1 Vector Calculus

1.1 Double integrals

Let $f(x, y)$ be a real valued function defined over a domain $\Omega \subset I\!R^2$. To start with, let us assume that Ω be the rectangle $R = (a, b) \times (c, d)$. We partition the rectangle with node points (x_k, y_k), where

$$a = x_1 < x_2 < ...., x_n = b, \text{ and } c = y_1 < y_2 < ...., y_n = d.$$

Let R_{nm} be the small sub-rectangle with above vertices. Now we can define Upper and lower Riemann sum as

$$U(P_n, f) = \sum_{n,m} \sup_{R_{nm}} f(x, y)|R_{nm}|$$

and

$$L(P_n, f) = \sum_{n,m} \inf_{R_{nm}} f(x, y)|R_{nm}|$$

where $|R_{nm}|$ is the area of the rectangle R_{nm}. Here we may define the norm of partition P_n as $\|P_n\| = \max_i \sqrt{|x_i - x_{i-1}|^2 + |y_i - y_{i-1}|^2}$. Then by our understanding of definite integral we can define the upper, lower integrals and $f(x, y)$ is integrable if and only if

$$\inf\{U(P, f) : P\} = \sup\{L(P, f) : P\}.$$

The definite integral is defined as

$$\iint_\Omega f(x, y)dA = \inf\{U(P, f) : P\} = \sup\{L(P, f) : P\}.$$

It can be shown that $\displaystyle\iint_\Omega f(x, y)dA = \lim_{\|P_n\|\to 0} S(P_n, f)$, where $S(P_n, f) = \displaystyle\sum_{n,m} f(x_k, y_k)|R_{nm}|$.

We have the following **Fubini's theorem** for rectangle:
Suppose $f(x, y)$ is integrable over $R = (a, b) \times (c, d)$, then

$$\iint_R f(x, y)dA = \int_a^b \left(\int_c^d f(x, y)dy \right) dx = \int_c^d \left(\int_a^b f(x, y)dx \right) dy$$

In case of $f(x, y) \geq 0$ we may interpret this as the volume of the solid formed by the surface $z = f(x, y)$ over the rectangle R. This is precisely the "sum" of areas of the cross section $A(x) = \int_c^d f(x, y)dy$ between $x = a$ and $x = b$. Since x varies over all of (a, b), this sum is nothing but the integral $\int_a^b A(x)dx$.

For any general bounded domain Ω, we can divide the domain into small sub domains Ω_k and consider the upper, lower sum exactly as above by replacing R_{nm} by Ω_k. Then a function $f(x,y) : \Omega \to I\!R$ is integrable if the supremum of lower sums and infimum of upper sums exist and are equal. We may define

$$\iint_\Omega f(x,y)dA = \lim_{\|P_n\|\to 0} \sum_k f(x_k, y_k)|\Omega_k|$$

The **basic properties** of the definite integral like integrability of $f \pm g$, kf and domain decomposition theorems holds in this case also.

We call a domain as y**-regular** if $\Omega = \{(x,y) : a \le x \le b, g_1(x) \le y \le g_2(x)\}$ for some continuous functions g_1, g_2. Similarly, Ω is x**-regular** if $\Omega = \{(x,y) : c \le y \le d, h_1(y) \le x \le h_2(y)\}$. A domain is called regular if it is either x-regular or y-regular. Then we have the following Fubini's theorem for regular domains.

Theorem 1.1 *Let $f(x,y)$ be continuous over Ω.*

1. *If Ω is y-regular, i.e.,$\Omega = \{(x,y) : a \le x \le b, g_1(x) \le y \le g_2(x)\}$. Then*

$$\iint_R f(x,y)dA = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)dy \right) dx.$$

2. *If Ω is x-regular, i.e.,$\Omega = \{(x,y) : c \le y \le d, h_1(y) \le x \le h_2(y)\}$. Then*

$$\iint_R f(x,y)dA = \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y)dx \right) dy.$$

This theorem basically says that if a function is integrable over a domain Ω, then the value of integral is does not depend on order of integration. That is we can integrate with respect to x first followed by y or vice versa.

Example 1: Evaluate the integral $\int_\Omega (x+y+xy)dA$ where Ω is the triangle bounded by $y = 0, x = 1$ and $y = x$.

Solution: The triangle is regular in both x and y. The given triangle is

$$\{(x,y) : 0 \le x \le 1, 0 \le y \le x\} = \{(x,y) : 0 \le y \le x, y \le x \le 1\}$$

Therefore, taking it as $y-$ regular we see that the domain is bounded below by $y = g_1(x) = 0$ and above by $y = g_2(x) = x$ over $x = 0$ to $x = 1$. Hence,

$$\begin{aligned}\iint_\Omega (x+y+xy)dA &= \int_{x=0}^{x=1} \left(\int_{y=0}^{x} (x+y+xy)dy \right) dx \\ &= \int_0^1 (x^2 + \frac{x^2}{2} + \frac{x^3}{2})dx = \frac{15}{24}\end{aligned}$$

Similarly, taking it as $x-$ regular, we see that the domain is bounded below y $x = h_1(y) = y$ and above by $x = h_2(y) = 1$ over $y = 0$ to 1. Hence

$$\iint_\Omega (x+y+xy)dA = \int_{y=0}^{1} \left(\int_{x=y}^{1} (x+y+xy)dx \right) dy = \frac{15}{24}$$

Example 2: Evaluate the integral $\iint_\Omega (2+4x)dA$ where Ω is the domain bounded by $y = x$ and $y = x^2$.
Solution:

$$\begin{aligned}\iint_\Omega (2+4x)dA &= \int_{x=0}^{1} \left(\int_{y=x^2}^{x} (2+4x)dy \right) dx \\ &= \int_0^1 (2x + 2x^2 - 4x^3)dx = 2/3\end{aligned}$$

