Classical Demand Theory*

Chapter 3 provides an overview of classical demand theory. We have already discussed some properties of Walrasian demand functions, but we never really made any assumptions about how consumers were making choices, just that they were making choices. In this chapter we will impose some additional assumptions on consumer behavior.

1 Consumer's Preference Relation

We have already assumed our preference relation, $\succsim$, to be rational, which by definition means it is complete and transitive. We now impose two more assumptions, desirability and convexity.

Definition 1 *The preference relation $\succsim$ on X is monotone if $x \in X$ and $y >> x$ implies $y \succ x$. It is strongly monotone if $y \geq x$ and $y \neq x$ imply $y \succ x$.*

What does this mean, and what is the difference between monotone and strongly monotone? Think in terms of vectors. For monotone, where we have $y >> x$, it means that every element of y is greater than every element of x. As an example, if $y = \begin{bmatrix} 12 \\ 9 \\ 6 \end{bmatrix}$ and $x = \begin{bmatrix} 7 \\ 2 \\ 5 \end{bmatrix}$, then $y >> x$. For strongly monotone, where we have $y \geq x$ and $y \neq x$, it means that every element of y is at least as large as every element of x, but that at least one element of y must be greater than one element of x. As an example, if $y = \begin{bmatrix} 12 \\ 9 \\ 6 \end{bmatrix}$ and $x = \begin{bmatrix} 12 \\ 9 \\ 5 \end{bmatrix}$, then $y \geq x$ and $y \neq x$. This assumption is satisfied if our commodities are economic goods, not economic bads. But economic bads can be redefined in terms of economic goods (amount of pollution, which is a bad, redefined as absence of pollution, which is a good).

A weaker desirability assumption is that of local nonsatiation.

Definition 2 *The preference relation $\succsim$ is locally nonsatiated if for every $x \in X$ and every $\varepsilon > 0$, there is $y \in X$ such that $\|y - x\| \leq \varepsilon$ and $y \succ x$.*

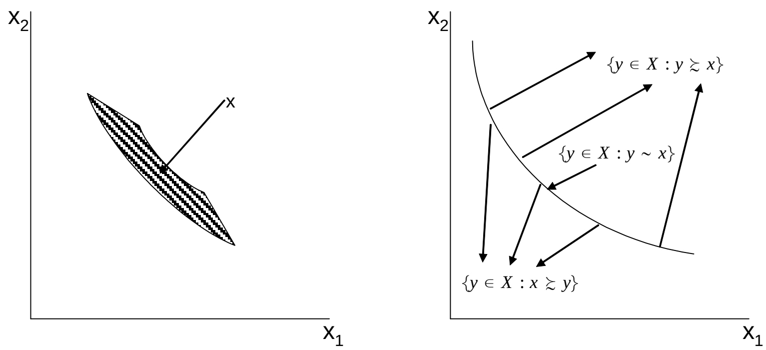
Note that $\|y - x\|$ is the Euclidean distance between y and x, defined as $\left[\sum_{\ell=1}^{L} (y_\ell - x_\ell)^2\right]^{1/2}$. Think about finding the distance between two points in the Cartesian plane, and then just extrapolate that to L dimensions.

What does local nonsatiation mean? It means that if you draw a "circle" (think $\mathbb{R}^2$) with any radius $\varepsilon > 0$ around any point $x \in X$, there must be some $y \in X$ inside that circle such that $y \succ x$. Note that in this case we can have $y << x$ and yet $y \succ x$ – with local nonsatiation, we are not necessarily assuming that "more is better".

Given $\succsim$ and bundle x, we can define 3 related sets of consumption bundles:

1. Indifference set: set of all bundles indifferent to x. $\{y \in X : y \sim x\}$

*These notes correspond to chapter 3 of Mas-Colell, Whinston, and Green.



A thick indifference set.

An indifference set, upper contour set, and lower contour set.

Figure 1: Indifference sets.

2. Upper contour set: set of all bundles at least as good as x. $\{y \in X : y \succsim x\}$. Note that the upper contour set includes the indifference set.

3. Lower contour set: set of all bundles that x is at least as good as. $\{y \in X : x \succsim y\}$. Note that the lower contour set includes the indifference set.

These sets are depicted in the right-hand side graph of Figure 1. Local nonsatiation does tell us that we cannot have "thick" indifference sets. An example of a thick indifference set is depicted in the left-hand side graph of Figure 1.

The second new assumption is convexity. We have already discussed convexity in terms of consumption sets and budget sets. Now we will discuss convexity in terms of the preference relation.

Definition 3 *The preference relation $\succsim$ on X is <u>convex</u> if for every $x \in X$, the upper contour set is convex; that is, if $y \succsim x, z \succsim x$, and $z \neq y$, then $\alpha y + (1-\alpha) z \succsim x$ for any $\alpha \in [0,1]$.*

Why impose convexity? Two reasons:

1. Consumers typically like to consume mixed bundles, that is they do not like having a bundle of goods that consist solely of one item.

2. With convexity we get diminishing marginal rates of substitution. That is, for any given consumption bundle x consisting of good x_1 and good x_2, a consumer will have to obtain increasingly larger amounts of x_1 to be persuaded to give up each additional unit of x_2.

Another useful assumption we will make regards strict convexity.

Definition 4 *The preference relation on $\succsim$ on X is <u>strictly convex</u> if for every x, we have that $y \succsim x, z \succsim x$, and $z \neq y$ implies that $\alpha y + (1-\alpha) z \succ x$ $\forall \alpha \in (0,1)$.*

Note the subtle difference between convexity and strict convexity. Convexity tells us that if y and z are both at least as good as x, then any weighted average of y and z is also at least as good as x. Strict convexity tells us that if y and z are at least as good as x, then any weighted average of y and z is PREFERRED to x. (Also note that we have closed brackets, [], with convexity and open brackets, (), with strict convexity. This is because with convexity, by assumption $z \succsim x$ and $y \succsim x$ and using the endpoints of 0 and 1 does not contradict the assumption. However, with strict convexity there is no guarantee that $y \succ x$ or $z \succ x$, which is what we would have if we included the endpoints.)

2 Preference and Utility

Life becomes much easier if we can represent our consumer's preference relation with a continuous utility function. We have already discussed the need for rationality of $\succsim$ in order to have a utility function that represents preferences. However, even adding local nonsatiation and convexity do not guarantee the existence of a continuous utility function. We will need to assume that $\succsim$ is continuous.

Definition 5 *The preference relation $\succsim$ on X is continuous if it is preserved under limits. That is, for any sequence of pairs $\{(x^n, y^n)\}_{n=1}^{\infty}$, with $x^n \succsim y^n \forall n$, $x = \lim_{n\to\infty} x^n$, and $y = \lim_{n\to\infty} y^n$, we have $x \succsim y$.*

The most intuitive concepts, like continuity, or "not picking your pencil up off the page", can have the damnedest definitions. All this definition says is that if we have a sequence of $\{x^n\}$ and a sequence of $\{y^n\}$, and if a consumer prefers every element of x^n to the corresponding element of y^n, that the consumer cannot change his mind at the limit and prefer y to x. Alternatively, we can say that $\succsim$ is continuous if the upper contour set and lower contour sets are both closed (we will discuss a formal definition of a closed set shortly – think about the closed unit interval, $[0,1]$, versus the open unit interval, $(0,1)$).

Proposition 6 *If the rational preference relation $\succsim$ on X is continuous, then there is a continuous utility function $u(x)$ that represents $\succsim$.*

Note that we are only assuming that $\succsim$ is rational and continuous, and no mention is made of convexity and nonsatiation. They are not needed to guarantee that a continuous utility function exists. The bottom line is that continuity is essentially a mathematically useful assumption, while completeness and transitivity impose rationality on the consumer's preferences, and monotonicity and convexity are assumptions about tastes. We will also add that $u(x)$ is twice continuously differentiable. Again, this makes the math much more tractable (always say tractable rather than easy whenever possible – it makes you sound sophisticated).

Note that the monotonicity of $\succsim$ implies that $u(\cdot)$ is increasing, that is $u(x) > u(y)$ if $x >> y$. The convexity of $\succsim$ implies that $u(\cdot)$ is quasiconcave.

Definition 7 *A function $u(\cdot)$ is quasiconcave if the set $\{y \in \mathbb{R}_+^L : u(y) \geq u(x)\}$ is convex for all x, or, equivalently, if $u(\alpha x + (1-\alpha) y) \geq \min(u(x), u(y))$ for any x, y and $\alpha \in [0,1]$.*

Think about what quasiconcavity means – if we take a weighted average of two bundles, then the utility of that weighted average is greater than or equal to the minimum of the two original bundles. For strict quasiconcavity replace $\geq$ with $>$ and $\alpha \in [0,1]$ with $\alpha \in (0,1)$. For a function to be concave, we need $f(\alpha x' + (1-\alpha) x'') > \alpha f(x') + (1-\alpha) f(x'')\ \ \forall \alpha \in (0,1)$. Concave simply says that for any two points in the function, any weighted average of the two points evaluated by the function is greater than the weighted average of the evaluated values of the two points. Thus, if we pick any two points on the function and draw a line between them the function will lie above that line. Note that this is a stronger assumption than quasiconcavity, and the goal is to provide the weakest possible assumptions to obtain the result.

Let's take a look at the difference between concave and quasiconcave using pictures. Note the similarities in the pictures of the concave and quasiconcave functions. The key is that both have a unique maximum, but that quasiconcave allows more functional forms (also note that a concave function is quasiconcave, but that a quasiconcave function may not be concave). In the picture that is neither, there are local maxima. The goal will be to find the consumer's optimal bundle, and with a function that is not quasiconcave we may be doing all the work only to find out we are at a local maximum (or worse yet we may not realize it is a local maximum).

