MAT 335 Notes

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Introduction

This document contains notes taken for the class MAT 335: Complex Analysis at Princeton University, taken in the Fall 2024 semester. These notes are primarily based on lectures by Professor Assaf Naor. Other references used in these notes include *Complex Analysis* by Elias Stein and Rami Shakarchi, *Complex Analysis* by Lars Ahlfors, *Visual Complex Analysis* by Tristan Needham, and *Real and Complex Analysis* by Walter Rudin. Since these notes were primarily taken live, they may contains typos or errors.

Chapter 1

Preliminaries

1.1 The Complex Number System

The set of complex numbers, denoted $\mathbb{C}$ is identified with ordered pairs $(x, y) \in \mathbb{R}^2$. We may alternately write this as $x + iy$, where the symbol i is currently undefined.

For a given complex number $z = x + iy$, $x = \text{Re}(z)$ is called the **real part** of z, $y = \text{Im}(z)$ is called the **imaginary part**, $|z| = \sqrt{x^2 + y^2}$ is the **modulus** of z, and the **argument** of z, $\theta = \arg(z)$, is the angle between (x, y) and the x-axis, defined up to integer multiples of 2π.

Definition 1.1

Let $\theta \in \mathbb{R}$. We define

$$e^{i\theta} = \cos\theta + i\sin\theta = (\cos\theta, \sin\theta)$$

One can observe using the identity $\cos^2 + \sin^2 = 1$ that $e^{i\theta}$ lies on the unit circle. Moreover, if $r = |z|$, then elementary geometry shows that we have $z = re^{i\theta}$ using the definition above.

Proposition 1.2

For any $z \in \mathbb{C}$, $|\text{Re}(z)| \leq |z|$ and $|\text{Im}(z)| \leq |z|$.

Proof. $|z| = \sqrt{\text{Re}(z)^2 + \text{Im}(z)^2}$. $\square$

One of the distinguishing features of $\mathbb{C}$ from the real plane $\mathbb{R}^2$ is the algebraic structure present on $\mathbb{C}$.

Definition 1.3

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then we define addition and multiplication on $\mathbb{C}$ by

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$
$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)$$

Taking $i = (0, 1)$, then we observe that $i^2 = -1 + 0i = -1$. Thus we recover the basic identity $i^2 = -1$. We also observe that Re and Im are both linear operators.

Proposition 1.4

Addition and multiplication over $\mathbb{C}$ are commutative and associative. Moreover, multiplication distributes over addition.

Proof. Commutative and associativity of addition is inherited from $\mathbb{R}$. □

Using the definition of $e^{i\theta}$, we can reinterpret complex multiplication in a much more pleasant manner than the definition above.

Proposition 1.5

If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then

$$z_1 z_2 = (r_1 r_2) e^{i(\theta_1 + \theta_2)}$$

Proof. We have proved commutativity. From here, we apply trig identities. □

Thus multiplication results in multiplication of lengths and addition of arguments.

Proposition 1.6

For $z_1, z_2 \in \mathbb{C}$, the **triangle inequality** holds:

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

Proof. Choose r, θ such that $z_1 + z_2 = re^{i\theta}$. Then

$$|z_1 + z_2| = r = (z_1 + z_2)e^{-i\theta} = z_1 e^{-i\theta} + z_2 e^{-i\theta} = \mathrm{Re}(z_1 e^{-i\theta} + z_2 e^{-i\theta})$$

Now note that $\mathrm{Re}(z + w) = \mathrm{Re}(z) + \mathrm{Re}(w)$. So

$$\mathrm{Re}(z_1 e^{-i\theta} + z_2 e^{-i\theta}) = \mathrm{Re}(z_1 e^{-i\theta}) + \mathrm{Re}(z_2 e^{-i\theta}) \leq \left|z_1 e^{-i\theta}\right| + \left|z_2 e^{-i\theta}\right| = |z_1| + |z_2| \quad \square$$

The above proof amounts to applying the real triangle inequality to the components of z_1, z_2 in the direction of $z_1 + z_2$.

Corollary 1.7

The reverse triangle inequality also holds:

$$||z| - |w|| \leq |z - w|$$

Proof. We have

$$\begin{cases} |z| \leq |z-w| + |w| \\ |w| \leq |w-z| + |z| \end{cases} \implies \begin{cases} |z| - |w| \leq |z-w| \\ -|z-w| \leq |z| - |w| \end{cases} \implies ||z| - |w|| \leq |z-w| \qquad \square$$

Definition 1.8

Let $z = x + iy \in \mathbb{C}$. Then the **complex conjugate** of z is defined as

$$\overline{z} = x - iy$$

Geometrically, this is reflection over the x axis.

Proposition 1.9

For $z \in \mathbb{C}$, $z\overline{z} = |z|^2$.

Proof. Let $z = x + iy$. Then

$$z\overline{z} = (x+iy)(x-iy) = x^2 + y^2 = |z|^2 \qquad \square$$

Definition 1.10

For $z \neq 0$, define

$$\frac{1}{z} = \frac{\overline{z}}{|z|^2}$$

The above proposition and definition show that

$$z \cdot \frac{1}{z} = 1$$

Definition 1.11

A **sequence** of complex numbers $\{z_n\}_{n=1}^{\infty}$ **converges** to $z \in \mathbb{C}$ (written $\lim_{n\to\infty} z_n = z$) if

$$\begin{cases} \lim_{n\to\infty} \mathrm{Re}(z_n) = \mathrm{Re}(z) \\ \lim_{n\to\infty} \mathrm{Im}(z_n) = \mathrm{Im}(z) \end{cases}$$

This equivalent to the familiar definition:

Proposition 1.12

A sequence $\{z_n\} \subseteq \mathbb{C}$ converges to z if and only for $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that whenever $n \geq N$ we have

$$|z_n - z| < \varepsilon$$

Proof. ($\implies$) Let $\varepsilon > 0$. Then pick N_1, N_2 such that

$$\begin{cases} n \geq N_1 \implies |\mathrm{Re}(z_n) - \mathrm{Re}(z)| < \frac{\varepsilon}{\sqrt{2}} \\ n \geq N_2 \implies |\mathrm{Im}(z_n) - \mathrm{Im}(z)| < \frac{\varepsilon}{\sqrt{2}} \end{cases}$$

Letting $N = \max\{N_1, N_2\}$, whenever $n \geq N$ we have

$$|z_n - z|^2 = \mathrm{Re}(z_n - z)^2 + \mathrm{Im}(z_n - z)^2 = |\mathrm{Re}(z_n) - \mathrm{Re}(z)|^2 + |\mathrm{Im}(z_n) - \mathrm{Im}(z)|^2 < \frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2}$$

Taking square roots on both sides we have

$$|z_n - z| < \varepsilon$$

($\impliedby$) $|\mathrm{Re}(z_n) - \mathrm{Re}(z)| = |\mathrm{Re}(z_n - z)| \leq |z_n - z|$ $\square$

We similarly define the limit of a complex function $\lim_{z \to a} f(z)$.

