MAL 100: Calculus
Lecture Notes

2 Continuity, Differentiability and Taylor's theorem

2.1 Limits of real valued functions

Let $f(x)$ be defined on (a, b) except possibly at x_0.

Definition 2.1.1. *We say that* $\lim_{x \to x_0} f(x) = L$ *if, for every real number* $\epsilon > 0$, *there exists a real number* $\delta > 0$ *such that*

$$0 < |x - x_0| < \delta \implies |f(x) - L| < \epsilon. \tag{2.1}$$

Equivalently,

Remark 2.1. *The above definition is equivalent to: for any sequence* $\{x_n\}$ *with* $x_n \to x_0$, *we have* $f(x_n) \to L$ *as* $n \to \infty$.

Proof. Suppose $lim_{x \to x_0} f(x)$ exists. Take $\epsilon > 0$ and let $\{x_n\}$ be a sequence converging to x_0. Then there exists N such that $|x_n - x_0| < \delta$ for $n \geq N$. Then by the definition $|f(x_n) - L| < \epsilon$. i.e., $f(x_n) \to L$.
For the other side, assume that $x_n \to c \implies f(x_n) \to L$. Suppose the limit does not exist. i.e., $\exists\, \epsilon_0 > 0$ such that for any $\delta > 0$, and all $|x - x_0| < \delta$, we have $|f(x) - L| \geq \epsilon_0$. Then take $\delta = \frac{1}{n}$ and pick x_n in $|x_n - x_0| < \frac{1}{n}$, then $x_n \to x_0$ but $|f(x_n) - L| \geq \epsilon_0$. Not possible.

Theorem 2.1.2. *If limit exists, then it is unique.*

Proof. Proof is easy.

Examples: (i) $\lim_{x \to 1}(\frac{3x}{2} - 1) = \frac{1}{2}$. Let $\epsilon > 0$. We have to find $\delta > 0$ such that (2.1) holds with $L = 1/2$. Working backwards,

$$\frac{3}{2}|x - 1| < \epsilon \text{ whenever } |x - 1| < \delta := \frac{2}{3}\epsilon.$$

(ii) Prove that $\lim_{x \to 2} f(x) = 4$, where $f(x) = \begin{cases} x^2 & x \neq 2 \\ 1 & x = 2 \end{cases}$

Problem: Show that $\lim_{x \to 0} \sin(\frac{1}{x})$ does not exist.

Consider the sequences $\{x_n\} = \{\frac{1}{n\pi}\}, \{y_n\} = \{\frac{1}{2n\pi + \frac{\pi}{2}}\}$. Then it is easy to see that

$x_n, y_n \to 0$ and $\sin\left(\frac{1}{x_n}\right) \to 0, \sin\left(\frac{1}{y_n}\right) \to 1$. In fact, for every $c \in [-1, 1]$, we can find a sequence z_n such that $z_n \to 0$ and $\sin(\frac{1}{z_n}) \to c$ as $n \to \infty$.

By now we are familiar with limits and one can expect the following:

Theorem 2.1.3. *Suppose* $\lim_{x\to c} f(x) = L$ *and* $\lim_{x\to c} g(x) = M$, *then*

1. $\lim_{x\to c}(f(x) \pm g(x)) = L \pm M$.
2. $f(x) \leq g(x)$ *for all* x *in an open interval containing* c. *Then* $L \leq M$.
3. $\lim_{x\to c}(fg)(x) = LM$ *and when* $M \neq 0$, $\lim_{x\to c} \frac{f}{g}(x) = \frac{L}{M}$.
4. *(Sandwich): Suppose that* $h(x)$ *satisfies* $f(x) \leq h(x) \leq g(x)$ *in an interval containing* c, *and* $L = M$. *Then* $\lim_{x\to c} h(x) = L$.

Proof. We give the proof of (iii). Proof of other assertions are easy to prove. Let $\epsilon > 0$. From the definition of limit, we have $\delta_1, \delta_2, \delta_3 > 0$ such that

$$|x - c| < \delta_1 \implies |f(x) - L| < \frac{1}{2} \implies |f(x)| < N \text{ for some } N > 0,$$

$$|x - c| < \delta_2 \implies |f(x) - L| < \frac{\epsilon}{2M + 1}, \text{ and}$$
$$|x - c| < \delta_3 \implies |g(x) - M| < \frac{\epsilon}{2N}.$$

Hence for $|x - c| < \delta = \min\{\delta_1, \delta_2, \delta_3\}$, we have

$$\begin{aligned} |f(x)g(x) - LM| &\leq |f(x)g(x) - f(x)M| + |f(x)M - LM| \\ &\leq |f(x)||g(x) - M| + M|f(x) - L| \\ &< \epsilon. \end{aligned}$$

To prove the second part, note that there exists an interval $(c-\delta, c+\delta)$ around c such that $g(x) \neq 0$ in $(c - \delta, c + \delta)$.

Examples: (i) $\lim_{x\to 0} x^m = 0$ $(m > 0)$. (ii) $\lim_{x\to 0} x \sin x = 0$.

Remark: Suppose $f(x)$ is bounded in an interval containing c and $\lim_{x\to c} g(x) = 0$. Then $\lim_{x\to c} f(x)g(x) = 0$.

