Full Model Notes: 9 May 2018

$$\text{Co-dimension 2 cusp bifurcation:} \quad \dot{x} \;=\; a + bx - x^3 \tag{1}$$

$$\text{FitzHugh-Nagumo bifurcation:} \quad \dot{x} \;=\; a - bx - x^3 + cy \;, \qquad \dot{y} \;=\; -x - y \tag{2}$$

$$\text{Plasma Dispersion Function:} \;\; X(\zeta) \;=\; i\sqrt{\pi}\, w(\zeta) \;=\; i\sqrt{\pi}\, e^{-\zeta^2} \mathrm{erfc}(-i\zeta) \;=\; \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{e^{-t^2}}{t - \zeta}\, \mathrm{d}t \;, \;\; \zeta \;\equiv\; \frac{\omega/k}{v_T} \tag{3}$$

$$\text{Approx. SI Units at Edge:} \quad n \;\approx\; 0.5 \times 10^{19}\ \mathrm{m}^{-3} \;, \;\; T \;\approx\; 100\ e\mathrm{V} \;, \;\; v_{T_i} \;\approx\; 10^5\ \mathrm{m/s} \;, \;\; \rho_{\theta i} \;\approx\; 10^{-4}\ \mathrm{m} \tag{4}$$

Plasma Parameters

All temperatures are in eV.

$$\epsilon \;=\; \frac{a_m}{R} \;, \quad q \;=\; \frac{\epsilon\, B_\phi}{B_\theta} \;, \quad v_{T_j} \;=\; \sqrt{\frac{2\, e\, T}{m_j}} \;, \quad \rho_{\theta j} \;=\; \frac{m_j\, v_{T_j}}{e\, B_\theta} \;=\; \sqrt{\frac{2\, m_j\, T}{e^2\, B_\theta^2}} \tag{5}$$

$$B_\theta \;=\; \frac{\mu_0\, I_\phi}{2\pi\, a_m} \;, \quad \omega_t \;=\; \frac{v_{T_i}}{q\, R} \;, \quad \omega_{bj} \;=\; \frac{\epsilon^{3/2}\, v_{T_j}}{q\, R} \;, \quad w_{bi} \;=\; \rho_{\theta i}\, \sqrt{\epsilon} \tag{6}$$

$$\nu_{ei} \;=\; 1.33 \times 10^5 \frac{n_{20}}{(T_{\mathrm{keV}})^{3/2}} \;=\; 4.2058 \times 10^{-5} \frac{n}{(T_{\mathrm{eV}})^{3/2}} \;, \quad \nu_{ii} \;=\; 1.2\, \nu_{ei} \sqrt{\frac{m_e}{m_i}} \;, \quad \nu_{*j} \;=\; \frac{\nu_{ij}}{\omega_{bj}} \tag{7}$$

$$\langle \sigma_{\mathrm{cx}} v \rangle \;=\; \underbrace{10^{-14} \sqrt[3]{100\, T}}_{\text{Itoh 1989}} \;=\; \underbrace{1.985 \times 10^{-14} \sqrt{T}}_{\text{Rozhansky 2001}} \;, \quad \text{Depricated: } \nu_{in_0} \;=\; a_{in_0}\, \omega_{bi} \;, \quad \nu_{\mathrm{eff}} \;=\; \nu_{ii} + \nu_{in_0} \tag{8}$$

$$\text{Non-formal:} \;\; n_0 \;=\; \frac{n_0(0)}{(1 + \exp\left[k(x - d)\right])} \;, \quad n_0(0) \;=\; \frac{\theta\, \Gamma_c}{v_{T_i}} \;\; \text{for} \;\; 0 < \theta \leq 1 \tag{9}$$

Model Forms

Domain with boundary:

$$\Omega \;=\; \left\{ x, t \;\in\; \mathbb{R}^2 \,|\, (0 \leq x \leq L) \text{ and } (t \geq 0) \right\} , \qquad \delta\Omega \;=\; \{ x, t \;\in\; \Omega \,|\, x = 0 \text{ and } x = L \} \tag{10}$$

Electric field normalization, energy definition, diffusivity relation, dielectric constant, and viscosity:

$$Z \;\equiv\; \frac{e\, \rho_\theta\, E_r}{T} \;, \quad U \;=\; \frac{n\, T}{\gamma - 1} \;, \quad \chi \;=\; \frac{D}{\zeta(\gamma - 1)} \;, \quad \epsilon \;=\; \frac{B_\theta^2}{B^2 \nu_i} \;, \quad \mu \approx m_i\, n\, v_{T_i}\, \lambda \tag{11}$$

