MAT 335 Notes

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Contents

Introduction

This document contains notes taken for the class MAT 335: Complex Analysis at Princeton University, taken in the Fall 2024 semester. These notes are primarily based on lectures by Professor Assaf Naor. Other references used in these notes include *Complex Analysis* by Elias Stein and Rami Shakarchi, *Complex Analysis* by Lars Ahlfors, *Visual Complex Analysis* by Tristan Needham, and *Real and Complex Analysis* by Walter Rudin. Since these notes were primarily taken live, they may contains typos or errors.

Chapter 1

Preliminaries

1.1 The Complex Number System

The set of complex numbers, denoted $\mathbb{C}$ is identified with ordered pairs $(x, y) \in \mathbb{R}^2$. We may alternately write this as $x + iy$, where the symbol i is currently undefined.

For a given complex number $z = x + iy$, $x = \mathrm{Re}(z)$ is called the **real part** of z, $y = \mathrm{Im}(z)$ is called the **imaginary part**, $|z| = \sqrt{x^2 + y^2}$ is the **modulus** of z, and the **argument** of z, $\theta = \arg(z)$, is the angle between (x, y) and the x-axis, defined up to integer multiples of 2π.

Definition 1.1

Let $\theta \in \mathbb{R}$. We define

$$e^{i\theta} = \cos\theta + i\sin\theta = (\cos\theta, \sin\theta)$$

One can observe using the identity $\cos^2 + \sin^2 = 1$ that $e^{i\theta}$ lies on the unit circle. Moreover, if $r = |z|$, then elementary geometry shows that we have $z = re^{i\theta}$ using the definition above.

Proposition 1.1

For any $z \in \mathbb{C}$, $|\mathrm{Re}(z)| \leq |z|$ and $|\mathrm{Im}(z)| \leq |z|$.

Proof. $|z| = \sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2}$. $\square$

One of the distinguishing features of $\mathbb{C}$ from the real plane $\mathbb{R}^2$ is the algebraic structure present on $\mathbb{C}$.

Definition 1.2

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then we define addition and multiplication on $\mathbb{C}$ by

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$
$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)$$

Taking $i = (0, 1)$, then we observe that $i^2 = -1 + 0i = -1$. Thus we recover the basic identity $i^2 = -1$. We also observe that Re and Im are both linear operators.

Proposition 1.2

Addition and multiplication over $\mathbb{C}$ are commutative and associative. Moreover, multiplication distributes over addition.

Proof. Commutative and associativity of addition is inherited from $\mathbb{R}$. $\square$

Using the definition of $e^{i\theta}$, we can reinterpret complex multiplication in a much more pleasant manner than the definition above.

Proposition 1.3

If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then

$$z_1 z_2 = (r_1 r_2) e^{i(\theta_1 + \theta_2)}$$

Proof. We have proved commutativity. From here, we apply trig identities. $\square$

Thus multiplication results in multiplication of lengths and addition of arguments.

Proposition 1.4

For $z_1, z_2 \in \mathbb{C}$, the **triangle inequality** holds:

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

Proof. Choose r, θ such that $z_1 + z_2 = re^{i\theta}$. Then

$$|z_1 + z_2| = r = (z_1 + z_2)e^{-i\theta} = z_1 e^{-i\theta} + z_2 e^{-i\theta} = \mathrm{Re}(z_1 e^{-i\theta} + z_2 e^{-i\theta})$$

Now note that $\mathrm{Re}(z + w) = \mathrm{Re}(z) + \mathrm{Re}(w)$. So

$$\mathrm{Re}(z_1 e^{-i\theta} + z_2 e^{-i\theta}) = \mathrm{Re}(z_1 e^{-i\theta}) + \mathrm{Re}(z_2 e^{-i\theta}) \leq \left|z_1 e^{-i\theta}\right| + \left|z_2 e^{-i\theta}\right| = |z_1| + |z_2| \quad \square$$

The above proof amounts to applying the real triangle inequality to the components of z_1, z_2 in the direction of $z_1 + z_2$.

Corollary 1.5

The reverse triangle inequality also holds:

$$||z| - |w|| \leq |z - w|$$

Proof. We have

$$\begin{cases} |z| \leq |z - w| + |w| \\ |w| \leq |w - z| + |z| \end{cases} \implies \begin{cases} |z| - |w| \leq |z - w| \\ -|z - w| \leq |z| - |w| \end{cases} \implies ||z| - |w|| \leq |z - w| \qquad \square$$

Definition 1.3

Let $z = x + iy \in \mathbb{C}$. Then the **complex conjugate** of z is defined as

$$\overline{z} = x - iy$$

Geometrically, this is reflection over the x axis.

Proposition 1.6

For $z \in \mathbb{C}$, $z\overline{z} = |z|^2$.

Proof. Let $z = x + iy$. Then

$$z\overline{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2 \qquad \square$$

Definition 1.4

For $z \neq 0$, define

$$\frac{1}{z} = \frac{\overline{z}}{|z|^2}$$

The above proposition and definition show that

$$z \cdot \frac{1}{z} = 1$$

Definition 1.5

A **sequence** of complex numbers $\{z_n\}_{n=1}^{\infty}$ **converges** to $z \in \mathbb{C}$ (written $\lim_{n\to\infty} z_n = z$) if

$$\begin{cases} \lim_{n\to\infty} \operatorname{Re}(z_n) = \operatorname{Re}(z) \\ \lim_{n\to\infty} \operatorname{Im}(z_n) = \operatorname{Im}(z) \end{cases}$$

This equivalent to the familiar definition:

Proposition 1.7

A sequence $\{z_n\} \subseteq \mathbb{C}$ converges to z if and only for $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that whenever $n \geq N$ we have

$$|z_n - z| < \varepsilon$$

Proof. ($\implies$) Let $\varepsilon > 0$. Then pick N_1, N_2 such that

$$\begin{cases} n \geq N_1 \implies |\mathrm{Re}(z_n) - \mathrm{Re}(z)| < \frac{\varepsilon}{\sqrt{2}} \\ n \geq N_2 \implies |\mathrm{Im}(z_n) - \mathrm{Im}(z)| < \frac{\varepsilon}{\sqrt{2}} \end{cases}$$

Letting $N = \max\{N_1, N_2\}$, whenever $n \geq N$ we have

$$|z_n - z|^2 = \mathrm{Re}(z_n - z)^2 + \mathrm{Im}(z_n - z)^2 = |\mathrm{Re}(z_n) - \mathrm{Re}(z)|^2 + |\mathrm{Im}(z_n) - \mathrm{Im}(z)|^2 < \frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2}$$

Taking square roots on both sides we have

$$|z_n - z| < \varepsilon$$

($\impliedby$) $|\mathrm{Re}(z_n) - \mathrm{Re}(z)| = |\mathrm{Re}(z_n - z)| \leq |z_n - z|$ $\square$

We similarly define the limit of a complex function $\lim_{z \to a} f(z)$.

Definition 1.6

A **Cauchy sequence** is a sequence $(z_n) \subseteq \mathbb{C}$ such that $(\mathrm{Re}(z_n))$ and $(\mathrm{Im}(z_n))$ are both Cauchy.

Again we can formulate this analogously to the single variable case:

Proposition 1.8

A sequence $\{z_n\} \subseteq \mathbb{C}$ is Cauchy if and only if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that whenever $m, n \geq N$ it follows that

$$|z_n - z_m| < \varepsilon$$

Proof. Same as the proof of Proposition 1.7. $\square$

Proposition 1.9

A Cauchy sequence is convergent.

Proof. Follows from completeness of $\mathbb{R}$:

$$\{z_n\} \text{ conv.} \iff \begin{cases} \{\mathrm{Re}(z_n)\} \text{ conv.} \\ \{\mathrm{Im}(z_n)\} \text{ conv.} \end{cases} \iff \begin{cases} \{\mathrm{Re}(z_n)\} \text{ Cauchy} \\ \{\mathrm{Im}(z_n)\} \text{ Cauchy} \end{cases} \iff \{z_n\} \text{ Cauchy} \quad \square$$

1.2 Topology of $\mathbb{C}$

The topological nature of $\mathbb{C}$ should not be unfamiliar to the reader, since it is essentially the same as that of $\mathbb{R}^2$, rephrased slightly using complex variables.

Definition 1.7

Let $r > 0$ and $z_0 \in \mathbb{C}$. Then the **open disk** of radius ε about z is the set

$$\mathbb{D}_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$$

and the **closed disk** as

$$\overline{\mathbb{D}_r}(z_0) = \{z \in \mathbb{C} : |z - z_0| \leq r\}$$

We also specify $\mathbb{D}_r = \mathbb{D}_r(0)$ and $\mathbb{D} = \mathbb{D}_1$.

Definition 1.8

An **interior point** $z_0 \in \Omega$ of a subset $\Omega \subseteq \mathbb{C}$ is a point such that there exists $r > 0$ where $\mathbb{D}_r(z_0) \subseteq \Omega$.

Definition 1.9

The set of interior point in Ω is the **interior** of Ω, denoted $\operatorname{int} \Omega$.

Definition 1.10

An **open set** in $\mathbb{C}$ is a subset $\Omega \subseteq \mathbb{C}$ such that for any $z_0 \in \Omega$ there exists $\varepsilon > 0$ such that $D_\varepsilon(z_0) \subseteq \Omega$.

It is immediate that Ω is open if and only if $\operatorname{int} \Omega = \Omega$.

Definition 1.11

Let $\Omega \in \mathbb{C}$ and let $z \in \mathbb{C}$. z is a **limit point** of Ω if there exists a sequence of points $\{z_n\}_{n=1}^{\infty} \subseteq \Omega$ such that $z_n \neq z$ for each n and $\lim z_n = z$.

We can equivalently define a limit point as a point z such that $\mathbb{D}_r(z) \setminus \{z\} \cap \Omega \neq \varnothing$ for each $r > 0$

Definition 1.12

$A \subseteq \mathbb{C}$ is a **closed set** if $\mathbb{C} \setminus A$ is open.

Proposition 1.10

A is closed if and only if it contains all its limit points.

Proof. ($\implies$) Suppose not. Then pick z which is a limit point of A that is not in A. Then there is no disk around z entirely contained in $\mathbb{C} \setminus A$. Thus A is not closed.

($\impliedby$) Suppose A is not closed. Then there exists $z \notin A$ such that each $\mathbb{D}_r(z) \setminus \{z\}$ intersects A. Then z is a limit point of A. □

Definition 1.13

The **closure** of $\Omega \subseteq \mathbb{C}$, denoted $\overline{\Omega}$, is the union of Ω with its limit points.

Definition 1.14

The **boundary** of $\Omega \subseteq \mathbb{C}$, denoted $\partial\Omega$, is defined as $\overline{\Omega} \setminus \operatorname{int}\Omega$.

Definition 1.15

$\Omega \subseteq \mathbb{C}$ is **bounded** if there exists $M > 0$ such that $|z| < M$ for each $z \in \Omega$ (or equivalently, $\Omega \subseteq \mathbb{D}_M$).

Definition 1.16

Let $\Omega \subseteq \mathbb{C}$ be bounded. Then the **diameter** of Ω is defined as

$$\operatorname{diam}\Omega = \sup_{z,w\in\Omega} |z - w|$$

The following definition, as in the real case, is critical:

Definition 1.17

$\Omega \subseteq \mathbb{C}$ is **compact** if it is closed and bounded.

Theorem 1.11: Bolzano-Weierstrass Theorem

Let $\Omega \subseteq \mathbb{C}$. Then the following conditions are equivalent:

1. Ω is compact.
2. Each sequence $\{z_n\}_{n=1}^{\infty} \subseteq \Omega$ has a subsequence $\{z_{n_k}\}_{k=1}^{\infty}$ which converges to some $z \in \Omega$.

We can treat $\mathbb{C}$ similarly to $\mathbb{R}^2$ to prove this.

Proof. ($1 \implies 2$) If Ω is compact, then $\{z_n\}_{n=1}^\infty \subseteq \mathbb{C}$ may be written as $\{x_n + iy_n\}_{n=1}^\infty \subseteq \mathbb{C}$. Since Ω is bounded, there exists $M > 0$ such that $|z| < M$ for all $z \in \Omega$. In particular $\sqrt{x_n^2 + y_n^2} = |z_n| < M$. So the real sequences $\{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty$ are bounded. Apply the real version of Bolzano-Weierstrass, there exists a convergent subsequence $\{x_{n_k}\}$. Then consider the sequence $\{y_{n_k}\}$. This is also bounded, so we apply Bolzano-Weierstrass again to produce $\{y_{n_{k_i}}\}$ convergent. Then the sequence $\{z_{n_{k_i}}\}$ is a convergent subsequence. If $z = z_n$ for some n, then $z \in \Omega$; otherwise it is a limit point. Since Ω is closed it contains its limit points so $z \in \Omega$.

($2 \implies 1$) Suppose each sequence has a convergent subsequence. Let z be a limit point and let $\{z_n\} \subseteq \Omega \setminus \{z\}$ be a sequence converging to z. Then there exists a subsequence $\{z_{n_k}\}$ which converges to $z' \in \Omega$. But subsequences converge to the same value as the original sequence, so $z = z' \in \Omega$. So Ω is closed. If Ω is not bounded, then we may take $\{z_n\}$ such that $|z_n| \geq n$, and that $|z_{n+1}| > |z_n| + 1$. But then $|z_m - z_{m-1}| > 1$ so no subsequence is Cauchy and thus no subsequence converges. So Ω is bounded. $\square$

Definition 1.18

An **open cover** of a set $\Omega \subseteq \mathbb{C}$ is a collection $\mathcal{O}$ of open sets such that each $z \in \Omega$ is contained in some $O \in \mathcal{O}$. A **subcover** of $\mathcal{O}$ is a subcollection which is still a cover.

Theorem 1.12: Heine-Borel Theorem

A set $\Omega \subseteq \mathbb{C}$ is compact if and only if every open cover has a finite subcover.

Proof. ($\implies$) Since Ω is bounded, it is a subset of a closed rectangle K. We showed in $\mathbb{R}^2$ that $X \times Y$ is compact when $X, Y \subseteq \mathbb{R}$ are, and the same is true here. So K is compact. Take an open cover $\mathcal{O}$ of Ω and add the (open) set $\mathbb{C} \setminus \Omega$. This is an open cover of $\mathbb{C}$ and thus one of K, so only finitely many are needed. Remove $\mathbb{C} \setminus \Omega$ if necessary and we still have an open cover of Ω.

($\impliedby$) Boundedness is immediate by covering Ω with balls of finite radius.

For closure, suppose not. Then take a limit point $w \notin \Omega$. Each $z \in \Omega$ has $|z - w| > 0$, so we may cover Ω with open balls $O_z = \mathbb{D}_\varepsilon(z)$ where $\varepsilon < |z - w|/2$. Then a finite number of them cover Ω but this implies that y is not a limit point. $\square$

For the sake of completeness, here is an independent proof that a set is sequentially compact if it is covering compact.

Proof that covering compactness $\implies$ sequential compactness. Let K be covering compact and pick a sequences $\{a_n\} \subseteq K$. Suppose for contradiction that a_n has no convergent subsequence in K. Then for each $x \in K$, there exists $\varepsilon_x > 0$ and $N_x \in \mathbb{N}$ such that whenever $n \geq N_x$ it follows that $a_n \notin \mathbb{D}_{\varepsilon_x}(x)$. Then the collection of $\mathbb{D}_{\varepsilon_x}(x)$ for $x \in K$ is an open cover of K, so we may pick a finite subcover

$$\mathbb{D}_{\varepsilon_{x_1}}(x_1), \ldots, \mathbb{D}_{\varepsilon_{x_m}}(x_m)$$

Then let $N = \max N_{x_i}$. For $n \geq N$ it follows that $a_n \notin K$, contradiction. $\square$

Proposition 1.13: Nested Compact Set Property

Suppose that $\Omega_1 \supseteq \Omega_2 \supseteq \dots$ is a nested sequence of compact, nonempty subsets of $\mathbb{C}$. Then

$$\bigcap_{n=1}^{\infty} \Omega_n \neq \varnothing$$

Moreover, if $\lim_{n\to\infty} \operatorname{diam} \Omega_n = 0$, then there is a unique point $z \in \mathbb{C}$ such that $z \in \Omega_n$ for all n.

Proof. Choose $z_n \in \Omega_n$ for each n. Then the sequence of points $\{z_n\} \subseteq \Omega_1$, and Ω_1 is compact, so there exists a convergent subsequence $\{z_{n_k}\}$ tending to $z \in \Omega_1$. Then for arbitrary Ω_n, there exists a subsequence $\{z_{n_{k+k_0}}\} \subseteq \Omega_n$ for sufficiently large k_0, which converges to z and we see that $z \in \Omega_n$. So the intersection is nonempty.

To show uniqueness, take $z, w \in \bigcap_{n=1}^{\infty} \Omega_n$. Then

$$|z - w| \leq \operatorname{diam} \Omega_n$$

for each n, but $\operatorname{diam} \Omega_n \to 0$ so $|z - w| = 0$ and thus $z = w$. $\square$

Remark

With the assumption that $\operatorname{diam} \Omega_n \to 0$, we need not take subsequences as $\{z_n\}$ itself is Cauchy. To see this, pick $\varepsilon > 0$ and let N be such that $\operatorname{diam} \Omega_n < \varepsilon$ for any $n \geq N$. Then for any $n, m \geq N$, $z_n, z_m \in \Omega_N$ and thus $|z_n - z_m| \leq \operatorname{diam} \Omega_N < \varepsilon$.

Definition 1.19

A set $\Omega \subseteq \mathbb{C}$ is **connected** if there are no nonempty disjoint sets $A, B \subseteq \Omega$ such that $\Omega = A \sqcup B$ such that $A \cap \overline{B} = \varnothing$ and $\overline{A} \cap B = \varnothing$.

The above definition may be rephrased as saying that Ω is not the disjoint union of nonempty sets which are open in the subspace topology of Ω:

Proposition 1.14

A topological space X is connected if and only if it cannot be written as $X = \Omega_1 \cup \Omega_2$ with Ω_1, Ω_2 nonempty, disjoint and open (in X).

Proof. ($\implies$) Suppose $X \subseteq \mathbb{C}$ is connected. Let $\Omega_1, \Omega_2 \subseteq X$ be nonempty, open and disjoint. Consider Ω_2. Then $\Omega_2 \subseteq X \setminus \Omega_1$. By definition $X \setminus \Omega_1$ is closed. $\overline{\Omega_2}$ is the smallest closed set containing Ω_2, so $\overline{\Omega_2} \subseteq X \setminus \Omega_1$ and thus $\Omega_1 \cap \overline{\Omega_2} = \varnothing$. Similarly $\overline{\Omega_1} \cap \Omega_2 = \varnothing$. Since X is connected, we conclude that $\Omega_1 \cup \Omega_2 \neq X$.

($\impliedby$) Pick A, B nonempty with $X = A \cup B$. Assume that $A \cap \overline{B} = B \cap \overline{A} = \varnothing$, so that $A \subseteq X \setminus \overline{B}$ and $B \subseteq X \setminus \overline{A}$. Define $\Omega_1 = X \setminus \overline{B}$ and $\Omega_2 = X \setminus \overline{A}$. Since $X = A \cup B$, we have $X = \Omega_1 \cup \Omega_2$. Ω_1, Ω_2 are both open in X, so it must not be the case that they are disjoint. So there exists some $x \in \Omega_1 \cap \Omega_2$. But this implies that $x \notin A$ and $x \notin B$. $\square$

Thus the above general definition can be simplified for nicer sets:

Proposition 1.15

If Ω is open, then it is connected if and only if it cannot be written as the union of disjoint open sets (in $\mathbb{C}$). Similarly if F is closed then it is connected if and only if it is not the union of disjoint closed sets.

We may also introduce another notion of connectedness, which involves functions into Ω.

Definition 1.20

Suppose $\Omega \subseteq \mathbb{C}$ and $f : \Omega \to \mathbb{C}$. f is **continuous** at z_0 if for every $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $z \in \Omega$ and $|z - z_0| < \delta$, it follows that $|f(z) - f(z_0)| < \varepsilon$.

Proposition 1.16

f is continuous at z_0 if and only if for every $\{z_n\} \subseteq \Omega$ with $z_n \to z_0$, it follows that $f(z_n) \to f(z_0)$. We say that f is continuous on Ω if it is continuous at each point in Ω.

Definition 1.21

A **path** is a function $f : [0, 1] \to \mathbb{C}$. A continuous path is a continuous such function.