Examples: (i) $\lim\limits_{x\to 0} |x| \sin \dfrac{1}{x} = 0.$ $\quad (ii)$ $\lim_{x\to 0} |x| \cos \frac{1}{x} = 0.$

One sided limits: Let $f(x)$ is defined on (c, b). The right hand limit of $f(x)$ at c is L, if given $\epsilon > 0$, there exists $\delta > 0$, such that

$$x - c < \delta \implies |f(x) - L| < \epsilon.$$

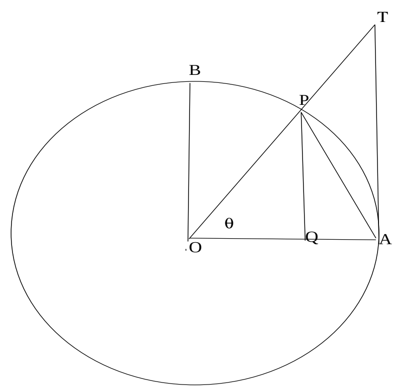
Notation: $\lim\limits_{x\to c^+} f(x) = L$. Similarly, one can define the left hand limit of $f(x)$ at b and is denoted by $\lim\limits_{x\to b^-} f(x) = L$.

Both theorems above holds for right and left limits. Proof is easy.

Problem: Show that $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$.
Solution: Consider the unit circle centred at $O(0,0)$ and passing through $A(1,0)$ and $B(0,1)$. Let Q be the projection of P on $x-$axis and let T be such that A is the projection of T. Let OT be the ray with $\angle AOT = \theta, 0 < \theta < \pi/2$. Let P be the point of intersection of OT and circle. Then ΔOPQ and ΔOTA are similar triangles and hence, Area of $\Delta OAP <$ Area of sector $OAP <$ area of ΔOAT. i.e.,

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$$

dividing by $\sin\theta$, we get $1 > \dfrac{\sin\theta}{\theta} > \cos\theta$. Now $\lim\limits_{\theta\to 0^+} \cos\theta = 1$ implies that $\lim\limits_{\theta\to 0^+} \dfrac{\sin\theta}{\theta} = 1$. Now use the fact that $\dfrac{\sin\theta}{\theta}$ is even function.



At this stage, it is not difficult to prove the following:

Theorem 2.1.4. $\lim_{x\to a} f(x) = L$ *exists* $\iff$ $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L.$

Limits at infinity and infinite limits

Definition 2.1.5. *$f(x)$ has limit L as x approaches $+\infty$, if for any given $\epsilon > 0$, there exists $M > 0$ such that*

$$x > M \implies |f(x) - L| < \epsilon.$$

Similarly, one can define limit as x approaches $-\infty$.

Problem: (i) $\lim\limits_{x\to\infty} \dfrac{1}{x} = 0$, $\quad (ii)$ $\lim_{x\to-\infty} \frac{1}{x} = 0$. $\quad (iii)$ $\lim\limits_{x\to\infty} \sin x$ does not exist.
Solution: (i) and (ii) are easy. For (iii), Choose $x_n = n\pi$ and $y_n = \frac{\pi}{2} + 2n\pi$. Then $x_n, y_n \to \infty$ and $\sin x_n = 0$, $\sin y_n = 1$. Hence the limit does not exist.

Above two theorems on limits hold in this also.

Definition 2.1.6. *(Horizontal Asymptote:) A line $y = b$ is a horizontal asymptote of $y = f(x)$ if either $\lim\limits_{x\to\infty} f(x) = b$ or $\lim\limits_{x\to-\infty} f(x) = b$.*

Examples: (i) $y = 1$ is a horizontal asymptote for $1 + \frac{1}{x+1}$
Definition 2.1.7. *(Infinite Limit): A function $f(x)$ approaches ∞ ($f(x) \to \infty$) as $x \to x_0$ if, for every real $B > 0$, there exists $\delta > 0$ such that*

$$0 < |x - x_0| < \delta \implies f(x) > B.$$

Similarly, one can define for $-\infty$. Also one can define one sided limit of $f(x)$ approaching ∞ or $-\infty$.

Examples (i) $\lim\limits_{x\to 0} \dfrac{1}{x^2} = \infty$, $\quad (ii)$ $\lim_{x\to 0} \frac{1}{x^2} \sin(\frac{1}{x})$ does not exist.

For (i) given $B > 0$, we can choose $\delta \le \frac{1}{\sqrt{B}}$. For (ii), choose a sequence $\{x_n\}$ such that $\sin \frac{1}{x_n} = 1$, say $\frac{1}{x_n} = \frac{\pi}{2} + 2n\pi$ and $\frac{1}{y_n} = n\pi$. Then $\lim\limits_{n\to\infty} f(x_n) = \dfrac{1}{x_n^2} \to \infty$ and $\lim\limits_{n\to\infty} f(y_n) = 0$, though $x_n, y_n \to 0$ as $n \to \infty$.

