Far Eastern Federal University

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Higher Mathematics

TextBook

Vladivostok
2020

Министерство науки и высшего образования Российской Федерации
Дальневосточный федеральный университет

Л. С. Ксендзенко

HIGHER MATHEMATICS

Учебное пособие

Для иностранных студентов, обучающихся
по нематематическим специальностям вузов

Учебное электронное издание

Владивосток
ИЗДАТЕЛЬСТВО ДВФУ
2020

ISBN 987-5-7444-4798-4

УДК 51(075.8)
ББК 22.1я73

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Ксендзенко, Л.С. Higher Mathematics : учебное пособие : для иностранных студентов, обучающихся по нематематическим специальностям вузов / Л.С. Ксендзенко. – Владивосток : Издательство Дальневосточного федерального университета, 2020. – [90 с.]. – ISBN 987-5-7444-4798-4. – URL: https://www.dvfu.ru/science/publishing-activities/catalogue-of-books-fefu/. – Дата публикации: 06.04.2020. – Текст : электронный.

В пособии содержатся теоретические сведения по линейной алгебре и аналитической геометрии, а также по математическому анализу (теория функций одной переменной). Для объяснения основных понятий и определений приводятся примеры и соответствующие рисунки.

Предназначено для англоязычных студентов первого курса нематематических специальностей, которые хотят понять и использовать основные положения аналитической геометрии и математического анализа.

The textbook contains theoretical information on linear algebra and analytical geometry, as well as mathematical analysis (theory of functions of one variable). To explain the basic concepts and definitions, examples and corresponding figures are given.

The training manual is intended for English-speaking first-year students of non-mathematical specialties who want to understand and use the main provisions of analytical geometry and mathematical analysis.

Текстовое электронное издание

Минимальные системные требования:
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Размещено на сайте 06.04.2020 г.
Объем 2,0 Мб

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INTRODUCTION

The preparation of foreign students studying in non-mathematical specialties of universities and not knowing the Russian language, especially at the beginning of training requires the creation of educational and methodical literature on mathematics in English, which could help the student successfully master the course of higher mathematics.

This tutorial is to some extent created in order to partially eliminate the lack of mathematical literature in English.

The manual provides theoretical information, as well as problems with solutions from various sections of higher mathematics, studied in the first year.

The content of the manual corresponds to the program of the course of higher mathematics for students of non-mathematical specialties of universities, and can be recommended to students of these specialties when studying the sections of higher mathematics described in the work.

Cartesian Rectangular and Polar Coordinates on the Plane

1. Cartesian Rectangular Coordinates on the Plane

Definition. The numerical axis is the line on which the starting point, the positive direction of the axis and the unit of scale are selected.

A real number x is called the coordinate of the point that represents it (fig. 1).

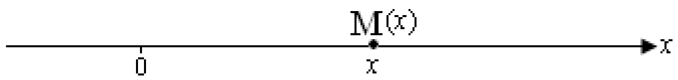


Fig. 1. Real number on the numerical axis

Let two mutually perpendicular numerical axes and, having a common origin and a common unit of scale, be given on the plane.

The set of coordinate axes Ox, Oy and the selected unit of scale is called the Cartesian rectangular coordinate system.

We assign two numbers to an arbitrary point M on the plane (fig. 2): the abscissa x, equal to the distance from the point M to the axis Oy, taken with the “+” sign if M it is to the right of Oy , and with the “–” sign if M it is to the left of Oy; the ordinate equal to the distance from the point M to the axis Ox, taken with the “+” sign if M lies above Ox, and with the “–” sign if M lies below Ox O.

The abscissa x and ordinate y are called the Cartesian rectangular coordinates of point M .

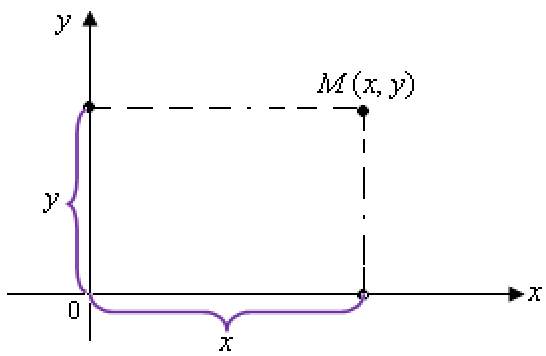


Fig. 2. Cartesian rectangular coordinates of a point on a plane

The plane in which the axes Ox and Oy are located is called the coordinate plane and is designated by Oxy .

Each point $M(x, y)$ on the coordinate plane has two coordinates x and y, where x is the abscissa of the point, y is the ordinate of the point.

Axis $Ox -$ is axis abscissa; $Oy -$ is axis ordinate. The coordinate axes divide the coordinate plane into four quarters (fig. 3).

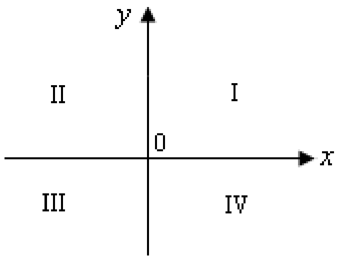


Fig. 3. Partitioning a plane into four quadrants

In figure 4, four points are plotted: $M_1(3,1)$, $M_2(-5,2)$, $M_3(-3,-2)$, $M_4(0,-1)$.

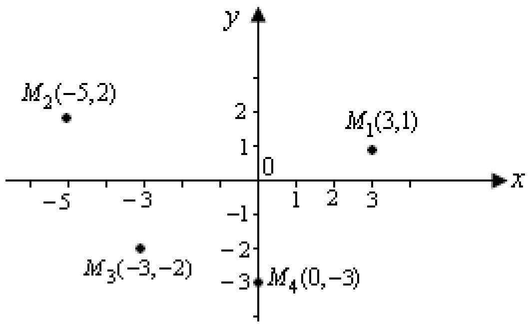


Fig. 4. Coordinates of points on the plane

2. Polar coordinate system

Definition. The polar coordinate system is the set of point O and the ray Or originating from it. The point O is called the pole of the polar coordinate system, the axis Or is called the polar axis (fig. 5).

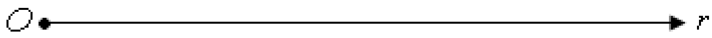


Fig. 5. Polar coordinate system

Let consider a point $M(r,\varphi)$ in the polar coordinate system (fig. 6).

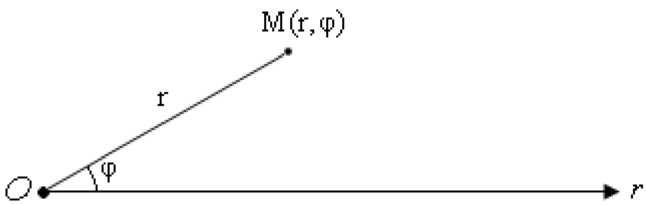


Fig. 6. Coordinates of a point in the polar coordinate system

Numbers r, φ are called the polar coordinates of the point M .

The length of the segment $OM = r$ is called the polar radius of the point M, and the angle φ measured from the polar axis to the segment OM, counterclockwise, is called the polar angle of the point M.

Example. Draw points in the polar coordinate system (fig. 7):

$$A(3,0);\ B\left(2,\frac{\pi}{4}\right);\ C\left(1,\frac{\pi}{2}\right);\ D(4,\pi);\ E\left(1{,}5;\frac{3\pi}{2}\right).$$

Solution.

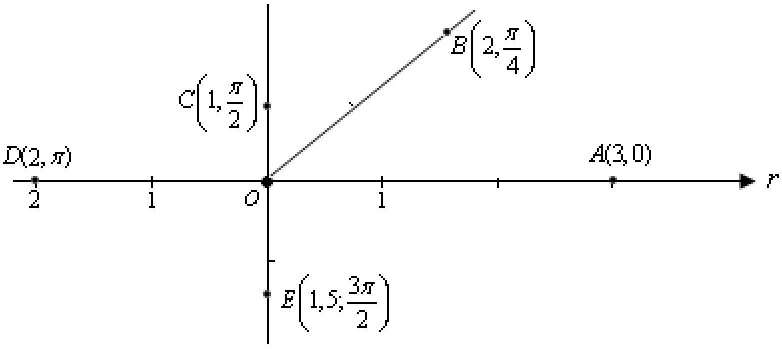


Fig. 7. Coordinates of points on a plane in the polar coordinate system

The relationship of the Cartesian coordinates of the point with the polar

Let us compatible Cartesian rectangular coordinate system with the polar system. The polar axis Or is compatible with the axis Ox, and the pole of the polar coordinate system is compatible with the origin of the Cartesian rectangular coordinate system (fig. 8.).

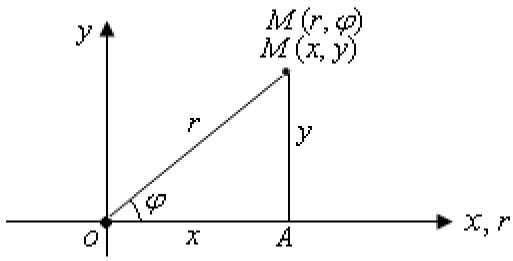


Fig. 8. Combination of Cartesian rectangular and polar coordinate systems

From the triangle OMA it follows that

$$x = R\cos\varphi,\ \ y = R\sin\varphi,\ \ r^2 = x^2 + y^2,\ \ tg\varphi = \frac{y}{x}. \tag{1}$$

Example. Build a cardioids $r = a(1+\cos\varphi)$ (*fig. 9*).

Solution. We compile a table of values of the polar angle and polar radius.

№	1	2	3	4	5	6	7
φ	0	$\pi/3$	$\pi/6$	$\pi/2$	$2\pi/3$	$5\pi/6$	π
$\cos\varphi$	1	0,5	0,866	0	-0,5	-0,866	-1
$1+\cos\varphi$	2	1,5	1,866	1	0,5	0,134	0
$r=a(1+\cos\varphi)$	2a	1,5a	1,866a	a	0,5a	0,134a	0

First, draw rays $\varphi=0$, $\varphi=\dfrac{\pi}{3}$, etc. in the polar coordinate system.

On each ray from the pole, we postpone the corresponding value of the polar radius. As a result, we obtain several points, connecting which we will sequentially obtain a cardioid.

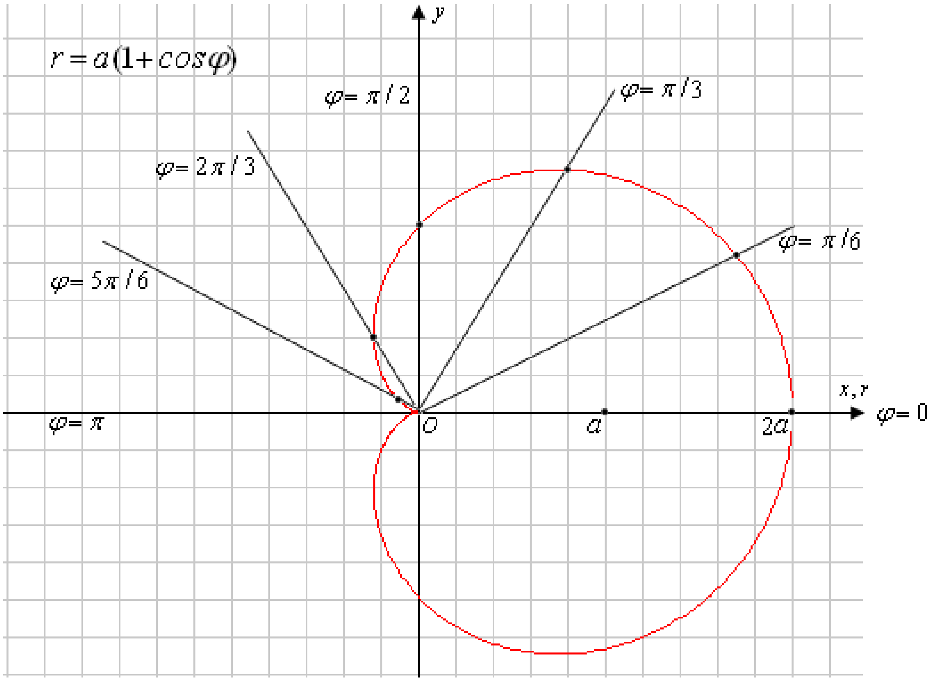


Fig. 9. Construction of cardioids in the polar coordinate system

LINEAR ALGEBRA

1. Matrices and Determinants

An order $m\times n$ matrix is a rectangular table of numbers containing m rows and n columns of the form

$$A=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}=\left(a_{ij}\right)_{m,n},$$

where $i-$ is the row number, $j-$ is the column number in which the element a_{ij} is located.

Example. The matrix $A=\begin{pmatrix} 2 & -3 & 0 \\ -1 & 0 & 4 \end{pmatrix}$ has a size 2×3;

$a_{11}=2,\ a_{12}=-3,\ a_{13}=0;\quad a_{21}=-1,\ a_{22}=0,\ a_{23}=4.$

If the number of rows is equal to the number of columns $(m=n)$, then the matrix is called a square matrix of order n $\left(A=\left(a_{ij}\right)_n\right)$.

Example. $A=\begin{pmatrix} 2 & -3 \\ 4 & 5 \end{pmatrix}$ – is a square matrix of the second order.

Two matrices $A=\left(a_{ij}\right)_{m,n}$ and $B=\left(b_{ij}\right)_{m,n}$ are called equal (A = B) if their respective elements are equal, i.e. $a_{ij}=b_{ij}$ $(i=1,2,\dots,m;\ j=1,2,\dots,n)$.

Determinants of the second and third order

Each square matrix can be associated with a number called its determinant.

The matrix $A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ corresponds to the determinant $\Delta=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}=a_{11}\cdot a_{22}-a_{12}\cdot a_{21}$.

The second-order determinant is a number equal to the product of the elements of the main diagonal without the product of the elements of the secondary diagonal (fig. 10).

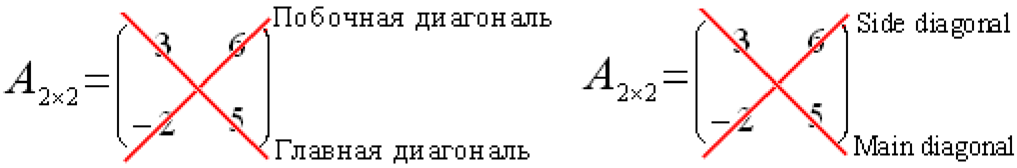


Fig. 10. Main and secondary diagonals of the determinant

Example. Find the determinant of the matrix $A=\begin{pmatrix} 2 & -3 \\ 5 & 7 \end{pmatrix}$.

Solution. $\Delta=\begin{vmatrix} 2 & -3 \\ 5 & 7 \end{vmatrix}=2\cdot 7-(-3)\cdot 5=14+15=29$.

The third-order determinant corresponding to the matrix $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

is called the number denoted by $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ or Δ and defined by the equality

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{21}a_{32}a_{13} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

On the right side there are six terms, three of which are associated with the main diagonal taken with a plus sign, and three terms associated with the side diagonal are taken with a minus sign.

On the main diagonal of the determinant there are elements a_{11}, a_{22}, a_{33}, on the secondary – a_{13}, a_{22}, a_{31}.

There is a “triangle rule” to remember this definition.

Each term on the right side with a plus sign is a product of three determinant elements taken, as shown in fig. 11a.

Each term with a minus sign is a product of three elements of the determinant, taken, as shown in fig. 11b.

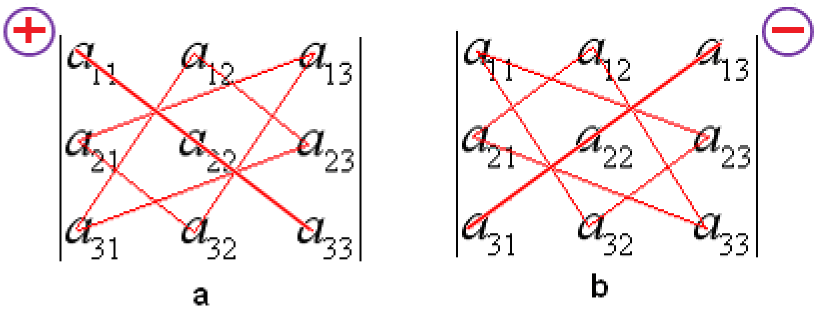


Fig. 11. "Rule of the triangle" of the calculation of the determinant of the third order

Example. Calculate the determinant of the third order: $\Delta = \begin{vmatrix} 1 & -1 & 7 \\ 2 & 3 & -3 \\ 3 & 2 & 5 \end{vmatrix}$.

Solution.

$$\Delta = \begin{vmatrix} 1 & -1 & 7 \\ 2 & 3 & -3 \\ 3 & 2 & 5 \end{vmatrix} = 1 \cdot 3 \cdot 5 + (-1) \cdot (-3) \cdot 3 + 2 \cdot 2 \cdot 7 - 3 \cdot 3 \cdot 7 - 2 \cdot (-1) \cdot 5 - 2 \cdot (-3) \cdot 1 = 68 - 63 = 5.$$

The minor of the element a_{ij} $(i=1,2,3;\ \ j=1,2,3)$ of the third-order determinant $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

the second-order determinant is called the resulting determinant of the third order by deleting the i – th row of the j-th column, at the intersection of which stands the element a_{ij}.

The minor of the element a_{ij} is denoted by M_{ij}.

The algebraic complement of an element a_{ij} of a third-order determinant is the product of the minor M_{ij} of this element by a number $(-1)^{i+j}$, where i is the row number, j is the column number at the intersection of which the element a_{ij} stands.

The algebraic complement of an element a_{ij} is denoted by A_{ij}.

$$A_{ij} = (-1)^{i+j} \cdot M_{ij}. \tag{2}$$

The following theorem holds.

The determinant is equal to the sum of the products of the elements of a row (or column) by their algebraic additions, i.e.

$$\Delta = a_{11} \cdot A_{11} + a_{12} \cdot A_{12} + a_{13} \cdot A_{13}; \quad \Delta = a_{21} \cdot A_{21} + a_{22} \cdot A_{22} + a_{23} \cdot A_{23};$$
$$\Delta = a_{31} \cdot A_{31} + a_{32} \cdot A_{32} + a_{33} \cdot A_{33}; \quad \Delta = a_{11} \cdot A_{11} + a_{21} \cdot A_{21} + a_{31} \cdot A_{31};$$
$$\Delta = a_{12} \cdot A_{12} + a_{22} \cdot A_{22} + a_{32} \cdot A_{32}; \quad \Delta = a_{13} \cdot A_{13} + a_{23} \cdot A_{23} + a_{33} \cdot A_{33}.$$

2. Systems of Linear Algebraic Equations

Let us consider a system of three linear algebraic equations with three unknowns:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3. \end{cases} \tag{3}$$

Here x_1, x_2, x_3 is unknown, $a_{11}, a_{12}, ..., a_{33}, b_1, b_2, b_3$ are given numbers.

The matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, composed of coefficients for unknowns, is called the matrix of the system, and the determinant $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ is called the determinant of system (3).

Cramer Formulas

If the determinant of system (3) is nonzero, then system (3) has a unique solution, which is found by the formulas:

$$x_1 = \frac{\Delta_1}{\Delta}, \ x_2 = \frac{\Delta_2}{\Delta}, \ x_3 = \frac{\Delta_3}{\Delta}, \tag{4}$$

where

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0, \quad \Delta_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

Example. The system is given

$$\begin{cases} x_1 - x_2 + 7x_3 = 6, \\ 2x_1 + 3x_2 - 3x_3 = 10, \\ 3x_1 + 2x_2 + 5x_3 = 17. \end{cases} \tag{5}$$

Solve it using Cramer's formulas.

Solution. 1. We calculate the determinant of the system (5): $\Delta = \begin{vmatrix} 1 & -1 & 7 \\ 2 & 3 & -3 \\ 3 & 2 & 5 \end{vmatrix} = 5$.

$\Delta \neq 0$, therefore, system (5) has a unique solution. We find it using Cramer's formulas (4). We calculate $\Delta_1 = \begin{vmatrix} 6 & -1 & 7 \\ 10 & 3 & -3 \\ 17 & 2 & 5 \end{vmatrix} = 90 + 51 + 140 - 357 + 50 + 36 = 10$;

$\Delta_2 = \begin{vmatrix} 1 & 6 & 7 \\ 2 & 10 & -3 \\ 3 & 17 & 5 \end{vmatrix} = 50 - 54 + 238 - 210 - 60 + 51 = 15$;

$\Delta_3 = \begin{vmatrix} 1 & -1 & 6 \\ 2 & 3 & 10 \\ 3 & 2 & 17 \end{vmatrix} = 51 - 30 + 24 - 54 + 34 - 20 = 5$. From here, using formulas (4), we find:

$x_1 = \frac{10}{5} = 2$, $x_2 = \frac{15}{5} = 3$, $x_3 = \frac{5}{5} = 1$.

Check. 1) $2 - 3 + 7 \cdot 1 = 6 -$ it's right; 2) $2 \cdot 2 + 3 \cdot 3 - 3 \cdot 1 = 10 -$ it's right;

3) $3 \cdot 2 + 2 \cdot 3 + 5 \cdot 1 = 17 -$ it's right. **Answer:** $x_1 = 2, \ x_2 = 3, \ x_3 = 1.$

ANALYTICAL GEOMETRY

1. Distance between Points

Two points are given $M_1(x_1, y_1)$ and $M_2(x_2, y_2)$. It is required to find the distance between them (fig. 12).

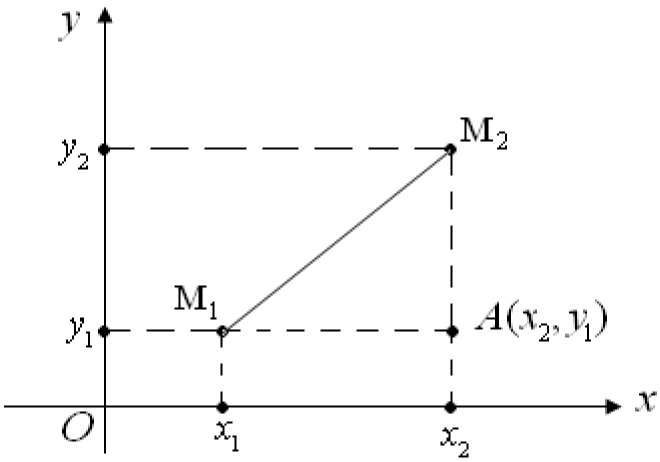


Fig. 12. Distance between points

By the Pythagorean theorem, the length of the hypotenuse is equal to the square root of the sum of the squares of the lengths of the legs:

$$|M_1M_2| = \sqrt{|M_1A|^2 + |M_2A|^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\,. \tag{5}$$

Example. Find the distance between points $A(2,-4)$ и $B(-1,3)$.

Solution. By the condition of the task $x_1 = 2,\ y_1 = -4;\ x_2 = -1,\ y_2 = 3$. According to the formula (5) we have: $|AB| = \sqrt{(-1-2)^2 + (3-(-4))^2} = \sqrt{9+49} = \sqrt{58}$.

2. Straight Line on the Plane

a) the canonical equation of a line on the plane

The point $M_0(x_0, y_0)$ lie on the line and the directing vector $\bar{S}(m,n)$ of line is known (fig. 13). A direction vector of a straight line is a vector parallel to the line.

Let $M(x,y)-$ is an arbitrary point on the line, then the vectors $\overline{M_0M} = (x - x_0, y - y_0)$ and $\bar{S}(m,n)$ are collinear, then the coordinates of these vectors are proportional:

$$\frac{x - x_0}{m} = \frac{y - y_0}{n}\,. \tag{6}$$

Thus, the canonical equation of a line in the $x, y-$plane has a form $\frac{x-x_0}{m}=\frac{y-y_0}{n}$, where $\bar{S}(m,n)$ is a direction vector of the line.

Example. Find an equation for a line passing through a point $A(2,-4)$ and being parallel to the vector $\bar{a}=3\bar{i}-5\bar{j}$.

Solution. By condition $x_0=2,\ y_0=-4,\ m=3,\ n=-5$. By the formula (6), the equation of the line takes the form $\frac{x-2}{3}=\frac{y+4}{-5}$.

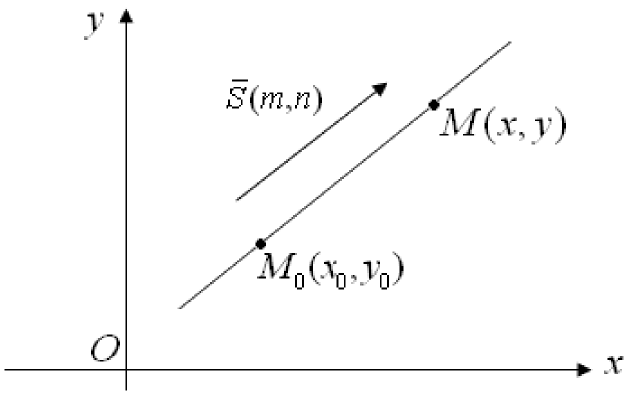


Fig. 13. Canonical equations of the line on the plane Oxy

b) The equation of a line passing through a given point in a given direction

The canonical equation of the line is given $\frac{x-x_0}{m}=\frac{y-y_0}{n}$. We rewrite it in the form $y-y_0=\frac{n}{m}\cdot(x-x_0)$. By parallel transfer, we move the direction vector of the line so that its beginning coincides with the origin (fig. 14).

