PYL100: EM Waves and Quantum Mechanics
Wave-Particle Duality
Problem set 1

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September 26, 2017

Exercise 1.
The classical description of the hydrogen atom
A couple of semesters ago a PYL100 student mentioned that an accelerating charged particle shall emit electromagnetic radiation and thus progressively "slow down". Well, it turns that a non-relativistic, accelerating electric charge does radiate energy at a rate given by the **Larmor formula**,

$$\frac{dE}{dt} = \frac{q^2 a^2}{6\pi\epsilon_0 c^3} \tag{1}$$

where q is the electric charge and a is the magnitude of the acceleration. Let's consider the case of the hydrogen atom with an electron in a *circular* orbit around the proton. Eq. 1 suggests that the hydrogen is thus *unstable*.

1. Even though on continually losing energy the electron ought to spiral into the proton, show that the approximation of a circular orbit is reasonable.
2. How long will it take for the electron to spiral into the nucleus? Assume reasonable values for the radii of the atoms and the nucleus.
3. Compare the velocity of the electron with the velocity of light c. Use an orbital radius of 0.5 Å.
4. As the electron approaches the proton, what happens to its energy? Is there a minimum value of the energy the electron can have?

Solution 1.

1. We begin with the assumption that the electron orbit is *circular*. The Coulomb force between the electron and proton provides the centripetal force keeping the electron in orbit:

$$\frac{q^2}{4\pi\epsilon_0 r^2} = \frac{mv^2}{r} \tag{2}$$

which gives us:

$$v = \sqrt{\frac{q^2}{4\pi\epsilon_0 mr}} \tag{3}$$

Now, let's find the time taken T for each orbit, assuming the electron is travelling at constant speed $|\, v\,|$ in its circular orbit.

$$T = \frac{2\pi r}{v} \tag{4}$$

The energy lost during a single orbit is:

$$\begin{aligned} \Delta E = \frac{dE}{dt}T &= \frac{q^2\left(\frac{v^2}{r}\right)^2}{6\pi\epsilon_0 c^3}T \\ &= \frac{q^2 v^3}{3\epsilon_0 c^3 r} \end{aligned} \tag{5}$$

The energy lost above needs to be compared with the kinetic energy K of the electron in orbit:

$$K = \frac{mv^2}{2} = \frac{q^2}{8\pi\epsilon_0 r} \tag{6}$$

where we have used Eq. 3. Thus,

$$\frac{\Delta E}{K} = \frac{8\pi}{3}\left(\frac{v}{c}\right)^3 \ll 1 \tag{7}$$

which is true as long as the particle is non-relativistic, or $v \ll c$. Thus the energy lost per orbit ΔE is much less than the kinetic energy K of the electron in orbit, making the approximation of circular orbit quite reasonable!

2. The total energy E is a sum of the potential U and kinetic energy K. Thus,

$$\begin{aligned} E &= U + K \\ &= \frac{-q^2}{4\pi\epsilon_0 r} + \frac{mv^2}{2} \quad = \frac{-q^2}{8\pi\epsilon_0 r} \end{aligned} \tag{8}$$

Using the Larmor formula we get:

$$\frac{dE}{dt} = \frac{d}{dt}\left(\frac{-q^2}{8\pi\epsilon_0 r}\right) = -\frac{q^2 a^2}{6\pi\epsilon_0 c^3} \tag{9}$$

or,

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{1}{r}\right) &= \frac{4v^4}{3r^2c^3} \\
-\frac{1}{r^2}\frac{dr}{dt} &= \frac{q^4}{12\pi^2\epsilon_0^2 r^4 m^2 c^3} \\
\int_{r_i}^{r_f} r^2 dr &= \frac{q^4 dt}{12\pi^2\epsilon_0^2 m^2 c^3} \\
\frac{1}{3}(r_f^3 - r_i^3) &= \frac{q^4 t}{12\pi^2\epsilon_0^2 m^2 c^3} \\
t &= \frac{4\pi^2\epsilon_0^2 m^2 c^3}{q^4}(r_f^3 - r_i^3) \approx 1 \times 10^{-10}\,\mathrm{s}
\end{aligned}
\tag{10}
$$

where we have used $r_i = 1 \times 10^{-10}$ Å and $r_f = 1\,\mathrm{fm}$.

3. Using,

$$
v = \sqrt{\frac{q^2}{4\pi\epsilon_0 r}} = 2.52 \times 10^6\,\mathrm{m/s} \tag{11}
$$

Thus,

$$
\frac{v}{c} = \frac{2.52 \times 10^6\,\mathrm{m/s}}{3 \times 10^8\,\mathrm{m/s}} = 0.84\% \tag{12}
$$

4. As the electron gets closer and closer to the proton ($r \longrightarrow 0$), its energy $E \longrightarrow -\infty$. Thus there is no minimum energy unlike in quantum mechanics where we will see the existence of well-defined *ground state* or minimum energy.

