MAT 335 Notes

Max Chien

Fall 2024

Contents

Introduction

This document contains notes taken for the class MAT 335: Complex Analysis at Princeton University, taken in the Fall 2024 semester. These notes are primarily based on lectures by Professor Assaf Naor. Other references used in these notes include *Complex Analysis* by Elias Stein and Rami Shakarchi, *Complex Analysis* by Lars Ahlfors, *Visual Complex Analysis* by Tristan Needham, and *Real and Complex Analysis* by Walter Rudin. Since these notes were primarily taken live, they may contains typos or errors.

Chapter 1

Preliminaries

1.1 The Complex Number System

The set of complex numbers, denoted $\mathbb{C}$ is identified with ordered pairs $(x, y) \in \mathbb{R}^2$. We may alternately write this as $x + iy$, where the symbol i is currently undefined.

For a given complex number $z = x + iy$, $x = \text{Re}(z)$ is called the **real part** of z, $y = \text{Im}(z)$ is called the **imaginary part**, $|z| = \sqrt{x^2 + y^2}$ is the **modulus** of z, and the **argument** of z, $\theta = \arg(z)$, is the angle between (x, y) and the x-axis, defined up to integer multiples of 2π.

Definition 1.1

Let $\theta \in \mathbb{R}$. We define

$$e^{i\theta} = \cos\theta + i\sin\theta = (\cos\theta, \sin\theta)$$

One can observe using the identity $\cos^2 + \sin^2 = 1$ that $e^{i\theta}$ lies on the unit circle. Moreover, if $r = |z|$, then elementary geometry shows that we have $z = re^{i\theta}$ using the definition above.

Proposition 1.1

For any $z \in \mathbb{C}$, $|\text{Re}(z)| \leq |z|$ and $|\text{Im}(z)| \leq |z|$.

Proof. $|z| = \sqrt{\text{Re}(z)^2 + \text{Im}(z)^2}$. $\square$

One of the distinguishing features of $\mathbb{C}$ from the real plane $\mathbb{R}^2$ is the algebraic structure present on $\mathbb{C}$.

Definition 1.2

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then we define addition and multiplication on $\mathbb{C}$ by

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$
$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)$$

Taking $i = (0, 1)$, then we observe that $i^2 = -1 + 0i = -1$. Thus we recover the basic identity $i^2 = -1$. We also observe that Re and Im are both linear operators.

Proposition 1.2

Addition and multiplication over $\mathbb{C}$ are commutative and associative. Moreover, multiplication distributes over addition.

Proof. Commutative and associativity of addition is inherited from $\mathbb{R}$. $\square$

Using the definition of $e^{i\theta}$, we can reinterpret complex multiplication in a much more pleasant manner than the definition above.

Proposition 1.3

If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then

$$z_1 z_2 = (r_1 r_2) e^{i(\theta_1 + \theta_2)}$$

Proof. We have proved commutativity. From here, we apply trig identities. $\square$

Thus multiplication results in multiplication of lengths and addition of arguments.

Proposition 1.4

For $z_1, z_2 \in \mathbb{C}$, the **triangle inequality** holds:

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

Proof. Choose r, θ such that $z_1 + z_2 = re^{i\theta}$. Then

$$|z_1 + z_2| = r = (z_1 + z_2)e^{-i\theta} = z_1 e^{-i\theta} + z_2 e^{-i\theta} = \mathrm{Re}(z_1 e^{-i\theta} + z_2 e^{-i\theta})$$

Now note that $\mathrm{Re}(z + w) = \mathrm{Re}(z) + \mathrm{Re}(w)$. So

$$\mathrm{Re}(z_1 e^{-i\theta} + z_2 e^{-i\theta}) = \mathrm{Re}(z_1 e^{-i\theta}) + \mathrm{Re}(z_2 e^{-i\theta}) \leq \left|z_1 e^{-i\theta}\right| + \left|z_2 e^{-i\theta}\right| = |z_1| + |z_2| \quad \square$$

The above proof amounts to applying the real triangle inequality to the components of z_1, z_2 in the direction of $z_1 + z_2$.

Corollary 1.5

The reverse triangle inequality also holds:

$$||z| - |w|| \leq |z - w|$$

Proof. We have

$$\begin{cases} |z| \leq |z-w| + |w| \\ |w| \leq |w-z| + |z| \end{cases} \implies \begin{cases} |z| - |w| \leq |z-w| \\ -|z-w| \leq |z| - |w| \end{cases} \implies ||z| - |w|| \leq |z-w| \qquad \square$$

Definition 1.3

Let $z = x + iy \in \mathbb{C}$. Then the **complex conjugate** of z is defined as

$$\overline{z} = x - iy$$

Geometrically, this is reflection over the x axis.

Proposition 1.6

For $z \in \mathbb{C}$, $z\overline{z} = |z|^2$.

Proof. Let $z = x + iy$. Then

$$z\overline{z} = (x+iy)(x-iy) = x^2 + y^2 = |z|^2 \qquad \square$$

Definition 1.4

For $z \neq 0$, define

$$\frac{1}{z} = \frac{\overline{z}}{|z|^2}$$

The above proposition and definition show that

$$z \cdot \frac{1}{z} = 1$$

Definition 1.5

A **sequence** of complex numbers $\{z_n\}_{n=1}^{\infty}$ **converges** to $z \in \mathbb{C}$ (written $\lim_{n\to\infty} z_n = z$) if

$$\begin{cases} \lim_{n\to\infty} \mathrm{Re}(z_n) = \mathrm{Re}(z) \\ \lim_{n\to\infty} \mathrm{Im}(z_n) = \mathrm{Im}(z) \end{cases}$$

This equivalent to the familiar definition:

Proposition 1.7

A sequence $\{z_n\} \subseteq \mathbb{C}$ converges to z if and only for $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that whenever $n \geq N$ we have

$$|z_n - z| < \varepsilon$$

Proof. ($\implies$) Let $\varepsilon > 0$. Then pick N_1, N_2 such that

$$\begin{cases} n \geq N_1 \implies |\mathrm{Re}(z_n) - \mathrm{Re}(z)| < \frac{\varepsilon}{\sqrt{2}} \\ n \geq N_2 \implies |\mathrm{Im}(z_n) - \mathrm{Im}(z)| < \frac{\varepsilon}{\sqrt{2}} \end{cases}$$

Letting $N = \max\{N_1, N_2\}$, whenever $n \geq N$ we have

$$|z_n - z|^2 = \mathrm{Re}(z_n - z)^2 + \mathrm{Im}(z_n - z)^2 = |\mathrm{Re}(z_n) - \mathrm{Re}(z)|^2 + |\mathrm{Im}(z_n) - \mathrm{Im}(z)|^2 < \frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2}$$

Taking square roots on both sides we have

$$|z_n - z| < \varepsilon$$

($\impliedby$) $|\mathrm{Re}(z_n) - \mathrm{Re}(z)| = |\mathrm{Re}(z_n - z)| \leq |z_n - z|$ $\square$

We similarly define the limit of a complex function $\lim_{z \to a} f(z)$.

Definition 1.6

A **Cauchy sequence** is a sequence $(z_n) \subseteq \mathbb{C}$ such that $(\mathrm{Re}(z_n))$ and $(\mathrm{Im}(z_n))$ are both Cauchy.

Again we can formulate this analogously to the single variable case:

Proposition 1.8

A sequence $\{z_n\} \subseteq \mathbb{C}$ is Cauchy if and only if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that whenever $m, n \geq N$ it follows that

$$|z_n - z_m| < \varepsilon$$

Proof. Same as the proof of Proposition 1.7. $\square$

Proposition 1.9

A Cauchy sequence is convergent.

Proof. Follows from completeness of $\mathbb{R}$:

$$\{z_n\} \text{ conv.} \iff \begin{cases} \{\mathrm{Re}(z_n)\} \text{ conv.} \\ \{\mathrm{Im}(z_n)\} \text{ conv.} \end{cases} \iff \begin{cases} \{\mathrm{Re}(z_n)\} \text{ Cauchy} \\ \{\mathrm{Im}(z_n)\} \text{ Cauchy} \end{cases} \iff \{z_n\} \text{ Cauchy} \quad \square$$

1.2 Topology of $\mathbb{C}$

The topological nature of $\mathbb{C}$ should not be unfamiliar to the reader, since it is essentially the same as that of $\mathbb{R}^2$, rephrased slightly using complex variables.

Definition 1.7

Let $r > 0$ and $z_0 \in \mathbb{C}$. Then the **open disk** of radius ε about z is the set

$$\mathbb{D}_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$$

and the **closed disk** as

$$\overline{\mathbb{D}_r}(z_0) = \{z \in \mathbb{C} : |z - z_0| \leq r\}$$

We also specify $\mathbb{D}_r = \mathbb{D}_r(0)$ and $\mathbb{D} = \mathbb{D}_1$.

Definition 1.8

An **interior point** $z_0 \in \Omega$ of a subset $\Omega \subseteq \mathbb{C}$ is a point such that there exists $r > 0$ where $\mathbb{D}_r(z_0) \subseteq \Omega$.

Definition 1.9

The set of interior point in Ω is the **interior** of Ω, denoted $\operatorname{int}\Omega$.

Definition 1.10

An **open set** in $\mathbb{C}$ is a subset $\Omega \subseteq \mathbb{C}$ such that for any $z_0 \in \Omega$ there exists $\varepsilon > 0$ such that $D_\varepsilon(z_0) \subseteq \Omega$.

It is immediate that Ω is open if and only if $\operatorname{int}\Omega = \Omega$.

Definition 1.11

Let $\Omega \in \mathbb{C}$ and let $z \in \mathbb{C}$. z is a **limit point** of Ω if there exists a sequence of points $\{z_n\}_{n=1}^{\infty} \subseteq \Omega$ such that $z_n \neq z$ for each n and $\lim z_n = z$.

We can equivalently define a limit point as a point z such that $\mathbb{D}_r(z) \setminus \{z\} \cap \Omega \neq \varnothing$ for each $r > 0$

Definition 1.12

$A \subseteq \mathbb{C}$ is a **closed set** if $\mathbb{C} \setminus A$ is open.

Proposition 1.10

A is closed if and only if it contains all its limit points.

Proof. ($\Longrightarrow$) Suppose not. Then pick z which is a limit point of A that is not in A. Then there is no disk around z entirely contained in $\mathbb{C} \setminus A$. Thus A is not closed.

($\Longleftarrow$) Suppose A is not closed. Then there exists $z \notin A$ such that each $\mathbb{D}_r(z) \setminus \{z\}$ intersects A. Then z is a limit point of A. $\square$

Definition 1.13

The **closure** of $\Omega \subseteq \mathbb{C}$, denoted $\overline{\Omega}$, is the union of Ω with its limit points.

Definition 1.14

The **boundary** of $\Omega \subseteq \mathbb{C}$, denoted $\partial\Omega$, is defined as $\overline{\Omega} \setminus \operatorname{int} \Omega$.

Definition 1.15

$\Omega \subseteq \mathbb{C}$ is **bounded** if there exists $M > 0$ such that $|z| < M$ for each $z \in \Omega$ (or equivalently, $\Omega \subseteq \mathbb{D}_M$).

Definition 1.16

Let $\Omega \subseteq \mathbb{C}$ be bounded. Then the **diameter** of Ω is defined as

$$\operatorname{diam} \Omega = \sup_{z,w \in \Omega} |z - w|$$

The following definition, as in the real case, is critical:

Definition 1.17

$\Omega \subseteq \mathbb{C}$ is **compact** if it is closed and bounded.

Theorem 1.11: Bolzano-Weierstrass Theorem

Let $\Omega \subseteq \mathbb{C}$. Then the following conditions are equivalent:

1. Ω is compact.
2. Each sequence $\{z_n\}_{n=1}^{\infty} \subseteq \Omega$ has a subsequence $\{z_{n_k}\}_{k=1}^{\infty}$ which converges to some $z \in \Omega$.

We can treat $\mathbb{C}$ similarly to $\mathbb{R}^2$ to prove this.

Proof. ($1 \implies 2$) If Ω is compact, then $\{z_n\}_{n=1}^{\infty} \subseteq \mathbb{C}$ may be written as $\{x_n + iy_n\}_{n=1}^{\infty} \subseteq \mathbb{C}$. Since Ω is bounded, there exists $M > 0$ such that $|z| < M$ for all $z \in \Omega$. In particular $\sqrt{x_n^2 + y_n^2} = |z_n| < M$. So the real sequences $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty}$ are bounded. Apply the real version of Bolzano-Weierstrass, there exists a convergent subsequence $\{x_{n_k}\}$. Then consider the sequence $\{y_{n_k}\}$. This is also bounded, so we apply Bolzano-Weierstrass again to produce $\{y_{n_{k_i}}\}$ convergent. Then the sequence $\{z_{n_{k_i}}\}$ is a convergent subsequence. If $z = z_n$ for some n, then $z \in \Omega$; otherwise it is a limit point. Since Ω is closed it contains its limit points so $z \in \Omega$.

($2 \implies 1$) Suppose each sequence has a convergent subsequence. Let z be a limit point and let $\{z_n\} \subseteq \Omega \setminus \{z\}$ be a sequence converging to z. Then there exists a subsequence $\{z_{n_k}\}$ which converges to $z' \in \Omega$. But subsequences converge to the same value as the original sequence, so $z = z' \in \Omega$. So Ω is closed. If Ω is not bounded, then we may take $\{z_n\}$ such that $|z_n| \geq n$, and that $|z_{n+1}| > |z_n| + 1$. But then $|z_m - z_{m-1}| > 1$ so no subsequence is Cauchy and thus no subsequence converges. So Ω is bounded. $\square$

Definition 1.18

An **open cover** of a set $\Omega \subseteq \mathbb{C}$ is a collection $\mathcal{O}$ of open sets such that each $z \in \Omega$ is contained in some $O \in \mathcal{O}$. A **subcover** of $\mathcal{O}$ is a subcollection which is still a cover.

Theorem 1.12: Heine-Borel Theorem

A set $\Omega \subseteq \mathbb{C}$ is compact if and only if every open cover has a finite subcover.

Proof. ($\implies$) Since Ω is bounded, it is a subset of a closed rectangle K. We showed in $\mathbb{R}^2$ that $X \times Y$ is compact when $X, Y \subseteq \mathbb{R}$ are, and the same is true here. So K is compact. Take an open cover $\mathcal{O}$ of Ω and add the (open) set $\mathbb{C} \setminus \Omega$. This is an open cover of $\mathbb{C}$ and thus one of K, so only finitely many are needed. Remove $\mathbb{C} \setminus \Omega$ if necessary and we still have an open cover of Ω.

($\impliedby$) Boundedness is immediate by covering Ω with balls of finite radius.

For closure, suppose not. Then take a limit point $w \notin \Omega$. Each $z \in \Omega$ has $|z - w| > 0$, so we may cover Ω with open balls $O_z = \mathbb{D}_\varepsilon(z)$ where $\varepsilon < |z - w|/2$. Then a finite number of them cover Ω but this implies that y is not a limit point. $\square$

For the sake of completeness, here is an independent proof that a set is sequentially compact if it is covering compact.

Proof that covering compactness $\implies$ sequential compactness. Let K be covering compact and pick a sequences $\{a_n\} \subseteq K$. Suppose for contradiction that a_n has no convergent subsequence in K. Then for each $x \in K$, there exists $\varepsilon_x > 0$ and $N_x \in \mathbb{N}$ such that whenever $n \geq N_x$ it follows that $a_n \notin \mathbb{D}_{\varepsilon_x}(x)$. Then the collection of $\mathbb{D}_{\varepsilon_x}(x)$ for $x \in K$ is an open cover of K, so we may pick a finite subcover

$$\mathbb{D}_{\varepsilon_{x_1}}(x_1), \ldots, \mathbb{D}_{\varepsilon_{x_m}}(x_m)$$

Then let $N = \max N_{x_i}$. For $n \geq N$ it follows that $a_n \notin K$, contradiction. $\square$

Proposition 1.13: Nested Compact Set Property

Suppose that $\Omega_1 \supseteq \Omega_2 \supseteq \dots$ is a nested sequence of compact, nonempty subsets of $\mathbb{C}$. Then

$$\bigcap_{n=1}^{\infty} \Omega_n \neq \varnothing$$

Moreover, if $\lim_{n\to\infty} \operatorname{diam} \Omega_n = 0$, then there is a unique point $z \in \mathbb{C}$ such that $z \in \Omega_n$ for all n.

Proof. Choose $z_n \in \Omega_n$ for each n. Then the sequence of points $\{z_n\} \subseteq \Omega_1$, and Ω_1 is compact, so there exists a convergent subsequence $\{z_{n_k}\}$ tending to $z \in \Omega_1$. Then for arbitrary Ω_n, there exists a subsequence $\{z_{n_{k+k_0}}\} \subseteq \Omega_n$ for sufficiently large k_0, which converges to z and we see that $z \in \Omega_n$. So the intersection is nonempty.

To show uniqueness, take $z, w \in \bigcap_{n=1}^{\infty} \Omega_n$. Then

$$|z - w| \leq \operatorname{diam} \Omega_n$$

for each n, but $\operatorname{diam} \Omega_n \to 0$ so $|z - w| = 0$ and thus $z = w$. $\square$

Remark

With the assumption that $\operatorname{diam} \Omega_n \to 0$, we need not take subsequences as $\{z_n\}$ itself is Cauchy. To see this, pick $\varepsilon > 0$ and let N be such that $\operatorname{diam} \Omega_n < \varepsilon$ for any $n \geq N$. Then for any $n, m \geq N$, $z_n, z_m \in \Omega_N$ and thus $|z_n - z_m| \leq \operatorname{diam} \Omega_N < \varepsilon$.

Definition 1.19

A set $\Omega \subseteq \mathbb{C}$ is **connected** if there are no nonempty disjoint sets $A, B \subseteq \Omega$ such that $\Omega = A \sqcup B$ such that $A \cap \overline{B} = \varnothing$ and $\overline{A} \cap B = \varnothing$.

The above definition may be rephrased as saying that Ω is not the disjoint union of nonempty sets which are open in the subspace topology of Ω:

Proposition 1.14

A topological space X is connected if and only if it cannot be written as $X = \Omega_1 \cup \Omega_2$ with Ω_1, Ω_2 nonempty, disjoint and open (in X).

Proof. ($\implies$) Suppose $X \subseteq \mathbb{C}$ is connected. Let $\Omega_1, \Omega_2 \subseteq X$ be nonempty, open and disjoint. Consider Ω_2. Then $\Omega_2 \subseteq X \setminus \Omega_1$. By definition $X \setminus \Omega_1$ is closed. $\overline{\Omega_2}$ is the smallest closed set containing Ω_2, so $\overline{\Omega_2} \subseteq X \setminus \Omega_1$ and thus $\Omega_1 \cap \overline{\Omega_2} = \varnothing$. Similarly $\overline{\Omega_1} \cap \Omega_2 = \varnothing$. Since X is connected, we conclude that $\Omega_1 \cup \Omega_2 \neq X$.

($\impliedby$) Pick A, B nonempty with $X = A \cup B$. Assume that $A \cap \overline{B} = B \cap \overline{A} = \varnothing$, so that $A \subseteq X \setminus \overline{B}$ and $B \subseteq X \setminus \overline{A}$. Define $\Omega_1 = X \setminus \overline{B}$ and $\Omega_2 = X \setminus \overline{A}$. Since $X = A \cup B$, we have $X = \Omega_1 \cup \Omega_2$. Ω_1, Ω_2 are both open in X, so it must not be the case that they are disjoint. So there exists some $x \in \Omega_1 \cap \Omega_2$. But this implies that $x \notin A$ and $x \notin B$. $\square$

Thus the above general definition can be simplified for nicer sets:

Proposition 1.15

If Ω is open, then it is connected if and only if it cannot be written as the union of disjoint open sets (in $\mathbb{C}$). Similarly if F is closed then it is connected if and only if it is not the union of disjoint closed sets.

We may also introduce another notion of connectedness, which involves functions into Ω.

Definition 1.20

Suppose $\Omega \subseteq \mathbb{C}$ and $f : \Omega \to \mathbb{C}$. f is **continuous** at z_0 if for every $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $z \in \Omega$ and $|z - z_0| < \delta$, it follows that $|f(z) - f(z_0)| < \varepsilon$.

Proposition 1.16

f is continuous at z_0 if and only if for every $\{z_n\} \subseteq \Omega$ with $z_n \to z_0$, it follows that $f(z_n) \to f(z_0)$. We say that f is continuous on Ω if it is continuous at each point in Ω.

Definition 1.21

A **path** is a function $f : [0, 1] \to \mathbb{C}$. A continuous path is a continuous such function.