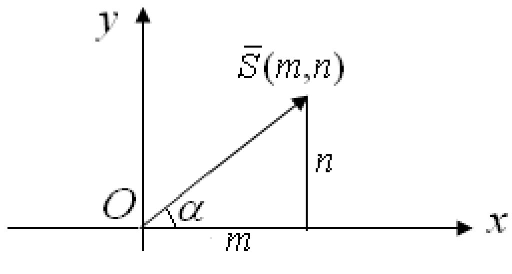


Fig. 14. Directing vector of the line

From figure 14 we can see that $tg\alpha=\frac{n}{m}$, or $k=\frac{n}{m}$. Then the equation of a line passing through a given point in a given direction will take the form:

$$y-y_0=k\cdot(x-x_0) \tag{7}$$

Example. Find an equation of a line passing through point A (-3.7) and forming an angle $\frac{\pi}{4}$ with the axis Ox.

Solution. By condition $x_0 = -3,\ y_0 = 7,\ tg\frac{\pi}{4} = 1$, then by the formula (6), $y - 7 = 1\cdot(x+3)$, or $x - y + 10 = 0$.

c) The general equation of the line

The equation of the first degree with respect to variables x and y, i.e. equation of the form

$$Ax + By + C = 0, \tag{8}$$

provided that the coefficients A and B at the same time are not equal to zero, is called the general equation of the line.

If the coordinates of the point satisfy the equation, i.e. turn it into an identity, then this point lie on a given line; if the coordinates of the point do not satisfy the equation, then the point does not lie on the line.

Example. Check if the points $A(1,9)$ and $B(5,6)$ lie on the line $7x - 2y + 11 = 0$.

Solution. Substituting the coordinates of the point A instead of the variables x and y in the equation $7x - 2y + 11 = 0$, we obtain the identity $7\cdot 1 - 2\cdot 9 + 11 \equiv 0$, therefore, the point $A(1,9)$ lies on this line. Similarly, we see that the point $B(5,6)$ does not lie on the line, since $7\cdot 5 - 2\cdot 6 + 11 = 34 \neq 0$.

d) Equation of a line with an angular coefficient

Let the line intersect the axis Oy at a point $B(0,b)$ and form an angle α with the positive direction of the axis Ox (fig. 15).

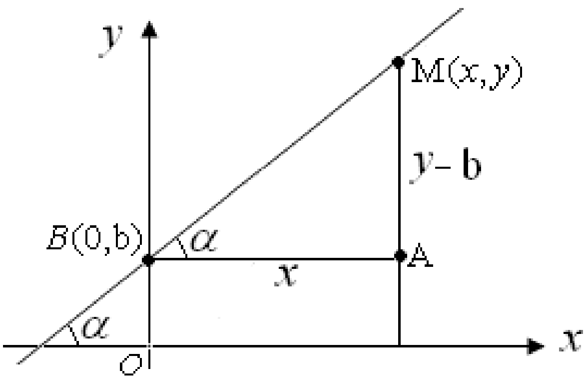


Fig. 15. Equation of the line with an angular coefficient

From the triangle of the BMA we find that the leg $(y-b)$ can be expressed through the leg x: $y - b = tg\alpha\cdot x$, where $k = tg\alpha$ – the angular coefficient of the straight BM.

Thus, the equation of a line with an angular coefficient has the form:

$$y = k \cdot x + b\,, \tag{9}$$

where $k -$ is the tangent of the angle between the straight line of the BM and the positive direction of the axis Ox.

Example. Find the equation of a line passing through point A (3,4) and cutting off a segment $b = 3$ on the axis Oy.

Solution. We substitute in the equation (9) $x = 3,\ y = 4,\ b = 3\ : 4 = 3k + 3$. From here $k = \dfrac{1}{3}$. Then the desired equation has the form: $y = \dfrac{1}{3} \cdot x + 3$.

e) The equation of a line passing through two given points

Two points are given $M_1(x_1, y_1)$ and $M_2(x_2, y_2)$. It is required to find up the equation of a line passing through these points (fig. 16).

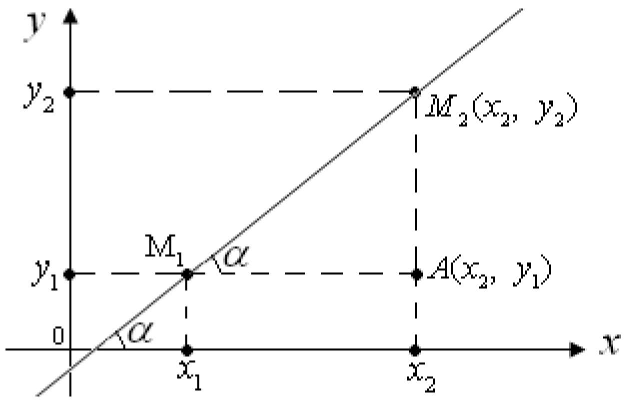


Fig. 16. Equation of the line passing through two given points

. First, we find the angular coefficient of a straight line passing through two given points. From the triangle M_1M_2A we find

$$k = tg\alpha = \frac{y_2 - y_1}{x_2 - x_1}\,. \tag{10}$$

In formula (7), instead of x_0, substitute x_1, and instead of y_0, substitute y_1. Instead of k, a fraction $\dfrac{y_2 - y_1}{x_2 - x_1}$. Then we get $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1} \cdot (x - x_1)$ or

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}\,. \tag{11}$$

Example. Find the equation of the line passing through two given points A(- 2,6) and B(4, 9).

Solution. By condition $x_1=-2,\ y_1=6,\ x_2=4,\ y_2=9$. Then by formula (11) we have $\frac{x-(-2)}{4-(-2)}=\frac{y-6}{9-6}$, or $\frac{x+2}{6}=\frac{y-6}{3}$.

f) Distance From a Point to a Line

Let a line L be given by the equation $Ax+By+C=0$ and a point $M_0(x_0,y_0)$ (fig. 17). It is required to find the distance from a point M_0 to a line L.

Solution. The distance d from the point M_0 to the line L is equal to the modulus of the projection of the vector $\overline{M_1M_0}$ on the direction of the normal vector $\bar{N}(A,B)$, where $M_1(x_1,y_1)$ is an arbitrary point of the line L.

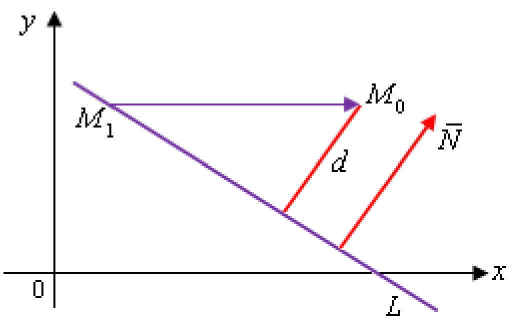


Fig. 17. Distance from the point to the line

Consequently, $d=\left|np_{\bar{N}}\overline{M_1M_0}\right|=\left|\frac{\overline{M_1M_0}\cdot\bar{N}}{|\bar{N}|}\right|=\frac{|(x_0-x_1)A+(y_0-y_1)B|}{\sqrt{A^2+B^2}}=\frac{|Ax_0+By_0-Ax_1-By_1|}{\sqrt{A^2+B^2}}$.

Since the point $M_1(x_1,y_1)$ belongs to the line L, then $Ax_1+By_1+C=0$, i.e. $C=-Ax_1-By_1$ so

the distance from a point $M_0(x_0,\ y_0)$ to a straight line $Ax+By+C=0$ is found by the formula:

$$d=\frac{|Ax_0+By_0+C|}{\sqrt{A^2+B^2}}. \tag{12}$$

g) The condition of parallelism and perpendicularity of two lines

Two parallel lines I and II are given (fig. 18 a). It can be seen from the figure 18a that if the lines are parallel, then their angular coefficients are equal, i.e.

$$k_1=k_2. \tag{13}$$

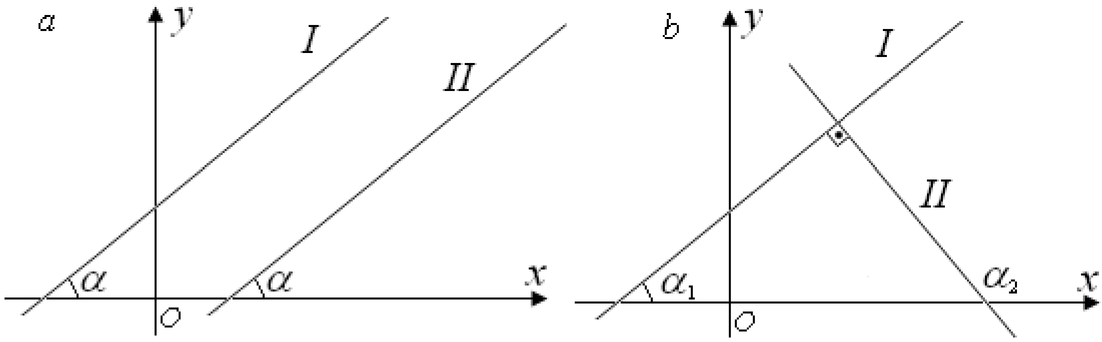


Fig. 18. Parallel lines – *a* and perpendicular lines – *b*

Let us straight line I and II are perpendicular (fig. 18 b), then their angular coefficients are inverse in magnitude and opposite in sign:

$$k_2 = -\frac{1}{k_1}. \qquad (14)$$

Since the angle $\alpha_2 = \frac{\pi}{2} + \alpha_1$ (fig. 18*b*). Than $tg\alpha_2 = \frac{\pi}{2} + \alpha_1 = tg\left(\frac{\pi}{2} + \alpha_1\right) = -ctg\alpha_1 = -\frac{1}{tg\alpha_1}$, or $k_2 = -\frac{1}{k_1}$.

Example. Find the equation of the line passing through point A (4, -7) and being perpendicular to the line $3x + 2y - 4 = 0$.

Solution. Find the angular coefficient of the line $3x + 2y - 4 = 0$: $2y = 4 - 3x$, $y = -\frac{3}{2}x + 2$. Thus, the slope of the line $3x + 2y - 4 = 0$ is $k_1 = -\frac{3}{2}$. Since the desired line is perpendicular to the line $3x + 2y - 4 = 0$, its angular coefficient is $k_2 = \frac{2}{3}$.

By condition $x_0 = 4$, $y_0 = -7$. Now, using formula (7) $y - y_0 = k \cdot (x - x_0)$, we find the equation of the desired line: $y + 7 = \frac{2}{3} \cdot (x - 4)$ or $2x - 3y - 29 = 0$.

3. Quadratic Curves

A line is called a curve of the 2nd order if it is determined by an equation of degree 2 with respect to the current coordinates and, i.e. equation of the form

$$Ax^2 + By^2 + Cxy + Dx + Ey + F = 0. \qquad (15)$$

CIRCLE

A **circle** is set of points in a plane that are equidistant from a fixed point. The fixed point is called the **center.** A line segment that joints the center with any point of the circle is called the **radius.**

In the $x, y-$ plane, the distance between two points $M(x, y)$ and $M_0(a, b)$ equals $\sqrt{(x-a)^2+(y-b)^2}$, and so the circle is described by the equation

$$(x-a)^2+(y-b)^2=R^2, \tag{16}$$

where a and b are the coordinates of the center, and R is the radius (fig. 19).

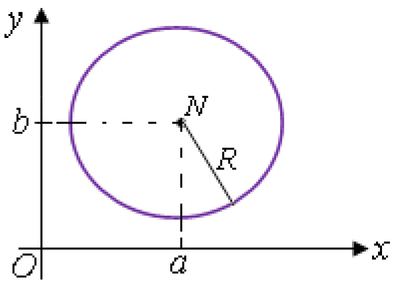


Fig. 19. Circle centered at point $N(a,b)$

Let the coefficients $B=A, \ C=0$ in equation (16) then

$$Ax^2+Ay^2+Dx+Ey+F=0, \ (A\neq 0). \tag{17}$$

After dividing by A and highlighting the full squares, equation (17) takes the form

$$(x-a)^2+(y-b)^2=k. \tag{18}$$

If k> 0, then equation (18) defines a circle of radius $R=\sqrt{k}$;

for k = 0, the coordinates of a single point $N(a,b)$ satisfy the equation,

for k <0, the coordinates of no point on the plane satisfy the equation.

If the center of the circle of radius R is at the origin at the point O (0; 0), then the equation of the circle has the form

$$x^2+y^2=R^2. \tag{19}$$

Example. Two points are given $M_1(-4;5)$, $M_2(2;-3)$. Make an equation of a circle whose diameter is a segment M_1M_2.

Solution. The center of the circle is the point $N(x;y)$ - the middle of the segment M_1M_2 : $x=\frac{-4+2}{2}=-1$; $y=\frac{5-3}{2}=1$; those, circle center is point $N(-1;1)$. The radius of the circle is half the length of the segment M_1M_2.

We find $|M_1M_2|=\sqrt{(2-(-4))^2+(-3-5)^2}=\sqrt{36+64}=\sqrt{100}=10$, from here $R=\frac{10}{2}=5$.

By the formula (18) we find the desired equation $(x+1)^2+(y-1)^2=25$.

Example. The circle is given by the equation $x^2+y^2+8x-12y-12=0$. Find the radius and the coordinates of the center.

Solution. Transform the quadratic polynomial on the left-hand side of the equation by adding and subtracting the corresponding constants to complete the perfect squares: $x^2+8x=(x^2+8x+16)-16=(x+4)^2-16$, $y^2-12y=(y^2-12y+36)-36=(y-6)^2-36$.

Then the given equation is reduced to the form

$$(x+4)^2+(y-6)^2=8^2,$$

which describes the circle centered at the point $M_0(-4;6)$ with radius 8.

Example. What set of points of the plane determines the equation $4x^2+4y^2-24x+4y+37=0$?

Solution. Dividing both sides of the equation by 4, we get:

$x^2+y^2-6x+y+\frac{37}{4}=0$. Selecting the full squares, we have

$\left(x^2-6x+9\right)-9+\left(y^2+2\cdot\frac{1}{2}y+\frac{1}{4}\right)-\frac{1}{4}+\frac{37}{4}=0$, or $(x-3)^2+\left(y+\frac{1}{2}\right)^2=0$.

This equation is satisfied only by a pair of values $x=3,\ y=-\frac{1}{2}$ i.e. single point coordinates $M(3;-0,5)$.

ELLIPSE

An ellipse is the set of all points in a plane, for each of it the sum of the distance to two given points (foci) of the same plane is a constant value equal to 2a. (fig. 20).

The canonical equation of the ellipse has the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \qquad (20)$$

where $a = |OA|$ is the major, $b = |OB|$ is the minor semiaxis. The positive quantities 2a and 2b are called the axes of the ellipse. One of them is said to be the major axis, while the other is the minor axis (fig. 20).

The intersection points of the ellipse with axes of symmetry are called the vertices of the ellipse. Hence, the points $(\pm a, 0)$ and $(0, \pm b)$ are the vertices of ellipse (20).

Two fixed points, $F_1(-c, 0)$ and $F_2(c, 0)$, are called the focuses of the ellipse, if equality $c^2 = a^2 - b^2$ is satisfied. Correspondingly, the distances r_1 and r_2 from any point $M(x, y)$ of the ellipse to the points F_1 and F_2 are called the focal distances. The ratio $\frac{c}{a} = \varepsilon$ is called the eccentricity of ellipse. Note that $0 < \varepsilon < 1$.

The coordinates of focuses $F_1(-c, 0)$, $F_2(c, 0)$, where $c = \sqrt{a^2 - b^2}$. $\sqrt{x^2 - y^2}$ the square root of x square minus y square;

The eccentricity of an ellipse is the ratio of the focal length 2c to the length of the major axis 2a: $\varepsilon = \frac{c}{a}$.

The distances of point $M(x, y)$ of the ellipse to the foci (focal radii) are determined by the formulas:

$$r_1 = a + \varepsilon x, \ r_2 = a - \varepsilon x \ . \tag{21}$$

Directrixes of an ellipse (20) are called lines defined by equations $x = \pm\frac{a}{\varepsilon}$, where ε is the eccentricity (fig. 21).

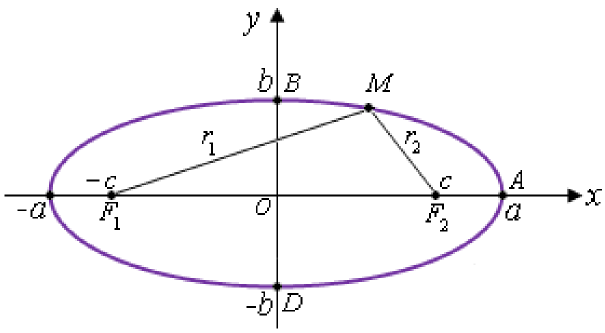


Fig. 20. Ellipse

The diameter of an ellipse is the chord passing through its center - the intersection point of the axes.

Example. Find the semi-axes, foci and eccentricity of the ellipse defined by the equation $9x^2 + 16y^2 = 36$.

Solution. Dividing both sides of the equation by 36, we get:

$$\frac{x^2}{4}+\frac{4y^2}{9}=1 \text{ or } \frac{x^2}{4}+\frac{y^2}{9/4}=1.$$

From here $a^2=4$, $a=2$; $b^2=\frac{9}{4}$, $b=\frac{3}{2}$. $c^2=a^2-b^2=4-\frac{9}{4}=\frac{7}{4}$; $c=\frac{\sqrt{7}}{2}$; $\varepsilon=\frac{c}{a}=\frac{\sqrt{7}}{4}$. In this way $a=2$; $b=\frac{3}{2}$; $F_1\left(-\frac{\sqrt{7}}{2},0\right)$; $F_2\left(\frac{\sqrt{7}}{2},0\right)$; $\varepsilon=\frac{\sqrt{7}}{4}$.

Example. Prove that the ratio of the distance from any point of the ellipse to the focus to the distance of this point to the corresponding directrix is a constant value equal to ε.

Proof. Let $M(x,y)$ be an arbitrary point of the ellipse (Fig. 21). ellipse directories директрисы эллипса;

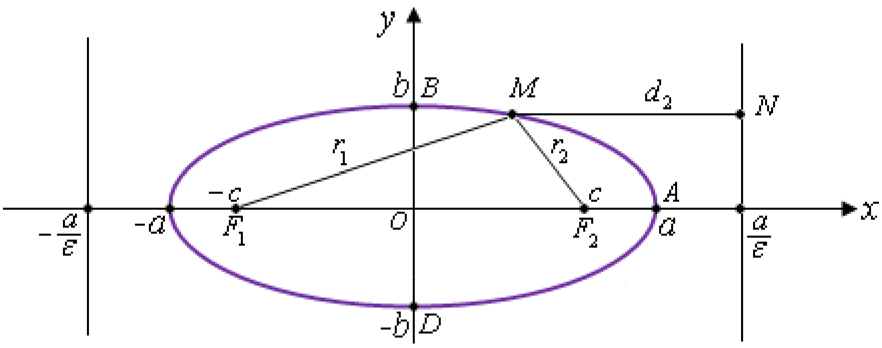


Fig. 21. Ellipse and his directresses

Расстояние $d_2=|MN|$ точки M до иректрисы $x=\frac{a}{\varepsilon}$ выразится формулой $d_2=\frac{a}{\varepsilon}-x$. The distance $d_2=|MN|$ of the point M to the directrix $x=\frac{a}{\varepsilon}$ is expressed by the formula $d_2=\frac{a}{\varepsilon}-x$.

The distance from this point to the focus is determined by the formula (6): $r_2=a-\varepsilon x$. The ratio

$\frac{r_2}{d_2}=\frac{a-\varepsilon x}{(a-\varepsilon x)/\varepsilon}=\varepsilon$, Q.E.D.

HYPERBOLA

A hyperbola is the set of all points of a plane, for each of which the modulus of the difference in distances to two given points (foci) of the same plane is a constant value equal to 2a (fig. 22).

The canonical hyperbola equation has the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 , \tag{22}$$

where $a = |OA|$ – real semi-axis; $b = |OB|$ – imaginary semi-axis.

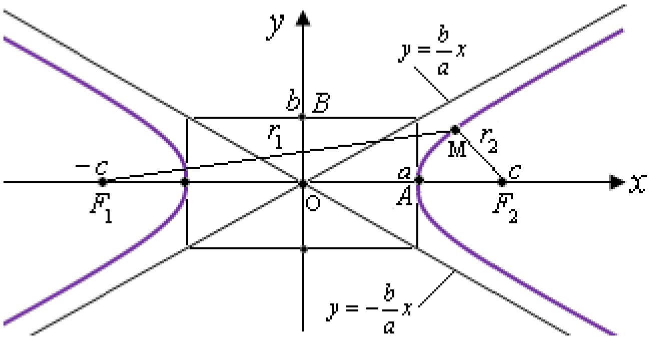


Fig. 22. Hyperbola

Two fixed points $F_1(-c, 0)$ and $F_2(c, 0)$, are called the focuses of the hyperbola, if the equality

$$c = \sqrt{a^2 + b^2} \tag{23}$$

is satisfied.

The eccentricity of hyperbola (22) is the ratio of the focal length $2c$ to the length of the real axis $2a$:

$$\varepsilon = \frac{c}{a}. \tag{24}$$

The asymptotes of hyperbola (22) are the lines defined by the equations:

$$y = \frac{b}{a}x, \ \ y = -\frac{b}{a}x . \tag{25}$$

On these lines lie the diagonals of the characteristic rectangle, the base of which is 2a, the height is 2b, and the center is at the origin.

Directrixes of hyperbola (22) are called lines defined by equations $x = -\frac{a}{\varepsilon}$; $x = \frac{a}{\varepsilon}$ where a is the real semi-axis; ε - eccentricity.

Equation

$$-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{26}$$

defines a hyperbola symmetric about the coordinate axes; its branches intersect the Oy axis, and the foci lie on the Oy axis.

Two hyperbolas defined by equations $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in the same coordinate system are called conjugate.

A hyperbola with equal semi-axes $a=b$ is called equilateral; its canonical equation has the form $x^2-y^2=a^2$ or $-x^2+y^2=a^2$.

Equations of the asymptotes of an equilateral hyperbola $y=x,\ y=-x$.

The focal radii of a point on the right branch of a hyperbola (22) are calculated by the formulas

$$r_1=\varepsilon x+a,\ r_2=\varepsilon x-a; \tag{27}$$

focal radii of a point of the left branch - according to formulas

$$r_1=-\varepsilon x-a,\ r_2=-\varepsilon x+a. \tag{28}$$

Comment. If the hyperbola is given by equation (26), then its directrixes are determined by the equations

$$y=-\frac{b}{\varepsilon};\ y=\frac{b}{\varepsilon} \tag{29}$$

Example. Find semi-axes, foci, eccentricity, equations of asymptotes and directrixes of a hyperbola $25x^2-144y^2=3600$.

Solution. Bringing this equation to canonical form (for which both parts of it must be divided by 3600), we obtain $\frac{x^2}{144}-\frac{y^2}{25}=1$. From here $a^2=144,\ b^2=25$, therefore, $a=12$ is the real semi-axis, $b=5$ is the imaginary semi-axis. By the formula (23) we find $c=\sqrt{144+25}=13$.

In this way, $F_1(-13,0)$; $F_2(13,0)$. Than eccentricity is equal $\varepsilon=\frac{13}{12}$.

By formulas (25) we find the equations of asymptotes $y=\frac{5}{12}x$; $y=-\frac{5}{12}x$.

Substituting the values of a and ε in the corresponding equations, we obtain:

$$x = -\frac{144}{13};\ x = \frac{144}{13}.$$

Example. Find the intersection points of the hyperbola $2x^2 - 9y^2 = 36$ and the line $x - 3y = 0$.

Solution. It follows from the equation of the line that $x = 3y$.

We substitute this value of x into the hyperbola equation: $2 \cdot 9 \cdot y^2 - 9y^2 = 36$, $9y^2 = 36$, $y^2 = 4$, $y_1 = 2$, $y_2 = -2$; $x_1 = 3 \cdot 2 = 6$; $x_2 = 3 \cdot (-2) = -6$; from here we get two points $M_1(6, 2)$ and $M_2(-6, -2)$.

Example. To compose the canonical equation for the hyperbola, the foci of which are located on the axis of the OA symmetrically with respect to the origin, if the distance between the foci is 10, the distance between the vertices is 8.

Solution. By the formula (26) $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, in this case the axis Oy is real, axis Ox is imaginary.

Since the distance between the vertices is 8, then $2b = 8$, $b = 4$; by condition $2c = 10$, from here $c = 5$;

for hyperbole $c = \sqrt{a^2 + b^2}$, or $c^2 = a^2 + b^2$; $a^2 = c^2 - b^2$. From here $a^2 = 5^2 - 4^2$; $a^2 = 25 - 16 = 9$.

Knowing a^2 and b^2, we find by formula (26) the desired equation: $-\frac{x^2}{9} + \frac{y^2}{16} = 1$.

PARABOLA

A parabola is the set of all points of the plane equidistant from a given point (focus) and a given line (directrix) lying in the same plane (fig. 23).

