On $\zeta(-1)$

Max Chien

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1 Introduction

These are notes written to accompany a presentation at the Albany High School Mathletes club on the "formula"

$$1 + 2 + 3 + \ldots = -\frac{1}{12} \tag{$*$}$$

A "proof" in terms of naive sums of sequences is given, followed by some discussion of the caveats of the interpretation of $(*)$. A brief discussion of the Riemann zeta function is given in relation to this formula. These notes are intended to be accessible to students with little to no formal math background, including calculus.

2 Making Sense of Infinite Sums

About 10 years ago, the Youtube channel Numberphile released a series of videos which claimed the following formula:

$$1 + 2 + 3 + 4 + \ldots = -\frac{1}{12}$$

This generated a lot of discussion and response videos. Of course, many people objected to the unintuitive result. The video was especially controversial among mathematicians, though, who had many arguments about whether the derivation was rigorous. These disagreements were because mathematicians have a standardized convention for what it means for an infinite sum to "equal" a number, and this formula uses the equals sign in a different way.

Before we understand how this equation comes about, we should first discuss what is usually meant when we say that an infinite sum equals a number. Consider the following sum:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$$

Let's look at what happens as we add up these terms. We have:

$$\begin{aligned} \frac{1}{2} &= \frac{1}{2} \\ \frac{1}{2} + \frac{1}{4} &= \frac{3}{4} \\ \frac{1}{2} + \frac{1}{4} + \frac{1}{8} &= \frac{7}{8} \\ \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} &= \frac{15}{16} \end{aligned}$$

As we add each number, we get closer and closer to the value 1. In fact, we can get as close as we want to 1, simply by adding up enough of the terms. Because the "partial sums" get closer and closer to 1, we say that the entire sum *equals* 1:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = 1$$

However, a sum might not approach any number at all. For instance, the sum that we are consider just gets larger and larger and approaches infinity. We say that this series **diverges**, which we sometimes write as

$$1 + 2 + 3 + 4 + \ldots = \infty$$

In any case, it is not proper for us to say that this sum equals any finite number.

Sums can also fail to converge in more subtle ways. Consider the following:

$$1 - 1 + 1 - 1 + \ldots$$

The partial sums alternate between 0 and 1, so they don't approach a single value. In this case, we also say that the sum diverges.

3 A "Proof"

Here I present the proof from the original Numberphile video. I use the symbol $\sim$ rather than $=$ to emphasize that the below manipulations are somewhat unrigorous and should not be taken as equalities.[1]

$$S \sim 1 + 2 + 3 + \ldots$$

Consider the sum

$$S_1 \sim 1 - 1 + 1 - 1 + \ldots$$

Again, the sum does not converge, so we can't properly use equality here. However, if we were to assign a value to S_1, what value would be most appropriate?

There are three values that make sense here:

[1] Of course, these manipulations are not completely unrigorous! There is a theoretical justification for each of these, but it is important to note that they require a very specific understanding of the equality symbol, and it would be better to avoid this here.

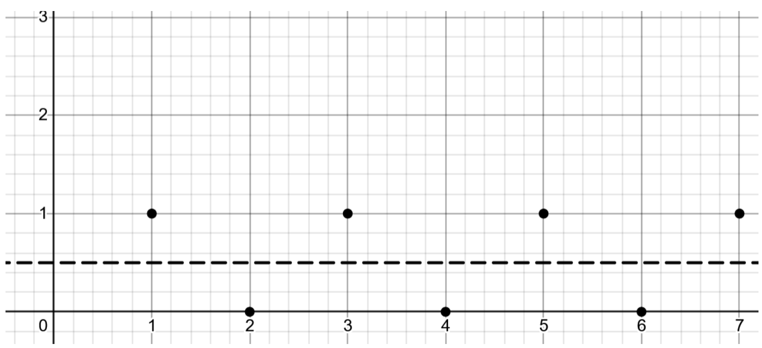
- Since the sum alternates between 1 and 0, we could set S_1 to be either 0 or 1.
- Since the sum oscillates, it is on average closest to $\frac{1}{2}$, so we can set $S_1 = \frac{1}{2}$.