Definition 1.22

A set $\Omega \subseteq \mathbb{C}$ is **path connected** if for any $z, w \in \Omega$ there exists a continuous path with $f(0) = z$ and $f(1) = w$ with $f(t) \in \Omega$ for each $t \in [0, 1]$.

Proposition 1.17

An open set Ω is path connected if and only if it is connected.

Definition 1.23

A nonempty open, connected set $\Omega \subseteq \mathbb{C}$ is called a **region**.

Corollary 1.18

A region is path connected.

1.3 Functions on $\mathbb{C}$

We now turn our attention to functions which map complex numbers to complex numbers, the primary object of study in this course. Continuing from the definition of continuity from the previous section, we have the following:

Proposition 1.19

If f is continuous at z_0 then $|f|$ is continuous at z_0.

Proof. By the reverse triangle inequality we have $||f(z)| - |f(z_0)|| \leq |f(z) - f(z_0)|$. The conclusion follows. $\square$

Definition 1.24

f attains its maximum on $\Omega \subseteq \mathbb{C}$ if there exists $z_0 \in \Omega$ such that

$$|f(z)| \leq |f(z_0)|$$

for each $z \in \Omega$. The minimum case is analogous.

Theorem 1.20

Suppose that $\Omega \subseteq \mathbb{C}$ is compact and $f : \Omega \to \mathbb{C}$ is continuous. Then f attains is maximum (and minimum) on Ω.

Proof. First we show that f is bounded on Ω. If not, then we may take a sequence of points $\{z_n\} \subseteq \Omega$ such that $|f(z_n)| \to \infty$. Then $\{z_n\}$ contains a convergent subsequence $\{z_{n_k}\}$ tending to some $z \in \Omega$. It follows that

$$|f(z_{n_k})| \to |f(z)|$$

But the left side diverges to ∞, contradiction. Thus $f(\Omega)$ is bounded.

Let $M = \sup |f|(\Omega)$. Then there exists a sequence $\{z_n\} \subseteq \Omega$ such that $|f(z_n)| \to M$. Then there exists a subsequence $\{z_{n_k}\}$ converging to $z \in \Omega$. By continuity we have

$$|f(z)| = \lim |f(z_{n_k})| = M$$

$\square$

We now make the most important definition of this course:

Definition 1.25

Let $\Omega \subseteq \mathbb{C}$ be open and let $z_0 \in \Omega$. Let $f : \Omega \to \mathbb{C}$. We say that f is **holomorphic** at z_0 (or complex differentiable) if the limit

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists. In this case, the limit is denoted $f'(z_0)$.

If f is holomorphic at every $z \in \Omega$, then we simply say it is holomorphic on Ω. If f is holomorphic on $\mathbb{C}$ it is said to be **entire**.

We will sometimes also say that f is **analytic** or **complex differentiable** when it is holomorphic.

Note that the specification that Ω is open ensures that the difference quotient is actually defined (for sufficiently small h). Moreover, although this definition appears similar to the real analogue, the structure of the complex numbers means that it has far-reaching implications.

We will prove the following theorems in this class:

- (Cauchy's Theorem) If f is holomorphic on Ω, then it has derivatives of all orders.
- (Liouville's Theorem) If f is entire and bounded, then it is constant.
- (Prime Number Theorem) If $\pi(n)$ denotes the number of prime numbers less than or equal to n, then
$$\lim_{n\to\infty} \pi(n) \cdot \frac{\ln n}{n} = 1$$
- (Hardy-Ramanujan Theorem) Define $p(n)$ (the partition function) to be the number of ways to write $n = k_1 + k_2 + \ldots + k_n$ where $k_1 \geq k_2 \geq \ldots \geq k_n$ are all integers. For instance, $p(4) = 5$. Then
$$p(n) \sim \frac{1}{n\sqrt{48}} e^{\pi\sqrt{\frac{2}{3}}\cdot\sqrt{n}}$$

Example 1.1

The function $f(z) = z$ is holomorphic:

$$\frac{f(z+h) - f(z)}{h} = \frac{z + h - z}{h} = \frac{h}{h} = 1$$

so $z' = 1$.

Definition 1.26

If $A \subseteq \mathbb{C}$ is closed and $f : A \to \mathbb{C}$, then we say f is holomorphic on A if there exists $\Omega \supseteq A$ open and $F : \Omega \to \mathbb{C}$ which is holomorphic, and $F|_A = f$.

We can rewrite the definition of holomorphicity similarly to the multivariable real case as the following:

Proposition 1.21

$f : \Omega \to \mathbb{C}$ (Ω open) is holomorphic at z_0 if and only if there exists $a \in \mathbb{C}$ and $\psi : \mathbb{C} \to \mathbb{C}$ with $\psi(h) \to 0$ as $h \to 0$ such that

$$f(z_0 + h) = f(z_0) + ah + h\psi(h)$$

on some $\mathbb{D}_r(z_0) \subseteq \Omega$.

Proof. We can rewrite the above as

$$\psi(h) = \frac{f(z_0 + h) - f(z_0)}{h} - a$$

which goes to 0 if and only if

$$\frac{f(z_0 + h) - f(z_0)}{h} \to a$$

so that $f'(z_0) = a$. □

This recharacterization allows for a simple proof of the following:

Proposition 1.22

If f is holomorphic at z_0, then it is continuous at z_0.

Proof. Let $\{z_n\}$ be a sequence with $z_n \to z_0$. We want to show that $f(z_n) \to f(z_0)$. Let $h_n = z_n - z_0$. Then

$$f(z_n) = f(z_0 + h_n) = f(z_0) + ah_n + h_n\psi(h_n)$$

by assumption, the second and third terms go to zero, so $f(z_n) \to f(z_0)$. □

Example 1.2

Let $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ be defined by $f(z) = \frac{1}{z}$. Then

$$\lim_{h\to 0} \frac{\frac{1}{z_0+h} - \frac{1}{z_0}}{h} = \lim_{h\to 0} \frac{-h}{h(z_0)(z_0 + h)} = -\frac{1}{z_0^2}$$

so $f'(z_0) = -\frac{1}{z_0^2}$.

Proposition 1.23

Let $\Omega \subseteq \mathbb{C}$ be open, and let $f, g : \Omega \to \mathbb{C}$ be holomorphic at z_0. Then

1. $f + g$ is holomophic at z_0, and $(f + g)' = f' + g'$.
2. fg is holomorphic at z_0, and $(fg)' = f'g + fg'$.

3. If $g(z_0) \neq 0$, then $\frac{f}{g}$ is well defined on an open disk aroud z_0, and $\frac{f}{g}$ is holomorphic at z_0 with $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$.

Proof. Let ψ, φ be such that

$$\begin{aligned} f(z_0+h) &= f(z_0) + f'(z_0)h + h\psi(h) \\ g(z_0+h) &= g(z_0) + g'(z_0)h + h\varphi(h) \end{aligned}$$

Then

$$f(z_0+h) + g(z_0+h) = f(z_0) + g(z_0) + [f'(z_0) + g'(z_0)]h + h(\psi(h) + \varphi(h))$$

$\lim_{h\to 0} \varphi + \psi = 0$, so the above shows that $(f+g)' = f' + g'$.

Letting

$$\phi(h) = f'(z_0)g'(z_0) + \psi(h)[g(z_0) + g'(z_0)] + \varphi(h)[f(z_0) + f'(z_0)] + \varphi(h)\psi(h)$$

which tends to 0 as $h \to 0$, we have

$$f(z_0+h)g(z_0+h) = f(z_0)g(z_0) + [f(z_0)g'(z_0) + f'(z_0)g(z_0)]h + h\phi(h)$$

so $(fg)' = f'g + g'f$.

The quotient rule may be derived from the Chain Rule using the fact that $\left(\frac{1}{z}\right)' = -\frac{1}{z^2}$ when $z \neq 0$. ☐

Proposition 1.24: Chain Rule

Let $\Omega, U \subseteq \mathbb{C}$ be open, and let $f : \Omega \to U$ and $g : U \to \mathbb{C}$. Then $g \circ f : \Omega \to \mathbb{C}$ is holomorphic and

$$(g \circ f)'(z) = g'(f(z))f'(z)$$

Proof. Using the alternative characterization of holomorphicity, we have

$$f(z_0+h) = f(z_0) + f'(z_0)h + h\psi_f(h)$$

where $\psi_f(h) \to 0$ as $h \to 0$. Similarly,

$$g(f(z_0)+w) = g(f(z_0)) + g'(f(z_0))w + w\psi_g(w)$$

Then

$$\begin{aligned} (g \circ f)(z_0+h) &= g(f(z_0) + f'(z_0)h + h\psi_f(h)) \\ &= g(f(z_0)) + g'(f(z_0))(f'(z_0)h + h\psi_f(h)) + (f'(z_0)h + h\psi_f(h))\psi_g(f'(z_0)h + h\psi_f(h)) \\ &= g(f(z_0)) + g'(f(z_0))f'(z_0)h + h[\psi_f(h)g'(f(z_0)) + (f'(z_0) + \psi_f(h))\psi_g(f'(z_0)h + h\psi_f(h))] \end{aligned}$$

Note that

$$\lim_{h\to 0} \underbrace{\psi_f(h)}_{=0} g'(f(z_0)) + (f'(z_0) + \underbrace{\psi_f(h)}_{=0})\psi_g(\underbrace{f'(z_0)h + h\psi_f(h)}_{=0}) = 0$$

so $(g \circ f)'(z_0) = g'(f(z_0))f'(z_0)$. ☐

Example 1.3

Let f be a constant function. Then f is entire and $f'(z) = 0$.

We showed in Example 1.1 that the identity $g(z) = z$ is entire with $g'(z) = 1$.

Combination of the two functions above, together with Proposition 1.23 gives

Corollary 1.25

Let $p(z) = a_0 + a_1 z + \ldots + a_n z^n$. Then p is entire and $p'(z) = a_1 + 2a_2 z + \ldots + n a_n z^{n-1}$.

Let us consider a non-example.

Example 1.4

Let $f(z) = \overline{z}$, so that $f(x+iy) = x - iy$. This is a smooth function in the case of $\mathbb{R}^2$; in fact since it is linear, $Df = f$, so that f has infinitely many derivatives.

However, in the complex case, we have

$$\frac{f(z+h) - f(z)}{h} = \frac{\overline{z+h} - \overline{z}}{h} = \frac{\overline{h}}{h}$$

But

$$\lim_{t \to 0} \frac{\overline{t}}{t} = 1$$

and

$$\lim_{t \to 0} \frac{\overline{it}}{it} = -1$$

so the limits disagree and f is not holomorphic at any z.

Consider some function $f : \Omega \to \mathbb{C}$. Let us denote its real and imaginary parts by u, v, respectively, so that

$$f(x+iy) = u(x,y) + iv(x,y)$$

(u, v are defined on $\Omega' \subseteq \mathbb{R}^2$ which is equivalent to Ω in the obvious way.) This allows us to consider f as a pair of functions from $\mathbb{R}^2 \to \mathbb{R}$, which are surfaces lying in $\mathbb{R}^3$. We will investigate which choices of u, v may be associated with a holomorphic f.

Let h be a (small) complex number and write $h = h_1 + ih_2$. Then write

$$\frac{f(z+h) - f(z)}{h} = \frac{u(x+h_1, y+h_2) - u(x,y)}{h_1 + ih_2} + \frac{v(x+h_1, y+h_2) - iv(x,y)}{h_1 + ih_2}$$

Let us consider what happens as h tends to 0 from different directions. For instance, suppose h is entirely real, so $h_2 = 0$. Then

$$\lim_{h_1 \to 0} \frac{f(z+h_1) - f(z)}{h_1} = \lim_{h_1 \to 0} \frac{(u+iv)(x+h_1, y) - (u+iv)(x,y)}{h_1} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

and similarly

$$\lim_{h_2 \to 0} \frac{f(z+h_2)-f(z)}{ih_2} = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Then if f is holomorphic, then we can match components to get the following:

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial v}{\partial y} = -\dfrac{\partial u}{\partial y} \end{cases} \tag{$*$}$$

The system of equations $(*)$ are known as the **Cauchy-Riemann equations**. We have shown that these are a necessary conditions for f to be holomorphic; below we will show that if we also assume that the partials are continuous, then we have a sufficient condition. Later, we will show that u, v are necessarily continuously differentiable, so that these are equivalent characterizations. For now we will content ourselves with one direction:

Theorem 1.26

Let $\Omega \subseteq \mathbb{C}$ be open and let $f : \Omega \to \mathbb{C}$. Let $f = u + iv$, where $u, v : \Omega \to \mathbb{R}$ are continuously differentiable and satisfy the Cauchy-Riemann equations. Then f is holomorphic.

Proof. Consider the first-order Taylor expansion in two variables, which says that

$$u(x+h_1, y+h_2) = u(x,y) + \frac{\partial u}{\partial x}h_1 + \frac{\partial u}{\partial y}h_2 + h\psi_u(h)$$

where $h = h_1 + ih_2$ and $\psi_u(h) \to 0$ as $h \to 0$. Similarly

$$v(x+h_1, y+h_2) = v(x,y) + \frac{\partial v}{\partial x}h_1 + \frac{\partial v}{\partial y}h_2 + h\psi_v(h)$$

Now by assumption, f satisfies the Cauchy-Riemann equations, so

$$v(x+h_1, y+h_2) = v(x,y) - \frac{\partial u}{\partial y}h_1 + \frac{\partial u}{\partial x}h_2 + h\psi_v(h)$$

so

$$\begin{aligned} f(x+h) &= u(x+h_1, y+h_2) + iv(x+h_1, y+h_2) \\ &= u(x,y) + \frac{\partial u}{\partial x}h_1 + \frac{\partial u}{\partial x}h_2 + h\psi_u(y) + iv(x,y) - i\frac{\partial u}{\partial y}h_1 + i\frac{\partial u}{\partial x}h_2 + ih\psi_v(h) \\ &= f(x,y) + (h_1 + ih_2)\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right) + h(\underbrace{\psi_u(h) + i\psi_v(h)}_{\psi(h)}) \\ &= f(z) + h\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right) + h\psi(h) \end{aligned}$$

so f is holomorphic using the alternative characterization and

$$f'(z) = f'(x,y) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

□

Let f be a complex valued function of the form $f(x+iy) = u(x,y) + iv(x,y)$. Associate with it a $\mathbb{R}^2$-valued function

$$F(x,y) = (u(x,y), v(x,y))$$

Recall that its Jacobian matrix is

$$J_F(x,y) = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \tag{$**$}$$

and that F is differentiable in the real sense if it is true that

$$\lim_{(h_1,h_2)\to(0,0)} \frac{F(x+h_1, y+h_2) - F(x,y) - J_F(x,y)\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}}{|(h_1,h_2)|} = 0$$

Comparing this to the complex condition

$$\lim_{h\to 0} \frac{f(z+h) - f(z) - f'(z)h}{h} = 0$$

we can see that complex differentiability requires $J_f(x,y)$ to be of the form of multiplying by some complex number. This happens if and only if

$$J_F(x,y) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

for $a, b \in \mathbb{R}$. By reconciling this with ($**$) we recover the Cauchy-Riemann equations ($*$).

Definition 1.27

Let $f : \Omega \to \mathbb{C}$ with Ω open. Then we define

$$\frac{\partial f}{\partial x} := \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

and

$$\frac{\partial f}{\partial y} := \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}$$

We further define

$$\frac{\partial f}{\partial z} := \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$$

and

$$\frac{\partial f}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

Proposition 1.27

Let $\Omega \subseteq \mathbb{C}$ be open and let $f : \Omega \to \mathbb{C}$ be of the form $f(x+iy) = u(x,y) + iv(x,y)$. If f is holomorphic on Ω, then

1. $\dfrac{\partial f}{\partial \overline{z}} = 0$.
2. $\dfrac{\partial f}{\partial z} = \dfrac{\partial f}{\partial x} = -i\dfrac{\partial f}{\partial y}$.
3. $\dfrac{\partial f}{\partial z} = f'$.
4. $f'(z_0) = 2\dfrac{\partial u}{\partial z}(z_0)$.
5. $F = (u(x,y), v(x,y))$ is differentiable in the real sense, and $\det J_F(x,y) = |f'(x+iy)|^2$.

Proof.

1. By Cauchy-Riemann,
$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\right) = 0$$

2. By part 1,
$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y} + \frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{\partial f}{\partial x}$$
Similarly,
$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial z} - \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y} - \frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) = -i\frac{\partial f}{\partial y}$$

3. Take h_1 real and $z_0 \in \Omega$. Then
$$\frac{\partial f}{\partial x}(z_0) = \lim_{h_1 \to 0} \frac{f(z_0 + h_1) - f(z_0)}{h_1} = \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h} = f'(z_0)$$
By part 2 the conclusion follows.

4. By parts 2 and 3,
$$f' = \frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = 2\frac{\partial u}{\partial z}$$

5. We have
$$\det J_F(x,y) = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$
By the Cauchy-Riemann equations and parts 2 and 3 this is
$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = |f'(x+iy)|^2$$
□

1.4 Power Series

We will now discuss power series, which are defined similarly to the real case. They will initially serve as a valuable example of holomorphic functions. Later we will see that they are actually the *only* example, which justifies particular attention in their study.

Definition 1.28

A **power series** (centered around z_0) is a function f of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n := \lim_{N\to\infty} \sum_{n=0}^{N} a_n (z - z_0)^n$$

where $\{a_n\} \subseteq \mathbb{C}$, which is defined wherever the right hand limit converges.

Note that it is certainly a necessary condition that $a_n z^n \to 0$ as $n \to \infty$, since

$$\lim_{n\to\infty} a_n z^n = \lim_{n\to\infty} \left[\sum_{k=0}^{n} a_k z^k - \sum_{k=0}^{n-1} a_k z^k \right] = \lim_{n\to\infty} \left[\sum_{k=0}^{n} a_k z^k \right] - \lim_{n\to\infty} \left[\sum_{k=0}^{n-1} a_k z^k \right] = 0$$

In this section, we will first consider only power series which are centered at 0.

Definition 1.29

We say that a power series $\sum_{n=0}^{\infty} a_n z^n$ **converges absolutely** (in the complex sense) if the series

$$\sum_{n=0}^{\infty} |a_n| \cdot |z|^n$$

converges in the real sense.

Proposition 1.28

If $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely, then it converges (in the complex sense).

Proof. Let $\varepsilon > 0$.

Write

$$S_N(z) = \sum_{n=0}^{N} a_n z^n$$

Since

$$\sum_{n=0}^{\infty} |a_n| \cdot |z|^n$$

converges, there exists $N_0 \in \mathbb{N}$ such that

$$\sum_{n=N_0+1}^{\infty} |a_n| \cdot |z|^n < \varepsilon$$

Then for $M, N \geq N_0$, assume without loss of generality that $M < N$. Then we have

$$|S_N(z) - S_M(z)| = \left| \sum_{n=M+1}^{N} a_n z^n \right| \leq \sum_{a=M+1}^{N} |a_n| \cdot |z|^n \leq \sum_{a=N_0+1}^{\infty} |a_n| \cdot |z|^n < \varepsilon$$

so $\{S_N(z)\}$ is a Cauchy sequence, and thus

$$\sum_{n=0}^{\infty} a_n z^n$$

converges. □

Recall that in the single real variable case, we found that power series converge on some interval (possibly open, closed, or half-open) which is centered around 0 (or any other point of expansion). An analogous statement is true here, with the interval replaced by a disk.

Theorem 1.29

For any power series $\sum_{n=0}^{\infty} a_n z^n$ there exists $0 \leq R \leq \infty$[a] (called the **radius of convergence**) such that for any $z \in \mathbb{C}$:

1. If $|z| < R$, then $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely.
2. If $|z| > R$, then $\sum_{n=0}^{\infty} a_n z^n$ does not converge.

Moreover, R is given by **Hadamard's formula**:

$$R = \frac{1}{\limsup_{n \to \infty} \sqrt[n]{|a_n|}}$$

[a] $R = \infty$ means the condition $|z| < R$ is satisfied for all $z \in \mathbb{C}$.

This theorem says that we have absolute convergence inside the disk $\mathbb{D}_R$ (called the **disk of convergence**), and divergence outside of it. As in the real case, this theorem makes no statement about convergence on the boundary of the disk.

Proof. Denote $L = \limsup_{n \to \infty} \sqrt[n]{|a_n|}$. Suppose that $L \neq 0, \infty$. If $|z| < \frac{1}{L} = R$, then $L|z| < 1$, so there exists $\varepsilon > 0$ such that $(L + \varepsilon)|z| = r < 1$. By the definition of lim sup, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\sqrt[n]{|a_n|} < L + \varepsilon \implies |a_n| < (L + \varepsilon)^n \implies |a_n||z|^n < ((L + \varepsilon)|z|)^n = r^n$$

so $\sum_{n=0}^{\infty} a_n z^n$ is dominated by the absolutely convergent geometric series $\sum_{n=0}^{\infty} r^n$ and thus converges absolutely.