Definition 2.1.8. *(Vertical Asymptote:) A line $x = a$ is a vertical asymptote of $y = f(x)$ if either $\lim\limits_{x\to a^+} f(x) = \pm\infty$ or $\lim\limits_{x\to a^-} f(x) = \pm\infty$.*

Example: $f(x) = \frac{x+3}{x+2}$.
$x = -2$ is a vertical asymptote and $y = 1$ is a horizontal asymptote.

2.2 Continuous functions

Definition 2.2.1. *A real valued function $f(x)$ is said to be continuous at $x = c$ if*
(i) $c \in domain(f)$
(ii) $\lim\limits_{x \to c} f(x)$ exists
(iii) The limit in (ii) is equal to $f(c)$.

In other words, for every sequence $x_n \to c$, we must have $f(x_n) \to f(c)$ as $n \to \infty$. i.e., for a given $\epsilon > 0$, there exists $\delta > 0$ such that

$$0 < |x - c| < \delta \implies |f(x) - f(c)| < \epsilon.$$

Examples: (i) $f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$ is continuous at 0.
Let $\epsilon > 0$. Then $|f(x) - f(0)| \leq |x^2|$. So it is enough to choose $\delta = \sqrt{\epsilon}$.

(ii) $g(x) = \begin{cases} \frac{1}{x} \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ is not continuous at 0.
Choose $\frac{1}{x_n} = \frac{\pi}{2} + 2n\pi$. Then $\lim x_n = 0$ and $f(x_n) = \frac{1}{x_n} \to \infty$.

The following theorem is an easy consequence of the definition.

Theorem 2.2.2. *Suppose f and g are continuous at c. Then*
(i) $f \pm g$ is also continuous at c
(ii) fg is continuous at c
(iii) $\dfrac{f}{g}$ is continuous at c if $g(c) \neq 0$.

Theorem 2.2.3. *Composition of continuous functions is also continuous i.e., if f is continuous at c and g is continuous at $f(c)$ then $g(f(x))$ is continuous at c.*

Corollary: If $f(x)$ is continuous at c, then $|f|$ is also continuous at c.

Theorem 2.2.4. *If f, g are continuous at c, then* $\max(f, g)$ *is continuous at c.*

Proof. Proof follows from the relation

$$\max(f, g) = \frac{1}{2}(f + g) + \frac{1}{2}|f - g|$$

and the above theorems.

Types of discontinuities

Removable discontinuity: $f(x)$ is defined every where in an interval containing a except at $x = a$ and limit exists at $x = a$ OR $f(x)$ is defined also at $x = a$ and limit is NOT equal to function value at $x = a$. Then we say that $f(x)$ has removable discontinuity at $x = a$. These functions can be extended as continuous functions by defining the value of f to be the limit value at $x = a$.

Example: $f(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$. Here limit as $x \to 0$ is 1. But $f(0)$ is defined to be 0.

Jump discontinuity: The left and right limits of $f(x)$ exists but not equal. This type of discontinuities are also called discontinuities of first kind.

Example: $f(x) = \begin{cases} 1 & x \leq 0 \\ -1 & x \geq 0 \end{cases}$. Easy to see that left and right limits at 0 are different.

Infinite discontinuity: Left or right limit of $f(x)$ is ∞ or $-\infty$.

Example: $f(x) = \frac{1}{x}$ has infinite discontinuity at $x = 0$.

Discontinuity of second kind: If either $\lim\limits_{x \to c^-} f(x)$ or $\lim\limits_{x \to c^+} f(x)$ does not exist, then c is called discontinuity of second kind.

Example: Let $f(x) = \begin{cases} 1 & x \in Q \\ 0 & x \notin Q \end{cases}$. Then lett and right limits does not exist at any point.

Example: Let $f(x) = \begin{cases} x & x \in Q \\ 0 & x \notin Q \end{cases}$. Then f is continuous only at $x = 0$.

Properties of continuous functions

Definition 2.2.5. *(Closed set): A subset A of $\mathbb{R}$ is called closed set if A contains all its limit points. (i.e., if $\{x_n\} \subset A$ and $x_n \to c$, then $c \in A$).*

Theorem 2.2.6. *Continuous functions on closed, bounded interval is bounded.*

Proof. Let $f(x)$ be continuous on $[a, b]$ and let $\{x_n\} \subset [a, b]$ be a sequence such that $f(x_n) \to \infty$. Then $\{x_n\}$ is a bounded sequence and hence there exists a subsequence $\{x_{n_k}\}$ which converges to c. Then $f(x_{n_k}) \to f(c)$, a contradiction.

Theorem 2.2.7. *Let $f(x)$ be a continuous function on closed, bounded interval $[a, b]$. Then maximum and minimum of functions are achieved in $[a, b]$.*

Proof. Let $\{x_n\}$ be a sequence such that $f(x_n) \to \max f$. Then $\{x_n\}$ is bounded and hence by Bolzano-Weierstrass theorem, there exists a subsequence x_{n_k} such that $x_{n_k} \to x_0$ for some x_0. $a \leq x_n \leq b$ implies $x_0 \in [a, b]$. Since f is continuous, $f(x_{n_k}) \to f(x_0)$. Hence $f(x_0) = \max f$. The attainment of minimum can be proved by noting that $-f$ is also continuous and $\min f = -\max(-f)$.