Definition 1.13

A **Cauchy sequence** is a sequence $(z_n) \subseteq \mathbb{C}$ such that $(\mathrm{Re}(z_n))$ and $(\mathrm{Im}(z_n))$ are both Cauchy.

Again we can formulate this analogously to the single variable case:

Proposition 1.14

A sequence $\{z_n\} \subseteq \mathbb{C}$ is Cauchy if and only if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that whenever $m, n \geq N$ it follows that

$$|z_n - z_m| < \varepsilon$$

Proof. Same as the proof of Proposition 1.12. $\square$

Proposition 1.15

A Cauchy sequence is convergent.

Proof. Follows from completeness of $\mathbb{R}$:

$$\{z_n\} \text{ conv.} \iff \begin{cases} \{\mathrm{Re}(z_n)\} \text{ conv.} \\ \{\mathrm{Im}(z_n)\} \text{ conv.} \end{cases} \iff \begin{cases} \{\mathrm{Re}(z_n)\} \text{ Cauchy} \\ \{\mathrm{Im}(z_n)\} \text{ Cauchy} \end{cases} \iff \{z_n\} \text{ Cauchy} \quad \square$$

1.2 Topology of $\mathbb{C}$

The topological nature of $\mathbb{C}$ should not be unfamiliar to the reader, since it is essentially the same as that of $\mathbb{R}^2$, rephrased slightly using complex variables.

Definition 1.16

Let $r > 0$ and $z_0 \in \mathbb{C}$. Then the **open disk** of radius ε about z is the set

$$\mathbb{D}_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$$

and the **closed disk** as

$$\overline{\mathbb{D}}_r(z_0) = \{z \in \mathbb{C} : |z - z_0| \leq r\}$$

We also specify $\mathbb{D}_r = \mathbb{D}_r(0)$ and $\mathbb{D} = \mathbb{D}_1$.

Definition 1.17

An **interior point** $z_0 \in \Omega$ of a subset $\Omega \subseteq \mathbb{C}$ is a point such that there exists $r > 0$ where $\mathbb{D}_r(z_0) \subseteq \Omega$.

Definition 1.18

The set of interior point in Ω is the **interior** of Ω, denoted $\operatorname{int} \Omega$.

Definition 1.19

An **open set** in $\mathbb{C}$ is a subset $\Omega \subseteq \mathbb{C}$ such that for any $z_0 \in \Omega$ there exists $\varepsilon > 0$ such that $D_\varepsilon(z_0) \subseteq \Omega$.

It is immediate that Ω is open if and only if $\operatorname{int} \Omega = \Omega$.

Definition 1.20

Let $\Omega \in \mathbb{C}$ and let $z \in \mathbb{C}$. z is a **limit point** of Ω if there exists a sequence of points $\{z_n\}_{n=1}^{\infty} \subseteq \Omega$ such that $z_n \neq z$ for each n and $\lim z_n = z$.

We can equivalently define a limit point as a point z such that $\mathbb{D}_r(z) \setminus \{z\} \cap \Omega \neq \varnothing$ for each $r > 0$

Definition 1.21

$A \subseteq \mathbb{C}$ is a **closed set** if $\mathbb{C} \setminus A$ is open.

Proposition 1.22

A is closed if and only if it contains all its limit points.

Proof. ($\implies$) Suppose not. Then pick z which is a limit point of A that is not in A. Then there is no disk around z entirely contained in $\mathbb{C} \setminus A$. Thus A is not closed.

($\impliedby$) Suppose A is not closed. Then there exists $z \notin A$ such that each $\mathbb{D}_r(z)\setminus\{z\}$ intersects A. Then z is a limit point of A. $\square$

Definition 1.23

The **closure** of $\Omega \subseteq \mathbb{C}$, denoted $\overline{\Omega}$, is the union of Ω with its limit points.

Definition 1.24

The **boundary** of $\Omega \subseteq \mathbb{C}$, denoted $\partial\Omega$, is defined as $\overline{\Omega} \setminus \operatorname{int}\Omega$.

Definition 1.25

$\Omega \subseteq \mathbb{C}$ is **bounded** if there exists $M > 0$ such that $|z| < M$ for each $z \in \Omega$ (or equivalently, $\Omega \subseteq \mathbb{D}_M$).

Definition 1.26

Let $\Omega \subseteq \mathbb{C}$ be bounded. Then the **diameter** of Ω is defined as

$$\operatorname{diam}\Omega = \sup_{z,w\in\Omega} |z - w|$$

The following definition, as in the real case, is critical:

Definition 1.27

$\Omega \subseteq \mathbb{C}$ is **compact** if it is closed and bounded.

Theorem 1.28: Bolzano-Weierstrass Theorem

Let $\Omega \subseteq \mathbb{C}$. Then the following conditions are equivalent:

1. Ω is compact.
2. Each sequence $\{z_n\}_{n=1}^{\infty} \subseteq \Omega$ has a subsequence $\{z_{n_k}\}_{k=1}^{\infty}$ which converges to some $z \in \Omega$.

We can treat $\mathbb{C}$ similarly to $\mathbb{R}^2$ to prove this.