On the other hand, if $|z| > R = \frac{1}{L}$, then $L|z| > 1$ so there exists a subsequence $\{a_{n_k}\}$ such that

$$\sqrt[n_k]{|a_{n_k}|} \cdot |z| > 1$$

for all k. Then

$$|a_{n_k}| \cdot |z|^{n_k} > 1$$

which does not tend to 0 as $k \to \infty$, so convergence is impossible.

If $L = 0$, then $R = \infty$. For any z there exists N such that $n \geq N$ implies that

$$\sqrt[n]{a_n} < \frac{1}{|z|}$$

Then

$$|a_n||z|^n < r < 1$$

So

$$\sum_{n=N}^{\infty} |a_n||z|^n$$

is dominated by $\sum_{n=N}^{\infty} r^n$, and thus converges for $|z| < R = \infty$.

If $L = \infty$, then $L|z| > 1$ and we apply the argument for the case $|z| > R$. $\square$

Example 1.5

The power series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}$$

has $R = \infty$, since the real-valued power series

$$\sum_{n=0}^{\infty} \frac{|z|^n}{n!} = e^{|z|}$$

converges absolutely everywhere. This also allows us to define the **exponential** of z as

$$e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Example 1.6

The power series

$$\sum_{n=0}^{\infty} z^n$$

has radius of convergence 1. This can be seen either by direct computation in the real case, or using Hadamard's formula and the fact that each a_n is 1:

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{1}} = \frac{1}{1} = 1$$

Moreover, this power series satisfies the equation

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

This can be seen using the identity for partial sums (which holds in all fields)

$$\sum_{n=0}^{N} z^n = \frac{1 - z^{N+1}}{1 - z}$$

so

$$\lim_{N \to \infty} \frac{1 - z^{N+1}}{1 - z} = \frac{1}{1 - z}$$

Definition 1.30

We define the **trigonometric functions** in terms of power series:

$$\cos z := \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$
$$\sin z := \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

Note that both of the above series converge with $R = \infty$. Moreover, we can observe that

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$
$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

and that this is consistent with our previous definition in terms of the identity

$$e^{iz} = \cos z + i \sin z$$

We now prove a fundamental fact about power series which, while analogous to the real case, will have farther reaching implications for us.

Theorem 1.30

The function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is holomorphic on the disk of convergence $\mathbb{D}_R$. Moreover, its derivative is given by

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n$$

which has the same radius of convergence.

Proof. To show that the radius of convergence is the same, simply apply Hadamard's formula to the new power series. Letting R' be the radius of convergence of $\sum_{n=1}^{\infty} n a_n z^{n-1}$, we have

$$\frac{1}{R'} = \limsup_{n \to \infty} \sqrt[n]{(n+1)|a_n|} = \limsup_{n \to \infty} \sqrt[n]{n+1} \sqrt[n]{|a_n|}$$

But $\sqrt[n]{n+1} \to 1$ so

$$\limsup_{n\to\infty} \sqrt[n]{n+1}\,\sqrt[n]{|a_n|} = \limsup_{n\to\infty} \sqrt[n]{|a_n|} = \frac{1}{R}$$

so $R' = R$.

Denote

$$g(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$$

We want to show that $f' = g$. It suffices to show that

$$\frac{f(z_0+h) - f(z_0)}{h} - g(z_0) \underset{h\to 0}{\to} 0$$

whenever $z_0 \in \mathbb{D}_R$.

To show this, pick any $z_0 \in \mathbb{D}_R$ and let $\varepsilon > 0$. Then there exists r such that $|z_0| < r < R$. Then whenever $|h| < r - |z_0|$ we have $|z_0 + h| \leq |z_0| + |h| < r < R$. So for small h, $z_0 + h \in \mathbb{D}_R$.

Denote the partial sums and error terms by

$$S_N(z) = \sum_{n=1}^{N} a_n z^n$$

$$E_N(z) = \sum_{n=N+1}^{\infty} a_n z^n$$

By our observation about Hadamard's formula,

$$\sum_{n=1}^{\infty} n|a_n|z^{n-1}$$

converges, so there exists $N_0 \in \mathbb{N}$ such that

$$\sum_{n=N_0+1}^{\infty} n|a_n||z|^{n-1} < \frac{\varepsilon}{3} \tag{1}$$

S_{N_0} is a polynomial, which we showed is holomorphic. So

$$\frac{S_{N_0}(z_0+h) - S_{N_0}(z_0)}{h} - S_N'(z_0) = \frac{S_{N_0}(z_0+h) - S_{N_0}(z_0)}{h} - \sum_{n=1}^{N_0} n a_n z^{n-1} \underset{h\to 0}{\to} 0$$

Thus there exists $\delta > 0$ such that whenever $|h| < \delta$,

$$\left| \frac{S_{N_0}(z_0+h) - S_{N_0}(z_0)}{h} - S_N'(z_0) \right| < \frac{\varepsilon}{3} \tag{2}$$

The difference quotient

$$\left|\frac{f(z_0+h)-f(z_0)}{h}-g(z_0)\right|$$

may be written as

$$\left|\frac{S_{N_0}(z_0+h)-S_{N_0}(z_0)}{h}-S'_{N_0}(z_0)+S'_{N_0}(z_0)-g(z_0)+\frac{E_{N_0}(z_0+h)-E_N(z_0)}{h}\right|$$

$$\leq\left|\frac{S_{N_0}(z_0+h)-S_{N_0}(z_0)}{h}-S'_{N_0}(z_0)\right|+\left|S'_{N_0}(z_0)-g(z_0)\right|+\left|\frac{E_{N_0}(z_0+h)-E_N(z_0)}{h}\right|$$

For small h, the first term is less than $\frac{\varepsilon}{3}$ by (2). Also,

$$\left|S'_{N_0}(z_0)-g(z_0)\right|=\left|-\sum_{n=N_0+1}^{\infty}na_nz^{n-1}\right|\leq\sum_{n=N_0+1}^{\infty}n|a_n|\cdot|z|^{n-1}<\frac{\varepsilon}{3}$$

by (1). Lastly,

$$\left|\frac{E_{N_0}(z_0+h)-E_{N_0}(z_0)}{h}\right|=\left|\sum_{n=N_0+1}^{\infty}a_n\frac{(z_0+h)^n-z_0^n}{h}\right|$$

Using the identity

$$a^n-b^n=(a-b)\sum_{k=0}^{n-1}a^kb^{n-1-k}$$

for $a=z_0+h, b=z_0$ we have

$$\left|\frac{\sum_{n=N_0+1}^{\infty}a_n\left((z_0+h)^n-z_0^n\right)}{h}\right|=\left|\sum_{n=N_0+1}^{\infty}a_n\left(\sum_{k=0}^{n-1}(z_0+h)^kz_0^{n-1-k}\right)\right|$$

$$\leq\sum_{n=N_0+1}^{\infty}|a_n|\sum_{k=0}^{n-1}|z_0+h|^k|z_0|^{n-1-k}<\sum_{n=N_0+1}^{\infty}|a_n|nr^n<\frac{\varepsilon}{3}$$

where the last inequality follows from (1). Thus

$$\left|\frac{f(z_0+h)-f(z_0)}{h}-g(z_0)\right|<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon$$

□

Corollary 1.31

A power series is infinitely complex differentiable on its disk of convergence.

Proof. Induct using Theorem 1.30. □

Corollary 1.32

A power series may be integrated term-by-term on its disk of convergence.

Proof. The integrated power series has some radius of convergence, and Theorem 1.30 says that this is the same radius as the original power series. □

We now note that for the case of power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ centered around arbitrary z_0, they converge on $\mathbb{D}_R(z_0)$, where R is the radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$. In other words, the behavior is identical, merely translated. Thus it suffices to consider power series centered around 0.

Definition 1.31

Let $\Omega \subseteq \mathbb{C}$ be open and $z_0 \in \Omega$. Then $f : \Omega \to \mathbb{C}$ is said to be **analytic** at z_0 if there exists $r > 0$ such that $\mathbb{D}_r(z_0) \subseteq \Omega$ and there exist coefficients $\{a_n\} \subseteq \mathbb{C}$ such that

$$\sum_{n=0}^{\infty} a_n z^n$$

converges absolutely on $\mathbb{D}_r$ and

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

on $\mathbb{D}_r(z_0)$.

We say that f is analytic on Ω if it is analytic at each $z_0 \in \Omega$.

By Theorem 1.30, each analytic function is (infinitely) holomorphic. We will show later that every holomorphic function is also analytic. Thus the terms are often used interchangably, but we will not do so until we have proved this fact.

1.5 Integration on Curves

We will soon show that the behavior of analytic functions can be well understood by studying the behavior of those functions when integrated over various curves in the plane. This will motivate the particular definitions of an integral over a curve that appear in this section.

Definition 1.32

A **parameterized curve** is a continuous function $z : [a, b] \to \mathbb{C}$. Writing $z(t) = z_1(t) + iz_2(t)$, we say that z is a **smooth curve** if $z_1'(t), z_2'(t)$ exist for all $t \in [a, b]$.

z is said to be **piecewise smooth** if there exist $a = a_0 < a_1 < \ldots < a_n = b$ such that z is smooth on $[a_k, a_{k+1}]$ for each k.

z is a **closed curve** if $z(a) = z(b)$.

Note that in our definition of piecewise smooth curves, z must be continuous at each a_k, but the left and right derivatives need not coincide.

We will adopt the convention that all curves are assumed to be piecewise smooth, unless stated otherwise.

A parameterized curve traces out a particular image in the complex plane, which is intuitively a curve in the plane. Thus it is useful to distinguish the function z and its image.

Definition 1.33

Two parameterized curves $z_1 : [a, b] \to \mathbb{C}$ and $z_2 : [c, d] \to \mathbb{C}$ are said to be **equivalent curves** if there exists a continuously differentiable bijection $t : [a, b] \to [c, d]$ such that $t'(s) > 0$ and $z_2(s) = z_1(t(s))$ for all $s \in [a, b]$.

The condition that $t'(s) > 0$ could be equivalently stated as saying that $t(a) = c$ and $(b) = d$ (so that the direction does not change). It follows that $z_1([a, b]) = z_2([c, d])$.

Definition 1.34

A **curve** is an equivalence class of parameterized curves.

Note that for a given curve, there is another curve which has the same image in $\mathbb{C}$, but with the opposite orientation.

Definition 1.35

If $\gamma \subseteq \mathbb{C}$ is a curve, and $z : [a, b] \to \mathbb{C}$ is a parameterization of γ, then γ^- is the curve parameterized by $z(a + b - t)$.

A commonly used curve is the circle, so it is convenient to define some conventional parameterizations of the circle:

Definition 1.36

The circle with radius $r > 0$ and center $z \in \mathbb{C}$ is

$$C_r(z_0) \coloneqq \partial \mathbb{D}_r(z_0)$$

The **positive orientation** of $C_r(z)$ is the curve parameterized by $z : [-\pi, \pi] \to \mathbb{C}$ and

$$z(\theta) = z_0 + re^{i\theta}$$

The **negative orientation** is the curve parameterized by

$$z^-(\theta) = z_0 + re^{-i\theta}$$

Now, let us define the manner in which we will integrate functions over smooth planar curves.

Definition 1.37

Let $\gamma \subseteq \mathbb{C}$ be a smooth curve. Let $f : \gamma \to \mathbb{C}$ be continuous. Let $z : [a, b] \to \mathbb{C}$ be a smooth parameterization of γ. Then define the **contour integral** of f along γ with respect to z to be

$$\int_{\gamma,z} f(z)\,\mathrm{d}z = \int_a^b f(z(t))z'(t)\,\mathrm{d}t = \int_a^b \mathrm{Re}(f(z(t))z'(t))\,\mathrm{d}t + i\int_a^b \mathrm{Im}(f(z(t))z'(t))\,\mathrm{d}t$$

As Proposition 1.33 shows, this value is independent of the choice of z. Thus, we take this common value to be the contour integral of f along γ.

Proposition 1.33

If γ is a smooth curve, $f : \gamma \to \mathbb{C}$ is continuous, and z_1, z_2 are two parameterizations of γ, then

$$\int_{\gamma,z_1} f(z)\,\mathrm{d}z = \int_{\gamma,z_2} f(z)\,\mathrm{d}z$$

Proof. Since z_1, z_2 are equivalent, we write $z_1(s) = z_2(t(s))$. Then

$$\begin{aligned}\int_{\gamma,z_1} f(z)\,\mathrm{d}z &= \int_a^b f(z_1(s))z_1'(s)\,\mathrm{d}s = \int_a^b f(z_2(t(s)))z_2'(t(s))t'(s)\,\mathrm{d}s \\ &= \int_{t(a)}^{t(b)} f(z_2(t))z_2'(t)\,\mathrm{d}t = \int_c^d f(z_2(t))z_2'(t)\,\mathrm{d}t = \int_{\gamma,z_2} f(z)\,\mathrm{d}z\end{aligned}$$ □

The above proof shows that we may define contour integrals with respect to a curve, without referring to a specific parameterization.

If γ is only piecewise smooth, and some parameterization $z : [a, b] \to \mathbb{C}$ is smooth on $[a_k, a_{k+1}]$ for $a = a_0 < a_1 < \ldots < a_n = b$, then we define the integral to be

$$\int_\gamma f(z)\,\mathrm{d}z = \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} f(z(t))z'(t)\,\mathrm{d}t$$

Similar work as above shows that this is also independent of the parameterization.

Definition 1.38

Let $\gamma \subseteq \mathbb{C}$ be a curve parameterized by $z : [a, b] \to \mathbb{C}$. Then the **length** of γ is

$$\mathrm{length}(\gamma) = \int_a^b |z'(t)|\,\mathrm{d}t$$

The following example will be extremely instructive for later applications.

Example 1.7

Let $\gamma = \partial\mathbb{D}$. Let $f : \gamma \to \mathbb{C}$ be continuous. Let z be the parameterization

$$z(\theta) = e^{i\theta}$$

Then

$$z'(\theta) = ie^{i\theta}$$

So

$$\int_\gamma f(z)\,\mathrm{d}z = \int_0^{2\pi} f(e^{i\theta})ie^{i\theta}\,\mathrm{d}\theta$$

For instance, take

$$f(z) = \frac{1}{z}$$

Then we have

$$\int_\gamma \frac{1}{z}\,\mathrm{d}z = \int_0^{2\pi} \frac{ie^{i\theta}}{e^{i\theta}}\,\mathrm{d}\theta = \int_0^{2\pi} i\,\mathrm{d}\theta = 2\pi i$$

The length of γ is

$$\int_0^{2\pi} \left|ie^{i\theta}\right|\mathrm{d}\theta = \int_0^{2\pi} \mathrm{d}\theta = 2\pi$$

as we expect.

Let us briefly cover some properties of the contour integral.

Proposition 1.34

Let $\gamma \subseteq \mathbb{C}$ be a curve, $f, g : \gamma \to \mathbb{C}$ be continuous, and $\alpha, \beta \in \mathbb{C}$. Then

1. $\int_\gamma$ is linear:
$$\int_\gamma (\alpha f + \beta g)\,\mathrm{d}z = \alpha \int_\gamma f\,\mathrm{d}z + \beta \int_\gamma g\,\mathrm{d}z$$
2. $\int_\gamma$ is reversed by orientation:
$$\int_\gamma f\,\mathrm{d}z = -\int_{-\gamma} f\,\mathrm{d}z$$
3. The following inequality holds analogous to the triangle inequality:
$$\left|\int_\gamma f\,\mathrm{d}z\right| \leq \left(\sup_{z\in\gamma}|f(z)|\right)\text{length}(\gamma)$$

Proof. 1. Follows from properties of the Riemann integral.

2. Homework (follows from reparameterizing).
3. First, we show the following:

Claim

Let $h : [a,b] \to \mathbb{C}$ be continuous. Then

$$\left|\int_a^b h(t)\,\mathrm{d}t\right| \le \int_a^b |h(t)|\,\mathrm{d}t$$

Proof. Write $\int_a^b h(t)\,\mathrm{d}t = re^{i\theta}$ for appropriate r, θ. Then

$$\begin{aligned} r = \left|\int_a^b h(t)\,\mathrm{d}t\right| &= e^{-i\theta}\int_a^b h(t)\,\mathrm{d}t \\ &= \int_a^b e^{-i\theta}h(t)\,\mathrm{d}t = \mathrm{Re}\left(\int_a^b e^{-i\theta}h(t)\,\mathrm{d}t\right) \\ &= \int_a^b \mathrm{Re}(e^{-i\theta}h(t))\,\mathrm{d}t \le \int_a^b |h(t)|\,\mathrm{d}t \end{aligned}$$

■

Now, suppose that $z : [a,b] \to \mathbb{C}$ is piecewise smooth with $a = a_0 < \ldots < a_n = b$. Then by definition,

$$\begin{aligned} \left|\int_\gamma f(z)\,\mathrm{d}z\right| &= \left|\sum_{k=0}^{n-1}\int_{a_k}^{a_{k+1}} f(z(t))z'(t)\,\mathrm{d}t\right| \le \sum_{k=0}^{n-1}\left|\int_{a_k}^{a_{k+1}} f(z(t))z'(t)\,\mathrm{d}t\right| \\ &\le \sum_{k=0}^{n-1}\int_{a_k}^{a_{k+1}} |f(z(t))|\cdot|z'(t)|\,\mathrm{d}t \le \sum_{k=0}^{n-1}\int_{a_k}^{a_{k+1}}\left(\sup_{z\in\gamma}|f(z(t))|\right)|z'(t)|\,\mathrm{d}t \\ &= \left(\sup_{z\in\gamma}|f(z(t))|\right)\sum_{k=0}^{n-1}\int_{a_k}^{a_{k+1}}|z'(t)|\,\mathrm{d}t = \left(\sup_{z\in\gamma}|f(z(t))|\right)\mathrm{length}(\gamma) \end{aligned}$$

□

In vector calculus, the fundamental theorem of line integrals shows that calculation of line integrals can often be reduced to evaluating an antiderivative at its endpoints. The same is true in $\mathbb{C}$.

Definition 1.39

Let $\Omega \subseteq \mathbb{C}$ be open. Let $f : \Omega \to \mathbb{C}$. Then we say that $F : \Omega \to \mathbb{C}$ is a **primitive** of f on Ω if F is holomorphic on Ω and

$$F'(z) = f(z)$$

for all $z \in \Omega$.

Theorem 1.35

Let $\Omega \subseteq \mathbb{C}$ be open. Let $f : \Omega \to \mathbb{C}$ be continuous and suppose $F : \Omega \to \mathbb{C}$ is a primitive of f. Then for any curve $\gamma \subseteq \Omega$ joining z_1, z_2, we have

$$\int_\gamma f(z)\,\mathrm{d}z = F(z_2) - F(z_1)$$

Proof. This is effectively the Fundamental Theorem of Calculus. Take a piecewise smooth parameterization $z : [a, b] \to \mathbb{C}$ of γ and pick $a = a_0 < \ldots < a_n = b$ so that z is smooth on each subinterval. Then

$$\int_\gamma f(z)\,\mathrm{d}z = \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} f(z(t))z'(t)\,\mathrm{d}t = \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} F(z(t))'\,\mathrm{d}t$$
$$= \sum_{k=0}^{n-1} (F(z(a_{k+1})) - F(z(a_k))) = F(z(a_n)) - F(z(a_0)) = F(z_2) - F(z_1) \qquad \square$$

Corollary 1.36

Let $\Omega \subseteq \mathbb{C}$ be open, $f : \Omega \to \mathbb{C}$ continuous, and F a primitive for f on Ω. Then for any *closed* curve $\gamma \subseteq \Omega$,

$$\int_\gamma f(z)\,\mathrm{d}z = 0$$

Although the above theorem is powerful, its assumptions sometimes fail to be satisfied in subtle ways.

Example 1.8

Let $\Omega = \mathbb{C} \setminus \{0\}$. Let $f : \Omega \to \mathbb{C}$ be $f(z) = \frac{1}{z}$. We showed that

$$\int_{\partial\mathbb{D}} \frac{1}{z}\,\mathrm{d}z = 2\pi i \neq 0$$

This shows that $\frac{1}{z}$ does not have a primitive on Ω. This is an interesting contrast to the real case, where $\ln x$ is a primitive for $\frac{1}{x}$ on $\mathbb{R} \setminus \{0\}$. (This is due to the fact that 0 is a branch point for the logarithm in $\mathbb{C}$.)

