Solution to Minor I : APL104

1 (b) is always symmetric

2 (d) fluid is viscous and moving

3 (b) is always shear-free

4 (c) has zero normal traction if two of principal stresses are equal and opposite.

(d) none of these → both option will fetch <u>mark</u>

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(5) $\underline{\nabla} \cdot \underline{\underline{\sigma}}$, We need to work in cylinderical co-ordinate

Let $\underline{e}_r = \underline{e}_1$, $\underline{e}_\theta = \underline{e}_2$, $\underline{e}_z = \underline{e}_3$

$r = x_1$, $\theta = x_2$, $z = x_3$

$$\underline{\nabla} \cdot \underline{\underline{\sigma}} = \frac{\partial}{\partial r}(\underline{\underline{\sigma}}) \cdot \underline{e}_r + \frac{1}{r}\frac{\partial}{\partial \theta}(\underline{\underline{\sigma}}) \cdot \underline{e}_\theta + \frac{\partial}{\partial z}(\underline{\underline{\sigma}}) \cdot \underline{e}_z$$

$$= \frac{\partial}{\partial x_1}(\underline{\underline{\sigma}}) \cdot \underline{e}_1 + \frac{1}{r}\frac{\partial}{\partial x_2}(\underline{\underline{\sigma}}) \cdot \underline{e}_2 + \frac{\partial}{\partial x_3}(\underline{\underline{\sigma}}) \cdot \underline{e}_3$$

This is tricky because co-ordinate axis $(\underline{e}_r, \underline{e}_\theta)$ both change due to change in 'θ'.

Infact $\frac{\partial \underline{e}_r}{\partial \theta} = \underline{e}_\theta \Rightarrow \frac{\partial \underline{e}_1}{\partial x_2} = \underline{e}_2$

2

$\frac{\partial \underline{e}_\theta}{\partial \theta} = -\underline{e}_r \Rightarrow \frac{\partial \underline{e}_2}{\partial x_2} = -\underline{e}_1$

Rest all derivatives of co-ordinate axis are zero.

In fact, $\dfrac{\partial \underline{e}_i}{\partial x_1} = 0$ for all $i = 1, 2, 3$

(2) $\dfrac{\partial \underline{e}_i}{\partial x_3} = 0$ for $i = 1, 2, 3$

Now, $\dfrac{\partial}{\partial x_1}\left(\underline{\underline{\sigma}}\right)\cdot \underline{e}_1 = \dfrac{\partial}{\partial x_1}\left(\sigma_{ij}\, \underline{e}_i \otimes \underline{e}_j\right)\cdot \underline{e}_1$

$$= \left(\frac{\partial \sigma_{ij}}{\partial x_1}\, \underline{e}_i \otimes \underline{e}_j\right)\cdot \underline{e}_1$$

(2)

$$= \frac{\partial \sigma_{ij}}{\partial x_1}\, \underline{e}_i\, \delta_{j1} = \frac{\partial \sigma_{i1}}{\partial x_1}\, \underline{e}_i \quad \text{--- (i)}$$

Similarly, $\dfrac{\partial}{\partial x_3}\left(\underline{\underline{\sigma}}\right)\cdot \underline{e}_3 = \dfrac{\partial \sigma_{i3}}{\partial x_3}\, \underline{e}_i$ --- (ii)

$\dfrac{1}{r}\dfrac{\partial}{\partial x_2}\left(\underline{\underline{\sigma}}\right)\cdot \underline{e}_2$ is most tricky which we pursue now!

$$\Downarrow$$

$$\frac{1}{r}\left[\frac{\partial \sigma_{ij}}{\partial x_2}\, \underline{e}_i \otimes \underline{e}_j + \sigma_{ij}\left(\frac{\partial \underline{e}_i}{\partial x_2} \otimes \underline{e}_j + \underline{e}_i \otimes \frac{\partial \underline{e}_j}{\partial x_2}\right)\right]\cdot \underline{e}_2$$

$$= \frac{1}{r}\frac{\partial \sigma_{ij}}{\partial x_2}\underline{e}_i\,\delta_{j2} + \frac{\sigma_{ij}}{r}\left(\frac{\partial \underline{e}_i}{\partial x_2}\otimes\underline{e}_j + \underline{e}_i\otimes\frac{\partial \underline{e}_j}{\partial x_2}\right)\cdot\underline{e}_2$$

$$= \frac{1}{r}\frac{\partial \sigma_{i2}\underline{e}_i}{\partial x_2} + \frac{1}{r}\Big[\sigma_{1j}\,\underline{e}_2\otimes\underline{e}_j + \sigma_{2j}(-\underline{e}_1)\otimes\underline{e}_j + \sigma_{i1}\,\underline{e}_i\otimes\underline{e}_2 - \sigma_{i2}\,\underline{e}_i\otimes\underline{e}_1\Big]\cdot\underline{e}_2$$

$$= \frac{1}{r}\frac{\partial \sigma_{i2}}{\partial x_2}\underline{e}_i + \frac{1}{r}\Big[\sigma_{12}\underline{e}_2 - \sigma_{22}\underline{e}_1 + \sigma_{i1}\underline{e}_i\Big]$$