Definition 1.22

A set $\Omega \subseteq \mathbb{C}$ is **path connected** if for any $z, w \in \Omega$ there exists a continuous path with $f(0) = z$ and $f(1) = w$ with $f(t) \in \Omega$ for each $t \in [0, 1]$.

Proposition 1.17

An open set Ω is path connected if and only if it is connected.

Definition 1.23

A nonempty open, connected set $\Omega \subseteq \mathbb{C}$ is called a **region**.

Corollary 1.18

A region is path connected.

1.3 Functions on $\mathbb{C}$

We now turn our attention to functions which map complex numbers to complex numbers, the primary object of study in this course. Continuing from the definition of continuity from the previous section, we have the following:

Proposition 1.19

If f is continuous at z_0 then $|f|$ is continuous at z_0.

Proof. By the reverse triangle inequality we have $||f(z)| - |f(z_0)|| \leq |f(z) - f(z_0)|$. The conclusion follows. $\square$

Definition 1.24

f attains its maximum on $\Omega \subseteq \mathbb{C}$ if there exists $z_0 \in \Omega$ such that

$$|f(z)| \leq |f(z_0)|$$

for each $z \in \Omega$. The minimum case is analogous.

Theorem 1.20

Suppose that $\Omega \subseteq \mathbb{C}$ is compact and $f : \Omega \to \mathbb{C}$ is continuous. Then f attains is maximum (and minimum) on Ω.

Proof. First we show that f is bounded on Ω. If not, then we may take a sequence of points $\{z_n\} \subseteq \Omega$ such that $|f(z_n)| \to \infty$. Then $\{z_n\}$ contains a convergent subsequence $\{z_{n_k}\}$ tending to some $z \in \Omega$. It follows that

$$|f(z_{n_k})| \to |f(z)|$$

But the left side diverges to ∞, contradiction. Thus $f(\Omega)$ is bounded.

Let $M = \sup |f|(\Omega)$. Then there exists a sequence $\{z_n\} \subseteq \Omega$ such that $|f(z_n)| \to M$. Then there exists a subsequence $\{z_{n_k}\}$ converging to $z \in \Omega$. By continuity we have

$$|f(z)| = \lim |f(z_{n_k})| = M$$

$\square$

We now make the most important definition of this course:

Definition 1.25

Let $\Omega \subseteq \mathbb{C}$ be open and let $z_0 \in \Omega$. Let $f : \Omega \to \mathbb{C}$. We say that f is **holomorphic** at z_0 (or complex differentiable) if the limit

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists. In this case, the limit is denoted $f'(z_0)$.

If f is holomorphic at every $z \in \Omega$, then we simply say it is holomorphic on Ω. If f is holomorphic on $\mathbb{C}$ it is said to be **entire**.

We will sometimes also say that f is **analytic** or **complex differentiable** when it is holomorphic.

Note that the specification that Ω is open ensures that the difference quotient is actually defined (for sufficiently small h). Moreover, although this definition appears similar to the real analogue, the structure of the complex numbers means that it has far-reaching implications.

We will prove the following theorems in this class:

- (Cauchy's Theorem) If f is holomorphic on Ω, then it has derivatives of all orders.
- (Liouville's Theorem) If f is entire and bounded, then it is constant.
- (Prime Number Theorem) If $\pi(n)$ denotes the number of prime numbers less than or equal to n, then
$$\lim_{n\to\infty} \pi(n) \cdot \frac{\ln n}{n} = 1$$
- (Hardy-Ramanujan Theorem) Define $p(n)$ (the partition function) to be the number of ways to write $n = k_1 + k_2 + \ldots + k_n$ where $k_1 \geq k_2 \geq \ldots \geq k_n$ are all integers. For instance, $p(4) = 5$. Then
$$p(n) \sim \frac{1}{n\sqrt{48}} e^{\pi\sqrt{\frac{2}{3}}\cdot\sqrt{n}}$$

Example 1.1

The function $f(z) = z$ is holomorphic:

$$\frac{f(z+h) - f(z)}{h} = \frac{z+h-z}{h} = \frac{h}{h} = 1$$

so $z' = 1$.

Definition 1.26

If $A \subseteq \mathbb{C}$ is closed and $f : A \to \mathbb{C}$, then we say f is holomorphic on A if there exists $\Omega \supseteq A$ open and $F : \Omega \to \mathbb{C}$ which is holomorphic, and $F|_A = f$.

We can rewrite the definition of holomorphicity similarly to the multivariable real case as the following:

Proposition 1.21

$f : \Omega \to \mathbb{C}$ (Ω open) is holomorphic at z_0 if and only if there exists $a \in \mathbb{C}$ and $\psi : \mathbb{C} \to \mathbb{C}$ with $\psi(h) \to 0$ as $h \to 0$ such that

$$f(z_0 + h) = f(z_0) + ah + h\psi(h)$$

on some $\mathbb{D}_r(z_0) \subseteq \Omega$.

Proof. We can rewrite the above as

$$\psi(h) = \frac{f(z_0 + h) - f(z_0)}{h} - a$$

which goes to 0 if and only if

$$\frac{f(z_0 + h) - f(z_0)}{h} \to a$$

so that $f'(z_0) = a$. □

This recharacterization allows for a simple proof of the following:

Proposition 1.22

If f is holomorphic at z_0, then it is continuous at z_0.

Proof. Let $\{z_n\}$ be a sequence with $z_n \to z_0$. We want to show that $f(z_n) \to f(z_0)$. Let $h_n = z_n - z_0$. Then

$$f(z_n) = f(z_0 + h_n) = f(z_0) + ah_n + h_n\psi(h_n)$$

by assumption, the second and third terms go to zero, so $f(z_n) \to f(z_0)$. □

Example 1.2

Let $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ be defined by $f(z) = \frac{1}{z}$. Then

$$\lim_{h \to 0} \frac{\frac{1}{z_0+h} - \frac{1}{z_0}}{h} = \lim_{h \to 0} \frac{-h}{h(z_0)(z_0 + h)} = -\frac{1}{z_0^2}$$

so $f'(z_0) = -\frac{1}{z_0^2}$.

Proposition 1.23

Let $\Omega \subseteq \mathbb{C}$ be open, and let $f, g : \Omega \to \mathbb{C}$ be holomorphic at z_0. Then

1. $f + g$ is holomophic at z_0, and $(f + g)' = f' + g'$.
2. fg is holomorphic at z_0, and $(fg)' = f'g + fg'$.

3. If $g(z_0) \neq 0$, then $\frac{f}{g}$ is well defined on an open disk aroud z_0, and $\frac{f}{g}$ is holomorphic at z_0 with $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$.

Proof. Let ψ, φ be such that

$$f(z_0 + h) = f(z_0) + f'(z_0)h + h\psi(h)$$
$$g(z_0 + h) = g(z_0) + g'(z_0)h + h\varphi(h)$$

Then

$$f(z_0 + h) + g(z_0 + h) = f(z_0) + g(z_0) + [f'(z_0) + g'(z_0)]h + h(\psi(h) + \varphi(h))$$

$\lim_{h\to 0} \varphi + \psi = 0$, so the above shows that $(f + g)' = f' + g'$.

Letting

$$\phi(h) = f'(z_0)g'(z_0) + \psi(h)[g(z_0) + g'(z_0)] + \varphi(h)[f(z_0) + f'(z_0)] + \varphi(h)\psi(h)$$

which tends to 0 as $h \to 0$, we have

$$f(z_0 + h)g(z_0 + h) = f(z_0)g(z_0) + [f(z_0)g'(z_0) + f'(z_0)g(z_0)]h + h\phi(h)$$

so $(fg)' = f'g + g'f$.

The quotient rule may be derived from the Chain Rule using the fact that $\left(\frac{1}{z}\right)' = -\frac{1}{z^2}$ when $z \neq 0$. □

Proposition 1.24: Chain Rule

Let $\Omega, U \subseteq \mathbb{C}$ be open, and let $f : \Omega \to U$ and $g : U \to \mathbb{C}$. Then $g \circ f : \Omega \to \mathbb{C}$ is holomorphic and

$$(g \circ f)'(z) = g'(f(z))f'(z)$$

Proof. Using the alternative characterization of holomorphicity, we have

$$f(z_0 + h) = f(z_0) + f'(z_0)h + h\psi_f(h)$$

where $\psi_f(h) \to 0$ as $h \to 0$. Similarly,

$$g(f(z_0) + w) = g(f(z_0)) + g'(f(z_0))w + w\psi_g(w)$$

Then

$$\begin{aligned}(g \circ f)(z_0 + h) &= g(f(z_0) + f'(z_0)h + h\psi_f(h)) \\ &= g(f(z_0)) + g'(f(z_0))(f'(z_0)h + h\psi_f(h)) + (f'(z_0)h + h\psi_f(h))\psi_g(f'(z_0)h + h\psi_f(h)) \\ &= g(f(z_0)) + g'(f(z_0))f'(z_0)h + h[\psi_f(h)g'(f(z_0)) + (f'(z_0) + \psi_f(h))\psi_g(f'(z_0)h + h\psi_f(h))]\end{aligned}$$

Note that

$$\lim_{h\to 0} \underbrace{\psi_f(h)}_{=0} g'(f(z_0)) + (f'(z_0) + \underbrace{\psi_f(h)}_{=0})\psi_g(\underbrace{f'(z_0)h + h\psi_f(h)}_{=0}) = 0$$

so $(g \circ f)'(z_0) = g'(f(z_0))f'(z_0)$. □

Example 1.3

Let f be a constant function. Then f is entire and $f'(z) = 0$.

We showed in Example 1.1 that the identity $g(z) = z$ is entire with $g'(z) = 1$.

Combination of the two functions above, together with Proposition 1.23 gives

Corollary 1.25

Let $p(z) = a_0 + a_1 z + \ldots + a_n z^n$. Then p is entire and $p'(z) = a_1 + 2a_2 z + \ldots + n a_n z^{n-1}$.

Let us consider a non-example.

Example 1.4

Let $f(z) = \overline{z}$, so that $f(x+iy) = x - iy$. This is a smooth function in the case of $\mathbb{R}^2$; in fact since it is linear, $Df = f$, so that f has infinitely many derivatives.

However, in the complex case, we have

$$\frac{f(z+h) - f(z)}{h} = \frac{\overline{z+h} - \overline{z}}{h} = \frac{\overline{h}}{h}$$

But

$$\lim_{t \to 0} \frac{\overline{t}}{t} = 1$$

and

$$\lim_{t \to 0} \frac{\overline{it}}{it} = -1$$

so the limits disagree and f is not holomorphic at any z.

Consider some function $f : \Omega \to \mathbb{C}$. Let us denote its real and imaginary parts by u, v, respectively, so that

$$f(x+iy) = u(x,y) + iv(x,y)$$

(u, v are defined on $\Omega' \subseteq \mathbb{R}^2$ which is equivalent to Ω in the obvious way.) This allows us to consider f as a pair of functions from $\mathbb{R}^2 \to \mathbb{R}$, which are surfaces lying in $\mathbb{R}^3$. We will investigate which choices of u, v may be associated with a holomorphic f.

Let h be a (small) complex number and write $h = h_1 + ih_2$. Then write

$$\frac{f(z+h) - f(z)}{h} = \frac{u(x+h_1, y+h_2) - u(x,y)}{h_1 + ih_2} + \frac{v(x+h_1, y+h_2) - iv(x,y)}{h_1 + ih_2}$$

Let us consider what happens as h tends to 0 from different directions. For instance, suppose h is entirely real, so $h_2 = 0$. Then

$$\lim_{h_1 \to 0} \frac{f(z+h_1) - f(z)}{h_1} = \lim_{h_1 \to 0} \frac{(u+iv)(x+h_1, y) - (u+iv)(x,y)}{h_1} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

and similarly

$$\lim_{h_2\to 0}\frac{f(z+h_2)-f(z)}{ih_2}=\frac{1}{i}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}=\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y}$$

Then if f is holomorphic, then we can match components to get the following:

$$\begin{cases}\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y}\\ \dfrac{\partial v}{\partial y}=-\dfrac{\partial u}{\partial y}\end{cases}\tag{$*$}$$

The system of equations $(*)$ are known as the **Cauchy-Riemann equations**. We have shown that these are a necessary conditions for f to be holomorphic; below we will show that if we also assume that the partials are continuous, then we have a sufficient condition. Later, we will show that u, v are necessarily continuously differentiable, so that these are equivalent characterizations. For now we will content ourselves with one direction:

Theorem 1.26

Let $\Omega \subseteq \mathbb{C}$ be open and let $f : \Omega \to \mathbb{C}$. Let $f = u + iv$, where $u, v : \Omega \to \mathbb{R}$ are continuously differentiable and satisfy the Cauchy-Riemann equations. Then f is holomorphic.

Proof. Consider the first-order Taylor expansion in two variables, which says that

$$u(x+h_1,y+h_2)=u(x,y)+\frac{\partial u}{\partial x}h_1+\frac{\partial u}{\partial y}h_2+h\psi_u(h)$$

where $h = h_1 + ih_2$ and $\psi_u(h) \to 0$ as $h \to 0$. Similarly

$$v(x+h_1,y+h_2)=v(x,y)+\frac{\partial v}{\partial x}h_1+\frac{\partial v}{\partial y}h_2+h\psi_v(h)$$

Now by assumption, f satisfies the Cauchy-Riemann equations, so

$$v(x+h_1,y+h_2)=v(x,y)-\frac{\partial u}{\partial y}h_1+\frac{\partial u}{\partial x}h_2+h\psi_v(h)$$

so

$$\begin{aligned}f(x+h)&=u(x+h_1,y+h_2)+iv(x+h_1,y+h_2)\\&=u(x,y)+\frac{\partial u}{\partial x}h_1+\frac{\partial u}{\partial x}h_2+h\psi_u(y)+iv(x,y)-i\frac{\partial u}{\partial y}h_1+i\frac{\partial u}{\partial x}h_2+ih\psi_v(h)\\&=f(x,y)+(h_1+ih_2)\left(\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}\right)+h(\underbrace{\psi_u(h)+i\psi_v(h)}_{\psi(h)})\\&=f(z)+h\left(\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}\right)+h\psi(h)\end{aligned}$$

so f is holomorphic using the alternative characterization and

$$f'(z)=f'(x,y)=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}$$

□

Let f be a complex valued function of the form $f(x+iy) = u(x,y) + iv(x,y)$. Associate with it a $\mathbb{R}^2$-valued function

$$F(x,y) = (u(x,y), v(x,y))$$

Recall that its Jacobian matrix is

$$J_F(x,y) = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \tag{$**$}$$

and that F is differentiable in the real sense if it is true that

$$\lim_{(h_1,h_2)\to(0,0)} \frac{F(x+h_1, y+h_2) - F(x,y) - J_F(x,y)\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}}{|(h_1,h_2)|} = 0$$

Comparing this to the complex condition

$$\lim_{h\to 0} \frac{f(z+h) - f(z) - f'(z)h}{h} = 0$$

we can see that complex differentiability requires $J_f(x,y)$ to be of the form of multiplying by some complex number. This happens if and only if

$$J_F(x,y) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

for $a, b \in \mathbb{R}$. By reconciling this with $(**)$ we recover the Cauchy-Riemann equations $(*)$.

Definition 1.27

Let $f : \Omega \to \mathbb{C}$ with Ω open. Then we define

$$\frac{\partial f}{\partial x} := \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

and

$$\frac{\partial f}{\partial y} := \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}$$

We further define

$$\frac{\partial f}{\partial z} := \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$$

and

$$\frac{\partial f}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

Proposition 1.27

Let $\Omega \subseteq \mathbb{C}$ be open and let $f : \Omega \to \mathbb{C}$ be of the form $f(x+iy) = u(x,y) + iv(x,y)$. If f is holomorphic on Ω, then

1. $\dfrac{\partial f}{\partial \overline{z}} = 0$.
2. $\dfrac{\partial f}{\partial z} = \dfrac{\partial f}{\partial x} = -i\dfrac{\partial f}{\partial y}$.
3. $\dfrac{\partial f}{\partial z} = f'$.
4. $f'(z_0) = 2\dfrac{\partial u}{\partial z}(z_0)$.
5. $F = (u(x,y), v(x,y))$ is differentiable in the real sense, and $\det J_F(x,y) = |f'(x+iy)|^2$.

Proof. 1. By Cauchy-Riemann,

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\right) = 0$$

2. By part 1,

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y} + \frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{\partial f}{\partial x}$$

Similarly,

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial z} - \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y} - \frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) = -i\frac{\partial f}{\partial y}$$

3. Take h_1 real and $z_0 \in \Omega$. Then

$$\frac{\partial f}{\partial x}(z_0) = \lim_{h_1 \to 0} \frac{f(z_0 + h_1) - f(z_0)}{h_1} = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = f'(z_0)$$

By part 2 the conclusion follows.

4. By parts 2 and 3,

$$f' = \frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = 2\frac{\partial u}{\partial z}$$

5. We have

$$\det J_F(x,y) = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

By the Cauchy-Riemann equations and parts 2 and 3 this is

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = |f'(x+iy)|^2$$

□

1.4 Power Series

We will now discuss power series, which are defined similarly to the real case. They will initially serve as a valuable example of holomorphic functions. Later we will see that they are actually the *only* example, which justifies particular attention in their study.

Definition 1.28

A **power series** (centered around z_0) is a function f of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n := \lim_{N\to\infty} \sum_{n=0}^{N} a_n (z - z_0)^n$$

where $\{a_n\} \subseteq \mathbb{C}$, which is defined wherever the right hand limit converges.

Note that it is certainly a necessary condition that $a_n z^n \to 0$ as $n \to \infty$, since

$$\lim_{n\to\infty} a_n z^n = \lim_{n\to\infty} \left[\sum_{k=0}^{n} a_k z^k - \sum_{k=0}^{n-1} a_k z^k\right] = \lim_{n\to\infty} \left[\sum_{k=0}^{n} a_k z^k\right] - \lim_{n\to\infty} \left[\sum_{k=0}^{n-1} a_k z^k\right] = 0$$

In this section, we will first consider only power series which are centered at 0.

Definition 1.29

We say that a power series $\sum_{n=0}^{\infty} a_n z^n$ **converges absolutely** (in the complex sense) if the series

$$\sum_{n=0}^{\infty} |a_n| \cdot |z|^n$$

converges in the real sense.

Proposition 1.28

If $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely, then it converges (in the complex sense).

Proof. Let $\varepsilon > 0$.

Write

$$S_N(z) = \sum_{n=0}^{N} a_n z^n$$

Since

$$\sum_{n=0}^{\infty} |a_n| \cdot |z|^n$$

converges, there exists $N_0 \in \mathbb{N}$ such that

$$\sum_{n=N_0+1}^{\infty} |a_n| \cdot |z|^n < \varepsilon$$

Then for $M, N \geq N_0$, assume without loss of generality that $M < N$. Then we have

$$|S_N(z) - S_M(z)| = \left| \sum_{n=M+1}^{N} a_n z^n \right| \leq \sum_{a=M+1}^{N} |a_n| \cdot |z|^n \leq \sum_{a=N_0+1}^{\infty} |a_n| \cdot |z|^n < \varepsilon$$

so $\{S_N(z)\}$ is a Cauchy sequence, and thus

$$\sum_{n=0}^{\infty} a_n z^n$$

converges. □

Recall that in the single real variable case, we found that power series converge on some interval (possibly open, closed, or half-open) which is centered around 0 (or any other point of expansion). An analogous statement is true here, with the interval replaced by a disk.

Theorem 1.29

For any power series $\sum_{n=0}^{\infty} a_n z^n$ there exists $0 \leq R \leq \infty$[a] (called the **radius of convergence**) such that for any $z \in \mathbb{C}$:

1. If $|z| < R$, then $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely.
2. If $|z| > R$, then $\sum_{n=0}^{\infty} a_n z^n$ does not converge.

Moreover, R is given by **Hadamard's formula**:

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}$$

[a] $R = \infty$ means the condition $|z| < R$ is satisfied for all $z \in \mathbb{C}$.

This theorem says that we have absolute convergence inside the disk $\mathbb{D}_R$ (called the **disk of convergence**), and divergence outside of it. As in the real case, this theorem makes no statement about convergence on the boundary of the disk.

Proof. Denote $L = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. Suppose that $L \neq 0, \infty$. If $|z| < \frac{1}{L} = R$, then $L|z| < 1$, so there exists $\varepsilon > 0$ such that $(L + \varepsilon)|z| = r < 1$. By the definition of lim sup, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\sqrt[n]{|a_n|} < L + \varepsilon \implies |a_n| < (L + \varepsilon)^n \implies |a_n||z|^n < ((L + \varepsilon)|z|)^n = r^n$$

so $\sum_{n=0}^{\infty} a_n z^n$ is dominated by the absolutely convergent geometric series $\sum_{n=0}^{\infty} r^n$ and thus converges absolutely.