Corollary 1.37

Let $\Omega \subseteq \mathbb{C}$ be a region. Let $f : \Omega \to \mathbb{C}$ be holomorphic on Ω and $f'(z) = 0$ for all $z \in \Omega$. Then f is constant.

Proof. Pick $z_1, z_2 \in \Omega$. Ω is path connected since it is a region, so let $\gamma \subseteq \mathbb{C}$ be a curve from

z_1 to z_2. f is a primitive for f', so

$$f(z_2) - f(z_1) = \int_\gamma f'(z)\,\mathrm{d}z = \int_\gamma 0\,\mathrm{d}z = 0$$

and thus $f(z_2) = f(z_1)$. So f is constant. $\square$

Chapter 2

Cauchy's Theorem and Applications

Up until this point, the results we have shown about contour integrals have been analogous to line integrals in $\mathbb{R}^n$. In this chapter, we will see that integrating in the complex setting results in unique results. The main theorem we will prove is as follows:

Theorem: Cauchy's Theorem for a Disk

Let $f : \mathbb{D} \to \mathbb{C}$ be holomorphic on $\mathbb{D}$. Then for any closed curve $\gamma \subseteq \mathbb{D}$,

$$\int_\gamma f(z)\,\mathrm{d}z = 0$$

In order to prove that, we will first prove some intermediate results.

2.1 Goursat's Theorem

We will first prove a version of Cauchy's Theorem for a specific type of curve.

Definition 2.1

A **triangle** T is a subset of $\mathbb{C}$ which consists of three line segments and the region between them. The boundary of T is the line segments between them, which by convention is taken with the counterclockwise parameterization.

Theorem 2.1: Cauchy-Goursat

Let $\Omega \subseteq \mathbb{C}$ be open. Let $f : \Omega \to \mathbb{C}$ be holomorphic on Ω, and let $T \subseteq \Omega$ be a triangle. Then

$$\int_{\partial T} f(z)\,\mathrm{d}z = 0$$

Proof. Let us bisect the line segments of the triangle and draw a triangle between them. This creates four subtriangles, which we denote S_1, S_2, S_3, S_4.

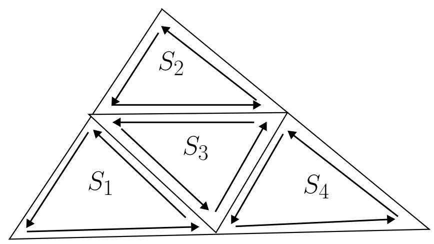


Figure 2.1: Subdivision of T

Each triangle made this way is similar to the original. Then consider the sum

$$\int_{\partial S_1} f(z)\,\mathrm{d}z + \int_{\partial S_2} f(z)\,\mathrm{d}z + \int_{\partial S_3} f(z)\,\mathrm{d}z + \int_{\partial S_4} f(z)\,\mathrm{d}z$$

Since we maintain the counterclockwise orientation, the edges of S_3 will be integrated along in both directions, which cancels out. Thus the above expression is equal to

$$\int_{\partial T} f(z)\,\mathrm{d}z$$

Thus

$$\left|\int_{\partial T} f(z)\,\mathrm{d}z\right| \leq 4 \max\left\{\left|\int_{\partial S_i} f(z)\,\mathrm{d}z\right|\right\}$$

Suppose that the largest integral occurs over S_j. Then set $T_1 = S_j$ and subdivide again.

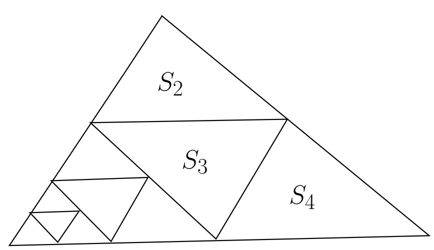


Figure 2.2: Continued subdivisions

Thus we get a nested sequence of triangles satisfying

$$\begin{cases} T = T_0 \supseteq T_1 \supseteq \dots \\ \left|\int_{\partial T_n} f(z)\,\mathrm{d}z\right| \le 4\left|\int_{\partial T_{n+1}} f(z)\,\mathrm{d}z\right| \\ \operatorname{diam}(T_{n+1}) = \frac{1}{2}\operatorname{diam}(T_n) \\ \operatorname{length}(\partial T_{n+1}) = \frac{1}{2}\operatorname{length}(\partial T_n) \end{cases}$$

It follows that

$$\begin{cases} \left|\int_{\partial T} f(z)\,\mathrm{d}z\right| \le 4^n\left|\int_{\partial T_n} f(z)\,\mathrm{d}z\right| \\ \operatorname{diam}(T_n) = \frac{1}{2^n}\operatorname{diam}(T) \\ \operatorname{length}(\partial T_n) = \frac{1}{2^n}\operatorname{length}(\partial T) \end{cases}$$

Since the diameters go to 0 and each T_i is compact, there exists a unique point ω in each of the T_i. Since f is holomorphic at ω, we can write

$$f(z) = f(\omega) + f'(\omega)(z-\omega) + \psi(z)(z-\omega)$$

where $\lim_{z\to\omega}\psi(z) = 0$. So for any n,

$$\int_{\partial T_n} f(z)\,\mathrm{d}z = \int_{\partial T_n} (f(\omega) + f'(\omega)(z-\omega))\,\mathrm{d}z + \int_{\partial T_n} \psi(z)(z-\omega)\,\mathrm{d}z$$

But notice that

$$\left(f(\omega)z + \frac{1}{2}f'(\omega)(z-\omega)^2\right)' = f(\omega) + f'(\omega)(z-\omega)$$

so we have a primitive, and it follows that

$$\int_{\partial T_n} (f(\omega) + f'(\omega)(z-\omega))\,\mathrm{d}z = 0$$

Now,

$$\begin{aligned} \left|\int_{\partial T} f(z)\,\mathrm{d}z\right| &\le 4^n\left|\int_{\partial T_n} f(z)\,\mathrm{d}z\right| \\ &= 4^n\left|\int_{\partial T_n} \psi(z)(z-\omega)\,\mathrm{d}z\right| \\ &\le 4^n\left(\sup_{z\in\partial T_n}[\psi(z)(z-\omega)]\right)\operatorname{length}(\partial T_n) \\ &= 2^n\operatorname{length}(\partial T)\left(\operatorname{diam}(T_n)\sup_{z\in\partial T_n}\psi(z)\right) \\ &= \operatorname{length}(\partial T)\operatorname{diam}(T)\sup_{z\in\partial T_n}\psi(z) \end{aligned}$$

Since

$$\lim_{z \to \omega} \psi(z) = 0$$

we can make $\sup_{z \in \partial T_n} \psi(z)$ arbitrarily small by considering large enough n. It follows that

$$\lim_{n \to \infty} \sup_{z \in \partial T_n} \psi(z) = 0$$

and thus

$$\left| \int_{\partial T} f(z) \, \mathrm{d}z \right| = 0$$

□

We will now show that holomorphic functions locally have primitives. We will adopt the notation that for $z, \omega \in \mathbb{C}$, $[z, \omega]$ represents the line segment joining z and ω. Specifically, it can be parameterized by

$$t \mapsto (1-t)z + t\omega, \quad t \in [0,1]$$

Theorem 2.2

Let $z_0 \in \mathbb{C}$, $r > 0$, and $f : \mathbb{D}_r(z_0) \to \mathbb{C}$ be holomorphic on $\mathbb{D}_r(z_0)$. Then f has a primitive on $\mathbb{D}_r(z_0)$.

Proof. Take some $z \in \mathbb{D}_r(z_0)$. Define

$$F(z) = \int_{[z_0, z]} f(\omega) \, \mathrm{d}\omega$$

We claim that F is holomorphic on $\mathbb{D}_r(z_0)$ and that it is a primitive for f. Let h be small and consider the triangle T_h between $z_0, z, z+h$.

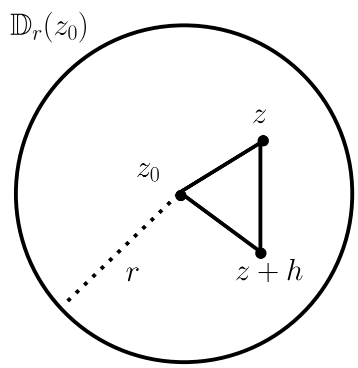


Figure 2.3: Auxiliary triangle construction

By Cauchy-Goursat,

$$0 = \int_{\partial T_h} f(\omega) \, \mathrm{d}\omega = \int_{[z_0, z]} f(\omega) \, \mathrm{d}\omega + \int_{[z, z+h]} f(\omega) \, \mathrm{d}\omega - \int_{[z_0, z+h]} f(\omega) \, \mathrm{d}\omega$$

Thus

$$F(z+h) - F(z) = F(z+h) - F(z) + \int_{\partial T_h} f(\omega)\,\mathrm{d}\omega = \int_{[z,z+h]} f(\omega)\,\mathrm{d}\omega$$
$$= h\int_0^1 f((1-t)z + t(z+h))\,\mathrm{d}t$$

and

$$\frac{F(z+h)-F(z)}{h} - f(z) = \int_0^1 (f((1+t)z + t(z+h)) - f(z)\,\mathrm{d}t)$$

Since f is continuous at z, for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $|\omega - z| < \delta$, then $|f(\omega) - f(z)| < \varepsilon$. So if $|h| < \delta$, then

$$\left|\frac{F(z+h)-F(z)}{h} - f(z)\right| \leq \int_0^1 |f((1-t)z + t(z+h)) - f(z)|\,\mathrm{d}t < \varepsilon$$

Thus $F' = f$ and f has a primitive. □

Later, we will see that holomorphic functions can all be represented by power series, which makes the above trivial by integrating term-by-term.

We now have the tools necessary to prove Cauchy's Theorem on a disk.

Corollary 2.3: Cauchy's Theorem on a Disk

If $f : \mathbb{D}_r(z_0) \to \mathbb{C}$ is holomorphic than for any closed curve $\gamma \subseteq \mathbb{D}_r(z_0)$,

$$\int_\gamma f(z)\,\mathrm{d}z = 0$$

Proof. By Theorem 2.2, f has a primitive, so the integral is zero. □

Corollary 2.4

If $\Omega \subseteq \mathbb{C}$ is open and contains $\overline{\mathbb{D}_r}(z_0)$, if $f : \Omega \to \mathbb{C}$ is holomorphic, then

$$\int_{\partial D_r(z_0)} f(z)\,\mathrm{d}z = 0$$

Proof. It suffices to pick r' such that $\mathbb{D}_{r'}(z_0) \supseteq \overline{\mathbb{D}_r}(z_0)$. To show that such an r' exists, suppose that it does not. Then for any $n \in \mathbb{N}$ there exists $z_n \in D_{r+\frac{1}{n}}(z_0)$ such that $z_n \in \mathbb{C} \setminus \Omega$. So $\{z_n\}$ has a convergent subsequence $\{z_{n_k}\}$, which converges to a limit $z \notin \Omega$ since $\mathbb{C} \setminus \Omega$ is closed. But $|z_n - z_0| < r + \frac{1}{n}$, which means that $|z - z_0| \leq r$. Thus $z \in \overline{\mathbb{D}_r}(z_0)$, contradiction. □

2.2 Homotopies and Simply Connected Domains

We will now take a short detour into some topology in order to investigate which other sets Cauchy's Theorem applies to. For some intuition, consider an open set Ω and two curves in it which share endpoints. Then if Ω is particularly nice, we should be able to continuously deform one curve into the other.

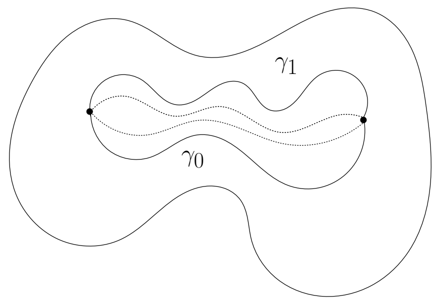


Figure 2.4: Continuous deformation of curves

We formalize this notion by considering a continuous family of curves which deform γ_0 into γ_1.

Definition 2.2

Let $\Omega \subseteq \mathbb{C}$ be open, $\alpha, \beta \in \Omega$, and $\gamma_0, \gamma_1 : [a, b] \to \Omega$ be two curves with the same endpoints, such that $\gamma_1(a) = \gamma_2(a) = \alpha$ and $\gamma_1(b) = \gamma_2(b) = \beta$. We say that γ_1, γ_2 are **homotopic** in Ω if for every $s \in [0, 1]$ there is a curve $\gamma_s : [a, b] \to \Omega$ with $\gamma_s(a) = \alpha, \gamma_s(b) = \beta$ such that the function $H : [0, 1] \times [a, b] \to \Omega$ defined by

$$H(s, t) = \gamma_s(t)$$

is continuous as a function of two variables.

Theorem 2.5

Let $\Omega \subseteq \mathbb{C}$ be open, and let $f : \Omega \to \mathbb{C}$ be holomorphic on Ω. If $\gamma_0, \gamma_1 : [a, b] \to \Omega$ are curves with the same endpoints that are homotopic in Ω, then

$$\int_{\gamma_0} f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z$$

Proof. We claim that there exists $\varepsilon > 0$ such that for any $s \in [0, 1]$ and $t \in [a, b]$, then

$$\mathbb{D}_{2\varepsilon}(\gamma_s(t)) \subseteq \Omega$$

Suppose not. Then for every $n \in \mathbb{N}$, there are $s_n \in [0,1]$ and $t_n \in [a,b]$ and $\omega_n \in \mathbb{C} \setminus \Omega$ such that

$$\omega_n \in \mathbb{D}_{2/n}(\gamma_{s_n}(t_n))$$

Now, the sequence $(s_n, t_n) \in [0,1] \times [a,b]$ is a sequence in a compact set, so there is a subsequence $\{(s_{n_k}, t_{n_k})\}$ tending to (s,t). But then

$$\left|\omega_{n_k} - \gamma_{s_{n_k}}(t_{n_k})\right| < \frac{2}{n_k}$$

which tends to 0. Since $\mathbb{C} \setminus \Omega$ is closed, $\lim \omega_{n_k} \in \mathbb{C} \setminus \Omega$. But $\omega_{n_k} \to \gamma_s(k)$ is the limit of $H(s_{n_k}, t_{n_k})$, which is continuous, and thus $\gamma_s(t) \in \Omega$, contradiction. So the claim is proved and such an ε exists.

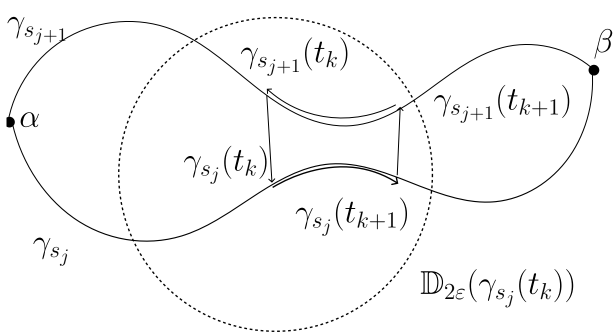
Now, note that $H(s,t)$ is continuous on the compact set $[0,1] \times [a,b]$, so it is also uniformly continuous. So for $\varepsilon > 0$ which is produced by the claim, there exists $\delta > 0$ such that if $|s_1 - s_2| < \delta$ and $|t_1 - t_2| < \delta$, then

$$|\gamma_{s_1}(t_1) - \gamma_{s_2}(t_2)| < \varepsilon$$

Then subdivide $[0,1] \times [a,b]$ using $0 = s_0 < s_1 < \ldots < s_n = 1$ and $a = t_0 < \ldots < t_n = b$ such that $|s_{j+1} - s_j| < \delta$ and $|t_{j+1} - t_j| < \delta$ for all j. We claim that

$$\int_{\gamma_{s_j}} f(z)\,\mathrm{d}z = \int_{\gamma_{s_{j+1}}} f(z)\,\mathrm{d}z$$

for all j. Clearly this suffices to prove the theorem. Consider some pair of points (s_j, t_k) and (s_{j+1}, t_{k+1}). Draw a circle of radius 2ε around $\gamma_{s_j}(t_k)$. Note that this circle also contains $\gamma_{s_{j+1}}(t_k), \gamma_{s_{j+1}}(t_{k+1}), \gamma_{s_j}(t_{k+1})$.



By Cauchy's Theorem

$$\int_{\gamma_{s_j}([t_k,t_{k+1}])} f + \int_{[\gamma_{s_j}(t_{k+1}),\gamma_{s_{j+1}}(t_{k+1})]} f - \int_{\gamma_{s_{j+1}}([t_k,t_{k+1}])} f - \int_{[\gamma_{s_j}(t_k),\gamma_{s_{j+1}}(t_k)]} f = 0$$

Thus

$$\int_{\gamma_{s_j}([t_k,t_{k+1}])} f = \int_{\gamma_{s_{j+1}}([t_k,t_{k+1}])} f + \left(\int_{[\gamma_{s_j}(t_k),\gamma_{s_{j+1}}(t_k)]} f - \int_{[\gamma_{s_j}(t_{k+1}),\gamma_{s_{j+1}}(t_{k+1})]} f\right)$$

Let us write

$$a_k = \int_{\gamma_{s_j}(t_k), \gamma_{s_{j+1}(t_k)}} f$$

Then summing over all k, we see that

$$\int_{\gamma_{s_j}} f = \int_{\gamma_{s_{j+1}}} f + a_0 - a_n$$

But $\gamma_{s_j}, \gamma_{s_{j+1}}$ have the same endpoints so $a_0 = a_n = 0$. Thus the theorem is proved. $\square$

Definition 2.3

An open connected set $\Omega \subseteq \mathbb{C}$ is called **simply connected** if any curves $\gamma_0, \gamma_1 : [a, b] \to \Omega$ which share endpoints are homotopic in Ω.

Example 2.1

Any convex set, which is a set such that the line between two points in the set is contained in the set, such as the disk, is simply connected.

Example 2.2

A "star-shaped" set, which is a set with a center point such that any two points can be joined by a curve passing through the center, is simply connected.

Example 2.3

$\mathbb{C} \setminus \{0\}$ is not simply connected. Consider the upper and lower halves of $\partial\mathbb{D}$. Intuitively, we see that one cannot be transformed to the other without passing through the origin. We will prove this later.

Theorem 2.6

Let $\Omega \subseteq \mathbb{C}$ be simply connected and $f : \Omega \to \mathbb{C}$ be holomorphic. Then f has a primitive on Ω.

Proof. Let Ω be simply connected and pick a point $z_0 \in \Omega$. For any $z \in \Omega$, choose some curve $\gamma_z \subseteq \Omega$ connecting z_0 to z. For consistency, let us demand that γ_{z_0} is a constant curve. Then define

$$F(z) := \int_{\gamma_z} f(\omega)\, \mathrm{d}\omega$$

Now, let h be small and let φ be the line segment connecting z to $z + h$. Then the curve $\gamma_z + \varphi$ (where $+$ means concatenation) shares endpoints with γ_{z+h}, so they are homotopic. By Theorem 2.5, integrating over either gives the same value, so

$$\int_{\gamma_{z+h}} f(\omega)\, \mathrm{d}\omega = \int_{\gamma_z} f(\omega)\, \mathrm{d}\omega + \int_{\varphi} f(\omega)\, \mathrm{d}\omega$$

We can then write

$$F(z+h) = F(z) + \int_{\varphi} f(\omega)\,\mathrm{d}\omega$$

EXERCISE: conclude the proof similarly to the proof of local existence of primitives. □

A corollary to this is a more general form of Cauchy's Theorem.

Corollary 2.7: Cauchy's Theorem

If Ω is simply connected and $f : \Omega \to \mathbb{C}$ is holomorphic, then

$$\int_{\gamma} f(z)\,\mathrm{d}z = 0$$

for every closed curve $\gamma \subseteq \Omega$.

Proof. f has a primitive, so the integral is zero. □

Corollary 2.8

$\mathbb{C} \setminus \{0\}$ is not simply connected.

Proof. We showed that $\int_{\partial\mathbb{D}} \frac{1}{z}\,\mathrm{d}z = 2\pi i \neq 0$. □

In fact, there are stronger versions of this theorem. We will not formally prove them here, but we will give an intuitive explanation.

Definition 2.4

A curve γ is **simple** if, given some parameterization $z : [a,b] \to \mathbb{C}$, $z(s) = z(t) \implies s = t$. A **simple closed** curve is the same, except that $z(a) = z(b)$.

Note that a simple closed curve is not technically simple, but the intuitive idea is the same.