Proof. ($1 \implies 2$) If Ω is compact, then $\{z_n\}_{n=1}^{\infty} \subseteq \mathbb{C}$ may be written as $\{x_n + iy_n\}_{n=1}^{\infty} \subseteq \mathbb{C}$. Since Ω is bounded, there exists $M > 0$ such that $|z| < M$ for all $z \in \Omega$. In particular $\sqrt{x_n^2 + y_n^2} = |z_n| < M$. So the real sequences $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty}$ are bounded. Apply the real version of Bolzano-Weierstrass, there exists a convergent subsequence $\{x_{n_k}\}$. Then consider the sequence $\{y_{n_k}\}$. This is also bounded, so we apply Bolzano-Weierstrass again to produce $\{y_{n_{k_i}}\}$ convergent. Then the sequence $\{z_{n_{k_i}}\}$ is a convergent subsequence. If $z = z_n$ for some n, then $z \in \Omega$; otherwise it is a limit point. Since Ω is closed it contains its limit points so $z \in \Omega$.

($2 \implies 1$) Suppose each sequence has a convergent subsequence. Let z be a limit point and let $\{z_n\} \subseteq \Omega \setminus \{z\}$ be a sequence converging to z. Then there exists a subsequence $\{z_{n_k}\}$ which converges to $z' \in \Omega$. But subsequences converge to the same value as the original sequence, so $z = z' \in \Omega$. So Ω is closed. If Ω is not bounded, then we may take $\{z_n\}$ such that $|z_n| \geq n$, and that $|z_{n+1}| > |z_n| + 1$. But then $|z_m - z_{m-1}| > 1$ so no subsequence is Cauchy and thus no subsequence converges. So Ω is bounded. $\square$

Definition 1.29

An **open cover** of a set $\Omega \subseteq \mathbb{C}$ is a collection $\mathcal{O}$ of open sets such that each $z \in \Omega$ is contained in some $O \in \mathcal{O}$. A **subcover** of $\mathcal{O}$ is a subcollection which is still a cover.

Theorem 1.30: Heine-Borel Theorem

A set $\Omega \subseteq \mathbb{C}$ is compact if and only if every open cover has a finite subcover.

Proof. ($\implies$) Since Ω is bounded, it is a subset of a closed rectangle K. We showed in $\mathbb{R}^2$ that $X \times Y$ is compact when $X, Y \subseteq \mathbb{R}$ are, and the same is true here. So K is compact. Take an open cover $\mathcal{O}$ of Ω and add the (open) set $\mathbb{C} \setminus \Omega$. This is an open cover of $\mathbb{C}$ and thus one of K, so only finitely many are needed. Remove $\mathbb{C} \setminus \Omega$ if necessary and we still have an open cover of Ω.

($\impliedby$) Boundedness is immediate by covering Ω with balls of finite radius.

For closure, suppose not. Then take a limit point $w \notin \Omega$. Each $z \in \Omega$ has $|z - w| > 0$, so we may cover Ω with open balls $O_z = \mathbb{D}_\varepsilon(z)$ where $\varepsilon < |z - w|/2$. Then a finite number of them cover Ω but this implies that y is not a limit point. $\square$

For the sake of completeness, here is an independent proof that a set is sequentially compact if it is covering compact.

Proof that covering compactness $\implies$ sequential compactness. Let K be covering compact and pick a sequences $\{a_n\} \subseteq K$. Suppose for contradiction that a_n has no convergent subsequence in K. Then for each $x \in K$, there exists $\varepsilon_x > 0$ and $N_x \in \mathbb{N}$ such that whenever $n \geq N_x$ it follows that $a_n \notin \mathbb{D}_{\varepsilon_x}(x)$. Then the collection of $\mathbb{D}_{\varepsilon_x}(x)$ for $x \in K$ is an open cover of K, so we may pick a finite subcover

$$\mathbb{D}_{\varepsilon_{x_1}}(x_1), \ldots, \mathbb{D}_{\varepsilon_{x_m}}(x_m)$$

Then let $N = \max N_{x_i}$. For $n \geq N$ it follows that $a_n \notin K$, contradiction. $\square$

Proposition 1.31: Nested Compact Set Property

Suppose that $\Omega_1 \supseteq \Omega_2 \supseteq \dots$ is a nested sequence of compact, nonempty subsets of $\mathbb{C}$. Then

$$\bigcap_{n=1}^{\infty} \Omega_n \neq \varnothing$$

Moreover, if $\lim_{n\to\infty} \operatorname{diam} \Omega_n = 0$, then there is a unique point $z \in \mathbb{C}$ such that $z \in \Omega_n$ for all n.

Proof. Choose $z_n \in \Omega_n$ for each n. Then the sequence of points $\{z_n\} \subseteq \Omega_1$, and Ω_1 is compact, so there exists a convergent subsequence $\{z_{n_k}\}$ tending to $z \in \Omega_1$. Then for arbitrary Ω_n, there exists a subsequence $\{z_{n_{k+k_0}}\} \subseteq \Omega_n$ for sufficiently large k_0, which converges to z and we see that $z \in \Omega_n$. So the intersection is nonempty.

To show uniqueness, take $z, w \in \bigcap_{n=1}^{\infty} \Omega_n$. Then

$$|z - w| \leq \operatorname{diam} \Omega_n$$

for each n, but $\operatorname{diam} \Omega_n \to 0$ so $|z - w| = 0$ and thus $z = w$. $\square$

Remark

With the assumption that $\operatorname{diam} \Omega_n \to 0$, we need not take subsequences as $\{z_n\}$ itself is Cauchy. To see this, pick $\varepsilon > 0$ and let N be such that $\operatorname{diam} \Omega_n < \varepsilon$ for any $n \geq N$. Then for any $n, m \geq N$, $z_n, z_m \in \Omega_N$ and thus $|z_n - z_m| \leq \operatorname{diam} \Omega_N < \varepsilon$.

Definition 1.32

A set $\Omega \subseteq \mathbb{C}$ is **connected** if there are no nonempty disjoint sets $A, B \subseteq \Omega$ such that $\Omega = A \sqcup B$ such that $A \cap \overline{B} = \varnothing$ and $\overline{A} \cap B = \varnothing$.

The above definition may be rephrased as saying that Ω is not the disjoint union of nonempty sets which are open in the subspace topology of Ω:

Proposition 1.33

A topological space X is connected if and only if it cannot be written as $X = \Omega_1 \cup \Omega_2$ with Ω_1, Ω_2 nonempty, disjoint and open (in X).