On the other hand, if $|z| > R = \frac{1}{L}$, then $L|z| > 1$ so there exists a subsequence $\{a_{n_k}\}$ such that

$$\sqrt[n_k]{|a_{n_k}|} \cdot |z| > 1$$

for all k. Then

$$|a_{n_k}| \cdot |z|^{n_k} > 1$$

which does not tend to 0 as $k \to \infty$, so convergence is impossible.

If $L = 0$, then $R = \infty$. For any z there exists N such that $n \geq N$ implies that

$$\sqrt[n]{a_n} < \frac{1}{|z|}$$

Then

$$|a_n||z|^n < r < 1$$

So

$$\sum_{n=N}^{\infty} |a_n||z|^n$$

is dominated by $\sum_{n=N}^{\infty} r^n$, and thus converges for $|z| < R = \infty$.

If $L = \infty$, then $L|z| > 1$ and we apply the argument for the case $|z| > R$. □

Example 1.5

The power series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}$$

has $R = \infty$, since the real-valued power series

$$\sum_{n=0}^{\infty} \frac{|z|^n}{n!} = e^{|z|}$$

converges absolutely everywhere. This also allows us to define the **exponential** of z as

$$e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Example 1.6

The power series

$$\sum_{n=0}^{\infty} z^n$$

has radius of convergence 1. This can be seen either by direct computation in the real case, or using Hadamard's formula and the fact that each a_n is 1:

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{1}} = \frac{1}{1} = 1$$

Moreover, this power series satisfies the equation

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

This can be seen using the identity for partial sums (which holds in all fields)

$$\sum_{n=0}^{N} z^n = \frac{1 - z^{N+1}}{1 - z}$$

so

$$\lim_{N\to\infty} \frac{1 - z^{N+1}}{1 - z} = \frac{1}{1 - z}$$

Definition 1.30

We define the **trigonometric functions** in terms of power series:

$$\cos z := \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$
$$\sin z := \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

Note that both of the above series converge with $R = \infty$. Moreover, we can observe that

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$
$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

and that this is consistent with our previous definition in terms of the identity

$$e^{iz} = \cos z + i \sin z$$

We now prove a fundamental fact about power series which, while analogous to the real case, will have farther reaching implications for us.

Theorem 1.30

The function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is holomorphic on the disk of convergence $\mathbb{D}_R$. Moreover, its derivative is given by

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n$$

which has the same radius of convergence.

Proof. To show that the radius of convergence is the same, simply apply Hadamard's formula to the new power series. Letting R' be the radius of convergence of $\sum_{n=1}^{\infty} n a_n z^{n-1}$, we have

$$\frac{1}{R'} = \limsup_{n\to\infty} \sqrt[n]{(n+1)|a_n|} = \limsup_{n\to\infty} \sqrt[n]{n+1}\, \sqrt[n]{|a_n|}$$

But $\sqrt[n]{n+1} \to 1$ so

$$\limsup_{n\to\infty} \sqrt[n]{n+1}\,\sqrt[n]{|a_n|} = \limsup_{n\to\infty} \sqrt[n]{|a_n|} = \frac{1}{R}$$

so $R' = R$.

Denote

$$g(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$$

We want to show that $f' = g$. It suffices to show that

$$\frac{f(z_0+h) - f(z_0)}{h} - g(z_0) \underset{h\to 0}{\to} 0$$

whenever $z_0 \in \mathbb{D}_R$.

To show this, pick any $z_0 \in \mathbb{D}_R$ and let $\varepsilon > 0$. Then there exists r such that $|z_0| < r < R$. Then whenever $|h| < r - |z_0|$ we have $|z_0 + h| \leq |z_0| + |h| < r < R$. So for small h, $z_0 + h \in \mathbb{D}_R$.

Denote the partial sums and error terms by

$$S_N(z) = \sum_{n=1}^{N} a_n z^n$$
$$E_N(z) = \sum_{n=N+1}^{\infty} a_n z^n$$

By our observation about Hadamard's formula,

$$\sum_{n=1}^{\infty} n|a_n|z^{n-1}$$

converges, so there exists $N_0 \in \mathbb{N}$ such that

$$\sum_{n=N_0+1}^{\infty} n|a_n||z|^{n-1} < \frac{\varepsilon}{3} \tag{1}$$

S_{N_0} is a polynomial, which we showed is holomorphic. So

$$\frac{S_{N_0}(z_0+h) - S_{N_0}(z_0)}{h} - S_N'(z_0) = \frac{S_{N_0}(z_0+h) - S_{N_0}(z_0)}{h} - \sum_{n=1}^{N_0} n a_n z^{n-1} \underset{h\to 0}{\to} 0$$

Thus there exists $\delta > 0$ such that whenever $|h| < \delta$,

$$\left| \frac{S_{N_0}(z_0+h) - S_{N_0}(z_0)}{h} - S_N'(z_0) \right| < \frac{\varepsilon}{3} \tag{2}$$

The difference quotient

$$\left|\frac{f(z_0+h)-f(z_0)}{h}-g(z_0)\right|$$

may be written as

$$\left|\frac{S_{N_0}(z_0+h)-S_{N_0}(z_0)}{h}-S'_{N_0}(z_0)+S'_{N_0}(z_0)-g(z_0)+\frac{E_{N_0}(z_0+h)-E_N(z_0)}{h}\right|$$
$$\leq\left|\frac{S_{N_0}(z_0+h)-S_{N_0}(z_0)}{h}-S'_{N_0}(z_0)\right|+\left|S'_{N_0}(z_0)-g(z_0)\right|+\left|\frac{E_{N_0}(z_0+h)-E_N(z_0)}{h}\right|$$

For small h, the first term is less than $\frac{\varepsilon}{3}$ by (2). Also,

$$\left|S'_{N_0}(z_0)-g(z_0)\right|=\left|-\sum_{n=N_0+1}^{\infty}na_nz^{n-1}\right|\leq\sum_{n=N_0+1}^{\infty}n|a_n|\cdot|z|^{n-1}<\frac{\varepsilon}{3}$$

by (1). Lastly,

$$\left|\frac{E_{N_0}(z_0+h)-E_{N_0}(z_0)}{h}\right|=\left|\sum_{n=N_0+1}^{\infty}a_n\frac{(z_0+h)^n-z_0^n}{h}\right|$$

Using the identity

$$a^n-b^n=(a-b)\sum_{k=0}^{n-1}a^kb^{n-1-k}$$

for $a=z_0+h, b=z_0$ we have

$$\left|\frac{\sum_{n=N_0+1}^{\infty}a_n\left((z_0+h)^n-z_0^n\right)}{h}\right|=\left|\sum_{n=N_0+1}^{\infty}a_n\left(\sum_{k=0}^{n-1}(z_0+h)^kz_0^{n-1-k}\right)\right|$$
$$\leq\sum_{n=N_0+1}^{\infty}|a_n|\sum_{k=0}^{n-1}|z_0+h|^k|z_0|^{n-1-k}<\sum_{n=N_0+1}^{\infty}|a_n|nr^n<\frac{\varepsilon}{3}$$

where the last inequality follows from (1). Thus

$$\left|\frac{f(z_0+h)-f(z_0)}{h}-g(z_0)\right|<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon$$

□

Corollary 1.31

A power series is infinitely complex differentiable on its disk of convergence.

Proof. Induct using Theorem 1.30. □

Corollary 1.32

A power series may be integrated term-by-term on its disk of convergence.

Proof. The integrated power series has some radius of convergence, and Theorem 1.30 says that this is the same radius as the original power series. □

We now note that for the case of power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ centered around arbitrary z_0, they converge on $\mathbb{D}_R(z_0)$, where R is the radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$. In other words, the behavior is identical, merely translated. Thus it suffices to consider power series centered around 0.

Definition 1.31

Let $\Omega \subseteq \mathbb{C}$ be open and $z_0 \in \Omega$. Then $f : \Omega \to \mathbb{C}$ is said to be **analytic** at z_0 if there exists $r > 0$ such that $\mathbb{D}_r(z_0) \subseteq \Omega$ and there exist coefficients $\{a_n\} \subseteq \mathbb{C}$ such that

$$\sum_{n=0}^{\infty} a_n z^n$$

converges absolutely on $\mathbb{D}_r$ and

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

on $\mathbb{D}_r(z_0)$.

We say that f is analytic on Ω if it is analytic at each $z_0 \in \Omega$.

By Theorem 1.30, each analytic function is (infinitely) holomorphic. We will show later that every holomorphic function is also analytic. Thus the terms are often used interchangably, but we will not do so until we have proved this fact.

1.5 Integration on Curves

We will soon show that the behavior of analytic functions can be well understood by studying the behavior of those functions when integrated over various curves in the plane. This will motivate the particular definitions of an integral over a curve that appear in this section.

Definition 1.32

A **parameterized curve** is a continuous function $z : [a, b] \to \mathbb{C}$. Writing $z(t) = z_1(t) + iz_2(t)$, we say that z is a **smooth curve** if $z_1'(t), z_2'(t)$ exist for all $t \in [a, b]$.

z is said to be **piecewise smooth** if there exist $a = a_0 < a_1 < \ldots < a_n = b$ such that z is smooth on $[a_k, a_{k+1}]$ for each k.

z is a **closed curve** if $z(a) = z(b)$.

Note that in our definition of piecewise smooth curves, z must be continuous at each a_k, but the left and right derivatives need not coincide.

We will adopt the convention that all curves are assumed to be piecewise smooth, unless stated otherwise.

A parameterized curve traces out a particular image in the complex plane, which is intuitively a curve in the plane. Thus it is useful to distinguish the function z and its image.

Definition 1.33

Two parameterized curves $z_1 : [a, b] \to \mathbb{C}$ and $z_2 : [c, d] \to \mathbb{C}$ are said to be **equivalent curves** if there exists a continuously differentiable bijection $t : [a, b] \to [c, d]$ such that $t'(s) > 0$ and $z_2(s) = z_1(t(s))$ for all $s \in [a, b]$.

The condition that $t'(s) > 0$ could be equivalently stated as saying that $t(a) = c$ and $(b) = d$ (so that the direction does not change). It follows that $z_1([a, b]) = z_2([c, d])$.

Definition 1.34

A **curve** is an equivalence class of parameterized curves.

Note that for a given curve, there is another curve which has the same image in $\mathbb{C}$, but with the opposite orientation.

Definition 1.35

If $\gamma \subseteq \mathbb{C}$ is a curve, and $z : [a, b] \to \mathbb{C}$ is a parameterization of γ, then γ^- is the curve parameterized by $z(a + b - t)$.

A commonly used curve is the circle, so it is convenient to define some conventional parameterizations of the circle:

Definition 1.36

The circle with radius $r > 0$ and center $z \in \mathbb{C}$ is

$$C_r(z_0) \coloneqq \partial\mathbb{D}_r(z_0)$$

The **positive orientation** of $C_r(z)$ is the curve parameterized by $z : [-\pi, \pi] \to \mathbb{C}$ and

$$z(\theta) = z_0 + re^{i\theta}$$

The **negative orientation** is the curve parameterized by

$$z^-(\theta) = z_0 + re^{-i\theta}$$

Now, let us define the manner in which we will integrate functions over smooth planar curves.

Definition 1.37

Let $\gamma \subseteq \mathbb{C}$ be a smooth curve. Let $f : \gamma \to \mathbb{C}$ be continuous. Let $z : [a, b] \to \mathbb{C}$ be a smooth parameterization of γ. Then define the **contour integral** of f along γ with respect to z to be

$$\int_{\gamma,z} f(z)\,\mathrm{d}z = \int_a^b f(z(t))z'(t)\,\mathrm{d}t = \int_a^b \mathrm{Re}(f(z(t))z'(t))\,\mathrm{d}t + i\int_a^b \mathrm{Im}(f(z(t))z'(t))\,\mathrm{d}t$$

As Proposition 1.33 shows, this value is independent of the choice of z. Thus, we take this common value to be the contour integral of f along γ.

Proposition 1.33

If γ is a smooth curve, $f : \gamma \to \mathbb{C}$ is continuous, and z_1, z_2 are two parameterizations of γ, then

$$\int_{\gamma,z_1} f(z)\,\mathrm{d}z = \int_{\gamma,z_2} f(z)\,\mathrm{d}z$$

Proof. Since z_1, z_2 are equivalent, we write $z_1(s) = z_2(t(s))$. Then

$$\begin{aligned}\int_{\gamma,z_1} f(z)\,\mathrm{d}z &= \int_a^b f(z_1(s))z_1'(s)\,\mathrm{d}s = \int_a^b f(z_2(t(s)))z_2'(t(s))t'(s)\,\mathrm{d}s \\ &= \int_{t(a)}^{t(b)} f(z_2(t))z_2'(t)\,\mathrm{d}t = \int_c^d f(z_2(t))z_2'(t)\,\mathrm{d}t = \int_{\gamma,z_2} f(z)\,\mathrm{d}z\end{aligned}$$ □

The above proof shows that we may define contour integrals with respect to a curve, without referring to a specific parameterization.

If γ is only piecewise smooth, and some parameterization $z : [a, b] \to \mathbb{C}$ is smooth on $[a_k, a_{k+1}]$ for $a = a_0 < a_1 < \ldots < a_n = b$, then we define the integral to be

$$\int_\gamma f(z)\,\mathrm{d}z = \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} f(z(t))z'(t)\,\mathrm{d}t$$

Similar work as above shows that this is also independent of the parameterization.

Definition 1.38

Let $\gamma \subseteq \mathbb{C}$ be a curve parameterized by $z : [a, b] \to \mathbb{C}$. Then the **length** of γ is

$$\mathrm{length}(\gamma) = \int_a^b |z'(t)|\,\mathrm{d}t$$

The following example will be extremely instructive for later applications.

Example 1.7

Let $\gamma = \partial\mathbb{D}$. Let $f : \gamma \to \mathbb{C}$ be continuous. Let z be the parameterization

$$z(\theta) = e^{i\theta}$$

Then

$$z'(\theta) = ie^{i\theta}$$

So

$$\int_\gamma f(z)\,\mathrm{d}z = \int_0^{2\pi} f(e^{i\theta})ie^{i\theta}\,\mathrm{d}\theta$$

For instance, take

$$f(z) = \frac{1}{z}$$

Then we have

$$\int_\gamma \frac{1}{z}\,\mathrm{d}z = \int_0^{2\pi} \frac{ie^{i\theta}}{e^{i\theta}}\,\mathrm{d}\theta = \int_0^{2\pi} i\,\mathrm{d}\theta = 2\pi i$$

The length of γ is

$$\int_0^{2\pi} \left|ie^{i\theta}\right|\mathrm{d}\theta = \int_0^{2\pi} \mathrm{d}\theta = 2\pi$$

as we expect.

Let us briefly cover some properties of the contour integral.

Proposition 1.34

Let $\gamma \subseteq \mathbb{C}$ be a curve, $f, g : \gamma \to \mathbb{C}$ be continuous, and $\alpha, \beta \in \mathbb{C}$. Then

1. $\int_\gamma$ is linear:

$$\int_\gamma (\alpha f + \beta g)\,\mathrm{d}z = \alpha \int_\gamma f\,\mathrm{d}z + \beta \int_\gamma g\,\mathrm{d}z$$

2. $\int_\gamma$ is reversed by orientation:

$$\int_\gamma f\,\mathrm{d}z = -\int_{-\gamma} f\,\mathrm{d}z$$

3. The following inequality holds analogous to the triangle inequality:

$$\left|\int_\gamma f\,\mathrm{d}z\right| \leq \left(\sup_{z\in\gamma}|f(z)|\right)\mathrm{length}(\gamma)$$

Proof. 1. Follows from properties of the Riemann integral.

2. Homework (follows from reparameterizing).
3. First, we show the following:

Claim

Let $h : [a, b] \to \mathbb{C}$ be continuous. Then

$$\left|\int_a^b h(t)\,\mathrm{d}t\right| \leq \int_a^b |h(t)|\,\mathrm{d}t$$

Proof. Write $\int_a^b h(t)\,\mathrm{d}t = re^{i\theta}$ for appropriate r, θ. Then

$$\begin{aligned} r = \left|\int_a^b h(t)\,\mathrm{d}t\right| &= e^{-i\theta}\int_a^b h(t)\,\mathrm{d}t \\ = \int_a^b e^{-i\theta}h(t)\,\mathrm{d}t &= \mathrm{Re}\left(\int_a^b e^{-i\theta}h(t)\,\mathrm{d}t\right) \\ = \int_a^b \mathrm{Re}(e^{-i\theta}h(t))\,\mathrm{d}t &\leq \int_a^b |h(t)|\,\mathrm{d}t \end{aligned}$$ ■

Now, suppose that $z : [a, b] \to \mathbb{C}$ is piecewise smooth with $a = a_0 < \ldots < a_n = b$. Then by definition,

$$\begin{aligned} \left|\int_\gamma f(z)\,\mathrm{d}z\right| &= \left|\sum_{k=0}^{n-1}\int_{a_k}^{a_{k+1}} f(z(t))z'(t)\,\mathrm{d}t\right| \leq \sum_{k=0}^{n-1}\left|\int_{a_k}^{a_{k+1}} f(z(t))z'(t)\,\mathrm{d}t\right| \\ &\leq \sum_{k=0}^{n-1}\int_{a_k}^{a_{k+1}} |f(z(t))|\cdot|z'(t)|\,\mathrm{d}t \leq \sum_{k=0}^{n-1}\int_{a_k}^{a_{k+1}}\left(\sup_{z\in\gamma}|f(z(t))|\right)|z'(t)|\,\mathrm{d}t \\ &= \left(\sup_{z\in\gamma}|f(z(t))|\right)\sum_{k=0}^{n-1}\int_{a_k}^{a_{k+1}}|z'(t)|\,\mathrm{d}t = \left(\sup_{z\in\gamma}|f(z(t))|\right)\mathrm{length}(\gamma) \end{aligned}$$ □

In vector calculus, the fundamental theorem of line integrals shows that calculation of line integrals can often be reduced to evaluating an antiderivative at its endpoints. The same is true in $\mathbb{C}$.

Definition 1.39

Let $\Omega \subseteq \mathbb{C}$ be open. Let $f : \Omega \to \mathbb{C}$. Then we say that $F : \Omega \to \mathbb{C}$ is a **primitive** of f on Ω if F is holomorphic on Ω and

$$F'(z) = f(z)$$

for all $z \in \Omega$.

Theorem 1.35

Let $\Omega \subseteq \mathbb{C}$ be open. Let $f : \Omega \to \mathbb{C}$ be continuous and suppose $F : \Omega \to \mathbb{C}$ is a primitive of f. Then for any curve $\gamma \subseteq \Omega$ joining z_1, z_2, we have

$$\int_\gamma f(z)\,\mathrm{d}z = F(z_2) - F(z_1)$$

Proof. This is effectively the Fundamental Theorem of Calculus. Take a piecewise smooth parameterization $z : [a, b] \to \mathbb{C}$ of γ and pick $a = a_0 < \ldots < a_n = b$ so that z is smooth on each subinterval. Then

$$\int_\gamma f(z)\,\mathrm{d}z = \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} f(z(t))z'(t)\,\mathrm{d}t = \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} F(z(t))'\,\mathrm{d}t$$
$$= \sum_{k=0}^{n-1} (F(z(a_{k+1})) - F(z(a_k))) = F(z(a_n)) - F(z(a_0)) = F(z_2) - F(z_1) \qquad \square$$

Corollary 1.36

Let $\Omega \subseteq \mathbb{C}$ be open, $f : \Omega \to \mathbb{C}$ continuous, and F a primitive for f on Ω. Then for any *closed* curve $\gamma \subseteq \Omega$,

$$\int_\gamma f(z)\,\mathrm{d}z = 0$$

Although the above theorem is powerful, its assumptions sometimes fail to be satisfied in subtle ways.

Example 1.8

Let $\Omega = \mathbb{C} \setminus \{0\}$. Let $f : \Omega \to \mathbb{C}$ be $f(z) = \frac{1}{z}$. We showed that

$$\int_{\partial\mathbb{D}} \frac{1}{z}\,\mathrm{d}z = 2\pi i \neq 0$$

This shows that $\frac{1}{z}$ does not have a primitive on Ω. This is an interesting contrast to the real case, where $\ln x$ is a primitive for $\frac{1}{x}$ on $\mathbb{R} \setminus \{0\}$. (This is due to the fact that 0 is a branch point for the logarithm in $\mathbb{C}$.)

