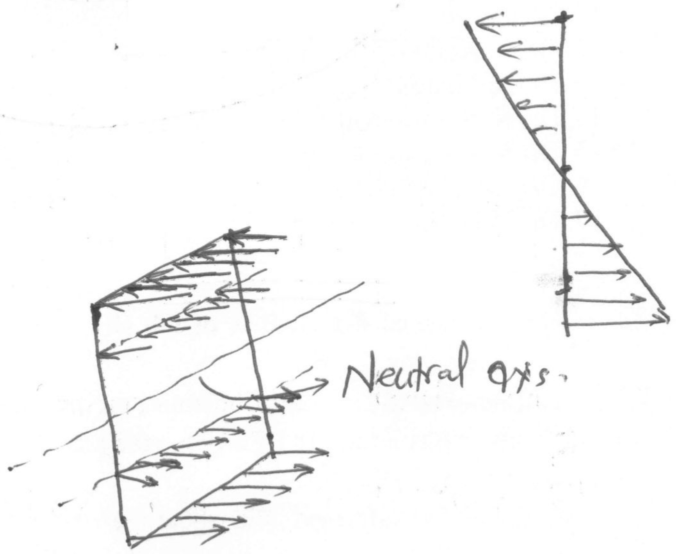
$$\Rightarrow \epsilon_{xx}(y) = \frac{(R-y)\theta - R\theta}{R\theta} = -y/R.$$

Let us further assume, $\sigma_{yy} = \sigma_{zz} = 0$

$$\Rightarrow \quad \epsilon_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu(\cancel{\sigma_{yy}} + \cancel{\sigma_{zz}})\right)$$

$$\Rightarrow \boxed{\sigma_{xx} = E\,\epsilon_{xx} = -Ey/R.}$$

So, σ_{xx} (bending stress or ~~flext~~ flexural stress) varies linearly with y.



Basically σ_{xx} only depends on y and is independent of z

$$\sigma_{xx}(y,z) = -Ey/R.$$

As, axial force is zero in case of pure bending

$$\Rightarrow \iint_{\Omega} \sigma_{xx}\, dA = 0$$

$$\Rightarrow -E/R \iint y\, dA = 0$$ $\Rightarrow$ Neutral axis must pass through the beam's centroid.

Let us now find the resultant moment (about 'z' axis) due to this stress distribution

$$\Rightarrow \quad M_z = -\iint y\,\sigma_{xx}\,dA$$

$$= \frac{E}{R}\iint y^2\,dA = \frac{E}{R} I_{zz}$$

$$\Rightarrow \quad \boxed{M_z = \frac{E I_{zz}}{R}}$$

$$M_y = \iint z\,\sigma_{xx}\,dA = -\frac{E}{R}\int yz\,dA = -\frac{E}{R} I_{yz}$$

$$= 0 \quad \left(\begin{array}{l}\text{due to symmetry of} \\ \text{cross-section, } I_{yz} = 0\end{array}\right)$$

Hence, if we apply bending moment M_z in a beam, the bending curvature of beam is given by

$$\boxed{\frac{1}{R} = \frac{M_z}{E I_{zz}}}$$

* $E I_{zz}$ is also called bending rigidity or flexural rigidity of a beam!

We can also see that

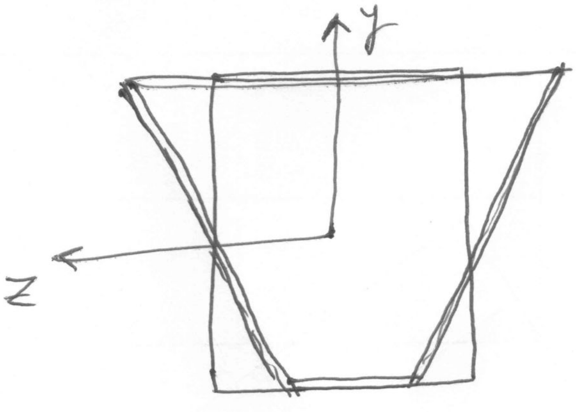
$$\epsilon_{yy} = \frac{1}{E}\left(\cancel{\sigma_{yy}} - \nu\left(\sigma_{xx} + \cancel{\sigma_{zz}}\right)\right)$$