Proof. ($\implies$) Suppose $X \subseteq \mathbb{C}$ is connected. Let $\Omega_1, \Omega_2 \subseteq X$ be nonempty, open and disjoint. Consider Ω_2. Then $\Omega_2 \subseteq X \setminus \Omega_1$. By definition $X \setminus \Omega_1$ is closed. $\overline{\Omega_2}$ is the smallest closed set containing Ω_2, so $\overline{\Omega_2} \subseteq X \setminus \Omega_1$ and thus $\Omega_1 \cap \overline{\Omega_2} = \varnothing$. Similarly $\overline{\Omega_1} \cap \Omega_2 = \varnothing$. Since X is connected, we conclude that $\Omega_1 \cup \Omega_2 \neq X$.

($\impliedby$) Pick A, B nonempty with $X = A \cup B$. Assume that $A \cap \overline{B} = B \cap \overline{A} = \varnothing$, so that $A \subseteq X \setminus \overline{B}$ and $B \subseteq X \setminus \overline{A}$. Define $\Omega_1 = X \setminus \overline{B}$ and $\Omega_2 = X \setminus \overline{A}$. Since $X = A \cup B$, we have $X = \Omega_1 \cup \Omega_2$. Ω_1, Ω_2 are both open in X, so it must not be the case that they are disjoint. So there exists some $x \in \Omega_1 \cap \Omega_2$. But this implies that $x \notin A$ and $x \notin B$. $\square$

Thus the above general definition can be simplified for nicer sets:

Proposition 1.34

If Ω is open, then it is connected if and only if it cannot be written as the union of disjoint open sets (in $\mathbb{C}$). Similarly if F is closed then it is connected if and only if it is not the union of disjoint closed sets.

We may also introduce another notion of connectedness, which involves functions into Ω.

Definition 1.35

Suppose $\Omega \subseteq \mathbb{C}$ and $f : \Omega \to \mathbb{C}$. f is **continuous** at z_0 if for every $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $z \in \Omega$ and $|z - z_0| < \delta$, it follows that $|f(z) - f(z_0)| < \varepsilon$.

Proposition 1.36

f is continuous at z_0 if and only if for every $\{z_n\} \subseteq \Omega$ with $z_n \to z_0$, it follows that $f(z_n) \to f(z_0)$. We say that f is continuous on Ω if it is continuous at each point in Ω.

Definition 1.37

A **path** is a function $f : [0, 1] \to \mathbb{C}$. A continuous path is a continuous such function.

Definition 1.38

A set $\Omega \subseteq \mathbb{C}$ is **path connected** if for any $z, w \in \Omega$ there exists a continuous path with $f(0) = z$ and $f(1) = w$ with $f(t) \in \Omega$ for each $t \in [0, 1]$.

Proposition 1.39

An open set Ω is path connected if and only if it is connected.

1.3 Functions on $\mathbb{C}$

We now turn our attention to functions which map complex numbers to complex numbers, the primary object of study in this course. Continuing from the definition of continuity from the previous section, we have the following:

Proposition 1.40

If f is continuous at z_0 then $|f|$ is continuous at z_0.

Proof. By the reverse triangle inequality we have $||f(z)| - |f(z_0)|| \leq |f(z) - f(z_0)|$. The conclusion follows. $\square$

Definition 1.41

f attains its maximum on $\Omega \subseteq \mathbb{C}$ if there exists $z_0 \in \Omega$ such that

$$|f(z)| \leq |f(z_0)|$$

for each $z \in \Omega$. The minimum case is analogous.

Theorem 1.42

Suppose that $\Omega \subseteq \mathbb{C}$ is compact and $f : \Omega \to \mathbb{C}$ is continuous. Then f attains is maximum (and minimum) on Ω.

Proof. First we show that f is bounded on Ω. If not, then we may take a sequence of points $\{z_n\} \subseteq \Omega$ such that $|f(z_n)| \to \infty$. Then $\{z_n\}$ contains a convergent subsequence $\{z_{n_k}\}$ tending to z. It follows that

$$|f(z_{n_k})| \to |f(z)|$$

But the left side diverges to ∞, contradiction. Thus $f(\Omega)$ is bounded.

Let $M = \sup |f|(\Omega)$. Then there exists a sequence $\{z_n\} \subseteq \Omega$ such that $|f(z_n)| \to M$. Then there exists a subsequence $\{z_{n_k}\}$ converging to $z \in \Omega$. By continuity we have

$$|f(z)| = \lim |f(z_{n_k})| = M$$

$\square$

We now make the most important definition of this course:

Definition 1.43

Let $\Omega \subseteq \mathbb{C}$ be open and let $z_0 \in \Omega$. Let $f : \Omega \to \mathbb{C}$. We say that f is **holomorphic** at z_0 (or complex differentiable) if the limit

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists. In this case, the limit is denoted $f'(z_0)$.

If f is holomorphic at every $z \in \Omega$, then we simply say it is holomorphic on Ω. If f is holomorphic on $\mathbb{C}$ it is said to be **entire**.

We will sometimes also say that f is **analytic** or **complex differentiable** when it is holomorphic.

Note that the specification that Ω is open ensures that the difference quotient is actually defined (for sufficiently small h). Moreover, although this definition appears similar to the real analogue, the structure of the complex numbers means that it has far-reaching implications.

We will prove the following theorems in this class:

- (Cauchy's Theorem) If f is holomorphic on Ω, then it has derivatives of all orders.
- (Liouville's Theorem) If f is entire and bounded, then it is constant.
- (Prime Number Theorem) If $\pi(n)$ denotes the number of prime numbers less than or equal to n, then
$$\lim_{n\to\infty} \pi(n) \cdot \frac{\ln n}{n} = 1$$
- (Hardy-Ramanujan Theorem) Define $p(n)$ (the partition function) to be the number of ways to write $n = k_1 + k_2 + \ldots + k_n$ where $k_1 \geq k_2 \geq \ldots \geq k_n$ are all integers. For instance, $p(4) = 5$. Then
$$p(n) \sim \frac{1}{n\sqrt{48}} e^{\pi\sqrt{\frac{2}{3}}\cdot\sqrt{n}}$$

Example 1.44

The function $f(z) = z$ is holomorphic:

$$\frac{f(z+h) - f(z)}{h} = \frac{z + h - z}{h} = \frac{h}{h} = 1$$

so $z' = 1$.