On the other hand, this is also equal to $\displaystyle\int_{y=0}^{1} \left(\int_{x=y}^{\sqrt{y}} (2+4x)dx \right) dx.$

Remark 1.1 *1. When$f(x,y) = 1$, then we approximate the area of Ω as $A \sim \sum_k \Omega_k = \sum_k f(x_k, y_k)|\Omega_k|$ where $f = 1$. By the definition of Riemann integral this sum converges to $\iint_\Omega dA$ as $\|P_n\| \to 0$.*

2. As discussed in the beginning, when $f(x,y) \geq 0$, the $\iint_\Omega f(x,y)dA$ is the volume of the solid bounded above by $z = f(x,y)$ and below by Ω.

Example: Find the area bounded by $y = 2x^2$ and $y^2 = 4x$.
Solution: The two parabola's intersect at $(0,0)$ and $(1,1)$. Hence the area is

$$A = \int_0^1 \int_{2x^2}^{2\sqrt{x}} dydx = \int_0^1 (2\sqrt{x} - 2x^2)dx = \frac{2}{3}.$$

Example: Find the volume of the solid under the paraboloid $z = x^2 + y^2$ over the bounded domain R bounded by $y = x, x = 0$ and $x + y = 2$.

Solution: The domain of integration R is Y-regular bounded above by $x+y=2$ and below by $y=x$ with x varying over $(0,1)$.

$$V=\iint_R (x^2+y^2)dA=\int_0^1\left(\int_{y=x}^{y=2-x}(x^2+y^2)dy\right)dx$$
$$=\int_0^1 \frac{y^3}{3}+yx^2\Big|_{y=x}^{y=2-x}dx=\int_0^1(\frac{1}{3}((2-x)^3-x^3)+x^2(2-2x))dx$$

Example: Find the volume of the solid bounded above by the surface $z=x^2$ and below by the plane region R bounded by the parabola $y=2-x^2, y=x$.

Solution: The points of intersection of $y=x, y=2-x^2$ are $x=-2,1$. So $R=\{(x,y): -2\le x\le 1, x\le y\le 2-x^2\}$. Therefore,

$$V=\int_{x=-2}^1\int_{y=x}^{2-x^2}x^2dydx=\int_{-2}^1 x^2(2-x^2-x)dx$$

Change of order

Consider the evaluation of integral $\iint_R \frac{\sin x}{x}dA$ over the triangle formed by $y=0, x=1$ and $y=x$. Since the can be extended as continuous function over R, by the basic properties of Riemann integral the function is integrable. Now by Fubini's theorem, the value of integral does not depend on the order of integration. As we noted earlier R is regular in x and y. If we take it as x regular, then $R=\{(x,y): 0\le y\le 1, y\le x\le 1\}$ and try to evaluate the integral, then

$$\iint_R \frac{\sin x}{x}dA=\int_0^1\left(\int_{x=y}^1\frac{\sin x}{x}dx\right)dy.$$

This is singular integral and difficult to evaluate.

But when we consider R to be y-regular, we see that $R=\{(x,y):\ 0\le x\le 1, 0\le y\le x\}$. Then the given integral is

$$\iint_R \frac{\sin x}{x}dA=\int_0^1\int_{y=0}^x\frac{\sin x}{x}dydx=\int_0^1 \sin x dx=1-\cos 1$$

At times this technique can be used to evaluate some complicated definite integrals, for example,

Example: Evaluate the integral $\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}dx,\ a,b>0$.

Solution: This integral is equivalent to

$$\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}dx=\int_0^\infty\left(\int_a^b e^{-xy}dy\right)dx$$

The domain of integration is the infinite strip $\{(x, y) : 0 \leq x \leq \infty,\ a \leq y \leq b\}$. Changing the order of integration, we get

$$\int_0^\infty \left(\int_a^b e^{-xy} dy \right) dx = \int_{y=a}^b \left(\int_0^\infty e^{-xy} dx \right) dy$$
$$= \ln \frac{b}{a}$$

Double integrals in Polar form

Suppose we are given a bounded region whose boundaries are given by polar equations, say $f_1(r, \theta) = 0, f_2(r, \theta) = 0$. Then we divide the region into smaller "polar rectangles" whose sides have constant r, θ values.

Suppose $f(r, \theta)$ is defined over a region R defined using the polar equations, $R : \alpha \leq \theta \leq \beta, g_1(\theta) \leq r \leq g_2(\theta)$. Then we divide the r range by $\Delta r, 2\Delta r, ..., m\Delta r$ and $\alpha, \alpha + \Delta\theta, ...\alpha + m'\Delta\theta = \beta$. Let ΔA be the polar rectangle with sides $r_k - \Delta r/2, r_k + \Delta r/2$ and $\alpha + k\Delta\theta, \alpha + (k+1)\Delta\theta$. Then we define the Riemann sum as

$$S_n = \sum_k f(r_k, \theta_k)\Delta A_k.$$

The area of small "polar rectangle" A_k is

$$\Delta A_k = \text{ area of outer sector} - \text{area of inner sector} = r_k \Delta r \Delta\theta.$$

As $\|P_n\| \to 0$,we get

$$S_n = \sum_k f(r_k, \theta) r_k \Delta r \Delta\theta \to \iint_R f(r, \theta) r dr d\theta.$$

Example: Find the area common to the cardioids $r = 1 + \cos\theta$ and $r = 1 - \cos\theta$.