Corollary 1.37

Let $\Omega \subseteq \mathbb{C}$ be a region. Let $f : \Omega \to \mathbb{C}$ be holomorphic on Ω and $f'(z) = 0$ for all $z \in \Omega$. Then f is constant.

Proof. Pick $z_1, z_2 \in \Omega$. Ω is path connected since it is a region, so let $\gamma \subseteq \mathbb{C}$ be a curve from

z_1 to z_2. f is a primitive for f', so

$$f(z_2) - f(z_1) = \int_\gamma f'(z)\,\mathrm{d}z = \int_\gamma 0\,\mathrm{d}z = 0$$

and thus $f(z_2) = f(z_1)$. So f is constant. $\square$

Chapter 2

Cauchy's Theorem and Applications

Up until this point, the results we have shown about contour integrals have been analogous to line integrals in $\mathbb{R}^n$. In this chapter, we will see that integrating in the complex setting results in unique results. The main theorem we will prove is as follows:

Theorem: Cauchy's Theorem for a Disk

Let $f : \mathbb{D} \to \mathbb{C}$ be holomorphic on $\mathbb{D}$. Then for any closed curve $\gamma \subseteq \mathbb{D}$,

$$\int_\gamma f(z)\,\mathrm{d}z = 0$$

In order to prove that, we will first prove some intermediate results.

2.1 Goursat's Theorem

We will first prove a version of Cauchy's Theorem for a specific type of curve.

Definition 2.1

A **triangle** T is a subset of $\mathbb{C}$ which consists of three line segments and the region between them. The boundary of T is the line segments between them, which by convention is taken with the counterclockwise parameterization.

Theorem 2.1: Cauchy-Goursat

Let $\Omega \subseteq \mathbb{C}$ be open. Let $f : \Omega \to \mathbb{C}$ be holomorphic on Ω, and let $T \subseteq \Omega$ be a triangle. Then

$$\int_{\partial T} f(z)\,\mathrm{d}z = 0$$

Proof. Let us bisect the line segments of the triangle and draw a triangle between them. This creates four subtriangles, which we denote S_1, S_2, S_3, S_4.

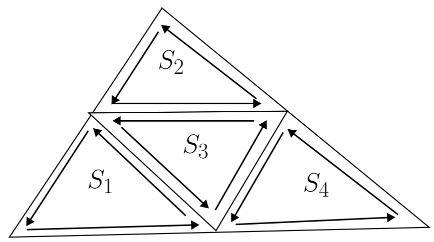


Figure 2.1: Subdivision of T

Each triangle made this way is similar to the original. Then consider the sum

$$\int_{\partial S_1} f(z)\,\mathrm{d}z + \int_{\partial S_2} f(z)\,\mathrm{d}z + \int_{\partial S_3} f(z)\,\mathrm{d}z + \int_{\partial S_4} f(z)\,\mathrm{d}z$$

Since we maintain the counterclockwise orientation, the edges of S_3 will be integrated along in both directions, which cancels out. Thus the above expression is equal to

$$\int_{\partial T} f(z)\,\mathrm{d}z$$

Thus

$$\left|\int_{\partial T} f(z)\,\mathrm{d}z\right| \leq 4\max\left\{\left|\int_{\partial S_i} f(z)\,\mathrm{d}z\right|\right\}$$

Suppose that the largest integral occurs over S_j. Then set $T_1 = S_j$ and subdivide again.

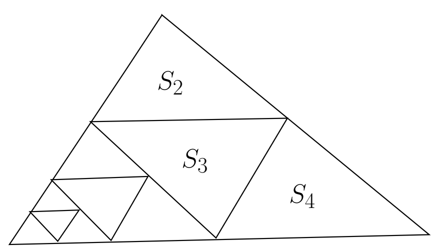


Figure 2.2: Continued subdivisions

Thus we get a nested sequence of triangles satisfying

$$\begin{cases} T = T_0 \supseteq T_1 \supseteq \dots \\ \left|\int_{\partial T_n} f(z)\,\mathrm{d}z\right| \leq 4\left|\int_{\partial T_{n+1}} f(z)\,\mathrm{d}z\right| \\ \operatorname{diam}(T_{n+1}) = \frac{1}{2}\operatorname{diam}(T_n) \\ \operatorname{length}(\partial T_{n+1}) = \frac{1}{2}\operatorname{length}(\partial T_n) \end{cases}$$

It follows that

$$\begin{cases} \left|\int_{\partial T} f(z)\,\mathrm{d}z\right| \leq 4^n\left|\int_{\partial T_n} f(z)\,\mathrm{d}z\right| \\ \operatorname{diam}(T_n) = \frac{1}{2^n}\operatorname{diam}(T) \\ \operatorname{length}(\partial T_n) = \frac{1}{2^n}\operatorname{length}(\partial T) \end{cases}$$

Since the diameters go to 0 and each T_i is compact, there exists a unique point ω in each of the T_i. Since f is holomorphic at ω, we can write

$$f(z) = f(\omega) + f'(\omega)(z-\omega) + \psi(z)(z-\omega)$$

where $\lim_{z\to\omega}\psi(z) = 0$. So for any n,

$$\int_{\partial T_n} f(z)\,\mathrm{d}z = \int_{\partial T_n} (f(\omega) + f'(\omega)(z-\omega))\,\mathrm{d}z + \int_{\partial T_n} \psi(z)(z-\omega)\,\mathrm{d}z$$

But notice that

$$\left(f(\omega)z + \frac{1}{2}f'(\omega)(z-\omega)^2\right)' = f(\omega) + f'(\omega)(z-\omega)$$

so we have a primitive, and it follows that

$$\int_{\partial T_n} (f(\omega) + f'(\omega)(z-\omega))\,\mathrm{d}z = 0$$

Now,

$$\begin{aligned} \left|\int_{\partial T} f(z)\,\mathrm{d}z\right| &\leq 4^n\left|\int_{\partial T_n} f(z)\,\mathrm{d}z\right| \\ &= 4^n\left|\int_{\partial T_n} \psi(z)(z-\omega)\,\mathrm{d}z\right| \\ &\leq 4^n\left(\sup_{z\in\partial T_n}[\psi(z)(z-\omega)]\right)\operatorname{length}(\partial T_n) \\ &= 2^n\operatorname{length}(\partial T)\left(\operatorname{diam}(T_n)\sup_{z\in\partial T_n}\psi(z)\right) \\ &= \operatorname{length}(\partial T)\operatorname{diam}(T)\sup_{z\in\partial T_n}\psi(z) \end{aligned}$$

Since

$$\lim_{z \to \omega} \psi(z) = 0$$

we can make $\sup_{z \in \partial T_n} \psi(z)$ arbitrarily small by considering large enough n. It follows that

$$\lim_{n \to \infty} \sup_{z \in \partial T_n} \psi(z) = 0$$

and thus

$$\left| \int_{\partial T} f(z)\, \mathrm{d}z \right| = 0$$

□

We will now show that holomorphic functions locally have primitives. We will adopt the notation that for $z, \omega \in \mathbb{C}$, $[z, \omega]$ represents the line segment joining z and ω. Specifically, it can be parameterized by

$$t \mapsto (1-t)z + t\omega, \quad t \in [0,1]$$

Theorem 2.2

Let $z_0 \in \mathbb{C}$, $r > 0$, and $f : \mathbb{D}_r(z_0) \to \mathbb{C}$ be holomorphic on $\mathbb{D}_r(z_0)$. Then f has a primitive on $\mathbb{D}_r(z_0)$.

Proof. Take some $z \in \mathbb{D}_r(z_0)$. Define

$$F(z) = \int_{[z_0, z]} f(\omega)\, \mathrm{d}\omega$$

We claim that F is holomorphic on $\mathbb{D}_r(z_0)$ and that it is a primitive for f. Let h be small and consider the triangle T_h between $z_0, z, z+h$.

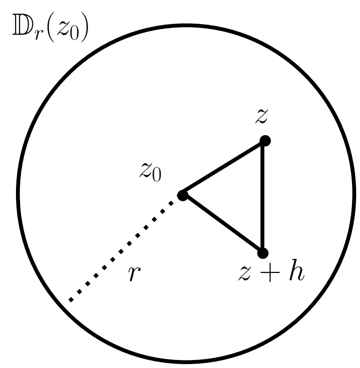


Figure 2.3: Auxiliary triangle construction

By Cauchy-Goursat,

$$0 = \int_{\partial T_h} f(\omega)\, \mathrm{d}\omega = \int_{[z_0, z]} f(\omega)\, \mathrm{d}\omega + \int_{[z, z+h]} f(\omega)\, \mathrm{d}\omega - \int_{[z_0, z+h]} f(\omega)\, \mathrm{d}\omega$$

Thus

$$F(z+h)-F(z)=F(z+h)-F(z)+\int_{\partial T_h} f(\omega)\,\mathrm{d}\omega=\int_{[z,z+h]} f(\omega)\,\mathrm{d}\omega$$
$$=h\int_0^1 f((1-t)z+t(z+h))\,\mathrm{d}t$$

and
$$\frac{F(z+h)-F(z)}{h}-f(z)=\int_0^1 (f((1+t)z+t(z+h))-f(z)\,\mathrm{d}t)$$

Since f is continuous at z, for every $\varepsilon>0$ there exists $\delta>0$ such that if $|\omega-z|<\delta$, then $|f(\omega)-f(z)|<\varepsilon$. So if $|h|<\delta$, then

$$\left|\frac{F(z+h)-F(z)}{h}-f(z)\right|\leq\int_0^1 |f((1-t)z+t(z+h))-f(z)|\,\mathrm{d}t<\varepsilon$$

Thus $F'=f$ and f has a primitive. □

Later, we will see that holomorphic functions can all be represented by power series, which makes the above trivial by integrating term-by-term.

We now have the tools necessary to prove Cauchy's Theorem on a disk.

Corollary 2.3: Cauchy's Theorem on a Disk

If $f:\mathbb{D}_r(z_0)\to\mathbb{C}$ is holomorphic than for any closed curve $\gamma\subseteq\mathbb{D}_r(z_0)$,

$$\int_\gamma f(z)\,\mathrm{d}z=0$$

Proof. By Theorem 2.2, f has a primitive, so the integral is zero. □

Corollary 2.4

If $\Omega\subseteq\mathbb{C}$ is open and contains $\overline{\mathbb{D}_r}(z_0)$, if $f:\Omega\to\mathbb{C}$ is holomorphic, then

$$\int_{\partial D_r(z_0)} f(z)\,\mathrm{d}z=0$$

Proof. It suffices to pick r' such that $\mathbb{D}_{r'}(z_0)\supseteq\overline{\mathbb{D}_r}(z_0)$. To show that such an r' exists, suppose that it does not. Then for any $n\in\mathbb{N}$ there exists $z_n\in D_{r+\frac{1}{n}}(z_0)$ such that $z_n\in\mathbb{C}\setminus\Omega$. So $\{z_n\}$ has a convergent subsequence $\{z_{n_k}\}$, which converges to a limit $z\notin\Omega$ since $\mathbb{C}\setminus\Omega$ is closed. But $|z_n-z_0|<r+\frac{1}{n}$, which means that $|z-z_0|\leq r$. Thus $z\in\overline{\mathbb{D}_r}(z_0)$, contradiction. □

2.2 Homotopies and Simply Connected Domains

We will now take a short detour into some topology in order to investigate which other sets Cauchy's Theorem applies to. For some intuition, consider an open set Ω and two curves in it which share endpoints. Then if Ω is particularly nice, we should be able to continuously deform one curve into the other.

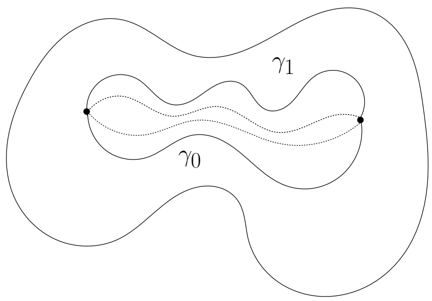


Figure 2.4: Continuous deformation of curves

We formalize this notion by considering a continuous family of curves which deform γ_0 into γ_1.

Definition 2.2

Let $\Omega \subseteq \mathbb{C}$ be open, $\alpha, \beta \in \Omega$, and $\gamma_0, \gamma_1 : [a, b] \to \Omega$ be two curves with the same endpoints, such that $\gamma_1(a) = \gamma_2(a) = \alpha$ and $\gamma_1(b) = \gamma_2(b) = \beta$. We say that γ_1, γ_2 are **homotopic** in Ω if for every $s \in [0, 1]$ there is a curve $\gamma_s : [a, b] \to \Omega$ with $\gamma_s(a) = \alpha, \gamma_s(b) = \beta$ such that the function $H : [0, 1] \times [a, b] \to \Omega$ defined by

$$H(s, t) = \gamma_s(t)$$

is continuous as a function of two variables.

Theorem 2.5

Let $\Omega \subseteq \mathbb{C}$ be open, and let $f : \Omega \to \mathbb{C}$ be holomorphic on Ω. If $\gamma_0, \gamma_1 : [a, b] \to \Omega$ are curves with the same endpoints that are homotopic in Ω, then

$$\int_{\gamma_0} f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z$$

Proof. We claim that there exists $\varepsilon > 0$ such that for any $s \in [0, 1]$ and $t \in [a, b]$, then

$$\mathbb{D}_{2\varepsilon}(\gamma_s(t)) \subseteq \Omega$$

Suppose not. Then for every $n \in \mathbb{N}$, there are $s_n \in [0,1]$ and $t_n \in [a,b]$ and $\omega_n \in \mathbb{C} \setminus \Omega$ such that

$$\omega_n \in \mathbb{D}_{2/n}(\gamma_{s_n}(t_n))$$

Now, the sequence $(s_n, t_n) \in [0,1] \times [a,b]$ is a sequence in a compact set, so there is a subsequence $\{(s_{n_k}, t_{n_k})\}$ tending to (s,t). But then

$$\left|\omega_{n_k} - \gamma_{s_{n_k}}(t_{n_k})\right| < \frac{2}{n_k}$$

which tends to 0. Since $\mathbb{C} \setminus \Omega$ is closed, $\lim \omega_{n_k} \in \mathbb{C} \setminus \Omega$. But $\omega_{n_k} \to \gamma_s(k)$ is the limit of $H(s_{n_k}, t_{n_k})$, which is continuous, and thus $\gamma_s(t) \in \Omega$, contradiction. So the claim is proved and such an ε exists.

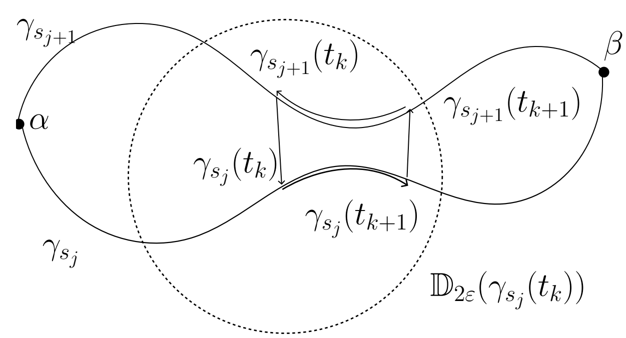
Now, note that $H(s,t)$ is continuous on the compact set $[0,1]\times[a,b]$, so it is also uniformly continuous. So for $\varepsilon > 0$ which is produced by the claim, there exists $\delta > 0$ such that if $|s_1 - s_2| < \delta$ and $|t_1 - t_2| < \delta$, then

$$|\gamma_{s_1}(t_1) - \gamma_{s_2}(t_2)| < \varepsilon$$

Then subdivide $[0,1] \times [a,b]$ using $0 = s_0 < s_1 < \ldots < s_n = 1$ and $a = t_0 < \ldots < t_n = b$ such that $|s_{j+1} - s_j| < \delta$ and $|t_{j+1} - t_j| < \delta$ for all j. We claim that

$$\int_{\gamma_{s_j}} f(z)\,\mathrm{d}z = \int_{\gamma_{s_{j+1}}} f(z)\,\mathrm{d}z$$

for all j. Clearly this suffices to prove the theorem. Consider some pair of points (s_j, t_k) and (s_{j+1}, t_{k+1}). Draw a circle of radius 2ε around $\gamma_{s_j}(t_k)$. Note that this circle also contains $\gamma_{s_{j+1}}(t_k), \gamma_{s_{j+1}}(t_{k+1}), \gamma_{s_j}(t_{k+1})$.



By Cauchy's Theorem

$$\int_{\gamma_{s_j}([t_k,t_{k+1}])} f + \int_{[\gamma_{s_j}(t_{k+1}),\gamma_{s_{j+1}}(t_{k+1})]} f - \int_{\gamma_{s_{j+1}}([t_k,t_{k+1}])} f - \int_{[\gamma_{s_j}(t_k),\gamma_{s_{j+1}}(t_k)]} f = 0$$

Thus

$$\int_{\gamma_{s_j}([t_k,t_{k+1}])} f = \int_{\gamma_{s_{j+1}}([t_k,t_{k+1}])} f + \left(\int_{[\gamma_{s_j}(t_k),\gamma_{s_{j+1}}(t_k)]} f - \int_{[\gamma_{s_j}(t_{k+1}),\gamma_{s_{j+1}}(t_{k+1})]} f\right)$$

Let us write

$$a_k = \int_{\gamma_{s_j}(t_k), \gamma_{s_{j+1}(t_k)}} f$$

Then summing over all k, we see that

$$\int_{\gamma_{s_j}} f = \int_{\gamma_{s_{j+1}}} f + a_0 - a_n$$

But $\gamma_{s_j}, \gamma_{s_{j+1}}$ have the same endpoints so $a_0 = a_n = 0$. Thus the theorem is proved. $\square$

Definition 2.3

An open connected set $\Omega \subseteq \mathbb{C}$ is called **simply connected** if any curves $\gamma_0, \gamma_1 : [a, b] \to \Omega$ which share endpoints are homotopic in Ω.

Example 2.1

Any convex set, which is a set such that the line between two points in the set is contained in the set, such as the disk, is simply connected.

Example 2.2

A "star-shaped" set, which is a set with a center point such that any two points can be joined by a curve passing through the center, is simply connected.

Example 2.3

$\mathbb{C} \setminus \{0\}$ is not simply connected. Consider the upper and lower halves of $\partial\mathbb{D}$. Intuitively, we see that one cannot be transformed to the other without passing through the origin. We will prove this later.

Theorem 2.6

Let $\Omega \subseteq \mathbb{C}$ be simply connected and $f : \Omega \to \mathbb{C}$ be holomorphic. Then f has a primitive on Ω.

Proof. Let Ω be simply connected and pick a point $z_0 \in \Omega$. For any $z \in \Omega$, choose some curve $\gamma_z \subseteq \Omega$ connecting z_0 to z. For consistency, let us demand that γ_{z_0} is a constant curve. Then define

$$F(z) := \int_{\gamma_z} f(\omega)\,\mathrm{d}\omega$$

Now, let h be small and let φ be the line segment connecting z to $z + h$. Then the curve $\gamma_z + \varphi$ (where $+$ means concatenation) shares endpoints with γ_{z+h}, so they are homotopic. By Theorem 2.5, integrating over either gives the same value, so

$$\int_{\gamma_{z+h}} f(\omega)\,\mathrm{d}\omega = \int_{\gamma_z} f(\omega)\,\mathrm{d}\omega + \int_{\varphi} f(\omega)\,\mathrm{d}\omega$$

We can then write

$$F(z+h) = F(z) + \int_{\varphi} f(\omega)\,\mathrm{d}\omega$$

so

$$F(z+h) - F(z) = h\int_0^{\infty} f((1-t)z + t(z+h))\,\mathrm{d}t$$

From here, we conclude identically to the proof of local existence of primitives. □

A corollary to this is a more general form of Cauchy's Theorem.

Corollary 2.7: Cauchy's Theorem

If Ω is simply connected and $f : \Omega \to \mathbb{C}$ is holomorphic, then

$$\int_{\gamma} f(z)\,\mathrm{d}z = 0$$

for every closed curve $\gamma \subseteq \Omega$.

Proof. f has a primitive, so the integral is zero. □

Corollary 2.8

$\mathbb{C} \setminus \{0\}$ is not simply connected.

Proof. We showed that $\int_{\partial\mathbb{D}} \frac{1}{z}\,\mathrm{d}z = 2\pi i \neq 0$. □

In fact, there are stronger versions of this theorem. We will not formally prove them here, but we will give an intuitive explanation.