3 The Utility Maximization Problem (UMP)

Finally, an actual problem. You all know this problem – the consumer's goal is to maximize utility, given that they have a budget constraint. We can be a little more formal, saying something like the consumer's goal is to choose the most preferred consumption bundle $x(p, w)$ given prices $p >> 0$ and $w > 0$ (and hence, budget constraint $p * x \leq w$). Write this out as a maximization problem:

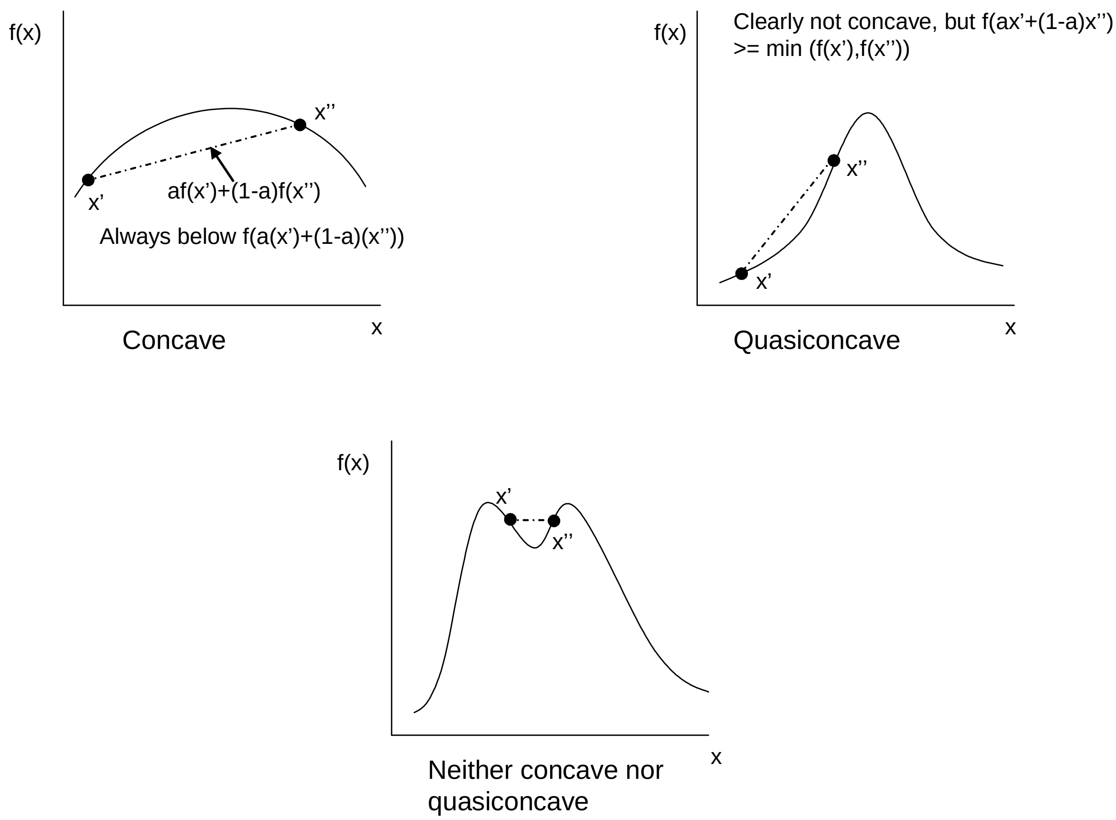


Figure 2: Depictions of concave and quasiconcave functions.

$$\max_{x \geq 0} u(x) \text{ subject to (or s.t.) } p * x \leq w$$

This is a nice problem, but does it have a solution? There are two questions that you will want to ask when setting up your models. The first is, Does a solution exist? The second is, Is the solution unique? So existence and uniqueness are two concepts that model builders, at least in the sense of classical demand theory, strive for.[1]

Proposition 8 *If* $p >> 0$ *and* $u(\cdot)$ *is continuous, then the UMP has a solution.*

This is great, because it gives us existence. But we will need a few definitions and "prior knowledge" to show this.

Definition 9 *A set* S *is compact if it is closed and bounded.*

That's fine, but what are closed and bounded sets?

Definition 10 *A set* S *is closed if its complement* S^c *is open.*

Again, nice, but what is an open set?

Definition 11 *A set* S *is open if for all of its elements* $x \in S$*, there exists some* $\varepsilon > 0$ *such that the open ball* $B_\varepsilon(x)$ *is in the set.*

Since the definition is longer we must be getting somewhere, but what is an open ball?

Definition 12 *The* <u>*open ball*</u> *with center* x^0 *and radius* $\varepsilon > 0$ *(a real number) is the subset of points in* $\mathbb{R}^n$:
$B_\varepsilon(x^0) \equiv \{x \in \mathbb{R}^n : d(x^0, x) < \varepsilon\}$, *where* $d(x^0, x)$ *is the distance from* x^0 *to* x
The closed ball with center x^0 *and radius* $\varepsilon > 0$ *(a real number) is the subset of points in* $\mathbb{R}^n$:
$B^*_\varepsilon(x^0) \equiv \{x \in \mathbb{R}^n : d(x^0, x) \leq \varepsilon\}$, *where* $d(x^0, x)$ *is the distance from* x^0 *to* x

Note that the difference between an open ball and a closed ball (besides the asterisk) is in the inequality. So, an open ball contains all the points within a circle of given radius ε but NOT the boundary of the circle, while a closed ball contains all the points within a a circle of given radius ε including the boundary points (circle is for $\mathbb{R}^2$). So if a set is open then any element of the set can be contained within SOME open ball (not every, but at least one) centered at that point. Think of the open unit interval $(0, 1)$ (the term open kind of gives it away). Even for points very close to 0 and 1 we can draw SOME open ball around them and all the points in the open ball will be in the set. Now, think of the closed unit interval $[0, 1]$. A set is closed if its complement (the complement is the set of all elements in the universal set – in our current example, the real number line – that are not in our set) is open. The complement to the closed unit interval is $(-\infty, 0) \cup (1, \infty)$, which is an open set.

Definition 13 *A set* S *is bounded if it is entirely contained within some open or closed ball. That is,* S *is bounded if there exists some* $\varepsilon > 0$ *such that* $S \subset B_\varepsilon(x)$ *or* $S \subset B^*_\varepsilon(x)$ *for some* $x \in \mathbb{R}^n$.

Basically, if I can draw a circle around the set, then it is bounded.

Why do we need this information? It will be useful to show that our budget set, $B_{p,w}$ is a compact set. First, consider whether or not $B_{p,w}$ is bounded. Since $p >> 0$, $B_{p,w}$ is bounded (also, recall that we are restricting our consumption set X to be $\mathbb{R}^L_+$). Why? Think about the question in two dimensions, for goods x_1 and x_2. If both prices are strictly greater than zero and w is finite, then there must be a maximum amount of w that I can spend on either x_1 and x_2, denoted by $\frac{w}{p_1}$ and $\frac{w}{p_2}$ respectively. Take whichever is larger, $\frac{w}{p_1}$ or $\frac{w}{p_2}$, and draw a closed ball of radius equal to the larger value plus 1. Then our consumer's entire budget set will fall in that closed ball. If we had any price equal to zero then our budget set would

[1] When we discuss game theory there are some game theorists who believe the fact that multiple equilibria exist in theory is useful because multiple equilibria exist in the real world. The question then becomes how one of those equilibria was selected in one case and how another was selected in a second case. So uniqueness is not necessarily that important to some game theorists, but they still strive for existence.

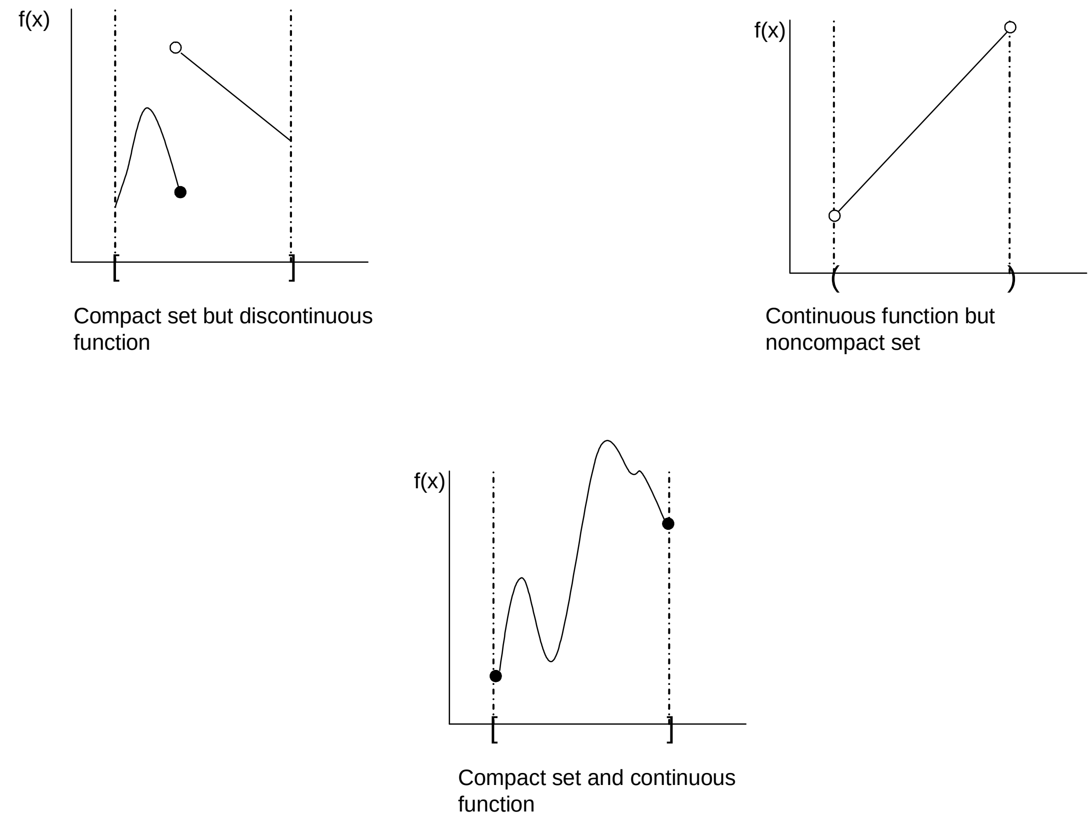


Figure 3: Examples that show a maximum may not be guaranteed without certain assumptions.

not be bounded as we could consume an infinite amount of the good that had price zero (it's hard to draw a ball around infinity).

Now, consider whether or not $B_{p,w}$ is closed. Our budget set is $\{B_{p,w} : p * x \leq w\} \forall x \in \mathbb{R}^L_+$. The complement of that is the set $B^C_{p,w} = \{p * x > w\} \forall x \in \mathbb{R}^L_+ \cup x \notin \mathbb{R}^L_+$. This set $B^C_{p,w}$ is open, so $B_{p,w}$ is closed. Thus, because $B_{p,w}$ is closed and bounded it is compact.

There is a nice result called the Weierstrass Theorem (extreme value theorem) that states that a continuous function attains a maximum (as well as a minimum) on any compact set. While we will not go through the details, it guarantees that there is actually a maximum to our problem, so that a solution does exist. Figure 3 shows examples of why we need both a continuous function and a compact set to guarantee the existence of a maximum. If the set is compact but the function is discontinuous then it is possible to have the function be open where the maximum would be. Thus, the maximum would never be reached. The same is true if the function is continuous but the set is not compact. The maximum may be at the boundary of the set, but since that boundary is never reached the maximum is never reached. The bottom picture provides an example where a maximum is attained, although it is only an example and not a proof. It would be simple to construct examples for the other cases where a maximum is attained, but these counterexamples are sufficient to disprove the suggestion that a maximum would be guaranteed without a continuous function or a compact set. So we now have existence, and shortly we will discuss uniqueness.

3.1 Walrasian Demand Correspondence

Note: On page 53 of the text, equation 3.D.1 is incorrect. There is a partial with respect to c_ℓ, but there is no c anywhere else on page 52, 53, or 54. The partial should be with respect to x_ℓ.

We have already discussed the Walrasian demand correspondence, $x(p, w)$, in general terms. Now, it is the rule that assigns the set of *optimal* consumption vectors.

Proposition 14 *Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. Then, the Walrasian demand correspondence $x(p,w)$ possesses the following properties:*

1. *Homogeneity of degree zero in $(p,w) : x(\alpha p, \alpha w) = x(p,w)$*
2. *Walras' law: $p * x = w \ \forall x \in x(p,w)$*
3. *Convexity (additional assumption)/uniqueness: If $\succsim$ is convex, so that $u(\cdot)$ is quasiconcave, then $x(p,w)$ is a convex set. Moreover, if $\succsim$ is strictly convex, so that $u(\cdot)$ is strictly quasiconcave, then $x(p,w)$ consists of a single element (i.e. is unique).*

We have already discussed homogeneity of degree zero and Walras' law in relation to the Walrasian demand correspondence. However, in chapter 2 these were assumptions. Now, they are properties of the Walrasian demand correspondence that result from our additional assumptions of continuity and local nonsatiation.