The canonical equation of a parabola symmetric about an axis Ox and passing through the origin has the form

$$y^2 = 2px. \tag{30}$$

Parabola directrix equation (15):

$$x = -\frac{p}{2}. \tag{31}$$

The parabola defined by equation (30) has focus $F\left(\frac{p}{2};0\right)$. The corresponding ordinates of the points of the parabola are equal $\pm p$; parabola goes through the points $O(0,0)$; $P\left(\frac{p}{2};p\right)$; $Q\left(\frac{p}{2};-p\right)$. The focal radius of point $M(x,y)$ of this parabola is calculated by the formula

$$r = x + \frac{p}{2}. \tag{32}$$

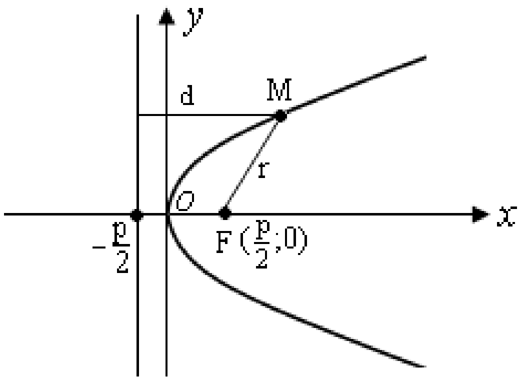


Fig. 23. Parabola symmetrical about the axis Ox

A parabola symmetric about the axis Oy and passing through the origin is determined by the equation (fig. 24)

$$x^2 = 2qy. \tag{33}$$

The focus of the parabola (33) is at $F\left(0,\frac{q}{2}\right)$, her director's equation $y = -\frac{q}{2}$. The focal radius of its point $M(x,y)$ is expressed by the formula $r = y + \frac{q}{2}$.

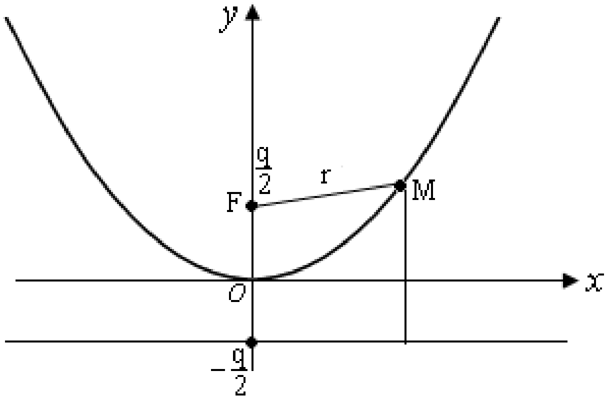


Fig. 24. Parabola symmetric about the axis Oy

Comment 1. Equation $y^2 = -2px, \ (p > 0)$ defines a parabola that is symmetrical about the Ox axis, passing through the origin and completely located to the left of the Oy axis (fig. 25 a).

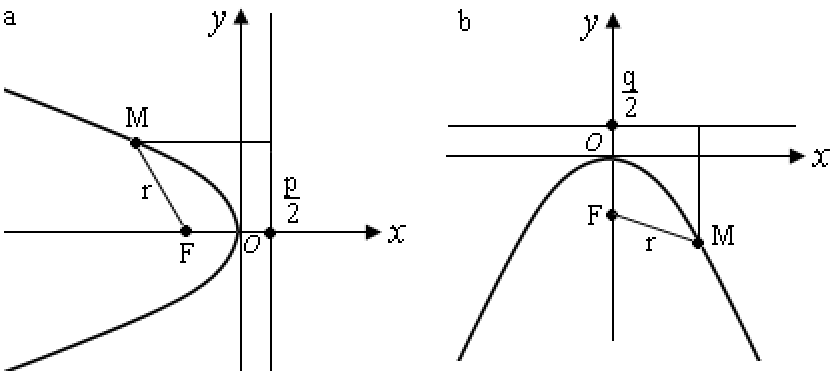


Fig. 25. Parabolas symmetric about axis Ox – (a) and about axis Oy – (b)

Comment 2. Equation $x^2 = -2qy, \ q > 0$ defines the parabola shown in the last figure on the right (fig. 25b).

Example. Find the coordinates of the focus and the equation of the directrix of the parabola $y^2 = 16x$. Calculate the distance of the point M (1,4) to the focus.

Solution. Comparing equation $y^2 = 16x$ with equation (30), we find that $2p = 16$, where from $p = 8$; $\frac{p}{2} = 4$.

In accordance with formula (31), we obtain the equation $x = -4$ of the directrix of the parabola, the focus of the parabola is at a point $F(4, 0)$.

The point M(1, 4) lies on a parabola, since its coordinates satisfy the equation $y^2 = 16x$. By the formula (32) we find the focal radius of the point M: $r = 1 + 4 = 5$.

4. Strait Line in a Space $\square^3$

The equations of the line passing through the point $M_0(x_0, y_0, z_0)$ and being parallel to the vector $\bar{S} = (l, m, n)$, are written as follows:

$$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n} \qquad (34)$$

Equations (34) are called the canonical equations of the line, and the vector that is collinear to this line is called the directing vector of the line (fig. 26).

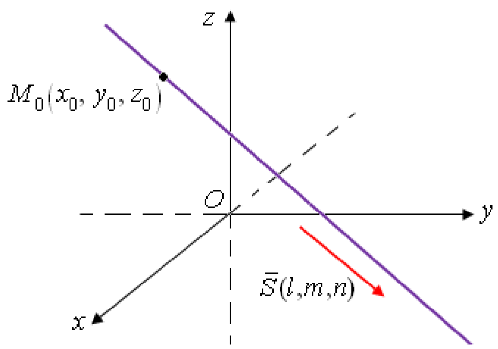


Fig. 26. Line passing through the point $M_0(x_0,\ y_0,\ z_0)$ and being parallel to the vector $\bar{S}=(l,m,n)$

If each of relations (34) is equated to the parameter t, then we obtain the parametric equations of the line:

$$x = x_0 + l\cdot t,\quad y = y_0 + m\cdot t,\quad z = z_0 + n\cdot t\,, \tag{35}$$

The equations of the line passing through two given points $M_1(x_1,\ y_1,\ z_1)$, $M_2(x_2,\ y_2,\ z_2)$:

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1} \tag{36}$$

Example. Make the equations of the line passing through points $A(1,-2,5)$ and $B(3,4,0)$.

Solution. By condition $x_1=1,\ y_1=-2,\ z_1=5$; $x_2=3,\ y_2=4,\ z_2=0$. Than by formula (36) we find $\frac{x-1}{3-1}=\frac{y-(-2)}{4-(-2)}=\frac{z-5}{0-5}$, or $\frac{x-1}{2}=\frac{y+2}{6}=\frac{z-5}{-5}$.

Example: Compose a canonical equations for the line passing through the point $M_0(-1,\ 3,\ 2)$ and being parallel to the axis Ox.

Solution. Let us find the equations of the line by the formula (34):

$$\frac{x-x_0}{l}=\frac{y-y_0}{m}=\frac{z-z_0}{n}.$$

Since the desired line passes through point $M_0(-1,\ 3,\ 2)$, its coordinates satisfy the equations of the line:

$$\frac{x+1}{l}=\frac{y-3}{m}=\frac{z-2}{n}.$$

Since the line is parallel to the Ox axis, the basis vector $\bar{i}\,(1,0,0)$ can be taken as the directing vector of this line.

From here the desired equations will take the form:

$$\frac{x+1}{1}=\frac{y-3}{0}=\frac{z-2}{0}.$$

5. Planes

Space plane is given by the equation $Ax+By+Cz+D=0$ in a rectangular coordinate system of the E^3 space.

General Equation of the Plane

The general equation of plane in a rectangular Cartesian coordinate system of the E^3 space has the following form:

$$Ax+By+Cz+D=0, \tag{37}$$

where $A^2+B^2+C^2\neq 0$; x, y and z are running coordinates of a point in the plane.

Vector $\overline{N}=(A,B,C)$ is perpendicular to the plane and is called the normal vector to the plane.

The equation of the plane passing through point $M_0(x_0,\ y_0,\ z_0)$ and being perpendicular to the vector $\overline{N}=(A,B,C)$ has the form (fig. 27):

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0. \tag{38}$$

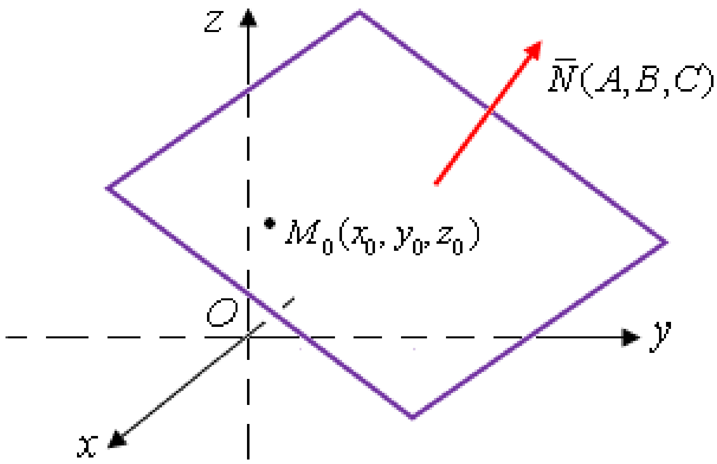


Fig. 27. Plane passing through the point $M_0(x_0,\ y_0,\ z_0)$
and being perpendicular to the vector $\overline{N}=(A,B,C)$

The Equation of the Plane in the Intercept Form

The equation of the plane in the intercept form:

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1, \tag{39}$$

where a, b, c – are the lengths of the segments cut off on the coordinate axes, taken with the corresponding signs (fig. 28).

Where, the quantities a, b and c are respectively, the $x-$intercept, $y-$intercept and $z-$intercept of the plane.

Equation (39) is called the equation of a plane in the intercept form.

For instance, the equation $\frac{x}{2}+\frac{y}{-5}+\frac{z}{4}=1$ describes the plane with the $x-$, $y-$, $z-$intercepts equal 2, -5 and 4, respectively.

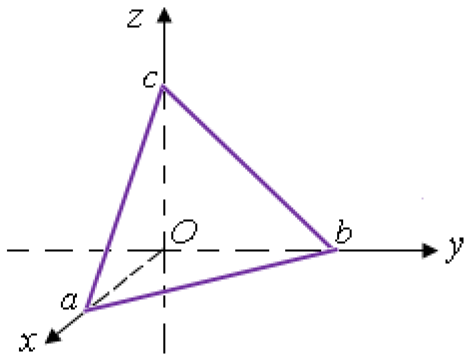


Fig. 28. Plane cutting off the segments along the coordinate axes a, b and c

Equation of a Plane Passing through Three Points $M_1(x_1, y_1, z_1)$, $M_2(x_2, y_2, z_2)$, $M_3(x_3, y_3, z_3)$

Let $M_1(x_1, y_1, z_1)$, $M_2(x_2, y_2, z_2)$, $M_3(x_3, y_3, z_3)$ be three given points in a plane P, and $M(x, y, z)$ be an arbitrary point in P. The equation of a plane passing through three given points has the form

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0 . \tag{40}$$

Distance from Point $M_0(x_0, y_0, z_0)$ to a Plane $Ax+By+Cz+D=0$

The distance from the point $M_0(x_0, y_0, z_0)$ to the plane $Ax+By+Cz+D=0$ is found by the formula:

$$d = \frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}} . \tag{41}$$

Example. Find the distance from point $A(1,-3,4)$ to the plane $5x-y+4z-31=0$.

Solution. By condition $x_0=1,\ \ y_0=-3,\ \ z_0=4$, $A=5,\ B=-1,\ C=4,\ D=-31$. Then by formula (41) we obtain

$$d=\frac{|5\cdot 1+(-1)\cdot(-3)+4\cdot 4-31|}{\sqrt{5^2+(-1)^2+4^2}}=\frac{|5+3+16-31|}{\sqrt{42}}=\frac{|-7|}{\sqrt{42}}=\frac{7}{\sqrt{42}}.$$

Example. Make an equation for the plane passing through point $B(2,1,5)$ and being perpendicular to the vector $\bar{N}=(7,2,-3)$.

Solution. By condition $x_0=2,\ \ y_0=1,\ \ z_0=5$, $A=7,\ B=2,\ C=-3$. Then by formula (38) we obtain $7(x-2)+2(y-1)-3(z-5)=0$, or $7x+2y-3z-1=0$.

Example. Write the equation of the plane passing through the axis Ox and point $M(-2,3,5)$.

Solution. Because the desired plane passes through the axis Ox, then it passes through the origin, point $O(0,0,0)$, therefore, the coordinates $x=0,\ y=0,\ z=0$ satisfy this equation.

Then by the formula (37) we get

$$A\cdot 0+B\cdot 0+C\cdot 0+D=0\Rightarrow D=0$$

Since the desired plane passes through the axis Ox, its normal vector $\bar{N}(A,B,C)$ is perpendicular to the axis Ox, i.e. the projection of the vector $\bar{N}$ onto the axis Ox is zero, i.e. , $A=0$, then the equation of the desired plane takes the form

$$By+Cz=0. \tag{42}$$

Since the plane passes through point $M(-2,3,5)$, the coordinates of this point satisfy equation (42): $B\cdot 3+C\cdot 5=0$, from here $B=\frac{-5C}{3}$.

We substitute the obtained value of B into (6), then $\frac{-5Cy}{3}+C\cdot z=0$, or after dividing by C: $\frac{-5y}{3}+z=0$, from here $5y-3z=0$.

Example. Find the distance between parallel planes $2x-3y+6z-14=0$ and $4x-6y+12z+21=0$.

Solution. We write the normal vectors of the first and second planes: $\bar{N}_1=(2-3,6)$ $\bar{N}_2=(4,-6,12)$.

We calculate the modules of these vectors:

$$|\bar{N}_1|=\sqrt{2^2+(-3)^2+6^2}=\sqrt{4+9+36}=7;\ |\bar{N}_1|=\sqrt{4^2+(-6)^2+12^2}=\sqrt{16+36+144}=\sqrt{196}=14.$$

Find the distance from point $O(0,0,0)$ to the first and second planes by the formula (41):

$$d_1 = \frac{|2\cdot 0 - 3\cdot 0 + 6\cdot 0 - 14|}{7} = \frac{14}{7} = 2, \quad d_2 = \frac{|4\cdot 0 - 6\cdot 0 + 12\cdot 0 + 21|}{14} = \frac{21}{14} = \frac{3}{2}.$$

Since the free terms D equal to –14 and 21 in the equations of the planes have different signs, the planes are located on different sides of the point $O(0,0,0)$, then $d = d_1 + d_2 = 2 + 1{,}5 = 3{,}5$.

Answer: 3.5.

SECTION II. MATHEMATICAL ANALYSIS

Introduction to the Analysis

Sequence. Sequence Limit. Function. Function Limit

1. The Real Numbers. Constant and Variable Values

Number is a main concept of Mathematics, used for quantative characteristic, comparison, numbering of objects and their parts.

Natural numbers are the numbers such as $1, 2, 3, 4, \ldots$; integers are the following numbers $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$.

One can also say that the set of integers consist of all natural numbers, the negative of the natural numbers, and the number zero. The set of all integers can be subdivided into the two classes of numbers: either even or odd.

An even number is integer that is divisible by the number two. If n is any integer, than the number $m = 2n$ is even. A number m is called an odd number, if only m is an integer but not $m/2$. If n is any integer, than the number $m = 2n + 1$ is odd.

A number is called a rational number, if it can be expressed exactly as the quotient of two integers m and n, where $n \neq 0$. All integers are also rational numbers, because any integer m can be expressed by the quotient of two integers m and 1. In addition, a rational number can be also represented: either by a terminating decimal, for instance, $\frac{3}{4} = 0,75$; or a recurring decimal, for example, $\frac{15}{11} = 1,3636(36)\ldots$.

Conversely, irrational numbers are the numbers that can be expressed by non-repeating and non-terminating decimals. Any irrational number is not capable of being expressed as the quotient of two integers. Some examples of irrational numbers: $\sqrt{2} = 1,4142\ldots$, $\pi \approx 3,141592\ldots$, $e \approx 2,718281\ldots$. Real numbers are either rational or irrational

$\mathbb{N} = (1, 2, 3, \ldots, n, \ldots)$ – the set of all natural numbers;

$\mathbb{Z} = (\ldots, -n, -n+1, \ldots, -1, 0, 1, 2, \ldots, n, \ldots)$ – the set of all integers;

$\mathbb{Q} -$ the set of all rational numbers; $I -$ the set of all irrational numbers;

The set of all rational and irrational numbers form the set of real numbers: $\mathbb{R} = \mathbb{Q} \cup \mathrm{I}$,

$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

Example. Number 7 is natural. It is also positive, integer, odd, rational and real.

There is a one-to-one correspondence between the set of real numbers and points on the real number line, that is, every point on this line corresponds to unique real number, and every real number can be paired with a unique point on this number line.

In practice, one has to deal with different quantities: volume, weight, temperature, etc. Values come in two forms - constant and variable.

Definition. A quantity is called a variable if it can take on different numerical values.

Definition. A value is called constant if it takes the same value.

Variables are denoted by letters $x, y, z, \ldots$. Constant values are indicated by letters $a, b, c, \ldots$. The set of all values that a variable can take is called the range of its.

Example. A circle of radius R is given (fig. 29).

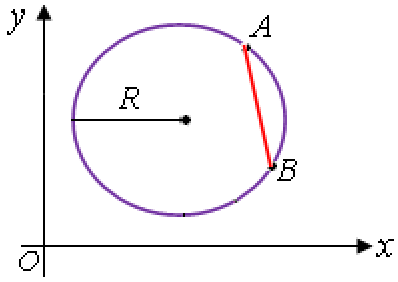


Fig. 29. Constant and variable values in a circle

Here $R,\ 2\pi R,\ \pi R^2 -$ values are constant; $AB-$chord. Let the point $A-$ be fixed and the point B move in a circle. Then the length of the chord AB will be a variable.

Often the ranges of a variable are the intervals: a closed interval [a,b]; the open interval (a,b); $x \to \infty$ x tends to infinity;

A finite interval is a set of real numbers represented by a line segment of the number line between the two endpoints, a and b. An interval whose endpoints are not included in the interval is called open.

In particular, the open interval (a,b) is the set of all real number x such that $a < x < b$. If both endpoints, a and b, are included in a finite set, then the interval is called closed and denoted by the symbol [a,b]. Thus, the closed interval [a,b] is the set of all real numbers x such that $a \le x \le b$. Examples of open and closed intervals are shown in diagram form in fig. 30.

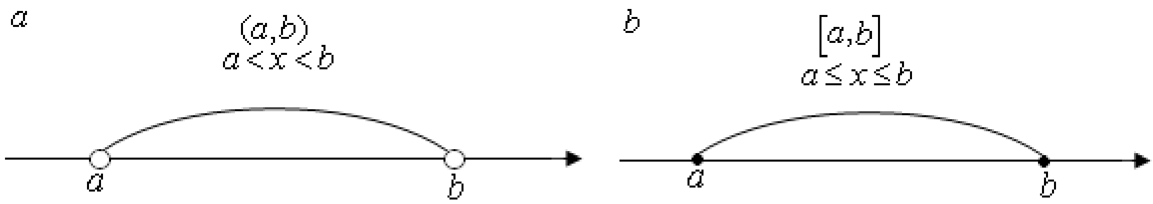


Fig. 30. Image of open and closed interval

An interval is called half-open if only one endpoint is included, that is, half-open interval contains all points between a and b, and either a or b but not both. A half-open interval is denoted by $(a,b]$ if the point b included in the interval. Therefore, $(a,b] = \{x / a < x \le b\}$; while $[a,b)$ is a set of real number x such that $a \le x < b$: $[a,b) = \{x / a \le x < b\}$. Half-open intervals are represented in diagram form in fig. 31.

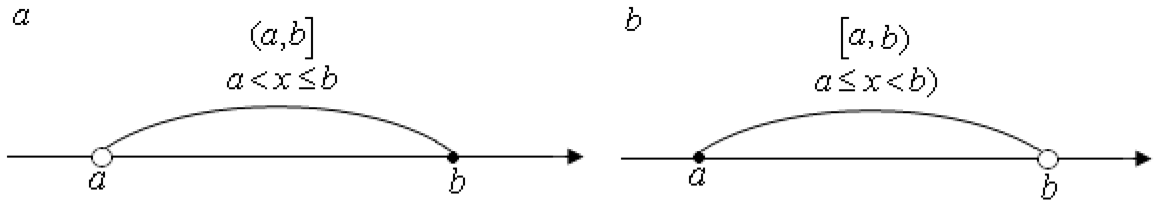


Fig. 31. Image half-open interval

The half-infinity interval is a set of real numbers represented by a part of the number line. This part is bounded from one side and unbounded from the other in the direction of positive or negative infinity. The infinity interval $(-\infty,+\infty)$ has no endpoints and represents the set of all real numbers.

Example. The infinity interval $[a,+\infty)$ is a set of real numbers x with $a\le x$ (fig. 32).

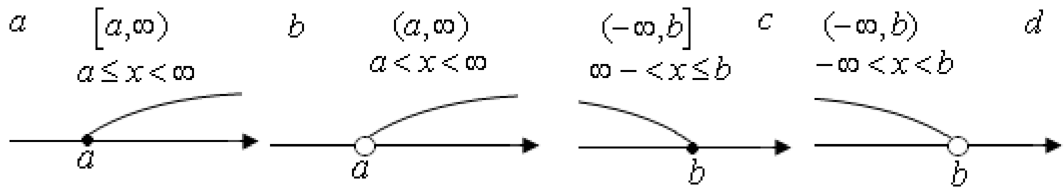


Fig. 32. Image of semi-infinite intervals

1) $a\le x\le b$; $X=[a,b]$ –a closed interval
2) $a<x<b$; $X=(a,b)$ –the open interval
3) $a\le x<b$; $X=[a,b)$; a half-open interval
4) $a\le x<+\infty$; $X=[a,+\infty)$; The half-infinity interval
5) $-\infty<x<+\infty$; $X=(-\infty,+\infty)$. The infinity interval

2. Function

Definition. A variable y is called a function of a variable x, if each numerical value of a variable x from the range of its corresponds to a single value of a variable y:

$$y=f(x). \tag{42}$$

A variable x is argument; a variable y is a function.
The range of argument x is called the domain of definition of the function y $-D(y)$.
The range of variable y is called the domain of values of the function y $-\mathrm{E}(y)$.
Definition. A graph of a function $y=f(x)$ is a set of points of a plane of the form $(x,f(x))$.

There are various ways of setting a function: analytical, graphic, tabular, verbal, etc.

Example. The function is determined by means of table.

x	-3	-2	-1	0	1	2	3
$f(x)$	5	2	0	-1	3	4	5

The Basic elementary functions (figures 33-41)

1) $y(x) \equiv C$ – constant function (Fig. 32); $D(y) = (-\infty, +\infty)$, $E(y) = C$.

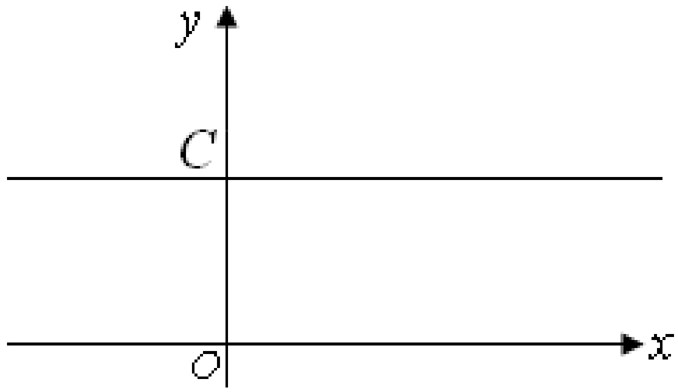


Fig. 33. Constant function graph

2) $y(x) = x^{\alpha}$ is a power function , $x > 0$, α is a real number.

Special cases of a power function (fig .34-35)

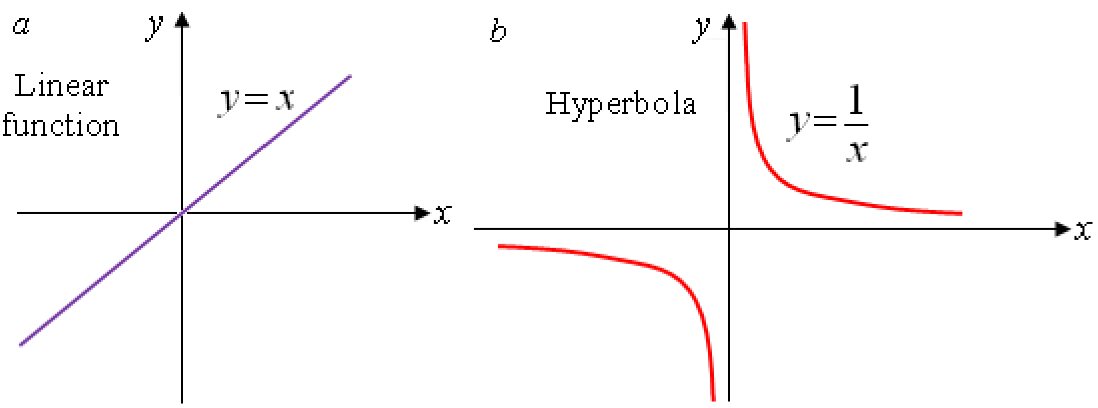


Fig. 34. Graphs of the linear function $y = x$ (a) and hyperbola $y = \frac{1}{x}$ (*b*)

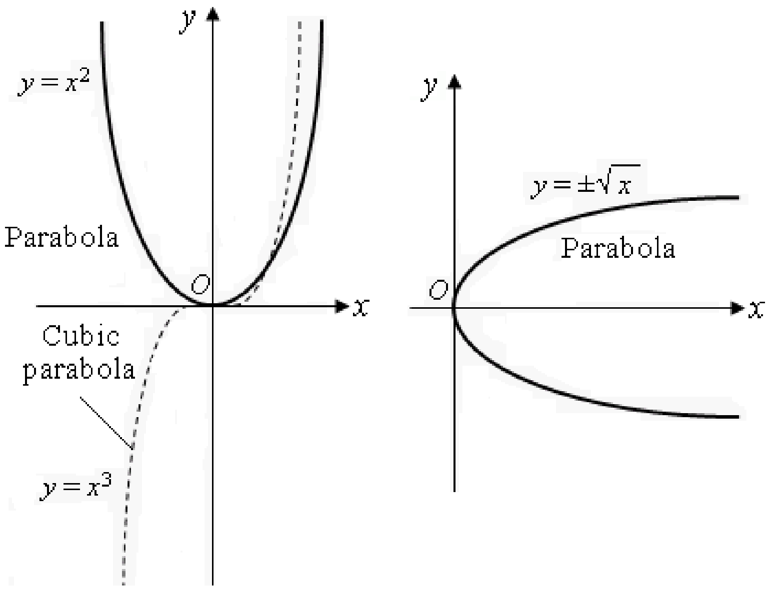


Fig. 35. Graphs of Parabols $y = x^2$, $y = x^3$ (*a*) and $y = \pm\sqrt{x}$ (*b*)

3) $y(x) = a^x$ – the exponential function, $a > 0$, $a \neq 1$. $D(y) = (-\infty, +\infty)$, $E(y) = (0, +\infty)$ (fig. 36). The exponential function has a follow form, $y(x) = a^x$ where a is a base $a > 0$, $a \neq 1$. The domain of any exponential function consists of all real numbers while its range consists of positive real numbers only.