Definition 2.4

A curve γ is **simple** if, given some parameterization $z : [a,b] \to \mathbb{C}$, $z(s) = z(t) \implies s = t$. A **simple closed** curve is the same, except that $z(a) = z(b)$.

Note that a simple closed curve is not technically simple, but the intuitive idea is the same.

Theorem: Jordan Curve Theorem

Let $\gamma \subseteq \mathbb{C}$ be a simple closed curve. Then $\mathbb{C} \setminus \gamma = \Omega \cup U$, where Ω, U are open, connected, and disjoint. Moreover, Ω is bounded and simply connected, and U is unbounded and connected. In this case, Ω is called the **interior** of γ (denoted $\operatorname{int}\gamma$), and U is called the **exterior** (denoted $\operatorname{ext}\gamma$.)

Proof. The proof of this theorem is omitted. Note that it is true in the piecewise smooth case (see Stein & Shakarchi appendix), but it also true for continuous curves. □

Theorem: General Cauchy's Theorem

Let $\Omega \subseteq \mathbb{C}$ be open and let $f : \Omega \to \mathbb{C}$ be holomorphic on Ω. Let $\gamma \subseteq \mathbb{C}$ be a simple closed curve in Ω such that $\operatorname{int}\gamma \subseteq \Omega$. Then

$$\int_\gamma f(z)\,\mathrm{d}z = 0$$

Proof. Draw a curve inside of $\operatorname{int}\gamma$ which is a slight pertubation of γ and is homotopic to γ. Then this follows since $\operatorname{int}\gamma$ is simply connected. See Stein & Shakarchi appendix for a complete proof. $\square$

2.3 Consequences of Cauchy's Theorem

Aside from being a powerful theorem about complex integrals, Cauchy's Theorem also allows us to solve many difficult real integrals using complex integrals.

Example 2.4

Consider the integral

$$\int_0^\infty \frac{1-\cos x}{x^2}\,\mathrm{d}x$$

Note that this is integrable since

$$\lim_{x\to\infty}\left|\frac{1-\cos x}{x^2}\right| \leq \lim_{x\to\infty}\frac{2}{x^2}$$

and near 0, this behaves like 0. So the integral is indeed a real number.

Now, consider the function $f : \mathbb{C}\setminus\{0\} \to \mathbb{C}$ defined by

$$f(z) = \frac{1-e^{iz}}{z^2}$$

which is holomorphic. For each $R > 0, \varepsilon > 0$, consider the integral of $f(z)$ over $\Gamma_{R,\varepsilon}$, which is defined below as $\gamma_R + [-R, -\varepsilon] + \gamma_\varepsilon + [\varepsilon, R]$:

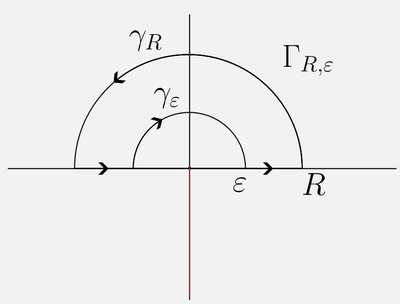


Figure 2.5: Indented semicircle

This function is holomorphic on the star-shaped domain $\mathbb{C} \setminus \{it : t \le 0\}$ (which is simply connected), so

$$\begin{aligned} 0 &= \int_{\Gamma_{R,\varepsilon}} f(z)\,\mathrm{d}z \\ &= \underbrace{\int_{-R}^{-\varepsilon} \frac{1-e^{-ix}}{x^2}\,\mathrm{d}x}_{\mathrm{I}} + \underbrace{\int_{\gamma_\varepsilon} \frac{1-e^{iz}}{z^2}\,\mathrm{d}z}_{\mathrm{II}} + \underbrace{\int_{\varepsilon}^{R} \frac{1-e^{ix}}{x^2}\,\mathrm{d}x}_{\mathrm{III}} + \underbrace{\int_{\gamma_R} \frac{1-e^{iz}}{z^2}\,\mathrm{d}z}_{\mathrm{IV}} \end{aligned}$$

Now,

$$\mathrm{I} + \mathrm{III} = \int_{\varepsilon}^{R} \frac{2-(e^{ix}+e^{-ix})}{x^2}\,\mathrm{d}x = 2\int_{\varepsilon}^{R} \frac{1-\cos x}{x^2}\,\mathrm{d}x$$

So the problem reduces to calculating II and IV. We claim that

$$\lim_{\varepsilon \to 0} \int_{\gamma_\varepsilon^-} \frac{1-e^{iz}}{z^2}\,\mathrm{d}z - \pi = 0$$

To see this, parameterize γ_ε^- as $t \mapsto \varepsilon e^{it}$ for $t \in [0, \pi]$. Then

$$\begin{aligned} \int_{\gamma_\varepsilon^-} \frac{1-e^{iz}}{z^2}\,\mathrm{d}z - \pi &= \int_0^\pi \frac{1-e^{i\varepsilon e^{it}}}{\varepsilon^2 e^{2it}} (\varepsilon i e^{it})\,\mathrm{d}t - \pi \\ &= \frac{i}{\varepsilon} \int_0^\pi \left(\frac{1-e^{i\varepsilon e^{it}}}{e^{it}} + i\varepsilon \right) \mathrm{d}t \\ &= \frac{i}{\varepsilon} \int_0^\pi \frac{1-e^{i\varepsilon e^{it}} + i\varepsilon e^{it}}{e^{it}}\,\mathrm{d}t \end{aligned}$$

We may write

$$e^{i\varepsilon e^{it}} = \sum_{n=0}^{\infty} \frac{\left(i\varepsilon e^{it}\right)^n}{n!}$$

so

$$\left| e^{i\varepsilon e^{it}} - (1 + i\varepsilon e^{it}) \right| \le \sum_{n=2}^{\infty} \frac{\left|i\varepsilon e^{it}\right|^n}{n!} = \sum_{n=2}^{\infty} \frac{\varepsilon^n}{n!} \le 10\varepsilon^2$$

(10 is an overbound here). Returning to the original integral,

$$\left| \frac{i}{\varepsilon} \int_0^\pi \frac{1-e^{i\varepsilon e^{it}} + i\varepsilon e^{it}}{e^{it}}\,\mathrm{d}t \right| \le \frac{1}{\varepsilon} \pi 10\varepsilon^2 = 10\pi\varepsilon$$

which tends to 0 as $\varepsilon \to 0$.

Lastly, let us compute IV. We parameterize γ_R by $t \mapsto Re^i t$ on $[0, \pi]$, and see that

$$\left| \int_{\gamma_R} \frac{1-e^{iz}}{z^2}\,\mathrm{d}z \right| = \left| \int_0^\pi \frac{1-e^{iRe^{it}}}{R^2 e^{2it}} \right| iRe^{it}\,\mathrm{d}t$$

Observe that

$$\begin{aligned} \left|1 - e^{iRe^{it}}\right| &= \left|1 - e^{iR(\cos t + i \sin t)}\right| \\ &= \left|1 - e^{iR\cos t} e^{-R\sin t}\right| \\ &\leq 1 + e^{R \sin t} \\ &\leq 2 \end{aligned}$$

where the last inequality follows since $\sin t > 0$ on $[0, \pi]$. Returning to IV,

$$\left| \int_0^\pi \frac{1 - e^{iRe^{it}}}{R^2 e^{2it}} \right| iRe^{it} \, \mathrm{d}t \leq \frac{1}{R} 2\pi$$

which tends to 0 as $R \to \infty$. Then

$$\int_0^\infty \frac{1 - \cos x}{x^2} \, \mathrm{d}x = \frac{1}{2}(\mathrm{II} + \mathrm{IV}) = \frac{\pi}{2}$$

The following theorem is an important formula which is a powerful example of the local-global properties present in complex analysis.

Theorem 2.9: Cauchy's Integral Formula

Let $\Omega \subseteq \mathbb{C}$ be open, $\overline{\mathbb{D}_R}(z_0) \subseteq \Omega$, and let $f : \Omega \to \mathbb{C}$ be holomorphic on Ω. Then for $z \in \mathbb{D}_R(z_0)$,

$$f(z) = \frac{1}{2\pi i} \int_{\partial \mathbb{D}_R(z_0)} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta$$

Proof. Define

$$F(\zeta) = \frac{f(\zeta)}{\zeta - z}$$

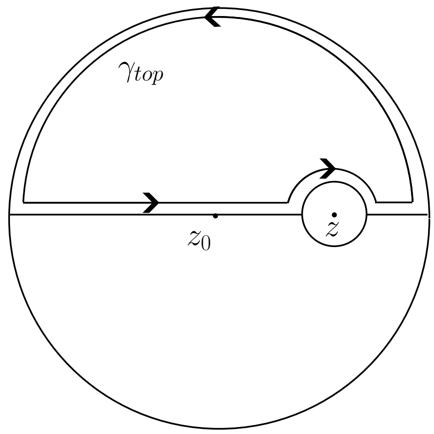
Define the curve γ_{top} as shown in the diagram below:

Define γ_{bottom} similarly, using the counterclockwise orientation so that the integrals over the line segment cancel. Both are holomorphic on star-shaped regions, so their integrals are zero and

$$\begin{aligned} 0 &= \frac{1}{2\pi i} \left(\int_{\gamma_{top}} F(\zeta) \, \mathrm{d}\zeta + \int_{\gamma_{bottom}} F(\zeta) \, \mathrm{d}\zeta \right) \\ &= \frac{1}{2\pi i} \left(\int_{\partial \mathbb{D}_R(z_0)} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta - \int_{\partial \mathbb{D}_\varepsilon(z)} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta \right) \end{aligned}$$

so

$$\frac{1}{2\pi i} \int_{\partial \mathbb{D}_R(z_0)} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta = \frac{1}{2\pi i} \int_{\partial \mathbb{D}_\varepsilon(z)} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta$$



We want to show that the right side quantity is equal to $f(z)$. Calculating,

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}_\varepsilon(z)}\frac{f(\zeta)}{\zeta-z}\,\mathrm{d}\zeta=\frac{1}{2\pi i}\int_{\partial\mathbb{D}_\varepsilon(z)}\left(\frac{f(z)}{\zeta-z}+\frac{f(\zeta)-f(z)}{\zeta-z}\right)\mathrm{d}\zeta$$

Notice that the first term of the integral is the integral of $\frac{1}{z}$ on a circle around 0, but translated and multiplied by $f(z)$. So this becomes

$$f(z)+\frac{1}{2\pi i}\int_{\partial\mathbb{D}_\varepsilon(z)}\frac{f(\zeta)-f(z)}{\zeta-z}\,\mathrm{d}\zeta$$

The integrand is bounded since f is holomorphic, so it vanishes as $\varepsilon\to 0$. Thus

$$f(z)=\frac{1}{2\pi i}\int_{\partial\mathbb{D}_R(z_0)}\frac{f(\zeta)}{\zeta-z}\,\mathrm{d}\zeta \qquad \square$$

We now arrive at perhaps the most important theorem in all of complex analysis. This theorem shows that holomorphic functions are also analytic. We showed that analytic functions are holomorphic; we now see that they are the same. Moreover, we see that all holomorphic functions are infinitely differentiable.

Theorem 2.10

Let $\Omega\subseteq\mathbb{C}$ be open, and suppose $\overline{\mathbb{D}_R}(z_0)\subseteq\Omega$. Suppose $f:\Omega\to\mathbb{C}$ is holomorphic on Ω. Then for every $z\in\mathbb{D}_R(z_0)$,

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n$$

where

$$a_n=\frac{1}{2\pi i}\int_{\partial\mathbb{D}_R(z_0)}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\,\mathrm{d}\zeta$$

In particular, the radius of convergence is at least R.

Proof. By the Cauchy Integral formula, when $|z - z_0| < R$ we have

$$f(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}_R(z_0)} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}z = \frac{1}{2\pi i}\int_{\partial\mathbb{D}_R(z_0)} \frac{f(\zeta)}{\zeta - z_0}\cdot\frac{1}{1-\frac{z-z_0}{\zeta-z_0}}\,\mathrm{d}\zeta$$

Now, if $|z - z_0| = r < R$, then $\left|\frac{z-z_0}{\zeta-z_0}\right| = \frac{r}{R}$ (since ζ is on the boundary of $\mathbb{D}_R(z_0)$). So we may write this as the sum of a geometric series:

$$\begin{aligned}\frac{1}{2\pi i}\int_{\partial\mathbb{D}_R(z_0)} \frac{f(\zeta)}{\zeta - z_0}\cdot\frac{1}{1-\frac{z-z_0}{\zeta-z_0}}\,\mathrm{d}\zeta &= \frac{1}{2\pi i}\int_{\partial\mathbb{D}_R(z_0)} \frac{f(\zeta)}{\zeta - z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{\zeta-z_0}\right)^n \mathrm{d}\zeta \\ &= \frac{1}{2\pi i}\int_{\partial\mathbb{D}_R(z_0)} \left(\sum_{n=0}^{\infty}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}(z-z_0^n)\right)\mathrm{d}\zeta\end{aligned}$$

Thus the proof reduces to justifying the interchange of the sum and integral above. To prove this, for $N \in \mathbb{N}$ we have

$$\begin{aligned}&\frac{1}{2\pi i}\left[\int_{\partial\mathbb{D}_R(z_0)}\left(\sum_{n=0}^{\infty}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}(z-z_0)^n\right)\mathrm{d}\zeta - \underbrace{\int_{\partial\mathbb{D}_R(z_0)}\left(\sum_{n=0}^{N}\frac{f(\zeta)}{(\zeta-z)^{n+1}}(z-z_0)^n\right)\mathrm{d}\zeta}_{=0}\right] \\ &\qquad = \frac{1}{2\pi i}\int_{\partial\mathbb{D}_R(z_0)}\left(\sum_{n=N+1}^{\infty}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}(z-z_0)^n\right)\mathrm{d}\zeta\end{aligned}$$

So

$$\begin{aligned}\left|f(z) - \sum_{n=0}^{N}a_n(z-z_0)^n\right| &= \left|\frac{1}{2\pi i}\int_{\partial\mathbb{D}_R(z_0)}\left(\sum_{n=N+1}^{\infty}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}(z-z_0)^n\right)\mathrm{d}\zeta\right| \\ \le \frac{1}{2\pi}2\pi R\sum_{n=N+1}^{\infty}\frac{r^n}{R^{n+1}}\sup_{\zeta\in\partial\mathbb{D}_R(z_0)}|f(\zeta)| &\le \left(\sup_{\zeta\in\partial\mathbb{D}_R(z_0)}|f(\zeta)|\right)\sum_{n=N+1}^{\infty}\left(\frac{r}{R}\right)^n \\ = \left(\sup_{\zeta\in\partial\mathbb{D}_R(z_0)}|f(\zeta)|\right)\cdot\left(\frac{r}{R}\right)^{N+1}\frac{1}{1-\frac{r}{R}} &\overset{N\to\infty}{\longrightarrow} 0\end{aligned}$$

□

We can use this to extend Cauchy's integral formula to higher order derivatives, still exhibiting local-global behavior:

Theorem 2.11: Cauchy integral formulas

Let $\Omega \subseteq \mathbb{C}$ be open, and let $f : \Omega \to \mathbb{C}$ be holomorphic on Ω. Let $\overline{\mathbb{D}_R}(z_0) \subseteq \Omega$. Then f has n complex derivatives for every $n \ge 0$ and for $z \in \mathbb{D}_R(z_0)$,

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{\partial\mathbb{D}_R(z_0)}\frac{f(\zeta)}{(\zeta-z)^{n+1}}\,\mathrm{d}\zeta$$

Note that this is equivalent to Theorem 2.9 for $n = 0$.

Proof. From Theorem 2.10, we can represent f as a power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

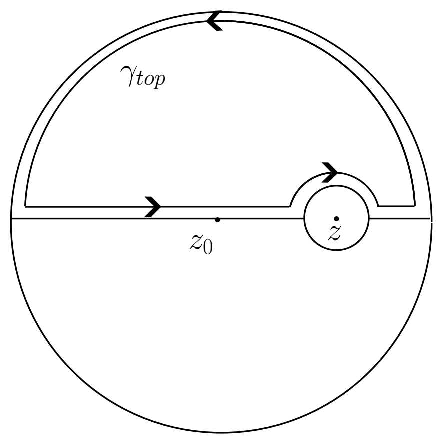
moreover, we know that we may differentiate the series term by term to see that

$$f^{(n)}(z) = \sum_{k=n}^{\infty} \frac{n!}{(n-k)!} a_k (z - z_0)^{k-n}$$

Evaluating at z_0, all the terms cancel out except $k = n$, so

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\partial \mathbb{D}_R(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, \mathrm{d}\zeta$$

Now, we use the same contours $\gamma_{top}, \gamma_{bottom}$ as in the original Cauchy Integral formula proof:



Defining $F(\zeta) = \frac{f(\zeta)}{(\zeta - z)^{n+1}}$, which is holomorphic on the simply connected domains containing γ_{top} and γ_{bottom}, we have

$$\frac{1}{2\pi i} \int_{\partial \mathbb{D}_R(z_0)} \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, \mathrm{d}\zeta = \frac{1}{2\pi i} \int_{\partial \mathbb{D}_\varepsilon(z)} \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, \mathrm{d}\zeta = \frac{f^{(n)}(z)}{n!}$$

The last equality follows by our above work, since z is the center of $\mathbb{D}_\varepsilon(z)$. $\square$

As an easy corollary we have:

Corollary 2.12: Cauchy Inequality

Let $\Omega \subseteq \mathbb{C}$ be open, $f : \Omega \to \mathbb{C}$ holomorphic on Ω, and $\overline{\mathbb{D}_R}(z_0) \subseteq \Omega$. Then

$$\left| f^{(n)}(z_0) \right| \leq \frac{n!}{R^n} \sup_{\zeta \in \partial \mathbb{D}_R(z_0)} |f(\zeta)|$$

Proof. By the previous theorem,

$$\left|f^{(n)}(z_0)\right| = \left|\frac{n!}{2\pi i}\int_{\partial\mathbb{D}_R(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\,\mathrm{d}\zeta\right| \le \frac{n!}{2\pi i}2\pi R \cdot \frac{1}{R^{n+1}} \sup_{\zeta\in\partial\mathbb{D}_R(z_0)} |f(\zeta)| \qquad \square$$

Theorem 2.13: Liouville's Theorem

If $f : \mathbb{C} \to \mathbb{C}$ is entire and bounded, then f is constant.

Proof. Let $z_0 \in \mathbb{C}$. Let $|f(\zeta)| \le M$ for all $\zeta \in \mathbb{C}$. Then as $R \to \infty$ we have

$$|f'(z_0)| \le \frac{1}{R}M \to 0$$

so $f' \equiv 0$ and f is constant. $\square$

As a corollary to this we may prove the Fundamental Theorem of Algebra:

Corollary 2.14: Fundamental Theorem of Algebra

Let $p \in \mathbb{C}[x]$ be non-constant. Then p has a zero in $\mathbb{C}$.

Proof. Suppose p has no root. Then $\frac{1}{p}$ is entire. We write

$$\left|\frac{1}{p(z)}\right| = \frac{1}{|z^n|} \cdot \frac{1}{\left|a_n + \frac{a_{n-1}}{z} + \ldots + \frac{a_0}{z^n}\right|}$$

(we may assume $a_0 \neq 0$ and $n \ge 1$). We bound the denominator from below for large z by

$$\left|a_n + \frac{a_{n-1}}{z} + \ldots + \frac{a_0}{z^n}\right| \ge |a_n| - \frac{|a_{n-1}|}{|z|} - \ldots - \frac{|a_0|}{|z|^n}$$

so

$$\left|\frac{1}{p(z)}\right| \le \frac{1}{|z|^n} \cdot \frac{1}{|a_n| - \frac{|a_{n-1}|}{|z|} - \ldots - \frac{|a_0|}{|z|^n}} \xrightarrow{|z|\to\infty} 0$$

Thus we pick $R > 0$ such that $\frac{1}{|p(z)|} \le 1$ when $|z| > R$. On $\overline{\mathbb{D}_R}(z_0)$, $\frac{1}{p(z)}$ is continuous and thus bounded. So $\frac{1}{p}$ is bounded everywhere and thus constant by Liouville's Theorem. But we supposed p was not constant, so p must have a root. $\square$

We now provide a proof of the method known as analytic continuation.

Definition 2.5

Let $\Omega \subseteq \mathbb{C}$ be open and let $\Omega' \subseteq \Omega$ be open. Let $f : \Omega' \to \mathbb{C}$ and $F : \Omega \to \mathbb{C}$ be holomorphic, with $F|_{\Omega'} = f$. Then we call F an **analytic continuation** of f.