Theorem: Jordan Curve Theorem

Let $\gamma \subseteq \mathbb{C}$ be a simple closed curve. Then $\mathbb{C} \setminus \gamma = \Omega \cup U$, where Ω, U are open, connected, and disjoint. Moreover, Ω is bounded and simply connected, and U is unbounded and connected. In this case, Ω is called the **interior** of γ (denoted $\operatorname{int}\gamma$), and U is called the **exterior** (denoted $\operatorname{ext}\gamma$.)

Proof. The proof of this theorem is omitted. Note that it is true in the piecewise smooth case (see Stein & Shakarchi appendix), but it also true for continuous curves. □

Theorem: General Cauchy's Theorem

Let $\Omega \subseteq \mathbb{C}$ be open and let $f : \Omega \to \mathbb{C}$ be holomorphic on Ω. Let $\gamma \subseteq \mathbb{C}$ be a simple closed curve in Ω such that $\operatorname{int}\gamma \subseteq \Omega$. Then

$$\int_\gamma f(z)\,\mathrm{d}z = 0$$

Proof. Draw a curve inside of $\operatorname{int}\gamma$ which is a slight pertubation of γ and is homotopic to γ. Then this follows since $\operatorname{int}\gamma$ is simply connected. See Stein & Shakarchi appendix for a complete proof. $\square$

2.3 Consequences of Cauchy's Theorem

Aside from being a powerful theorem about complex integrals, Cauchy's Theorem also allows us to solve many difficult real integrals using complex integrals.

Example 2.4

Consider the integral

$$\int_0^\infty \frac{1-\cos x}{x^2}\,\mathrm{d}x$$

Note that this is integrable since

$$\lim_{x\to\infty}\left|\frac{1-\cos x}{x^2}\right| \leq \lim_{x\to\infty}\frac{2}{x^2}$$

and near 0, this behaves like 0. So the integral is indeed a real number.

Now, consider the function $f : \mathbb{C}\setminus\{0\} \to \mathbb{C}$ defined by

$$f(z) = \frac{1-e^{iz}}{z^2}$$

which is holomorphic. For each $R > 0, \varepsilon > 0$, consider the integral of $f(z)$ over $\Gamma_{R,\varepsilon}$, which is defined below as $\gamma_R + [-R, -\varepsilon] + \gamma_\varepsilon + [\varepsilon, R]$:

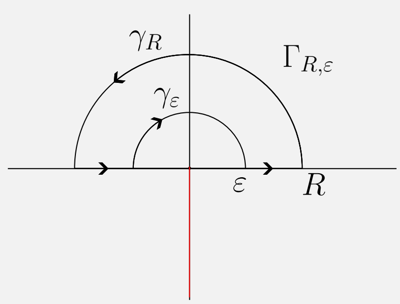


Figure 2.5: Indented semicircle

This function is holomorphic on the star-shaped domain $\mathbb{C} \setminus \{it : t \le 0\}$ (which is simply connected), so

$$\begin{aligned}
0 &= \int_{\Gamma_{R,\varepsilon}} f(z)\,\mathrm{d}z \\
&= \underbrace{\int_{-R}^{-\varepsilon} \frac{1-e^{-ix}}{x^2}\,\mathrm{d}x}_{\mathrm{I}} + \underbrace{\int_{\gamma_\varepsilon} \frac{1-e^{iz}}{z^2}\,\mathrm{d}z}_{\mathrm{II}} + \underbrace{\int_{\varepsilon}^{R} \frac{1-e^{ix}}{x^2}\,\mathrm{d}x}_{\mathrm{III}} + \underbrace{\int_{\gamma_R} \frac{1-e^{iz}}{z^2}\,\mathrm{d}z}_{\mathrm{IV}}
\end{aligned}$$

Now,

$$\mathrm{I} + \mathrm{III} = \int_\varepsilon^R \frac{2-(e^{ix}+e^{-ix})}{x^2}\,\mathrm{d}x = 2\int_\varepsilon^R \frac{1-\cos x}{x^2}\,\mathrm{d}x$$

So the problem reduces to calculating II and IV. We claim that

$$\lim_{\varepsilon\to 0} \int_{\gamma_\varepsilon^-} \frac{1-e^{iz}}{z^2}\,\mathrm{d}z - \pi = 0$$

To see this, parameterize γ_ε^- as $t \mapsto \varepsilon e^{it}$ for $t \in [0,\pi]$. Then

$$\begin{aligned}
\int_{\gamma_\varepsilon^-} \frac{1-e^{iz}}{z^2}\,\mathrm{d}z - \pi &= \int_0^\pi \frac{1-e^{i\varepsilon e^{it}}}{\varepsilon^2 e^{2it}}(\varepsilon i e^{it})\,\mathrm{d}t - \pi \\
&= \frac{i}{\varepsilon}\int_0^\pi \left(\frac{1-e^{i\varepsilon e^{it}}}{e^{it}} + i\varepsilon\right)\mathrm{d}t \\
&= \frac{i}{\varepsilon}\int_0^\pi \frac{1-e^{i\varepsilon e^{it}}+i\varepsilon e^{it}}{e^{it}}\,\mathrm{d}t
\end{aligned}$$

We may write

$$e^{i\varepsilon e^{it}} = \sum_{n=0}^\infty \frac{\left(i\varepsilon e^{it}\right)^n}{n!}$$

so

$$\left|e^{i\varepsilon e^{it}} - (1+i\varepsilon e^{it})\right| \le \sum_{n=2}^\infty \frac{\left|i\varepsilon e^{it}\right|^n}{n!} = \sum_{n=2}^\infty \frac{\varepsilon^n}{n!} \le 10\varepsilon^2$$

(10 is an overbound here). Returning to the original integral,

$$\left|\frac{i}{\varepsilon}\int_0^\pi \frac{1-e^{i\varepsilon e^{it}}+i\varepsilon e^{it}}{e^{it}}\,\mathrm{d}t\right| \le \frac{1}{\varepsilon}\pi 10\varepsilon^2 = 10\pi\varepsilon$$

which tends to 0 as $\varepsilon \to 0$.

Lastly, let us compute IV. We parameterize γ_R by $t \mapsto Re^it$ on $[0,\pi]$, and see that

$$\left|\int_{\gamma_R} \frac{1-e^{iz}}{z^2}\,\mathrm{d}z\right| = \left|\int_0^\pi \frac{1-e^{iRe^{it}}}{R^2 e^{2it}}\right| iRe^{it}\,\mathrm{d}t$$

Observe that

$$\begin{aligned}\left|1-e^{iRe^{it}}\right| &= \left|1-e^{iR(\cos t+i\sin t)}\right| \\ &= \left|1-e^{iR\cos t}e^{-R\sin t}\right| \\ &\leq 1+e^{R\sin t} \\ &\leq 2\end{aligned}$$

where the last inequality follows since $\sin t > 0$ on $[0,\pi]$. Returning to IV,

$$\left|\int_0^\pi \frac{1-e^{iRe^{it}}}{R^2e^{2it}}\right| iRe^{it}\,\mathrm{d}t \leq \frac{1}{R}2\pi$$

which tends to 0 as $R\to\infty$. Then

$$\int_0^\infty \frac{1-\cos x}{x^2}\,\mathrm{d}x = \frac{1}{2}(\mathrm{II}+\mathrm{IV}) = \frac{\pi}{2}$$

The following theorem is an important formula which is a powerful example of the local-global properties present in complex analysis.

Theorem 2.9: Cauchy's Integral Formula

Let $\Omega\subseteq\mathbb{C}$ be open, $\overline{\mathbb{D}_R}(z_0)\subseteq\Omega$, and let $f:\Omega\to\mathbb{C}$ be holomorphic on Ω. Then for $z\in\mathbb{D}_R(z_0)$,

$$f(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}_R(z_0)} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta$$

Proof. Define

$$F(\zeta) = \frac{f(\zeta)}{\zeta - z}$$

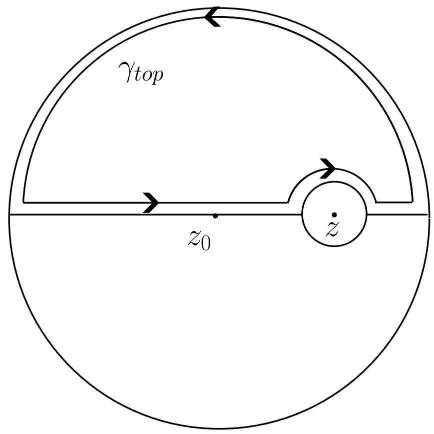
Define the curve γ_{top} as shown in the diagram below:

Define γ_{bottom} similarly, using the counterclockwise orientation so that the integrals over the line segment cancel. Both are holomorphic on star-shaped regions, so their integrals are zero and

$$\begin{aligned}0 &= \frac{1}{2\pi i}\left(\int_{\gamma_{top}} F(\zeta)\,\mathrm{d}\zeta + \int_{\gamma_{bottom}} F(\zeta)\,\mathrm{d}\zeta\right) \\ &= \frac{1}{2\pi i}\left(\int_{\partial\mathbb{D}_R(z_0)} \frac{f(\zeta)}{\zeta-z}\,\mathrm{d}\zeta - \int_{\partial\mathbb{D}_\varepsilon(z)} \frac{f(\zeta)}{\zeta-z}\,\mathrm{d}\zeta\right)\end{aligned}$$

so

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}_R(z_0)} \frac{f(\zeta)}{\zeta-z}\,\mathrm{d}\zeta = \frac{1}{2\pi i}\int_{\partial\mathbb{D}_\varepsilon(z)} \frac{f(\zeta)}{\zeta-z}\,\mathrm{d}\zeta$$



We want to show that the right side quantity is equal to $f(z)$. Calculating,

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}_\varepsilon(z)}\frac{f(\zeta)}{\zeta-z}\,\mathrm{d}\zeta = \frac{1}{2\pi i}\int_{\partial\mathbb{D}_\varepsilon(z)}\left(\frac{f(z)}{\zeta-z}+\frac{f(\zeta)-f(z)}{\zeta-z}\right)\mathrm{d}\zeta$$

Notice that the first term of the integral is the integral of $\frac{1}{z}$ on a circle around 0, but translated and multiplied by $f(z)$. So this becomes

$$f(z)+\frac{1}{2\pi i}\int_{\partial\mathbb{D}_\varepsilon(z)}\frac{f(\zeta)-f(z)}{\zeta-z}\,\mathrm{d}\zeta$$

The integrand is bounded since f is holomorphic, so it vanishes as $\varepsilon\to 0$. Thus

$$f(z)=\frac{1}{2\pi i}\int_{\partial\mathbb{D}_R(z_0)}\frac{f(\zeta)}{\zeta-z}\,\mathrm{d}\zeta \qquad \square$$

We now arrive at perhaps the most important theorem in all of complex analysis. This theorem shows that holomorphic functions are also analytic. We showed that analytic functions are holomorphic; we now see that they are the same. Moreover, we see that all holomorphic functions are infinitely differentiable.

Theorem 2.10

Let $\Omega\subseteq\mathbb{C}$ be open, and suppose $\overline{\mathbb{D}_R}(z_0)\subseteq\Omega$. Suppose $f:\Omega\to\mathbb{C}$ is holomorphic on Ω. Then for every $z\in\mathbb{D}_R(z_0)$,

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n$$

where

$$a_n=\frac{1}{2\pi i}\int_{\partial\mathbb{D}_R(z_0)}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\,\mathrm{d}\zeta$$

In particular, the radius of convergence is at least R.

Proof. By the Cauchy Integral formula, when $|z-z_0|<R$ we have

$$f(z)=\frac{1}{2\pi i}\int_{\partial\mathbb{D}_R(z_0)}\frac{f(\zeta)}{\zeta-z}\,\mathrm{d}z=\frac{1}{2\pi i}\int_{\partial\mathbb{D}_R(z_0)}\frac{f(\zeta)}{\zeta-z_0}\cdot\frac{1}{1-\frac{z-z_0}{\zeta-z_0}}\,\mathrm{d}\zeta$$

Now, if $|z-z_0|=r<R$, then $\left|\frac{z-z_0}{\zeta-z_0}\right|=\frac{r}{R}$ (since ζ is on the boundary of $\mathbb{D}_R(z_0)$). So we may write this as the sum of a geometric series:

$$\begin{aligned}\frac{1}{2\pi i}\int_{\partial\mathbb{D}_R(z_0)}\frac{f(\zeta)}{\zeta-z_0}\cdot\frac{1}{1-\frac{z-z_0}{\zeta-z_0}}\,\mathrm{d}\zeta&=\frac{1}{2\pi i}\int_{\partial\mathbb{D}_R(z_0)}\frac{f(\zeta)}{\zeta-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{\zeta-z_0}\right)^n\mathrm{d}\zeta\\&=\frac{1}{2\pi i}\int_{\partial\mathbb{D}_R(z_0)}\left(\sum_{n=0}^{\infty}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}(z-z_0^n)\right)\mathrm{d}\zeta\end{aligned}$$

Thus the proof reduces to justifying the interchange of the sum and integral above. To prove this, for $N\in\mathbb{N}$ we have

$$\begin{aligned}&\frac{1}{2\pi i}\left[\int_{\partial\mathbb{D}_R(z_0)}\left(\sum_{n=0}^{\infty}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}(z-z_0)^n\right)\mathrm{d}\zeta-\underbrace{\int_{\partial\mathbb{D}_R(z_0)}\left(\sum_{n=0}^{N}\frac{f(\zeta)}{(\zeta-z)^{n+1}}(z-z_0)^n\right)\mathrm{d}\zeta}_{=0}\right]\\&\qquad=\frac{1}{2\pi i}\int_{\partial\mathbb{D}_R(z_0)}\left(\sum_{n=N+1}^{\infty}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}(z-z_0)^n\right)\mathrm{d}\zeta\end{aligned}$$

So

$$\begin{aligned}\left|f(z)-\sum_{n=0}^{N}a_n(z-z_0)^n\right|&=\left|\frac{1}{2\pi i}\int_{\partial\mathbb{D}_R(z_0)}\left(\sum_{n=N+1}^{\infty}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}(z-z_0)^n\right)\mathrm{d}\zeta\right|\\&\le\frac{1}{2\pi}2\pi R\sum_{n=N+1}^{\infty}\frac{r^n}{R^{n+1}}\sup_{\zeta\in\partial\mathbb{D}_R(z_0)}|f(\zeta)|\le\left(\sup_{\zeta\in\partial\mathbb{D}_R(z_0)}|f(\zeta)|\right)\sum_{n=N+1}^{\infty}\left(\frac{r}{R}\right)^n\\&=\left(\sup_{\zeta\in\partial\mathbb{D}_R(z_0)}|f(\zeta)|\right)\cdot\left(\frac{r}{R}\right)^{N+1}\frac{1}{1-\frac{r}{R}}\xrightarrow{N\to\infty}0\end{aligned}$$

□

We can use this to extend Cauchy's integral formula to higher order derivatives, still exhibiting local-global behavior:

Theorem 2.11: Cauchy integral formulas

Let $\Omega\subseteq\mathbb{C}$ be open, and let $f:\Omega\to\mathbb{C}$ be holomorphic on Ω. Let $\overline{\mathbb{D}_R}(z_0)\subseteq\Omega$. Then f has n complex derivatives for every $n\ge0$ and for $z\in\mathbb{D}_R(z_0)$,

$$f^{(n)}(z)=\frac{n!}{2\pi i}\int_{\partial\mathbb{D}_R(z_0)}\frac{f(\zeta)}{(\zeta-z)^{n+1}}\,\mathrm{d}\zeta$$

Note that this is equivalent to Theorem 2.9 for $n=0$.

Proof. From Theorem 2.10, we can represent f as a power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

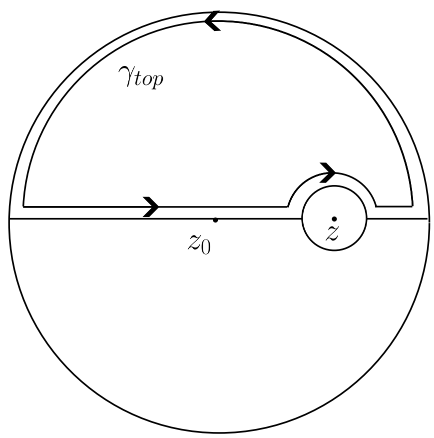
moreover, we know that we may differentiate the series term by term to see that

$$f^{(n)}(z) = \sum_{k=n}^{\infty} \frac{n!}{(n-k)!} a_k (z - z_0)^{k-n}$$

Evaluating at z_0, all the terms cancel out except $k = n$, so

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\partial \mathbb{D}_R(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, \mathrm{d}\zeta$$

Now, we use the same contours $\gamma_{top}, \gamma_{bottom}$ as in the original Cauchy Integral formula proof:



Defining $F(\zeta) = \frac{f(\zeta)}{(\zeta - z)^{n+1}}$, which is holomorphic on the simply connected domains containing γ_{top} and γ_{bottom}, we have

$$\frac{1}{2\pi i} \int_{\partial \mathbb{D}_R(z_0)} \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, \mathrm{d}\zeta = \frac{1}{2\pi i} \int_{\partial \mathbb{D}_\varepsilon(z)} \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, \mathrm{d}\zeta = \frac{f^{(n)}(z)}{n!}$$

The last equality follows by our above work, since z is the center of $\mathbb{D}_\varepsilon(z)$. $\square$

As an easy corollary we have:

Corollary 2.12: Cauchy Inequality

Let $\Omega \subseteq \mathbb{C}$ be open, $f : \Omega \to \mathbb{C}$ holomorphic on Ω, and $\overline{\mathbb{D}_R}(z_0) \subseteq \Omega$. Then

$$\left| f^{(n)}(z_0) \right| \leq \frac{n!}{R^n} \sup_{\zeta \in \partial \mathbb{D}_R(z_0)} |f(\zeta)|$$

Proof. By the previous theorem,

$$\left|f^{(n)}(z_0)\right| = \left|\frac{n!}{2\pi i}\int_{\partial\mathbb{D}_R(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\,\mathrm{d}\zeta\right| \leq \frac{n!}{2\pi i} 2\pi R \cdot \frac{1}{R^{n+1}} \sup_{\zeta\in\partial\mathbb{D}_R(z_0)} |f(\zeta)| \qquad \square$$

Theorem 2.13: Liouville's Theorem

If $f : \mathbb{C} \to \mathbb{C}$ is entire and bounded, then f is constant.

Proof. Let $z_0 \in \mathbb{C}$. Let $|f(\zeta)| \leq M$ for all $\zeta \in \mathbb{C}$. Then as $R \to \infty$ we have

$$|f'(z_0)| \leq \frac{1}{R} M \to 0$$

so $f' \equiv 0$ and f is constant. $\square$

As a corollary to this we may prove the Fundamental Theorem of Algebra:

Corollary 2.14: Fundamental Theorem of Algebra

Let $p \in \mathbb{C}[x]$ be non-constant. Then p has a zero in $\mathbb{C}$.

Proof. Suppose p has no root. Then $\frac{1}{p}$ is entire. We write

$$\left|\frac{1}{p(z)}\right| = \frac{1}{|z^n|} \cdot \frac{1}{\left|a_n + \frac{a_{n-1}}{z} + \ldots + \frac{a_0}{z^n}\right|}$$

(we may assume $a_0 \neq 0$ and $n \geq 1$). We bound the denominator from below for large z by

$$\left|a_n + \frac{a_{n-1}}{z} + \ldots + \frac{a_0}{z^n}\right| \geq |a_n| - \frac{|a_{n-1}|}{|z|} - \ldots - \frac{|a_0|}{|z|^n}$$

so

$$\left|\frac{1}{p(z)}\right| \leq \frac{1}{|z|^n} \cdot \frac{1}{|a_n| - \frac{|a_{n-1}|}{|z|} - \ldots - \frac{|a_0|}{|z|^n}} \xrightarrow{|z|\to\infty} 0$$

Thus we pick $R > 0$ such that $\frac{1}{|p(z)|} \leq 1$ when $|z| > R$. On $\overline{\mathbb{D}_R}(z_0)$, $\frac{1}{p(z)}$ is continuous and thus bounded. So $\frac{1}{p}$ is bounded everywhere and thus constant by Liouville's Theorem. But we supposed p was not constant, so p must have a root. $\square$

We now provide a proof of the method known as analytic continuation.

Definition 2.5

Let $\Omega \subseteq \mathbb{C}$ be open and let $\Omega' \subseteq \Omega$ be open. Let $f : \Omega' \to \mathbb{C}$ and $F : \Omega \to \mathbb{C}$ be holomorphic, with $F|_{\Omega'} = f$. Then we call F an **analytic continuation** of f.

The key result is that, under minor restrictions, this choice of F is actually unique; that is, the value of f on a small set completely determine its extension to Ω.