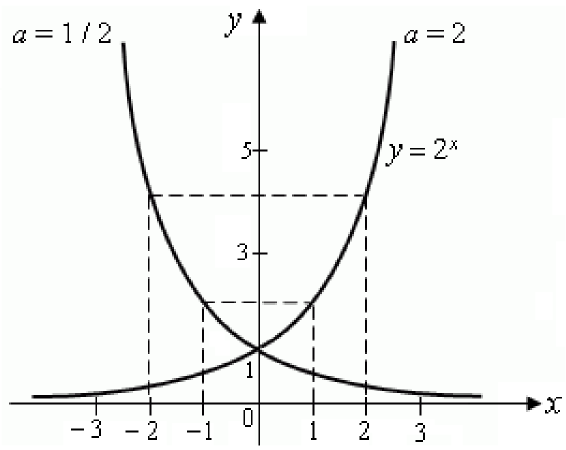


Fig. 36. Graphs of exponential functions $y = 2^x$ and $y = \left(\frac{1}{2}\right)^x$

Let a and be positive real numbers but $a \neq 1$.Then the logarithm of x to the base a is the exponent of the power to which the base a must be raised to equal a given number x, that is, $y = \log_a x$ whenever, $x = a^y$. This definition implies the following useful identities: $x = a^{\log_a x}$ $y = \log_a a^y$.

4) $y(x) = \log_a x$ – the logarithmic function, where a is a base of logarithm $a > 0$, $a \neq 1$. $D(y) = (0, +\infty)$, $E(y) = (-\infty, +\infty)$ (fig. 37).

$\log_a x$ Logarithm of x to base a

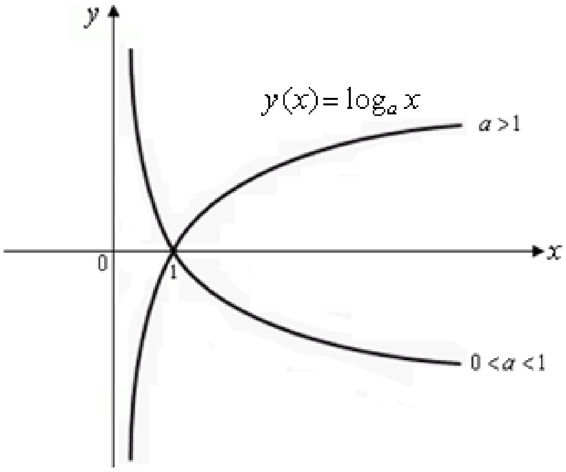
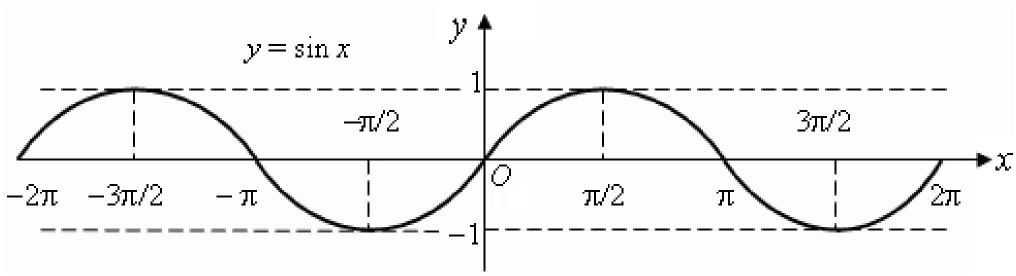


Fig. 37. Graphs of logarithmic functions $y(x) = \log_a x$

5) Trigonometric functions (fig. 38-39).



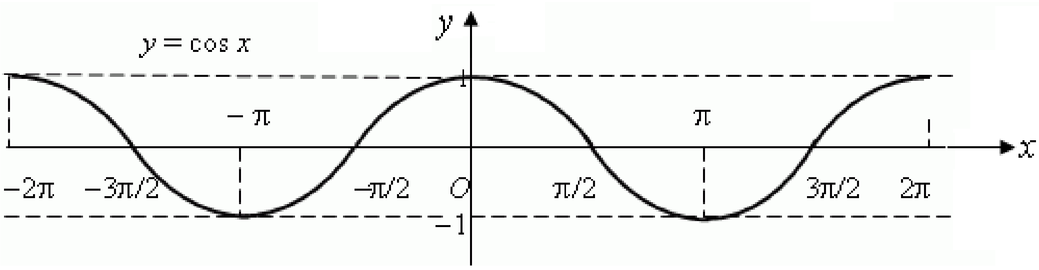


Fig. 38. Graphs of trigonometric functions $y(x) = \sin x$ and $y(x) = \cos x$

Sine wave is a graph of odd function $y(x) = \sin x$.Cosine curve is a graph of even function $y(x) = \cos x$.

For functions $y(x) = \sin x$ and $y(x) = \cos x$ $D(y) = (-\infty, +\infty)$, $E(y) = [-1, 1]$. This functions are periodical. Period of functions $\sin x$ and $\cos x$ is equal $T = 2\pi$.

For functions $y(x) = \operatorname{tg} x$ and $y(x) = \operatorname{ctg} x$ are periodical. Period is equal $T = \pi$.

Tangensoid is a graph of an odd function $y(x) = \operatorname{tg} x$. Points of gap of the tangensoid are $x = \frac{\pi}{2} + \pi n,\ n \in Z$. Cotangent curve is a graph of odd function $y(x) = ctgx$. Gap points of contangent curve are $x = \pi n,\ n \in Z$.

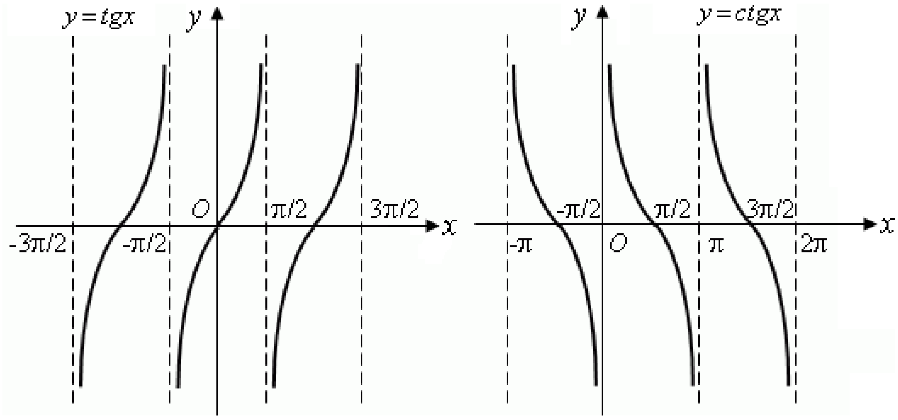


Fig. 39. Graphs of trigonometric functions $y(x) = \text{tg}\, x$ and $y(x) = \text{ctg}\, x$

6) Inverse trigonometricfunctions (fig. 40-41).

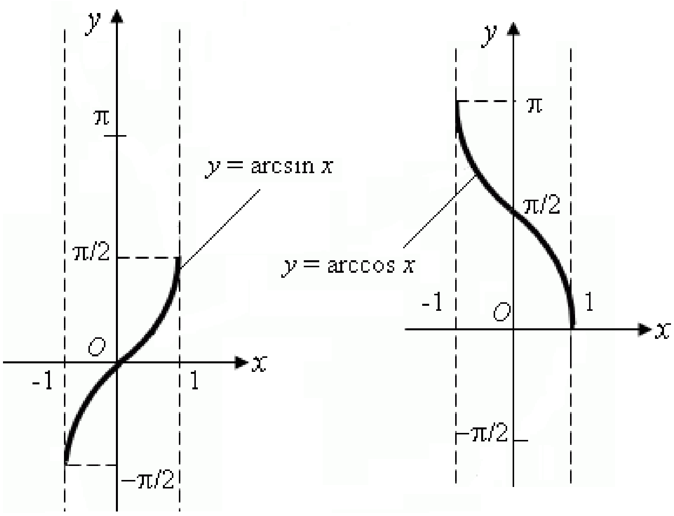


Fig. 40. Graphs of inverse trigonometric functions $y(x) = \arcsin x$ and $y(x) = \arccos x$

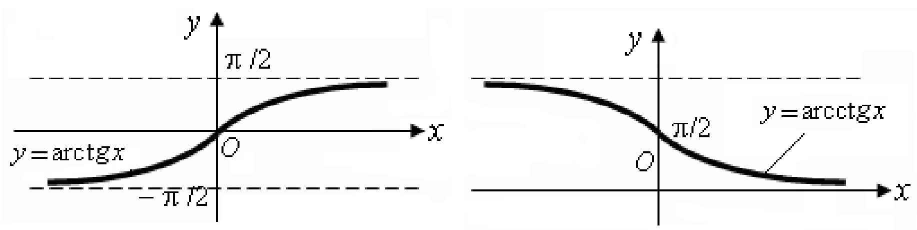


Fig. 41. Graphs of inverse trigonometric functions $y(x) = \text{arctg}\, x$ and $y(x) = arcctgx$

Complex Function

A complex function is a function of a function.

$y = y(u(x))-$ superposition of functions y and u .

Elementary functions are functions that can be obtained from the basic elementary functions using the operations of addition, subtraction, multiplication, division and superposition.

Example. $y = \dfrac{tg^3 \lg\sqrt{x^2+1}}{x+\sin(\cos x)}$.

3. Modulus of Number

The absolute value of nonnegative number is the number itself, while absolute value of negative number is the number itself, multiplied by minus one.

$$|a| = \begin{cases} a, & \text{если } a \ge 0, \\ -a, & \text{если } a < 0. \end{cases}$$

Geometric interpretation

The absolute value of real number is the distance between the corresponding point on the number line and the zero-point regardless of the direction (fig. 42).

For any numbers a and b, the distance between points a and b on the number line is $|a-b|$.

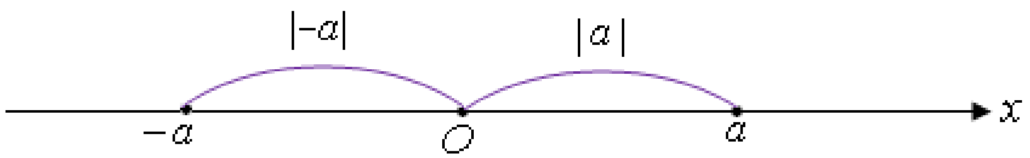


Fig. 42. Geometric interpretation of the modulus of a number

Properties of modulus

1. $|a| \ge 0$;

2. $|-a| = |a|$;

3. $|a-b| = |b-a|$;

4. $|ab| = |a|\cdot|b|$;

5. $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$, $b \ne 0$;

6. $|a| = 0$ *тогда и только тогда*, *когда* $a = 0$;

7. $|a+b| \le |a| + |b|$;

8. $|a|^2 = a^2$.

Comment. If $|x| < \varepsilon$, then $-\varepsilon < x < \varepsilon$.

4. Limit of Numerical Sequence

Definition. $\delta-$ neighborhood of a point x_0 is an interval $(x_0-\delta, x_0+\delta)$ centered at a point x_0 of radius δ (fig. 43).

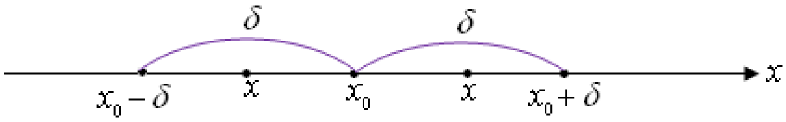


Fig. 43. $\delta-$neighborhood of point x_0

All points x belonging to a $\delta-$ neighborhood of point x_0 satisfy the inequality: $x_0-\delta<x<x_0+\delta$ or $-\delta<x-x_0<\delta$, or $|x-x_0|<\delta$, those the distance from points x to points x_0 is less then δ.

Definition. $\varepsilon-$ neighborhood of a point y_0 is an interval $(y_0-\varepsilon, y_0+\varepsilon)$ centered at a point y_0 of radius ε (fig. 44).

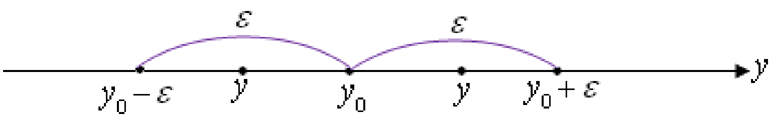


Fig. 44. $\varepsilon-$neighborhood of point y_0

All points y belonging to a $\varepsilon-$neighborhood of a point y_0 satisfy the inequality: $|y-y_0|<\varepsilon$ those the distance from points y to points y_0 is less then ε.

Definition. Sequence of elements in a given set (u_n) is a function, defined on the set of natural numbers $n\in\mathbb{N}$ which is given by its members $u_1, u_2, ..., u_n, ...$ or by the formula of general term $u_n=f(n)$.

When $n=1$ $u_1=f(1)$; at $n=2$ $u_2=f(2)$ …

Example. Let given the sequence $u_n=\dfrac{1}{n}$. Here $u_1=1$, $u_2=\dfrac{1}{2}$, $u_3=\dfrac{1}{3}, ...$

$1, \dfrac{1}{2}, \dfrac{1}{3}, \dfrac{1}{4}, ..., \dfrac{1}{n}, ...$

It can be noted that for $n\to\infty$ implies $\dfrac{1}{n}\to 0$.

Definition. The number A is called the limit of the number sequence (u_n) where $n \to \infty$,

if for any ($\forall\ \varepsilon > 0$), we can indicate the number N_ε , that for $n > N_\varepsilon$, the inequality $|u_n - A| < \varepsilon$ is carried out, i.e., the distance from the members of the sequence to point A will be less ε .

$$\lim_{n\to\infty} u_n = A.$$

Definition. Limit of sequence (u_n) is a number A, if a natural number N exists for any positive number ε , and the inequality $|u_n - A| < \varepsilon$ is carried out for all $n > N$.

Example. Proof that $\lim\limits_{n\to\infty} \dfrac{n-1}{n} = 1$.

Solution. By definition, the number 1 will be the limit of the sequence $x_n = \dfrac{n-1}{n}$, $n \in \mathbb{N}$,

if $\forall \varepsilon > 0$ there is a natural number N such that for all $n > N$ the inequality holds $\left|\dfrac{n-1}{n} - 1\right| < \varepsilon$,

i.e. $\dfrac{1}{n} < \varepsilon$. It is valid for all $n > \dfrac{1}{\varepsilon}$, i.e. for all $n > N = \left[\dfrac{1}{\varepsilon}\right]$, where $\left[\dfrac{1}{\varepsilon}\right]$ - is the integer part of the number $\dfrac{1}{\varepsilon}$ (the integer part of number x is denoted by $[x]$, is the largest integer not exceeding x; for example, the integer part of the number $6{,}3$ is $[6{,}3] = 6$). If $\varepsilon > 1$, then for N we can take $\left[\dfrac{1}{\varepsilon}\right] + 1$.So, $\forall \varepsilon > 0$, the corresponding value of N is indicated. This proves that $\lim\limits_{n\to\infty} \dfrac{n-1}{n} = 1$.

Properties Limit of Sequence

1. A Limit of constant is equal constant:

$$\lim_{n\to\infty} C = C;$$

2. A constant factor can be taken out of the sign of the limit:

$$\lim_{n\to\infty} C \cdot u_n = C \lim_{n\to\infty} u_n;$$

3. If both limits, $\lim\limits_{n\to\infty} u_n$ and $\lim\limits_{n\to\infty} v_n$, esxist then the limit of the sum of the sequences equals the sum of the limits of the sequences:

$$\lim_{n\to\infty} (u_n \pm v_n) = \lim_{n\to\infty} u_n \pm \lim_{n\to\infty} v_n;$$

4. If the limits of the sequences (u_n) and (v_n) esxist as $n \to \infty$, then the limit of the product of these sequences equals the product of the limits of the sequences:

$$\lim_{n\to\infty} (u_n \cdot v_n) = \lim_{n\to\infty} u_n \cdot \lim_{n\to\infty} v_n;$$

5. If the limits of the sequences (u_n) and (v_n) esxist as $n \to \infty$, then the limit of the quotient of these sequences equals the quotient of the limits of the sequences (provided that $\lim\limits_{n\to\infty} v_n \neq 0$) :

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{\lim\limits_{n\to\infty} u_n}{\lim\limits_{n\to\infty} v_n}.$$

Definition. If a sequence (u_n) has a finite limit, then it is called convergent.

If the limit of a numerical sequence does not exist $(\exists)$ or is equal the infinite (∞), then it is called divergent.

Definition. A sequence (u_n) is called bounded if exists $(\exists)$ a number C is such that $|u_n| \leq C$ for any n $(\forall n)$.

Example. The sequence $u_n = \dfrac{n}{2n+1}$ is limited because $\left|\dfrac{n}{2n+1}\right| < \dfrac{1}{2}$ $(\forall n)$ for any $n \in \mathbb{N}$.

Theorem 1. If there is a limit to a sequence, then it is unique.

Theorem 2. If a sequence monotonically increases (decreases) and is bounded above (below), then it has a finite limit.

Second Remarkable Limit

Consider a numerical sequence with a common term $u_n = \left(1+\dfrac{1}{n}\right)^n$.

$n=1,\ u_1 = (1+1)^1 = 2$;

$n=2,\ u_2 = \left(1+\dfrac{1}{2}\right)^2 = \left(\dfrac{3}{2}\right)^2 = \dfrac{9}{4} = 2,25$;

$n=3,\ u_3 = \left(1+\dfrac{1}{3}\right)^3 = \left(\dfrac{4}{3}\right)^3 = \dfrac{64}{27} = 2,37$;

$n=4,\ u_4 = \left(1+\dfrac{1}{4}\right)^4 = \left(\dfrac{5}{4}\right)^4 = \dfrac{625}{256} \approx 2,44$; …

$n=100,\ u_{100} = \left(1+\dfrac{1}{100}\right)^{100} = \left(\dfrac{5}{4}\right)^4 \approx 2,704$ и т.д. etc.

You can see that the sequence (u_n) is monotonically increasing, it is proved that it is bounded above $u_n = \left(1+\dfrac{1}{n}\right)^n < 3$ for any n $(\forall n)$.

Then, by Theorem 2, this sequence has a finite limit:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = e \approx 2,718 .$$

This is the second (II) wonderful limit.

This formula gives the definition of the number e.

5. Function Limit of Continuous Argument

Example. Let $y=(x-1)^2$. Consider a sequence of x values converging to 1, and y will tend to 0.

x	0	0,5	0,75	0,99	0,999	$\ldots\to 1$	a
y	1	0,25	0,0625	0,0001	0,000001	$\ldots\to 0$	A

It often happens that when x tends to a $(x\to a)$, then y tends to A $(y\to A)$.

In this case, it is said that the number A is the limit of the function $y=f(x)$ as $x\to a$.

Definition. Limit of function $f(x)$ where x is tending to a, is a number A, if for any $\varepsilon>0$ it can be found $\delta>0$, such as from $|x-a|<\delta$ follows $|f(x)-A|<\varepsilon$:

$$\lim_{x\to a} f(x) = A .$$

Comment. When finding the limit of a continuous function $\lim\limits_{x\to a} f(x)$, it is necessary to substitute the number x on the number a in the function.

Example. $\lim\limits_{x\to 3}(2x+1)=2\cdot 3+1=7$.

Limit from the left and limit from the right

1) Limit from the left (fig. 45).

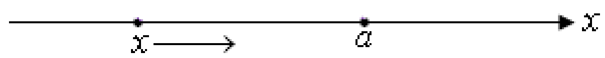


Fig. 45. Limit from the left

Let x tends towards for a, remaining smaller a $(x<a)$.

$\lim\limits_{\substack{x\to a\\ x<a}} f(x) = \lim\limits_{x\to a-0} f(x) = f(a-0) -$ limit from the left at the point a.

2) Limit from the right (fig. 46).

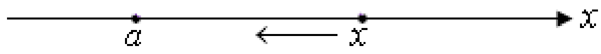


Fig.46. Limit from the right

Let x is tend for a, staying more than a ($x > a$).

$\lim\limits_{\substack{x\to a\\x>a}} f(x) = \lim\limits_{x\to a+0} f(x) = f(a+0) -$ limit to the right at the point a.

Example. Find the limit of the function from the left and from the right at a point $x=1$ if
$$y(x)=\begin{cases} x^2, & x\le 1,\\ 3-x, & x>1.\end{cases}$$

Solution. First find the limit of the function from the left: $\lim\limits_{\substack{x\to 1\\x<1}} f(x) = \lim\limits_{x\to 1-0} x^2 = 1$; and from the right $\lim\limits_{\substack{x\to 1\\x>1}} f(x) = \lim\limits_{x\to 1+0}(3-x) = 2$, we conclude from here: $y(1-0) \ne y(1+0)$ (fig. 47).

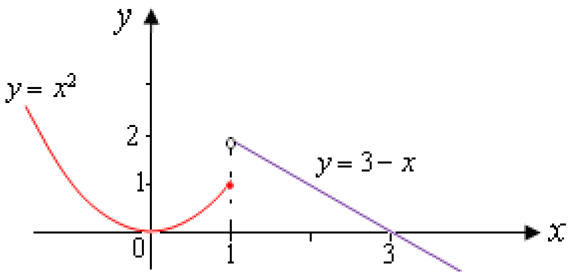


Fig. 47. Limit of the function from the left and from the right at the point $x=1$

If $\lim\limits_{x\to a} f(x) = A$ exists, then $f(a-0)$, $f(a+0)$ exist, and $f(a-0) = f(a+0) = A$.

Example. Let $f(x) = 2x+4$. Find limit from the left and from the right at the point $x=1$.

Solution. $\lim\limits_{\substack{x\to 1\\x<1}}(2x+4) = 2\cdot 1+4 = 6$, $\lim\limits_{\substack{x\to 1\\x>1}}(2x+4) = 2\cdot 1+4 = 6$.

If $f(a-0) \ne f(a+0)$, then $\lim\limits_{x\to a} f(x)$ does not exist.

Example. Make sketch of the function $y = 2^{1/x}$.

Solution. The function is not defined at the point $x=0$. Find the limit from the left and from the right at this point: $\lim\limits_{\substack{x\to 0\\x<0}} 2^{1/x} = 2^{\frac{1}{-0}} = 2^{-\infty} = \frac{1}{2^{+\infty}} = \frac{1}{\infty} = 0$, $\lim\limits_{\substack{x\to 0\\x>0}} 2^{1/x} = 2^{\frac{1}{+0}} = 2^{+\infty} = \infty$. Because $f(0-0) \ne f(0+0)$, then $\lim\limits_{x\to 0} 2^{1/x}$ does not exist. Find the horizontal asymptote of the graph of the function: $\lim\limits_{x\to\infty} 2^{1/x} = 2^{\frac{1}{\infty}} = 2^0 = 1$. Hence $y=1$ is the horizontal asymptote of the function graph (fig. 48).

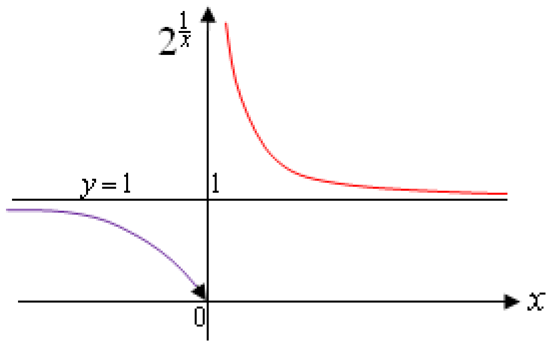


Fig. 48. Graph of the function $y = 2^{1/x}$

Limit of Function as x Tends to Infinity

Предел функции при $x \to \infty$ The limit of function as x becomes infinite

Definition. The number A is called the limit of the function $y = f(x)$ as x tends to infinity ($x \to \infty$), if for any arbitrarily small number $\varepsilon > 0$, you can specify a sufficiently large number $M > 0$, such that for all x exceeding M in absolute value ($|x| > M$), the inequality $|f(x) - A| < \varepsilon$ is carried out:

$$\lim_{x\to\infty} f(x) = A.$$

Example. $\lim\limits_{x\to\infty} \frac{1}{x} = 0$, $\lim\limits_{x\to+\infty} arctgx = +\frac{\pi}{2}$, $\lim\limits_{x\to-\infty} arctgx = -\frac{\pi}{2}$.