Solution: Since the region is symmetric with respect to x -axis and y-axis, the required area is

$$A = 4 \int_{\theta=0}^{\pi/2} \int_{r=0}^{1-\cos\theta} r dr d\theta$$
$$= 4 \int_0^{\pi/2} \frac{1}{2}(1 - 2\cos\theta + \cos^2\theta) d\theta = 2(\frac{\pi}{2} - 2 + \int_0^{\pi/2} \cos^2\theta d\theta)$$

Example: Evaluate $\displaystyle\iint_R 3y dA$ where R is the region bounded below by x-axis and above by the cardioid $r = 1 - \cos\theta$.

Solution: The given integral is equivalent to

$$\iint_R 3y dA = \int_{\theta=0}^{\pi} \int_{r=0}^{1-\cos\theta} r\sin\theta r dr d\theta.$$

Example: Evaluate $I = \displaystyle\int_0^\infty e^{-x^2} dx$.

Solution: Recall that $2I = \Gamma(\frac{1}{2})$. Using the Fubini's theorem, we may write

$$I^2 = \left(\int_0^\infty e^{-x^2} dx\right)\left(\int_0^\infty e^{-y^2} dy\right) = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dA$$

Where the integration is over the first quadrant $(0, \infty) \times (0, \infty)$. So representing this in polar form we integrate over $\{(r, \theta) : 0 \leq r < \infty, 0 \leq \theta \leq \frac{\pi}{2}\}$. Therefore, the above integral becomes

$$\int_0^{\pi/2} \int_0^\infty e^{-r^2} r dr d\theta = \frac{\pi}{4}.$$

1.2 Triple (Volume) integrals

Let $f(x, y, z)$ be a real valued function defined over a closed and bounded region of space $\mathbb{R}^3$. For example the solid ball or rectangular box. Now we want to define the definite integral of $f(x, y, z)$ over such regions.

We partition the region by small planes parallel to the coordinate axes. Then we obtain small rectangular cubes over which the function will be approximated by $f(x_k y_k, z_k)$. We form the Riemann sum

$$S_n = \sum_k f(x_k, y_k, z_k)|\Omega_k|,$$

where $|\Omega_k|$ is the volume of the small rectangle. Now by our understanding of Riemann sums we choose refinement of partitions in such way that $\max_k |\Omega_k| \to 0$. Then we obtain the definite integral as

$$\iiint_\Omega f(x, y, z) dV = \lim_{n\to\infty} S_n.$$

Evaluation of integrals in three dimensions is done again using Fubini's theorem. In this case again Fubini's theorem states

Theorem 1.2 *Suppose $f(x, y, z)$ is integrable over $\Omega \subset \mathbb{R}^3$, then*

$$\begin{aligned}\iiint_\Omega f(x, y, z) dV = \int_x \int_y \int_z f(x, y, z)\ dz\ dy\ dx &= \int_x \int_z \int_y f(x, y, z)\ dx\ dz\ dy \\ \int_z \int_x \int_y f(x, y, z)\ dy\ dx\ dz &= \int_z \int_y \int_x f(x, y, z)\ dx\ dy\ dz \\ = \int_y \int_x \int_z f(x, y, z)\ dz\ dx\ dy &= \int_y \int_z \int_x f(x, y, z)\ dy\ dz\ dx\end{aligned}$$

To evaluate the triple integrals we follow the following steps:

1. Draw a line parallel to z axis that passes through the point (x, y) of R where R is the projection of Ω onto $I\!R^2$.

2. Identify the upper surface and lower surface through which this line passes at most once.

3. Identify the upper curve and lower curve of the projection R and limits of integration.

It is easy to see from the definition, the volume of Ω is

$$V = \lim_{k\to\infty} \sum_k |\Omega_k| = \lim_{k\to\infty} \sum_k 1\ |\Omega_k| = \iiint_\Omega 1\ dV$$

Example: Find the volume of the region bounded by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.

Solution: The volume is $V = \iiint_\Omega dzdydx$, where Ω is bounded above by the surface $z = 8 - x^2 - y^2$ and below by the surface $z = x^2 + 3y^2$. Therefore, the limits of z are from $z = x^2 + 3y^2$ to $z = 8 - x^2 - y^2$.

The Projection of Ω on xy-plane is the solution of

$$8 - x^2 - y^2 = x^2 + 3y^2 \quad\Longrightarrow\quad x^2 + 2y^2 = 4.$$

Therefore the limits of x and y are to be determined by $R : x^2 + 2y^2 = 4$. Hence

$$\begin{aligned}
V &= \iint_R \int_{y=x^2+3y^2}^{8-x^2-y^2} dzdA \\
&= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} (8 - 2x^2 - 4y^2)dydx \\
&= \int_{-2}^{2} \left((8 - x^2)y - \frac{4}{3}y^3\right)\Bigg|_{y=-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \\
&= \frac{4\sqrt{2}}{3} \int_{-2}^{2} (4 - x^2)^{3/2}dx = 8\pi\sqrt{2}.
\end{aligned}$$

Example: Find the volume of the region bounded by $x + z = 1$, $y + 2z = 2$ in the first quadrant.