Definition 1.45

If $A \subseteq \mathbb{C}$ is open and $f : A \to \mathbb{C}$, then we say f is holomorphic on A if there exists $\Omega \supseteq A$ open and $F : \Omega \to \mathbb{C}$ which is holomorphic, and $F|_A = f$.

We can rewrite the definition of holomorphicity similarly to the multivariable real case as the following:

Proposition 1.46

$f : \Omega \to \mathbb{C}$ (Ω open) is holomorphic at z_0 if and only if there exists $a \in \mathbb{C}$ and $\psi : \mathbb{C} \to \mathbb{C}$ with $\psi(h) \to 0$ as $h \to 0$ such that

$$f(z_0 + h) = f(z_0) + ah + h\psi(h)$$

on some $\mathbb{D}_r(z_0) \subseteq \Omega$.

We can rewrite the above as

$$\psi(h) = \frac{f(z_0 + h) - f(z_0)}{h} - a$$

which goes to 0 if and only if

$$\frac{f(z_0 + h) - f(z_0)}{h} \to a$$

so that $f'(z_0) = a$.

This recharacterization allows for a simple proof of the following:

Proposition 1.47

If f is holomorphic at z_0, then it is continuous at z_0.

Proof. Let $\{z_n\}$ be a sequence with $z_n \to z_0$. We want to show that $f(z_n) \to f(z_0)$. Let $h_n = z_n - z_0$. Then

$$f(z_n) = f(z_0 + h_n) = f(z_0) + ah_n + h_n\psi(h_n)$$

by assumption, the second and third terms go to zero, so $f(z_n) \to f(z_0)$. ☐

Proposition 1.48

Let $\Omega \subseteq \mathbb{C}$ be open, and let $f, g : \Omega \to \mathbb{C}$ be holomorphic at z_0. Then

1. $f + g$ is holomophic at z_0, and $(f + g)' = f' + g'$.
2. fg is holomorphic at z_0, and $(fg)' = f'g + fg'$.
3. If $g(z_0) \neq 0$, then $\frac{f}{g}$ is well defined on an open disk aroud z_0, and $\frac{f}{g}$ is holomorphic at z_0 with $(\frac{f}{g})' = \frac{f'g - fg'}{g^2}$.

Proposition 1.49: Chain Rule

Let $\Omega, U \subseteq \mathbb{C}$ be open, and let $f : \Omega \to U$ and $g : U \to \mathbb{C}$. Then $g \circ f : \Omega \to \mathbb{C}$ is holomorphic and

$$(g \circ f)'(z) = g'(f(z))f'(z)$$

Proof. Using the alternative characterization of holomorphicity, we have

$$f(z_0 + h) = f(z_0) + f'(z_0)h + h\psi_f(h)$$

where $\psi_f(h) \to 0$ as $h \to 0$. Similarly,

$$g(f(z_0) + w) = g(f(z_0)) + g'(f(z_0))w + w\psi_g(w)$$

Then

$$
\begin{gathered}
(g \circ f)(z_0 + h) = g(f(z_0) + f'(z_0)h + h\psi_f(h)) \\
= g(f(z_0)) + g'(f(z_0))(f'(z_0)h + h\psi_f(h)) + (f'(z_0)h + h\psi_f(h))\psi_g(f'(z_0)h + h\psi_f(h)) \\
= g(f(z_0)) + g'(f(z_0))f'(z_0)h + h[\psi_f(h)g'(f(z_0)) + (f'(z_0) + \psi_f(h))\psi_g(f'(z_0)h + h\psi_f(h))]
\end{gathered}
$$

Note that

$$
\lim_{h \to 0} \underbrace{\psi_f(h)}_{=0} g'(f(z_0)) + (f'(z_0) + \underbrace{\psi_f(h)}_{=0})\psi_g(\underbrace{f'(z_0)h + h\psi_f(h)}_{=0}) = 0
$$

so $(g \circ f)'(z_0) = g'(f(z_0))f'(z_0)$. □

Example 1.50

Let f be a constant function. Then f is entire and $f'(z) = 0$.

We showed in Example 1.44 that the identity $g(z) = z$ is entire with $g'(z) = 1$.

Combination of the two functions above, together with Proposition 1.48 gives

Corollary 1.51

Let $p(z) = a_0 + a_1 z + \ldots + a_n z^n$. Then p is entire and $p'(z) = a_1 + 2a_2 z + \ldots + n a_n z^{n-1}$.

Let us consider a non-example.

Example 1.52

Let $f(z) = \overline{z}$, so that $f(x + iy) = x - iy$. This is a smooth function in the case of $\mathbb{R}^2$; in fact since it is linear, $Df = f$, so that f has infinitely many derivatives.

However, in the complex case, we have

$$
\frac{f(z+h) - f(z)}{h} = \frac{\overline{z+h} - \overline{z}}{h} = \frac{\overline{h}}{h}
$$

But

$$
\lim_{t \to 0} \frac{\overline{t}}{t} = 1
$$

and

$$
\lim_{t \to 0} \frac{\overline{it}}{it} = -1
$$

so the limits disagree and f is not holomorphic at any z.

Here is one more example.

Example 1.53

Let $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ be defined by $f(z) = \frac{1}{z}$. Then the denominator of f is nonzero and holomorphic. By Proposition 1.48, it follows that f is holomorphic and

$$f'(z) = \frac{-1}{z^2}$$

Consider some function $f : \Omega \to \mathbb{C}$. Let us denote its real and imaginary parts by u, v, respectively, so that

$$f(x + iy) = u(x, y) + iv(x, y)$$

(u, v are defined on $\Omega' \subseteq RR^2$ which is equivalent to Ω in the obvious way.) This allows us to consider f as a pair of functions from $\mathbb{R}^2 \to \mathbb{R}$, which are surfaces lying in $\mathbb{R}^3$. We will investigate which choices of u, v produce a valid holomorphic f.