Proof. If $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$, then, the Walrasian demand correspondence $x(p,w)$ is homogeneous of degree zero.

Statement	Reason
1. $\{x \in \mathbb{R}^L_+ : p * x \leq w\}$	1. Our set of feasible consumption bundles under prices p and wealth w
2. $x(p,w)$ is the optimal consumption vector for p and w	2. Assumption
3. $\{x \in \mathbb{R}^L_+ : \alpha p * x \leq \alpha w\}$	3. Our set of feasible consumption bundles under prices αp and wealth αw
4. $x(\alpha p, \alpha w)$ is the optimal consumption vector for αp and αw	4. Assumption
5. $\{x \in \mathbb{R}^L_+ : \alpha p * x \leq \alpha w\} = \{x \in \mathbb{R}^L_+ : p * x \leq w\}$	5. Division by α
6. $x(p,w) = x(\alpha p, \alpha w)$	6. Since the feasible sets are the same (step 5) and $x(p,w)$ and $x(\alpha p, \alpha w)$ are the optimal bundles from the feasible sets, they must be equal
7. $x(p,w)$ is homogeneous of degree zero	7. From 6 and def. of homogeneous of degree zero

■

Proof of Walras law.

Proof. If $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$, then, the Walrasian demand correspondence $x(p,w)$ satisfies Walras' law.

We will prove this by contradiction, so we will assume that $p * x < w$

Statement	Reason
1. $p * x < w$	1. Our contradictory statement
2. There is another another bundle y in $B_{p,w}$ such that $p * y < w$ and $y \succ x$	2. Local nonsatiation

But this contradicts the fact that x is the optimal consumption bundle because y is preferred to x. Thus, Walras' law must hold. ■

Proof of convexity of $x(p,w)$ when $\succsim$ is convex.

Proof. If $\succsim$ is convex, so that $u(\cdot)$ is quasiconcave, then $x(p,w)$ is a convex set. We have two bundles, x and x', with $x \neq x'$, and $x, x' \in x(p,w)$. Need to show that $x'' = \alpha x + (1-\alpha)x' \in x(p,w) \forall \alpha \in [0,1]$. To show that, we need to show that (1) x'' is feasible and (2) that $u(x'') \geq u(x) = u(x')$.

Statement	Reason
1. $u(x) = u(x')$	1. $x(p,w)$ assigns the optimal bundle(s), so these must have the same utility
2. $u(x) \equiv u^*$	2. Useful definition
3. $x'' = \alpha x + (1-\alpha) x' \forall \alpha \in [0,1]$	3. Defining our new bundle
4. $x'' \in B_{p,w}$	4. Convexity of $B_{p,w}$
5. $u(x'') \geq u^*$	5. Quasiconcavity of $u(\cdot)$
6. x'' is feasible and has utility at least as large as x and x'	6. Steps 4 and 5
7. $x'' \in x(p,w)$	7. Definition of $x(p,w)$

■

Now, prove that if $\succsim$ is strictly convex, so that $u(\cdot)$ is strictly quasiconcave, then $x(p,w)$ consists of a single element (uniqueness).

Proof. If $\succsim$ is strictly convex, so that $u(\cdot)$ is strictly quasiconcave, then $x(p,w)$ consists of a single element (uniqueness). We can start by assuming that $x(p,w)$ is NOT unique, so that we have two bundles x and x', with $x \neq x'$, and $x, x' \in x(p,w)$. Define $x'' = \alpha x + (1-\alpha) x' \in x(p,w) \forall \alpha \in [0,1]$.

Statement	Reason
1. $u(x) = u(x')$	1. $x(p,w)$ assigns the optimal bundle(s), so these must have the same utility
2. $u(x) \equiv u^*$	2. Useful definition
3. $x'' = \alpha x + (1-\alpha) x' \forall \alpha \in [0,1]$	3. Defining our new bundle
4. $x'' \in B_{p,w}$	4. Convexity of $B_{p,w}$
5. $u(x'') > u^*$	5. Strict quasiconcavity of $u(\cdot)$
6. x'' is feasible and has utility greater than x and x'	6. Steps 4 and 5
7. $x(p,w)$ is unique	7. Step 6 contradicts the fact that x and $x' \in x(p,w)$ because we have a feasible bundle with strictly higher utility in x''

■

3.2 Inequality Constrained Optimization

We now know that given our consumer's problem there is a solution and it is unique (provided the assumptions we made on $\succsim$ and $u(\cdot)$ hold). Now we will discuss the mechanics of actually solving the consumer's problem and finding $x(p,w)$.

Consider a general 2-good problem with goods x_1 and x_2. We assume that $\succsim$ is rational, continuous, monotone, and strictly convex, so that $u(x_{1,}x_2)$ is continuous, increasing, and strictly quasiconcave. The consumer faces prices $p_1 > 0$ and $p_2 > 0$ for goods x_1 and x_2 respectively, and has a level of wealth $w > 0$, and that $p_1x_1 + p_2x_2 \leq w$. We will also assume (for the current example) that $x_1^*(p,w) > 0$ and $x_2^*(p,w) > 0$, where $x_1^*(p,w)$ and $x_2^*(p,w)$ are the consumer's optimal consumption levels of x_1 and x_2. This means there is an interior solution, and not a corner solution. The consumer's problem is then:

$$\max_{x_1,x_2} u(x_1,x_2) \text{ s.t. } p_1x_1 + p_2x_2 \leq w$$

Some steps:

1. Rewrite $p_1x_1 + p_2x_2 \leq w$ as $w - p_1x_1 - p_2x_2 \geq 0$ (there is a reason for this which we will discuss later).
2. Form the Lagrangian,
$$\mathcal{L}(x_1,x_2,\lambda) = u(x_{1,}x_2) + \lambda[w - p_1x_1 - p_2x_2]$$
3. Ponder where this λ came from ... (again, we will discuss this shortly ... mechanics right now)

If $x_1^*(p,w) > 0$ and $x_2^*(p,w) > 0$, we get the following Kuhn-Tucker conditions:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_1} &= \frac{\partial u\left(x_1^*, x_2^*\right)}{\partial x_1} - \lambda^* p_1 = 0 \\
\frac{\partial \mathcal{L}}{\partial x_2} &= \frac{\partial u\left(x_1^*, x_2^*\right)}{\partial x_2} - \lambda^* p_2 = 0 \\
w - p_1 x_1 - p_2 x_2 &\geq 0 \\
\lambda^* \left[w - p_1 x_1 - p_2 x_2\right] &= 0
\end{aligned}$$

The first 2 conditions are the first order conditions (FOCs) with respect to our consumer's 2 choice variables, x_1 and x_2 (it is a maximization problem after all). The 3^{rd} condition is our inequality constraint (it is an *inequality* constrained maximization problem after all). The last condition is called the complementary slackness condition. The consumer's goal is to maximize $u\left(x_1, x_2\right)$, NOT $\mathcal{L}\left(x_1, x_2, \lambda\right)$. This complementary slackness condition assures us that $u\left(x_1, x_2\right) = \mathcal{L}\left(x_1, x_2, \lambda\right)$. This means that either $\lambda^* = 0$ or $w - p_1 x_1 - p_2 x_2 = 0$. But we know from Walras' law that $w - p_1 x_1 - p_2 x_2 = 0$, so we also know that condition 3 holds with equality. Now we have a system of 3 equations (the FOCs and the constraint which is now an equality) and 3 unknowns $\left(x_1, x_2, \lambda\right)$.

Something we can see is that at the optimum,

$$\begin{aligned}
\frac{\partial u\left(x_1^*, x_2^*\right)}{\partial x_1} &= \lambda^* p_1 \\
\frac{\partial u\left(x_1^*, x_2^*\right)}{\partial x_2} &= \lambda^* p_2.
\end{aligned}$$

Note that $\frac{\partial u\left(x_1^*, x_2^*\right)}{\partial x_1}$ is the marginal utility of good x_1, or MU_{x_1} and that $\frac{\partial u\left(x_1^*, x_2^*\right)}{\partial x_2}$ is the marginal utility of good x_2, or MU_{x_2}. If we take the ratio of those 2 equations, we get: $\frac{MU_{x_1}}{MU_{x_2}} = \frac{p_1}{p_2}$, or $\frac{MU_{x_1}}{p_1} = \frac{MU_{x_2}}{p_2}$. These equations should look familiar from principles or intermediate economics classes as the "conditions" for a consumer's optimization problem. Note that $\frac{p_1}{p_2}$ is the slope of the budget line (the negative of the slope) and that $\frac{MU_{x_1}}{MU_{x_2}}$ is the marginal rate of substitution, or the slope of the indifference curve at x_1^* and x_2^*, so that the slope of the indifference curve is equal to the slope of the budget line at that point, or, in very technical terms, the budget line is tangent to the indifference curve at that point.

Now, what is λ? This variable λ tells us the marginal or shadow value of relaxing the constraint in the UMP. When applied to the budget constraint, it is the marginal value of wealth. Think about when $\lambda > 0$ and $\lambda = 0$. If $\lambda > 0$, then wealth has a positive marginal value, and more w will increase $u(\cdot)$ or $\mathcal{L}(\cdot)$, if we hold the other variables constant. If $\lambda = 0$ (and this is only for this example), then additional w is worthless to the consumer, even holding the other variables constant. That is because the constraint would be non-binding, and the consumer would have chosen x_1^* and x_2^* such that $w - p_1 x_1^* - p_2 x_2^* > 0$. Our concern as of right now is not with a specific value of λ, but whether or not $\lambda > 0$ or $\lambda = 0$.

Here is an actual example with a utility function. Let $u_B\left(x_1, x_2\right) = \alpha \ln x_1 + (1-\alpha) \ln x_2$. The consumer's problem is:

$$\max_{x_1, x_2} u_B\left(x_1, x_2\right) \text{ s.t. } p_1 x_1 + p_2 x_2 \leq w$$

We know that we will need to formulate the Lagrangian,

$$\mathcal{L}\left(x_1, x_2, \lambda\right) = u_B\left(x_{1,} x_2\right) + \lambda\left[w - p_1 x_1 - p_2 x_2\right]$$

and to obtain the Kuhn-Tucker conditions:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_1} &= \frac{\partial u_B\left(x_1^*, x_2^*\right)}{\partial x_1} - \lambda^* p_1 = 0 \\
\frac{\partial \mathcal{L}}{\partial x_2} &= \frac{\partial u_B\left(x_1^*, x_2^*\right)}{\partial x_2} - \lambda^* p_2 = 0 \\
w - p_1 x_1 - p_2 x_2 &\geq 0 \\
\lambda^* \left[w - p_1 x_1 - p_2 x_2\right] &= 0
\end{aligned}$$

We know that if $u_B(x_1, x_2) = \alpha \ln x_1 + (1-\alpha)\ln x_2$, then

$$\begin{aligned} \frac{\partial u_B(x_1^*, x_2^*)}{\partial x_1} &= \frac{\alpha}{x_1} \\ \frac{\partial u_B(x_1^*, x_2^*)}{\partial x_2} &= \frac{1-\alpha}{x_2} \end{aligned}$$

We also know that $w - p_1x_1 - p_2x_2 = 0$ because of Walras' law, so we have 3 equations with 3 unknowns. I won't type out all the rearranging of terms for this term to find the solution, but you should be able to verify that

$$\begin{aligned} x_1^*(p, w) &= \frac{\alpha w}{p_1} \\ x_2^*(p, w) &= \frac{(1-\alpha) w}{p_2} \\ \lambda^* &= \frac{1}{w} > 0. \end{aligned}$$

Now suppose that $p_1 = 10$, $p_2 = 5$, $\alpha = \frac{1}{3}$, and $w = 100$. We can actually find our consumer's optimal bundle in terms of a number. Plugging in those values we get that $x_1^*(p, w) = \frac{10}{3}$, $x_2^*(p, w) = \frac{40}{3}$, and $\lambda^* = \frac{1}{100} > 0$. Moreover, $\frac{MU_{x_1}}{MU_{x_2}} = \frac{1/3}{10/3} / \frac{2/3}{40/3} = 2$. Also, $\frac{p_1}{p_2} = 2$. So, $\frac{MU_{x_1}}{MU_{x_2}} = \frac{p_1}{p_2}$.