The key result is that, under minor restrictions, this choice of F is actually unique; that is, the value of f on a small set completely determine its extension to Ω.

Theorem 2.15: Uniqueness of Analytic Continuation

Let $\Omega \subseteq \mathbb{C}$ be open and connected, and consider some collection of distinct points $\{\omega_n\}_{n=1}^{\infty} \subseteq \Omega$ such that $\lim \omega_n = \omega \in \Omega$ exists. Suppose that $f : \Omega \to \mathbb{C}$ is holomorphic and the ω_i are zeroes of f. Then $f \equiv 0$ on Ω.

Proof. Let $A = \{z \in \Omega : f(z) = 0\}$. Let $U = \operatorname{int} A$. I claim the following:

Claim 1: U is nonempty.

Claim 2: U is closed in Ω.

The theorem follows from the above, as if they are true, we have $\Omega = U \cup (\Omega \setminus U)$. By Claim 2, these are disjoint open sets, so one must be nonempty. By Claim 1, it must be $\Omega \setminus U$ so $U = \Omega$.

To prove these claims, we show the following:

Claim

If $\{\zeta_n\} \subseteq A$ are distinct and $\zeta_n \to \zeta_0 \in \Omega$, then there is $R > 0$ such that f vanishes on $\mathbb{D}_R(\zeta_0)$.

Proof. Ω is open and $\zeta_0 \in \Omega$, so there exists $R > 0$ such that $\overline{\mathbb{D}_R}(\zeta_0) \subseteq \Omega$. Since f is holomorphic, it is analytic, so we may write

$$f(\zeta) = \sum_{n=0}^{\infty} a_n (\zeta - \zeta_0)^n$$

on $\mathbb{D}_R(\zeta_0)$. For large enough n, $\zeta_n \in \mathbb{D}_R(\zeta_0)$, and thus

$$f(\zeta_0) = \lim f(\zeta_n) = 0$$

so $a_0 = 0$. If all the a_i are zero, then we are done. Suppose not. Then let $m \geq 1$ be the smallest index such that $a_m \neq 0$. Then

$$f(\zeta) = a_m (\zeta - \zeta_0)^m \left(1 + \underbrace{\frac{a_{m+1}}{a_m}(\zeta - \zeta_0) + \frac{a_{m+2}}{a_m}(\zeta - \zeta_0)^2 + \dots}_{=g(\zeta)} \right)$$

Now, $g(\zeta)$ is analytic on $\mathbb{D}_R(z_0)$ as it is a power series, and $g(\zeta_0) = 0$. For large enough n we then have

$$f(\zeta_n) = a_m (\zeta_n - \zeta_0)^m (1 + g(\zeta_n))$$

By assumption, $f(\zeta_n) = 0$. But $a_m \neq 0$ by assumption. $(\zeta_n - \zeta_0)$ is only zero for at most one ζ_n, so we conclude that $1 + g(\zeta_n) = 0$ for all large enough n. As $n \to \infty$ $g(\zeta_n) \to 0$, so $1 = 0$, contradiction. Thus no a_m is nonzero and f is identically zero. ∎

Now we apply the above to our $\{\omega_n\}$ and arrive at Claim 1. For Claim 2, if there is some sequence of points in U, then the Claim also shows that their limit is in U. So we are done. $\square$

Corollary 2.16

Let $f : \Omega' \to \mathbb{C}$ be holomorphic with $\Omega' \subseteq \Omega \subseteq \mathbb{C}$ both open. If Ω is connected, then there is at most one analytic continuation $F : \Omega \to \mathbb{C}$ of f.

Proof. Suppose $F_1, F_2 : \Omega \to \mathbb{C}$ are analytic continuations of f. Then $F_1 - F_2$ is zero on Ω' open, and we immediately conclude that $F_1 - F_2 \equiv 0$ on Ω. So $F_1 = F_2$. $\square$

We now prove a converse to Goursat's Theorem.

Theorem 2.17: Morera's Theorem

Let $f : \mathbb{D}_R(z_0) \to \mathbb{C}$ is continuous and for any triangle $T \subseteq \mathbb{D}_R(z_0)$ we have

$$\int_{\partial T} f(z)\,\mathrm{d}z = 0$$

then f is holomorphic on $\mathbb{D}_R(z_0)$.

Proof. We imitate the proof that holomorphic functions locally have primitives, which only assumed Goursat's Theorem. In this case we do not know that f is holomorphic but we assume the conclusion of Goursat's Theorem. Thus f has a primitive F. So F is holomorphic, and thus infinitely differentiable. In particular, $f = F'$ is holomorphic. $\square$

2.4 Sequences of Functions

In this section we develop the theory of the convergence of sequences of functions, in particular holomorphic ones. This is analogous to the discussion of sequences of functions in real variables.

Definition 2.6

Let $\Omega \subseteq \mathbb{C}$. Let $\{f_n : \Omega \to \mathbb{C}\}$ be a sequence of functions and $f : \Omega \to \mathbb{C}$. We say that $\{f_n\}$ **converges uniformly** to f (denoted $f_n \rightrightarrows f$) on a subset $A \subseteq \Omega$ if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq n, z \in A$, $|f_n(z) - f(z)| < \varepsilon$. We say that $\{f_n\}$ converges uniformly on compact subsetes if for every $K \subseteq \Omega$ compact, $f_n \rightrightarrows f$ on K.

Equivalently, $f_n \rightrightarrows f$ on compact subsets if and only if for all K, $\sup_{z \in K}|f_n(z) - f(z)| \to 0$ uniformly.

As in the real case, uniform convergence is distinguished from pointswise convergence since our N must work for all $z \in A$. Convergence on compact subsets is particularly important for us, so we should note that the N may depend on the choice of K; that is, f_n need not converge uniformly on all of Ω.

Theorem 2.18

Let $\Omega \subseteq \mathbb{C}$ be open. Let $f_n : \Omega \to \mathbb{C}$ be holomorphic and $f_n \rightrightarrows f$ on compact subsets. Then f is holomorphic on Ω.

Note that this theorem is false in the real case, if holomorphicity is replaced by, say, continuous differentiability. For instance, let $f_n : \mathbb{R} \to \mathbb{R}$ be equal to the absolute value function outside of $[-\frac{1}{n}, \frac{1}{n}]$ and be some smooth interpolating function between them. This is continuously differentiable and the convergence is uniform, but the limit is the absolute value function which is not even differentiable.

Proof. We prove this using Morera's theorem. Fix $z_0 \in \Omega$ and let $\overline{\mathbb{D}_r}(z_0) \subseteq \Omega$. Recall from real analysis that the uniform limit of continuous function is continuous. Applying this to the real and imaginary parts of f, f is also continuous. By Morera's theorem, it is enough to show that for any triangle $T \subseteq \mathbb{D}_r(z_0)$, the integral of f along ∂T vanishes. f_n is holomorphic, so

$$\left|\int_{\partial T} f(z)\,\mathrm{d}z\right| = \left|\int_{\partial T} f(z)\,\mathrm{d}z - \int_{\partial T} f_n(z)\,\mathrm{d}z\right|$$
$$\leq \operatorname{length}(\partial T) \sup_{z\in\partial T}|f(z) - f_n(z)|$$

As $n \to \infty$, $\sup_{z\in\partial T}|f(z) - f_n(z)| \to 0$ and $\operatorname{length}(\partial T)$ is constant, so

$$\int_{\partial T} f(z)\,\mathrm{d}z = 0$$

and by Morera's f is holomorphic. □

We use the above result to prove a related result which essentially says that we may interchange the derivative and limit operators:

Theorem 2.19

Let $\Omega \subseteq \mathbb{C}$ be open and let $f_n : \Omega \to \mathbb{C}$ be holomorphic with $f_n \rightrightarrows f$ on compact subsets. Then $f_n' \rightrightarrows f'$ on compact subsets.

Proof. To show that $f_n' \rightrightarrows f'$ on compact subsets, let $K \subseteq \Omega$ be compact.

Claim 1: There exists $\delta > 0$ such that for all $z \in K$, $\overline{\mathbb{D}_\delta}(z) \subseteq \Omega$.

To see this, suppose there exists sequences $z_n \subseteq K$ and $w_n \notin \Omega$ such that $w_n \in \overline{\mathbb{D}_{\frac{1}{n}}}(z_n)$. Take a convergent subsequence z_{n_k} which converges to some limit $z \in K$. But then $w_{n_k} \to z$ so there is no open disk around z contained in Ω, contradicting the fact that Ω is open. Thus such a δ exists.

Let

$$K_\delta = \bigcup_{z \in K} \overline{\mathbb{D}_\delta}(z) \subseteq \Omega$$

Claim 2: K_δ is compact.

Clearly K_δ is bounded since K is. To show closure, take a sequence $z_n \subseteq K_\delta$ such that $z_n \to z \in \mathbb{C}$. For each n, there exists $w_n \in K$ such that $|z_n - w_n| \leq \delta$. Pick a subsequence $\{w_{n_k}\}$ which converges to $w \in K$. Then $|z - w| = \lim|z_{n_k} - w_{n_k}| \leq \delta$ so $z \in K_\delta$.

Let $z \in K$. Then by Cauchy's Integral formula for derivatives,

$$\begin{aligned} |f_n'(z) - f'(z)| &= \left| \frac{1}{2\pi i} \int_{\partial \mathbb{D}_\delta(z_0)} \frac{f_n(\zeta) - f(\zeta)}{(\zeta - z)^2} \, \mathrm{d}\zeta \right| \\ &\leq \frac{1}{2\pi} 2\pi\delta \frac{1}{(\delta - |z - z_0|)^2} \sup_{\zeta \in K_\delta} |f_n(\zeta) - f(\zeta)| \end{aligned}$$

As $n \to \infty$, the above tends to 0 uniformly since K_δ is compact, so $f_n' \rightrightarrows f'$ on K. □

The above theorems allow us to produce important holomorphic functions.

Example 2.5

Consider the **Riemann zeta function** $\zeta : \{z \in \mathbb{C} : \mathrm{Re}(z) > 1\} \to \mathbb{C}$ defined by

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

To show that this is holomorphic, we write

$$\zeta_N(z) = \sum_{n=1}^{N} \frac{1}{n^z} = \sum_{n=1}^{N} e^{-z \ln n}$$

So each ζ_N is holomorphic. We want to show that $\zeta_N \rightrightarrows \zeta$ on compact subsets. Let $K \subseteq \{z \in \mathbb{C} : \mathrm{Re}(z) > 1\}$ be compact.

Since K is compact there exists a $\delta > 0$ such that $K \subseteq \{z \in \mathbb{C} : \mathrm{Re}(z) > 1 + \delta\}$. Then

$$\begin{aligned} |\zeta_N(z) - \zeta(z)| &= \left| \sum_{n=N+1}^{\infty} \frac{1}{n^z} \right| \leq \sum_{n=N+1}^{\infty} \left| e^{-z \ln n} \right| = \sum_{n=N+1}^{\infty} \left| e^{-\mathrm{Re}(z) \ln n} \right| \\ &= \sum_{n=N+1}^{\infty} \frac{1}{n^{\mathrm{Re}(z)}} \leq \sum_{n=N+1}^{\infty} \frac{1}{n^{1+\delta}} \end{aligned}$$

Since $1 + \delta > 1$ this is a convergent series and therefore tends to 0. Moreover it does so independent of z so $\zeta_N \rightrightarrows \zeta$ on compact subsets and ζ is holomorphic on the indicated half plane.

We now develop analogous results for integration.

Theorem 2.20

Let $\Omega \subseteq \mathbb{C}$ be open and $F : \Omega \times [0,1] \to \mathbb{C}$ be such that

1. For fixed $s \in [0,1]$, $F_s(z) = F(z,s)$ is holomorphic on Ω.
2. F is continuous.

Define
$$f(z) = \int_0^1 F(z,s)\,\mathrm{d}s$$
Then f is holomorphic on Ω.

Proof. Define
$$f_N(z) = \frac{1}{N}\sum_{n=1}^{N} F\left(z, \frac{n}{N}\right)$$
This is a finite Riemann sum approximation of f obtained by uniformly partitioning $[0,1]$ into N subintervals. f_N is holomorphic, so we want to show that $f_n \rightrightarrows f$ on compact subsets. Now, take some $K \subseteq \Omega$ compact, and note $K \times [0,1]$ is compact. Then f is uniformly continuous on $K \times [0,1]$. So for $\varepsilon > 0$ pick $\delta > 0$ such that for $z_1, z_2 \in K, s_1, s_2 \in [0,1]$ with $|z_1 - z_2| < \delta, |s_1 - s_2| < \delta$, then $|F(z_1,s_1) - F(z_2,s_2)| < \varepsilon$.

Now, take N large enough that $N > \frac{1}{\delta}$. Then
$$\begin{aligned}
|f(z) - f_n(z)| &= \left| \int_0^1 F(z,s)\,\mathrm{d}s - \frac{1}{N}\sum_{n=1}^{N} F\left(z, \frac{n}{N}\right) \right| \\
&= \sum_{n=1}^{N} \int_{\frac{n-1}{N}}^{\frac{n}{N}} \left| \left(F(z,s) - F\left(z, \frac{n}{N}\right) \right) \right| \mathrm{d}s \\
&< \sum_{n=1}^{N} \varepsilon \frac{1}{N} \\
&= \varepsilon
\end{aligned}$$
so $f_n \rightrightarrows f$ on compact subsets and thus f is holomorphic on Ω. $\square$

Next, we discuss the Schwarz reflection principle, which is another technique for constructing holomorphic functions.

Definition 2.7

A set $\Omega \subseteq \mathbb{C}$ is **symmetric** about the real axis if
$$z \in \Omega \iff \overline{z} \in \Omega$$

In this case, we write the following:

$$\Omega^+ := \Omega \cap \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$$
$$\Omega^- := \Omega \cap \{z \in \mathbb{C} : \mathrm{Im}(z) < 0\}$$
$$I := \Omega \cap \mathbb{R}$$

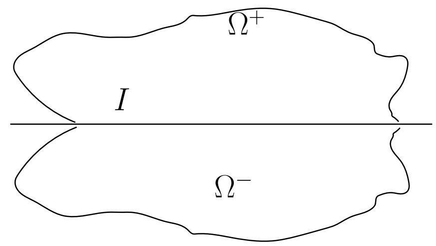


Figure 2.6: Symmetric Sets

Theorem 2.21: Symmetry Principle

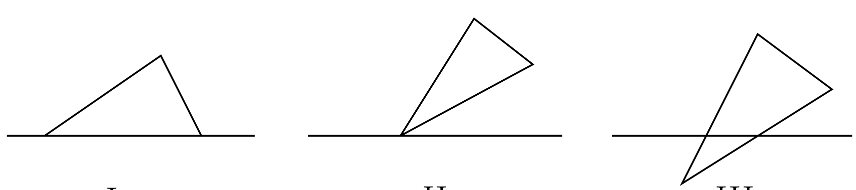
Let $\Omega \subseteq \mathbb{C}$ be open and symmetric about the real axis. Suppose that $f^+ : \Omega^+ \cup I \to \mathbb{C}$ is continuous on $\Omega^+ \cup I$ and holomorphic on Ω^+, and that f^- is the same (with Ω^+ replaced by Ω^-). Moreover, suppose that $f^+ = f^-$ on I. Define $f : \Omega \to \mathbb{C}$ by

$$f(z) = \begin{cases} f^+(z), & z \in \Omega^+ \\ f^+(z) = f^-(z), & z \in I \\ f^-(z), & z \in \Omega^- \end{cases}$$

Then f is holomorphic on Ω.

Again, observe that this is false in the real case: consider the absolute value function on $\mathbb{R}$, or similar linear functions on $\mathbb{R}^n$.

Proof. Let $z \in \Omega$. Clearly f is holomorphic at points in Ω^+, Ω^- so we only consider $z \in I$. Now, Ω is open so we let $\overline{\mathbb{D}_r}(z) \subseteq \Omega$. We want to use Morera's. Consider any triangle $T \in \overline{\mathbb{D}_r}(z_0)$. If T is entirely on one side of the line then we are done. Otherwise, there are threee possible cases:



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Case 1: Consider the triangle T_ε which is T, except translated upward by ε.

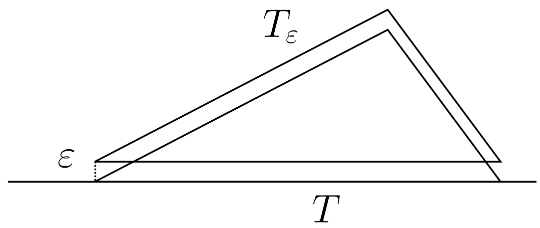


Figure 2.7: Lifting of Case 1 triangle

By Cauchy-Goursat the integral around ∂T_ε is zero, and in particular the only parts which do not cancel are the baselines. So

$$\int_{\partial T} f(z)\,\mathrm{d}z - \int_{\partial T_\varepsilon} f(z)\,\mathrm{d}z = \left|\int_a^b [f(t) - f(t+i\varepsilon)]\,\mathrm{d}t\right| \leq (b-a) \sup_{t\in[a,b]} |f(t) - f(t+i\varepsilon)|$$

This tends to 0 uniformly as $\varepsilon \to 0$ since f is continuous.

Case 2: Similar to Case 1. EXERCISE: prove this.

Case 3: We split T into three triangles, two of which satisfy Case 1 and one of which satisfies Case 2:

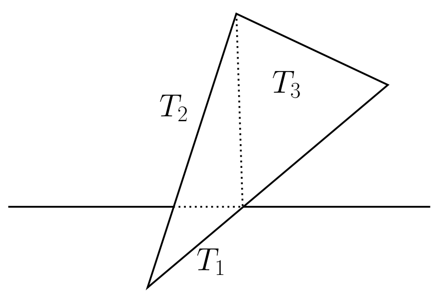


Figure 2.8: Subdivision of Case 3 triangle into Case 1 and Case 2 triangles

Choosing the right orientation shows that the integral around ∂T is zero.

So f is holomorphic on Ω. $\square$

Theorem 2.22: Schwarz Reflection Principle

Let $\Omega \subseteq \mathbb{C}$ be open and symmetric, with $\Omega \cap \mathbb{R} \neq \varnothing$, and let $f^+ : \Omega \cup I \to \mathbb{C}$ be continuous on $\Omega^+ \cup I$ and holomorphic on Ω^+. Also, suppose $f^+(z) \in \mathbb{R}$ for $z \in I$. Then there exists a unique $f : \Omega \to \mathbb{C}$ holomorphic on Ω which coincides with f^+ on $\Omega^+ \cup I$.

We could alternately demand that Ω be connected rather than $\Omega \cap \mathbb{R} \neq \varnothing$, this just gives us uniqueness.

Proof. For uniqueness, this is immediate by analytic continuation.

For $z \in \Omega^-$, define

$$f(z) = \overline{f^+(\overline{z})}$$

We want to show that this is holomorphic, and we will do this using the Symmetry principle. Because f^+ is real on I, f coincides with f^+ on I. Now we just need to show f^- is holomorphic. Let $z_0 \in \Omega^-$. Then there exists $r > 0$ such that

$$f^+(z) = \sum_{n=0}^{\infty} a_n (z - \overline{z_0})^n$$

in some disk $\mathbb{D}_r(\overline{z_0}) \subseteq \Omega^+$. Now take $z \in \mathbb{D}_r(z_0)$. Then we have

$$f^+(\overline{z}) = \sum_{n=0}^{\infty} a_n (\overline{z} - \overline{z_0})^n = \overline{\sum_{n=0}^{\infty} \overline{a_n} (z - z_0)^n}$$

so $f^-(z) = \overline{f^+(\overline{z})}$ is holomorphic at z_0 and thus on Ω^-. Now conclude by the symmetry principle. $\square$

Chapter 3

Meromorphic Functions and Poles

In this chapter we study those functions which are holomorphic except possibly at some isolated points. Together, understanding these functions together with holomorphic functions will allow us to understand a wide variety of functions for practical use.

3.1 Classification of Singularities

In this section, it is of interest to understand the different ways that a function might fail to be holomorphic at a certain point. One way that this may happen is if a function is the ratio of two functions, and the denominator vanishes at a point. Thus, we briefly consider the zeroes of certain functions.

Definition 3.1

If $f : \Omega \to \mathbb{C}$ and $z_0 \in \Omega$, then z_0 is called a **zero** of f if $f(z_0) = 0$. It is called an **isolated zero** if there is $r > 0$ such that

$$f((\mathbb{D}_r(z_0) \setminus \{z_0\}) \cap \Omega) \subseteq \mathbb{C} \setminus \{0\}$$

that is, if f is nonzero around z_0.