Let h be a (small) complex number and write $h = h_1 + ih_2$. Then write

$$\frac{f(z+h) - f(z)}{h} = \frac{u(x + h_1, y + h_2) - u(x, y)}{h_1 + ih_2} + \frac{v(x + h_1, y + h_2) - iv(x, y)}{h_1 + ih_2}$$

Let us consider what happens as h tends to 0 from different directions. For instance, suppose h is entirely real, so $h_2 = 0$. Then

$$\lim_{h_1 \to 0} \frac{f(z + h_1) - f(z)}{h_1} = \lim_{h_1 \to 0} \frac{u(x + h_1, y) - u(x, y)}{h_1} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

and similarly

$$\lim_{h_2 \to 0} \frac{f(z + h_2) - f(z)}{ih_2} = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Then if f is holomorphic, then we can match components to get the following:

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial v}{\partial y} = -\dfrac{\partial u}{\partial y} \end{cases} \tag{$*$}$$

The system of equations ($*$) are known as the **Cauchy-Riemann equations**. We have shown that these are a necessary conditions for f to be holomorphic; below we will show that if we also assume that the partials are continuous, then we have a sufficient condition. Later, we will show that u, v are necessarily continuously differentiable, so that these are equivalent characterizations. For now we will content ourselves with one direction:

Theorem 1.54

Let $\Omega \subseteq \mathbb{C}$ be open and let $f : \Omega \to \mathbb{C}$. Let $f = u + iv$, where $u, v : \Omega \to \mathbb{R}$ are continuously differentiable and satisfy the Cauchy-Riemann equations. Then f is holomorphic.

Proof. Consider Taylor's expansion, which says that

$$u(x+h_1, y+h_2) = u(x,y) + \frac{\partial u}{\partial x}h_1 + \frac{\partial u}{\partial y}h_2 + h\psi_u(h)$$

where $h = h_1 + ih_2$ and $\psi_u(h) \to 0$ as $h \to 0$. Similarly

$$v(x+h_1, y+h_2) = v(x,y) + \frac{\partial v}{\partial x}h_1 + \frac{\partial v}{\partial y}h_2 + h\psi_v(h)$$

Now by assumption, f satisfies the Cauchy-Riemann equations, so

$$v(x+h_1, y+h_2) = v(x,y) - \frac{\partial u}{\partial y}h_1 + \frac{\partial u}{\partial x}h_2 + h\psi_v(h)$$

so

$$\begin{aligned} f(x+h) &= u(x+h_1, y+h_2) + iv(x+h_1, y+h_2) \\ &= u(x,y) + \frac{\partial u}{\partial x}h_1 + \frac{\partial u}{\partial x}h_2 + h\psi_u(y) + iv(x,y) - i\frac{\partial u}{\partial y}h_1 + i\frac{\partial u}{\partial x}h_2 + ih\psi_v(h) \\ &= f(x,y) + (h_1 + ih_2)\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right) + h(\underbrace{\psi_u(h) + i\psi_v(h)}_{\psi(h)}) \\ &= f(z) + h\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right) + h\psi(h) \end{aligned}$$

so f is holomorphic using the alternative characterization and

$$f'(z) = f'(x,y) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$ □

Let f be a complex valued function of the form $f(x+iy) = u(x,y) + iv(x,y)$. Associate with it a $\mathbb{R}^2$-valued function

$$F(x,y) = (u(x,y), v(x,y))$$

Recall that its Jacobian matrix is

$$J_F(x,y) = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} \tag{$**$}$$

and that F is differentiable in the real sense if it is true that

$$\lim_{(h_1,h_2)\to(0,0)} \frac{F(x+h_1, y+h_2) - F(x,y) - J_F(x,y)\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}}{|(h_1,h_2)|} = 0$$

Comparing this to the complex condition

$$\lim_{h\to 0} \frac{f(z+h) - f(z) - f'(z)h}{h} = 0$$

we can see that complex differentiability requires $J_f(x,y)$ to be of the form of multiplying by some complex number. This happens if and only if

$$J_F(x,y) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

for $a, b \in \mathbb{R}$. By reconciling this with $(**)$ we recover the Cauchy-Riemann equations $(*)$.

Definition 1.55

Let $f : \Omega \to \mathbb{C}$ with Ω open. Then we define

$$\frac{\partial f}{\partial x} \coloneqq \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

and

$$\frac{\partial f}{\partial y} \coloneqq \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y}$$

We further define

$$\frac{\partial f}{\partial z} \coloneqq \frac{1}{2} \left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right)$$

and

$$\frac{\partial f}{\partial \overline{z}} \coloneqq \frac{1}{2} \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right)$$

Proposition 1.56

Let $\Omega \subseteq \mathbb{C}$ be open and let $f : \Omega \to \mathbb{C}$ be of the form $f(x+iy) = u(x,y) + iv(x,y)$. If f is holomorphic on Ω, then

1. $\dfrac{\partial f}{\partial \overline{z}} = 0$.
2. $f'(z_0) = 2\frac{\partial u}{\partial z}(z_0)$.
3. $F = (u(x,y), v(x,y))$ is differentiable in the real sense, and $\det J_F(x,y) = |f'(x+iy)|^2$.

Proof. 1. Follows from Cauchy-Riemann.

2. We have

$$\begin{aligned}2\frac{\partial u}{\partial z}(z_0) &= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \\ &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ &= \lim_{h_1\to 0}\frac{u(x+h_1,y)-u(x,y)}{h_1} + i\frac{v(x+h_1,y)-v(x,y)}{h_1} \\ &= \lim_{h_1\to 0}\frac{f(z+h_1)-f(z)}{h_1} \\ &= f'(z)\end{aligned}$$

3. We have

$$\det J_F(x,y) = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

By the Cauchy-Riemann equations this is

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = |f'(x+iy)|^2$$

□

1.4 Power Series

We will now discuss power series, which are defined similarly to the real case. They will initially serve as a valuable example of holomorphic functions. Later we will see that they are actually the *only* example, which justifies particular attention in their study.

Definition 1.57

A **power series** (centered around z_0) is a function f of the form

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n := \lim_{N\to\infty}\sum_{n=0}^{N} a_n(z-z_0)^n$$

where $\{a_n\} \subseteq \mathbb{C}$, which is defined wherever the right hand limit converges.

Note that it is certainly a necessary condition that $a_n z^n \to 0$ as $n \to \infty$, since

$$\lim_{n\to\infty} a_n z^n = \lim_{n\to\infty}\left[\sum_{k=0}^{n} a_k z^k - \sum_{k=0}^{n-1} a_k z^k\right] = \lim_{n\to\infty}\left[\sum_{k=0}^{n} a_k z^k\right] - \lim_{n\to\infty}\left[\sum_{k=0}^{n-1} a_k z^k\right] = 0$$

In this section, we will first consider only power series which are centered at 0.