Exercise 2.
Light as quanta/particles

1. Visible light has a wavelength in the range 400-700 nm. What are the energy and frequency of a photon of visible light?

2. Microwave ovens operate at roughly 2.5 GHz at a max power of 7.5×10^2 J/s. How many photons per second can they emit? What about a cell phone (4×10^{-1} J/s)?

3. How many such microwave photons does it take to warm a 200 mL glass of water by 10 °C? (The heat capacity of water is roughly 4184 $\mathrm{Jkg}^{-}1\mathrm{K}^{-}1$)

4. At a given power of an electromagnetic wave, do you expect a classical wave description to work better for radio frequencies, or for X-rays?

Matter as waves
Why doesn't the wave-like nature of matter manifest in our everyday experience? In order to answer this question, calculate the de Broglie wave lengths for:

1. A car of mass one metric ton and travelling at 60 km/h.
2. A jellybean weighing 10 g and moving with a speed of 20 km/h toward an open mouth.
3. An ^{87}Rb atom that has been cooled to a temperature of $T = 100\,\mu\text{K}$. Assume $KE = 1.5k_BT$.

Solution 2.
Light as quanta/particles

1. Using $E = \frac{hc}{\lambda}$ we get: 1.77 eV - 3.10 eV.
2. At 2.5 GHz each photon carries $E = h\nu = 1.66 \times 10^{-24}$ J, thus emitting 7.5×10^2 J/s$/1.66 \times 10^{-24}$ J $= 4.53 \times 10^{26}\,\text{s}^{-1}$. For the cell phone: $7.10 \times 10^{23}\,\text{s}^{-1}$.
3. $\Delta Q = mC_v\Delta T = 0.2 * 4184 * 10 = 8368$ J. Since each photon carries 1.66×10^{-24} J from above, we require: 5.04×10^{27} photons.
4. At the same power radio waves contain many more photons than an X-rays, and thus the former is much more amenable to a classical (or *statistical*) description.

Matter as waves

1. $\lambda_{dB} = \frac{h}{p} = 3.978 \times 10^{-38}$ m.
2. $\lambda_{dB} = 1.934 \times 10^{-32}$ m.
3. We first calculate $v = \sqrt{\frac{3k_BT}{m}} = \sqrt{\frac{3 \times 1.38 \times 10^{-23} \times 100 \times 10^{-6}}{87 * 1.67 \times 10^{-27}}} = 0.169$ m/s. $\lambda_{dB} = \frac{6.63 \times 10^{-34}}{87 * 1.67 \times 10^{-27} \times 0.169} \approx 27$ nm.

Exercise 3.
Double-slit interference with electrons (instead of light)

1. Electrons of momentum p fall normally on a pair of slits separated by a distance d. What is the distance, w, between adjacent maxima of the interference fringe pattern formed on a screen a distance D beyond the slits? (*You may assume that the width of the slits is much less than the electron de Broglie wavelength.*)
2. In an experiment performed by Jönsson in 1961, electrons were accelerated through a 50 kV potential towards two slits separated by a distance $d = 2 \times 10^{-6}$ m then detected on a screen $D = 0.35$ m beyond the slits. Calculate the electron's de Broglie wavelength, λ_{dB}, and the fringe spacing, w.

3. What values would d, D, and w take if Jönsson's apparatus were simply scaled up for use with visible light rather than electrons?

Solution 3.

1. The condition for constructive interference in double-slit interference is:

$$d \sin \theta_m = m\lambda \tag{13}$$

where m is the order of the maximum. If y_m is the position of maximum of order m, then:

$$\sin \theta_m = \frac{y_m}{D} \tag{14}$$

Thus, $w = y_{m+1} - y_m = \lambda \frac{D}{d}$ and using the de Broglie relation:

$$w = \frac{hD}{pd} \tag{15}$$

2. Using the energy of a free particle $\frac{p^2}{2m}$ and equating it to $q\Delta V$ we get $p = \sqrt{2mq\Delta V}$ giving us $\lambda_{dB} = \frac{h}{\sqrt{2mq\Delta V}}$. Plugging in the numbers we get $\lambda_{dB} = 5.5 \times 10^{-12}\,\mathrm{m}$, and $w = 9.6 \times 10^{-7}\,\mathrm{m}$.

3. The ratio of the wavelengths a given that light $\lambda \approx 550\,\mathrm{nm}$ is:

$$a = \frac{550\,\mathrm{nm}}{5.5 \times 10^{-12}\,\mathrm{m}} = 100000. \tag{16}$$

Thus,

$$\begin{aligned} d' &= ad &&= 20\,\mathrm{cm}. \\ D' &= aD &&= 35\,\mathrm{km}. \\ w' &= aw &&= 9.6\,\mathrm{cm}. \end{aligned} \tag{17}$$

and thus the screen would have to be at a distance D which is clearly impractical!