Remark: Closed and boundedness of the interval is important in the above theorem. Consider the examples (i) $f(x) = \frac{1}{x}$ on $(0, 1)$ (ii) $f(x) = x$ on $I\!R$.

Theorem 2.2.8. *Let $f(x)$ be a continuous function on $[a, b]$ and let $f(c) > 0$ for some $c \in (a, b)$, Then there exists $\delta > 0$ such that $f(x) > 0$ in $(c - \delta, c + \delta)$.*

Proof. Let $\epsilon = \frac{1}{2} f(c) > 0$. Since $f(x)$ is continuous at c, there exists $\delta > 0$ such that

$$|x - c| < \delta \implies |f(x) - f(c)| < \frac{1}{2} f(c)$$

i.e., $-\frac{1}{2} f(c) < f(x) - f(c) < \frac{1}{2} f(c)$. Hence $f(x) > \frac{1}{2} f(c)$ for all $x \in (c - \delta, c + \delta)$.

Corollary: Suppose a continuous functions $f(x)$ satisfies $\int_a^b f(x)\phi(x)dx = 0$ for all continuous functions $\phi(x)$ on $[a, b]$. Then $f(x) \equiv 0$ on $[a, b]$.

Proof. Suppose $f(c) > 0$. Then by above theorem $f(x) > 0$ in $(c - \delta, c + \delta)$. Choose $\phi(x)$ so that $\phi(x) > 0$ in $(c - \delta/2, c + \delta/2)$ and is 0 otherwise. Then $\int_a^b f(x)\phi(x) > 0$. A contradiction.
Alternatively, one can choose $\phi(x) = f(x)$.

Theorem 2.2.9. *Let $f(x)$ be a continuous function on $I\!R$ and let $f(a)f(b) < 0$ for some a, b. Then there exits $c \in (a, b)$ such that $f(c) = 0$.*

Proof. Assume that $f(a) < 0 < f(b)$. Let $S = \{x \in [a, b] : f(x) < 0\}$. Then $[a, a + \delta) \subset S$ for some $\delta > 0$ and S is bounded. Let $c = \sup S$. We claim that $f(c) = 0$. Take $x_n = c + \frac{1}{n}$, then $x_n \notin S$, $x_n \to c$. Therefore, $f(c) = \lim f(x_n) \geq 0$. On the otherhand, taking $y_n = c - \frac{1}{n}$, we see that $y_n \in S$ for n large and $y_n \to c$, $f(c) = \lim f(y_n) \leq 0$. Hence $f(c) = 0$.

Corollary: Intermediate value theorem: Let $f(x)$ be a continuous function on $[a, b]$ and let $f(a) < y < f(b)$. Then there exists $c \in (a, b)$ such that $f(c) = y$

Remark: A continuous function assumes all values between its maximum and minimum.

Problem: (fixed point): Let $f(x)$ be a continuous function from $[0,1]$ into $[0,1]$. Then show that there is a point $c \in [0,1]$ such that $f(c) = c$.

Define the function $g(x) = f(x) - x$. Then $g(0) \geq 0$ and $g(1) \leq 0$. Now Apply Intermediate value theorem.

Application: Root finding: To find the solutions of $f(x) = 0$, one can think of defining a new function g such that $g(x)$ has a fixed point, which in turn satisfies $f(x) = 0$.
Example: (1) $f(x) = x^3 + 4x^2 - 10$ in the interval $[1,2]$. Define $g(x) = \left(\frac{10}{4+x}\right)^{1/2}$. We can check that g maps $[1,2]$ into $[1,2]$. So g has fixed point in $[1,2]$ which is also solution of $f(x) = 0$. Such fixed points can be obtained as limit of the sequence $\{x_n\}$, where $x_{n+1} = g(x_n), x_0 \in (1,2)$. Note that

$$g'(x) = \frac{\sqrt{10}}{(4+x)^{3/2}} < \frac{1}{2}.$$

By Mean Value Theorem, $\exists\, z$ (see next section)

$$|x_{n+1} - x_n| = |g'(z)||x_n - x_{n-1}| \leq \frac{1}{2}|x_n - x_{n-1}|$$

Iterating this, we get

$$|x_{n+1} - x_n| < \frac{1}{2^n}|x_1 - x_0|.$$

Therefore, $\{x_n\}$ is a Cauchy sequence. (see problem after Theorem 1.4.4).

Uniformly continuous functions

Definition: A function $f(x)$ is said to be uniformly continuous on a set S, if for given $\epsilon > 0$, there exists $\delta > 0$ such that

$$x, y \in S, \;\; |x - y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

Here δ depends only on ϵ, not on x or y.

Proposition: If $f(x)$ is uniformly continuous function $\iff$ for ANY two sequences $\{x_n\}, \{y_n\}$ such that $|x_n - y_n| \to 0$, we have $|f(x_n) - f(y_n)| \to 0$ as $n \to \infty$.

Proof. Suppose not. Then there exists $\{x_n\}, \{y_n\}$ such that $|x_n - y_n| \to 0$ and $|f(x_n) -$

$f(y_n)| > \eta$ for some $\eta > 0$. Then it is clear that for $\epsilon = \eta$, there is no δ for which $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$. Because the above sequence satisfies $|x - y| < \delta$, but its image does not.