Solution: Draw line parallel to z-axis and note that the upper surfaces are: $2z + y = 2$ over triangle bounded by $x = 0, y = 1y = 2x$ and $z = 1 - x$ over the triangle bounded by $y = 0, x = 1, y = 2x$. Therefore,

$$V = \int_{y=0}^{2} \int_{x=0}^{y/2} \int_{z=0}^{\frac{2-y}{2}} dz\ dx\ dy + \int_{x=0}^{1} \int_{y=0}^{2x} \int_{z=0}^{1-x} dz\ dy\ dx$$

On the other hand, by first drawing the line parallel to x-axis, we get

$$V = \int_{z=0}^{1} \int_{y=0}^{2-2z} \int_{x=0}^{1-z} dx\ dy\ dz$$

Taking the line parallel to y-axis we get

$$V = \int_{x=0}^{1} \int_{z=0}^{1-x} \int_{y=0}^{2-2z} dy\ dz\ dx$$

Example: (Order of integration) Evaluate $\displaystyle\int_{z=0}^{4} \int_{y=0}^{1} \int_{x=2y}^{2} \frac{2\cos(x^2)}{\sqrt{z}} dx\ dy\ dz.$

Solution: Note that the projection of Ω onto xyplane is the triangle bounded by $y = 0, x = 2$ and $x = 2y$. So changing the order of integration in x and y, we get

$$\begin{aligned} I &= \int_{z=0}^{4} \int_{x=0}^{2} \int_{y=0}^{x/2} \frac{2\cos(x^2)}{\sqrt{z}} dy\ dx\ dz. \\ &= \int_{z=0}^{4} \int_{x=0}^{2} \frac{x\cos(x^2)}{\sqrt{z}} dx\ dz = 2\sin 4. \end{aligned}$$

Substitutions in multiple integrals

Suppose a domain G in uv-plane is transformed onto a domain Ω of xy-plane by a transformation $x = g(u, v), y = h(u, v)$. Then any function of x, y may be written as a function of u, v. Then the relation between the double integral over G and Ω is

$$\iint_{\Omega} f(x, y) dx\ dy = \iint_{G} f(g(u, v), h(u, v)) |J(u, v)| du\ dv$$

where J is the Jacobian given by

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

Example: Evaluate the integral $I = \displaystyle\int_{0}^{4} \int_{y/2}^{1+\frac{y}{2}} \frac{2x - y}{2} dxdy.$

Solution: The domain of integration is a parallelogram with vertices $(0, 0), (1, 0), (3, 4)$ and $(2, 4)$. One has to divide the domain into 3 domains. Instead we can take the transformation $u = \frac{2x-y}{2}, v = \frac{y}{2}$. Then the inverse transformation is $x = u + v, y = 2v$. Then

$$J = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2.$$

Under this transformation, the parallelogram is transformed into cube with vertices $(0,0)$, $(1,0)(1,2)$ and $(0,2)$. Now by change of variable formula

$$I = \iint f(u+v, 2v)2dudv = \int_0^2 \int_0^1 2udu\ dv = 2.$$

Example: Evaluate the integral $I = \displaystyle\int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2 dA.$

Solution: The given domain is the triangle bounded by $x=0, y=0$ and $x+y=1$. In this case the integrand is complicated....so we can take transformation $u = x+y$ and $v = y-2x$. Under this transformation, the given triangle will be transformed into triangle bounded by $v = u, v = -2u$ and $u = 1$. The inverse of this transformation is $x = \frac{u-v}{3}$ and $y = \frac{2u+v}{3}$. Hence the Jacobian

$$J = \begin{vmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{vmatrix} = 1/3.$$

Hence

$$I = \int_0^1 \int_{v=-2u}^{u} \sqrt{u} v^2 dv\ du$$

Example: Evaluate the integral $I = \iint_R \frac{dA}{(2-x^2-y^2)^2}$ over $R : x^2+y^2 \leq 1$.

Solution: Taking the transformation $x = r\cos\theta, y = r\sin\theta$, we get

$$J = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

By substitution formula,

$$I = \int_0^{2\pi} \int_{r=0}^{1} \frac{r\ dr\ d\theta}{(2-r^2)^2} = 2\pi \int_1^2 \frac{dt}{2t^2} = \pi/2.$$

Substitution formula for triple integrals

As discussed above suppose a three dimensional domain G is transformed onto a domain D with a transformation $x = x(u,v,w), y = y(u,v,w), z = z(u,v,w)$, then

$$\iiint_D f(x,y,z)dV = \iiint_G F(u,v,w)|J(u,v,w)|dV$$

where

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}.$$

The main idea of the proof is as follows. Let $(u,v),(u+\Delta u,v),(u+\Delta u,v+\Delta v)$ and $(u,v+\Delta v)$ be the vertices of the rectangle in the $uv-$plane. Let ΔA_k be its area element. Under the transformation this points are mapped to $(x_1,y_1)=(g(u,v),h(u,v)),(x_2,y_2)=(g(u+\Delta u,v),h(u+\Delta u,v)),(x_3,y_3)=(g(u+\Delta u,v+\Delta v)$ and $(x_4,y_4)=(g(u,v+\Delta v),h(u,v+\Delta v))$. Then by Taylor's theorem

$$g(u+\Delta u,v)=g(u,v)+\frac{\partial g}{\partial u}\Delta u+o((\Delta u)^2)$$

$$g(u+\Delta u,v+\Delta v)=g(u,v)+\frac{\partial g}{\partial u}\Delta u+\frac{\partial g}{\partial v}\Delta v+o((\Delta u)^2)+o((\Delta v)^2)$$

Then the area of the "rectangle" in xy-plane $\Delta\tilde{A}_k$ is

$$\begin{aligned}\Delta\tilde{A}_k &\approx |(x_3-x_1)(y_3-y_1)-(x_3-x_2)(y_3-y_2)|\\ &\approx |J|\Delta u\Delta v+o((\Delta u)^2)+o((\Delta v)^2)\end{aligned}$$

Taking this as the area in the Riemann sum of $f(x,y)$ we get the required formula.