Theorem 3.1

Let $\Omega \subseteq \mathbb{C}$ be a region and let $f : \Omega \to \mathbb{C}$ be holomorphic and nonconstant. Let $z_0 \in \Omega$ be a zero of f. Then there exists $\delta > 0$ such that $\mathbb{D}_\delta(z_0) \subseteq \Omega$ and $n \in \mathbb{N}$ and $g : \mathbb{D}_\delta(z_0) \to \mathbb{C} \setminus \{0\}$ holomorphic such that $f(z) = (z - z_0)^n g(z)$ for all $z \in \mathbb{D}_\delta(z_0)$. Moreover, n, g are unique.

In other words, this theorem says that a function which vanishes at a point may be locally factored as a product of $(z - z_0)^n$ and a nonzero function.

Proof. Pick $r > 0$ such that $\mathbb{D}_r(z_0) \subseteq \Omega$. Since f is holomorphic on the disk we expand it as a power series

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

on $\mathbb{D}_r(z_0)$. We know $f(z_0) = a_0 = 0$. f cannot be zero on the disk because then it would be zero (and thus constant) overwhere by analytic continuation. Let n be the smallest integer such that $a_n \neq 0$. Then

$$f(z) = a_n(z - z_0)^n + \ldots = (z - z_0)^n \left(a_n + a_{n+1}(z - z_0) + \ldots\right)$$

Define

$$g(z) = \sum_{k=0}^{\infty} a_{n+k} z^k$$

This is nonzero at z_0 so by continuity there is some $\delta > 0$ such that g is nonzero on $\mathbb{D}_\delta(z_0)$. Thus we have shown existence.

For uniqueness, suppose that

$$f(z) = (z - z_0)^{n_1} g_1(z) = (z - z_0)^{n_2} g_2(z)$$

Suppose that $n_1 \neq n_2$ and without loss of generality suppose $n_1 < n_2$. Then

$$g_2(z) = (z - z_0)^{n_1 - n_2} g_1(z)$$

which is zero at z_0, contradiction. Thus $n_1 = n_2$. It follows by division that $g_1 = g_2$ except possibly at z_0, and since g_1, g_2 are holomorphic they are equal there as well. $\square$

Definition 3.2

Suppose $f : \Omega \to \mathbb{C}$ is holomorphic and nonconstant with Ω a region, and suppose $z_0 \in \Omega$ is a zero of f. Then the unique n referred to Theorem 3.1 is called the **order** or **multiplicity** of the zero of f at z_0. If $n = 1$ then z_0 is called a **simple zero** of f.

Definition 3.3

Let $\Omega \subseteq \mathbb{C}$ be open and suppose $z_0 \in \Omega$. Then if $f : \Omega \setminus \{z_0\} \to \mathbb{C}$ is holomorphic on $\Omega \setminus \{z_0\}$ we say that f has a **pole** at z_0 if

1. There is $\delta > 0$ such that $\mathbb{D}_\delta(z_0) \subseteq \Omega$ and f does not vanish on $\mathbb{D}_\delta(z_0) \setminus \{z_0\}$;
2. If we define
$$g(z) = \begin{cases} \frac{1}{f(z)}, & z \in \mathbb{D}_\delta(z_0) \setminus \{z_0\} \\ 0, & z = z_0 \end{cases}$$
then g is holomorphic.

Intuitively, a pole is a point at which the denominator vanishes and f tends to infinity.

Theorem 3.2

Let $\Omega \subseteq \mathbb{C}$ be open, $z_0 \in \Omega$, and $f : \Omega \setminus \{z_0\} \to \mathbb{C}$ holomorphic on $\Omega \setminus \{z_0\}$. Moreover, suppose f has a pole at z_0. Then there exists $r > 0$ such that for all $z \in \mathbb{D}_r(z_0) \setminus \{z_0\}$,

$$f(z) = \frac{h(z)}{(z - z_0)^n}$$

where $h : \mathbb{D}_r(z_0) \to \mathbb{C} \setminus \{0\}$ is holomorphic and nonvanishing, $n \in \mathbb{N}$, and n, h are unique.

Proof. By Theorem 3.1, there exists ψ such that

$$g(z) = (z - z_0)^n \psi(z)$$

where $\psi : \mathbb{D}_\delta(z_0) \to \mathbb{C} \setminus \{0\}$ is holomorphic and nonvanishing. Now, for $z \in \mathbb{D}_\delta(z_0) \setminus \{z_0\}$,

$$f(z) = \frac{1}{g(z)} = \frac{\frac{1}{\psi(z)}}{(z - z_0)^n}$$

Now let $h(z) = \frac{1}{\psi(z)}$. EXERCISE: prove uniqueness. $\square$

Definition 3.4

In the setting of Theorem 3.2 f is said to have a pole of **order** (or **multiplicity**) n at z_0, and if $n = 1$ then z_0 is called a **simple pole**.

Theorem 3.3

If $\Omega \subseteq \mathbb{C}$ is open and f has a pole of order n at $z_0 \in \Omega$, then there exists $\delta > 0$ such that for $z \in \mathbb{D}_\delta(z_0)$ we may write

$$f(z) = \frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-n+1}}{(z - z_0)^{n-1}} + \ldots + \frac{a_{-1}}{z - z_0} + G(z)$$

where $a_{-n}, \ldots, a_{-1} \in \mathbb{C}$ and G is holomorphic on $\mathbb{D}_\delta(z_0)$.

Proof. Let δ be as in Theorem 3.2, and write

$$f(z) = \frac{h(z)}{(z - z_0)^n}$$

h is holomorphic on $\mathbb{D}_\delta(z_0)$, so we expand it as a power series:

$$\frac{h(z)}{(z - z_0)^n} = \frac{A_0 + A_1(z - z_0) + \ldots}{(z - z_0)^n} = \frac{A_0}{(z - z_0)^n} + \ldots + \frac{A_{n-1}}{(z - z_0)} + A_n + A_{n+1}(z - z_0) + \ldots$$

Then relabel this by letting $A_0, \ldots, A_{n-1}$ be $a_{-n}, \ldots, a_{-1}$ respectively, and $A_n, \ldots$ be $a_0, \ldots$. Letting $G(z)$ be the right side, we are done. $\square$

Definition 3.5

The term
$$\frac{a_{-n}}{(z-z_0)^n} + \ldots + \frac{a_{-1}}{(z-z_0)}$$
in Theorem 3.3 is known as the **principal part** of f.

Moreover, the number a_{-1} is known as the **residue** of f at the pole z_0, denoted $\mathrm{res}_{z_0}(f)$.

The residue term ends up being the most important piece of data about f at a pole. This is because if we integrate on a circle around the pole, all the terms integrate to zero except the a_{-1} term.

Theorem 3.4

If f has a pole of order n at z_0, then
$$\mathrm{res}_{z_0}(f) = \lim_{z\to z_0} \frac{1}{(n-1)!}\left((z-z_0)^n f(z)\right)^{(n-1)}$$
In particular, if the pole is simple then
$$\mathrm{res}_{z_0}(f) = \lim_{z\to z_0} (z-z_0)f(z)$$

Proof. We write
$$(z-z_0)^n f(z) = a_{-n} + (z-z_0)a_{n+1} + \ldots + (z-z_0)^{n-1}a_{-1} + (z-z_0)^n G(z)$$
Differentiate $n-1$ times and take the limit as $z \to z_0$. Then every term of the principal part vanishes, and G is holomorphic multiplied by an nth power so the $(n-1)$th derivative will also be zero. Thus
$$\left((z-z_0)^{n-1} f(z)\right)^{(n-1)}\bigg|_{z\to z_0} = (n-1)!a_{-1} \qquad \square$$

This work leads us to the residue theorem, which is a powerful theorem that expands on Cauchy's Theorem for non-holomorphic functions.

Theorem 3.5: Residue Theorem

Let $\Omega \subseteq \mathbb{C}$ be simply connected and suppose $U \subseteq \Omega$ is open such that $\overline{U} \subseteq \Omega$ and $\gamma = \partial U$ is a simple closed curve. Suppose there exist a finite number of points $z_1, \ldots, z_N \in U$ (see Figure 3.1). Suppose $f : \Omega \setminus \{z_1, \ldots, z_N\} \to \mathbb{C}$ is holomorphic with poles at $z_1, \ldots, z_N$. Then
$$\int_\gamma f(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^{N} \mathrm{res}_{z_k}(f)$$

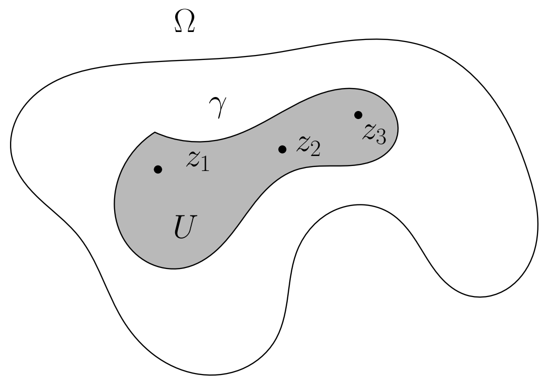


Figure 3.1: Setting for the Residue Theorem

Proof. We induct on N. For $N = 0$ this is true by Cauchy's Theorem for simply connected domains.

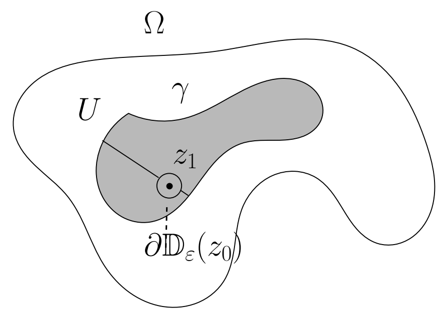
For $N = 1$, apply Theorem 3.2 and suppose the pole at z_1 has order n. Then for some small $\varepsilon > 0$ we have

$$\int_{\mathbb{D}_\varepsilon(z_1)} f(z)\,\mathrm{d}z = \int_{\mathbb{D}_\varepsilon(z_1)} \left(\frac{a_{-n}}{(z-z_1)^n} + \ldots + \frac{a_{-1}}{z-z_1} + G(z) \right) \mathrm{d}z$$

By Cauchy's Theorem, the $G(z)$ part drops out. Now, by Cauchy's integral formula for derivatives,

$$\frac{1}{2\pi i} \int_{\partial\mathbb{D}_\varepsilon(z_1)} \frac{a_{-k}}{(z-z_1)^{k-1+1}} = \frac{1}{(k-1)!}(a_{-k})^{(k-1)} = \begin{cases} 0, & k \geq 2 \\ a_{-1}, & k = 1 \end{cases}$$

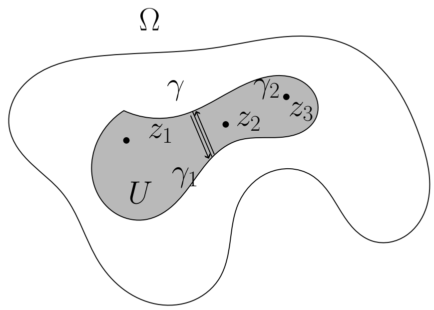
So the entire integral becomes $2\pi i \operatorname{res}_{z_1}(f)$. Now, similar to the proof of Cauchy's integral formula, we set up the following:



so

$$\int_\gamma f(z)\,\mathrm{d}z = \int_{\mathbb{D}_\varepsilon(z_1)} f(z)\,\mathrm{d}z = 2\pi i \operatorname{res}_{z_1}(f)$$

For the inductive step, for any N we draw a line through U which passes through none of the z_i, but so that not all the points are on the same side (this is possible since N is finite). Let γ_1 be the boundary of one of the subdomains created and γ_2 the other.



We see that

$$\int_{\gamma_1} f(z)\,\mathrm{d}z + \int_{\gamma_2} f(z)\,\mathrm{d}z = \int_{\gamma} f(z)\,\mathrm{d}z$$

Each of γ_1, γ_2 contain fewer than N of the z_i, and in particular contain all of them except z_N, so by the inductive hypothesis

$$\int_{\gamma_1} f(z)\,\mathrm{d}z + \int_{\gamma_2} f(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^{N} \mathrm{res}_{z_k}(f)$$ □

The residue theorem allows us to compute many integrals easily, including those of rational real functions:

Example 3.1

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^4 + x^2 + 1}\,\mathrm{d}x = \frac{\pi}{e^{\frac{\sqrt{3}}{2}}} \left(\frac{\cos\left(\frac{1}{2}\right)}{\sqrt{3}} + \sin\left(\frac{1}{2}\right) \right)$$

To see why this is, consider the function

$$f(z) = \frac{e^{iz}}{z^4 + z^2 + 1}$$

and note that since sin is odd,

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^4 + x^2 + 1}\,\mathrm{d}x = \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$$

We integrate this along the upper semicircle of radius R. Denote the arc section of its boundary by γ_R. Notice that the poles are the zeroes of the denominator, which cocur precisely when

$$z = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i = a_{\pm}$$
$$z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i = b_{\pm}$$

all of which have order 1 since the denominator is a polynomial of degree 4. We only care about a_+, b_+ since those are the only poles in our curve. Thus

$$\int_{-R}^{R} f(z)\,\mathrm{d}z + \int_{\gamma_R} f(z)\,\mathrm{d}z = 2\pi i\left(\mathrm{res}_{a_+}(f) + \mathrm{res}_{b_+}(f)\right)$$

We want to show that the integral over γ_R tends to zero. Parameterize this by $t \mapsto Re^{it}$ for $t \in [0, \pi]$. So

$$|f(z)| = \left|\frac{e^{i(R\cos\theta + i\sin\theta)}}{z^4 + z^2 + 1}\right| \leq \frac{e^{-R\sin\theta}}{R^4 - R^2 - 1} \leq \frac{1}{R^4 - R^2 - 1}$$

so

$$\left|\int_{\gamma_R} f(z)\,\mathrm{d}z\right| \leq \frac{\pi R}{R^4 - R^2 - 1}$$

which tends to 0 as $R \to \infty$. So

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^4 + x^2 + 1}\,\mathrm{d}x = 2\pi i\left(\mathrm{res}_{a_+}(f) + \mathrm{res}_{b_+}(f)\right)$$

To calculate the residues (noting they are simple poles), we have

$$\begin{aligned}\mathrm{res}_{a_+}(f) &= \lim_{z \to a_+}(z - a_+)\frac{e^{iz}}{(z - a_+)(z - a_-)(z - b_+)(z - b_-)} \\ &= \frac{e^{ia_+}}{(a_+ - a_-)(a_+ - b_+)(a_+ - b_-)}\end{aligned}$$

and similarly for $\mathrm{res}_{b_+}(f)$. Explicitly calculating these gives the result.

Although poles are the most important kind of singularity, we now briefly consider other kinds of singularities.

Definition 3.6

Let $\Omega \subseteq \mathbb{C}$ be open and $z_0 \in \omega$. Let $f : \Omega \setminus \{z_0\} \to \mathbb{C}$ be holomorphic. We say that f has a **removable singularity** at z_0 if there is $\tilde{f} : \Omega \to \mathbb{C}$ holomorphic with $\tilde{f}(z) = f(z)$ for all $z \in \Omega \setminus \{z_0\}$.

Theorem 3.6: Riemann's Theorem on Removable Singularities

Let $\Omega \subseteq \mathbb{C}$ be open and $z_0 \in \Omega$. Then $f : \Omega \setminus \{z_0\} \to \mathbb{C}$ has a removable singularity at z_0 if f is bounded on Ω.

Proof. There exists $r > 0$ such that $\overline{\mathbb{D}_r}(z_0) \subseteq \Omega$. Then for $z \in \mathbb{D}_r(z_0)$ define

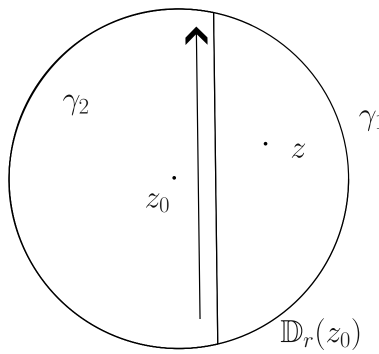
$$\tilde{f}(z) = \frac{1}{2\pi i} \int_{\partial \mathbb{D}_r(z_0)} \frac{f(\omega)}{\omega - z} \, \mathrm{d}\omega$$

Observe that

$$\tilde{f}(z) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{it})}{z_0 + re^{it} - z} rie^{it} \, \mathrm{d}t$$

The denominator is never zero since z is in the interior of $\mathbb{D}_r(z_0)$, so by Theorem 2.20, $\tilde{f}$ is holomorphic. So we need to show that f agrees to $\tilde{f}$ on $\mathbb{D}_r(z_0) \setminus \{z_0\}$.

Draw a line segment through $\mathbb{D}_r(z_0)$ such that z is separated from z_0. Let γ_1, γ_2 be the curves created this way.



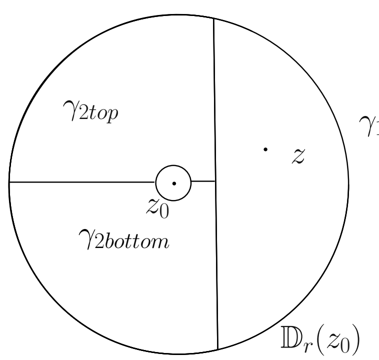
Then

$$\tilde{f}(z) = \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(\omega)}{\omega - z} \, \mathrm{d}\omega + \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(\omega)}{\omega - z} \, \mathrm{d}\omega$$

Now, by the Cauchy Integral Formula

$$\frac{1}{2\pi i} \int_{\gamma_1} \frac{f(\omega)}{\omega - z} \, \mathrm{d}\omega = f(z)$$

Now, divide γ_2 into $\gamma_{2top}, \gamma_{2bottom}$ to avoid z_0.



Then

$$0 = \int_{\gamma_{2top}} \frac{f(\omega)}{\omega - z} \, d\omega + \int_{\gamma_{2bottom}} \frac{f(\omega)}{\omega - z} \, d\omega = \int_{\gamma_2} \frac{f(\omega)}{\omega - z} \, d\omega - \int_{\partial \mathbb{D}_\varepsilon(z_0)} \frac{f(\omega)}{\omega - z} \, d\omega$$

Then we have

$$\left| \tilde{f}(z) - f(z) \right| \leq \left| \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(\omega)}{\omega - z} \, d\omega \right| = \frac{1}{2\pi} \left| \int_{\partial \mathbb{D}_\varepsilon(z_0)} \frac{f(\omega)}{\omega - z} \, d\omega \right| \leq \frac{1}{2\pi} 2\pi\varepsilon \frac{M}{|z - z_0| - \varepsilon}$$

where M is a bound for f. Then as $\varepsilon \to 0$, this quantity goes to zero and thus $f = \tilde{f}$. So f has a removable singularity. □

Note that this statement becomes an if and only if statement if we change the conclusion to "f is bounded on a neighborhood of z_0." In fact, there is a more powerful statement, which is that the following conditions are all equivalent:

- f has a removable singularity at z_0.
- f is bounded on a neighborhood of z_0.
- f is continuously extendable at z_0.

Though it is perhaps surprising that being continuously extendable is equivalent to be holomorphically extendable, this can be proved by using a fourth equivalent characterization, which is the condition that $\lim_{z \to z_0} (z - z_0) f(z) = 0$. As a corollary to this, we can formalize the intuition that a pole is a point where f is becomes unbounded.

Corollary 3.7

Let $f : \Omega \setminus \{z_0\} \to \mathbb{C}$ be holomorphic. Then f has a pole at z_0 if and only if

$$\lim_{z \to z_0} |f(z)| = \infty$$

Proof. ($\Longrightarrow$) If z_0 is a pole, then by the local description of poles,

$$f(z) = \frac{h(z)}{(z - z_0)^n}$$

near z_0, where h is nonzero. But then

$$\lim_{z \to z_0} |f(z)| = \lim_{z \to z_0} \frac{|h(z)|}{|z - z_0|^n} = \lim_{z \to z_0} \frac{|h(z_0)|}{0} = \infty$$

($\Longleftarrow$) If $\lim_{z \to z_0} |f(z)| = \infty$ then

$$\lim_{z \to z_0} \frac{1}{|f(z)|} = 0$$

so $\frac{1}{|f(z)|}$ is bounded near z_0. Then by Riemann's Theorem, there is $g : \mathbb{D}_r(z_0) \to \mathbb{C}$ holomorphic such that $g(z) = \frac{1}{f(z)}$ for all $z \in \mathbb{D}_r(z_0) \setminus \{z_0\}$. In particular, by continuity we must have $g(z_0) = 0$. So f has a pole at z_0. □

Thus we have classified two types of isolated singularities. We give a name to the other kinds now:

Definition 3.7

Let $f : \Omega \setminus \{z_0\} \to \mathbb{C}$ be holomorphic. Then the singularity of f at z_0 is said to be a **essential singularity** if it is neither a pole nor removable.