In this case, we are going to choose to set $S_1 = \frac{1}{2}$. There are a couple reasons this choice is somewhat natural, which are discussed in the appendix. Here is one argument:

$$\begin{cases} \begin{array}{l} S_1 \sim 1 - 1 + 1 - 1 \dots \\ \underline{S_1 \sim \quad\quad 1 - 1 + 1 \dots} \\ 2S_1 \sim 1 \end{array} \end{cases}$$

As some brief optional justification, though:

- If we choose to set $S_1 = 0$ or 1, then we have to pick one of the two, which is arguably more arbitrary than picking $\frac{1}{2}$.
- The "mean square error" is minimized by setting $S_1 = \frac{1}{2}$ (we can think of this as a linear regression on the infinite dataset of partial sums $\{(1,1), (2,0), (3,1), (4,0), \dots\}$: the line of best fit is $y = \frac{1}{2}$):



- The Cesaro sum of a sequence $\{a_n\}$ is defined to be the limit of the averages of a sequence:

$$C = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \sum_{i=1}^{k} a_n$$

It can be shown that if $\sum a_n$ is convergent in the usual sense, then C is the sum. However, the Cesaro sum may exist even when the sum is not convergent. In this case, we have

$$C = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \sum_{i=1}^{k} (-1)^i = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1-(-1)^n}{2} = \lim_{n\to\infty} \frac{1}{n} \left\lceil \frac{n}{2} \right\rceil = \frac{1}{2}$$

We will take the Cesaro sum as "good enough" justification for taking $S_1 = \frac{1}{2}$[2].

[2] There is actually an even more powerful technique than Cesaro summation, which is called Ramanujan summation, which also agrees with the assignment $S_1 = \frac{1}{2}$. I'm not going to provide details here, partially because it requires significantly more background to define, and also because we can just directly compute $1 + 2 + 3 + \dots = -\frac{1}{12}$ using Ramanujan summation anyway, so there is no point in trying to justify the intermediate steps with Ramanujan summation.

Now, consider the sum

$$S_2 \sim 1 - 2 + 3 - 4 + 5 - \dots$$

We can perform the following algebraic manipulation[3]:

$$\begin{cases} \begin{array}{r} S_2 \sim 1 - 2 + 3 - 4 + \dots \\ S_2 \sim \quad\;\; 1 - 2 + 3 - \dots \\ \hline 2S_2 \sim 1 - 1 + 1 - 1 + \dots \end{array} \end{cases}$$

So

$$2S_2 \sim S_1 \sim \frac{1}{2} \implies S_2 \sim \frac{1}{4}$$

Now, we can use S_2 to assign a value to S. Notice that

$$\begin{cases} \begin{array}{rl} S \sim & 1 + 2 + 3 + 4 + \dots \\ -S_2 \sim & -1 + 2 - 3 + 4 - \dots \\ \hline S - S_2 \sim & 0 + 4 + 0 + 8 + \dots \end{array} \end{cases}$$

This formula is exactly

$$S - S_2 \sim 4 + 8 + 12 + \dots \sim 4S$$

So

$$3S \sim -S_2 \implies S = -\frac{S_2}{3} = -\frac{1}{12}$$

4 The Riemann Zeta Function

As another method of assigning a value to $1+2+3+4+\dots$, we employ a common technique, where we first generalize the problem in order to spot patterns. Consider the following sums:

$$\begin{gathered} \vdots \\ 1^3 + 2^3 + 3^3 + \dots \\ 1^2 + 2^2 + 3^2 + \dots \\ 1 + 2 + 3 + \dots \\ 1 + 1 + 1 + \dots \\ \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots \\ \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \\ \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \dots \\ \vdots \end{gathered}$$

[3] This manipulation actually doesn't quite show what we want. What this really shows is that, for any choice of a method of summation (i.e. a way of assigning the values to these sums), a method of summation which is *linear* and *stable*, and assigns a value to S_2, will always assign the value $S_2 \sim \frac{1}{4}$. However, it could be the case that no such method exists! Luckily, the method of Abel summation does indeed work, so we are fine to make the conclusion $S_2 \sim \frac{1}{4}$.

Generally, we can write this as $1^n + 2^n + 3^n + \ldots$ for an integer n. When $n \geq 0$, then sum still tends to infinity, so we don't really gain anything by working with these sums over our original sum. When $n = -1$ the sum is still infinite, although this is a bit less obvious.

More importantly, when $n \leq -2$, the sum actually converges to a very well defined value. For instance, if $n = -2$, then

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{\pi^2}{6}$$

The above equality is known the **Basel problem**. While I will not derive this result, the point is that for $n \leq -2$, the sum is well defined and no issues will occur. We can be more general and define the following function, called the Riemann zeta function:

$$\zeta(t) = \frac{1}{1^t} + \frac{1}{2^t} + \frac{1}{3^t} + \ldots$$

for any number $t > 1$.