Definition 1.58

We say that a power series $\sum_{n=0}^{\infty} a_n z^n$ **converges absolutely** (in the complex sense) if the series

$$\sum_{n=0}^{\infty} |a_n| \cdot |z|^n$$

converges in the real case.

Proposition 1.59

If $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely, then it converges (in the complex sense).

Proof. Let $\varepsilon > 0$.

Write

$$S_N(z) = \sum_{n=0}^{N} a_n z^n$$

Since

$$\sum_{n=0}^{\infty} |a_n| \cdot |z|^n$$

converges, there exists $N_0 \in \mathbb{N}$ such that

$$\sum_{n=N_0+1}^{\infty} |a_n| \cdot |z|^n < \varepsilon$$

Then for $M, N \geq N_0$, assume without loss of generality that $M < N$. Then we have

$$|S_N(z) - S_M(z)| = \left| \sum_{n=M+1}^{N} a_n z^n \right| \leq \sum_{a=M+1}^{N} |a_n| \cdot |z|^n \leq \sum_{a=N_0+1}^{\infty} |a_n| \cdot |z|^n < \varepsilon$$

so $\{S_N(z)\}$ is a Cauchy sequence, and thus

$$\sum_{n=0}^{\infty} a_n z^n$$

converges. $\square$

Recall that in the single real variable case, we found that power series converge on some interval (possibly open, closed, or half-open) which is centered around 0 (or any other point of expansion). An analogous statement is true here, with the interval replaced by a disk.

Theorem 1.60

For any power series $\sum_{n=0}^{\infty} a_n z^n$ there exists $0 \leq R \leq \infty$[a] (called the **radius of convergence**) such that for any $z \in \mathbb{C}$:

1. If $|z| < R$, then $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely.
2. If $|z| > R$, then $\sum_{n=0}^{\infty} a_n z^n$ does not converge.

Moreover, R is given by **Hadamard's formula**:

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}$$

[a]If $R = \infty$ then we take $|z| < R$ for all $z \in \mathbb{C}$.

This theorem says that we have absolute convergence inside the disk $\mathbb{D}_R$ (called the **disk of convergence**), and divergence outside of it. As in the real case, this theorem makes no statement about convergence on the boundary of the disk.

Proof. Denote $L = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. Suppose that $L \neq 0, \infty$. If $|z| < \frac{1}{L} = R$, then $L|z| < 1$, so there exists $\varepsilon > 0$ such that $(L+\varepsilon)|z| = r < 1$. By the definition of lim sup, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\sqrt[n]{|a_n|} < L + \varepsilon \implies |a_n| < (L+\varepsilon)^n \implies |a_n||z|^n < ((L+\varepsilon)|z|)^n = r^n$$

so $\sum_{n=0}^{\infty} a_n z^n$ is dominated by the absolutely convergent geometric series $\sum_{n=0}^{\infty} r^n$ and thus converges absolutely.

On the other hand, if $|z| > R = \frac{1}{L}$, then $L|z| > 1$ so there exists a subsequence $\{a_{n_k}\}$ such that

$$\sqrt[n_k]{|a_{n_k}|} \cdot |z| > 1$$

for all k. Then

$$|a_{n_k} \cdot |z|^{n_k} > 1$$

which does not tend to 0 as $k \to \infty$, so convergence is impossible.

EXERCISE: Complete the proof for $L = 0, \infty$. $\square$

Example 1.61

The power series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}$$

has $R = \infty$, since the real-valued power series

$$\sum_{n=0}^{\infty} \frac{|z|^n}{n!} = e^{|z|}$$

converges absolutely everywhere. This also allows us to define the **exponential** of

z as

$$e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Example 1.62

The power series

$$\sum_{n=0}^{\infty} z^n$$

has radius of convergence 1. This can be seen either by direct computation in the real case, or using Hadamard's formula and the fact that each a_n is 1:

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{1}} = \frac{1}{1} = 1$$

Moreover, this power series satisfies the equation

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

This can be seen using the identity for partial sums (which holds in all fields)

$$\sum_{n=0}^{N} z^n = \frac{1-z^{N+1}}{1-z}$$

so

$$\lim_{N\to\infty} \frac{1-z^{N+1}}{1-z} = \frac{1}{1-z}$$

Definition 1.63

We define the **trigonometric functions** in terms of power series:

$$\cos z := \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$
$$\sin z := \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

Note that both of the above series converge with $R = \infty$. Moreover, we can observe that

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$
$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

and that this is consistent with our previous definition in terms of the identity

$$e^{iz} = \cos z + i \sin z$$

We now prove a fundamental fact about power series which, while analogous to the real case, will have farther reaching implications for us.

Theorem 1.64

The function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is holomorphic on the disk of convergence $\mathbb{D}_R$. Moreover, its derivative is given by

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n$$

which has the same radius of convergence.

Proof. To show that the radius of convergence is the same, simply apply Hadamard's formula to the new power series. Letting R' be the radius of convergence of $\sum_{n=1}^{\infty} n a_n z^{n-1}$, we have

$$\frac{1}{R'} = \limsup_{n\to\infty} \sqrt[n]{(n+1)|a_n|} = \limsup_{n\to\infty} \sqrt[n]{n+1}\,\sqrt[n]{|a_n|}$$

But $\sqrt[n]{n+1} \to 1$ so

$$\limsup_{n\to\infty} \sqrt[n]{n+1}\,\sqrt[n]{|a_n|} = \limsup_{n\to\infty} \sqrt[n]{|a_n|} = \frac{1}{R}$$

so $R' = R$.

Denote

$$g(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$$

We want to show that $f' = g$. It suffices to show that

$$\frac{f(z_0+h) - f(z_0)}{h} - g(z_0) \underset{h\to 0}{\to} 0$$

whenever $z_0 \in \mathbb{D}_R$.

To show this, pick any $z_0 \in \mathbb{D}_R$ and let $\varepsilon > 0$. Then there exists r such that $|z_0| < r < R$. Then whenever $|h| < r - |z_0|$ we have $|z_0 + h| \leq |z_0| + |h| < r < R$. So for small h, $z_0 + h \in \mathbb{D}_R$.