The following form of the model references particle and heat fluxes, Γ and q:

$$\frac{\partial n}{\partial t} \;=\; -\frac{\partial \Gamma}{\partial x} \;, \qquad \frac{\partial U}{\partial t} \;=\; -\frac{\partial q}{\partial x} \;, \qquad \epsilon \frac{\partial Z}{\partial t} \;=\; \mu \frac{\partial^2 Z}{\partial x^2} + \frac{c_n\, T}{n^2} \frac{\partial n}{\partial x} + \frac{c_T}{n} \frac{\partial T}{\partial x} + G(Z) \tag{12}$$

$$\Gamma \;=\; -D\, \frac{\partial n}{\partial x} \;, \qquad q \;=\; -\chi\, n \frac{\partial T}{\partial x} + \frac{\Gamma\, T}{\gamma - 1} \;, \qquad G \;=\; a + b(Z - Z_S) + c(Z - Z_S)^3 \tag{13}$$

Reducing down to equations of only n and T (of 2 possible forms): $\dfrac{\partial n}{\partial t} \;=\; \dfrac{\partial}{\partial x} \left[D\, \dfrac{\partial n}{\partial x} \right]$

$$\frac{\partial (n\, T)}{\partial t} \;=\; \frac{\partial}{\partial x} \left[\frac{D\, n}{\zeta} \frac{\partial T}{\partial x} \right] + \frac{\partial}{\partial x} \left[D\, T\, \frac{\partial n}{\partial x} \right] , \qquad \frac{\partial T}{\partial t} \;=\; \frac{\partial}{\partial x} \left[\frac{D}{\zeta} \frac{\partial T}{\partial x} \right] + \left(\frac{1}{\zeta} + 1 \right) \frac{D}{n} \frac{\partial n}{\partial x} \frac{\partial T}{\partial x} \tag{14}$$

The diffusivity function $D(\mathcal{E})$ is given in a few forms:

$$
\begin{aligned}
D(Z) &= \frac{D_{\max} + D_{\min}}{2} + \frac{(D_{\max} - D_{\min}) \tanh(Z)}{2} && \text{Zohm} && (15)\\
D(Z') &= D_{\min} + \frac{D_{\max} - D_{\min}}{1 + \alpha_{\sup} (Z')^{\beta}} \ , \quad \beta \approx 2 && \text{Staps} && (16)\\
D(Z, Z') &= D_{\min} + \frac{D_{\max} - D_{\min}}{1 + a_1 \, Z^2 + a_2 \, Z Z' + a_3 \, (Z')^2} && \text{Flow-Shear} && (17)
\end{aligned}
$$

Boundary Conditions, Initial Conditions, and Steady-State Solutions

Generalized versions for boundary conditions at the plasma edge ($x = 0$):

$$
n'(0) = \frac{n}{\lambda_n} \ , \quad T'(0) = \frac{T}{\lambda_T} \ , \quad Z'(0) = \frac{Z}{\lambda_Z} \quad \text{or} \quad Z'(0) = 0; \tag{18}
$$

...towards the core ($x = L$):

$$
\Gamma(L) = \Gamma_c \longrightarrow n'(L) = -\frac{\Gamma_c}{D} \ ; \quad q(L) = q_c \longrightarrow T'(L) = \frac{\zeta \, (T\Gamma_c - (\gamma - 1) q_c)}{n \, D} \ ; \quad Z'(L) = 0 \tag{19}
$$

Paquay's initial conditions for density and temperature, and Staps' initial condition for Z:

$$
n(x, 0) = -\frac{\Gamma_\infty \lambda_n}{D} \left(1 + \frac{x}{\lambda_n} \right) \ , \quad T(x, 0) = q_\infty \, \frac{\gamma - 1}{\Gamma_\infty} \left[1 - \frac{\lambda_n}{\zeta \lambda_T + \lambda_n} \left(1 + \frac{x}{\lambda_n} \right)^{-\zeta} \right] \ . \tag{20}
$$

Steady-State Solutions

$$
\bar{n}(x) = \bar{n}(0) - \int_0^x \frac{\Gamma_c}{D(\bar{Z}(x))} \, \mathrm{d}x \ , \quad \bar{T}(x) = \frac{(\gamma - 1) q_c}{\Gamma_c} \left(1 - \lambda_g \left(\frac{\bar{n}(x)}{\bar{n}(0)} \right)^{-\zeta} \right) \tag{21}
$$

$$
\bar{n}(0) = -\frac{\Gamma_c \lambda_n}{D(\bar{Z}(0))}, \quad \lambda_g = \frac{\frac{\lambda_n}{\zeta \lambda_T}}{1 + \frac{\lambda_n}{\zeta \lambda_T}} \ , \quad c_g = \frac{\zeta c_T - c_n}{1 + \zeta \frac{\lambda_T}{\lambda_n}} \ , \quad G(\bar{Z}(x)) = \theta \, D(\bar{Z}(x)) \tag{22}
$$

$$
\text{Staps': } \theta = \frac{(\gamma - 1) q_c}{\Gamma_c^2 \lambda_n^2} (c_n + c_g), \quad \text{Paquay's: } \theta = \frac{\Gamma_c^2 \lambda_n^2}{q_c \zeta (\gamma - 1)} \frac{\lambda_n + \zeta \lambda_T}{c_n \lambda_T + c_T \lambda_n} = \frac{1}{\text{Staps}} = \text{Weymiens} \tag{23}
$$

