MAT 335 Notes

Max Chien

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Contents

Introduction

This document contains notes taken for the class MAT 335: Complex Analysis at Princeton University, taken in the Fall 2024 semester. These notes are primarily based on lectures by Professor Assaf Naor. Other references used in these notes include *Complex Analysis* by Elias Stein and Rami Shakarchi, *Complex Analysis* by Lars Ahlfors, *Visual Complex Analysis* by Tristan Needham, and *Real and Complex Analysis* by Walter Rudin. Since these notes were primarily taken live, they may contains typos or errors.

Chapter 1

Preliminaries

1.1 The Complex Number System

The set of complex numbers, denoted $\mathbb{C}$ is identified with ordered pairs $(x, y) \in \mathbb{R}^2$. We may alternately write this as $x + iy$, where the symbol i is currently undefined.

For a given complex number $z = x + iy$, $x = \mathrm{Re}(z)$ is called the **real part** of z, $y = \mathrm{Im}(z)$ is called the **imaginary part**, $|z| = \sqrt{x^2 + y^2}$ is the **modulus** of z, and the **argument** of z, $\theta = \arg(z)$, is the angle between (x, y) and the x-axis, defined up to integer multiples of 2π.

Definition 1.1

Let $\theta \in \mathbb{R}$. We define

$$e^{i\theta} = \cos\theta + i\sin\theta = (\cos\theta, \sin\theta)$$

One can observe using the identity $\cos^2 + \sin^2 = 1$ that $e^{i\theta}$ lies on the unit circle. Moreover, if $r = |z|$, then elementary geometry shows that we have $z = re^{i\theta}$ using the definition above.

Proposition 1.1

For any $z \in \mathbb{C}$, $|\mathrm{Re}(z)| \leq |z|$ and $|\mathrm{Im}(z)| \leq |z|$.

Proof. $|z| = \sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2}$. $\square$

One of the distinguishing features of $\mathbb{C}$ from the real plane $\mathbb{R}^2$ is the algebraic structure present on $\mathbb{C}$.

Definition 1.2

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then we define addition and multiplication on $\mathbb{C}$ by

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$
$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)$$

Taking $i = (0, 1)$, then we observe that $i^2 = -1 + 0i = -1$. Thus we recover the basic identity $i^2 = -1$.

Proposition 1.2

Addition and multiplication over $\mathbb{C}$ are commutative and associative. Moreover, multiplication distributes over addition.

Proof. Commutative and associativity of addition is inherited from $\mathbb{R}$. $\square$

Using the definition of $e^{i\theta}$, we can reinterpret complex multiplication in a much more pleasant manner than the definition above.

Proposition 1.3

If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then

$$z_1 z_2 = (r_1 r_2) e^{i(\theta_1 + \theta_2)}$$

Proof. We have proved commutativity. From here, we apply trig identities. $\square$

Thus multiplication results in multiplication of lengths and addition of arguments.

Proposition 1.4

For $z_1, z_2 \in \mathbb{C}$, the **triangle inequality** holds:

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

Proof. Choose r, θ such that $z_1 + z_2 = re^{i\theta}$. Then

$$|z_1 + z_2| = r = (z_1 + z_2)e^{-i\theta} = z_1 e^{-i\theta} + z_2 e^{-i\theta} = \mathrm{Re}(z_1 e^{-i\theta} + z_2 e^{-i\theta})$$

Now note that $\mathrm{Re}(z + w) = \mathrm{Re}(z) + \mathrm{Re}(w)$. So

$$\mathrm{Re}(z_1 e^{-i\theta} + z_2 e^{-i\theta}) = \mathrm{Re}(z_1 e^{-i\theta}) + \mathrm{Re}(z_2 e^{-i\theta}) \leq |z_1 e^{-i\theta}| + |z_2 e^{-i\theta}| = |z_1| + |z_2| \quad \square$$

Corollary 1.5

The reverse triangle inequality also holds:

$$||z| - |w|| \leq |z - w|$$

Definition 1.3

Let $z = x + iy \in \mathbb{C}$. Then the **complex conjugate** of z is defined as

$$\overline{z} = x - iy$$

Geometrically, this is reflection over the x axis.

Proposition 1.6

For $z \in \mathbb{C}$, $z\overline{z} = |z|^2$.

Definition 1.4

For $z \neq 0$, define

$$\frac{1}{z} = \frac{\overline{z}}{|z|^2}$$

The above proposition and definition show that

$$z \cdot \frac{1}{z} = 1$$

Definition 1.5

A **sequence** of complex numbers $\{z_n\}_{n=1}^{\infty}$ **converges** to $z \in \mathbb{C}$ (written $\lim_{n\to\infty} z_n = z$) if

$$\begin{cases} \lim_{n\to\infty} \operatorname{Re}(z_n) = \operatorname{Re}(z) \\ \lim_{n\to\infty} \operatorname{Im}(z_n) = \operatorname{Im}(z) \end{cases}$$

We similarly define the limit of a complex function $\lim_{z\to a} f(z)$.

Definition 1.6

A **Cauchy sequence** is a sequence $(z_n) \subseteq \mathbb{C}$ such that $(\operatorname{Re}(z_n))$ and $(\operatorname{Im}(z_n))$ are both Cauchy.

Proposition 1.7

A Cauchy sequence is convergent.

Definition 1.7

Let $r > 0$ and $z_0 \in \mathbb{C}$. Then the **open disk** of radius ε about z is the set

$$\mathbb{D}_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$$

and the **closed disk** as

$$\overline{\mathbb{D}}_r(z_0) = \{z \in \mathbb{C} : |z - z_0| \leq r\}$$

We also specify $\mathbb{D}_r = \mathbb{D}_r(0)$ and $\mathbb{D} = \mathbb{D}_1$.

Definition 1.8

An **open set** in $\mathbb{C}$ is a subset $\Omega \subseteq \mathbb{C}$ such that for any $z_0 \in \Omega$ there exists $\varepsilon > 0$ such that $D_\varepsilon(z_0) \subseteq \Omega$.

We will now define the object that will become the main focus of this course:

Definition 1.9

Let $\Omega \subseteq \mathbb{C}$ be open and let $z_0 \in \Omega$. Let $f : \Omega \to \mathbb{C}$. We say that f is **holomorphic** at z_0 (or complex differentiable) if the limit

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists. In this case, the limit is denoted $f'(z_0)$.

If f is holomorphic at every $z \in \Omega$, then we simply say it is holomorphic on Ω. If f is holomorphic on $\mathbb{C}$ it is said to be **entire**.

Note that the specification that Ω is open ensures that the difference quotient is actually defined (for sufficiently small h). Moreover, although this definition appears similar to the real analogue, the structure of the complex numbers means that it has far-reaching implications.

We will prove the following theorems in this class:

- (Cauchy's Theorem) If f is holomorphic on Ω, then it has derivatives of all orders.
- (Liouville's Theorem) If f is entire and boudned, then it is constant.
- (Prime Number Theorem) If $\pi(n)$ denotes the number of prime numbers less than or equal to n, then
$$\lim_{n\to\infty} \pi(n) \cdot \frac{\ln n}{n} = 1$$
- (Hardy-Ramanujan Theorem) Define $p(n)$ (the partition function) to be the number of ways to write $n = k_1 + k_2 + \ldots + k_n$ where $k_1 \geq k_2 \geq \ldots \geq k_n$ are all integers. For instance, $p(4) = 5$. Then
$$p(n) \sim \frac{1}{n\sqrt{48}} e^{\pi\sqrt{\frac{2}{3}}\cdot\sqrt{n}}$$

Definitions