$$= -\frac{\nu}{E}\sigma_{xx} = -\nu\, y/R.$$

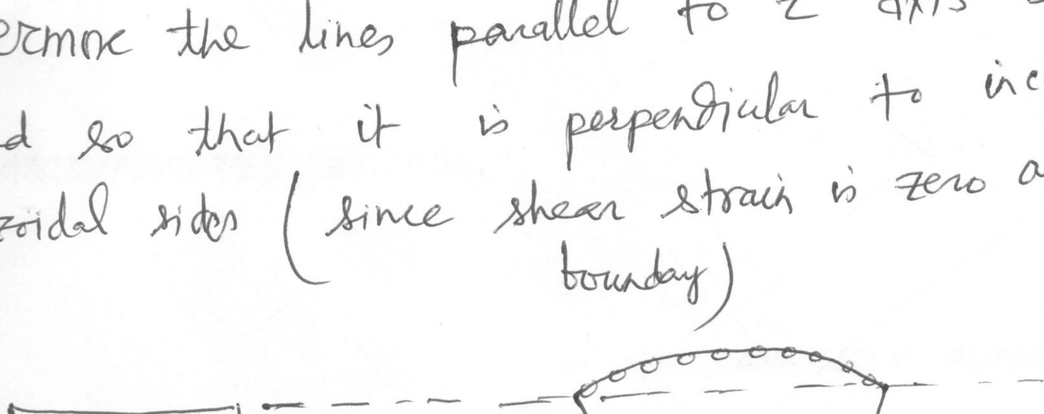
$$\epsilon_{zz} = -\nu \frac{y}{R}.$$

$\Downarrow$

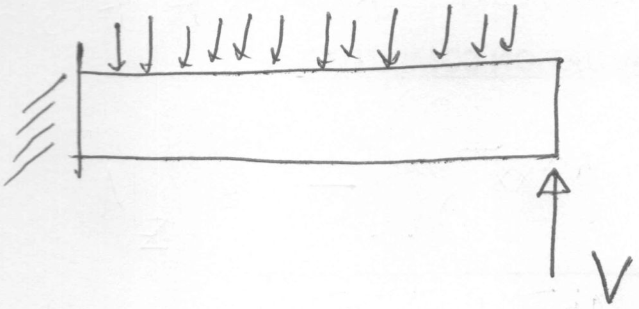
rectangular cross section becomes trapezoidal.



Furthermore the lines parallel to 'Z' axis become curved so that it is perpendicular to inclined trapezoidal sides (since shear strain is zero at the boundary)



Bending of beams under transverse load



In this case, shear force varies along the beam and also the bending moment

$$\Rightarrow \boxed{\frac{dM_z}{dx} = -V_y}$$

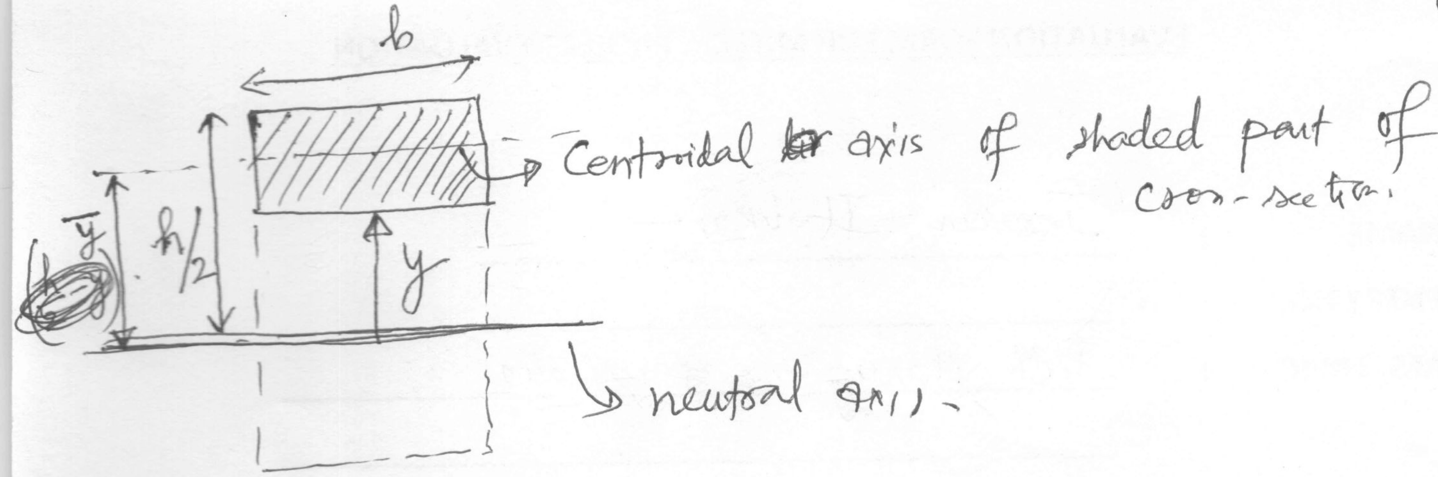
$\Rightarrow$ Bending curvature is no more constant and beam does not bend into a <u>perfect circle</u>.

In such cases, one can show that

$$\tau_{xy} = \frac{V_y\, Q(y)}{I_{zz}\, b(y)}$$

$Q(y) \rightarrow$ first moment of a portion of cross-section

first moment of this area $= Q(y)$
$= \bar{y} A$



$\bar{y}$ of shaded portion $= \cancel{\tfrac{h}{2}-y}\; y + \frac{1}{2}\left(\frac{h}{2} - y\right)$

$= \frac{1}{2}\left(y + \frac{h}{2}\right)$

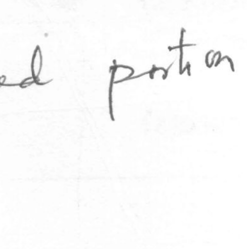
Area of shaded portion $= b\left(\frac{h}{2} - y\right)$

$Q(y)$ of shaded portion $= \bar{y} A$

$= \frac{b}{2}\left(\frac{h^2}{4} - y^2\right)$

$$\Rightarrow \tau_{xy}(y) = \frac{VQ}{I_{zz}\, b} = \frac{V}{2 I_{zz}}\left(\frac{h^2}{4} - y^2\right)$$

parabolic profile



$$\frac{V h^2}{8 I_{zz}} = \frac{V}{bh}\,\frac{3}{2}$$

$I_{zz} = \frac{bh^3}{12}$

$\frac{V}{bh}$: Avg. shear stress