Example: Evaluate $\iiint_\Omega (x^2y+3xyz)dV$ where $R=\{(x,y,z):1\le x\le 2,0\le xy\le 2,0\le z\le 1\}$.
solution: We take the transformation $u=x,v=xy$ and $w=z$. Then the planes $x=1,2$ transforms to $u=1,2$. The plane $y=0$ transforms to $v=0$. The surface $xy=2$ transforms to $v=2$. Then the Jacobian J is

$$\frac{1}{J}=\begin{vmatrix}1&0&0\\ y&x&0\\ 0&0&1\end{vmatrix}=x=u.$$

Now by substitution formula,

$$\begin{aligned}I&=\int_{u=1}^{2}\int_{v=0}^{2}\int_{w=0}^{1}(uv+3vw)\frac{1}{u}dw\ dv\ du\\ &=\int_1^2\int_0^2(v+\frac{3v}{2u})dv\ du\\ &=\int_1^2(2+\frac{3}{u})du=2+3\ln 2.\end{aligned}$$

Cylindrical coordinates: A point P in the space $(\mathbb{R}^3)$ is represented by (r,θ,z) where r,θ are polar coordinates of the projection of P on to xy-plane and z is the z distance of the projection from P. When we take the transformation $x = r\cos\theta, y = r\sin\theta, z = z$, the Jacobian is

$$J = \begin{vmatrix} x_r & x_\theta & x_z \\ y_r & y_\theta & y_z \\ z_r & z_\theta & z_z \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

Example: Find the volume of the cylinder $x^2 + (y-1)^2 = 1$ bounded by $z = x^2 + y^2$ and $z = 0$.

Solution: Drawing a line parallel to z axis, we see that the limits of z are from 0 to x^2+y^2 and the projection onto xy-plane is the disc:$R : x^2 + (y-1)^2 \le 1$. Therefore,

$$V = \iint_R \int_{z=0}^{x^2+y^2} dzdA$$

Now taking the cylindrical coordinates $x = r\cos\theta, y = r\sin\theta, z = z$ we get the projection to be

$$r^2\cos^2\theta + r^2\sin^2\theta - 2r\sin\theta = 0$$

$$i.e., \;\; r(r - 2\sin\theta) = 0 \implies r = 0 \;\text{ to } r = 2\sin\theta$$

$$\begin{aligned} V &= \int_{\theta=0}^{\pi}\int_{r=0}^{2\sin\theta}\int_{z=0}^{r^2} rdz\ dr\ d\theta \\ &= \int_{\theta=0}^{\pi}\int_{r=0}^{2\sin\theta} r^3 dr\ d\theta \\ &= 4\int_0^{\pi}\sin^4\theta d\theta = \frac{5\pi}{4} \end{aligned}$$

Spherical polar coordinates: A point P in the space is represented by (ρ, θ, ϕ) where ρ is the distance of P from the origin, ϕ is the angle made by the ray OP with positive z axis and θ is the angle made by the projection of P (onto xy-plane) with positive x-axis. So it is not difficult to see that the relation with cartesian coordinates: The projection of P on xy-plane has polar representation: $x = r\cos\theta, y = r\sin\theta$ where r is the distance of the projected point to origin. Therefore $r = \rho\sin\phi$. From the definition of ϕ it is easy to see that $z = \rho\cos\phi$ and

$$x = \rho\sin\phi\cos\theta,\ y = \rho\sin\phi\sin\theta,\ z = \rho\cos\phi$$

The Jacobian in this case is

$$J = \begin{vmatrix} x_\rho & x_\theta & x_\phi \\ y_\rho & y_\theta & y_\phi \\ z_\rho & z_\theta & z_\phi \end{vmatrix} = \begin{vmatrix} \rho\sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \rho\sin\phi\sin\theta & r\cos\theta\sin\phi & 0 \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix} = \rho^2\sin\phi$$

To find limits of integration in spherical coordinates,

1. Draw a ray from the origin to find the surfaces $\rho = g(\theta, \phi), \rho = g_2(\theta, \phi)$ where it enters the region and leaves the region.

2. Rotate this ray away and towards z-axis to find the limits of ϕ

3. Identify the projection R of the domain on the xy-plane and polar form of R to write the limits of θ.

Example: Evaluate $\displaystyle\iiint_\Omega \frac{dV}{\sqrt{1+x^2+y^2+z^2}}$ where Ω is the unit ball $x^2+y^2+z^2 \leq 1$.
Solution: Going to spherical polar coordinates $x = \rho \sin\phi \cos\theta,\ y = \rho \sin\phi \sin\theta,\ z = \rho \cos\phi$, we get

$$
\begin{aligned}
I =& \int_{\rho=0}^{1} \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \frac{\rho^2}{\sqrt{1+\rho^2}} \sin\phi \ d\phi \ d\theta \ d\rho \\
=& (2\pi \times 2) \int_0^1 \frac{\rho^2}{\sqrt{1+\rho^2}} d\rho = 4\pi(\sqrt{2} - \frac{1}{2}\ln(\sqrt{2}+1)).
\end{aligned}
$$

Example: Evaluate $I = \displaystyle\iiint_\Omega x dV$ where Ω is the part of the ball $x^2+y^2+z^2 \leq 4$ in the first octant.
Solution: Going to cylindrical coordinates, $x = \rho \sin\phi \cos\theta,\ y = \rho \sin\phi \sin\theta,\ z = \rho \cos\phi$. Since the domain is the ball of radius 4, we see that the limits of ρ are from 0 to 2. Again since it is cut by the xy-plane below, ϕ varies from 0 to $\pi/2$. The projection is the circle in the first quadrant with radius 2. So θ varies from 0 to $\pi/2$. Hence,

$$
\begin{aligned}
I =& \int_{\rho=0}^{2} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi/2} \rho \sin\phi \cos\theta \ (\rho^2 \sin\phi) \ d\rho \ d\theta \ d\phi \\
=& 4 \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi/2} \sin^2\phi \cos\theta d\phi \ d\theta = \pi
\end{aligned}
$$

Surface Area and Surface integrals
Consider a surface S defined with $f(x, y, z) = c$. Let R be its projection on xy-plane. Assume that this projection is **one-one, onto**. Let R_k be a small rectangle with area ΔA_k and let $\Delta\sigma_k$ be the piece of surface above this rectangle. Let ΔP_k be the tangent plane at (x_k, y_k, z_k) of the surface $\Delta\sigma_k$. Now consider the parallelogram with ΔP_k and ΔA_k as upper and lower planes of the parallelogram. We approximate the area of the surface with the area of the tangent plane ΔP_k.