3.2.1 Slightly more general notation

We may not have a guarantee of an interior solution, but we still want to restrict $x_1 \geq 0$ and $x_2 \geq 0$. So, our consumer's problem is still to maximize utility subject to his budget constraint, but now we have the additional constraints that $x_1 \geq 0$ and $x_2 \geq 0$. Writing this out for a two good problem we have:

$$\max_{x_1, x_2} u(x_1, x_2) \text{ s.t. } p_1x_1 + p_2x_2 \leq w,\ x_1 \geq 0,\ x_2 \geq 0.$$

We can still follow the same steps as before, making sure that all our constraints are written as $\geq$ constraints. Since $x_1 \geq 0$ and $x_2 \geq 0$ are already written in this manner, that just leaves rewriting the budget constraint as $w - p_1x_1 - p_2x_2 \geq 0$. Now we can form the Lagrangian:

$$\mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3) = u(x_1, x_2) + \lambda_1 [w - p_1x_1 - p_2x_2] + \lambda_2 [x_1] + \lambda_3 [x_2]$$

We will now have a full set of Kuhn-Tucker conditions for both our choice variables and our Lagrange multipliers:

$$\begin{array}{lll} \frac{\partial \mathcal{L}}{\partial x_1} \leq 0, & x_1 \geq 0, & x_1 * \frac{\partial \mathcal{L}}{\partial x_1} = 0 \\ \frac{\partial \mathcal{L}}{\partial x_2} \leq 0, & x_2 \geq 0, & x_2 * \frac{\partial \mathcal{L}}{\partial x_2} = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda_1} \geq 0, & \lambda_1 \geq 0, & \lambda_1 * \frac{\partial \mathcal{L}}{\partial \lambda_1} = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda_2} \geq 0, & \lambda_2 \geq 0, & \lambda_2 * \frac{\partial \mathcal{L}}{\partial \lambda_2} = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda_3} \geq 0, & \lambda_3 \geq 0, & \lambda_3 * \frac{\partial \mathcal{L}}{\partial \lambda_3} = 0 \end{array}.$$

Note that in this case we have complementary slackness conditions for the choice variables because we are uncertain as to whether or not the constraint is binding. If we end up at a corner solution, then either $x_1 = 0$ or $x_2 = 0$, so one of the constraints will be binding. Notice why when we assumed that we had an interior solution that we did not have $\frac{\partial \mathcal{L}}{\partial x_1} \leq 0$ and $\frac{\partial \mathcal{L}}{\partial x_2} \leq 0$,[2] but $\frac{\partial \mathcal{L}}{\partial x_1} = 0$ and $\frac{\partial \mathcal{L}}{\partial x_2} = 0$. If $x_1 > 0$ and $x_2 > 0$, then those partial derivatives must be zero. Technically, these Kuhn-Tucker conditions are one piece of the necessary and sufficient conditions for a general maximization problem with no guarantee of an interior solution. The theorem is called the Arrow-Enthoven Theorem. While we will not go through the theorem in detail, the list of the conditions needed is listed below.

1. The Kuhn-Tucker conditions, provided above.

[2] Note that we have $\frac{\partial \mathcal{L}}{\partial x_i} \leq 0$. The reason for this is that we are not allowing the objective function to increase in any direction.

2. That the utility function $u(x)$ is quasiconcave.

3. That the constraint functions are quasiconvex.

4. That the gradient of $u(x)$ evaluated at the optimum not equal zero – if it did we would be at the unconstrained maximum, and have no need for the Arrow-Enthoven theorem.

5. That the Slater condition holds – there is a viable interior in the feasible set – if all the constraints hold with equality, then we can use equality constrained maximization methods.

6. The gradients, evaluated at the optimum, of the constraint functions which are active at the optimum are linearly independent (essentially, there are no redundant constraints). This ensures that at the optimum the constraint functions are locally different, so that a solution exists.

3.2.2 An example with a second binding constraint

Given our original problem, with $u(x_1, x_2) = \alpha \ln x_1 + (1-\alpha)\ln x_2$, budget constraint $w - p_1x_1 - p_2x_2 \geq 0$, and an interior solution (**note**: think $x_1 > 0$ and $x_2 > 0$), we know that $x_1^*(p,w) = \frac{\alpha w}{p_1}$ and $x_2^*(p,w) = \frac{(1-\alpha)w}{p_2}$. Furthermore, when $\alpha = \frac{1}{3}$, $p_1 = 10$, $p_2 = 5$, and $w = 100$, $x_1^*(p_1 = 10, p_2 = 5, w = 100) = \frac{10}{3}$ and $x_2^*(p_1 = 10, p_2 = 5, w = 100) = \frac{40}{3}$. Now we will add the constraint that $x_1 \geq 4$, which forces the consumer to consume 4 units of good x_1. Additionally, we will make one more assumption, that $w > 4p_1$ rather than $w > 0$. This ensures that our consumer can actually afford 4 units of x_1. Now, before we even start, will this new constraint be binding using the parameters of $\alpha = \frac{1}{3}$, $p_1 = 10$, $p_2 = 5$, and $w = 100$? Of course it will, since the consumer only chose to consume $\frac{10}{3} < 4$ units of x_1 when the constraint was not imposed. Now, let's set up the Lagrangian:

$$\mathcal{L}(x_1, x_2, \lambda_1, \lambda_2) = \alpha \ln x_1 + (1-\alpha)\ln x_2 + \lambda_1[w - p_1x_1 - p_2x_2] + \lambda_2[x_1 - 4].$$

We get:

$$\begin{array}{cc} \frac{\partial \mathcal{L}}{\partial x_1} = \frac{\alpha}{x_1} - \lambda_1 p_1 + \lambda_2 = 0 & \\ \frac{\partial \mathcal{L}}{\partial x_2} = \frac{(1-\alpha)}{x_2} - \lambda_1 p_2 = 0 & \\ w - p_1x_1 - p_2x_2 \geq 0 & \lambda_1 * [w - p_1x_1 - p_2x_2] = 0 \\ x_1 - 4 \geq 0 & \lambda_2 * [x_1 - 4] = 0 \end{array}.$$

We know that the budget constraint will hold with equality, so that $w - p_1x_1 - p_2x_2 = 0$. Now focus on our last equation, $x_1 - 4 \geq 0$. Either $x_1 - 4 = 0$ or $\lambda_2 = 0$ (there is the remote possibility that both occur, which would happen using our numbers if we changed w from 100 to 120. However, $\lambda_2 = 0$ in this case because the constraint does not bind, meaning that the consumer would choose LESS than 4 units at his optimal consumption bundles). Now, if $\lambda_2 = 0$ then the constraint is not binding and we are right back to where we started, with $x_1^*(p,w) = \frac{\alpha w}{p_1}$ and $x_2^*(p,w) = \frac{(1-\alpha)w}{p_2}$ (just impose $\lambda_2 = 0$ in $\frac{\partial \mathcal{L}}{\partial x_1}$ to see this). If the constraint is binding, then $x_1^*(p,w) = 4$. If we know that $x_1^*(p,w) = 4$, then from the budget constraint we know that $x_2^*(p,w) = \frac{w-4p_1}{p_2}$. So, our Walrasian demand function would be:

$$\begin{array}{c} x_1^*(p,w) = \begin{cases} 4 \text{ if } \lambda_2 > 0 \\ \frac{\alpha w}{p_1} \text{ if } \lambda_2 = 0 \end{cases} \\ x_2^*(p,w) = \begin{cases} \frac{w-4p_1}{p_2} \text{ if } \lambda_2 > 0 \\ \frac{(1-\alpha)w}{p_2} \text{ if } \lambda_2 = 0 \end{cases} \end{array}.$$

Now, how do we check for our specific problem? From $\frac{\partial \mathcal{L}}{\partial x_2}$ we know that $\lambda_1 = \frac{(1-\alpha)}{x_2p_2}$. Substituting this back into $\frac{\partial \mathcal{L}}{\partial x_1}$ we can see that $\lambda_2 = \frac{(1-\alpha)p_1}{x_2p_2} - \frac{\alpha}{x_1}$ (**Note**: if $w = 120$, and the remaining parameters are kept as before, then $x_1 = 4$ and $x_2 = 16$. Plug those values into the equation for λ_2 and this illustrates that $\lambda_2 = 0$ despite the fact that $x_1 = 4$. Since the consumer would have chosen $x_1 = 4$ without the constraint, the constraint is not binding.). Now, if $\lambda_2 = 0$, then we are right back to the original utility maximization problem because $x_1 = \frac{x_2\alpha p_2}{(1-\alpha)p_1}$. When we plug this back into the budget constraint and solve for x_2 we find that $x_2^* = \frac{(1-\alpha)w}{p_2}$ and $x_1^* = \frac{\alpha w}{p_1}$ if $\lambda_2 = 0$. However, once we substitute in our original parameters of $\alpha = \frac{1}{3}$,

$p_1 = 10$, $p_2 = 5$, and $w = 100$, we see that $x_1^* = \frac{10}{3}$, which violates the constraint that $x_1 \geq 4$, and so we know that $\lambda_2 > 0$.

In the general problem, which is to maximize $u(x_1, x_2)$ subject to the budget constraint, you would typically have to check 8 different cases.

x_1	x_2	λ
$+$	$+$	$+$
$+$	$+$	0
$+$	0	$+$
0	$+$	$+$
0	0	$+$
0	$+$	0
$+$	0	0
0	0	0

You would have to check the cases that all of the choice variables are positive, all are equal to 0, or some variables are positive and some are equal to zero. With 3 choice variables we would have $2^3 = 8$ possibilities, with 4 we would have 16, with 5 we would have 32, etc. However, in our general two good problem with a budget constraint we know that $\lambda > 0$, and we know that since $w > 0$ and $p_1x_1 + p_2x_2 = w$, and x_1 and x_2 are both greater than or equal to zero that we cannot have λ positive with both $x_1 = x_2 = 0$, so we are now down to 3 cases. Either $\lambda, x_1, x_2 > 0$, or $\lambda, x_1 > 0$ and $x_2 = 0$, or $\lambda, x_2 > 0$ and $x_1 = 0$. So basically all you would have to do is check to see if either $x_1 = 0$ or $x_2 = 0$. You can do this by checking whether utility is higher at either $x_1 = \frac{w}{p_1}$ and $x_2 = 0$, or $x_1 = 0$ and $x_2 = \frac{w}{p_2}$, or at the interior solution you found. For our parameters of $\alpha = \frac{1}{3}$, $p_1 = 10$, $p_2 = 5$, and $w = 100$, and our optimal bundles $x_1 = \frac{10}{3}$ and $x_2 = \frac{40}{3}$, we had $u\left(\frac{10}{3}, \frac{40}{3}\right) = \frac{1}{3}\ln\frac{10}{3} + \frac{2}{3}\ln\frac{40}{3} = 2.1282$. For $u(10, 0) = \frac{1}{3}\ln 10 + \frac{2}{3}\ln 0$ we get something undefined since $\ln 0$ is undefined, and the same for $u(0, 20) = \frac{1}{3}\ln 0 + \frac{2}{3}\ln 20$. So we "know" that we have an interior solution, at least with this problem. It is easier to see this if we use $u(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$. For our parameters the optimal consumption bundle is still $x_1 = \frac{10}{3}$ and $x_2 = \frac{40}{3}$, so that $u\left(\frac{10}{3}, \frac{40}{3}\right) = \frac{10}{3}^{1/3}\frac{40}{3}^{2/3} = 8.3995$. If x_1 or x_2 equal 0, then our utility will be 0, which is less than 8.3995.