Comment. A function does not always have a limit, for example, $\lim\limits_{x\to\infty} cosx$ does not exist.

6. The Infinitesimal and infinite functions

Definition. The function $\alpha(x)$ is called infinitesimal for $x \to a$ if $\lim\limits_{x\to a} \alpha(x) = 0$.

Example. The function $y = \sin x$ is infinitesimal for $x \to 0$ because $\lim\limits_{x\to 0} \sin x = \sin 0 = 0$.

Definition. A function $f(x)$ is said to be bounded (restricted function) in some region of variation of x if there exists a number $C > 0$ such that $|f(x)| \le C$ for all x from this region

Example. $y = \sin x$ is limited on the entire numerical axis: $|\sin x| \le 1$ for any x.

Comment. The function $\alpha(x)$ is infinitesimal at a point $x=a$ bounded in a neighborhood of the point $x=a$.

Definition. A function $\mathrm{f}(x)$ is called infinite function for $x \to a$ if, for any sufficiently large number $M>0$, there exists a number $\delta>0$ such that when $|x-a|<\delta$ the inequality $|f(x)|>M$ is carried out :

$$\lim_{x\to a}\mathrm{f}(x)=\infty .$$

Example. $\lim\limits_{x\to 2}\frac{1}{x-2}=\infty$. $\lim\limits_{x\to 2}$ the limit when x tending to two

Let us $M=100$, then $\delta=\frac{1}{100}$, $|x-2|<\frac{1}{100}$, therefore $\frac{1}{|x-2|}>100$.

Theorem 1. The algebraic sum of a finite number of infinitesimal functions at a point a is a function infinitely small at this point.

Let $\alpha(x),\beta(x)-$are infinitesimal functions at a point a, i.e., $\lim\limits_{x\to a}\alpha(x)=0,\ \lim\limits_{x\to a}\beta(x)=0$.

Then a sum $\gamma(x)=\alpha(x)+\beta(x)-$ is an infinitesimal function at the point a, and a difference $\delta(x)=\alpha(x)-\beta(x)-$ is an infinitesimal function at this point a.

Symbolic notation: $0+0=0,\ 0-0=0$.

Theorem 2. Let a function $\alpha(x)$ be an infinitesimal function at a point a, and $|f(x)|\le C$ in some neighborhood of the point a. Then a product $f(x)\cdot\alpha(x)-$ is an infinitesimal function at a point a.

Consequence 1 of the theorem. The product of a constant value by an infinitesimal function at a point a, is a function infinitesimal at a point.

Symbolically: $C\cdot 0=0$.

Consequence 2 of the theorem. The product of two infinitesimal functions at a point a is a function infinitesimal at this point.

Symbolically: $0\cdot 0=0$.

Consequence 3 of the theorem. The whole positive degree of an infinitesimal function at a point a is a function infinitesimal at this point.

Symbolically: $0^k=0$.

The relationship between infinitesimal and infinitely large functions

Theorem 3. The quotient of dividing a constant by a function that is infinitesimal at a point a is a infinite function at this point.

Let a function $\alpha(x)$ be an infinitesimal function at a point a; then $\lim\limits_{x\to a}\frac{C}{\alpha(x)}=\infty$.

Symbolically $\frac{C}{0} = \infty$.

Example. $\lim_{x\to 0}\frac{5}{x} = \frac{5}{0} = \infty$.

Theorem 4. The quotient of dividing a constant by a infinite function at a point a is an infinitesimal function at this point.

Let a function $f(x)$ be an infinite function at a point a, then $\lim_{x\to a}\frac{C}{f(x)} = 0$.

Symbolically: $\frac{C}{\infty} = 0$.

Example. $\lim_{x\to\infty}\frac{7}{x^2+10} = \frac{7}{\infty} = 0$. –

Note that the case of the division of two infinitesimal functions has not been considered. We consider three infinitesimal functions at the point 0: $\alpha(x) = x^2$, $\beta(x) = x$, $\gamma(x) = 2x$.

We calculate the limit of the following ratios for $x \to 0$:

$\lim_{x\to 0}\frac{x^2}{x} = \lim_{x\to 0} x = 0$. This means that ratio $\frac{x^2}{x}$ is an infinitesimal function; $\lim_{x\to 0}\frac{x}{x^2} = \lim_{x\to 0}\frac{1}{x} = \frac{1}{0} = \infty$. This means that ratio $\frac{x}{x^2}$ is an infinite function; $\lim_{x\to 0}\frac{2x}{x} = 2$. This means that ratio $\frac{2x}{x}$ is const.

Thus, ratio of two an infinitesimal functions may be an infinitesimal function, or an infinite function, or const.

So we say that ratio two an infinitesimal functions at a point a is uncertainty of the form $\frac{0}{0}$.

Symbolically writing theorems about an infinitesimal functions at point a

1. $0 + 0 = 0$;

2. $0 - 0 = 0$;

3. $C \cdot 0 = 0$;

4. $0 \cdot 0 = 0$;

5. $0^k = 0$, $k -$ is integer greater than zero.

6. $\frac{0}{C} = 0$;

7. $\frac{C}{0} = \infty$;

8. $\frac{0}{0}-$ is uncertainty (indeterminacy).

Symbolically writing theorems about an infinite functions at point a

1. $\infty+\infty=\infty$;

2. $(\infty-\infty)-$ is uncertainty (indeterminacy);

3. $C\cdot\infty=\infty$;

4. $\infty^{k}=\infty$, $k-$ is integer greater than zero.

5. $\frac{\infty}{C}=\infty$;

6. $\frac{\infty}{0}=\frac{1}{0}\cdot\infty=\infty\cdot\infty=\infty$;

7. $\frac{C}{\infty}=0$;

8. $\frac{\infty}{\infty}-$ is uncertainty (indeterminacy).

9. $0\cdot\infty-$ is uncertainty (indeterminacy).

7. Main Theorems about Limits

Theorem 1. If function has limit at a point a then it may be represented as sum this limit and an infinitesimal function at point a.

Let $\lim\limits_{x\to a} f(x)=A$, then $f(x)=A+\alpha(x)$, where $\lim\limits_{x\to a}\alpha(x)=0$.

Proof. Let $\lim\limits_{x\to a} f(x)=A$, consequently $(\forall\varepsilon>0\ \ \exists\delta>0: 0<|x-a|<\delta)\Rightarrow|f(x)-A|<\varepsilon$, i.e. $|f(x)-A-0|<\varepsilon$. This means that function $f(x)-A$ has a limit of zero, i.e., it is an infinitesimal function, which we denote by $\alpha(x)$: $f(x)-A=\alpha(x)$. Hence, $f(x)=A+\alpha(x)$.

Inverse theorem.

If $f(x)=A+\alpha(x)$, where $\lim\limits_{x\to a}\alpha(x)=0$, then $\lim\limits_{x\to a} f(x)=A$.

Theorem 2. A constant factor can be taken of the sign of the limit: $\lim\limits_{x\to a} cf(x)=c\lim\limits_{x\to a} f(x)$.

Theorem 3. Let both limits, $\lim\limits_{x\to a} u(x)$ and $\lim\limits_{x\to a} v(x)$, exist then the limit of the algebraic sum of a finite number of functions is equal to the algebraic sum of the limits of the terms:

$\lim\limits_{x\to a}(u(x)\pm v(x))=\lim\limits_{x\to a} u(x)\pm\lim\limits_{x\to a} v(x)$.

Theorem 4. Let both limits, $\lim\limits_{x\to a} u(x)$ and $\lim\limits_{x\to a} v(x)$, exist then the limit of the product of any finite number of functions is equal to the product of the limits of the multipliers:

$\lim\limits_{x\to a}(u(x)\cdot v(x)) = \lim\limits_{x\to a} u(x)\cdot \lim\limits_{x\to a} v(x)$.

Theorem 5. If both limits, $\lim\limits_{x\to a} u(x)$ and $\lim\limits_{x\to a} v(x)$ exist as $x\to a$, then the limit of the quotient of these functions equals the quotient of the limits of the functions (provided that $\lim\limits_{x\to a} v(x)\neq 0$):

$\lim\limits_{x\to a}\dfrac{u(x)}{v(x)} = \dfrac{\lim\limits_{x\to a} u(x)}{\lim\limits_{x\to a} v(x)}$.

First Remarkable Limit

$$\lim_{x\to 0}\frac{\sin x}{x} = 1. \qquad (42)$$

The first remarkable limit reveals the indeterminacy of the form $\dfrac{0}{0}$.

Example. $\lim\limits_{x\to 0}\dfrac{\sin 5x}{x} = \cancel{\dfrac{0}{0}} = \lim\limits_{x\to 0}\dfrac{5\sin 5x}{5x} = 5\lim\limits_{x\to 0}\dfrac{\sin 5x}{5x} = 5\cdot 1 = 5$.

8. Comparison of Infinitesimal Functions

Let given two infinitesimal functions at a point a $\alpha(x)$ and $\beta(x)$, i.e. $\lim\limits_{x\to a}\alpha(x)=0$, $\lim\limits_{x\to a}\beta(x)=0$.

To compare two infinitesimal functions, one must find the limit of their ratio.

1. $\lim\limits_{x\to a}\dfrac{\alpha(x)}{\beta(x)} = C\neq 0$. Then functions $\alpha(x)$ and $\beta(x)$ are called infinitesimal of the same order of smallness.

If $\lim\limits_{x\to a}\dfrac{\alpha(x)}{\beta(x)} = 1$, then infinitesimal $\alpha(x)$ and $\beta(x)$ are called equivalent:

$\alpha(x)$ is equivalent to $\beta(x)$ as $x\to a$ or $\alpha(x)\sim\beta(x)$ as $x\to a$

2.If $\lim\limits_{x\to a}\dfrac{\alpha(x)}{\beta(x)} = 0$, then the infinitesimal $\alpha(x)$ is called the infinitesimal higher order of smallness compared to the infinitesimal $\beta(x)$: $\alpha(x) = o(\beta(x))$ for $x\to a$, where $o(\beta(x))$ means the infinitesimal function higher order of smallness compared to $\beta(x)$.

Table of Equivalent Infinitesimal Functions

Let function $\alpha(x)\to 0$, as $x\to a$ then

1. $\sin\alpha(x)\sim\alpha(x)$;

2. $tg\alpha(x) \sim \alpha(x)$;

3. $\arcsin \alpha(x) \sim \alpha(x)$;

4. $arctg\alpha(x) \sim \alpha(x)$;

5. $\ln(1+\alpha(x)) \sim \alpha(x)$;

6. $(e^{\alpha(x)} - 1) \sim \alpha(x)$;

7. $(a^{\alpha(x)} - 1) \sim \alpha(x) \cdot \ln a$;

9. $\left[(1+\alpha(x))^n - 1\right] \sim n \cdot \alpha(x)$.

Theorem 1. The limit of the ratio of two infinitesimal functions does not change if one of them or both is replaced by equivalent ones.

Let $\alpha(x),\ \beta(x),\ \alpha_1(x),\ \beta_1(x) -$ are the infinitesimal functions at point a and $\alpha(x) \sim \alpha_1(x),\ \beta(x) \sim \beta_1(x)$ for $x \to a$, then

$$\lim_{x\to a}\frac{\alpha(x)}{\beta(x)} = \lim_{x\to a}\frac{\alpha_1(x)}{\beta(x)} = \lim_{x\to a}\frac{\alpha(x)}{\beta_1(x)} = \lim_{x\to a}\frac{\alpha_1(x)}{\beta_1(x)}.$$

Proof. Since $\alpha(x) \sim \alpha_1(x),\ \beta(x) \sim \beta_1(x)$ for $x \to a$, then $\lim\limits_{x\to a}\frac{\alpha(x)}{\alpha_1(x)} = \lim\limits_{x\to a}\frac{\beta_1(x)}{\beta(x)} = 1$ and then we get

$$\lim_{x\to a}\frac{\alpha(x)}{\beta(x)} = \lim_{x\to a}\frac{\alpha(x)}{\beta(x)}\cdot\frac{\alpha_1(x)}{\alpha_1(x)}\cdot\frac{\beta_1(x)}{\beta_1(x)} = \lim_{x\to a}\frac{\alpha(x)}{\alpha_1(x)}\cdot\lim_{x\to a}\frac{\beta_1(x)}{\beta(x)}\cdot\lim_{x\to a}\frac{\alpha_1(x)}{\beta_1(x)} = \lim_{x\to a}\frac{\alpha_1(x)}{\beta_1(x)}.$$

Example. $\lim\limits_{x\to 0}\frac{\arcsin 4x}{e^{2x}-1} = \frac{0}{0} = \lim\limits_{x\to 0}\frac{4x}{2x} = 2$.

Theorem 2. Let $\alpha(x)$ be an infinitesimal function of a higher order of smallness compared to an infinitely small $\beta(x)$: $\alpha(x) = o(\beta(x))$ as $x \to a$. Then $(\alpha(x)+\beta(x)) \sim \beta(x)$ as $x \to a$.

Example. $\lim\limits_{x\to 0}\frac{\sin x + x^2}{\ln(1+x) - x^3} = \frac{0}{0} = \lim\limits_{x\to 0}\frac{\sin x}{\ln(1+x)} = \lim\limits_{x\to 0}\frac{x}{x} = 1$.

9. Function Continuity

Let an argument x_0 receive an increment of Δx and take on value $x = x_0 + \Delta x$. Then the function $y(x)$ will increment (fig. 49):

$$\Delta y = f(x) - f(x_0). \tag{43}$$

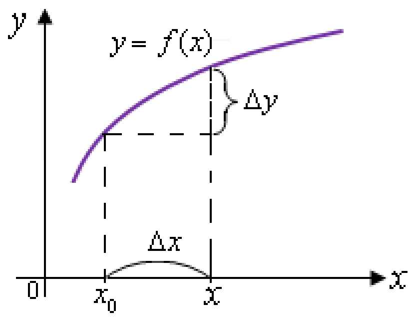


Fig. 49. Illustration of argument increment and function increment

Definition. A function $y = f(x)$ is called continuous at the point x_0 if

$$\lim_{\Delta x \to 0} \Delta y = 0. \qquad (44)$$

That is, a function is continuous at a point x_0 if an infinitesimal increment of the argument at this point corresponds to an infinitesimal increment of the function.
If $\Delta x \to 0$, then $x \to x_0$ and $\Delta y \to 0$.

Then it follows from (44) that

$$\lim_{x \to x_0} f(x) = f(x_0) \quad . \qquad (45)$$

Because $x_0 = \lim_{x \to x_0} x$, then from (45) it follows

$$\lim_{x \to x_0} f(x) = f(\lim_{x \to x_0} x) \quad . \qquad (46)$$

Thus, for a continuous function, you can swap the limit symbol and the characteristics of the function.

It follows from (45) that when finding the limit of a continuous function, substitute x_0 in the function instead of x.

Comment. All elementary functions are continuous in their domains.

Example. $\lim_{x \to 0}(2x + 3\cos x) = 2 \cdot 0 + 3\cos 0 = 3 \cdot 1 = 3$.

Continuous Function Operations

Theorem 1. If the functions $u(x)$ and $v(x)$ are continuous at the point x_0, then their sum $u(x) + v(x)$, product $u(x) \cdot v(x)$, and the quotient $\frac{u(x)}{v(x)}$, $(v(x_0) \neq 0)$ are also continuous at this point.

Theorem 2. If $f(u)$ is a continuous function of u, and $u(x)$ a continuous function of x, then their superposition $f(u(x))$ is also a continuous function of x.

Proof. According to (46) $\lim\limits_{x\to x_0} f(u(x)) = f(\lim\limits_{x\to x_0} u(x)) = f(u(\lim\limits_{x\to x_0} x)) = f(u(x_0))$ thus, the function $f(u(x))$ is continuous at a point x_0 .

Definition. If the function is continuous at each point of a certain domain, then they say that it is continuous in this domain.

The first Bolzano-Cauchy theorem.

Let a function $f(x)$ be defined and continuous on a closed interval $[a,b]$ and at its ends takes values of different signs, then inside the segment $[a,b]$ there exists at least one point $c\in(a,b)$ at which the function $f(x)$ vanishes (fig. 50).

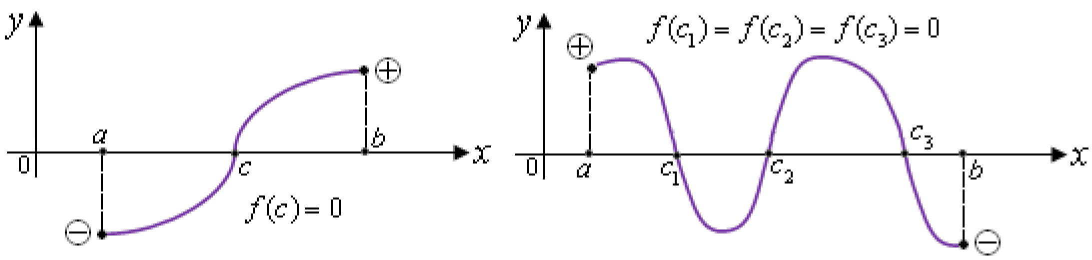


Fig. 50. Geometric interpretation of the first Bolzano-Cauchy theorem

The second Bolzano-Cauchy theorem. The function $f(x)$ being defined and continuous on a closed interval $[a,b]$ and at its ends taking the values of $f(a)=A,\ f(b)=B$ takes all intermediate values between A and B on the segment $[a,b]$ (fig. 51).

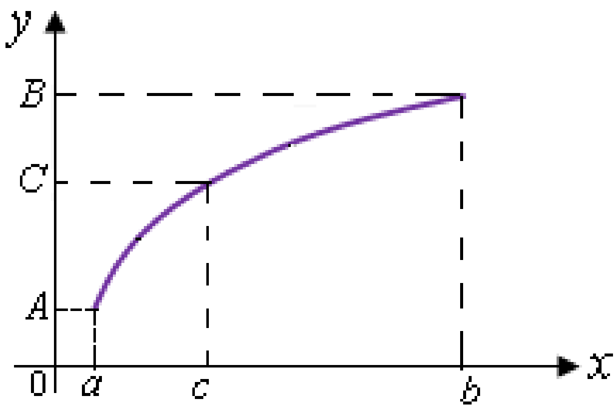


Fig. 51. Geometric interpretation of the second Bolzano-Cauchy theorem

Weierstrass theorem. The function $f(x)$ is continuous on a closed interval $[a,b]$, is bounded on this segment, and reaches its largest and smallest values on it (fig. 52).

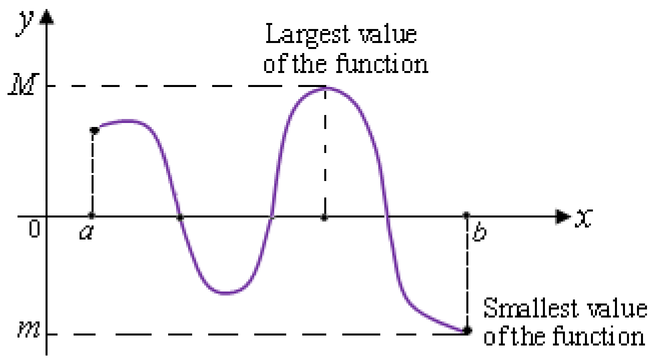


Fig. 52. Geometric interpretation of Weierstrass theorem

10. Points of Gap of Function

For the function $f(x)$ to be continuous at the point a, it is necessary and sufficient that the following three conditions are satisfied:

1.Let a function $f(x)$ is defined at the point a and some of its neighborhood.

2. There are limits of the function $f(x)$ from the left and from the right at the point a, i.e. exist $f(a-0)$ and $f(a+0)$.

3. These limits are equal to each other and equal to the value of the function at the point a, i.e. $f(a-0)=f(a+0)=f(a)$.

There are points of gap of two kinds.

1) If the limits from the left and from the right at point a exist and are finite, but either are not equal to each other, or are not equal to the value of the function at point a, then point a is called a point of gap of the first kind.

– limit from the left

Example. Fid the points of gap function $f(x)=\begin{cases} x^2, & if\ \ x\le 0, \\ x+2, & if\ \ x>0. \end{cases}$

Solution. The limit from the left at point $x=0$ is zero: $f(0-0)=0$, the limit from the right at point $x=0$ is two: $f(0+0)=2$, $f(0-0)=0$. I.e. $f(0-0)\ne f(0+0)$, then $x=0$ is a point of gap of the first kind (fig. 53).

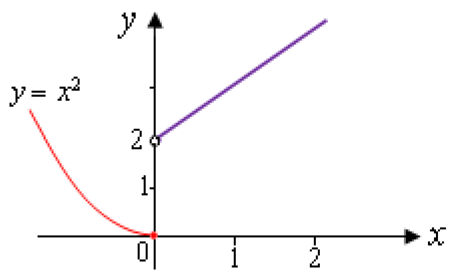


Fig. 53. Graph of a function having a point of gap of the first kind

Example. Fid the points of gap function $f(x)=\begin{cases} x+2, & if \;\; x\neq 1, \\ 1, & if \;\; x=1. \end{cases}$

Solution. The limit from the left at point $x=1$ is three: $f(1-0)=3$, the limit from the right at point $x=3$ is three: $f(1+0)=3$. I.e. $f(1-0)=f(1+0)\neq f(1)$, then $x=1$ is a point of removable gap (fig. 54).

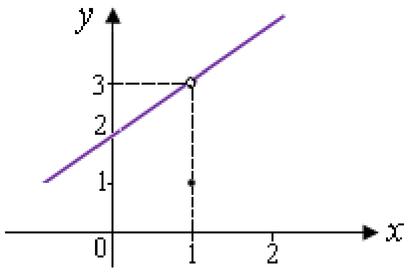


Fig. 54. Graph of a function having a removable gap

Definition. If the limit from the left and the limit from the right at point $x=a$, or at least one of them does not exist or is equal to infinity, then point $x=a$ is called a point of gap of the second kind.

Example. Plot a schematic graph of the function $y=2^{1/x}$.

Solution. The function $y=2^{1/x}$ at point $x=0$ is not defined. Find the limits from the left and from the right at this point: $\lim\limits_{\substack{x\to 0\\ x<0}} 2^{1/x}=2^{\frac{1}{-0}}=2^{-\infty}=\frac{1}{2^{+\infty}}=\frac{1}{\infty}=0$, $\lim\limits_{\substack{x\to 0\\ x>0}} 2^{1/x}=2^{\frac{1}{+0}}=2^{+\infty}=\infty$. I.e. limit from the right at point 0 equals the infinite, then point 0 is point of gap of the second kind. Let us find the horizontal asymptote the graph of function. Because $\lim\limits_{x\to\infty} 2^{1/x}=2^{\frac{1}{\infty}}=2^{0}=1$, then straight $y=1$ is the horizontal asymptote the graph of function (fig. 55).

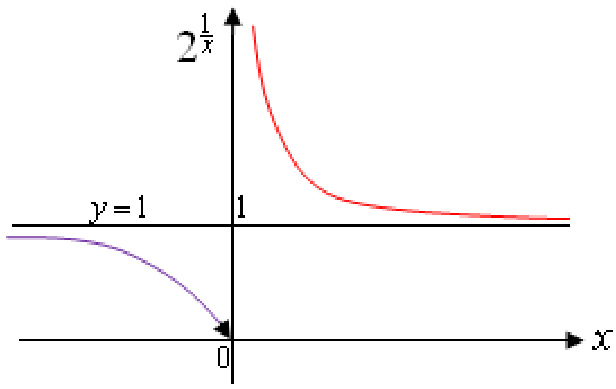


Fig. 55: Schematic graph of a function $y=2^{1/x}$

Derivative of Function of a Single Variable

1. Rate Motion of Material Point

The derivative of a function is the rate at which the function changes relative to the argument.

Let the point move non-uniformly along the axis with variable rate (fig. 56).
The law of motion of the material point is $s = s(t)$, where t is the time, s is the coordinate of the moving point.

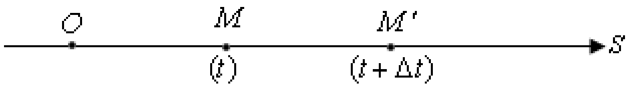


Fig. 56. Uneven movement of the material point along the axis

At time t , the point occupies the position $M: OM = s(t)$;

at moment $t + \Delta t$, the material point will occupy position M': $OM' = s(t + \Delta t)$.

Thus, during time Δt, the material point will pass the path
$MM' = OM' - OM = s(t + \Delta t) - s(t) = \Delta s(t)$.

Ratio $\dfrac{\Delta s(t)}{\Delta t}$ is the average rate move of a material point over a period of time from t to $t + \Delta t$.

$$v(t) = \lim_{\Delta t \to 0} \frac{\Delta s(t)}{\Delta t}. \qquad (47)$$

Here $v(t)$ is instantaneous rate move of a material point at a time t.

Instantaneous[ˌinstən'teiniəs] – мгновенный;

Derivative of function is a limit of the ratio of increment of function Δy to the increment of argument Δx, when the increment of argument Δx tends to zero.

2. Derivatives of Functions

For some values of x, the function increases (decreases) slowly, while for others it quickly.

This suggests that the rate of change of function at different points is different (fig. 57).