For converse, assume that for any two sequences $\{x_n\}, \{y_n\}$ such that $|x_n - y_n| \to 0$ we have $|f(x_n) - f(y_n)| \to 0$. Suppose f is not uniformly continuous. Then by the definition there exists ϵ_0 such that for any $\delta > 0$, $|x - y| < \delta \implies |f(x) - f(y)| > \epsilon_0$. Now take $\delta = \frac{1}{n}$ and choose x_n, y_n such that $|x_n - y_n| < 1/n$. Then $|f(x_n) - f(y_n)| > \epsilon_0$. A contradiction.

Examples: *(i)* $f(x) = x^2$ is uniformly continuous on bounded interval $[a, b]$.
Note that $|x^2 - y^2| \leq |x + y||x - y| \leq 2b|x - y|$. So one can choose $\delta < \frac{\epsilon}{2b}$.

(ii) $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0, 1)$.
Take $x_n = \frac{1}{n+1}, y_n = \frac{1}{n}$, then for n large $|x_n - y_n| \to 0$ but $|f(x_n) - f(y_n)| = 1$.

(iii) $f(x) = x^2$ is not uniformly continuous on $I\!R$.
Take $x_n = n + \frac{1}{n}$ and $y_n = n$. Then $|x_n - y_n| = \frac{1}{n} \to 0$, but $|f(x_n) - f(y_n)| = 2 + \frac{1}{n^2} > 2$.

Remarks:

1. It is easy to see from the definition that if f, g are uniformly continuous, then $f \pm g$ is also uniformly continuous.

2. If f, g are uniformly continuous, then fg need not be uniformly continuous. This can be seen by noting that $f(x) = x$ is uniformly continuous on $I\!R$ but x^2 is not uniformly continuous on $I\!R$.

Theorem 2.2.10. *A continuous function $f(x)$ on a closed, bounded interval $[a, b]$ is uniformly continuous.*

Proof. Suppose not. Then there exists $\epsilon > 0$ and sequences $\{x_n\}$ and $\{y_n\}$ in $[a, b]$ such that $|x_n - y_n| < \frac{1}{n}$ and $|f(x_n) - f(y_n)| > \epsilon$. But then by Bolzano-Weierstrass theorem, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges to x_0. Also $y_{n_k} \to x_0$. Now since f is continuous, we have $f(x_0) = \lim f(x_{n_k}) = \lim f(y_{n_k})$. Hence $|f(x_{n_k}) - f(y_{n_k})| \to 0$, a contradiction.

Corollary: Suppose $f(x)$ has only removable discontinuities in $[a, b]$. Then $\tilde{f}$, the extension of f, is uniformly continuous.
Example: $f(x) = \frac{\sin x}{x}$ for $x \neq 0$ and 0 for $x = 0$ on $[0, 1]$.

Theorem 2.2.11. *Let f be a uniformly continuous function and let $\{x_n\}$ be a cauchy sequence. Then $\{f(x_n)\}$ is also a Cauchy sequence.*

Proof. Let $\epsilon > 0$. As f is uniformly continuous, there exists $\delta > 0$ such that

$$|x - y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

Since $\{x_n\}$ is a Cauchy sequence, there exists N such that

$$m, n > N \implies |x_n - x_m| < \delta.$$

Therefore $|f(x_n) - f(x_m)| < \epsilon$.

Example: $f(x) = \frac{1}{x^2}$ is not uniformly continuous on $(0, 1)$.

The sequence $x_n = \frac{1}{n}$ is cauchy but $f(x_n) = n^2$ is not. Hence f cannot be uniformly continuous.

2.3 Differentiability

Definition 2.3.1. *A real valued function $f(x)$ is said to be differentiable at x_0 if*

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \quad \textit{exists.}$$

This limit is called the derivative of f at x_0, denoted by $f'(x_0)$.

Example: $f(x) = x^2$

$$f'(x) = \lim_{h \to 0} \frac{2xh + h^2}{h} = 2x.$$

Theorem 2.3.2. *If $f(x)$ is differentiable at a, then it is continuous at a.*

Proof. For $x \neq a$, we may write,

$$f(x) = (x - a)\frac{f(x) - f(a)}{(x - a)} + f(a).$$

Now taking the limit $x \to a$ and noting that $\lim(x - a) = 0$ and $\lim \frac{f(x) - f(a)}{(x - a)} = f'(a)$, we get the result.