Theorem 2.15: Uniqueness of Analytic Continuation

Let $\Omega \subseteq \mathbb{C}$ be open and connected, and consider some collection of distinct points $\{\omega_n\}_{n=1}^{\infty} \subseteq \Omega$ such that $\lim \omega_n = \omega \in \Omega$ exists. Suppose that $f : \Omega \to \mathbb{C}$ is holomorphic and the ω_i are zeroes of f. Then $f \equiv 0$ on Ω.

Proof. Let $A = \{z \in \Omega : f(z) = 0\}$. Let $U = \operatorname{int} A$. I claim the following:

Claim 1: U is nonempty.

Claim 2: U is closed in Ω.

The theorem follows from the above, as if they are true, we have $\Omega = U \cup (\Omega \setminus U)$. By Claim 2, these are disjoint open sets, so one must be nonempty. By Claim 1, it must be $\Omega \setminus U$ so $U = \Omega$.

To prove these claims, we show the following:

Claim

If $\{\zeta_n\} \subseteq A$ are distinct and $\zeta_n \to \zeta_0 \in \Omega$, then there is $R > 0$ such that f vanishes on $\mathbb{D}_R(\zeta_0)$.

Proof. Ω is open and $\zeta_0 \in \Omega$, so there exists $R > 0$ such that $\overline{\mathbb{D}_R}(\zeta_0) \subseteq \Omega$. Since f is holomorphic, it is analytic, so we may write

$$f(\zeta) = \sum_{n=0}^{\infty} a_n (\zeta - \zeta_0)^n$$

on $\mathbb{D}_R(\zeta_0)$. For large enough n, $\zeta_n \in \mathbb{D}_R(\zeta_0)$, and thus

$$f(\zeta_0) = \lim f(\zeta_n) = 0$$

so $a_0 = 0$. If all the a_i are zero, then we are done. Suppose not. Then let $m \geq 1$ be the smallest index such that $a_m \neq 0$. Then

$$f(\zeta) = a_m(\zeta - \zeta_0)^m \left(1 + \underbrace{\frac{a_{m+1}}{a_m}(\zeta - \zeta_0) + \frac{a_{m+2}}{a_m}(\zeta - \zeta_0)^2 + \dots}_{=g(\zeta)} \right)$$

Now, $g(\zeta)$ is analytic on $\mathbb{D}_R(z_0)$ as it is a power series, and $g(\zeta_0) = 0$. For large enough n we then have

$$f(\zeta_n) = a_m(\zeta_n - \zeta_0)^m (1 + g(\zeta_n))$$

By assumption, $f(\zeta_n) = 0$. But $a_m \neq 0$ by assumption. $(\zeta_n - \zeta_0)$ is only zero for at most one ζ_n, so we conclude that $1 + g(\zeta_n) = 0$ for all large enough n. As $n \to \infty$ $g(\zeta_n) \to 0$, so $1 = 0$, contradiction. Thus no a_m is nonzero and f is identically zero. ■

Now we apply the above to our $\{\omega_n\}$ and arrive at Claim 1. For Claim 2, if there is some sequence of points in U, then the Claim also shows that their limit is in U. So we are done. $\square$

Corollary 2.16

Let $f : \Omega' \to \mathbb{C}$ be holomorphic with $\Omega' \subseteq \Omega \subseteq \mathbb{C}$ both open. If Ω is connected, then there is at most one analytic continuation $F : \Omega \to \mathbb{C}$ of f.

Proof. Suppose $F_1, F_2 : \Omega \to \mathbb{C}$ are analytic continuations of f. Then $F_1 - F_2$ is zero on Ω' open, and we immediately conclude that $F_1 - F_2 \equiv 0$ on Ω. So $F_1 = F_2$. $\square$

We now prove a converse to Goursat's Theorem.

Theorem 2.17: Morera's Theorem

Let $f : \mathbb{D}_R(z_0) \to \mathbb{C}$ is continuous and for any triangle $T \subseteq \mathbb{D}_R(z_0)$ we have

$$\int_{\partial T} f(z)\,\mathrm{d}z = 0$$

then f is holomorphic on $\mathbb{D}_R(z_0)$.

Proof. We imitate the proof that holomorphic functions locally have primitives, which only assumed Goursat's Theorem. In this case we do not know that f is holomorphic but we assume the conclusion of Goursat's Theorem. Thus f has a primitive F. So F is holomorphic, and thus infinitely differentiable. In particular, $f = F'$ is holomorphic. $\square$

2.4 Sequences of Functions

In this section we develop the theory of the convergence of sequences of functions, in particular holomorphic ones. This is analogous to the discussion of sequences of functions in real variables.

Definition 2.6

Let $\Omega \subseteq \mathbb{C}$. Let $\{f_n : \Omega \to \mathbb{C}\}$ be a sequence of functions and $f : \Omega \to \mathbb{C}$. We say that $\{f_n\}$ **converges uniformly** to f (denoted $f_n \rightrightarrows f$) on a subset $A \subseteq \Omega$ if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq n, z \in A$, $|f_n(z) - f(z)| < \varepsilon$. We say that $\{f_n\}$ converges uniformly on compact subsetes if for every $K \subseteq \Omega$ compact, $f_n \rightrightarrows f$ on K.

Equivalently, $f_n \rightrightarrows f$ on compact subsets if and only if for all K, $\sup_{z \in K}|f_n(z) - f(z)| \to 0$ uniformly.

As in the real case, uniform convergence is distinguished from pointswise convergence since our N must work for all $z \in A$. Convergence on compact subsets is particularly important for us, so we should note that the N may depend on the choice of K; that is, f_n need not converge uniformly on all of Ω.

Theorem 2.18

Let $\Omega \subseteq \mathbb{C}$ be open. Let $f_n : \Omega \to \mathbb{C}$ be holomorphic and $f_n \rightrightarrows f$ on compact subsets. Then f is holomorphic on Ω.

Note that this theorem is false in the real case, if holomorphicity is replaced by, say, continuous differentiability. For instance, let $f_n : \mathbb{R} \to \mathbb{R}$ be equal to the absolute value function outside of $[-\frac{1}{n}, \frac{1}{n}]$ and be some smooth interpolating function between them. This is continuously differentiable and the convergence is uniform, but the limit is the absolute value function which is not even differentiable.

Proof. We prove this using Morera's theorem. Fix $z_0 \in \Omega$ and let $\overline{\mathbb{D}_r}(z_0) \subseteq \Omega$. Recall from real analysis that the uniform limit of continuous function is continuous. Applying this to the real and imaginary parts of f, f is also continuous. By Morera's theorem, it is enough to show that for any triangle $T \subseteq \mathbb{D}_r(z_0)$, the integral of f along ∂T vanishes. f_n is holomorphic, so

$$\begin{aligned}\left|\int_{\partial T} f(z)\,\mathrm{d}z\right| &= \left|\int_{\partial T} f(z)\,\mathrm{d}z - \int_{\partial T} f_n(z)\,\mathrm{d}z\right| \\ &\le \operatorname{length}(\partial T) \sup_{z\in\partial T} |f(z) - f_n(z)|\end{aligned}$$

As $n \to \infty$, $\sup_{z\in\partial T}|f(z) - f_n(z)| \to 0$ and $\operatorname{length}(\partial T)$ is constant, so

$$\int_{\partial T} f(z)\,\mathrm{d}z = 0$$

and by Morera's f is holomorphic. □

We use the above result to prove a related result which essentially says that we may interchange the derivative and limit operators:

Theorem 2.19

Let $\Omega \subseteq \mathbb{C}$ be open and let $f_n : \Omega \to \mathbb{C}$ be holomorphic with $f_n \rightrightarrows f$ on compact subsets. Then $f_n' \rightrightarrows f'$ on compact subsets.

Proof. To show that $f_n' \rightrightarrows f'$ on compact subsets, let $K \subseteq \Omega$ be compact.

Claim 1: There exists $\delta > 0$ such that for all $z \in K$, $\overline{\mathbb{D}_\delta}(z) \subseteq \Omega$.

To see this, suppose there exists sequences $z_n \subseteq K$ and $w_n \notin \Omega$ such that $w_n \in \overline{\mathbb{D}_{\frac{1}{n}}}(z_n)$. Take a convergent subsequence z_{n_k} which converges to some limit $z \in K$. But then $w_{n_k} \to z$ so there is no open disk around z contained in Ω, contradicting the fact that Ω is open. Thus such a δ exists.

Let

$$K_\delta = \bigcup_{z \in K} \overline{\mathbb{D}_\delta}(z) \subseteq \Omega$$

Claim 2: K_δ is compact.

Clearly K_δ is bounded since K is. To show closure, take a sequence $z_n \subseteq K_\delta$ such that $z_n \to z \in \mathbb{C}$. For each n, there exists $w_n \in K$ such that $|z_n - w_n| \leq \delta$. Pick a subsequence $\{w_{n_k}\}$ which converges to $w \in K$. Then $|z - w| = \lim |z_{n_k} - w_{n_k}| \leq \delta$ so $z \in K_\delta$.

Let $z \in K$. Then by Cauchy's Integral formula for derivatives,

$$\begin{aligned}
|f_n'(z) - f'(z)| &= \left| \frac{1}{2\pi i} \int_{\partial \mathbb{D}_\delta(z_0)} \frac{f_n(\zeta) - f(\zeta)}{(\zeta - z)^2} \, \mathrm{d}\zeta \right| \\
&\leq \frac{1}{2\pi} 2\pi\delta \frac{1}{(\delta - |z - z_0|)^2} \sup_{\zeta \in K_\delta} |f_n(\zeta) - f(\zeta)|
\end{aligned}$$

As $n \to \infty$, the above tends to 0 uniformly since K_δ is compact, so $f_n' \rightrightarrows f'$ on K. $\square$

The above theorems allow us to produce important holomorphic functions.

Example 2.5

Consider the **Riemann zeta function** $\zeta : \{z \in \mathbb{C} : \mathrm{Re}(z) > 1\} \to \mathbb{C}$ defined by

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

To show that this is holomorphic, we write

$$\zeta_N(z) = \sum_{n=1}^{N} \frac{1}{n^z} = \sum_{n=1}^{N} e^{-z \ln n}$$

So each ζ_N is holomorphic. We want to show that $\zeta_N \rightrightarrows \zeta$ on compact subsets. Let $K \subseteq \{z \in \mathbb{C} : \mathrm{Re}(z) > 1\}$ be compact.

Since K is compact there exists a $\delta > 0$ such that $K \subseteq \{z \in \mathbb{C} : \mathrm{Re}(z) > 1 + \delta\}$. Then

$$\begin{aligned}
|\zeta_N(z) - \zeta(z)| &= \left| \sum_{n=N+1}^{\infty} \frac{1}{n^z} \right| \leq \sum_{n=N+1}^{\infty} \left| e^{-z \ln n} \right| = \sum_{n=N+1}^{\infty} \left| e^{-\mathrm{Re}(z) \ln n} \right| \\
&= \sum_{n=N+1}^{\infty} \frac{1}{n^{\mathrm{Re}(z)}} \leq \sum_{n=N+1}^{\infty} \frac{1}{n^{1+\delta}}
\end{aligned}$$

Since $1 + \delta > 1$ this is a convergent series and therefore tends to 0. Moreover it does so independent of z so $\zeta_N \rightrightarrows \zeta$ on compact subsets and ζ is holomorphic on the indicated half plane.

We now develop analogous results for integration.

Theorem 2.20

Let $\Omega \subseteq \mathbb{C}$ be open and $F : \Omega \times [0,1] \to \mathbb{C}$ be such that

1. For fixed $s \in [0,1]$, $F_s(z) = F(z,s)$ is holomorphic on Ω.
2. F is continuous.

Define

$$f(z) = \int_0^1 F(z,s)\,\mathrm{d}s$$

Then f is holomorphic on Ω.

Proof. Define

$$f_N(z) = \frac{1}{N}\sum_{n=1}^{N} F\left(z, \frac{n}{N}\right)$$

This is a finite Riemann sum approximation of f obtained by uniformly partitioning $[0,1]$ into N subintervals. f_N is holomorphic, so we want to show that $f_n \rightrightarrows f$ on compact subsets. Now, take some $K \subseteq \Omega$ compact, and note $K \times [0,1]$ is compact. Then f is uniformly continuous on $K \times [0,1]$. So for $\varepsilon > 0$ pick $\delta > 0$ such that for $z_1, z_2 \in K, s_1, s_2 \in [0,1]$ with $|z_1 - z_2| < \delta, |s_1 - s_2| < \delta$, then $|F(z_1, s_1) - F(z_2, s_2)| < \varepsilon$.

Now, take N large enough that $N > \frac{1}{\delta}$. Then

$$\begin{aligned}
|f(z) - f_n(z)| &= \left| \int_0^1 F(z,s)\,\mathrm{d}s - \frac{1}{N}\sum_{n=1}^{N} F\left(z, \frac{n}{N}\right) \right| \\
&= \sum_{n=1}^{N} \int_{\frac{n-1}{N}}^{\frac{n}{N}} \left| \left(F(z,s) - F\left(z, \frac{n}{N}\right) \right) \right| \mathrm{d}s \\
&< \sum_{n=1}^{N} \varepsilon \frac{1}{N} \\
&= \varepsilon
\end{aligned}$$

so $f_n \rightrightarrows f$ on compact subsets and thus f is holomorphic on Ω. □

Next, we discuss the Schwarz reflection principle, which is another technique for constructing holomorphic functions.

Definition 2.7

A set $\Omega \subseteq \mathbb{C}$ is **symmetric** about the real axis if

$$z \in \Omega \iff \overline{z} \in \Omega$$

In this case, we write the following:

$$\Omega^+ := \Omega \cap \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$$
$$\Omega^- := \Omega \cap \{z \in \mathbb{C} : \mathrm{Im}(z) < 0\}$$
$$I := \Omega \cap \mathbb{R}$$

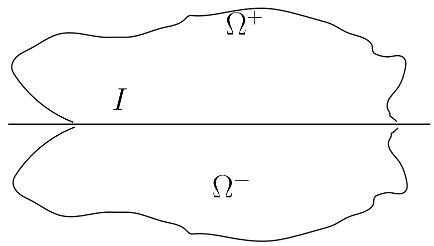


Figure 2.6: Symmetric Sets

Theorem 2.21: Symmetry Principle

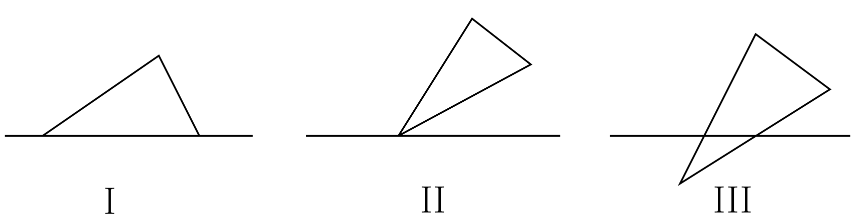
Let $\Omega \subseteq \mathbb{C}$ be open and symmetric about the real axis. Suppose that $f^+ : \Omega^+ \cup I \to \mathbb{C}$ is continuous on $\Omega^+ \cup I$ and holomorphic on Ω^+, and that f^- is the same (with Ω^+ replaced by Ω^-). Moreover, suppose that $f^+ = f^-$ on I. Define $f : \Omega \to \mathbb{C}$ by

$$f(z) = \begin{cases} f^+(z), & z \in \Omega^+ \\ f^+(z) = f^-(z), & z \in I \\ f^-(z), & z \in \Omega^- \end{cases}$$

Then f is holomorphic on Ω.

Again, observe that this is false in the real case: consider the absolute value function on $\mathbb{R}$, or similar linear functions on $\mathbb{R}^n$.

Proof. Let $z \in \Omega$. Clearly f is holomorphic at points in Ω^+, Ω^- so we only consider $z \in I$. Now, Ω is open so we let $\overline{\mathbb{D}_r}(z) \subseteq \Omega$. We want to use Morera's. Consider any triangle $T \in \overline{\mathbb{D}_r}(z_0)$. If T is entirely on one side of the line then we are done. Otherwise, there are threee possible cases:



Case 1: Consider the triangle T_ε which is T, except translated upward by ε.

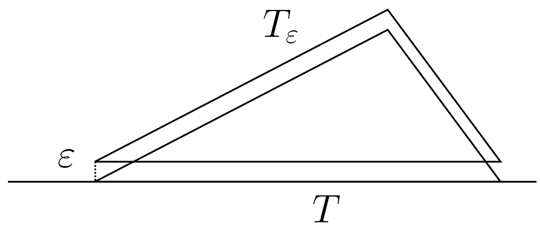


Figure 2.7: Lifting of Case 1 triangle

By Cauchy-Goursat the integral around ∂T_ε is zero, and in particular the only parts which do not cancel are the baselines. So

$$\int_{\partial T} f(z)\,\mathrm{d}z - \int_{\partial T_\varepsilon} f(z)\,\mathrm{d}z = \left| \int_a^b [f(t) - f(t + i\varepsilon)]\,\mathrm{d}t \right| \le (b-a) \sup_{t \in [a,b]} |f(t) - f(t + i\varepsilon)|$$

This tends to 0 uniformly as $\varepsilon \to 0$ since f is continuous.

Case 2: Similar to Case 1. EXERCISE: prove this.

Case 3: We split T into three triangles, two of which satisfy Case 1 and one of which satisfies Case 2:

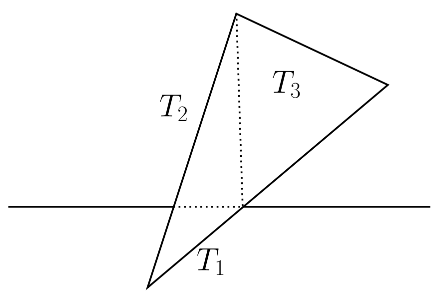


Figure 2.8: Subdivision of Case 3 triangle into Case 1 and Case 2 triangles

Choosing the right orientation shows that the integral around ∂T is zero.

So f is holomorphic on Ω. □

Theorem 2.22: Schwarz Reflection Principle

Let $\Omega \subseteq \mathbb{C}$ be open and symmetric, with $\Omega \cap \mathbb{R} \neq \varnothing$, and let $f^+ : \Omega \cup I \to \mathbb{C}$ be continuous on $\Omega^+ \cup I$ and holomorphic on Ω^+. Also, suppose $f^+(z) \in \mathbb{R}$ for $z \in I$. Then there exists a unique $f : \Omega \to \mathbb{C}$ holomorphic on Ω which coincides with f^+ on $\Omega^+ \cup I$.

We could alternately demand that Ω be connected rather than $\Omega \cap \mathbb{R} \neq \varnothing$, this just gives us uniqueness.

Proof. For uniqueness, this is immediate by analytic continuation.

For $z \in \Omega^-$, define

$$f(z) = \overline{f^+(\overline{z})}$$

We want to show that this is holomorphic, and we will do this using the Symmetry principle. Because f^+ is real on I, f coincides with f^+ on I. Now we just need to show f^- is holomorphic. Let $z_0 \in \Omega^-$. Then there exists $r > 0$ such that

$$f^+(z) = \sum_{n=0}^{\infty} a_n (z - \overline{z_0})^n$$

in some disk $\mathbb{D}_r(\overline{z_0}) \subseteq \Omega^+$. Now take $z \in \mathbb{D}_r(z_0)$. Then we have

$$f^+(\overline{z}) = \sum_{n=0}^{\infty} a_n (\overline{z} - \overline{z_0})^n = \overline{\sum_{n=0}^{\infty} \overline{a_n} (z - z_0)^n}$$

so $f^-(z) = \overline{f^+(\overline{z})}$ is holomorphic at z_0 and thus on Ω^-. Now conclude by the symmetry principle. $\square$

Chapter 3

Meromorphic Functions and Poles

In this chapter we study those functions which are holomorphic except possibly at some isolated points. Together, understanding these functions together with holomorphic functions will allow us to understand a wide variety of functions for practical use.

3.1 Classification of Singularities

In this section, it is of interest to understand the different ways that a function might fail to be holomorphic at a certain point. One way that this may happen is if a function is the ratio of two functions, and the denominator vanishes at a point. Thus, we briefly consider the zeroes of certain functions.

Definition 3.1

If $f : \Omega \to \mathbb{C}$ and $z_0 \in \Omega$, then z_0 is called a **zero** of f if $f(z_0) = 0$. It is called an **isolated zero** if there is $r > 0$ such that

$$f((\mathbb{D}_r(z_0) \setminus \{z_0\}) \cap \Omega) \subseteq \mathbb{C} \setminus \{0\}$$

that is, if f is nonzero around z_0.

