问题

一、验证：$k_a \cdot d_a = k_b \cdot d_b$时，第二个gap消失（同时，第一个gap达到最大），消失的gap只是一个特殊情况，对理解拓扑能带特性极其重要；

$k_a \cdot d_a = k_b \cdot d_b$也即$n_b \cdot d_b = n_b \cdot d_b$，即两种介质中光程相等，所以$cos(Ka)$的表达式可以化为下式：

$$cos(k_1h_1)cos(k_2h_2) - \frac{1}{2}\left(\frac{k_1}{k_2} + \frac{k_2}{k_1}\right)sin(k_1h_1)sin(k_2h_2)$$

$$= cos(\frac{2\pi n_1 h_1}{\lambda})cos(\frac{2\pi n_2 h_2}{\lambda}) - \frac{1}{2}\left(\frac{n_1}{n_2} + \frac{n_2}{n_1}\right)sin(\frac{2\pi n_1 h_1}{\lambda})sin(\frac{2\pi n_2 h_2}{\lambda})$$

出现禁带的**必要条件**是上式对λ的导数出现零点，令
$A = 2\pi n_1 h_1, B = 2\pi n_2 h_2, C = \frac{1}{2}\left(\frac{n_1}{n_2} + \frac{n_2}{n_1}\right)$

对上式求导:

$$\frac{A}{\lambda^2}sin(\frac{A}{\lambda})cos(\frac{B}{\lambda}) + \frac{B}{\lambda^2}sin(\frac{B}{\lambda})cos(\frac{A}{\lambda}) + \frac{AC}{\lambda^2}cos(\frac{A}{\lambda})sin(\frac{B}{\lambda}) + \frac{BC}{\lambda^2}sin(\frac{A}{\lambda})cos(\frac{B}{\lambda})$$

设零点处$\lambda = \lambda_0$，出现零点的充要条件为$n_1 h_1 = \frac{\lambda_0}{4}m \;\&\; n_2 h_2 = \frac{\lambda_0}{4}m$，$m$为正整数。

所以出现禁带的必要条件是：$n_1h_1 + n_2h_2 = m\lambda_0/2$，只是必要条件是因为根据之前对单层介质的推导，当$n_1h_1$和$n_2h_2$是$\lambda/2$的整数倍时，透射率始终为1。如下分别考虑$n_1h_1 = n_2h_2, n_1h_1 = 2n_2h_2, n_1h_1 = 3n_2h_2$禁带的情形:

1. 取$n_1h_1 = n_2h_2 = \lambda_0/4$，此时$n_1h_1 + n_2h_2 = \lambda_0/2 = m\lambda_0/2$，因此$m = 1, 2, 3....$，又因为当$m = 2, 4, 6\ldots$时透射率始终为1，所以出现禁带的位置为：$\lambda_0, \lambda_0/3, \lambda_0/5...$
2. 取$n_1h_1 = \lambda_0/3, n_2h_2 = \lambda_0/6$，此时禁带出现的位置是：$\lambda_0, \lambda_0/2, \lambda_0/4, \lambda_0/5, \lambda_0/7, \lambda_0/8...$
3. 取$n_1h_1 = 3\lambda_0/8, n_2h_2 = \lambda_0/8$，此时禁带出现的位置是：$\lambda_0, \lambda_0/2, \lambda_0/3, \lambda_0/5, \lambda_0/6, \lambda_0/7...$

所以$n_1h_1 = n_2h_2$时第二个gap消失是因为此时光子晶体相当于全透膜

二、当$\epsilon_1 = \epsilon_2 + \delta$，$\delta$为小量时，$cosKa$的等式可以进行线性简化，能否推导。

化简：$cos(Ka) = cos(k_1h_1)cos(k_2h_2) - \frac{1}{2}\left(\frac{k_1}{k_2} + \frac{k_2}{k_1}\right)sin(k_1h_1)sin(k_2h_2)$

令δ为二阶小量，所以：

$$\frac{1}{2}\left(\frac{k_1}{k_2} + \frac{k_2}{k_1}\right) = \frac{1}{2}\left(\frac{\sqrt{\epsilon}_1}{\sqrt{\epsilon}_2} + \frac{\sqrt{\epsilon}_2}{\sqrt{\epsilon}_1}\right) = \frac{1}{2}\left(\sqrt{\frac{(2\epsilon_2 + \delta)^2}{\epsilon_2(\epsilon_2 + \delta)}}\right) \approx \frac{1}{2}\left(\sqrt{\frac{4(\epsilon_2^2 + \delta)}{\epsilon_2^2 + \delta}}\right) = 1$$

所以：

$$cos(Ka) \approx cos(k_1h_1)cos(k_2h_2) - sin(k_1h_1)sin(k_2h_2) = cos(k_1h_1 + k_2h_2)$$

$$Ka \approx k_1h_1 + k_2h_2$$