Theorem 2.3.3. *Let f, g be differentiable at $c \in (a, b)$. Then $f \pm g, fg$ and $\dfrac{f}{g}$ $(g(c) \neq 0)$ is also differentiable at c*

Proof. We give the proof for product formula: First note that

$$\frac{(fg)(x) - (fg)(c)}{x - c} = f(x)\frac{g(x) - g(c)}{x - c} + g(c)\frac{f(x) - f(c)}{x - c}.$$

Now taking the limit $x \to c$, we get the product formula

$$(fg)'(c) = f(c)g'(c) + f'(c)g(c).$$

Since $g(c) \neq 0$ and g is continuous, we get $g(x) \neq 0$ in a small interval around c. Therefore

$$\frac{f}{g}(x) - \frac{f}{g}(c) = \frac{g(c)f(x) - g(c)f(c) + g(c)f(c) - g(x)f(c)}{g(x)g(c)}$$

Hence

$$\frac{(f/g)(x) - (f/g)(c)}{x - c} = \left\{ g(c)\frac{f(x) - f(c)}{x - c} - f(c)\frac{g(x) - g(c)}{x - c} \right\} \frac{1}{g(x)g(c)}$$

Now taking the limit $x \to c$, we get

$$(\frac{f}{g})'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{g^2(c)}.$$

Theorem 2.3.4. *(Chain Rule): Suppose $f(x)$ is differentiable at c and g is differentiable at $f(c)$, then $h(x) := g(f(x))$ is differentiable at c and*

$$h'(c) = g'(f(c))f'(c)$$

Proof. Define the function h as

$$h(y) = \begin{cases} \frac{g(y) - g(f(c))}{y - f(c)} & y \neq f(c) \\ g'(f(c)) & y = f(c) \end{cases}$$

Then the function h is continuous at $y = f(c)$ and $g(y) - g(f(c)) = h(y)(y - f(c))$, so

$$\frac{g(f(x)) - g(f(c))}{x - c} = h(f(x))\frac{f(x) - f(c)}{x - c}.$$

Now taking limit $x \to c$, we get the required formula.

Local extremum: A point $x = c$ is called local maximum of $f(x)$, if there exists $\delta > 0$ such that

$$0 < |x - c| < \delta \implies f(c) \geq f(x).$$

Similarly, one can define local minimum: $x = b$ is a local minimum of $f(x)$ if there exists $\delta > 0$ such that

$$0 < |x - b| < \delta \implies f(b) \leq f(x).$$

Theorem 2.3.5. *Let $f(x)$ be a differentiable function on (a, b) and let $c \in (a, b)$ is a local maximum of f. Then $f'(c) = 0$.*

Proof. Let δ be as in the above definition. Then

$$x \in (c, c + \delta) \implies \frac{f(x) - f(c)}{x - c} \leq 0$$

$$x \in (c - \delta, c) \implies \frac{f(x) - f(c)}{x - c} \geq 0.$$

Now taking the limit $x \to c$, we get $f'(c) = 0$.

Theorem 2.3.6. *Rolle's Theorem: Let $f(x)$ be a continuous function on $[a, b]$ and differentiable on (a, b) such that $f(a) = f(b)$. Then there exists $c \in (a, b)$ such that $f'(c) = 0$.*

Proof. If $f(x)$ is constant, then it is trivial. Suppose $f(x_0) > f(a)$ for some $x_0 \in (a, b)$, then f attains maximum at some $c \in (a, b)$. Other possibilities can be worked out similarly.

Theorem 2.3.7. *Mean-Value Theorem (MVT): Let f be a continuous function on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that*

$$f(b) - f(a) = f'(c)(b - a).$$

Proof. Let $l(x)$ be a straight line joining $(a, f(a))$ and $(b, f(b))$. Consider the function $g(x) = f(x) - l(x)$. Then $g(a) = g(b) = 0$. Hence by Rolle's theorem

$$0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}$$

Corollary: If f is a differentiable function on (a, b) and $f' = 0$, then f is constant.

Proof. By mean value theorem $f(x) - f(y) = 0$ for all $x, y \in (a, b)$.

Example: Show that $|\cos x - \cos y| \leq |x - y|$.
Use Mean-Value theorem and the fact that $|\sin x| \leq 1$.

Problem: If $f(x)$ is differentiable and $\sup |f'(x)| < C$ for some C. Then, f is uniformly continuous.
Apply mean value theorem to get $|f(x) - f(y)| \leq C|x - y|$ for all x, y.

Definition: A function $f(x)$ is strictly increasing on an interval I, if for $x, y \in I$ with $x < y$ we have $f(x) < f(y)$. We say f is strictly decreasing if $x < y$ in I implies $f(x) > f(y)$.

Theorem 2.3.8. *A differentiable function f is (i) strictly increasing in (a, b) if $f'(x) > 0$ for all $x \in (a, b)$. (ii) strictly decreasing in (a, b) if $f'(x) < 0$.*

Proof. Choose x, y in (a, b) such $x < y$. Then by MVT, for some $c \in (x, y)$

$$\frac{f(x) - f(y)}{x - y} = f'(c) > 0.$$

Hence $f(x) < f(y)$.