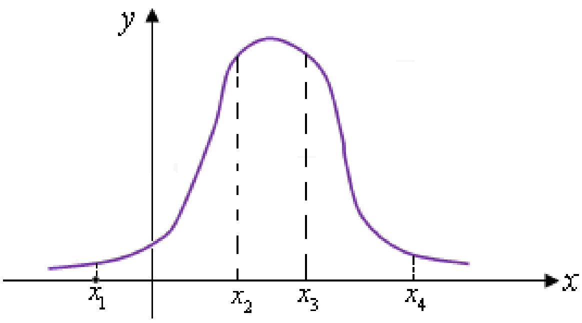


Fig. 57. Different rate of change of function at different points

Let x be a fixed point. The argument x received an increment of Δx.

Then the function will increment Δy.

Ratio a $\dfrac{\Delta y(x)}{\Delta x}$ is the average rate of change of $y(x)$ over the interval $[x, x+\Delta x]$ (fig. 58).

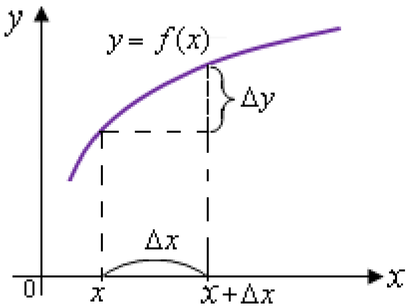


Fig. 58. Illustration of function increment and argument increment,

The smaller Δx, the more accurately ratio $\dfrac{\Delta y(x)}{\Delta x}$ characterizes the rate of change of the function at the point x itself.

If Δx approaches zero, then the average rate of change of $y(x)$ over the interval $[x, x+\Delta x]$ approaches the instantaneous rate of change of $y(x)$ at x.

The instantaneous rate of change of $y(x)$ is called the derivative of $y(x)$ and denoted by the symbol $\dfrac{dy(x)}{dx}$ or $y'(x)$:

$$y'(x) = \frac{dy(x)}{dx} = \lim_{\Delta x \to 0} \frac{y(x+\Delta x) - y(x)}{\Delta x}. \qquad (48)$$

Example. Find the function $y = \sqrt{x}$ increment, corresponding to the argument increment equal to Δx:

Solution. $\Delta y = \sqrt{x+\Delta x} - \sqrt{x}$.

Example. Calculate the derivative of a function $y = \sqrt{x}$ at a point $x_0 = 4$.

Solution. By definition of a derivative

$$y'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} = \lim_{\Delta x \to 0} \frac{x+\Delta x - x}{\Delta x\left(\sqrt{x+\Delta x} + \sqrt{x}\right)} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x\left(\sqrt{x+\Delta x} + \sqrt{x}\right)} =$$

$= \dfrac{1}{2\sqrt{x}}$. Then $y'\big|_{x=4x} = \dfrac{1}{2\sqrt{4}} = \dfrac{1}{4}$.

Example. Using the definition of a derivative, find the derivative of a function: $y = \cos x$.

Solution. Give the argument x an increment $\Delta x \neq 0$ and find the corresponding increment of the function: $\Delta y = \cos(x+\Delta x) - \cos x$, or $\Delta y = -2\sin\frac{\Delta x}{2}\cdot\sin\left(x+\frac{\Delta x}{2}\right)$. From here

$\frac{\Delta y}{\Delta x} = -\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}\cdot\sin\left(x+\frac{\Delta x}{2}\right)$. We pass to the limit at $\Delta x \to 0$:

$y' = \lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = -\lim\limits_{\Delta x\to 0}\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}\cdot\lim\limits_{\Delta x\to 0}\sin\left(x+\frac{\Delta x}{2}\right)$. By virtue of the first remarkable limit

$\lim\limits_{\Delta x\to 0}\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}} = 1$, and from the continuity of the function $\sin x$ it follows: $\lim\limits_{\Delta x\to 0}\sin\left(x+\frac{\Delta x}{2}\right) = \sin x$.
Then $y' = -\sin x$.

Physical Meaning of Derivative

to take a derivative - брать производную

From a physical point of view, the derivative of a function is the rate of change of a function respect to an argument.

From (47), (48) it follows that the rate of uneven motion is the derivative of the path with respect to time:

$$v(t) = s'(t). \tag{49}$$

3. Geometric Meaning of Derivative

Consider a continuous function $y = f(x)$ on an interval (a,b). Let $x -$ an arbitrary fixed point of the interval (a,b) and $\Delta x \neq 0 -$ a small increment of the argument, such that $x+\Delta x \in (a,b)$.

Consider the secant MN graph of the function passing through the points $M(x, y(x))$ and $M(x+\Delta x, y(x+\Delta x))$ (fig. 59).

Definition. If there is a limit position of the secant MM' when a point M' tends along a curve to a point M, then this limit position is called the tangent to the graph of the function $y = f(x)$ at a given fixed point M of this graph.

Let is given function $y = y(x)$. When the variable x received the increment Δx, then the function received the increment Δy.

Draw secant MM' through points $M(x, y(x))$ and $M'(x+\Delta x, y(x+\Delta x))$.

This secant forms the angle β with the positive direction of the axis Ox (fig. 59).

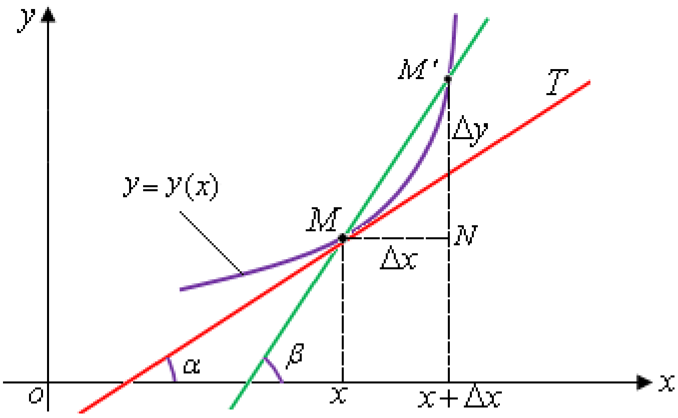


Fig. 59. Geometric meaning of the derivative

Consider the triangle $MM'N$.

Angle $\angle M'MN = \beta$ and tangent of angle β equals $\tan\beta = \dfrac{M'N}{MN} = \dfrac{\Delta y}{\Delta x}$.

If Δx approaches zero point M' along the curve tends to point M.

In this case, the secant MM' rotates and tends to occupy the position of the tangent MT to the graph of the function at point M with an abscissa x.

If Δx approaches zero angle β tends to angle α, where α is the angle between the tangent MT and the positive direction of the Ox axis.

Because angle β tends to angle α as $\Delta x \to 0$ then $\tan\beta$ tends to $\tan\alpha$ as $\Delta x \to 0$ or $\lim\limits_{\Delta x \to 0} \tan\beta = \tan\alpha$.

But $\tan\beta = \dfrac{\Delta y}{\Delta x}$, then $\lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} = \tan\alpha$, therefore

$$y'(x) = \tan\alpha. \tag{48}$$

From a geometric point of view, the derivative $y'(x)$ is the angular coefficient of the tangent drawn to the graph of the function at point M with abscissa x.

From a geometric point of view, the derivative $y'(x)$ can be interpreted as the slope of the tangent line of the graph of the function $y = f(x)$ at the point (x, y).

4 .Tangent and Tormal to a Flat Line

Let is given function $y = y(x)$ and point $M_0(x_0, y_0)$.

We draw the tangent and normal to the graph of the function at point $M_0(x_0, y_0)$ (fig. 60).

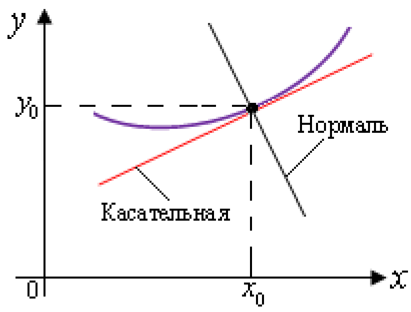


Fig. 60. Tangent and normal to the function graph

We write the equation of a line passing through a given point in a given direction: $y-y_0=k(x-x_0)$, where $k-$ is the angular coefficient of the line, i.e. $k=\tan\alpha$.

Where A $k-$ is the slope of the line, i.e. $k=\tan\alpha$.

Due to the geometric meaning of the derivative $k=y'(x_0)$, then according to equation (7) the tangent equation takes the form

$$y-y_0=y'(x_0)(x-x_0) \tag{49}$$

Example. Compose the equation of the tangent to the parabola $f(x)=\sqrt{x}$ at point $x=1$ (fig. 61).

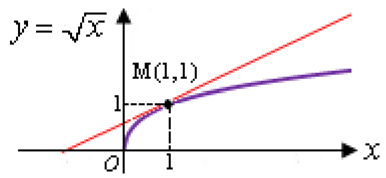


Fig. 61. Tangent to a parabola $f(x)=\sqrt{x}$ at a point $M(1,1)$

Solution. By condition $x_0=1$, then $y_0=\sqrt{1}=1$. We calculate $f'(x)=\dfrac{1}{2\sqrt{x}}$, $y'(1)=\dfrac{1}{2\sqrt{1}}=\dfrac{1}{2}-$ angular coefficient of tangent to parabola. Then accordingly to formula (49), the tangent equation takes the form: $y-1=\dfrac{1}{2}\cdot(x-1)$, or $y=\dfrac{x}{2}+\dfrac{1}{2}$.

Since the normal is perpendicular to the tangent, then its **angular coefficient** is $k_1=-\dfrac{1}{y'(x_0)}$, then the normal equation:

$$y-y_0=-\frac{1}{y'(x_0)}(x-x_0). \tag{50}$$

Example. Compose the equation of the tangent and the normal to the parabola $y=4-x^2$ at point $M_0(-1,3)$.

Solution. By condition $x_0=-1,\ y_0=3$. We calculate $y'=-2x$, $y'(x_0)=y'(-1)=2$. Then accordingly to formula (49), the tangent equation takes the form: $y-3=2(x+1)$, or $2x-y+5=0$. Accordingly to formula (50) the equation of a normal is: $y-3=-\frac{1}{2}(x+1)$, or $x+2y-5=0$ (fig. 62).

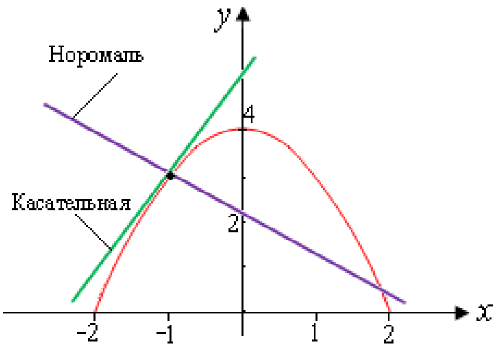


Fig. 62. Tangent and the normal to a parabola $y=4-x^2$ at a point $M_0(-1,3)$

Definition. If the derivative $y'(x)$ exists, then the function $y=y(x)$ is called differentiable.

Theorem. The differentiability of a function implies its continuity.

Proof. We calculate the limit of the increment of the function as $\Delta x\to 0$.

$\lim\limits_{\Delta x\to 0}\Delta y(x)=\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x}\cdot\Delta x=\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x}\cdot\lim\limits_{\Delta x\to 0}\Delta x=y'(x)\cdot 0=0$. It follows that the function $y=y(x)$ is continuous. The converse is not true: the continuity of a function does not imply its differentiability.

Example. The function $y=|x|$ is continuous at the point 0, but does not have a derivative at this point (fig. 63): $\lim\limits_{\substack{\Delta x\to 0\\ \Delta x>0}}\frac{\Delta y}{\Delta x}=\lim\limits_{\substack{\Delta x\to 0\\ \Delta x>0}}\frac{\Delta x}{\Delta x}=1$, $\lim\limits_{\substack{\Delta x\to 0\\ \Delta x<0}}\frac{\Delta y}{\Delta x}=\lim\limits_{\substack{\Delta x\to 0\\ \Delta x<0}}\frac{-\Delta x}{\Delta x}=-1$, then $\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$ does not exist.

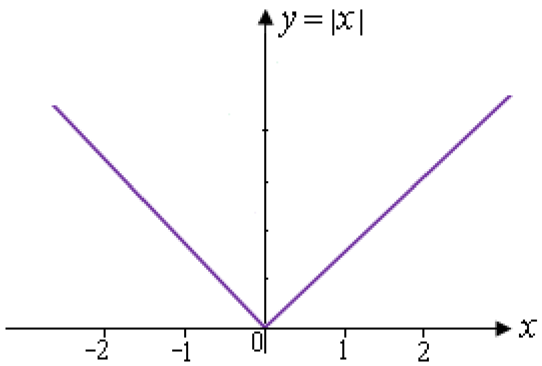


Fig. 63. Graph of function $y=|x|$

Definition. If a function has a derivative at each point of the interval (a,b), then they say that it is differentiable on the interval (a,b).

5. Common Table of Derivateves. Differentiation Rules

1. $C' = 0$;

2. $(x^{\alpha})' = \alpha \cdot x^{\alpha-1}$;

3. $(a^x)' = a^x \cdot \ln a, \quad a > 0, \ a \neq 1$;

4. $(e^x)' = e^x$;

5. $(\log_a x)' = \dfrac{1}{x \cdot \ln a}$;

6. $(\ln x)' = \dfrac{1}{x}$;

7. $(\sin x)' = \cos x$;

8. $(\cos x)' = -\sin x$;

9. $(\tan x)' = \dfrac{1}{\cos^2 x}$;

10. $(\cot x)' = -\dfrac{1}{\sin^2 x}$;

11. $(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}$;

12. $(\arccos x)' = -\dfrac{1}{\sqrt{1-x^2}}$;

13. $(\arctan x)' = \dfrac{1}{1+x^2}$;

14. $(\cot^{-1} x)' = -\dfrac{1}{1+x^2}$.

Let C be a constant and functions $f(x),\ g(x)$ have derivatives at some point x . Then the functions $f(x) \pm g(x),\ c \cdot f(x),\ f(x) \cdot g(x)$ and $\dfrac{f(x)}{g(x)}$, where $g(x) \neq 0$ also have a derivative at this point x, and:

1. If C is a constant, then the derivative of C times a function is C times the derivative of the function:

$$(C \cdot y(x))' = C \cdot y'(x). \tag{51}$$

2.If $f'(x)$ ahd $g'(x)$ exist, then the derivative of a sum is the sum of derivatives, and the derivative of a difference is the difference of the derivatives:

$$(f(x) \pm g(x))' = f'(x) \pm g'(x). \tag{52}$$

3. If $f'(x)$ ahd $g'(x)$ exist, then

$$(f(x)\cdot g(x))' = f'(x)\cdot g(x) + f(x)\cdot g'(x), \tag{53}$$

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{g^2(x)}. \tag{54}$$

Example. Find the derivative of the function $y(x) = 7\sin x - \dfrac{2-5x}{x^3+e^x}$.

Solution. Using the table of derivatives and differentiation rules we get:
$y'(x) = 7\cos x - \dfrac{(-5)\cdot(x^3+e^x) - (2-5x)\cdot(3x^2+e^x)}{(x^3+e^x)^2}$.

6. Derivative of Composite function

A function defined as a superposition of functions is called a composite function. Let us consider a superposition of two functions

$$y = f(u), \tag{55}$$

$$u = \varphi(x). \tag{56}$$

Function (12) is called an intermediate argument.

Theorem. If function (1) is a differentiable function of u, and $u = \varphi(x) -$ a differentiable function of x, then the complex function $f(\varphi(x))$ will be a differentiable function of x, and

$$y'_x = y'_u \cdot u'_x. \tag{57}$$

The first derivative of y with respect to x equals product of the first derivative of y with respect to u and the first derivative of u with respect to x.

Proof. An increment of argument Δx causes an increment of intermediate argument Δu. An increment of the intermediate argument Δu causes an increment of the function Δy. We represent the ratio of increments in the form

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u}\cdot\frac{\Delta u}{\Delta x}. \tag{58}$$

If $\Delta x \to 0$, then $\Delta u \to 0$, because $u = \varphi(x)$ is a differentiable function, and therefore continuous. We pass in (58) to the limit as $\Delta x \to 0$: $\lim\limits_{\Delta x\to 0}\dfrac{\Delta y}{\Delta x} = \lim\limits_{\Delta u\to 0}\dfrac{\Delta y}{\Delta u}\cdot\lim\limits_{\Delta x\to 0}\dfrac{\Delta u}{\Delta x}$. From here $y'_x = y'_u \cdot u'_x$

Example. Find the derivative of the function $y(x) = \sin^3 x$.

Solution. Denote $u = \sin x$, then $y(x) = u^3$. We take $y'_u = (u^3)' = 3u^2$; $u'_x = (\sin x)' = \cos x$. Then, by the formula (57): $y'_x = 3u^2\cdot\cos x = 3\sin^2 x\cdot\cos x$.

Table Derivatives of Complex Functions

Replace the variable x in the table of simple derivatives with the function $u(x)$.

Then each derivative is transformed by the formula (57):

1. $C' = 0$;

2. $(u^{\alpha})' = \alpha \cdot u^{\alpha-1} \cdot u'_x$;

3. $(a^u)' = a^u \cdot \ln a \cdot u'_x, \quad a > 0, \; a \neq 1$;

4. $(e^u)' = e^u \cdot u'_x$;

5. $(\log_a u)' = \dfrac{1}{u \cdot \ln a} \cdot u'_x$;

6. $(\ln u)' = \dfrac{1}{u} \cdot u'_x$;

7. $(\sin u)' = \cos u \cdot u'_x$;

8. $(\cos x)' = -\sin x \cdot u'_x$;

9. $(\tan u)' = \dfrac{1}{\cos^2 u} \cdot u'_x$;

10. $(\cot u)' = -\dfrac{1}{\sin^2 u} \cdot u'_x$;

11. $(\arcsin u)' = \dfrac{1}{\sqrt{1-u^2}} \cdot u'_x$;

12. $(\arccos u)' = -\dfrac{1}{\sqrt{1-u^2}} \cdot u'_x$;

13. $(\arctan u)' = \dfrac{1}{1+u^2} \cdot u'_x$;

14. $(\cot^{-1} u)' = -\dfrac{1}{1+u^2} \cdot u'_x$.

7. Derivative of Function Defined Parametrically

Let the variable A y depend on B x by the parameter C t, i.e. $\begin{cases} y = y(t), \\ x = x(t). \end{cases}$

Then the derivative of the function y with respect to the variable x is determined by the formula:

$$y'_x = \frac{y'_t}{x'_t}. \qquad (59)$$

Example. Find the derivative of the function y with respect to x, if $\begin{cases} y(t) = 4t^2, \\ x(t) = 5t. \end{cases}$

Solution. Find the derivatives of the functions y and x with respect to t:

$y'_t = (4t^2)' = 8t,\ x'_t = (5t)' = 5$. Then, by the formula (59) $y'_x = \frac{y'_t}{x'_t} = \frac{8t}{5}$.

8. Derivative of Function Specified Implicitly

$$\Phi(x, y) = 0. \qquad (60)$$

__The rule__ for finding a derivative of function implicitly defined

1. Find the derivative of the left and right sides of equation (16), not forgetting that y depends on x ($x' = 1,\ y' \neq 1$).

2. We open the brackets, and move the terms containing y' to the left, the rest to the right.

3. On the left side, the derivative of y' is put out of the bracket and then we find y'.

Example. Find the derivative of the function specified implicitly $\sin(x + y) = 2xy + 3$.

Solution. According to the rule, we find the derivative of the left and right sides of the equation:

$\cos(x + y)\cdot(1 + y') = 2(1\cdot y + x\cdot y')$, open the brackets $\cos(x + y) + y'\cdot\cos(x + y) = 2y + 2xy'$, then $y'\cdot\cos(x + y) - 2xy' = 2y - \cos(x + y)$, hence $y'\cdot\left[\cos(x + y) - 2x\right] = 2y - \cos(x + y)$. From here $y' = \frac{2y - \cos(x + y)}{\cos(x + y) - 2x}$.

9. Higher Derivative Functions

A second-order derivative or second derivative is a derivative of the first derivative:
$y'' = (y')',\ y''' = (y'')'$, etc.

Example. Find the derivative a thirdd-order of the function $y = x^3$.

Solution. $y'(x) = 3x^2,\ y''(x) = (3x^2)' = 6x,\ y'''(x) = (6x)' = 6$.

Differential Function of one Variable

1. Differential definition. Geometric Meaning. Properties

Definition. A differential of a function $y(x)$ is a product of a function with respect to an independent variable and an arbitrary increment of an independent variable:

$$dy = y' \cdot \Delta x. \tag{61}$$

Since $dx = x' \cdot \Delta x = \Delta x$, then (1) can be rewritten in the form

$$dy = y' \cdot dx, \tag{62}$$

From here

$$y' = \frac{dy}{dx}. \tag{63}$$

The derivative of a function with respect to an independent variable is equal to the ratio of the differential of the function to the differential of the independent variable.

Example. Find function differential $y(x) = \sin 5x$.

Solution. According to the formula (2) $dy(x) = 5\cos 5x \cdot dx$.

Geometric Meaning of Differential

Let is given the function $y = y(x)$.

Let the argument x receive the increment Δx, then the function will receive the increment Δy.

Draw the tangent MB to the function graph at point $M(x, y)$, forming an angle α with the positive direction of the Ox axis (fig. 64).

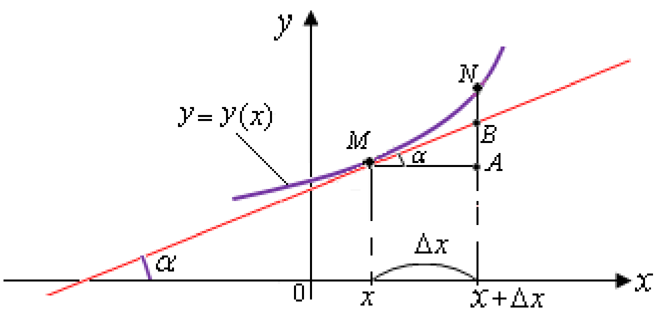


Fig. 64. Geometric meaning of the differential

In a curved triangle, MNA leg $MA = \Delta x$ and $NA = \Delta y$ leg. In a rectangular triangle MBA, the leg BA is equal to $BA = MA \cdot tg\alpha$ either $BA = \Delta x \cdot y'$ or $dy = BA -$ increment of the ordinate of the tangent.

Thus, from the geometric point of view, the differential of a function is the increment of the ordinate of the tangent.

Geometrically differential is equal to the increment of tangent ordinate to the curve $y = f(x)$.

Let $y = f(x)$ y is a function of x. Figure 64 shows that $NA = AB + NB$, or $\Delta y = dy + NB$. If Δx tends to zero, then BN tends to zero, those if Δx is small, then $\Delta y \approx dy$, or

$$y(x+\Delta x) - y(x) \approx y'(x)\cdot\Delta x. \tag{64}$$

Theorem. The differential of the function is the main part of the increment of the function, linear with respect to the increment of the argument and differs from it by an infinitely small higher order than Δx.

Proof. By definition of $y' = \lim\limits_{\Delta x\to 0}\dfrac{\Delta y}{\Delta x}$, then $\dfrac{\Delta y}{\Delta x} = y' + \alpha(x)$, where $\alpha(x)$ tends to zero at Δx tends to zero. From here $\Delta y = y'\cdot\Delta x + \alpha(x)\cdot\Delta x$, or $\Delta y = dy + \alpha(x)\cdot\Delta x$, where $\lim\limits_{\Delta x\to 0}\dfrac{\alpha(x)\cdot\Delta x}{\Delta x} = \lim\limits_{\Delta x\to 0}\alpha(x) = 0$. If Δx is small, then

$$\Delta y \approx y'(x)\cdot\Delta x. \tag{65}$$

Differential Properties

1.If C is a constant, then the differential of C is equal zero.

$$dC = 0. \tag{66}$$

2. If du and dv exist, then the differential of a sum is the sum of differentials, and the differential of a difference is the difference of the differentials:

$$d(u \pm v) = du \pm dv. \tag{67}$$

Corollary. $d(x \pm C) = dx \pm dC = dx$, or

$$dx = d(x \pm C), \tag{68}$$

3. If C is a constant, then the differential of C times a function is C times the differential of the function:

$$d(C\cdot u) = C\cdot du.$$

Corollary. $d(C\cdot x) = C\cdot dx$, or

$$dx = \frac{1}{C}d(C\cdot x). \tag{69}$$

3. If du ahd dv exist, then

$$d(u \cdot v) = udv + vdu, \tag{70}$$

$$d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2} . \tag{71}$$

2. Invariance Form of the first Differential

Let us consider a complex function $y = f(u)$, where $u = \varphi(x)$.

1)Let u be an independent variable, then

$$dy = y'_u \cdot du . \tag{72}$$

2)Let x be an independent variable, then

$$dy = y'_x \cdot dx = y'_u \cdot u'_x \cdot dx = y'_u \cdot du . \tag{73}$$

The right-hand sides of formulas (72) and (73) coincide, but have different meanings: in formula (72) $du = \Delta u$, and in formula (73) $du = u'_x \cdot dx$.

The invariance of the form of the first differential is $dy = y'_u \cdot du$, regardless of which variable is chosen as independent.

It follows from (73) that

$$y'_u = \frac{dy}{du} . \tag{74}$$

Thus, the derivative of a function with respect to any variable is equal to the ratio of the differential of the function to the differential of the variable.

Fundamental Theorems of Differential Calculus

1. Theorems of Roll, Lagrange, Cauchy

The Rolle Theorem (1652-1719) (French mathematician)

Theorem. Let a function $f(x)$ be defined and continuous on a closed interval $[a,b]$ and be differentiable at each point of open interval (a,b). If $f(a) = f(b)$, then there exists at least one point $c \in (a,b)$ such that $f'(c) = 0$ (fig. 65).