Theorem 3.1

Let $\Omega \subseteq \mathbb{C}$ be a region and let $f : \Omega \to \mathbb{C}$ be holomorphic and nonconstant. Let $z_0 \in \Omega$ be a zero of f. Then there exists $\delta > 0$ such that $\mathbb{D}_\delta(z_0) \subseteq \Omega$ and $n \in \mathbb{N}$ and $g : \mathbb{D}_\delta(z_0) \to \mathbb{C} \setminus \{0\}$ holomorphic such that $f(z) = (z - z_0)^n g(z)$ for all $z \in \mathbb{D}_\delta(z_0)$. Moreover, n, g are unique.

In other words, this theorem says that a function which vanishes at a point may be locally factored as a product of $(z - z_0)^n$ and a nonzero function.

Proof. Pick $r > 0$ such that $\mathbb{D}_r(z_0) \subseteq \Omega$. Since f is holomorphic on the disk we expand it as a power series

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

on $\mathbb{D}_r(z_0)$. We know $f(z_0) = a_0 = 0$. f cannot be zero on the disk because then it would be zero (and thus constant) overwhere by analytic continuation. Let n be the smallest integer such that $a_n \neq 0$. Then

$$f(z) = a_n (z - z_0)^n + \ldots = (z - z_0)^n \left(a_n + a_{n+1}(z - z_0) + \ldots\right)$$

Define

$$g(z) = \sum_{k=0}^{\infty} a_{n+k} z^k$$

This is nonzero at z_0 so by continuity there is some $\delta > 0$ such that g is nonzero on $\mathbb{D}_\delta(z_0)$. Thus we have shown existence.

For uniqueness, suppose that

$$f(z) = (z - z_0)^{n_1} g_1(z) = (z - z_0)^{n_2} g_2(z)$$

Suppose that $n_1 \neq n_2$ and without loss of generality suppose $n_1 < n_2$. Then

$$g_2(z) = (z - z_0)^{n_1 - n_2} g_1(z)$$

which is zero at z_0, contradiction. Thus $n_1 = n_2$. It follows by division that $g_1 = g_2$ except possibly at z_0, and since g_1, g_2 are holomorphic they are equal there as well. $\square$

Definition 3.2

Suppose $f : \Omega \to \mathbb{C}$ is holomorphic and nonconstant with Ω a region, and suppose $z_0 \in \Omega$ is a zero of f. Then the unique n referred to Theorem 3.1 is called the **order** or **multiplicity** of the zero of f at z_0. If $n = 1$ then z_0 is called a **simple zero** of f.

Definition 3.3

Let $\Omega \subseteq \mathbb{C}$ be open and suppose $z_0 \in \Omega$. Then if $f : \Omega \setminus \{z_0\} \to \mathbb{C}$ is holomorphic on $\Omega \setminus \{z_0\}$ we say that f has a **pole** at z_0 if

1. There is $\delta > 0$ such that $\mathbb{D}_\delta(z_0) \subseteq \Omega$ and f does not vanish on $\mathbb{D}_\delta(z_0) \setminus \{z_0\}$;
2. If we define
$$g(z) = \begin{cases} \frac{1}{f(z)}, & z \in \mathbb{D}_\delta(z_0) \setminus \{z_0\} \\ 0, & z = z_0 \end{cases}$$
then g is holomorphic.

Intuitively, a pole is a point at which the denominator vanishes and f tends to infinity.

Theorem 3.2

Let $\Omega \subseteq \mathbb{C}$ be open, $z_0 \in \Omega$, and $f : \Omega \setminus \{z_0\} \to \mathbb{C}$ holomorphic on $\Omega \setminus \{z_0\}$. Moreover, suppose f has a pole at z_0. Then there exists $r > 0$ such that for all $z \in \mathbb{D}_r(z-0) \setminus \{z_0\}$,

$$f(z) = \frac{h(z)}{(z-z_0)^n}$$

where $h : \mathbb{D}_r(z_0) \to \mathbb{C} \setminus \{0\}$ is holomorphic and nonvanishing, $n \in \mathbb{N}$, and n, h are unique.

Proof. By Theorem 3.1, there exists ψ such that

$$g(z) = (z-z_0)^n \psi(z)$$

where $\psi : \mathbb{D}_\delta(z_0) \to \mathbb{C} \setminus \{0\}$ is holomorphic and nonvanishing. Now, for $z \in \mathbb{D}_\delta(z_0) \setminus \{z_0\}$,

$$f(z) = \frac{1}{g(z)} = \frac{\frac{1}{\psi(z)}}{(z-z_0)^n}$$

Now let $h(z) = \frac{1}{\psi(z)}$. EXERCISE: prove uniqueness. $\square$

Definition 3.4

In the setting of Theorem 3.2 f is said to have a pole of **order** (or **multiplicity**) n at z_0, and if $n = 1$ then z_0 is called a **simple pole**.

Theorem 3.3

If $\Omega \subseteq \mathbb{C}$ is open and f has a pole of order n at $z_0 \in \Omega$, then there exists $\delta > 0$ such that for $z \in \mathbb{D}_\delta(z_0)$ we may write

$$f(z) = \frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-n+1}}{(z-z_0)^{n-1}} + \ldots + \frac{a_{-1}}{z-z_0} + G(z)$$

where $a_{-n}, \ldots, a_{-1} \in \mathbb{C}$ and G is holomorphic on $\mathbb{D}_\delta(z_0)$.

Proof. Let δ be as in Theorem 3.2, and write

$$f(z) = \frac{h(z)}{(z-z_0)^n}$$

h is holomorphic on $\mathbb{D}_\delta(z_0)$, so we expand it as a power series:

$$\frac{h(z)}{(z-z_0)^n} = \frac{A_0 + A_1(z-z_0) + \ldots}{(z-z_0)^n} = \frac{A_0}{(z-z_0)^n} + \ldots + \frac{A_{n-1}}{(z-z_0)} + A_n + A_{n+1}(z-z_0) + \ldots$$

Then relabel this by letting $A_0, \ldots, A_{n-1}$ be $a_{-n}, \ldots, a_{-1}$ respectively, and $A_n, \ldots$ be $a_0, \ldots$. Letting $G(z)$ be the right side, we are done. $\square$

Definition 3.5

The term

$$\frac{a_{-n}}{(z-z_0)^n}+\ldots+\frac{a_{-1}}{(z-z_0)}$$

in Theorem 3.3 is known as the **principal part** of f.

Moreover, the number a_{-1} is known as the **residue** of f at the pole z_0, denoted $\operatorname{res}_{z_0}(f)$.

The residue term ends up being the most important piece of data about f at a pole. This is because if we integrate on a circle around the pole, all the terms integrate to zero except the a_{-1} term.

Theorem 3.4

If f has a pole of order n at z_0, then

$$\operatorname{res}_{z_0}(f)=\lim_{z\to z_0}\frac{1}{(n-1)!}\left((z-z_0)^n f(z)\right)^{(n-1)}$$

In particular, if the pole is simple then

$$\operatorname{res}_{z_0}(f)=\lim_{z\to z_0}(z-z_0)f(z)$$

Proof. We write

$$(z-z_0)^n f(z)=a_{-n}+(z-z_0)a_{n+1}+\ldots+(z-z_0)^{n-1}a_{-1}+(z-z_0)^n G(z)$$

Differentiate $n-1$ times and take the limit as $z\to z_0$. Then every term of the principal part vanishes, and G is holomorphic multiplied by an nth power so the $(n-1)$th derivative will also be zero. Thus

$$\left((z-z_0)^{n-1}f(z)\right)^{(n-1)}\Big|_{z\to z_0}=(n-1)!a_{-1} \qquad \square$$

This work leads us to the residue theorem, which is a powerful theorem that expands on Cauchy's Theorem for non-holomorphic functions.

Theorem 3.5: Residue Theorem

Let $\Omega\subseteq\mathbb{C}$ be simply connected and suppose $U\subseteq\Omega$ is open such that $\overline{U}\subseteq\Omega$ and $\gamma=\partial U$ is a simple closed curve. Suppose there exist a finite number of points $z_1,\ldots,z_N\in U$ (see Figure 3.1). Suppose $f:\Omega\setminus\{z_1,\ldots,z_N\}\to\mathbb{C}$ is holomorphic with poles at $z_1,\ldots,z_N$. Then

$$\int_\gamma f(z)\,\mathrm{d}z=2\pi i\sum_{k=1}^N\operatorname{res}_{z_k}(f)$$

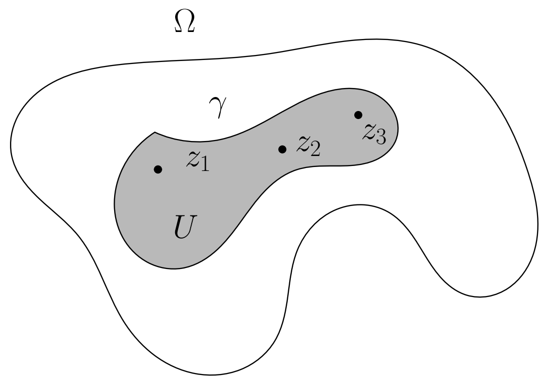


Figure 3.1: Setting for the Residue Theorem

Proof. We induct on N. For $N = 0$ this is true by Cauchy's Theorem for simply connected domains.

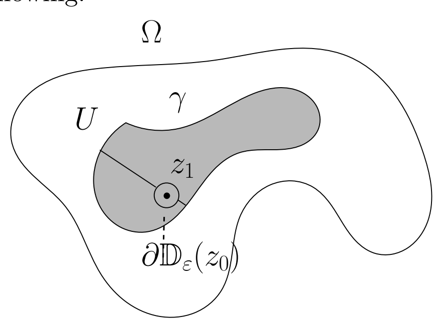
For $N = 1$, apply Theorem 3.2 and suppose the pole at z_1 has order n. Then for some small $\varepsilon > 0$ we have

$$\int_{\mathbb{D}_\varepsilon(z_1)} f(z)\,\mathrm{d}z = \int_{\mathbb{D}_\varepsilon(z_1)} \left(\frac{a_{-n}}{(z-z_1)^n} + \ldots + \frac{a_{-1}}{z-z_1} + G(z) \right) \mathrm{d}z$$

By Cauchy's Theorem, the $G(z)$ part drops out. Now, by Cauchy's integral formula for derivatives,

$$\frac{1}{2\pi i} \int_{\partial\mathbb{D}_\varepsilon(z_1)} \frac{a_{-k}}{(z-z_1)^{k-1+1}} = \frac{1}{(k-1)!} (a_{-k})^{(k-1)} = \begin{cases} 0, & k \geq 2 \\ a_{-1}, & k = 1 \end{cases}$$

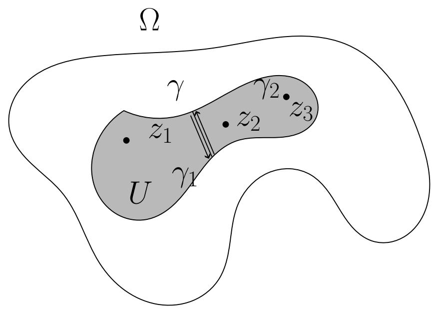
So the entire integral becomes $2\pi i \operatorname{res}_{z_1}(f)$. Now, similar to the proof of Cauchy's integral formula, we set up the following:



so

$$\int_\gamma f(z)\,\mathrm{d}z = \int_{\mathbb{D}_\varepsilon(z_1)} f(z)\,\mathrm{d}z = 2\pi i \operatorname{res}_{z_1}(f)$$

For the inductive step, for any N we draw a line through U which passes through none of the z_i, but so that not all the points are on the same side (this is possible since N is finite). Let γ_1 be the boundary of one of the subdomains created and γ_2 the other.



We see that

$$\int_{\gamma_1} f(z)\,\mathrm{d}z + \int_{\gamma_2} f(z)\,\mathrm{d}z = \int_{\gamma} f(z)\,\mathrm{d}z$$

Each of γ_1, γ_2 contain fewer than N of the z_i, and in particular contain all of them except z_N, so by the inductive hypothesis

$$\int_{\gamma_1} f(z)\,\mathrm{d}z + \int_{\gamma_2} f(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^{N} \mathrm{res}_{z_k}(f) \qquad \square$$

The residue theorem allows us to compute many integrals easily, including those of rational real functions:

Example 3.1

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^4 + x^2 + 1}\,\mathrm{d}x = \frac{\pi}{e^{\frac{\sqrt{3}}{2}}} \left(\frac{\cos\left(\frac{1}{2}\right)}{\sqrt{3}} + \sin\left(\frac{1}{2}\right) \right)$$

To see why this is, consider the function

$$f(z) = \frac{e^{iz}}{z^4 + z^2 + 1}$$

and note that since sin is odd,

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^4 + x^2 + 1}\,\mathrm{d}x = \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$$

We integrate this along the upper semicircle of radius R. Denote the arc section of its boundary by γ_R. Notice that the poles are the zeroes of the denominator, which cocur precisely when

$$z = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i = a_{\pm}$$
$$z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i = b_{\pm}$$

all of which have order 1 since the denominator is a polynomial of degree 4. We only care about a_+, b_+ since those are the only poles in our curve. Thus

$$\int_{-R}^{R} f(z)\,\mathrm{d}z + \int_{\gamma_R} f(z)\,\mathrm{d}z = 2\pi i\left(\mathrm{res}_{a_+}(f) + \mathrm{res}_{b_+}(f)\right)$$

We want to show that the integral over γ_R tends to zero. Parameterize this by $t \mapsto Re^{it}$ for $t \in [0,\pi]$. So

$$|f(z)| = \left|\frac{e^{i(R\cos\theta + i\sin\theta)}}{z^4+z^2+1}\right| \le \frac{e^{-R\sin\theta}}{R^4-R^2-1} \le \frac{1}{R^4-R^2-1}$$

so

$$\left|\int_{\gamma_R} f(z)\,\mathrm{d}z\right| \le \frac{\pi R}{R^4-R^2-1}$$

which tends to 0 as $R \to \infty$. So

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^4+x^2+1}\,\mathrm{d}x = 2\pi i\left(\mathrm{res}_{a_+}(f) + \mathrm{res}_{b_+}(f)\right)$$

To calculate the residues (noting they are simple poles), we have

$$\begin{aligned}\mathrm{res}_{a_+}(f) &= \lim_{z\to a_+}(z-a_+)\frac{e^{iz}}{(z-a_+)(z-a_-)(z-b_+)(z-b_-)} \\ &= \frac{e^{ia_+}}{(a_+-a_-)(a_+-b_+)(a_+-b_-)}\end{aligned}$$

and similarly for $\mathrm{res}_{b_+}(f)$. Explicitly calculating these gives the result.

Although poles are the most important kind of singularity, we now briefly consider other kinds of singularities.

Definition 3.6

Let $\Omega \subseteq \mathbb{C}$ be open and $z_0 \in \omega$. Let $f : \Omega \setminus \{z_0\} \to \mathbb{C}$ be holomorphic. We say that f has a **removable singularity** at z_0 if there is $\tilde{f} : \Omega \to \mathbb{C}$ holomorphic with $\tilde{f}(z) = f(z)$ for all $z \in \Omega \setminus \{z_0\}$.

Theorem 3.6: Riemann's Theorem on Removable Singularities

Let $\Omega \subseteq \mathbb{C}$ be open and $z_0 \in \Omega$. Then $f : \Omega \setminus \{z_0\} \to \mathbb{C}$ has a removable singularity at z_0 if f is bounded on Ω.

Proof. There exists $r > 0$ such that $\overline{\mathbb{D}_r}(z_0) \subseteq \Omega$. Then for $z \in \mathbb{D}_r(z_0)$ define

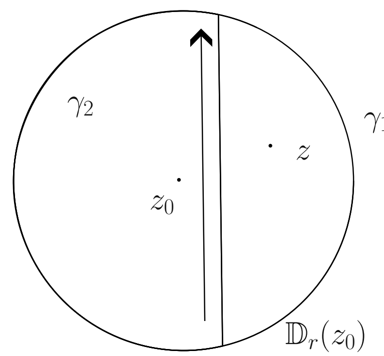
$$\tilde{f}(z) = \frac{1}{2\pi i} \int_{\partial \mathbb{D}_r(z_0)} \frac{f(\omega)}{\omega - z} \, \mathrm{d}\omega$$

Observe that

$$\tilde{f}(z) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{it})}{z_0 + re^{it} - z} rie^{it} \, \mathrm{d}t$$

The denominator is never zero since z is in the interior of $\mathbb{D}_r(z_0)$, so by Theorem 2.20, $\tilde{f}$ is holomorphic. So we need to show that f agrees to $\tilde{f}$ on $\mathbb{D}_r(z_0) \setminus \{z_0\}$.

Draw a line segment through $\mathbb{D}_r(z_0)$ such that z is separated from z_0. Let γ_1, γ_2 be the curves created this way.



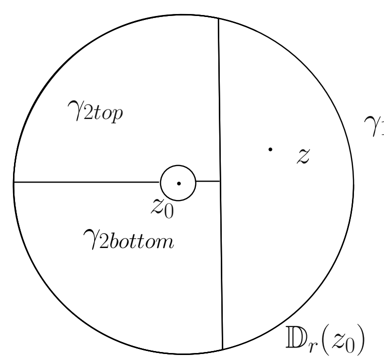
Then

$$\tilde{f}(z) = \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(\omega)}{\omega - z} \, \mathrm{d}\omega + \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(\omega)}{\omega - z} \, \mathrm{d}\omega$$

Now, by the Cauchy Integral Formula

$$\frac{1}{2\pi i} \int_{\gamma_1} \frac{f(\omega)}{\omega - z} \, \mathrm{d}\omega = f(z)$$

Now, divide γ_2 into $\gamma_{2top}, \gamma_{2bottom}$ to avoid z_0.



Then

$$0 = \int_{\gamma_{2top}} \frac{f(\omega)}{\omega - z}\, \mathrm{d}\omega + \int_{\gamma_{2bottom}} \frac{f(\omega)}{\omega - z}\, \mathrm{d}\omega = \int_{\gamma_2} \frac{f(\omega)}{\omega - z}\, \mathrm{d}\omega - \int_{\partial \mathbb{D}_\varepsilon(z_0)} \frac{f(\omega)}{\omega - z}\, \mathrm{d}\omega$$

Then we have

$$\left|\tilde{f}(z) - f(z)\right| \leq \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(\omega)}{\omega - z}\, \mathrm{d}\omega = \frac{1}{2\pi} \left| \int_{\partial \mathbb{D}_\varepsilon(z_0)} \frac{f(\omega)}{\omega - z}\, \mathrm{d}\omega \right| \leq \frac{1}{2\pi} 2\pi\varepsilon \frac{M}{|z - z_0| - \varepsilon}$$

where M is a bound for f. Then as $\varepsilon \to 0$, this quantity goes to zero and thus $f = \tilde{f}$. So f has a removable singularity. ☐

Note that this statement becomes an if and only if statement if we change the conclusion to "f is bounded on a neighborhood of z_0." As a corollary to this, we can formalize the intuition that a pole is a point where f is becomes bounded.

Corollary 3.7

Let $f : \Omega \setminus z_0 \to \mathbb{C}$ be holomorphic. Then f has a pole at z_0 if and only if

$$\lim_{z \to z_0} |f(z)| = \infty$$

Proof. ($\implies$) If z_0 is a pole, then by the local description of poles,

$$f(z) = \frac{h(z)}{(z - z_0)^n}$$

near z_0, where h is nonzero. But then

$$\lim_{z \to z_0} |f(z)| = \lim_{z \to z_0} \frac{|h(z)|}{|z - z_0|^n} = \lim_{z \to z_0} \frac{|h(0)|}{0} = \infty$$

($\impliedby$) If $\lim_{z \to z_0} |f(z)| = \infty$ then

$$\lim_{z \to z_0} \frac{1}{|f(z)|} = 0$$

so $\frac{1}{|f(z)|}$ is bounded near z_0. Then by Riemann's Theorem, there is $g : \mathbb{D}_r(z_0) \to \mathbb{C}$ holomorphic such that $g(z) = \frac{1}{f(z)}$ for all $z \in \mathbb{D}_r(z_0) \setminus \{z_0\}$. In particular, by continuity we must have $g(z_0) = 0$. So f has a pole at z_0. ☐

Thus we have classified two types of isolated singularities. We give a name to the other kinds now:

Definition 3.7

Let $f : \Omega \setminus \{z_0\} \to \mathbb{C}$ be holomorphic. Then the singularity of f at z_0 is said to be a **essential singularity** if it is neither a pole nor removable.

Example 3.2

Consider $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ given by $f(z) = e^{\frac{1}{z}}$. Then 0 is an essential signularity. To see this, first note that

$$\lim_{t \to 0^+} f(t) = \lim_{t \to 0^+} e^{\frac{1}{t}} = \infty$$

so the singularity is not bounded, and thus not a pole. But similarly,

$$\lim_{t \to 0^-} f(t) = 0$$

so the singularity is not a pole either.