4

iii

Hence,
$$\underline{\nabla}\cdot\underline{\underline{\sigma}} = \frac{\partial \sigma_{i1}}{\partial x_1}\underline{e}_i + \frac{\partial \sigma_{i3}}{\partial x_3}\underline{e}_i + \frac{1}{r}\frac{\partial \sigma_{i2}}{\partial x_2}\underline{e}_i + \frac{1}{r}\Big[\sigma_{12}\underline{e}_2 - \sigma_{22}\underline{e}_1 + \sigma_{i1}\underline{e}_i\Big]$$

To get radial component $\Rightarrow$

$$(\underline{\nabla}\cdot\underline{\underline{\sigma}})\cdot\underline{e}_1 = \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{13}}{\partial x_3} + \frac{1}{r}\frac{\partial \sigma_{12}}{\partial x_2} + \frac{\sigma_{11}-\sigma_{22}}{r}$$

$$= \frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}$$

Similarly, get 'θ' and 'z' components.

$\downarrow$ $\quad$ $\downarrow$

$(\underline{\nabla}\cdot\underline{\underline{\sigma}})\cdot\underline{e}_2 \qquad (\underline{\nabla}\cdot\underline{\underline{\sigma}})\cdot\underline{e}_3$

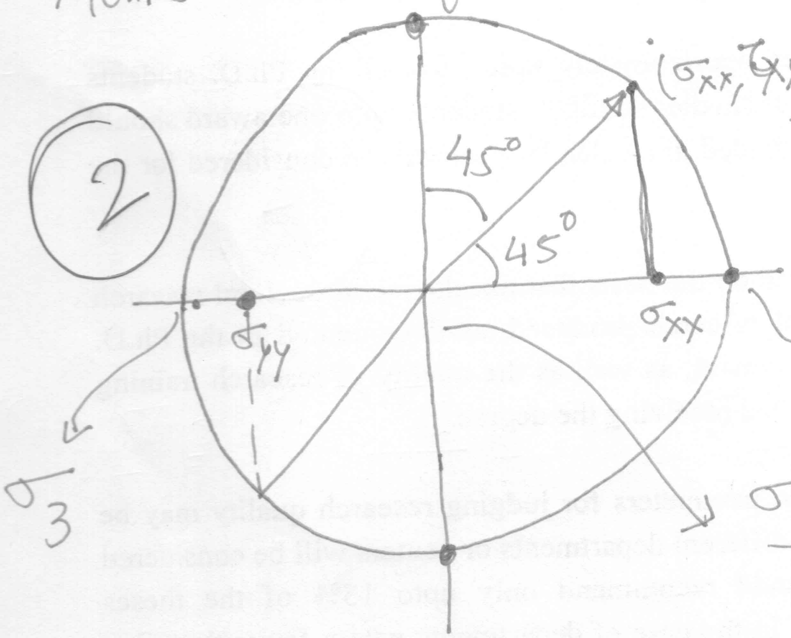
6 $\underline{\underline{\sigma}} = \begin{bmatrix} 4 & 4 & 0 \\ 4 & -4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ $\rightarrow$ this is already a principal stress ($\underline{e}_3$ being the principal plane)

1

We can thus use Mohr's circle idea!

(a) Mohr's circle for x-y plane.

2



$$\frac{\sigma_{xx}+\sigma_{yy}}{2} = \frac{4-4}{2} = 0.$$

radius $= 4\sqrt{2}$

We can very easily read from the circle:

$\sigma_1 = 4\sqrt{2}$, $\sigma_3 = -4\sqrt{2}$, $\sigma_2 = 3$

evident from matrix,

3

normal at an angle of 22.5° anti-clockwise from x axis (in x-y plane)