3.3 The Indirect Utility Function

For each $(p, w) >> 0$, the utility value of the UMP is denoted $v(p, w)$. It is equal to $u(x^*)$ for every $x^* \in x(p, w)$, where x^* denotes the optimal consumption bundle. So, for our two good example of $u(x_1, x_2)$, we have $v(p, w) = u(x_1^*(p, w), x_2^*(p, w))$. That is, we can write utility as a function of prices and wealth. The benefits should become clear momentarily. As an example, we know that if $u(x_1, x_2) = \alpha \ln x_1 + (1-\alpha)\ln x_2$, then $x_1^*(p, w) = \frac{\alpha w}{p_1}$ and $x_2^*(p, w) = \frac{(1-\alpha)w}{p_2}$. Substituting these back into the utility function we get:

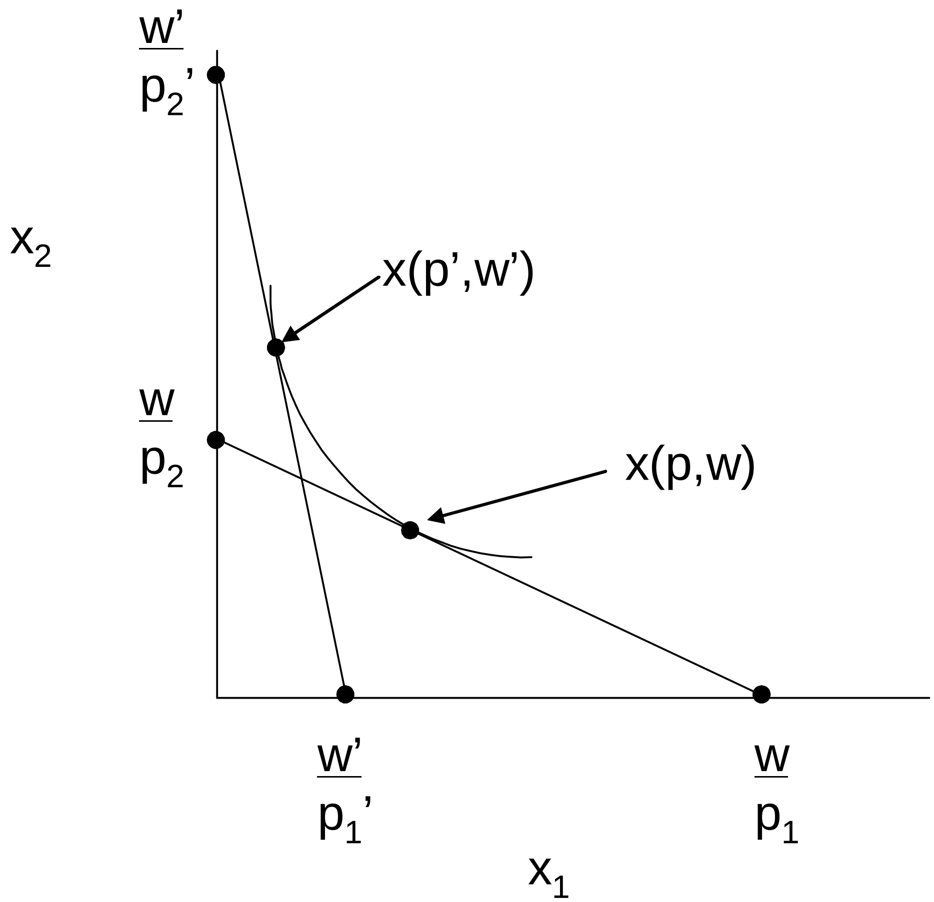
$$v(p, w) = \alpha \ln\left(\frac{\alpha w}{p_1}\right) + (1-\alpha)\ln\left(\frac{(1-\alpha)w}{p_2}\right)$$

so that utility is now a function of prices and wealth.

Proposition 15 *Suppose that $u(\cdot)$ is a continuous utility function representing locally nonsatiated preference relation $\succsim$ on $X = \mathbb{R}_+^L$. The indirect utility function $v(p, w)$ is*

1. *homogeneous of degree zero*
2. *strictly increasing in w and nonincreasing in p_ℓ for any $\ell = 1, ..., L$*
3. *Quasiconvex: that is the set $\{(p, w) : v(p, w) \leq \overline{v}\}$ is convex for any $\overline{v}$*
4. *Continuous in p and w*
5. *Roy's Identity holds: If $v(p, w)$ is differentiable at (p, w) and $\frac{\partial v(p,w)}{\partial w} \neq 0$, then*

$$x_\ell(p, w) = -\frac{\partial v(p, w)/\partial p_\ell}{\partial v(p, w)/\partial w}$$



For homogeneity, consider whether $u\left(x_1^*\left(p,w\right),x_2^*\left(p,w\right)\right) = u\left(x_1^*\left(\alpha p,\alpha w\right),x_2^*\left(\alpha p,\alpha w\right)\right)$ for any $\alpha > 0$. Well, $x_1^*\left(\alpha p,\alpha w\right) = x_1^*\left(p,w\right)$ and $x_2^*\left(\alpha p,\alpha w\right) = x_2^*\left(p,w\right)$ so that $u\left(x_1^*\left(\alpha p,\alpha w\right),x_2^*\left(\alpha p,\alpha w\right)\right) = u\left(x_1^*\left(p,w\right),x_2^*\left(p,w\right)\right)$ for any $\alpha > 0$. Since $u\left(x_1^*\left(p,w\right),x_2^*\left(p,w\right)\right) = v\left(p,w\right)$ and $u\left(x_1^*\left(\alpha p,\alpha w\right),x_2^*\left(\alpha p,\alpha w\right)\right) = v\left(\alpha p,\alpha w\right)$, then $v\left(p,w\right) = v\left(\alpha p,\alpha w\right)$. For strictly increasing in w, just consider that if you have more w then you will purchase more of at least one good, and given the restrictions we put on $\succsim$ and $u\left(\cdot\right)$ this means that we will obtain higher utility. For nonincreasing in p_ℓ consider that if p_ℓ increases then I will have to either decrease my consumption of x_ℓ or some other good, UNLESS I am at a corner solution where I am already consuming $x_\ell = 0$, in which case my utility will be no higher than it was before because my consumption bundle will not change (hence NONINCREASING versus strictly decreasing). We will skip the proof of continuity, and show quasiconvexity with a picture. The idea is to show that given budget sets (p,w) and (p',w'), both of which obtain the same utility level $\overline{v}$, any convex combination of those budget sets, $(p'',w'') = (\alpha p + (1-\alpha)p', \alpha w + (1-\alpha)w')$ for any $\alpha \in [0,1]$ will obtain AT MOST utility of $\overline{v}$. So, if $v\left(p,w\right) \leq \overline{v}$ and $v\left(p',w'\right) \leq \overline{v}$, then $v\left(p'',w''\right) \leq \overline{v}$. In Figure 3.3 we can see that the budget set (p,w) is given by the budget constraint with endpoints $\frac{w}{p_2}$ and $\frac{w}{p_1}$, and that (p',w') is given by the budget constraint with endpoints $\frac{w'}{p_1'}$ and $\frac{w'}{p_2'}$. Now, consider (p'',w'') when $\alpha = 1$. This will just be (p,w), and the maximum utility the consumer can attain as $\overline{v}$ at $x\left(p,w\right)$. When $\alpha = 0$, then $(p'',w'') = (p',w')$, and the maximum utility the consumer can attain is also $\overline{v}$ at $x\left(p',w'\right)$. If we look at how the budget constraint changes as α changes along the unit interval from 1 to 0, then we will just start at (p,w) and move to (p',w'). In other words, if we focus on $\frac{w''}{p_2''}$, then that will just range between $\frac{w'}{p_2'}$ and $\frac{w}{p_2}$. Also, $\frac{w''}{p_1''}$ must range between $\frac{w'}{p_1'}$ and $\frac{w}{p_1}$. Furthermore, the budget constraint for (p'',w'') must cross through the intersection of (p,w) and (p',w'). This is because at the intersection of those points all convex combinations are of that single intersection (if you take a convex combination over the same point you simply get back that point).

As for Roy's Identity, consider that the indirect utility function is simply the Lagrangian evaluated at the optimal choices of x_1, x_2, and λ, or $v(p,w) = \mathcal{L}(x^*, \lambda^*)$. If we differentiate $v(p,w)$ with respect to p_1, we get

$$\frac{\partial v(p,w)}{\partial p_\ell} = \frac{\partial \mathcal{L}(x^*, \lambda^*)}{\partial p_\ell} = -\lambda^* x_\ell^*$$

because p_ℓ only enters the Lagrangian in the budget constraint. So now we know that

$$\frac{\partial v(p,w)}{\partial p_\ell} = -\lambda^* x_\ell^*$$

but we also know that

$$\lambda^* = \frac{\partial \mathcal{L}(x^*, \lambda^*)}{\partial w} = \frac{\partial v(p,w)}{\partial w}.$$

To see this, just differentiate the Lagrangian with respect to w. Now we have

$$\frac{\partial v(p,w)}{\partial p_\ell} = -\frac{\partial v(p,w)}{\partial w} x_\ell^* \text{ or } x_\ell^* = -\frac{\partial v(p,w)/\partial p_\ell}{\partial v(p,w)/\partial w}.$$

Thus, if we know the indirect utility function $v(p,w)$, then we can derive the Walrasian demand functions directly from $v(p,w)$ without doing the UMP.

4 Expenditure Minimization Problem (EMP)

With the UMP, our goal is to find the maximum utility level given fixed prices and wealth. With the EMP, our goal is to find the minimum level of wealth necessary that will allow the consumer to attain a utility level of $\overline{u} > u(0)$, where $u(0)$ is simply the utility from consuming zero of all goods, given prices $p >> 0$. So our constraint from the UMP now becomes our objective function, and our objective function from the UMP now becomes our constraint. There are certain benefits to reframing the problem in this manner.

Proposition 16 *Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}_+^L$ and that the price vector is $p >> 0$. We have*

1. *if x^* is optimal in the UMP when $w > 0$, then x^* is optimal in the EMP when the required utility level is $u(x^*)$. Moreover, the minimized expenditure level in this EMP is exactly w.*

2. *if x^* is optimal in the EMP when the required utility level is $\overline{u} > u(0)$, then x^* is optimal in the UMP when wealth is $p * x^*$. Moreover, the maximized utility level in this UMP is exactly $\overline{u}$.*

Thus, the UMP and the EMP are related, and we will discuss this relationship in more detail.