Moreover, if we consider $z = it$,

$$f(it) = \cos\left(\frac{1}{t}\right) - i \sin\left(\frac{1}{t}\right)$$

so we see that the real and imaginary parts have oscillating discontinuities at 0.

The following theorem gives us some insight into the behavior of functions near essential singularities.

Theorem 3.8: Casorati-Weierstrass Theorem

Let $r > 0$ and suppose $f : \mathbb{D}_r(z_0) \setminus \{z_0\} \to \mathbb{C}$ is holomorphic with an essential singularity at z_0. Then the image $f(\mathbb{D}_r(z_0) \setminus \{z_0\})$ is dense in $\mathbb{C}$.

Proof. Suppose for contradiction that there exists $\omega \in \mathbb{C}$ and $\delta > 0$ such that $f(\mathbb{D}_r(z_0) \setminus \{z_0\}) \cap \mathbb{D}_\delta(\omega) = \varnothing$. Then

$$|f(z) - \omega| \geq \delta$$

for all $z \in \mathbb{D}_r(z_0) \setminus \{z_0\}$. Then

$$g(z) = \frac{1}{f(z) - \omega}$$

is well defined and holomorphic on $\mathbb{D}_r(z_0) \setminus \{z_0\}$, and moreover

$$|g(z)| \leq \frac{1}{\delta}$$

By Riemann's Theorem, g can be defined at z_0 so that g is holomorphic on all of $\mathbb{D}_r(z_0)$.

If $g(z_0) \neq 0$, then $\frac{1}{g(z)}$ is holmorphic near z_0. But then since $f(z) = \frac{1}{g(z)} + \omega$, we may define f at z_0 so that f is holomorphic, and therefore z_0 is a removable singularity, contradiction.

Otherwise, if $g(z_0) = 0$, then

$$\lim_{z \to z_0} \frac{1}{|g(z_0)|} = \infty$$

so

$$\lim_{z \to z_0} |f(z)| = \left| \frac{1}{g(z)} + \omega \right| \geq \lim_{z \to z_0} \frac{1}{|g(z)|} - |\omega| = \infty$$

so by Corollary 3.7 f has a pole, contradiction. Thus the image is dense in $\mathbb{C}$. $\square$

3.2 Meromorphic Functions

Now that we have classified the types of (isolated) singularities that a function may have, we generalize our study of holomorphic functions to those which have isolated singularities.

Definition 3.8

If $\Omega \subseteq \mathbb{C}$ is open, then f is **meromorphic** on Ω if there is a collection of points $\{z_1, z_2, \ldots\} \subseteq \Omega$ (either infinite or finite) such that:

1. The restriction of f to $\Omega \setminus \{z_1, z_2, \ldots\}$ is holomorphic;
2. f has a pole at each z_i;
3. $\{z_1, z_2, \ldots\}$ does not have a limit point in Ω.

The last condition here ensures that each pole is isolated from the others.

Definition 3.9

We say that f is **meromorphic at** ∞ if there is some radius $R > 0$ such that $f : \mathbb{C} \setminus \mathbb{D}_R \to \mathbb{C}$ is holomorphic and the function $F : \mathbb{D}_{\frac{1}{r}} \setminus \{0\} \to \mathbb{C}$ defined by

$$F(z) = f\left(\frac{1}{z}\right)$$

has a pole at 0. We can similarly define what it means for f to have a **removable singularity at** ∞ or an **essential singularity at** ∞.
f is said to be **meromorphic on the extended complex plane** if it is meromorphic on $\mathbb{C}$ and at ∞

Note that if f is meromorphic on the extended complex plane, then it has only finitely many points. Moreover, the following result allows us to transfer our knowledge of holomorphic functions onto meromorphic functions:

Theorem 3.9

Any function that is meromorphic on the extended complex plane is a rational function

$$f(z) = \frac{P(z)}{Q(z)}$$

where $P, Q \in \mathbb{C}[z]$.

Proof. If f is meromorphic on the extended complex plane then it has finitely many poles $z_1, \ldots, z_n$ (not including the pole at ∞). By Theorem 3.3, for each $k = 1, 2, \ldots, n$ there exists $\delta_k > 0$ such that for $z \in \mathbb{D}_{\delta_k}(z_0)$,

$$f(z) = f_k(z) + g_k(z)$$

where f is the principal part (a polynomial in $\frac{1}{z-z_k}$) and g is the holomorphic part. Since f is meromorphic at ∞, there exists $R > 0$ such that $f : \mathbb{C} \setminus \overline{\mathbb{D}_R} \to \mathbb{C}$ is characterized by

$$f\left(\frac{1}{z}\right) = \tilde{f}_\infty(z) + \tilde{g}_\infty(z)$$

Then define

$$f_\infty(z) = \tilde{f}_\infty\left(\frac{1}{z}\right), g_\infty = \tilde{g}_\infty\left(\frac{1}{z}\right)$$

Then f_∞ is a polynomial in z and g_∞ is holomorphic.

Define

$$H = f - f_1 - f_2 - \ldots - f_n - f_\infty$$

Intuitively, we have removed all of the principal parts of f.

Claim: H is entire.

At any point other than the z_i, f is holomorphic and each of the f_i is holomorphic as well. f_∞ is a polynomial, so H is certainly holomorphic at every point that is not a pole. Consider z_k. Then on $\mathbb{D}_{\delta_k}(z_k)$ we have

$$H(z) = (f - f_k) - f_1 - \ldots - f_{k-1} - f_{k+1} - \ldots - f_n - f_\infty = g_k - f_1 - \ldots - f_{k-1} - f_{k+1} - \ldots - f_n - f_\infty$$

The other f_i are holomorphic at z_k, and so is g_k, so H is holomorphic at z_k.

Now, we claim that H is bounded. To see this, recall that f_k is a polynomial in $\frac{1}{z-z_k}$. So

$$\lim_{z \to \infty} |f_k(z)| = 0$$

Moreover, for $z \in \mathbb{D}_{1/R}$,

$$f\left(\frac{1}{z}\right) = \tilde{f}_\infty(z) + \tilde{g}_\infty(z)$$

Now,

$$H = (f - f_\infty) - (f_1 + \ldots + f_n) = g_\infty - (f_1 + \ldots + f_n)$$

$g_\infty(z) = \tilde{g}_\infty\left(\frac{1}{z}\right)$ is bounded on $\mathbb{D}_{\frac{1}{R}}$, so g_∞ is bounded on $\mathbb{C} \setminus \mathbb{D}_R$.
and therefore constant. So

$$f = f_1 + f_2 + \ldots + f_n + f_\infty + c$$

???? □

Theorem 3.10: Argument Principle

Let $\Omega \subseteq \mathbb{C}$ be simply connected. Suppose $U \subseteq \Omega$ is open and $\overline{U} \subseteq \Omega$, with ∂U a simple closed curve γ. Suppose there exist $p_1, \ldots, p_M \in U$ such that $f : \Omega \setminus \{p_1, \ldots, p_M\} \to \mathbb{C}$ is holomorphic. Assume that the zeroes of f are $z_1, \ldots, z_N \in U$ and f has poles at each p_i. Moreover, suppose that n_k denotes the order of the zero at z_k, and that m_k denotes the order of the pole at p_k. Then

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \,\mathrm{d}z = n_1 + \ldots + n_N - (m_1 + \ldots + m_M)$$

Roughly speaking, the right side is the number of zeroes minus the number poles, with multiplicity.

Proof. Near each z_k, we know that

$$f(z) = (z - z_k)^{n_k} g(z)$$

for $g(z)$ holomorphic and nonvanishing near z_k. So

$$f'(z) = n_k (z - z_k)^{n_k - 1} g(z) + (z - z_k)^{n_k} g'(z)$$

so

$$\frac{f'(z)}{f(z)} = \frac{n_k}{(z - z_k)} + \frac{g'(z)}{g(z)}$$

since g is nonvanshing, the function g'/g is holomorphic near z_k. So

$$\mathrm{res}_{z_k}\left(\frac{f'}{f}\right) = n_k$$

Similarly, near each p_k we may write

$$f(z) = \frac{h(z)}{(z - z_k)^{m_k}}$$

Again we have

$$f'(z) = -m_k \frac{h(z)}{(z - z_k)^{m_k + 1}} + \frac{h'(z)}{(z - z_k)^{m_k}}$$

so

$$\frac{f'(z)}{f(z)} = \frac{-m_k}{(z - z_k)} + \frac{h'(z)}{h(z)}$$

and

$$\operatorname{res}_{p_k}\left(\frac{f'}{f}\right) = -m_k$$

By the residue formula,

$$\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = 2\pi i\left[\sum_{k=1}^{N} \operatorname{res}_{z_k}\left(\frac{f'}{f}\right) + \sum_{k=1}^{M} \operatorname{res}_{p_k}\left(\frac{f'}{f}\right)\right] = 2\pi i \sum n_k - 2\pi i \sum m_k \qquad \square$$

In particular, note that the right side is an integer. This fact allows us to prove the following:

Theorem 3.11: Rouche's Theorem

Let $\Omega \subseteq \mathbb{C}$ be simply connected, $U \subseteq \Omega$ open with $\overline{U} \subseteq \Omega$, and $\partial U = \gamma$ a simple closed curve. Let $f, g : \Omega \to \mathbb{C}$ be holomorphic such that

$$|f(z)| > |g(z)|$$

for every $z \in \gamma$. Then f and $f + g$ have the same number of zeroes in U (counted with multiplicity).

Intuitively, g is a small perturbation of f (small at the boundary), and we see that f has the same number of zeroes.

Proof. For $t \in [0, 1]$, define $f_t(z) = f(z) + tg(z)$. By our assumption, $f_t \neq 0$ on γ. Let n_t be the number of zeroes of f_t in U. f_t is holomorphic so it has no poles. Now, we apply the argument principle, so that

$$n_t = \frac{1}{2\pi i}\int_\gamma \frac{f_t(z)}{f(z)}\,\mathrm{d}z = \frac{1}{2\pi i}\int_\gamma \frac{f'(z) + tg'(z)}{f(z) + tg(z)}\,\mathrm{d}z$$

This is jointly continuous in z, t, so by real variable analysis we know that n_t is continuous in t. But n_t takes integer values so it must be constant. Thus $n_0 = n_1$. $\square$

Example 3.3

Consider the polynomial $z^5 + 3z^3 + 7$. We already know this has 5 roots in $\mathbb{C}$. We show that all of the zeroes lie in $\mathbb{D}_2$.

Let $f(z) = z^5$ and $g(z) = 3z^3 + 7$. If $|z| = 2$ then

$$|g(z)| = \left|3z^3 + 7\right| \leq 3|z|^3 + 7 = 31 < 32 = |z|^5 = |f(z)|$$

So by Rouche's theorem, $f, f + g$ have the same number of zeroes in $\mathbb{D}_2$. f has five zeroes in $\mathbb{D}_2$, so $z^5 + 3z^3 + 7$ has five roots in $\mathbb{D}_2$.

Definition 3.10

Let $\Omega \subseteq \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$. f is an **open mapping** if it is the case that for all $U \subseteq \Omega$ open, $f(U)$ is also open.

In other words, an open mapping is one that preserves open sets in the forward direction. Recall that continuous functions, both real and complex, preserve open sets in the reverse direction. However, in the real case we generally do not have open mappings, even for nice functions (consider $x \mapsto x^2$). This is completely different in the case of complex variables:

Theorem 3.12: Open Mapping Theorem

Let $\Omega \subseteq \mathbb{C}$ be open and connected. Let $f : \Omega \to \mathbb{C}$ be holomorphic on Ω and nonconstant. Then f is an open mapping.

Proof. Let $U \subseteq \Omega$. $f(U)$ if and only if for all $\omega_0 \in f(U)$ there exists $\mathbb{D}_\varepsilon(\omega_0) \subseteq f(U)$. This is the case if and only if for any $\omega \in \mathbb{D}_\varepsilon(\omega_0)$ there exists $z \in U$ such that $f(z) = \omega$.

Let $\omega_0 \in f(U)$ and let $z_0 \in U$ such that $f(z_0) = \omega_0$. Let $\overline{\mathbb{D}_\delta}(z_0) \subseteq U$. By uniqueness of analytic continuation, we may pick r small enough so that $f(z) \neq f(z_0) = \omega_0$ for every $z \in \overline{\mathbb{D}_r}(z_0) \setminus \{z_0\}$ (otherwise f would be identically ω_0 on all of Ω). $f(z) - f(z_0)$ is continuous on $\partial\mathbb{D}_r(z_0)$, so it achieves a minimum which cannot be zero. Thus there exists $\varepsilon > 0$ such that

$$|f(z) - f(z_0)| \geq \varepsilon$$

for all $z \in \partial\mathbb{D}_r(z_0)$. For each $\omega \in \mathbb{D}_\varepsilon(\omega_0)$, define $F(z) - f(z) - \omega_0$ and $G(z) = \omega_0 - \omega$. For $z \in \partial\mathbb{D}_r(z_0)$,

$$|G(z)| = |\omega_0 - \omega| < \varepsilon \leq |f(z) - \omega_0|$$

so we can apply Rouche's Theorem to conclude that F and $F + G = f(z) - \omega$ have the same number of zeroes. In particular, $F(z_0) = 0$, so there exists z such that $(F + G)(z) = f(z) - \omega = 0$. Thus $f(z) = \omega$. So $\omega \in f(U)$. Thus $\mathbb{D}_\varepsilon(\omega_0) \subseteq f(U)$. So $f(U)$ is open. $\square$

An easy but important result of of the Open Mapping Theorem is the following:

Theorem 3.13: Maximum Modulus Principle

Let $\Omega \subseteq \mathbb{C}$ be open and connected, and let $f : \Omega \to \mathbb{C}$ be holomorphic and nonconstant. Then f does not attain its maximum on Ω.

Recall that we say f attains a maximum on Ω if there exists $z_0 \in \Omega$ such that $|f(z)| \leq |f(z_0)|$ for all $z \in \Omega$.

Proof. Suppose not. Then there exists $z_0 \in \Omega$ such that $|f(z_0)| \geq |f(z)|$ for all $z \in \Omega$. By the open mapping theorem, $f(U)$ is open, so there exists $r > 0$ such that $\mathbb{D}_r(f(z_0)) \subseteq f(U)$. So there exists $z \in U$ such that $f(z) = \left(1 + \frac{r}{2}\right) f(z_0)$. Then

$$|f(z)| = \left(1 + \frac{r}{2}\right) |f(z_0)| > |f(z_0)|$$

contradiction. So no such z_0 exists. $\square$

Corollary 3.14

Let $\Omega \subseteq \mathbb{C}$ be open and bounded, and suppose $f : \overline{\Omega} \to \mathbb{C}$ is continuous on $\overline{\Omega}$ and holomorphic on Ω. Then

$$\max_{z\in\overline{\Omega}} |f(z)| = \max_{z\in\partial\Omega} |f(z)|$$

That is, f attains its maximum on the boundary of Ω.

Proof. This is obvious if f is constant, so assume f is nonconstant. $\overline{\Omega}$ is compact, so f attains its maximum on $\overline{\Omega}$. But by the Maximum Modulus Principle it does not attain the maximum on Ω, so it must be on $\partial\Omega$. □

3.3 Holomorphic Logarithms

The reader may have noted that although we have made extensive use of the complex exponential to this point, we have not yet used or even defined a complex logarithm. To see why this is the case, consider some complex number of the form

$$z = re^{i\theta}$$

with $r > 0$. If we were to define a logarithm, we would expect that $\log z = \log r + i\theta$. However, θ is only defined up to multiples of 2π. Thus, a naive definition of the logarithm is a multivalued function. However, by carefully restricting the logarithm to subsets of the complex plane, we can obtain sections of the logarithm (known as branches) which are proper, single-valued, holomorphic functions.

Theorem 3.15

Let $\Omega \subseteq \mathbb{C}$ be simply connected. Let $f : \Omega \to \mathbb{C} \setminus \{0\}$ be a nonvanishing holomorphic function. Moreover, suppose there exists $z_0 \in \Omega$ and $c_0 \in \mathbb{C}$ such that

$$f(z_0) = e^{c_0}$$

Then there exists a unique holomorphic function $g : \Omega \to \mathbb{C}$ such that

$$e^{g(z)} = f(z)$$

for all $z \in \Omega$ and

$$g(z_0) = c_0$$

Proof. For uniqueness, suppose that g_1, g_2 both satisfy the conclusion. Then

$$e^{g_1} = e^{g_2} = f$$

Differentiating the first equation, we have

$$g_1' e^{g_1} = g_2' e^{g_2}$$

so

$$g_1' = g_2'$$

Thus $g_1 = g_2 + c$ for some constant c, but $g_1(z_0) = g_2(z_0)$ so $g_1 = g_2$.

Let $z \in \Omega$. Fix some path γ_z joining z_0 to z. Define

$$g(z) = c_0 + \int_{\gamma_z} \frac{f'(\omega)}{f(\omega)} \,\mathrm{d}\omega$$

(Note that this is well defined since $f \neq 0$. The motivation for this choice of function is that it should integrate to the logarithm based on what we know.) By Cauchy's Theorem, this does not depend on our choice of γ_z. Observe that g is a primitve for f'/f by the proof for local existence of primitives on simply connected regions. Thus g is holomorphic and

$$g' = \frac{f'}{f}$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}z} f e^{-g} = f' e^{-g} - f g' e^{-g} = e^{-g}\left(f' - f g'\right) = e^{-g}\left(f' - f\frac{f'}{f}\right) = 0$$

Thus fe^{-g} is constant and equals 1 at z_0. So

$$e^{-g} = f$$

on all of Ω. □

Definition 3.11

For a function f and the corresponding function g produced as in Theorem 3.15, we write

$$g(z) = \log_{\Omega, z_0, c_0}(f)$$

and say that g is the **logarithm** of f with respect to Ω, z_0, c_0.

This allows us to define a branch of the logarithm without respect to a certain function, by picking a logarithm with respect to the identity $z \mapsto z$.

Corollary 3.16

Let $\Omega \subseteq \mathbb{C}$ be simply connected and suppose $0 \notin \Omega$, $1 \in \Omega$. Let $\mathbb{D}_r(1) \subseteq \Omega$ for some $0 < r < 1$. Then there exists $F : \Omega \to \mathbb{C}$ holomorphic such that

$$e^{F(z)} = z$$

for all $z \in \Omega$ and for $1 - r < t < 1 + r$, $F(t) = \log t$.

The first conclusion says that F is a function which matches our intution for what the logarithm should do, and the second says that it also coincides with the real version of the logarithm.

Proof. We take $F = \log_{\Omega,1,0}(f)$, where $f(z) = z$ is the identity. Note that this coincides with the real logarithm since for any $1 - r < t < 1 + r$, we have

$$F(t) = \int_1^t \frac{1}{s} \, \mathrm{d}s = \log t$$

□

Definition 3.12

The function F produced in Corollary 3.16 is denoted $F(z) = \log_\Omega z$.

Intuitively, Corollary 3.16 says we can define a logarithm so long as it is not possible to wrap around the origin, so that the argument of a function is actually well-defined.

Definition 3.13

The **principal branch** of the logarithm is the function $\operatorname{Log} = \log_\Omega$ where $\Omega = \mathbb{C} \setminus (-\infty, 0]$ is the slit complex plane.

Example 3.4

Let us verify that Log satisfies the property

$$\operatorname{Log} re^{i\theta} = \log r + i\theta$$

Let $z = re^{i\theta} \in \mathbb{C} \setminus (-\infty, 0]$. Then integrating first along the real axis and then along an arc, we have

$$\operatorname{Log} z = \int_1^r \frac{\mathrm{d}s}{s} + \int_0^\theta \frac{1}{e^{it}} ie^{it} \, \mathrm{d}t = \ln r + i\theta$$

Thus the principal branch of the logarithm preserves some of the properties of the real logarithm. However, it does not conserve all of these properties. For instance, let $z_1 = z_2 = e^{2\pi i/3}$. Then

$$\operatorname{Log} z_1 = \operatorname{Log} z_2 = \frac{2\pi i}{3}$$

But $z_1 z_2 = e^{4\pi i/3} = e^{-2\pi i/3}$ so

$$\operatorname{Log} z_1 z_2 = -\frac{2\pi i}{3} \neq \operatorname{Log} z_1 + \operatorname{Log} z_2$$

Definition 3.14

For $\alpha \in \mathbb{C}$, $z \notin (-\infty, 0]$, we define

$$z^\alpha := e^{\alpha \operatorname{Log} z}$$

Definitions