Now let $\hat{p}$ be the unit normal to the plane containing R_k and ∇f is the normal to the surface. Let u_k, v_k be the vectors along the sides of the tangent plane ΔP_k. Then the area of ΔP_k is $|u_k \times v_k|$ and $u_k \times v_k$ is the normal vector to ΔP_k. Thus ∇f and $u_k \times v_k$ are both normals to the tangent plane ΔP_k.

The angle between the plane ΔA_k and ΔP_k is same as the angle between their normals. i.e., the angle between $\hat{p}$ and $u_k \times v_k$. From the geometry, the area of the projection of this tangent plane is $|(u_k \times v_k) \cdot \hat{p}|$(proof of this can be seen in Thomas calculus Appendix 8). i.e.,

$$\Delta A_k = |(u_k \times v_k) \cdot \hat{p}| = |(u_k \times v_k)||\hat{p}| \cos(\text{angle between } (u_k \times v_k) \text{ and } \hat{p})$$

In other words,

$$\Delta P_k|\cos\gamma_k| = \Delta A_k \;\; or \; \Delta P_k = \frac{\Delta A_k}{|\cos\gamma_k|}$$

where γ_k = angle between $(u_k \times v_k)$ and $\hat{p}$. This angle can be calculated easily by noting that ∇f and $u_k \times v_k$ are both normals to the tangent plane.

(This formula is simple in case of straight lines: Let OP be the line from origin and let R be the projection of P on x-axis. Then $OR = OP\cos\gamma$ where γ is the angle between OP and OR. Now imagine the Area of plane is nothing but "sum" of lengths of lines.)

So

$$|\nabla f \cdot \hat{p}| = |\nabla f||\hat{p}||\cos\gamma_k|$$

Therefore,

$$\text{Surface Area} \approx \sum_k \Delta P_k = \sum_k \frac{|\nabla f|}{|\nabla f \cdot \hat{p}|}\Delta A_k$$

This sum converges to

$$\text{Surface Area} = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \hat{p}|} dA$$

where R is the projection of S on to one of the planes and $\hat{p}$ is the unit normal to the plane of projection.

Example: Find the surface area of the curved surface of paraboloid $z = x^2 + y^2$ that is cut by the plane $z = 2$.

Solution: The equation of surface is $f(x, y, z) = z - x^2 - y^2 = 0$. Clearly this is one-one from xy-plane to $\mathbb{R}^3$. So the projection of the surface $\{(x, y, x^2 + y^2) : x^2 + y^2 \leq 2\}$ is the disc $R : x^2 + y^2 \leq 2$. Since the plane of projection is xy-plane, $\hat{p} = \hat{k}$. Hence

$$\nabla f = -2x\hat{i} - 2y\hat{j} + \hat{k}$$

$$\begin{aligned} S &= \iint_R \frac{|\nabla f|}{|\nabla f \cdot \hat{p}} dA \\ &= \iint_R \sqrt{4x^2 + 4y^2 + 1}\;\; dA \end{aligned}$$

Going to polar coordinates $x = r\cos\theta, y = r\sin\theta$,

$$S = \int_0^{2\pi} \int_{r=0}^{\sqrt{2}} \sqrt{1+4r^2}r\ dr\ d\theta = 13\pi$$

Example: Find the surface of the cap obtained by cutting the hemisphere $x^2+y^2+z^2 = 2$ by the cone $z = \sqrt{x^2+y^2}$.

Solution: The equation of surface is $f(x,y,z) = x^2+y^2+z^2-2 = 0$ and we can take the projection onto xy-plane. So $\hat{p} = \hat{k}$. The projection is obtained by solving $x^2+y^2+z^2 = 2, z = \sqrt{x^2+y^2}$. i.e., $R = x^2+y^2 = 1$.

$$\nabla f = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$|\nabla f \cdot \hat{p}| = 2z = 2\sqrt{2-x^2-y^2}$$

Therefore, using polar coordinates $x = r\cos\theta, y = r\sin\theta$,

$$\begin{aligned} S &= \iint_R \frac{\sqrt{2}}{\sqrt{2-x^2-y^2}} dA \\ &= \int_0^{2\pi} \int_0^1 \frac{\sqrt{2}}{\sqrt{2-r^2}} r\ dr\ d\theta = 2\pi(2-\sqrt{2}). \end{aligned}$$

Surface Integrals

Let $g(x,y,z)$ be a function defined over a surface S. Then we can think of integration of g over S. Suppose, a surface S is heated up, we have a temperature distributed over this surface. Let $T(x,y,z)$ be the temperature at (x,y,z) of the surface. Then we can calculate the total temperature on S using the Riemann integration.