Physical Model

As a starting point, the nonambipolar currents are used, rather than some cubic polynomial. Start with Staps' derivation, now known to be slightly incorrect, since the polarization current is defined slightly incorrectly in his version:

$$
\frac{m_i \, n \, T}{e^2 \, \rho_{\theta i} \, B^2} \frac{\partial Z}{\partial t} = \frac{m_i \, \mu \, n \, T}{e^2 \rho_{\theta i} \, B_\theta^2} \frac{\partial^2 Z}{\partial x^2} + \Gamma_e^{\text{an}} - \Gamma_i^{\pi \parallel} - \Gamma_i^{\text{cx}} - \Gamma_i^{\text{ol}} \tag{24}
$$

Corrected version, which is used:

$$\text{Currents:} \quad \frac{e\,n\,\rho_{\theta i}}{2}\,\frac{\partial Z}{\partial t} \;=\; \frac{e\,\rho_{\theta i}}{2}\,\frac{\partial}{\partial x}\left[\mu\,n\,\frac{\partial Z}{\partial x}\right] \;+\; \sum_{\mathrm{k}} e\,\Gamma^{\mathrm{k}} \tag{25}$$

$$\text{Reduced:} \quad n\,\frac{\partial Z}{\partial t} \;=\; \frac{\partial}{\partial x}\left[\mu\,n\,\frac{\partial Z}{\partial x}\right] \;+\; \frac{2}{\rho_{\theta i}}\sum_{\mathrm{k}} \Gamma^{\mathrm{k}}\;, \quad \sum_{\mathrm{k}} \Gamma^{\mathrm{k}} \;=\; \Gamma_e^{\mathrm{an}} \;-\; \Gamma_i^{\mathrm{cx}} \;-\; \Gamma_i^{\pi\parallel} \;-\; \Gamma_i^{\mathrm{ol}} \tag{26}$$

Some of the fluxes take this form: $\Gamma_j^{\mathrm{k}} \;=\; g_n^{\mathrm{k}}\,\dfrac{n'}{n} \;+\; g_T^{\mathrm{k}}\,\dfrac{T'}{T} \;+\; g_Z^{\mathrm{k}}\,Z$

- Ion Bulk Viscosity:

$$D_{\pi\parallel} \;=\; \frac{\epsilon^2}{(x-a_m)\sqrt{\pi}}\,\frac{\rho_{\theta i}\,T}{B}\;, \quad e\Gamma_i^{\pi\parallel} \;=\; e\,n_e\,D_{\pi\parallel}\left(\frac{n'}{n} + \frac{Z}{\rho_{\theta i}}\right)\,\mathrm{Im}\left[X\left(Z + \frac{i\,\nu_{ii}}{\omega_t}\right)\right] \tag{27}$$

- Electron Anomalous Diffusion:

$$D_{\mathrm{an}} \;=\; \frac{\epsilon^2\sqrt{\pi}}{2a_m}\,\frac{\rho_{\theta e}T}{B}\;, \quad g_n^{\mathrm{an}} \;=\; -e\,n\,D_{\mathrm{an}}\;, \quad g_T^{\mathrm{an}} \;=\; \alpha_{\mathrm{an}}\,g_n^{\mathrm{an}}\;, \quad g_Z^{\mathrm{an}} \;=\; \frac{g_n^{\mathrm{an}}}{\rho_{\theta i}} \tag{28}$$

- Charge Exchange Friction:

$$g_n^{\mathrm{cx}} \;=\; -\frac{m_i\,n_0\langle\sigma_{\mathrm{cx}}v\rangle\,n\,T}{B_\theta^2}\left[\frac{B_\theta^2}{\epsilon^2 B_\phi^2} + 2\right]\;, \quad g_T^{\mathrm{cx}} \;=\; \alpha_{\mathrm{cx}}\,g_n^{\mathrm{cx}}\;, \quad g_Z^{\mathrm{cx}} \;=\; \pm\frac{g_n^{\mathrm{cx}}}{\rho_{\theta i}}\;? \tag{29}$$

- Ion Orbit Loss:

$$g^{\mathrm{ol}} \;=\; e\,n\,\nu_{ii}\,\nu_{*i}\,\rho_{\theta i}\;, \quad e\Gamma_i^{\mathrm{ol}} \;=\; g^{\mathrm{ol}}\,\frac{\exp\left[-\sqrt{\nu_{*i} + Z^4 + \frac{x^4}{w_{bi}^4}}\right]}{\sqrt{\nu_{*i} + Z^4 + \frac{x^4}{w_{bi}^4}}} \tag{30}$$