The goal with the EMP is to minimize expenditure, subject to meeting a certain utility. Formally we can define the expenditure function as:

$$e(p,u) = \min_{x \in \mathbb{R}_+^n} px \text{ subject to } u(x) \geq \overline{u}$$

Proposition 17 *Suppose that $u(\cdot)$ is a continuous utility function representing locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}_+^L$. The expenditure function $e(p,u)$ is:*

1. *Homogeneous of degree ONE in prices*

2. *Strictly increasing in u and nondecreasing in p_ℓ for any $\ell = 1, ..., L$*

3. *Concave in p*

4. *Continuous in p and u*

As for homogeneity of degree one in prices, $e(p,u) = p_1 x_1^* + p_2 x_2^*$. So $e(\alpha p, u) = \alpha p_1 x_1^* + \alpha p_2 x_2^* = \alpha (p_1 x_1^* + p_2 x_2^*) = \alpha e(p,u)$ for all $\alpha > 0$.

For strictly increasing in u, recall that $e(p,u)$ will be the minimum expenditure that achieves u. Alternatively, u is the maximum utility level achieved by $e(p,u)$. If we want more u, we will need to spend more money.

For nondecreasing in p_ℓ, picture the two-good case. If the price of good x_2, denoted p_2, increases, and we are at a unique interior solution with $u = \overline{u}$, we will no longer be able to afford that bundle. To return to $\overline{u}$ we will need to increase our expenditure. However, it is possible that p_2 increases and we do not change our expenditure if we are at a corner solution consuming all x_1. That is why $e(p,u)$ is nondecreasing in p_ℓ, and not strictly increasing.

For concavity in p, define three expenditure functions, $e(p,\overline{u})$, $e(p',\overline{u})$, and $e(p'',\overline{u})$, where $p'' = \alpha p + (1-\alpha) p'$ for $\alpha \in [0,1]$. Note that they all achieve the same utility level $\overline{u}$. Our expenditure function will be concave in p if $\alpha e(p,u) + (1-\alpha) e(p',u) \leq e(p'',u)$. We know the following:

1. x minimizes expenditure to achieve $\overline{u}$ when prices are p
2. x' minimizes expenditure to achieve $\overline{u}$ when prices are p'
3. x'' minimizes expenditure to achieve $\overline{u}$ when prices are p''

The cost of x at p must be no greater than the cost of any other bundle that achieves $\overline{u}$. The same is true of x' and p'. So

$$\begin{aligned} px &\leq px'' \\ p'x' &\leq p'x'' \end{aligned}$$

because x'' achieves the same utility level as x and x'. Multiply the first inequality by $\alpha \in [0,1]$ and the second by $(1-\alpha)$ to get:

$$\begin{aligned} \alpha px &\leq \alpha px'' \\ (1-\alpha) p'x' &\leq (1-\alpha) p'x'' \end{aligned}$$

Now add the two inequalities to get

$$\begin{aligned} \alpha px + (1-\alpha) p'x' &\leq \alpha px'' + (1-\alpha) p'x'' \\ \alpha px + (1-\alpha) p'x' &\leq (\alpha p + (1-\alpha) p') x'' \end{aligned}$$

Or

$$\alpha e(p,\overline{u}) + (1-\alpha) e(p',\overline{u}) \leq e(p'',\overline{u})$$

Note what this says – it says that for two expenditure functions that achieve the same utility level, any convex combination of those expenditure functions will require MORE expenditure (or at least the same) to achieve the same utility level.

4.1 A minimization problem

The expenditure minimization function is given by:

$$e(p,u) = \min_{x_1, x_2} \; p_1 x_1 + p_2 x_2 \text{subject to } u(x_1, x_2) \geq \overline{u}$$

This looks fairly familiar, and the expenditure minimization function is simply equal to the Lagrangian evaluated at the optimal values of x_1^*, x_2^*, and λ^*. We can set up a Lagrangian, and we will get:

$$\mathcal{L}(x_1, x_2, \lambda) = p_1 x_1 + p_2 x_2 + \lambda [\overline{u} - u(x_1, x_2)]$$

Note that there is a difference in the minimization problem and the maximization problem. In the maximization problem we needed the constraint, $w - p_1 x_1 - p_2 x_2 \geq 0$. In the minimization problem, we need the constraint $\overline{u} - u(x_1, x_2) \leq 0$. Again, this is due to technical details. We can work out first order conditions, constraints and complementary slackness conditions for the minimization problem as well. With an interior solution, all first order conditions and constraints will hold with equality.

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_1} &= p_1 - \lambda \frac{\partial u(x_1, x_2)}{\partial x_1} = 0 \\
\frac{\partial \mathcal{L}}{\partial x_2} &= p_2 - \lambda \frac{\partial u(x_1, x_2)}{\partial x_2} = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= \overline{u} - u(x_1, x_2) = 0
\end{aligned}$$

Note that $\frac{\partial e(p,u)}{\partial x_1} = p_1$ because $e(p,u) = \mathcal{L}(x_1^*, x_2^*, \lambda^*)$. If we had a utility function we could then calculate the partial derivatives of $u(x_1, x_2)$ and solve for λ^*, x_1^*, and x_2^*. Plugging x_1^* and x_2^* back into $p_1 x_1^* + p_2 x_2^*$ would yield the expenditure function. NOTE that the x_1^* in the EMP is DIFFERENT THAN the x_1^* in the UMP. In the UMP, we had $x(p,w)$. In the EMP, we will have $x(p,u)$. To distinguish between the two we will call the demand functions derived from the EMP the Hicksian demand functions, and denote them $h(p,u)$. At this time you should also note the relationship between $e(p,u)$ and $v(p,w)$. From the initial propositions relating $e(p,u)$ and $v(p,w)$, we know that $e(p, v(p,w)) = w$ and $v(p, e(p,u)) = u$, so that both problems capture the underlying aspects of the consumer's problem.

4.2 Hicksian or Compensated demand functions

As mentioned, we will call our demand functions from the EMP the Hicksian demand functions, and denote them $h(p,u)$. Note that they may be correspondences, although if $\succsim$ is strictly convex then $h(p,u)$ will be unique.

Proposition 18 *Suppose that $u(\cdot)$ is a continuous utility function representing locally nonsatiated preference relation $\succsim$ on the consumption set $X = \mathbb{R}_+^L$. Then for any $p >> 0$, the Hicksian demand correspondence*

1. *is homogeneous of degree zero in prices, so that $h(\alpha p, u) = h(p,u)$ for all $\alpha > 0$*
2. *has no excess utility: For any $x \in h(p,u)$, $u(x) = u$*
3. *convexity/uniqueness: If $\succsim$ is convex, then $h(p,u)$ is a convex set. If $\succsim$ is strictly convex, then $h(p,u)$ consists of a single element.*

For homogeneity, consider minimizing px subject to $u(x) \geq u$. Now consider minimizing αpx subject to $u(x) \geq u$. The constant has no effect on the minimizing bundle, although it will effect the wealth level necessary to achieve that bundle (much like a constant multiplied by the utility function had no effect on the utility maximizing bundle, although it would affect the utility level).

We rely on continuity of u for the proof of no excess utility. This proof is similar to those we have done for Walras' law holding, except we use continuity rather than local nonsatiation because $h(p,u)$ is a function of u.

For convexity and uniqueness, we can use arguments similar to those for establishing convexity and uniqueness of $x(p,w)$.

At this point we can relate the Hicksian demands to the Walrasian demands, given the relationship between the EMP and the UMP. We have that:

$$\begin{aligned}
h(p,u) &= x(p, e(p,u)) \\
x(p,w) &= h(p, v(p,w))
\end{aligned}$$

Note that we use the indirect utility function, which is a function of prices and wealth, as opposed to the direct utility function, which is a function of the actual consumption goods x.

Why call the Hicksian demand function a compensated demand function? As prices vary, $h(p,u)$ gives precisely the level of demand that would arise if the consumer's wealth were simultaneously adjusted to keep the utility level of $\overline{u}$.

Hicksian demand will satisfy the compensated law of demand. Demand and price move in opposite directions for price changes that are accompanied by Hicksian wealth compensation. So, as p_2 increases, x_2 decreases. This is different from Walrasian demand, which allows for Giffen goods.

Proposition 19 *Suppose that $u(\cdot)$ is a continuous utility function representing locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$ and that $h(p,u)$ consists of a single element for all $p >> 0$. Then the Hicksian demand function $h(p,u)$ satisfies the compensated law of demand for all p and p'*

$$(p'' - p') \cdot [h(p'', u) - h(p', u)] \leq 0$$

Note what this says: if the price change is positive, then the change in Hicksian demand must be negative (or 0). If the price change is negative, then the change in Hicksian demand must be positive (or 0).

Proof. By assumption consumption bundles $h(p', u)$ and $h(p'', u)$ are the optimal consumption bundles when prices are p' and p'' respectively, and utility is at level u.

Statement	Reason
1. $p'' \cdot h(p'', u) \leq p'' \cdot h(p', u)$	1. $h(p'', u)$ minimizes expenditure for price p'' and utility u
2. $p' \cdot h(p'', u) \geq p' \cdot h(p', u)$	2. $h(p', u)$ minimizes expenditure for price p' and utility u
3. $(p'' - p') \cdot [h(p'', u) - h(p', u)] \leq 0$	3. Subtraction (we have a nonpositive number minus a nonnegative one)

Thus, the Hicksian demand function satisfies the compensated law of demand. ∎

As with the indirect utility function and Walrasian demand, we can derive the Hicksian demand directly from the expenditure function, $e(p,u)$. Since $e(p,u) = \mathcal{L}(x_1^*, x_2^*, \lambda^*)$, and price only enters the Lagrangian in the expenditure function itself, then

$$\frac{\partial e(p,u)}{\partial p_\ell} = \frac{\partial \mathcal{L}(x_1^*, x_2^*, \lambda^*)}{\partial p_\ell} = x_\ell^* = h_\ell(p,u).$$

Thus, the partial derivative of the expenditure function with respect to any price p_ℓ is the Hicksian demand function.

5 Impact of price changes

While finding the solution to the UMP or the EMP is an important step, many economists focus on what happens when something changes in the economic system. We will begin by discussing price changes in Hicksian demand, as Hicksian demand satisfies the law of demand (price increases, quantity demanded decreases) while Walrasian demand may or may not. However, Hicksian demand is a function of an unobservable variable, utility. Walrasian demand, however, is a function of the observable variables (or at least variables that we might be able to observe) price and wealth (or income).

We have that the own-price derivatives of Hicksian demand are negative (or 0) because Hicksian demand follows the compensated law of demand. This means

$$\frac{\partial h_\ell(p,u)}{\partial p_\ell} \leq 0.$$

Now consider the cross-price derivative of Hicksian demand $h_\ell(p,u)$ with respect to the price of good k, p_k. If $\frac{\partial h_\ell(p,u)}{\partial p_k} \leq 0$ then goods ℓ and k are complements or complementary goods, because as the price of good k increases the Hicksian demand for good ℓ decreases. Thus we are consuming less of good k and less of good ℓ when p_k increases. If $\frac{\partial h_\ell(p,u)}{\partial p_k} \geq 0$ then goods ℓ and k are substitutes because as the price of good k increases the Hicksian demand for good ℓ increases. Note that if the cross-price derivative is equal to zero then the goods could be classified as either substitutes or complements. However, consider what it means if the cross-price derivative truly is zero – a change in p_k has no effect on the Hicksian demand for good ℓ. Thus the two goods could be classified as independent. We know that there must be at least one good which is a substitute for any specific good in the economy.[3] To see this, consider the 2-good case. If the price of good k increases, then the consumption of good k will decrease (unless the consumer is at a corner solution) because Hicksian demand follows the compensated law of demand. Now, if the consumer is to remain at the same utility level, and he is consuming less of good k, then he must consume more of good ℓ.