Let R be the projection of S on the plane. We partition R into small rectangles A_k. Let ΔS_k be the surface above the ΔA_k. We approximate this surface area element with its tangent plane ΔP_k. As we refine the rectangular partition this ΔP_K approximated the ΔS_k. Then the total temperature may be approximated as

$$\sum_k g(x_k, y_k, z_k)\Delta P_k = \sum_k g(x_k, y_k, z_k)\frac{\Delta A_k}{|\cos\gamma_k|} = \sum_k g(x_k, y_k, z_k)\frac{|\nabla f|}{|\nabla f \cdot \hat{p}|} dA$$

where $\hat{p}$ is the unit normal to R or the plane of projection. Now taking limit $n \to \infty$, we get

$$\iint_S g(x,y,z) dS = \iint_R g(x,y,z)\frac{|\nabla f|}{|\nabla f \cdot \hat{p}|} dA$$

If the surface is defined as $f = z - h(x,y) = 0$, then

$$\iint_S g(x,y,z) dS = \iint_R g(x,y,h(x,y))\frac{|\nabla f|}{|\nabla f \cdot \hat{k}|} dA.$$

Example: Integrate $g(x,y,z) = z$ over the surface S cut from the cylinder $y^2+z^2 = 1, z \geq 0$, by the planes $x = 0$ and $x = 1$.
Solution: $f = y^2 + z^2$ and this surface can be projected 1-1, onto to R of xy plane. This projection is the rectangle with vertices $(1,-1),(1,1),(0,1),(0,-1)$. So $\hat{p} = \hat{k}$

$$\frac{|\nabla f|}{|\nabla f \cdot \hat{p}} = \frac{2\sqrt{y^2+z^2}}{|2z|} = \frac{1}{z}$$

Therefore,

$$\iint_S z dS = \iint_R z\frac{1}{z}\, dA = Area(R) = 2$$

1.3 Line integrals and Green's theorem

In many physical phenomena, the integrals over paths through vector filed plays important role. For example, work done in moving an object along a path against a variable force or to find work done by a vector field in moving an object along a path through the field. A **vector field** on a domain in the plane or in the space is a vector valued function $f : \mathbb{R}^3 \to \mathbb{R}^3$ with components say M, N and P, for example

$$F(x,y,z) = M(x,y,z)\hat{i} + N(x,y,z)\hat{j} + P(x,y,z)\hat{k}$$

We assume that M, N, P are continuous functions. Suppose F represents a force throughout a region in space and let $r(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}, a \leq t \leq b$ is a smooth curve in the region. Then we introduce the partition $a = t_1 < t_2 ..... < t_n = b$ of $[a,b]$.

If F_k denotes the value of F at the point on the corresponding to t_k on the curve and T_k denotes the curve's unit tangent vector a this point. Then $F_k \cdot T_k$ is the scalar component of F in the director of T at t_k. Then the work done by F along the curve is approximately

$$\sum_{k=1}^{n} F_k \cdot T_k \Delta s_k,$$

where Δs_k is the length of the curve between t_{k-1}, t_k. As the norm of the partition approaches zero, these sum's approaches

$$\int_{t=a}^{b} F \cdot T ds = \int_a^b \overrightarrow{F} \cdot \overrightarrow{T} \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt.$$

Now substituting $T(t) = \frac{\overrightarrow{r}'(t)}{|\overrightarrow{r}'(t)|}$, we get

$$\int_a^b \overrightarrow{F} \cdot \overrightarrow{r}'(t) dt.$$

Example: Find the work done by $F = 3x^2\hat{\imath} + (2xz - y)\hat{\jmath} - z\hat{k}$ over the curve $r(t) = t\hat{\imath} + t^2\hat{\jmath} + t^3\hat{k}, 0 \leq t \leq 1$ from origin to $(1, 1, 1)$
Solution: The tangent along the curve T is $\frac{dr}{dt}$. Therefore,

$$\begin{aligned}\int_0^1 F \cdot T ds &= \int_0^1 F \cdot \frac{d\overrightarrow{r}}{dt} dt \\ &= \int_0^1 3t^2 + t^5 - 2t^3 dt = \frac{2}{3}.\end{aligned}$$

Green's theorem in the plane
Let R be a closed bounded region in $I\!R^2$ whose boundary $\mathcal{C}$ consists of finitely many smooth curves. Let $\overrightarrow{F}(x, y) = F_1(x, y)\hat{\imath} + F_2(x, y)\hat{\jmath}$ be continuous and has continuous partial derivatives everywhere in some domain containing R. Then

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA = \oint_C \overrightarrow{F} \cdot d\overrightarrow{r},$$

where the line integral is along the boundary $\mathcal{C}$ of R such that R is on the left as we advance on the boundary.
Example: Evaluate $\int_C \overrightarrow{F} \cdot d\overrightarrow{r}$ for $\overrightarrow{F} = (y^2 - 7y)\hat{\imath} + (2xy + 2x)\hat{\jmath}$ and $C : x^2 + y^2 = 1$.
Solution:

$$\int_C \overrightarrow{F} \cdot d\overrightarrow{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA = 9 \iint_R dA = 9\pi.$$

Remark: The region R is such that the boundary consists of smooth curves. The theorem does not hold true for regions like punctured disc $\{(x, y) : x^2 + y^2 \leq 1\} \setminus \{0\}$. For example, take $\overrightarrow{F} = -\dfrac{y}{x^2 + y^2}\hat{\imath} + \dfrac{x}{x^2 + y^2}\hat{\jmath}$. Then $\dfrac{\partial F_2}{\partial x} = \dfrac{\partial F_1}{\partial y}$ and $\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) = 0$. But the line integral $\displaystyle\int_C \overrightarrow{F} \cdot d\overrightarrow{r} = \int_0^{2\pi} \sin^2\theta + \cos^2\theta = 2\pi$.