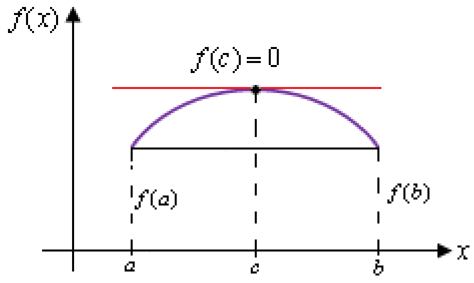


Fig. 65. Illustration of Rolle's theorem

The Mean Value Theorem

Lagrange Theorem (1763-1813) (French mathematician and mechanic)

Theorem. Let a function $f(x)$ be defined and continuous on a closed interval $[a,b]$ and be differentiable at each point of open interval (a,b). Then there exists a point $c \in (a,b)$ (fig. 66) such that

$$f(b)-f(a)=f'(c)(b-a). \tag{75}$$

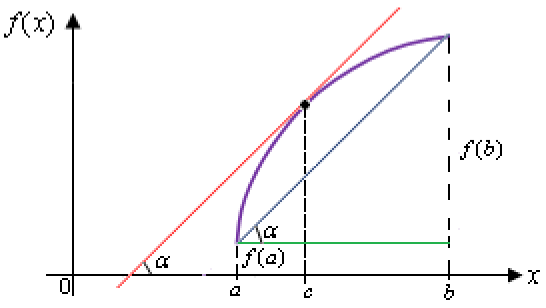


Fig. 66. Illustration to the Lagrange theorem

Geometric Interpretation

The quotient $\dfrac{f(b)-f(a)}{b-a}$ equals the slope of the secant line through the points $(a, f(a))$ and $(b, f(b))$. The derivative $f'(c)$ equals the slope of the tangent passing through the point $(c, f(c))$. Hence, the theorem asserts that the secant line through $(a, f(a))$ and $(b, f(b))$ is parallel to the tangent at some point $(c, f(c))$, where $a<c<b$ (fig. 66).

Proof. Consider the auxiliary function $F(x)=f(x)-f(a)-\dfrac{f(b)-f(a)}{b-a}(x-a)$, which satisfies the conditions of the Rolle Theorem. Indeed, function $F(x)$ is sum of the functions defined and continuous on $[a,b]$ and differentiable on (a,b). Moreover, $F(a)=F(b)=0$. Therefore, by the Rolle Theorem, there exists some point $c\in(a,b)$ such that

$F'(c)=f'(c)-\dfrac{f(b)-f(a)}{b-a}=0$, then $f'(c)=\dfrac{f(b)-f(a)}{b-a}$. Which was to be proved (Q.E.D.).

Corollary 1: The Theorem is a special case of the Mean Value Theorem: If $f(a)=f(b)$, then $f'(c)=0$ for some $c\in(a,b)$.

Corollary 2: If $f'(x)=0$ for all points of some interval (a,b), then $f(x)$ is a constant on (a,b).

Proof. Let x and x_0 be any points on (a,b). Then by the theorem, $f(x)-f(x_0)=f'(c)(x-x_0)$, where c is some point between x_0 and x. But $f'(c)=0$ and hence, $f(x)=f(x_0)$ for any $x\in(a,b)$.

Corollary 3: If functions $f(x)$ and $g(x)$ are such that $f'(x)=g'(x)$ for all $x\in(a,b)$, then $f(x)=g(x)+C$, where C is a constant.

Proof: Noting that $(f(x)-g(x))'=f'(x)-g'(x)=0$, by Corollary 2, we obtain $f(x)-g(x)=C$.

The Cauchy Theorem (1763-1813) (French mathematician)

Theorem. Let a functions $f(x)$ and $g(x)$ be defined and continuous on a closed interval $[a,b]$ and be differentiable at each point of open interval (a,b), and $g'(x)\neq 0$ for $a<x<b$. Then there exists a point $c\in(a,b)$ such that $\dfrac{f(b)-f(a)}{g(b)-g(a)}=\dfrac{f'(c)}{g'(c)}$.

2. The L'Hopital Rule

The L'Hopital Rule for an uncertainty form $\dfrac{0}{0}$.

Let functions $f(x)$ and $g(x)$ be defined and differentiable on (a,b), and $g'(x)\neq 0$ for all $a<x<b$. Assume that

1) $\lim\limits_{x\to a} f(x)=0=\lim\limits_{x\to a} g(x)$, and

2)there exists $\lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$ (finite or not).

Then there exists $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$ and the equality is true:

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}=\lim_{x\to a}\frac{f'(x)}{g'(x)}. \qquad (76)$$

Example 1: The expression $\frac{\ln(1+5x)}{x}$ is the **uncertainty** form $\frac{0}{0}$ as $x \to 0$.

By making use of the L'Hopital Rule we obtain
$\lim\limits_{x\to 0}\frac{\ln(1+5x)}{x} = \lim\limits_{x\to 0}\frac{(\ln(1+5x))'}{x'} = \lim\limits_{x\to 0}\frac{5/(1+5x)}{1} = 5$.

Example 2: $\lim\limits_{x\to 0}\frac{\sin 3x}{\sin 5x} = \cancel{\frac{0}{0}} = \lim\limits_{x\to a}\frac{3\cos 3x}{5\cos 5x} = \frac{3}{5}$.

The L'Hopital Rule for an Uncertainty form $\frac{\infty}{\infty}$.

Let functions $f(x)$ and $g(x)$ be defined and differentiable on (a,b), and let $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to a$.

Assume that there exists there exists $\lim\limits_{x\to a}\frac{f'(x)}{g'(x)}$ (finite or not).

Then there exists also $\lim\limits_{x\to a}\frac{f(x)}{g(x)}$ and the equality is true:

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \cancel{\frac{\infty}{\infty}} = \lim_{x\to a}\frac{f'(x)}{g'(x)}. \qquad (77)$$

Example 3: In order to find $\lim\limits_{x\to\infty}\frac{\ln x}{x}$ one has to expand the **uncertainty** form $\frac{\infty}{\infty}$.

Applying the L'Hopital Rule we obtain: $\lim\limits_{x\to\infty}\frac{\ln x}{x} = \cancel{\frac{\infty}{\infty}} = \lim\limits_{x\to\infty}\frac{(\ln x)'}{x'} = \lim\limits_{x\to\infty}\frac{1/x}{1} = 0$.

Application of Derivative to study of Functions

1. An Increase and decrease of a function. Necessary and sufficient conditions for the monotonicity of the function

Definition. A function $y = f(x)$ is called increasing on the interval (a,b) if for $x_2 > x_1$ the inequality $f(x_2) > f(x_1)$, $x_1, x_2 \in (a,b)$ holds (fig. 67a).

Definition. A function $y = f(x)$ is called decreasing on the interval (a,b) if for $x_2 > x_1$ the inequality $f(x_2) < f(x_1)$, $x_1, x_2 \in (a,b)$ is true (fig. 67b).

Increasing and decreasing functions are called monotonic.

Therefore, the given functions is monotonically increasing on each of the intervals $(-4,-1)$ and $(1,+\infty)$, and monotonically decreasing on $(-\infty,-4)$ and $(-1,1)$.

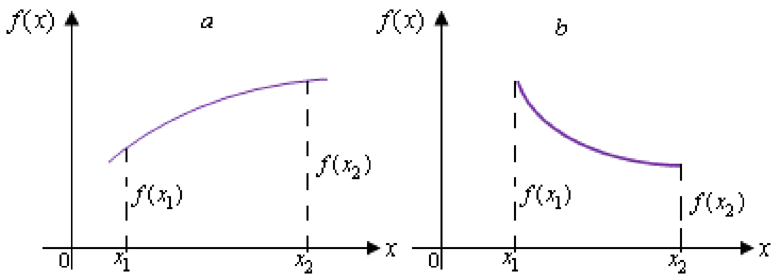


Fig. 67. Function increase and decrease

Theorem 1. Let a function $y = f(x)$ increase on the interval (a,b) and there exists the first derivative $f'(x)$ for any $x \in (a,b)$. Then the first derivative is greater than or equals 0 ($f'(x) \geq 0$) for any $x \in (a,b)$ (fig. 68 *a*, *b*).

$\angle\alpha$ is a acute angle, $\tan\alpha > 0$ $y'(x) = \tan\alpha > 0$

Theorem 2. Let the function $y = f(x)$ decrease on the interval (a,b) and there exists the first derivative $f'(x)$ for any $x \in (a,b)$. Then the first derivative is less than or equals 0 ($f'(x) \leq 0$) for any $x \in (a,b)$ (fig. 68 *c*).

$\angle\alpha$ is obtuse angle $\tan\alpha < 0$ $y'(x) = \tan\alpha < 0$.

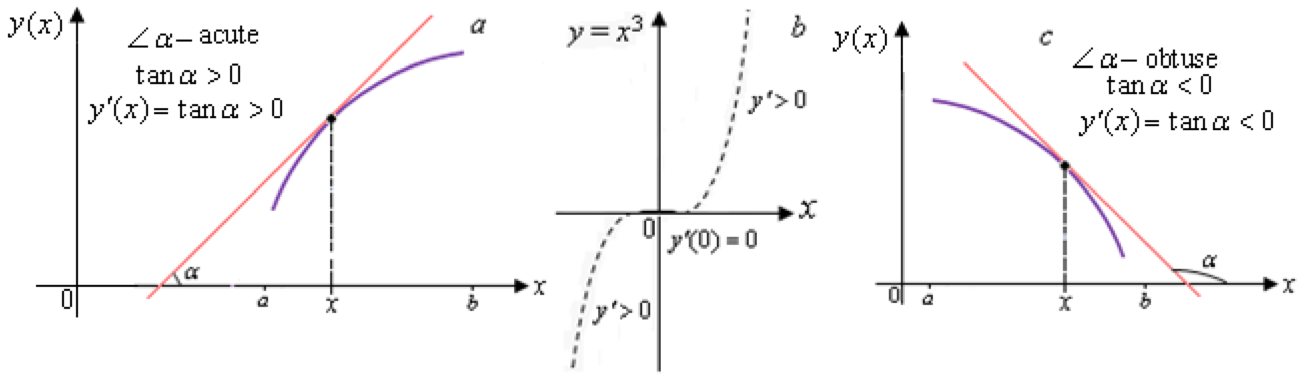


Fig. 68. Necessary condition for the monotonicity of a function

$a \geq b$ a is greater than or equals b;

Theorem. 1) If for any $x \in (a,b)$ the first derivative $f'(x)$ is positive ($f'(x) > 0$), then the function $y = f(x)$ increases on the interval (a,b).

2) If for any $x \in (a,b)$ the first derivative $f'(x)$ is negative ($f'(x) < 0$), then the function $y = f(x)$ decreases on the interval (a,b).

3) If for any $x \in (a,b)$ the first derivative $f'(x)$ is equal 0 ($f'(x) = 0$), then the function $y = f(x)$ equals const on the interval (a,b).

Example. Find the intervals of increasing and decreasing of the function $y = -x^2 + 4x + 1$.

Solution. The domain of the function is the entire numerical axis: $(-\infty, +\infty)$. We calculate the derivative $y' = -2x + 4$; $y' = 0$, $-2x + 4 = 0$, $x = 2$. We find out the sign of the first derivative at intervals $(-\infty, 2)$ and $(2, +\infty)$. Since $y'(0) = 4 > 0$, then the function $y = f(x)$ increases in the interval $(-\infty, 2)$; since $y'(4) = -4 < 0$, then the function $y = f(x)$ decreases on the interval $(2, +\infty)$ (fig. 69).

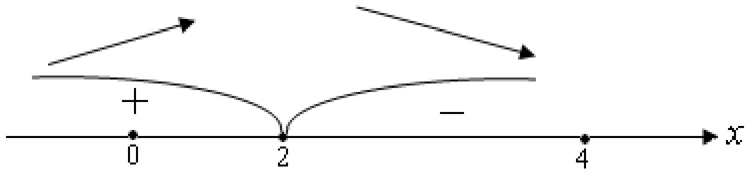


Fig. 69. Intervals of monotony of function $y = -x^2 + 4x + 1$

2. Maxima and Minima of the Function. Necessary and Sufficient Conditions for the Extremum of a Function

Definition. Let the function $y = f(x)$ be continuous at a point a. A point a is called a maximum point of the function $y = f(x)$ if there exists a neighborhood of point a such that for any x of this neighborhood the inequality $f(a) > f(x)$ hold (fig. 70*a*).

Definition. Let the function $y = f(x)$ be continuous at a point a. A point a is called the minimum point of the function $y = f(x)$ if there exists a neighborhood of point a such that for any x of this neighborhood the inequality $f(a) < f(x)$ hold (fig. 70*b*).

The maximum and minimum points are called extreme points of the function.

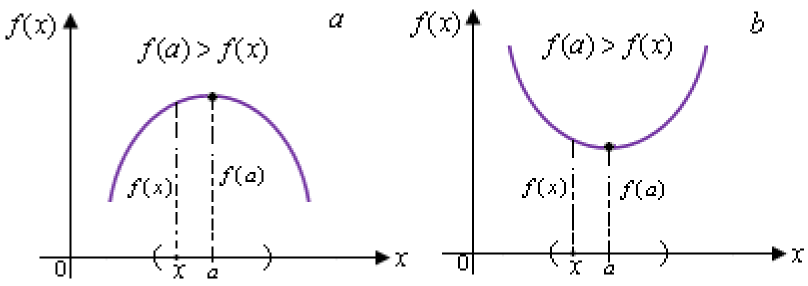


Fig. 70. Maximum and minimum points of the function

Necessary Conditions for the Extremum of a Function

The Fermat Theorem (1601-1665). Let a be the extreme point of the function. Then in this point the first derivative of the function equals zero ($f'(a) = 0$), or infinity ($f'(a) = \infty$), or $f'(a)$ does not exist (fig. 71).

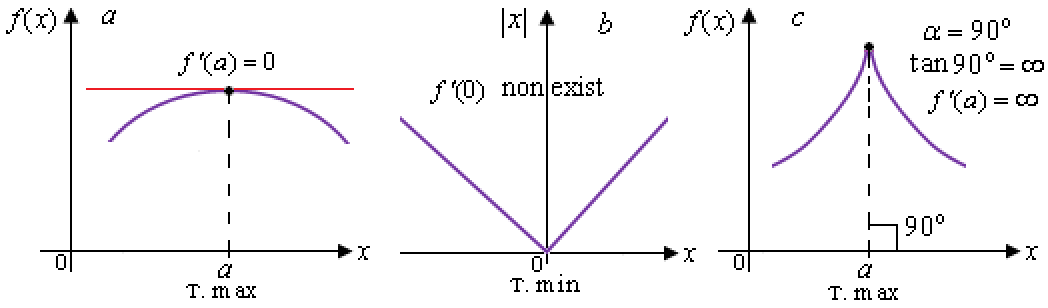


Fig. 71. Illustration of Fermat's theorem

Corollary. A function can have a maximum or minimum only for those values of the argument for which the first derivative is zero, infinity, or does not exist.

The reverse is wrong. If $f'(a)=0$, this does not mean that the point a is the extremum point of the function.

Example. For the function $y=x^3$, the first derivative at point $x=0$ is equal zero, but the point $x=0$ is not the extremum point of the function $y=x^3$ (fig. 72).

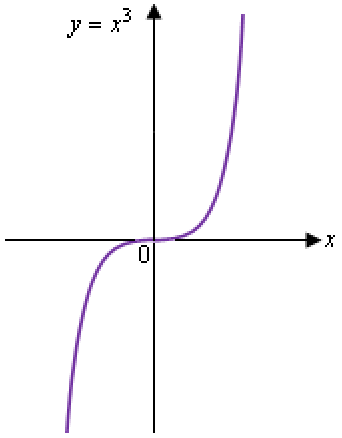


Fig. 72. The function has no extremum at point $x=0$

Definition. The internal points of the domain the function, in which the first derivative is zero, infinity, or does not exist, are called critical points of the first kind or points suspicious of an extremum.

The first sufficient condition for the existence of an extremum of a function

Theorem. Let the function $y=f(x)$ be continuous at point a and point a is a critical point first kind. If, when passing through this point, the first derivative $f'(x)$ changes sign from “+” to “-”, then point a is the maximum point; if from “-” to “+”, then point a is the minimum point. If it does not change sign, then at point a there is no extremum.

Example. Find points of function extremum $y=x^3-3x+5$.

Solution. The domain of the function is the entire numerical axis: $(-\infty, +\infty)$. We calculate the derivative and find the critical points of the first kind $y'(x) = 3x^2 - 3$;
$y'(x) = 0,\ x^2 = 1,\ x_1 = 1,\ x_2 = -1$. We mark points 1 and -1 on the numerical axis and divide the domain into 3 intervals. The derivative is positive for $-\infty < x < -1$ and for $x > 1$. The derivative is negative for $-1 < x < 1$.When passing through point -1, the first derivative changes sign from "+" to "-", so point -1 is a maximum point. When passing through point 1, the first derivative changes sign from "-" to "+", so point 1 is the minimum point (fig. 73): $y_{\max} = y(-1) = 7$, $y_{\min} = y(1) = 3$.

Find the intervals of monotonicity of the function

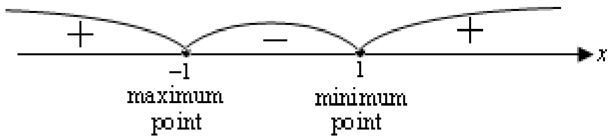


Fig. 73. Maximum and minimum points of the function $y(x) = x^3 - 3x + 5$

The second sufficient condition for the existence of an extremum of a function

If for a differentiable function at some point a its first derivative $f'(x)$ is equal to zero, and the second derivative $f''(x)$ exists and is negative, then the point a is the maximum point of the function $f(x)$.
If for a differentiable function at some point a its first derivative $f'(x)$ is equal to zero, and the second derivative $f''(x)$ exists and is positive, then the point a is the minimum point of the function $f(x)$ (fig. 74).

The rain rule (fig. 74).

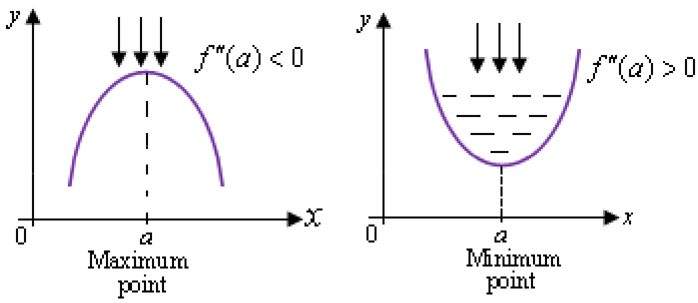


Fig. 74. Illustration of the second sufficient condition for the existence of an extremum of a function

Example. Find all extrema of function $y(x) = \dfrac{2 - x^2 - 2x}{x + 3}$.

Solution. Function domain all numerical axis except point $x = -3$. Let us calculate the derivative and and find the critical points of the first kind: $y' = -\dfrac{(x+2)(x+4)}{(x+3)^2}$, $y' = 0,\ x_1 = -2,\ x_2 = -4$. We find the second derivative $y'' = \dfrac{-2}{(x+3)^3}$. Let us calculate the second derivative at points -2 and -4: $y''(-2) < 0$, $y''(-4) > 0$. Тогда -2 точка максимума и $y_{\max} = y(-2) = 2$; point -4 minimum point and $y_{\min} = y(-4) = 6$.

3. Convexity, concavity of the curve. Inflection points. Necessary and sufficient conditions for the existence of an inflection point

Definition. The curve $y = f(x)$ on the interval (a,b) is concave (fig. 75 a) if it is located above its tangent for any $x \in (a,b)$, i.e. The ordinate of the function is greater than the ordinate of the tangent: $y(x) > Y(x)$.

Definition. The curve $y = f(x)$ on the interval (a,b) is convex (Fig. 75 b) if it is located under its tangent for any $x \in (a,b)$, i.e. ordinate tangent more ordinate functions: $\mathrm{Y}(x) > y(x)$.

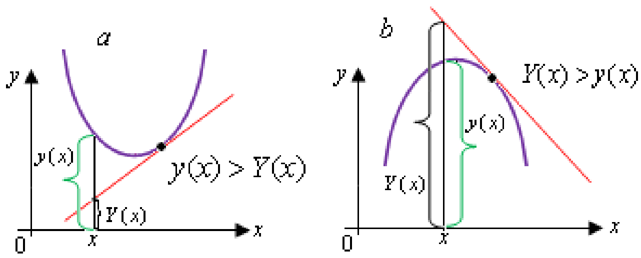


Fig. 75. Convexity and concavity of the curve

Definition. Point M on the curve is called the inflection point of the function $y = f(x)$ if, when passing through this point, the curve changes its direction of curvature (fig. 76).

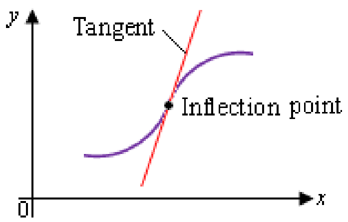


Fig. 76. The inflection point of the curve

A sufficient sign of convexity and concavity of a curve

Theorem. If the second derivative is positive at all points of the interval (a,b), then the curve $y = f(x)$ is convex on the interval (a,b). If the second derivative is negative at all points of the interval (a,b), then the curve is concave on the interval (a,b) (fig. 77).

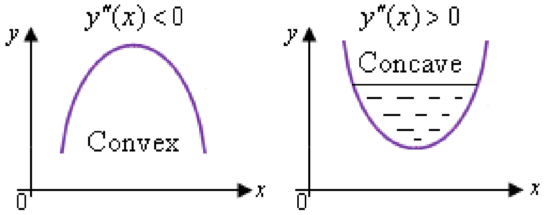


Fig. 77. Convex and concave curves

Definition. The interior points of the domain function $y = f(x)$, in which the second derivative is zero, infinity, or does not exist, are called critical points of the second kind or points suspicious of inflection.

A sufficient sign of the existence of an inflection point

Theorem. Let a function $y = f(x)$ be continuous at a point a and a point a a critical point of the second kind. If the second derivative $f''(x)$ changes sign when passing through this point, then the point $(a, y(a))$ will be an inflection point. If it does not change sign, then at point $(a, y(a))$ there is no inflection.

Example. Find the intervals of convexity, concavity and inflection points of the function $y = x^4 + 2x^3 - 5x + 2$.

Solution. The domain of the function is the entire numerical axis: $(-\infty, +\infty)$. To find critical points of the second kind, we calculate the derivatives of the first and second orders: $y' = 4x^3 + 6x^2 - 24x - 5$, $y'' = 12x^2 + 12x - 24$. Equate the second derivative to zero: $y'' = 0,\ 12x^2 + 12x - 24 = 0,\ x^2 + x - 2 = 0$; thus, $x_1 = 1,\ x_2 = -2$ are critical points of the second kind. We mark these points on the domain of definition. These points will break the domain into three intervals: $(-\infty,-2)\cup(-2,1)\cup(1,+\infty)$. Find out the sign of the second derivative in each interval. In the interval $(-\infty,-2)$, the second derivative is positive, because $y''(-3) = 9 - 3 - 2 = 4 > 0$ .Therefore, in this interval, the function is concave. Similarly, since $y''(0) = -2 < 0$, then in the interval $(-2,1)$ the function is convex. In the interval $(1,+\infty)$, the second derivative is positive, because $y''(4) = 16 + 4 - 2 = 18 > 0$. Therefore, in this interval, the function is concave (fig. 78).

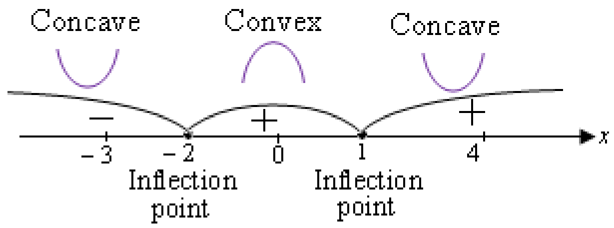


Fig. 78. Intervals of convexity, concavity and inflection points of the function $y = x^4 + 2x^3 - 5x + 2$

4. Asymptotes of the function graph

Definition. An asymptote is a straight line approached by a given curve as one of the variables in the equation of the curve approaches infinity.

Asymptotes can be vertical, horizontal or inclined. If $f(x) \to \infty$ as $x \to a$, then there exists the vertical asymptote, which is described by the equation $x = a$. In this case they say about the asymptotic behavior of the curve as $x = a$. If $f(x) \to b$ as $x \to \infty$, then there exists the horizontal asymptote, whose equation is $y = b$.

The general equation of an inclined asymptote is the following (fig. 79):

$$y = kx + b. \tag{78}$$

Assume that a curve $y = f(x)$ asymptotically approaches line (78) as $x \to \infty$, that is $f(x) \approx kx + b$ as $x \to \infty$.

Therefore, $k \approx \frac{f(x)}{x} + \frac{b}{x}$, and we obtain by the linit process

$$k = \lim_{x \to \infty} \frac{f(x)}{x}. \tag{79}$$

Likewise, $b \approx f(x) - kx$ inplies

$$b = \lim_{x \to \infty} (f(x) - kx). \tag{80}$$

Thus, if there exist finite limits (79) and (80), then the curve $y = f(x)$ has a inclined asymptote $y = kx + b$.