[3] With the exception of the 2-good case of perfect complements which we discussed in class, in which there was no substitution effect.

5.1 Decomposing Hicksian demand changes

The purpose of using Hicksian demand is because Hicksian demand follows the compensated law of demand. But, we cannot observe Hicksian demand because one of its arguments is unobservable (utility level). We can exploit the relationship between Hicksian demand and Walrasian demand to obtain information on price effects.[4]

Proposition 20 *(The Slutsky Equation) Suppose that $u(\cdot)$ is a continuous utility function representing locally nonsatiated preference relation $\succsim$ defined on $X = \mathbb{R}^L_+$. Then for all (p, w) and $u = v(p, w)$ we have*

$$\frac{\partial h_\ell (p,u)}{\partial p_k} = \frac{\partial x_\ell (p,w)}{\partial p_k} + \frac{\partial x_\ell (p,w)}{\partial w} x_k (p,w) \text{ for all } \ell, k.$$

Or, rewriting in terms of the cross-price effect of the Walrasian demand:

$$\frac{\partial x_\ell (p,w)}{\partial p_k} = \frac{\partial h_\ell (p,u)}{\partial p_k} - \frac{\partial x_\ell (p,w)}{\partial w} x_k (p,w) \text{ for all } \ell, k.$$

Proof. We know that $h_\ell (p,u) = x_\ell (p, e(p,u))$ at the optimal solution to the consumer's problem. We can differentiate with respect to p_k and evaluate at $\overline{p}$ and $\overline{u}$.

Statement	Reason
1. $\frac{\partial h_\ell(\overline{p},\overline{u})}{\partial p_k} = \frac{\partial x_\ell(\overline{p},e(\overline{p},\overline{u}))}{\partial p_k} + \frac{\partial x_\ell(\overline{p},e(\overline{p},\overline{u}))}{\partial w}\frac{\partial e(\overline{p},\overline{u})}{\partial p_k}$	1. Chain rule for differentiation
2. $\frac{\partial h_\ell(\overline{p},\overline{u})}{\partial p_k} = \frac{\partial x_\ell(\overline{p},e(\overline{p},\overline{u}))}{\partial p_k} + \frac{\partial x_\ell(\overline{p},e(\overline{p},\overline{u}))}{\partial w} h_k(\overline{p},\overline{u})$	2. Earlier result on relation of $e(p,u)$ to $h(p,u)$
3. $h_k(\overline{p},\overline{u}) = x_k(\overline{p}, e(\overline{p},\overline{u})) = x_k(\overline{p},\overline{w})$	3. Earlier result on relation of $h_k(p,u)$ to $x_k(p,w)$
4. $e(\overline{p},\overline{u}) = \overline{w}$	4. Earlier result on relation of $e(p,u)$ to w
5. $\frac{\partial h_\ell(\overline{p},\overline{u})}{\partial p_k} = \frac{\partial x_\ell(\overline{p},\overline{w})}{\partial p_k} + \frac{\partial x_\ell(\overline{p},\overline{w})}{\partial w} x_k(\overline{p},\overline{w})$	5. Substitution

■

For the Walrasian demand, the change in quantity of good ℓ with respect to a change in the price of good k is known as the Total Effect of the change in price of good k. The total effect is decomposed into the Substitution Effect $\left(\frac{\partial h_\ell(p,u)}{\partial p_k}\right)$ and the Income (or Wealth) Effect $\left(\frac{\partial x_\ell(p,w)}{\partial w} x_k(p,w)\right)$. The Substitution Effect is the change in quantity demanded of good ℓ due to the fact that good ℓ is now relatively more (less) expensive if the price of another good (say good k) increases (decreases) when the prices of all other goods stay the same. Thus, if the price of a good k increases, we would expect that a consumer would purchase more of a second good ℓ because ℓ is now a relatively less expensive substitute (unless of course the goods are complements). The Income Effect is the change in quantity demanded of good ℓ due to the fact that the consumer has control over how he spends his wealth. There need be no actual change in w for there to be an income effect, but if the price of good k increases, then the consumer may not just decide to reduce consumption of good k at the rate of the price increase. For example, if p_k doubles, the consumer may keep consumption of good ℓ the same and simply reduce consumption of good k to $\frac{1}{2}$ its previous level, but is not required to act in this manner. The consumer may cut consumption by more (or less) than $\frac{1}{2}$ and adjust consumption of good ℓ accordingly. The consumer may even increase the amount of good k when a price increase occurs – this is the case of a Giffen good, and it occurs because the Income Effect overwhelms the Substitution Effect. Consider the Slutsky equation for a change in the own-price of a good:

$$\frac{\partial x_\ell (p,w)}{\partial p_\ell} = \frac{\partial h_\ell (p,u)}{\partial p_\ell} - \frac{\partial x_\ell (p,w)}{\partial w} x_\ell (p,w)$$

We know that if p_ℓ increases that the Substitution Effect $\left(\frac{\partial h_\ell(p,u)}{\partial p_\ell}\right)$ will be negative. However, there is no such restriction on the Total Effect $\left(\frac{\partial x_\ell(p,w)}{\partial p_\ell}\right)$ as it may be positive or negative (recall the case of Giffen goods). It will be positive if the Income Effect is more negative than the Substitution Effect (remember,

[4] For a recent reference on using the Slutsky equation in empirical work, see Fisher, Shively, and Buccola (2005). Activity Choice, Labor Allocation, and Forest Use in Malawi. *Land Economics*, Vol. 81:4

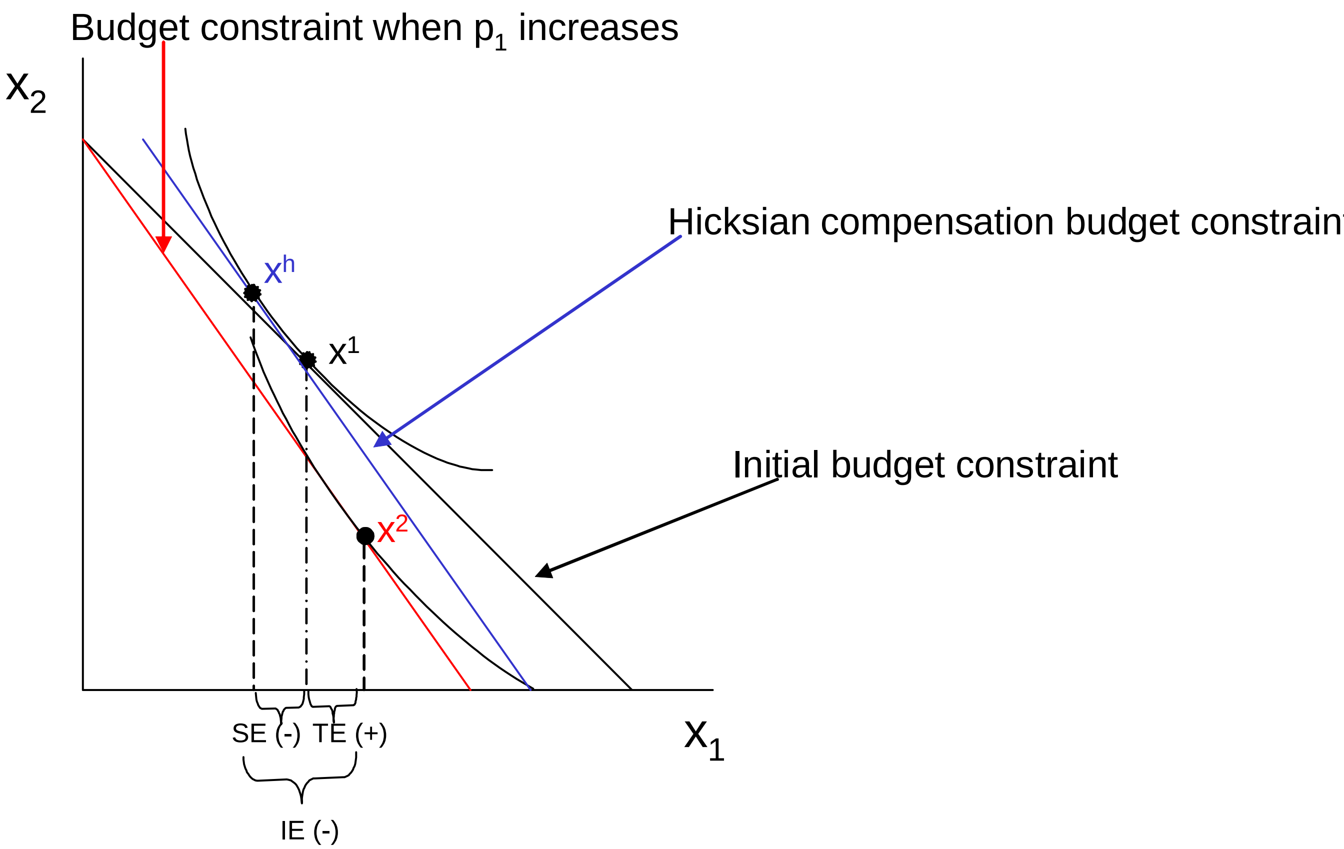


Figure 4: Decomposing the effect of a price change on a Giffen good.

this is an OWN-price equation, so the Hicksian demand must decrease when its own price increases). In this case, we have a Giffen good because $\frac{\partial x_\ell(p,w)}{\partial p_\ell} > 0$. Thus, it is an usually large negative income effect that is driving the Giffen good result. Typically one would think that income effects would be positive (we know that $x_\ell(p,w) \geq 0$, so focus on $\frac{\partial x_\ell(p,w)}{\partial w}$). This is just the derivative of the Walrasian demand function with respect to wealth, and usually if wealth increases consumers consume more of a good (hence the reason we call these goods "normal goods"). However, if a good is a Giffen good then it must have a wealth effect negative enough to overwhelm the negative substitution effect. Thus any good that is a Giffen good must be an inferior good. However, this does not mean that all inferior goods are Giffen goods – if the derivative of the Walrasian demand is negative (so that the good is inferior), it is possible that the wealth effect is LESS negative than the substitution effect. In this case, while the good is inferior, its total effect will still be negative. Figure 4 shows the effect of a price change of a Giffen good decomposed into its total, substitution, and income effects. The initial budget constraint is in black and the optimal bundle is represented by x^1. The new budget constraint after an increase in the price of good x_1 is given in red and its optimal consumption bundle is represented by x^2. The blue budget constraint is the budget constraint that returns the consumer to his original utility after the price change and the optimal bundle is represented by x^h. Now, the total effect is simply the change in good x_1 when its price changes, so we compare the quantity of x_1 consumed under the initial budget constraint with the quantity consumed under the budget constraint when p_1 increases. Note that there is an INCREASE in consumption of x_1 when p_1 increases – thus we have a Giffen good (the exact "equation" to find this is quantity of x_1 consumed at bundle x^2 minus quantity of x_1 consumed at bundle x^1). To find the income effect, compare the quantity of x_1 consumed under the new budget constraint with the quantity of x_1 consumed under the budget constraint with the new relative prices that returns the consumer to his initial utility level (the Hicksian compensation budget constraint as it is labeled). Again, to find this take the quantity of x_1 consumed at x^h and subtract the quantity of x_1 consumed at x^2. The substitution effect is simply the change in consumption of x_1 at x^1 to consumption of x_1 at x^h (take the amount of x_1 consumed at x^h and subtract the amount of x_1 consumed at x^1). Note that since this is an own-price effect on Hicksian demand it must be negative. We can do the exact same analysis for good x_2 when the price of good x_1 increases. For good x_2, its total effect is negative, while its substitution effect is positive (only two goods so they must be substitutes) but its income effect is MORE positive than its substitution effect, leading to the negative total effect.[5]