Area of plane region:
Using Green's theorem, we can write area of a plane region as a line integral over the boundary. Choose $F_1 = 0, F_2 = x$ and then $F_= -y, F_2 = 0$. This gives

$$\iint_R dA = \int_C xdy \quad \text{and} \quad \iint_R dA = -\int_C ydx$$

respectively. The double integral is the area A of R. By addition we have

$$A = \frac{1}{2}(xdy - ydx)$$

Example: Area bounded by ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
solution: Take $x = a\cos t, y = b\sin t, 0 \le t \le 2\pi$. Then by above formula

$$A = \frac{1}{2}\int_0^{2\pi} (xy' - yx')dt = \frac{1}{2}\left(ab\cos^2 t - (-ab\sin^2 t)\right) dt = \pi ab$$

1.4 Gauss and Stokes theorems

Let S be a smooth surface and we may choose unit normal $\hat{n}$ at P of S. The direction of $\hat{n}$ is called positive normal direction of S at P. We call a smooth surface S orientable surface if the positive normal at P can be continued in a unique and continuous way to the entire surface. For example the Mobius strip is not orientable. A normal at a point P of this strip is displaced continuously along a closed curve C, the resulting normal upon returning to P is opposite to the original vector at P.

Gauss Divergence Theorem

Let Ω be a closed, bounded region in $\mathbb{R}^3$ whose boundary is a piecewise smooth orientable surface S. Let $\overrightarrow{F}(x, y, z)$ be a continuous function that has continuous partial derivatives in some domain containing Ω. Then

$$\iiint_\Omega divF dV = \iint_S \overrightarrow{F} \cdot \hat{n} dS$$

where $\hat{n}$ is the outer unit normal vector of S.

Example: Evaluate $\displaystyle\iint_{\partial\Omega} \overrightarrow{F}.\hat{n} dA$ where $\partial\Omega$ is the boundary of the domain inside the cylinder $x^2 + y^2 = 1$ and between the planes $z = 0, z = x + 2$ and $\overrightarrow{F} = (x^2 + ye^z)\hat{\imath} + (y^2 + ze^2)\hat{\jmath} + (z^2 + xe^y)\hat{k}$.

Solution: With the given $\overrightarrow{F}$, it is not difficult to obtain, $\nabla \cdot \overrightarrow{F} = 2x + 2y + 2z$. By Divergence theorem

$$\iint_{\partial\Omega} \overrightarrow{F} \cdot \hat{n} dS = \iiint_\Omega 2(x + y + z)dV = 2\iint_{x^2+y^2\le 1} \left(\int_{z=0}^{x+2} (x + y + z)dz\right) dxdy$$

Stokes's theorem

Let S be a piecewise smooth oriented surface with boundary and let boundary $\mathcal{C}$ be a simple closed curve. Let $\overrightarrow{F}$ be a continuous function which has continuous partial derivatives in a domain containing S. Then

$$\iint_S (\nabla \times \overrightarrow{F}) \cdot \hat{n} dS = \int_{\mathcal{C}} \overrightarrow{F} \cdot d\overrightarrow{r}$$

where $\hat{n}$ is a unit normal vector of S and, depending on $\hat{n}$, the integration around C is taken in the way that S lies in the left of $\mathcal{C}$. Here $\hat{n}$ is the direction of your head while moving along the boundary with surface on your left.

Example: Evaluate $\int_C \overrightarrow{F} \cdot d\overrightarrow{r}$ where $\overrightarrow{F} = x^2y^3\hat{i} + \hat{j} + z\hat{k}$ and C The intersection of the cylinder $x^2 + y^2 = 4$ and the hemisphere $x^2 + y^2 + z^2 = 16, z \geq 0$.

Solution: The intersection of cylinder and sphere is the boundary of cylinder on the plane $z = \sqrt{12}$. The unit normal to the surface is $\hat{n} = \frac{1}{4}(x\hat{i} + y\hat{j} + z\hat{k})$. The projection R of S on the xy-plane is the disc $x^2 + y^2 \leq 2$, $\nabla \times \overrightarrow{F} = -3x^2y^2\hat{k}$ and $\frac{|\nabla f|}{|\nabla f \cdot \hat{p}|} = \frac{4}{z}$. Hence by Stoke's theorem

$$\begin{aligned}\oint_C \overrightarrow{F} \cdot \overrightarrow{dr} &= \iint_R (-\frac{3}{4})x^2y^2z\frac{4}{z}dA \\ &= -3\int_{\theta=0}^{2\pi}\int_{r=0}^{2}(r^2\cos^2\theta)(r^2\sin^2\theta)rdrd\theta = -8\pi.\end{aligned}$$

Suppose S_1, S_2 be two surfaces having the same boundary curve C. An important consequence of Stoke's theorem is that flux through S_1 or S_2 is same.

Example: Suppose S is a surface of a light bulb over the unit disc $x^2 + y^2 = 1$ oriented with outward pointing normal. Suppose $\overrightarrow{F} = e^{z^2-2z}x\hat{i} + (\sin(xyz) + y + 1)\hat{j} + e^{z^2}\sin(z^2)\hat{k}$. Compute $\iint_S (\nabla \times \overrightarrow{F}) \cdot \hat{n}dS$.

Solution: Enough to take any surface with boundary $x^2 + y^2 = 1$. So we take the unit disc $x^2 + y^2 \leq 1, z = 0$. Then $\overrightarrow{F}$ on this is $\overrightarrow{F} = x\hat{i} + (y + 1)\hat{j}$. Then $\nabla \times \overrightarrow{F} = 0$. Hence $\int_C \overrightarrow{F} \cdot d\overrightarrow{r} = 0$.

References

1. Thomas' Calculus, Chapters 15, 16

2. Advanced engineering Mathematics, E.Kreyszig, Chapter 9