2.4 Taylor's theorem and Taylor Series

Let f be a k times differentiable function on an interval I of $\mathbb{R}$. We want to approximate this function by a polynomial $P_n(x)$ such that $P_n(a) = f(a)$ at a point a. Moreover, if the derivatives of f and P_n also equal at a then we see that this approximation becomes more accurate in a neighbourhood of a. So the best coefficients of the polynomial can be calculated using the relation $f^{(k)}(a) = P_n^{(k)}(a), k = 0, 1, 2, ..., n$. The best is in the sense that if $f(x)$ itself is a polynomial of degree less than or equal to n, then both f and P_n are equal. This implies that the polynomial is $\displaystyle\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$. Then we write $f(x) = P_n(x) + R_n(x)$ in a neighbourhood of a. From this, we also expect the $R_n(x) \to 0$ as $x \to a$. In fact, we have the following theorem known as **Taylor's theorem**:

Theorem 2.4.1. *Let $f(x)$ and its derivatives of order m are continuous and $f^{(m+1)}(x)$ exists in a neighbourhood of $x = a$. Then there exists $c \in (a, x)$(or $c \in (x, a)$) such that*

$$f(x) = f(a) + f'(a)(x-a) + ... + f^{(m)}(a)\frac{(x-a)^m}{m!} + R_m(x)$$

where $R_m(x) = \dfrac{f^{(m+1)}(c)}{(m+1)!}(x-a)^{m+1}$.

Proof. Define the functions F and g as

$$F(y) = f(x) - f(y) - f'(y)(x-y) - .... - \frac{f^{(m)}(y)}{m!}(x-y)^m,$$

$$g(y) = F(y) - \left(\frac{x-y}{x-a}\right)^{m+1} F(a).$$

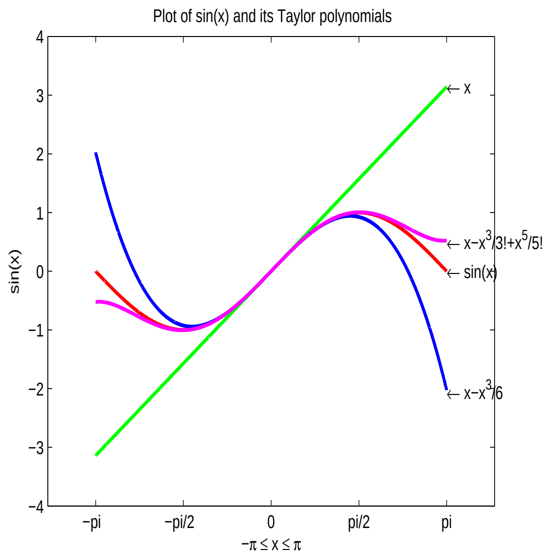


Figure 1: Approximation of sin(x) by Taylor's polynomials

Then it is easy to check that $g(a) = 0$. Also $g(x) = F(x) = f(x) - f(x) = 0$. Therefore, by Rolle's theorem, there exists some $c \in (a, x)$ such that

$$g'(c) = 0 = F'(c) + \frac{(m+1)(x-c)^m}{(x-a)^{m+1}} F(a).$$

On the other hand, from the definition of F,

$$F'(c) = -\frac{f^{(m+1)}(c)}{m!}(x-c)^m.$$

Hence $F(a) = \dfrac{(x-a)^{m+1}}{(m+1)!} f^{(m+1)}(c)$ and the result follows.

Examples:

(i) $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + ... + \dfrac{x^n}{n!}e^c, c \in (0, x)$ or $(x, 0)$ depending on the sign of x.

(ii) $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + ... + \dfrac{x^n}{n!}\sin(c + \dfrac{n\pi}{2}), c \in (0, x)$ or $(x, 0)$.

(iii) $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + ... + \dfrac{x^n}{n!}\cos(c + \dfrac{n\pi}{2}), c \in (0, x)$ or $(x, 0)$.

Problem: Find the order n of Taylor Polynomial P_n, about $x = 0$ to approximate e^x in $(-1, 1)$ so that the error is not more than 0.005

Solution: We know that $p_n(x) = 1 + x + ... + \frac{x^n}{n!}$. The maximum error in [-1,1] is

$$|R_n(x)| \leq \frac{1}{(n+1)!} \max_{[-1,1]} |x|^{n+1} e^x \leq \frac{e}{(n+1)!}.$$

So n is such that $\frac{e}{(n+1)!} \leq 0.005$ or $n \geq 5$.

Problem: Find the interval of validity when we approximate $\cos x$ with 2nd order polynomial with error tolerance 10^{-4}.

Solution: Taylor polynomial of degree 2 for $\cos x$ is $1 - \frac{x^2}{2}$. So the remainder is $(\sin c)\frac{x^3}{3!}$. Since $|\sin c| \leq 1$, the error will be atmost 10^{-4} if $|\frac{x^3}{3!}| \leq 10^{-4}$. Solving this gives $|x| < 0.084$

Taylor's Series

Suppose f is infinitely differentiable at a and if the remainder term in the Taylor's formula, $R_n(x) \to 0$ as $n \to \infty$. Then we write

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

This series is called Taylor series of $f(x)$ about the point a.
Suppose there exists $C = C(x) > 0$, independent of n, such that $|f^{(n)}(x)| \leq C(x)$. Then $|R_n(x)| \to 0$ if $\lim\limits_{n\to\infty} \dfrac{|x-a|^{n+1}}{(n+1)!} = 0$. For any fixed x and a, we can always find N such that $|x-a| < N$. Let $q := \frac{|x-a|}{N} < 1$. Then

$$\begin{aligned} \left|\frac{(x-a)^{n+1}}{(n+1)!}\right| &= \left|\frac{|x-a|}{1}\right|\left|\frac{|x-a|}{2}\right| ... \left|\frac{|x-a|}{N-1}\right|\left|\frac{|x-a|}{N}\right| ... \left|\frac{|x-a|}{n+1}\right| \\ &< \left|\frac{|x-a|^{N-1}}{(N-1)!}\right| q^{n-N+2} \end{aligned}$$

$\to 0$ as $n \to \infty$ thanks to $q < 1$.