Now, there are a few additional results that rely on the Hessian matrix of the expenditure function $e(p,u)$. A Hessian matrix is simply a matrix of 2^{nd} partial derivatives. If we take the derivative of the $e(p,u)$ once with respect to p we will obtain a row vector of length L, where L is the number of goods (we will have one derivative for each of the L goods). Recall that our Hicksian demand without a subscript, $h(p,u)$ is really a vector of Hicksian demands, one for each good, or $h(p,u) = \begin{bmatrix} h_1(p,u) & h_2(p,u) \end{bmatrix}$ for the two-good world. Alternatively, we could write $\frac{\partial e(p,u)}{\partial p} = \begin{bmatrix} \frac{\partial e(p,u)}{\partial p_1} & \frac{\partial e(p,u)}{\partial p_2} \end{bmatrix}$. The technical term for the first partial derivative of a vector is the gradient, so the Hicksian demand function is nothing more than the gradient of the expenditure function in pure math terms. Note that the vectors are the same because $\frac{\partial e(p,u)}{\partial p} = h(p,u)$. The Hessian matrix is simply an LxL matrix of second partial derivatives. For our two-good world, we would have:

$$\frac{\partial^2 e(p,u)}{\partial p^2} = \begin{bmatrix} \frac{\partial^2 e(p,u)}{\partial p_1 \partial p_1} & \frac{\partial^2 e(p,u)}{\partial p_1 \partial p_2} \\ \frac{\partial^2 e(p,u)}{\partial p_2 \partial p_1} & \frac{\partial^2 e(p,u)}{\partial p_2 \partial p_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial h_1(p,u)}{\partial p_1} & \frac{\partial h_1(p,u)}{\partial p_2} \\ \frac{\partial h_2(p,u)}{\partial p_1} & \frac{\partial h_2(p,u)}{\partial p_2} \end{bmatrix}.$$

Now, a proposition:

Proposition 21 *Suppose that $u(\cdot)$ is a continuous utility function representing locally nonsatiated preference relation $\succsim$ on the consumption set $X = \mathbb{R}^L_+$. Suppose also that $h(\cdot,u)$ is continuously differentiable at (p,u) and denote its LxL derivative matrix by $D_p h(p,u)$. Then*

1. $D_p h(p,u) = D_p^2 e(p,u)$
2. $D_p h(p,u)$ *is a negative semidefinite matrix.*
3. $D_p h(p,u)$ *is a symmetric matrix.*
4. $D_p h(p,u)\, p = 0$.

We have already discussed the first result. The second and third results have to do with the fact that since $e(p,u)$ is a twice continuously differentiable concave function, it has a symmetric and negative semidefinite Hessian matrix. The fourth result follows from Euler's formula since $h(p,u)$ is homogeneous of degree zero in prices. Recall from chapter 2 that homogeneity of degree zero implies that price and wealth derivatives of Walrasian demand for any good ℓ, when weighted by these prices and wealth, sum to zero. For Hicksian demand, we have only that $h(p,u)$ is homogeneous in p, so we will only consider the effect of prices. Euler's formula from chapter 2 said if $x(p,w)$ is homogeneous of degree zero in prices and wealth, then:

$$\left(\sum_{k=1}^{L} \frac{\partial x_\ell(p,w)}{\partial p_k} p_k\right) + \frac{\partial x_\ell(p,w)}{\partial w} w = 0 \text{ for } \ell = 1, ..., L$$

Or:

$$\begin{aligned} \frac{\partial x_1(p,w)}{\partial p_1} p_1 + \frac{\partial x_1(p,w)}{\partial p_2} p_2 + \frac{\partial x_1(p,w)}{\partial w} w &= 0 \text{ for } \ell = 1 \\ \frac{\partial x_2(p,w)}{\partial p_1} p_1 + \frac{\partial x_2(p,w)}{\partial p_2} p_2 + \frac{\partial x_2(p,w)}{\partial w} w &= 0 \text{ for } \ell = 2. \end{aligned}$$

[5] Obtaining the correct sign for these effects may be a little confusing. The key is to take the amount of the good at the NEW bundle, and subtract the amount of the good at the original bundle.

Well,

$$D_p h\left(p,u\right) p = \begin{bmatrix} \frac{\partial h_1(p,u)}{\partial p_1} & \frac{\partial h_1(p,u)}{\partial p_2} \\ \frac{\partial h_2(p,u)}{\partial p_1} & \frac{\partial h_2(p,u)}{\partial p_2} \end{bmatrix} * \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial h_1(p,u)}{\partial p_1} p_1 + \frac{\partial h_1(p,u)}{\partial p_2} p_2 \\ \frac{\partial h_2(p,u)}{\partial p_1} p_1 + \frac{\partial h_2(p,u)}{\partial p_2} p_2 \end{bmatrix}$$

which is the same Euler's formula as we had before. As for the terms "symmetric" and "negative semidefinite" matrix, a symmetric matrix is simply a matrix that equals its transpose (to transpose a matrix simply take the first column of the original matrix and make that the first row of the transpose, then take the second column of the matrix and make that the second row of the transpose, etc.). So, for our two-good world, if $D_p h\left(p,u\right)$ is our matrix and $D_p^T h\left(p,u\right)$ is its transpose,

$$D_p h\left(p,u\right) = D_p^T h\left(p,u\right)$$

Or

$$\begin{bmatrix} \frac{\partial h_1(p,u)}{\partial p_1} & \frac{\partial h_1(p,u)}{\partial p_2} \\ \frac{\partial h_2(p,u)}{\partial p_1} & \frac{\partial h_2(p,u)}{\partial p_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial h_1(p,u)}{\partial p_1} & \frac{\partial h_2(p,u)}{\partial p_1} \\ \frac{\partial h_1(p,u)}{\partial p_2} & \frac{\partial h_2(p,u)}{\partial p_2} \end{bmatrix}$$

Notice that the two off diagonal elements are switched. Thus, since $D_p h\left(p,u\right)$ is symmetric, $\frac{\partial h_1(p,u)}{\partial p_2} = \frac{\partial h_2(p,u)}{\partial p_1}$, or using the expenditure function notation, $\frac{\partial^2 e(p,u)}{\partial p_1 \partial p_2} = \frac{\partial^2 e(p,u)}{\partial p_2 \partial p_1}$. Thus, it does not matter which price you use to differentiate with first – the result will be the same. For a refresher on the definition of negative semidefiniteness, take a look at the mathematical appendix. That $D_p h\left(p,u\right)$ is negative semidefinite ensures us that the own-price derivatives of the Hicksian demand function are less than or equal to zero (note that the own-price derivatives of the Hicksian demand function are the elements along the diagonal of $D_p h\left(p,u\right)$).

6 Parting comments

Chapter 3 contains a wealth of material. We began by adding some additional assumptions about the consumer's preference relation (local nonsatiation, convexity, and continuity). We then moved into the consumer's problem, which is to maximize utility. We examined Lagrangian methods for this problem, which work when the problem is well-defined (in other words, would you consider using Lagrange's method if the objective function is not differentiable). We were able to derive Walrasian demands from the UMP and state some results about those correspondences/functions. We then examined the expenditure minimization problem, and obtained the Hicksian demands from that problem. We related Walrasian and Hicksian demands to one another, and then examined the impact of a price change on the demand functions.

Why bother with all of these assumptions and results? When you are building a model you need to consider how the assumptions you make impact the behavior of your consumer – specifically, you need to know what you are allowing your consumer to do and what the consumer is not allowed to do. As an example, with the EMP we know that own-price effects on Hicksian demand must be negative, but in "the real-world" we sometimes see the case of Giffen goods. By looking solely at the EMP one cannot see positive own-price effects, but when relating this back to the Walrasian demands we see how it is possible to observe Giffen goods.

Finally, remember that this chapter represents only one possible way in which a consumer may be solving his or her problem. We assume that the consumer is a utility maximizer. That may not be the case. It is possible that the consumer is a utility satisficer, meaning that once a consumer hits a certain level of utility he/she is content (consider the case of a firm attempting to maximize profits and attain a profit goal – those are two different objectives, and analogous to a consumer being a utility maximizer or a utility satisficer). Suppose that the optimal bundle for a consumer is 500 of x_1 and 630 of x_2. What if the consumer is at a bundle of 502 of x_1 and 627 of x_2, and is just barely away from the optimal bundle (this is in the "real-world" – if the optimal bundle is 500, 630 for a problem, and you write down 502, 627, then you are wrong (!!!)). It may be the case that the increase in utility is so infinitesimal to the consumer that he doesn't bother to change consumption to the optimal level, or doesn't even notice that there is much difference between 500,630 and 502,627. In that case, the consumer is satisfied with their utility level and they are not optimizing. Now that I think of it, if this interests you then you might want to consider looking at Herb Simon and the notion of bounded rationality, which says that there are limits to the computations that the human mind can make. Further, the entire analysis we have done so far has been static – if the consumer sees prices p and has wealth w, then the consumer chooses a consumption bundle x. We have not considered a dynamic problem where a consumer might have excess of a good that is storable. Think about it: do you buy the exact same bundle every time you go to the grocery store – of course not, because unless (1) you use an awful lot of toothpaste each time you brush your teeth or (2) you brush your teeth 90 times a day using a regular quantity of toothpaste or (3) you buy an incredibly small amount of toothpaste each time you go to the store, then you usually have some toothpaste left over from a purchase of toothpaste last week, and you do not buy toothpaste this week. The only changes we have considered are changes in prices, and even then we have done this statically – hence the term "comparative statics". There has been no transition while consumers attempt to move from one optimal bundle to the next – they just "know" what their optimal bundles are with the new price, and they purchase that bundle. I'm am sure there are plenty of other objections you could raise to consumer theory.

So why do all this? First, it is the standard model of consumer theory in the literature. If you want to change the standard, you need to understand it. Second, as a first approximation utility maximization isn't horrible. If there are only minor errors (like the example with 500,630 and 502,627) then it provides a good first approximation (of course, we wouldn't know if there were large errors because all we would observe is the actual purchases at p and w, and not an individual's utility function). Third, the math with maximization is relatively easy compared to the math necessary for other concepts of a consumer's objectives. Finally, it provides a starting place for analysis. If things seem inconsistent with the model, then perhaps the model needs to be changed. Maybe the change is only a minor tweak, or maybe it needs to be completely revamped. But that is why you all will be doing research – to investigate the pros and cons of various specifications (if not specifically of the consumer's problem then of some other problem).