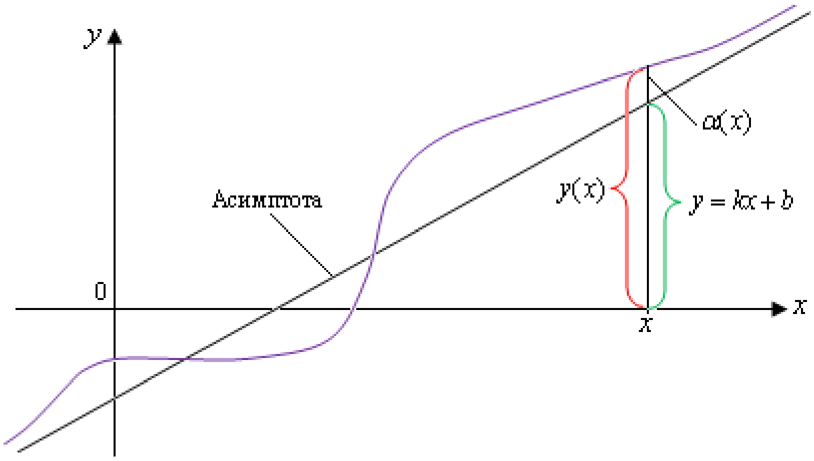


Fig. 79. Inclined asymptote of the function graph

Do not forget that the short form $x \to \infty$ describes two cases: $x \to +\infty$ and $x \to -\infty$.

For instance, if $f(x) = e^x$, then there exists the asymptote $y = 0$ as $x \to -\infty$ but there is no any asymptote as $x \to +\infty$.

Example. Find the asymptotes for the functions $f(x) = \dfrac{x^2+1}{x-2}$.

Solution: The function $f(x) \to \infty$ as $x \to 2$. Therefore, there is the vertical asymptote $x = 2$. One can easily get that $k = \lim\limits_{x\to\infty} \dfrac{f(x)}{x} = \lim\limits_{x\to\infty} \dfrac{x^2+1}{(x-2)x} = 1$,

$b = \lim\limits_{x\to\infty}(f(x) - kx) = \lim\limits_{x\to\infty}\left(\dfrac{x^2+1}{x-2} - x\right) = \lim\limits_{x\to\infty} \dfrac{x^2+1-x^2+2x}{x-2} = 2$. Therefore, there is the inclined asymptote $y = x + 2$.

Investigation of the Function and Plot its Graph

Graph of function of one variable $y = f(x)$ is a set of points on the plane with the coordinates that satisfy this equation.

1. Scheme of function ivestigation

In order to investigate a function $f(x)$ you can follow the scheme below:

1. Find the domain of the function, the point of intersection by the coordinate axes.

2. Determine whether the function has any symmetry (parity, oddness, periodicity of the function).

3. Determine critical points of the first order by solving the equation $f'(x) = 0$ and finding the points in which the derivative $f'(x)$ does not exist. Each critical point must be checked whether it is an extreme point or not.

4. Find the intervals of monotonicity of the function.

5. Determine the points of inflection, that is, find the solution of the equation $f''(x)=0$. It is necessary also to find the points where the second derivative $f''(x)$ does not exist. Each of these points must be checked whether it is a point of inflection or not.

6. Find the intervals where the curve $y=f(x)$ is concave, and where it is convex.

7. Find the asymptotes for the function.

2 .Example

Example. Perform a full study and plot the function $f(x)=-\dfrac{3+10x+3x^2}{4\left(x+1\right)^2}$.

Solution. 1. The domain of the function is the whole numerical axis: $(-\infty,+\infty)$ except for the point $x=-1$.

2.Points of intersection with axis Oy: $x=0$, then $y(0)=-\dfrac{3+10\cdot 0+3\cdot 0^2}{4\left(0+1\right)^2}=-\dfrac{3}{4}$; points of intersection with axis Ox: $y=0$, then $3+10x+3x^2=0,\ x_1=-3,\ x_2=-\dfrac{1}{3}$.

3. Intervals of increase, decrease, points of extremum of function:
$y'=-\dfrac{1}{4}\dfrac{(10+6x)\cdot(x+1)^2-(3+10x+3x^2)\cdot 2(x+1)}{(x+1)^4}$, or $y'=-\dfrac{1}{4}\cdot\dfrac{-4x+4}{(x+1)^3}=\dfrac{x-1}{(x+1)^3}$.

We find critical points of the first kind: $y'=0$ or, $x-1=0$ then the point 1 is the critical point. We mark this point on the domain of definition of the function and divide the domain of definition into intervals $(-\infty,-1)\cup(-1,1)\cup(1,+\infty)$.

Find out the sign of the first derivative in each interval. In the interval $(-\infty,-1)$, the function increases, because in this interval the first derivative is positive ($y'(-2)=3>0$). Because $y'(0)=-1<0$, then in the interval $(-1,1)$ the first derivative is negative and function decreases in this interval. Because $y'(2)>0$, then in the interval $(1,+\infty)$ the first derivative is positive and function increases in this interval (fig. 80). When passing through point 2, the derivative changes sign from minus to plus, so point 2 is the minimum point.

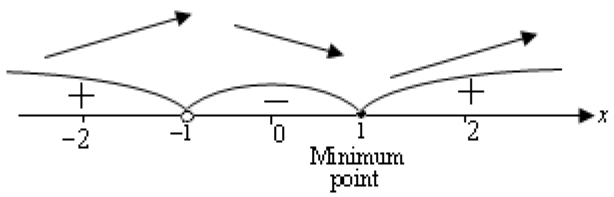


Fig. 80. Intervals of increase, decrease, extremum points of the function $f(x)=-\dfrac{3+10x+3x^2}{4\left(x+1\right)^2}$

При переходе через точку $x=2$ вторая производная меняет знак, поэтому $x=2-$ точка перегиба функции; $y(2)=-\dfrac{35}{36}$. Получаем точку $B\left(2,-\dfrac{35}{36}\right)$ на графике функции.

4. Intervals of convexity, concavity, inflection point.

We find the second-order derivative $y''=\left(\dfrac{x-1}{(x+1)^3}\right)'=\dfrac{1\cdot(x+1)^3-(x-1)\cdot 3(x+1)^2}{(x+1)^6}=\dfrac{4-2x}{(x+1)^4}$.

We find critical points of the second kind: $y''=0$, or $4-2x=0$, then $x=2-$is the critical point of the second-order. We mark this point on the domain of the function and divide the domain of the definition of the function into intervals $(-\infty,-1)\cup(-1,2)\cup(2,+\infty)$.

Find out the sign of the second derivative in each interval. To do this, we calculate the sign of the second derivative at some point in each interval: $y''(-2)>0,\ y''(0)>0,\ y''(3)<0$. Then in the intervals $(-\infty,-1)$ and $(-1,2)$ the second derivative is also positive, and in the interval $(2,+\infty)$ it is negative. From here, the curve is concave in the first and second intervals, and convex in the third interval $(2,+\infty)$ (fig. 81).

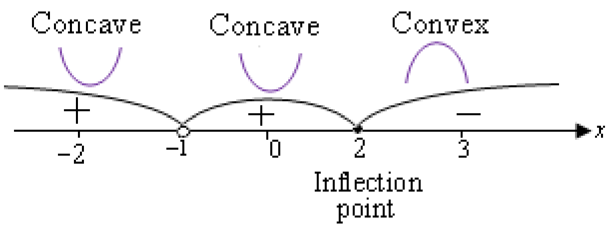


Fig. 81. Intervals of convexity, concavity and inflection points of the function $y=-\dfrac{3+10x+3x^2}{4(x+1)^2}$

When passing through a point 2, the second derivative changes sign, therefore $x=2$ is the inflection point of the function; $y(2)=-\dfrac{35}{36}$. We get the point $B\left(2,-\dfrac{35}{36}\right)$ on the function graph.

5. Asymptotes of the function graph.

Make sure that $x=-1$ is the equation of the vertical asymptote of the graph of the function. Let us calculate the limit left and right at the point -1: $\lim\limits_{\substack{x\to-1\\x<-1}} y(x)=-\lim\limits_{\substack{x\to-1\\x<-1}}\dfrac{3+10x+3x^2}{4(x+1)^2}=-\dfrac{3-10+3}{+0}=\dfrac{4}{+0}=+\infty$ and

$\lim\limits_{\substack{x\to-1\\x>-1}} y(x)=-\lim\limits_{\substack{x\to-1\\x>-1}}\dfrac{3+10x+3x^2}{4(x+1)^2}=-\dfrac{3-10+3}{+0}=\dfrac{4}{+0}=+\infty$. Hence $x=-1$ is the equation of the vertical asymptote of the graph of the function. We find the inclined asymptote, where k and b are found by formulas (79) and (80). We get: $k=\lim\limits_{x\to\infty}\dfrac{y}{x}=-\lim\limits_{x\to\infty}\dfrac{3+10x+3x^2}{4x(x+1)^2}=\cancel{\dfrac{\infty}{\infty}}=-\lim\limits_{x\to\infty}\dfrac{3x^2}{4x^3}=-\dfrac{3}{4}\lim\limits_{x\to\infty}\dfrac{1}{x}=0$;

$b = \lim_{x\to\infty}(y(x)-kx) = \lim_{x\to\infty}\left(-\frac{3+10x+3x^2}{4(x+1)^2}-0\right) = -\lim_{x\to\infty}\frac{3x^2}{4x^2} = -\frac{3}{4}$. Therefore, there is the horizontal asymptote $y=-\frac{3}{4}$. We calculate the values of the function at some points: $y(-4)=-\frac{11}{36}$, $y(-2)=\frac{5}{4}$, $y(-5)=-\frac{28}{64}$. We begin the construction of the graph by constructing the asymptotes (fig. 82). Then we mark the points of intersection with the coordinate axes, the extremum point A (-1.1) and the inflection point $B\left(2,\frac{-35}{36}\right)$, etc.

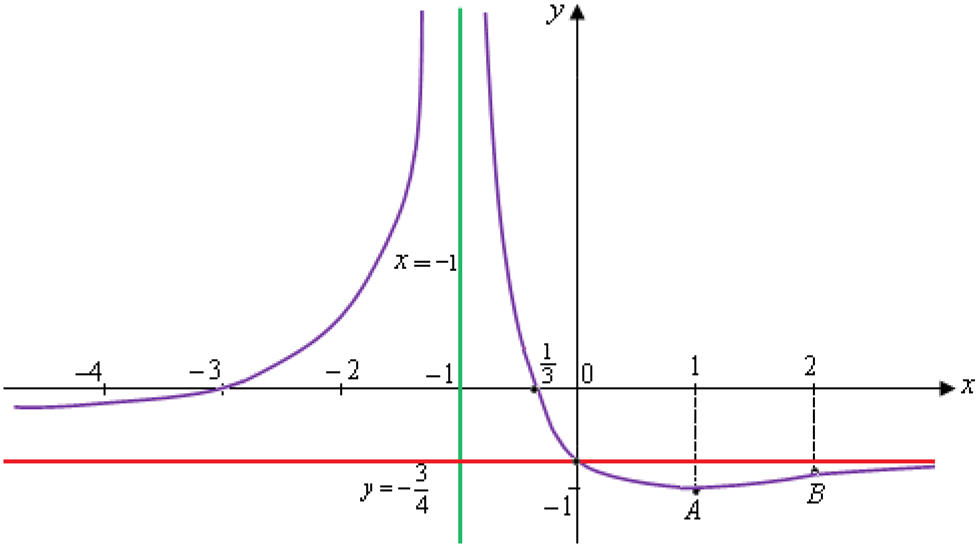


Fig. 82. Graph of function $y=-\frac{3+10x+3x^2}{4(x+1)^2}$

3. List of "non-English words and phrases"

ab initio – first сначала;
ab ovo – from the very beginning с самого начала;
addendum – for adding; для добавления;
ad hoc – к этому случаю; for this occasion;
ad infinitum – до бесконечности; to infinity;
a posteriori – на основании опыта; based on experience;
a priori – заранее, независимо от опыта; in advance, regardless of experience;
caeteris paribus – при прочих равных условиях; ceteris paribus;
corrigenda – список ошибок; list of errors;
erratum –опечатка; typo;
in loc (in loco) – на своем листе; on your sheet;
in parvo – в незначительной мере; to a small extent;
in re – относительно, по ворпосу; relatively, by question;
in situ – на месте; in place;
in toto – в целом; generally;
ipso facto – в силу очевидности; by virtue of evidence;
loc. cit. – в указанном сочинении: in the above essay:
modus operandi – способ действия; mode of action;
mutatis mutandis – сделав сооcтветствующие изменения; by making appropriate changes;
par example – например; eg;
par excellence – по преимуществу; for the most part;
per se – по-существу, сам по себе; essentially by itself;
pro et con – за и против; pros and cons;
pro forma – формально; formally;
pro rata – пропорционально; proportionally;
quantum libet – сколько угодно; as much as you like;
quod erat demonstrandum (q.e.d.) – что и требовалось доказать; which was to be proved
quod vide – смотри (там-то); look (there);
Sic! (Так!) указывает на важность данного места; indicates the importance of the place;
sui generis – своего рода, своеобразный; kind, peculiar;
ut sup (ut supra) – как указано выше; as mentioned above;
vice versa – наоборот; vice versa;
vide infra – ниже; below;
vide sura – выше; higher;

Appendix. Trigonometry

Determination of trigonometric functions of the acute angle of a right triangle.

Consider the right triangle ABC (fig. 1).

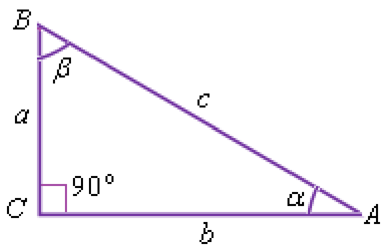


Fig. 1. The Right triangle

AB = c - hypotenuse (lying against an angle of 90°), AC = b - leg, BC = a - leg.

The acute angle is $\angle BAC = \alpha$, $\angle ABC = \beta$. Since the sum of the internal angles of the triangle is equal 180°, then $\angle BAC + \angle ABC = 90^\circ$ or $\alpha + \beta = 90^\circ$. From here we have $\beta = 90^\circ - \alpha$.

Sine of an acute angle in right triangle is a ratio of the length of cathetus opposite to this angle for the length of hypotenuse:

$$\sin\alpha = \frac{a}{c}, \tag{1}$$

$$\sin\beta = \frac{b}{c}. \tag{2}$$

Cosine of an acute angle in right triangle is a ratio of the cathetus adjacent to this angle to the length of hypotenuse:

$$\cos\alpha = \frac{b}{c}, \tag{3}$$

$$\cos\beta = \frac{a}{c}. \tag{4}$$

From formulas (1), (4) we obtain that $\sin\alpha = \cos\beta$ or $\sin\alpha = \cos(90^\circ - \alpha)$. From formulas (2), (3) we obtain that $\sin\beta = \cos\alpha$ or $\sin\beta = \cos(90^\circ - \beta)$.

Tangent of an acute angle in a right triangle is a ratio of opposite cathetus to the given angle to the length of adjacent cathetus:

$$tg\alpha = \frac{a}{b}, \qquad (5)$$

$$tg\beta = \frac{b}{a}. \qquad (6)$$

Cotangent of an acute angle in right triangle is a ratio of the cathetus adjacent to this corner to the length of the opposite cathetus:

$$ctg\alpha = \frac{b}{a}, \qquad (7)$$

$$ctg\beta = \frac{a}{b}. \qquad (8)$$

From (5), (8) follows $tg\alpha = ctg\beta$ $tg\alpha = ctg(90^\circ - \alpha)$ or; from (6), (7) it follows $tg\beta = ctg\alpha$ or $ctg\alpha = tg(90^\circ - \alpha)$; from (5), (7) it follows $tg\alpha \cdot ctg\alpha = \frac{a}{b} \cdot \frac{b}{a} = 1$ or

$$ctg\alpha = \frac{1}{tg\alpha}. \qquad (9)$$

Expression of a cathetus through a hypotenuse and hypotenuse through a cathetus

By definition $\sin\alpha = \frac{a}{c}$, then

$$a = c\sin\alpha, \qquad (10)$$

i.e., the leg of a right triangle is equal to the hypotenuse times the sine of the angle opposite the leg.

By definition $\cos\alpha = \frac{b}{c}$, then

$$b = c \cdot \cos\alpha, \qquad (11)$$

i.e., the leg of a right triangle is equal to the hypotenuse times the cosine of the angle adjacent to the leg.

From (10) it follows that the hypotenuse is equal $c = \frac{a}{\sin\alpha}$, from (11) we get: $c = \frac{b}{\cos\alpha}$.

In other words, the hypotenuse of a right-angled triangle is equal to the leg that is divided by the sine of the angle opposite to that leg or the cosine of the angle adjacent to it.

Expression of one leg of a right triangle through another leg

It follows from (5) $a = b \cdot tg\alpha$, i.e. the leg of a right triangle is equal to another leg, times the tangent of the angle opposite the leg.

Table 1

Table of natural values of acute angle trigonometric functions

градусы α	$sin\alpha$	$tg\alpha$	$ctg\alpha$	$cos\alpha$	
0	0,0000	0,0000	∞	1,0000	90
1	0,0175	0,0175	57,2900	0,9998	89
2	0,0349	0,0349	28,6363	0,9994	88
3	0,0523	0,0524	19,0811	0,9986	87
4	0,0698	0,0699	14,3007	0,9976	86
5	0,0872	0,0875	11,4301	0,9962	85
6	0,1045	0,1051	9,5144	0,9945	84
7	0,1219	0,1228	8,1443	0,9925	83
8	0,1392	0,1405	7,1154	0,9903	82
9	0,1564	0,1584	6,3138	0,9877	81
10	0,1736	0,1763	5,6713	0,9848	80
11	0,1908	0,1944	5,1446	0,9816	79
12	0,2079	0,2126	4,7046	0,9781	78
13	0,2250	0,2309	4,3315	0,9744	77
14	0,2419	0,2493	4,0108	0,9703	76
15	0,2588	0,2679	3,7321	0,9659	75
16	0,2756	0,2867	3,4874	0,9613	74
17	0,2924	0,3057	3,2709	0,9563	73
18	0,3090	0,3249	3,0777	0,9511	72
19	0,3256	0,3443	2,9042	0,9455	71
20	0,3420	0,3640	2,7475	0,9397	70
21	0,3584	0,3839	2,6051	0,9336	69
22	0,3746	0,4040	2,4751	0,9272	68
23	0,3907	0,4245	2,3559	0,9205	67
24	0,4067	0,4452	2,2460	0,9135	66

Continuation of table 1

градусы α	$sin\alpha$	$tg\alpha$	$ctg\alpha$	$cos\alpha$	
25	0,4226	0,4663	2,1445	0,9063	65
26	0,4384	0,4877	2,0503	0,8988	64
27	0,4540	0,5095	1,9626	0,8910	63
28	0,4695	0,5317	1,8807	0,8829	62
29	0,4848	0,5543	1,8040	0,8746	61
30	0,5000	0,5774	1,7321	0,8660	60
31	0,5150	0,6009	1,6643	0,8572	59
32	0,5299	0,6249	1,6003	0,8480	58
33	0,5446	0,6494	1,5399	0,8387	57
34	0,5592	0,6745	1,4826	0,8290	56
35	0,5736	0,7002	1,4281	0,8192	55
36	0,5878	0,7265	1,3764	0,8090	54
37	0,6018	0,7536	1,3270	0,7986	53
38	0,6157	0,7813	1,2799	0,7880	52
39	0,6293	0,8098	1,2349	0,7771	51
40	0,6428	0,8391	1,1918	0,7660	50
41	0,6561	0,8693	1,1504	0,7547	49
42	0,6691	0,9004	1,1106	0,7431	48
43	0,6820	0,9325	1,0725	0,7314	47
44	0,6947	0,9657	1,0355	0,7193	46
45	0,7071	1,0000	1,0000	0,7071	
	$cos\alpha$	$ctg\alpha$	$tg\alpha$	$sin\alpha$	градусы α

Table 2

Values of trigonometric functions at various angles α

α	радианы	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π	$3\pi/2$
	градусы	0°	30°	45°	60°	90°	120°	150°	180°	270°
$sin\alpha$		0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	-1
$cos\alpha$		1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	0
$tg\alpha$		0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$+\infty$	$-\sqrt{3}$	$-\frac{1}{\sqrt{3}}$	0	$-\infty$
$ctg\alpha$		$+\infty$	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	$-\sqrt{3}$	$-\infty$	0

Signs of trigonometric functions

Consider the unit trigonometric circle (fig. 2). The positive angle is counted from the Ox axis counterclockwise. The negative angle is counted from the Ox axis clockwise. In the first quarter (I), the angle α changes within $0 \le \alpha \le \frac{\pi}{2}$, in the second quarter (II) $\frac{\pi}{2} \le \alpha \le \pi$, in the third (III) $\pi \le \alpha \le \frac{3\pi}{2}$, in the fourth (IV) $\frac{3\pi}{2} \le \alpha \le 2\pi$.

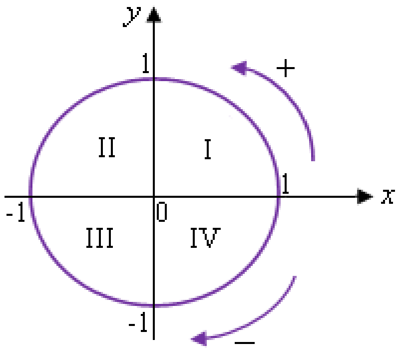


Fig. 2. Unit trigonometric circle

Signs of trigonometric functions in different quarters

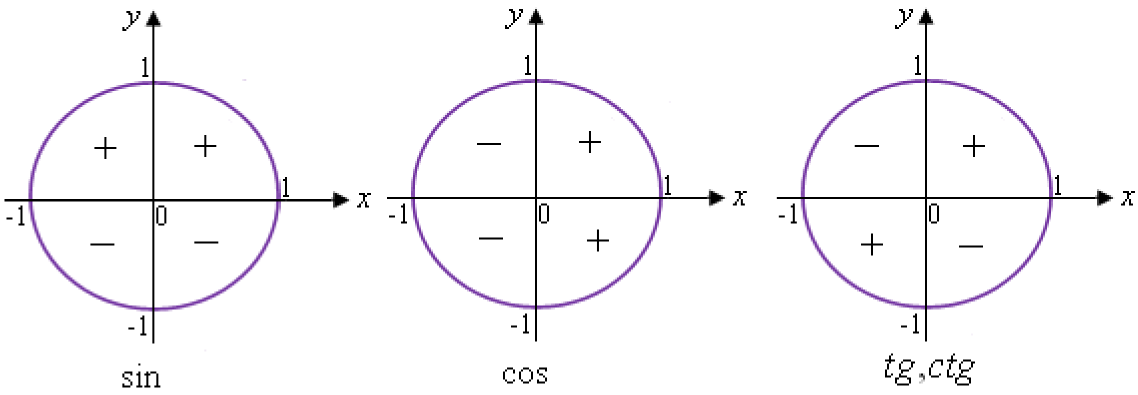


Fig. 3. Signs of trigonometric functions in different quarters

The period of trigonometric functions.

Functions $\sin x, \cos x$ have a period 2π : $\sin(x \pm 2\pi) = \sin x$, $\cos(x \pm 2\pi) = \cos x$ .Functions $tgx, ctgx$ have a period π : $tg(x \pm \pi) = tgx$, $ctg(x \pm \pi) = ctgx$.

Reduction formulas

The reduction formulas are such formulas (Table 3) by which the trigonometric functions of an arbitrary angle can be reduced to the functions of an acute angle.

Table 3

Reduction formulas

β	$\frac{\pi}{2} \pm \alpha$	$\pi \pm \alpha$	$\frac{3\pi}{2} \pm \alpha$	$2\pi - \alpha$
$\sin\beta$	$\cos\alpha$	$\mp\sin\alpha$	$-\cos\alpha$	$-\sin\alpha$
$\cos\beta$	$\mp\sin\alpha$	$-\cos\alpha$	$\pm\sin\alpha$	$\cos\alpha$
$tg\beta$	$\mp ctg\alpha$	$\pm tg\alpha$	$\mp ctg\alpha$	$-tg\alpha$
$ctg\beta$	$\mp tg\alpha$	$\pm ctg\alpha$	$\mp tg\alpha$	$-ctg\alpha$

Here the angle α varies within $0 \le \alpha \le \frac{\pi}{2}$.

Trigonometric functions of double, triple and half angles

$$\sin 2\alpha = 2\sin\alpha\cos\alpha, \qquad \cos 2\alpha = \cos^2\alpha - \sin^2\alpha,$$

$$tg\,2\alpha = \frac{2tg\,\alpha}{1 - tg^2\alpha}, \qquad ctg\,2\alpha = \frac{ctg^2\alpha - 1}{2ctg\,\alpha},$$

$$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha, \qquad \cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha,$$

$$tg\,3\alpha = \frac{3tg\,\alpha - tg^3\alpha}{1 - 3tg^2\alpha}, \qquad ctg\,3\alpha = \frac{ctg^3\alpha - 3ctg\,\alpha}{3ctg^2\alpha - 1},$$

$$\sin^2\frac{\alpha}{2} = \frac{1 - \cos\alpha}{2}, \qquad \cos^2\frac{\alpha}{2} = \frac{1 + \cos\alpha}{2},$$

$$tg\,\frac{\alpha}{2} = \frac{\sin\alpha}{1 + \cos\alpha} = \frac{1 - \cos\alpha}{\sin\alpha}, \qquad ctg\,\frac{\alpha}{2} = \frac{\sin\alpha}{1 - \cos\alpha} = \frac{1 + \cos\alpha}{\sin\alpha}.$$