In case of $a = 0$, the formula obtained in Taylor's theorem is known as *Maclaurin's formula* and the corresponding series that one obtains is known as *Maclaurin's series.*

Example: (i) $f(x) = e^x$.
In this case $R_n(x) = \dfrac{x^{n+1}}{(n+1)!} f^{(n+1)}(c) = \dfrac{x^{n+1}}{(n+1)!} e^c = \dfrac{x^{n+1}}{(n+1)!} e^{\theta x}$, for some $\theta \in (0,1)$.

Therefore for any given x fixed, $\lim_{n\to\infty} |R_n(x)| = \lim_{n\to\infty} \left(\frac{x^{n+1}}{(n+1)!} \right) e^{\theta x} = 0.$

Example: (ii) $f(x) = \sin x$.

In this case it is easy to see that $|R_n(x)| \leq \frac{|x|^{2n+1}}{(2n+1)!} |\sin(c + \frac{n\pi}{2})|$. Now use the fact that $|\sin x| \leq 1$ and follow as in example (i).

Maxima and Minima: Derivative test

Definition 2.4.2. *A point $x = a$ is called critical point of the function $f(x)$ if $f'(a) = 0$.*

Second derivative test: A point $x = a$ is a local maxima if $f'(a) = 0, f''(a) < 0$.

Suppose $f(x)$ is continuously differentiable in an interval around $x = a$ and let $x = a$ be a critical point of f. Then $f'(a) = 0$. By Taylor's theorem around $x = a$, there exists, $c \in (a, x)$ (or $c \in (x, a)$),

$$f(x) - f(a) = \frac{f''(c)}{2}(x - a)^2.$$

If $f''(a) < 0$. Then by Theorem2.2.8, $f''(c) < 0$ in $|x - a| < \delta$. Hence $f(x) < f(a)$ in $|x - a| < \delta$, which implies that $x = a$ is a local maximum.

Similarly, one can show the following for local minima: $x = a$ is a local minima if $f'(a) = 0, f''(a) > 0$.

Also the above observations show that if $f'(a) = 0, f''(a) = 0$ and $f^{(3)}(a) \neq 0$, then the sign of $f(x) - f(a)$ depends on $(x - a)^3$. i.e., it has no constant sign in any interval containing a. Such point is called point of inflection or saddle point.

We can also derive that if $f'(a) = f''(a) = f^{(3)}(a) = 0$, then we again have $x = a$ is a local minima if $f^{(4)}(a) > 0$ and is a local maxima if $f^{(4)}(a) < 0$.

Summarizing the above, we have:

Theorem 2.4.3. *Let f be a real valued function that is differentiable $2n$ times and $f^{(2n)}$ is continuous at $x = a$. Then*

1. *If $f^{(k)}(a) = 0$ for $k = 1, 2, ...2n - 1$ and $f^{(2n)}(a) > 0$ then a is a point of local minimum of $f(x)$*

2. *If $f^{(k)}(a) = 0$ for $k = 1, 2, ...2n-1$ and $f^{(2n)}(a) < 0$ then a is a point local maximum of $f(x)$.*

3. *If* $f^{(k)} = 0$ *for* $k = 1, 2, ...2n-2$ *and* $f^{(2n-1)}(a) \neq 0$*, then* a *is point of inflection. i.e.,* f *has neither local maxima nor local minima at* $x = a$*.*

L'Hospitals Rule:
Suppose $f(x)$ and $g(x)$ are differentiable n times, $f^{(n)}, g^{(n)}$ are continuous at a and $f^{(k)}(a) = g^{(k)}(a) = 0$ for $k = 0, 1, 2, ..., n-1$. Also if $g^{(n)}(a) \neq 0$. Then by Taylor's theorem,

$$\begin{aligned} \lim_{x \to a} \frac{f(x)}{g(x)} &= \lim_{x \to a} \frac{f^{(n)}(c)}{g^{(n)}(c)} \\ &= \frac{f^{(n)}(a)}{g^{(n)}(a)} \end{aligned}$$

In the above, we used the fact that $g^{(n)}(x) \neq 0$ "near $x = a$" and $g^{(n)}(c) \to g^{(n)}(a)$ as $x \to a$.
Similarly, we can derive a formula for limits as x approaches infinity by taking $x = \frac{1}{y}$.

$$\begin{aligned} \lim_{x \to \infty} \frac{f(x)}{g(x)} &= \lim_{y \to 0} \frac{f(1/y)}{g(1/y)} \\ &= \lim_{y \to 0} \frac{(-1/y^2) f'(1/y)}{(-1/y^2) g'(1/y)} \\ &= \lim_{x \to \infty} \frac{f'(x)}{g'(x)} \end{aligned}$$

References

[1] Elementary Analysis: The Theory of Calculus, K. A. Ross.

[2] Calculus, G. B. Thomas and R. L. Finney, Pearson .