Denote the partial sums and error terms by

$$S_N(z) = \sum_{n=1}^{N} a_n z^n$$

$$E_N(z) = \sum_{n=N+1}^{\infty} a_n z^n$$

By the observation above,

$$\sum_{n=1}^{\infty} n|a_n|z^{n-1}$$

so there exists $N_0 \in \mathbb{N}$ such that

$$\sum_{n=N_0+1}^{\infty} n|a_n||z|^{n-1} < \frac{\varepsilon}{3} \tag{1}$$

Then S_{N_0} is a polynomial, which we showed is holomorphic. So

$$\frac{S_{N_0}(z_0+h) - S_{N_0}(z_0)}{h} - S'_N(z_0) = \frac{S_{N_0}(z_0+h) - S_{N_0}(z_0)}{h} - \sum_{n=1}^{N_0} na_n z^{n-1} \underset{h\to 0}{\to} 0$$

Thus there exists $\delta > 0$ such that whenever $|h| < \delta$,

$$\left|\frac{S_{N_0}(z_0+h) - S_{N_0}(z_0)}{h} - S'_N(z_0)\right| < \frac{\varepsilon}{3} \tag{2}$$

It follows that the difference quotient

$$\left|\frac{f(z_0+h) - f(z_0)}{h} - g(z_0)\right|$$

may be written as

$$\begin{aligned}
&\left|\frac{S_{N_0}(z_0+h) - S_{N_0}(z_0)}{h} - S'_{N_0}(z_0) + S'_{N_0}(z_0) - g(z_0) + \frac{E_{N_0}(z_0+h) - E_N(z_0)}{h}\right| \\
\le &\left|\frac{S_{N_0}(z_0+h) - S_{N_0}(z_0)}{h} - S'_{N_0}(z_0)\right| + \left|S'_{N_0}(z_0) - g(z_0)\right| + \left|\frac{E_{N_0}(z_0+h) - E_N(z_0)}{h}\right|
\end{aligned}$$

For small h, the first term is less than $\frac{\varepsilon}{3}$ by (2). Also,

$$\left|S'_{N_0}(z_0) - g(z_0)\right| = \left|-\sum_{n=N_0+1}^{\infty} na_n z^{n-1}\right| \le \sum_{n=N_0+1}^{\infty} n|a_n| \cdot |z|^{n-1} < \frac{\varepsilon}{3}$$

by (1). Lastly,

$$\left|\frac{E_{N_0}(z_0+h) - E_{N_0}(z_0)}{h}\right| = \left|\sum_{n=N_0+1}^{\infty} a_n \frac{(z_0+h)^n - z_0^n}{h}\right|$$

Using the identity

$$a^n - b^n = (a-b)\sum_{k=0}^{n-1} a^k b^{n-1-k}$$

for $a = z_0 + h, b = z_0$ we have

$$\left| \sum_{n=N_0+1}^{\infty} a_n \left((z_0 + h)^n - z_0^n \right) \right| = \left| \sum_{n=N_0+1}^{\infty} a_n \left(\sum_{k=0}^{n-1} a^k b^{n-1-k} \right) \right|$$
$$\leq \sum_{n=N_0+1}^{\infty} |a_n| \sum_{k=0}^{n-1} a^k b^{n-1-k} < \sum_{n=N_0+1}^{\infty} |a_n| n r^n < \frac{\varepsilon}{3}$$

where the last inequality follows from (1). Thus

$$\left| \frac{f(z_0 + h) - f(z_0)}{h} - g(z_0) \right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

□

Corollary 1.65

A power series is infinitely complex differentiable on its disk of convergence.

Proof. Induct using Theorem 1.64. □

We now note that for the case of power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ centered around arbitrary z_0, they converge on $\mathbb{D}_R(z_0)$, where R is the radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$. In other words, the behavior is identical, merely translated. Thus it suffices to consider power series centered around 0.

Definition 1.66

Let $\Omega \subseteq \mathbb{C}$ be open and $z_0 \in \Omega$. Then $f : \Omega \to \mathbb{C}$ is said to be **analytic** at z_0 if there exists $r > 0$ such that $\mathbb{D}_r(z_0) \subseteq \Omega$ and there exist coefficients $\{a_n\} \subseteq \mathbb{C}$ such that

$$\sum_{n=0}^{\infty} a_n z^n$$

converges absolutely on $\mathbb{D}_r$ and

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

on $\mathbb{D}_r(z_0)$.

We say that f is analytic on Ω if it is analytic at each $z_0 \in \Omega$.

By Theorem 1.64, each analytic function is (infinitely) holomorphic. We will show later that every holomorphic function is also analytic. Thus the terms are often used interchangably, but we will not do so until we have proved this fact.

1.5 Integration on Curves

We will soon show that the behavior of analytic functions can be well understood by studying the behavior of those functions when integrated over various curves in the plane. This will motivate the particular definitions of an integral over a curve that appear in this section.

Definition 1.67

A **parameterized curve** is a continuous function $z : [a, b] \to \mathbb{C}$. Writing $z(t) = z_1(t) + iz_2(t)$, we say that z is a **smooth curve** if $z_1'(t), z_2'(t)$ exist for all $t \in [a, b]$.

z is said to be **piecewise smooth** if there exist $a = a_0 < a_1 < \ldots < a_n = b$ such that z is smooth on $[a_k, a_{k+1}]$ for each k.

Note that in our definition of piecewise smooth curves, z must be continuous at each a_k, but the left and right derivatives need not coincide.

We will adopt the convention that all curves are assumed to be piecewise smooth, unless stated otherwise.

A parameterized curve traces out a particular image in the complex plane, which is intuitively a curve in the plane. Thus it is useful to distinguish the function z and its image.

Definition 1.68

Two parameterized curves $z_1 : [a, b] \to \mathbb{C}$ and $z_2 : [c, d] \to \mathbb{C}$ are said to be **equivalent curves** if there exists a continuously differentiable bijection $t : [a, b] \to [c, d]$ such that $t'(s) > 0$ and $z_2(s) = z(t(s))$ for all $s \in [a, b]$.

The condition that $t'(s) > 0$ could be equivalently stated as saying that $t(a) = c$ and $(b) = d$. Thus an equivalence class of curves defines a subset of $\mathbb{C}$ together with a particular orientation.

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