We can go one step further and allow t to be a complex number as well. I won't go into what it means to raise a number to a complex power here. The reason that considering complex powers is helpful is that due to a (rather magical) property of complex functions known as **analytic continuation**, we can "extend" ζ to define it even when the sum itself doesn't properly converge. Moreover, this definition is unique, meaning that there is no *other* value that we could define for ζ. When we do this, the unique extended value for -1 is

$$1 + 2 + 3 + 4 + \ldots \sim \zeta(-1) = -\frac{1}{12}$$

Thus, if we consider $1+2+3+4+\ldots$ as part of a family of sums of the form $1^t + 2^t + 3^t + \ldots$, the only value that makes sense to assign to the sum is $-\frac{1}{12}$.[4]

Appendix: Methods of Summation

TODO: add some discussion of what methods of summation are In our derivations of the value of S_1, S_2, S, we made use of term-by-term addition and shifting of sequences. In other words, we implicitly assume that whatever method of summation we are using is linear and stable (as well as regular).

A technical point is also that linearity and regularity only require that the properties hold when a method of summation actually assigns a value to all the sequences involved. Thus, the manipulations are valid only as long as there exists a method of summation which is linear, regular, and stable, and assigns a value to S_1, S_2, S. Below, I discuss a few methods of summation that do this. TODO: note that Ramanujan summation, and actually any summation method which assigns a finite value to $1 + 2 + 3 + 4 + \ldots$, cannot be linear and stable.

[4] This is called zeta function regularization. See Appendix C.

A.1 Cesaro Sums

When we considered Grandi's sum,

$$1 - 1 + 1 - 1 + \ldots$$

We observed that the sum is, "on average," about $\frac{1}{2}$. Cesaro summation is a method of summation which formalizes this idea.

For a sequence $a_1, a_2, a_3, \ldots$, the **Cesaro mean** of the sequence is defined to be the limit of the running averages:

$$C = \lim_{n\to\infty} \frac{a_1 + a_2 + \ldots + a_n}{n}$$

It can be shown that if the sequence converges, then the Cesaro mean is the same as the limit of the sequence. So if we define the **Cesaro sum** of a sum $a_1 + a_2 + a_3 + \ldots$ to be the Cesaro mean of the partial sums:

$$C^\Sigma = \lim_{n\to\infty} \frac{s_1 + s_2 + \ldots + s_n}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \sum_{i=1}^{k} a_i$$

then C^Σ is indeed regular. It can also be shown that C^Σ is linear and stable.

Using Cesaro summation, it is easy to see that $S_1 \sim \frac{1}{2}$. The partial sums alternate between 0 and 1, so the Cesaro sum is

$$S_1 = \lim_{n\to\infty} \frac{1 + 0 + 1 + 0 + \ldots + 0}{n} = \frac{1}{n}\left\lceil \frac{n}{2} \right\rceil = \frac{1}{2}$$

A.2 Abel Sums

While Cesaro summation does help us assign a value to Grandi's series, it fails when we consider the series

$$1 - 2 + 3 - 4 + \ldots$$

Abel summation is a method of summation which uses power series from calculus. If you have learned about power series, consider the MacLaurin expansion of $\frac{1}{(1+x)^2}$. This is given by

$$\frac{1}{(x+1)^2} = \sum_{n=0}^{\infty} (n+1)(-1)^n x^n = 1 - 2x + 3x^2 - 4x^3 + \ldots$$

Notice that if we naively plug in $x = 1$, we get

$$\frac{1}{4} \sim 1 - 2 + 3 - 4 + \ldots \sim S_2$$

However, we cannot simply plug in $x = 1$. The radius of convergence of this power series is 1, but it is divergent at 1. Instead, we can let x get closer and closer to 1, and see what value it approaches.

Given a series $a_0 + a_1 + a_2 + \ldots$, the Abel sum A^Σ is defined to be this limiting value:

$$A^\Sigma = \lim_{x\to 1^-} \sum_{n=0}^{\infty} a_n x^n$$

whenever the limit exists. As with Cesaro summation, it can be shown that if the series converges in the usual sense, then the Abel sum is equal to the normal sum. Moreover, any Cesaro summable series is also Abel summable, and it has the same sum. By the properties of limits and infinite sums from calculus, A^{Σ} is linear and stable.

Using Abel summation, we can avoid the issues with just plugging in $x = 1$ by taking the limit:

$$S_2 = \lim_{x \to 1^-} \sum_{n=0}^{\infty} (n+1)(-1)^n x^n = \lim_{x \to 1^-} \frac{1}{(x+1)^2} = \frac{1}{4}$$