angle of $(90+22.5)^\circ$ from x axis

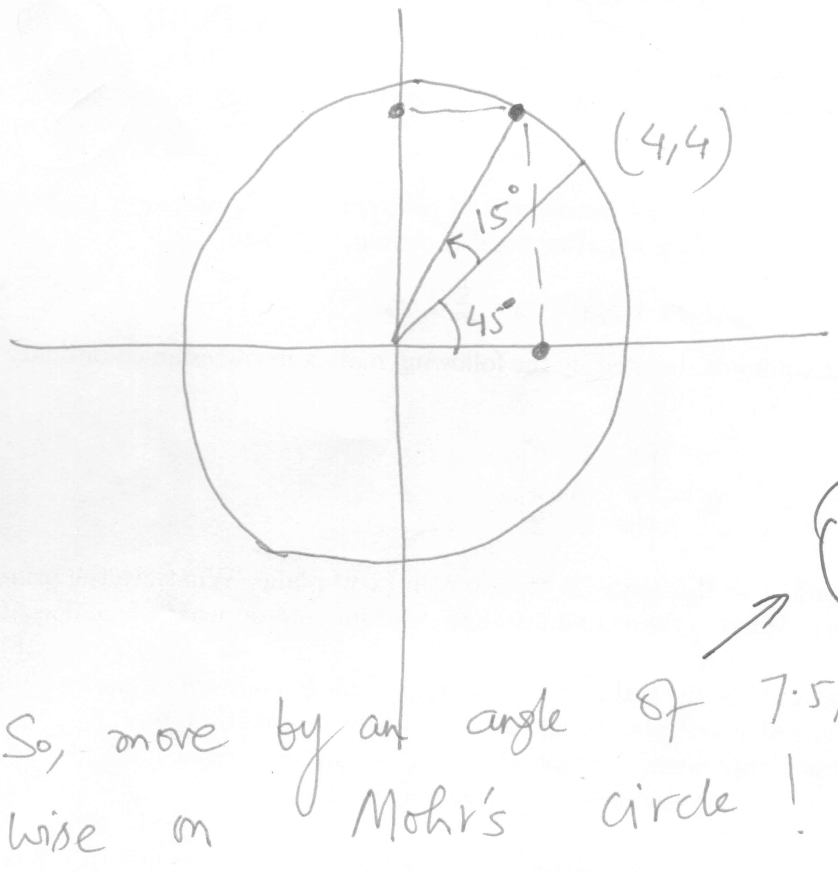
Z axis

Maximum shear can also be read directly from circle!

= radius of circle 2

= 45° (normal at an angle of 22.5° but in clock-wise direction)

(b) To find traction on plane at an angle of 7.5° (clockwise).



2

So, move by an angle of $7.5 \times 2 = 15°$ in anti-clock wise on Mohr's circle!

Thus,

$$\sigma = 4\sqrt{2}\cos 60° = 2\sqrt{2}$$ 2

$$\tau = 4\sqrt{2}\sin 60° = 2\sqrt{6}$$

(c) $\sigma_{oct} = I_1/3 = 1$ 1

$$\tau_{oct} = \frac{\sqrt{2}}{3}\left[I_1^2 - 3I_2 \right] \quad 1 \qquad I_2 = 32$$

$$\sqrt{2}/3\sqrt{9+96} = \sqrt{2/3}\sqrt{35}$$

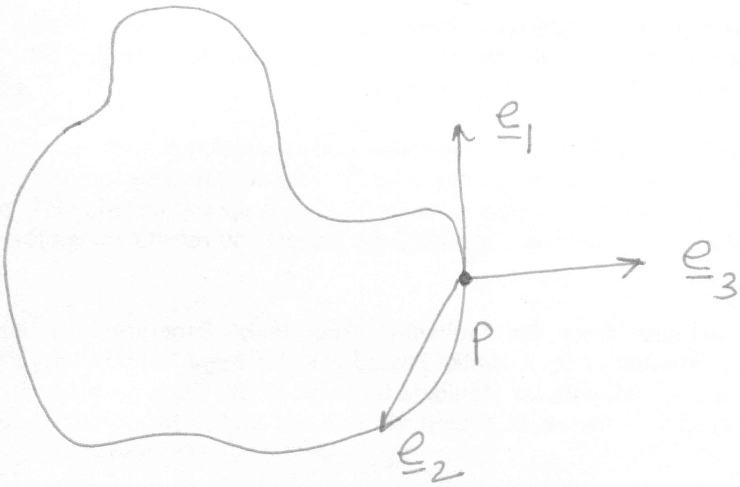
You can't get these values from Mohr's circle because the normal does not lie in x-y plane.

(d) $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}_{hydro} + \underline{\underline{\sigma}}_{dev}$.

$\underline{\underline{\sigma}}_{hydro} \rightarrow \frac{1}{3} I_1 \underline{\underline{I}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ 1

$\underline{\underline{\sigma}}_{dev} \rightarrow \underline{\underline{\sigma}} - \underline{\underline{\sigma}}_{hydro} = \begin{bmatrix} 3 & 4 & 0 \\ 4 & -5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ 1

(7)



Let us suppose $\underline{e}_3$ is the surface normal at 'P' then we already know traction on $\underline{e}_3$ plane,

3 $\underline{t}_3 = -P_{atom} \underline{e}_3 = \begin{bmatrix} 0 \\ 0 \\ -P_{atm} \end{bmatrix}$

However we have no knowledge about traction on $\underline{e}_1$ and $\underline{e}_2$ plane

So, $\underline{\underline{\sigma}} = \begin{bmatrix} X & X & 0 \\ X & X & 0 \\ 0 & 0 & -P_{atm} \end{bmatrix}$ → 2

2 → these are zero because cross-shears are equal!