Example 3.2

Consider $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ given by $f(z) = e^{\frac{1}{z}}$. Then 0 is an essential signularity. To see this, first note that

$$\lim_{t \to 0^+} f(t) = \lim_{t \to 0^+} e^{\frac{1}{t}} = \infty$$

so the singularity is not bounded, and thus not a pole. But similarly,

$$\lim_{t \to 0^-} f(t) = 0$$

so the singularity is not a pole either.

Moreover, if we consider $z = it$,

$$f(it) = \cos\left(\frac{1}{t}\right) - i \sin\left(\frac{1}{t}\right)$$

so we see that the real and imaginary parts have oscillating discontinuities at 0.

The following theorem gives us some insight into the behavior of functions near essential singularities.

Theorem 3.8: Casorati-Weierstrass Theorem

Let $r > 0$ and suppose $f : \mathbb{D}_r(z_0) \setminus \{z_0\} \to \mathbb{C}$ is holomorphic with an essential singularity at z_0. Then the image $f(\mathbb{D}_r(z_0) \setminus \{z_0\})$ is dense in $\mathbb{C}$.

Proof. Suppose for contradiction that there exists $\omega \in \mathbb{C}$ and $\delta > 0$ such that $f(\mathbb{D}_r(z_0) \setminus \{z_0\}) \cap \mathbb{D}_\delta(\omega) = \varnothing$. Then

$$|f(z) - \omega| \geq \delta$$

for all $z \in \mathbb{D}_r(z_0) \setminus \{z_0\}$. Then

$$g(z) = \frac{1}{f(z) - \omega}$$

is well defined and holomorphic on $\mathbb{D}_r(z_0) \setminus \{z_0\}$, and moreover

$$|g(z)| \leq \frac{1}{\delta}$$

By Riemann's Theorem, g can be defined at z_0 so that g is holomorphic on all of $\mathbb{D}_r(z_0)$.

If $g(z_0) \neq 0$, then $\frac{1}{g(z)}$ is holomorphic near z_0. But then since $f(z) = \frac{1}{g(z)} + \omega$, we may define f at z_0 so that f is holomorphic, and therefore z_0 is a removable singularity, contradiction.

Otherwise, if $g(z_0) = 0$, then
$$\lim_{z \to z_0} \frac{1}{|g(z_0)|} = \infty$$
so
$$\lim_{z \to z_0} |f(z)| = \left| \frac{1}{g(z)} + \omega \right| \geq \lim_{z \to z_0} \frac{1}{|g(z)|} - |\omega| = \infty$$
so by Corollary 3.7 f has a pole, contradiction. Thus the image is dense in $\mathbb{C}$. $\square$

We note here that a substantially stronger fact is true: the image of any punctured neighborhood of z_0 is not just dense, but is actually all of $\mathbb{C}$, excluding at most one point. This is known as Picard's Great Theorem but we cannot prove it here.

3.2 Meromorphic Functions

Now that we have classified the types of (isolated) singularities that a function may have, we generalize our study of holomorphic functions to those which have isolated singularities.

Definition 3.8

If $\Omega \subseteq \mathbb{C}$ is open, then f is **meromorphic** on Ω if there is a collection of points $\{z_1, z_2, \ldots\} \subseteq \Omega$ (either infinite or finite) such that:

1. The restriction of f to $\Omega \setminus \{z_1, z_2, \ldots\}$ is holomorphic;
2. f has a pole at each z_i;
3. $\{z_1, z_2, \ldots\}$ does not have a limit point in Ω.

The last condition here ensures that each pole is isolated from the others.

Definition 3.9

We say that f is **meromorphic at** ∞ if there is some radius $R > 0$ such that $f : \mathbb{C} \setminus \mathbb{D}_R \to \mathbb{C}$ is holomorphic and the function $F : \mathbb{D}_{\frac{1}{r}} \setminus \{0\} \to \mathbb{C}$ defined by
$$F(z) = f\left(\frac{1}{z}\right)$$
has a pole at 0. We can similarly define what it means for f to have a **removable singularity at** ∞ or an **essential singularity at** ∞.
f is said to be **meromorphic on the extended complex plane** if it is meromorphic on $\mathbb{C}$ and at ∞

Note that if f is meromorphic on the extended complex plane, then it has only finitely many poles (this follows by a compactness argument since the Riemann sphere is compact, and we assume the poles don't have a limit point). Moreover, the following result allows us to transfer our knowledge of holomorphic functions onto meromorphic functions:

Theorem 3.9

Any function that is meromorphic on the extended complex plane is a rational function

$$f(z) = \frac{P(z)}{Q(z)}$$

where $P, Q \in \mathbb{C}[z]$.

Proof. If f is meromorphic on the extended complex plane then it has finitely many poles $z_1, \ldots, z_n$ (not including the pole at ∞). By Theorem 3.3, for each $k = 1, 2, \ldots, n$ there exists $\delta_k > 0$ such that for $z \in \mathbb{D}_{\delta_k}(z_0)$,

$$f(z) = f_k(z) + g_k(z)$$

where f is the principal part (a polynomial in $\frac{1}{z-z_k}$) and g is the holomorphic part. Since f is meromorphic at ∞, there exists $R > 0$ such that $f : \mathbb{C} \setminus \overline{\mathbb{D}_R} \to \mathbb{C}$ is characterized by

$$f\left(\frac{1}{z}\right) = \tilde{f}_\infty(z) + \tilde{g}_\infty(z)$$

Then define

$$f_\infty(z) = \tilde{f}_\infty\left(\frac{1}{z}\right), g_\infty = \tilde{g}_\infty\left(\frac{1}{z}\right)$$

Then f_∞ is a polynomial in z and g_∞ is holomorphic.

Define

$$H = f - f_1 - f_2 - \ldots - f_n - f_\infty$$

Intuitively, we have removed all of the principal parts of f.

Claim: H is entire.

At any point other than the z_i, f is holomorphic and each of the f_i is holomorphic as well. f_∞ is a polynomial, so H is certainly holomorphic at every point that is not a pole. Consider z_k. Then on $\mathbb{D}_{\delta_k}(z_k)$ we have

$$H(z) = (f - f_k) - f_1 - \ldots - f_{k-1} - f_{k+1} - \ldots - f_n - f_\infty = g_k - f_1 - \ldots - f_{k-1} - f_{k+1} - \ldots - f_n - f_\infty$$

The other f_i are holomorphic at z_k, and so is g_k, so H is holomorphic at z_k.

Now, we claim that H is bounded. To see this, recall that f_k is a polynomial in $\frac{1}{z-z_k}$. So

$$\lim_{z \to \infty} |f_k(z)| = 0$$

Moreover, for $z \in \mathbb{D}_{1/R}$,

$$f\left(\frac{1}{z}\right) = \tilde{f}_\infty(z) + \tilde{g}_\infty(z)$$

Now,

$$H = (f - f_\infty) - (f_1 + \ldots + f_n) = g_\infty - (f_1 + \ldots + f_n)$$

$g_\infty(z) = \tilde{g}_\infty\left(\frac{1}{z}\right)$ is bounded on $\mathbb{D}_{\frac{1}{R}}$, so g_∞ is bounded on $\mathbb{C} \setminus \mathbb{D}_R$, and therefore constant. So

$$f = f_1 + f_2 + \ldots + f_n + f_\infty + c$$

is a rational function. □

Note that the above theorem also holds when f has a removable singularity at ∞. It only fails when f has an essential singularity at ∞ (consider e^z). This is seen most easily with the following alternate proof, that requires the uniqueness of Laurent series:

Alternate Proof. **Claim:** If f is entire and has a nonessential singularity at ∞ then it is a polynomial.

To see this, if f has a removable singularity then it is bounded and therefore constant by Liouville's theorem.

If it has a pole, then let $g(z) = f\left(\frac{1}{z}\right)$. Then we can write

$$g(z) = \sum_{n=-m}^{\infty} b_n z^n$$

for appropriate b_n. It follows that on $\mathbb{C} \setminus \{0\}$,

$$f(z) = \sum_{n=-\infty}^{m} b_n z^n$$

But if f is entire then it can be written as a power series starting at $n = 0$. By uniqueness of Laurent series, these are the same, so

$$f(z) = \sum_{n=0}^{m} b_n z^n \in \mathbb{C}[z]$$

Now, let $z_1, \ldots, z_m$ be the poles (excluding the one at ∞), and suppose $n_1, \ldots, n_m$ are their respective orders. Then

$$f(z) \prod_{k=1}^{m} (z - z_k)^{n_k}$$

has removable singularities at the poles and is holomorphic everywhere else. It has a nonessential singularity at ∞ since f does. So by the claim it is a polynomial and thus f is a rational function. □

Theorem 3.10: Argument Principle

Let $\Omega \subseteq \mathbb{C}$ be simply connected. Suppose $U \subseteq \Omega$ is open and $\overline{U} \subseteq \Omega$, with ∂U a simple closed curve γ. Suppose there exist $p_1, \ldots, p_M \in U$ such that $f : \Omega \setminus \{p_1, \ldots, p_M\} \to \mathbb{C}$ is holomorphic. Assume that the zeroes of f are $z_1, \ldots, z_N \in U$ and f has poles at each p_i. Moreover, suppose that n_k denotes the order of the zero at z_k, and that m_k denotes the order of the pole at p_k. Then

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \, \mathrm{d}z = n_1 + \ldots + n_N - (m_1 + \ldots + m_M)$$

Roughly speaking, the right side is the number of zeroes minus the number of poles, with multiplicity.

Proof. Near each z_k, we know that

$$f(z) = (z - z_k)^{n_k} g(z)$$

for $g(z)$ holomorphic and nonvanishing near z_k. So

$$f'(z) = n_k (z - z_k)^{n_k - 1} g(z) + (z - z_k)^{n_k} g'(z)$$

so

$$\frac{f'(z)}{f(z)} = \frac{n_k}{(z - z_k)} + \frac{g'(z)}{g(z)}$$

since g is nonvanshing, the function g'/g is holomorphic near z_k. So

$$\mathrm{res}_{z_k} \left(\frac{f'}{f} \right) = n_k$$

Similarly, near each p_k we may write

$$f(z) = \frac{h(z)}{(z - z_k)^{m_k}}$$

Again we have

$$f'(z) = -m_k \frac{h(z)}{(z - z_k)^{m_k + 1}} + \frac{h'(z)}{(z - z_k)^{m_k}}$$

so

$$\frac{f'(z)}{f(z)} = \frac{-m_k}{(z - z_k)} + \frac{h'(z)}{h(z)}$$

and

$$\mathrm{res}_{p_k} \left(\frac{f'}{f} \right) = -m_k$$

By the residue formula,

$$\int_\gamma \frac{f'(z)}{f(z)} \, \mathrm{d}z = 2\pi i \left[\sum_{k=1}^{N} \mathrm{res}_{z_k} \left(\frac{f'}{f} \right) + \sum_{k=1}^{M} \mathrm{res}_{p_k} \left(\frac{f'}{f} \right) \right] = 2\pi i \sum n_k - 2\pi i \sum m_k \qquad \square$$

The interpretation of $\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z$ is i times the total change in argument of $f(z)$ over γ. Using what we learn about logarithms later, we pick up winding numbers if we wind around 0, which is why we care about zeroes. It turns out that winding numbers from poles work in the exact opposite direction, so they cancel out a zero.

In particular, note that the right hand side is an integer. This fact allows us to prove equality between two functions in the form of the argument principle simply by estimating their difference and bounding it below $\frac{1}{2}$. For instance, we can do the following:

Theorem 3.11: Rouche's Theorem

Let $\Omega \subseteq \mathbb{C}$ be simply connected, $U \subseteq \Omega$ open with $\overline{U} \subseteq \Omega$, and $\partial U = \gamma$ a simple closed curve. Let $f, g : \Omega \to \mathbb{C}$ be holomorphic such that

$$|f(z)| > |g(z)|$$

for every $z \in \gamma$. Then f and $f+g$ have the same number of zeroes in U (counted with multiplicity).

Intuitively, g is a small perturbation of f (small at the boundary), and we see that f has the same number of zeroes.

Proof. For $t \in [0,1]$, define $f_t(z) = f(z) + tg(z)$. By our assumption, $f_t \neq 0$ on γ. Let n_t be the number of zeroes of f_t in U. f_t is holomorphic so it has no poles. Now, we apply the argument principle, so that

$$n_t = \frac{1}{2\pi i}\int_\gamma \frac{f_t'(z)}{f_t(z)}\,\mathrm{d}z = \frac{1}{2\pi i}\int_\gamma \frac{f'(z) + tg'(z)}{f(z) + tg(z)}\,\mathrm{d}z$$

This is jointly continuous in z, t, so by real variable analysis we know that n_t is continuous in t. But n_t takes integer values so it must be constant. Thus $n_0 = n_1$. □

Example 3.3

Consider the polynomial $z^5 + 3z^3 + 7$. We already know this has 5 roots in $\mathbb{C}$. We show that all of the zeroes lie in $\mathbb{D}_2$.

Let $f(z) = z^5$ and $g(z) = 3z^3 + 7$. If $|z| = 2$ then

$$|g(z)| = \left|3z^3 + 7\right| \leq 3|z|^3 + 7 = 31 < 32 = |z|^5 = |f(z)|$$

So by Rouche's theorem, $f, f+g$ have the same number of zeroes in $\mathbb{D}_2$. f has five zeroes in $\mathbb{D}_2$, so $z^5 + 3z^3 + 7$ has five roots in $\mathbb{D}_2$.

Definition 3.10

Let $\Omega \subseteq \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$. f is an **open mapping** if it is the case that for all $U \subseteq \Omega$ open, $f(U)$ is also open.

In other words, an open mapping is one that preserves open sets in the forward direction. Recall that continuous functions, both real and complex, preserve open sets in the reverse direction. However, in the real case we generally do not have open mappings, even for nice functions (consider $x \mapsto x^2$). This is completely different in the case of complex variables:

Theorem 3.12: Open Mapping Theorem

Let $\Omega \subseteq \mathbb{C}$ be open and connected. Let $f : \Omega \to \mathbb{C}$ be holomorphic on Ω and nonconstant. Then f is an open mapping.

Proof. Let $U \subseteq \Omega$. $f(U)$ is open if and only if for all $\omega_0 \in f(U)$ there exists $\mathbb{D}_\varepsilon(\omega_0) \subseteq f(U)$. This is the case if and only if for any $\omega \in \mathbb{D}_\varepsilon(\omega_0)$ there exists $z \in U$ such that $f(z) = \omega$.

Let $\omega_0 \in f(U)$ and let $z_0 \in U$ such that $f(z_0) = \omega_0$. Let $\overline{\mathbb{D}_\delta}(z_0) \subseteq U$. By uniqueness of analytic continuation, we may pick r small enough so that $f(z) \neq f(z_0) = \omega_0$ for every $z \in \overline{\mathbb{D}_r}(z_0) \setminus \{z_0\}$ (otherwise f would be identically ω_0 on all of Ω). $f(z) - f(z_0)$ is continuous on $\partial\mathbb{D}_r(z_0)$, so it achieves a minimum which cannot be zero. Thus there exists $\varepsilon > 0$ such that

$$|f(z) - f(z_0)| \geq \varepsilon$$

for all $z \in \partial\mathbb{D}_r(z_0)$. For each $\omega \in \mathbb{D}_\varepsilon(\omega_0)$, define $F(z) = f(z) - \omega_0$ and $G(z) = \omega_0 - \omega$. For $z \in \partial\mathbb{D}_r(z_0)$,

$$|G(z)| = |\omega_0 - \omega| < \varepsilon \leq |f(z) - \omega_0|$$

so we can apply Rouche's Theorem to conclude that F and $F + G = f(z) - \omega$ have the same number of zeroes. In particular, $F(z_0) = 0$, so there exists z such that $(F + G)(z) = f(z) - \omega = 0$. Thus $f(z) = \omega$. So $\omega \in f(U)$. Thus $\mathbb{D}_\varepsilon(\omega_0) \subseteq f(U)$. So $f(U)$ is open. $\square$

An easy but important result of of the Open Mapping Theorem is the following:

Theorem 3.13: Maximum Modulus Principle

Let $\Omega \subseteq \mathbb{C}$ be open and connected, and let $f : \Omega \to \mathbb{C}$ be holomorphic and nonconstant. Then f does not attain its maximum on Ω.

Recall that we say f attains a maximum on Ω if there exists $z_0 \in \Omega$ such that $|f(z)| \leq |f(z_0)|$ for all $z \in \Omega$.

Proof. Suppose not. Then there exists $z_0 \in \Omega$ such that $|f(z_0)| \geq |f(z)|$ for all $z \in \Omega$. By the open mapping theorem, $f(U)$ is open, so there exists $r > 0$ such that $\mathbb{D}_r(f(z_0)) \subseteq f(U)$. So there exists $z \in U$ such that $f(z) = \left(1 + \frac{r}{2}\right) f(z_0)$. Then

$$|f(z)| = \left(1 + \frac{r}{2}\right) |f(z_0)| > |f(z_0)|$$

contradiction. So no such z_0 exists. $\square$

Corollary 3.14

Let $\Omega \subseteq \mathbb{C}$ be open and bounded, and suppose $f : \overline{\Omega} \to \mathbb{C}$ is continuous on $\overline{\Omega}$ and holomorphic on Ω. Then

$$\max_{z \in \overline{\Omega}} |f(z)| = \max_{z \in \partial\Omega} |f(z)|$$

That is, f attains its maximum on the boundary of Ω.

Proof. This is obvious if f is constant, so assume f is nonconstant. $\overline{\Omega}$ is compact, so f attains its maximum on $\overline{\Omega}$. But by the Maximum Modulus Principle it does not attain the maximum on Ω, so it must be on $\partial\Omega$. □

3.3 Holomorphic Logarithms

The reader may have noted that although we have made extensive use of the complex exponential to this point, we have not yet used or even defined a complex logarithm. To see why this is the case, consider some complex number of the form

$$z = re^{i\theta}$$

with $r > 0$. If we were to define a logarithm, we would expect that $\log z = \log r + i\theta$. However, θ is only defined up to multiples of 2π. Thus, a naive definition of the logarithm is a multivalued function. However, by carefully restricting the logarithm to subsets of the complex plane, we can obtain sections of the logarithm (known as branches) which are proper, single-valued, holomorphic functions.

Theorem 3.15

Let $\Omega \subseteq \mathbb{C}$ be simply connected. Let $f : \Omega \to \mathbb{C} \setminus \{0\}$ be a nonvanishing holomorphic function. Moreover, suppose there exists $z_0 \in \Omega$ and $c_0 \in \mathbb{C}$ such that

$$f(z_0) = e^{c_0}$$

Then there exists a unique holomorphic function $g : \Omega \to \mathbb{C}$ such that

$$e^{g(z)} = f(z)$$

for all $z \in \Omega$ and

$$g(z_0) = c_0$$

Proof. For uniqueness, suppose that g_1, g_2 both satisfy the conclusion. Then

$$e^{g_1} = e^{g_2} = f$$

Differentiating the first equation, we have

$$g_1' e^{g_1} = g_2' e^{g_2}$$

so

$$g_1' = g_2'$$

Thus $g_1 = g_2 + c$ for some constant c, but $g_1(z_0) = g_2(z_0)$ so $g_1 = g_2$.

Let $z \in \Omega$. Fix some path γ_z joining z_0 to z. Define

$$g(z) = c_0 + \int_{\gamma_z} \frac{f'(\omega)}{f(\omega)} \, \mathrm{d}\omega$$

(Note that this is well defined since $f \neq 0$. The motivation for this choice of function is that it should integrate to the logarithm based on what we know.) By Cauchy's Theorem, this does not depend on our choice of γ_z. Observe that g is a primitve for f'/f by the proof for local existence of primitives on simply connected regions. Thus g is holomorphic and

$$g' = \frac{f'}{f}$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}z} f e^{-g} = f' e^{-g} - f g' e^{-g} = e^{-g}(f' - f g') = e^{-g}\left(f' - f\frac{f'}{f}\right) = 0$$

Thus fe^{-g} is constant and equals 1 at z_0. So

$$e^g = f$$

on all of Ω. □

Definition 3.11

For a function f and the corresponding function g produced as in Theorem 3.15, we write

$$g(z) = \log_{\Omega, z_0, c_0}(f)$$

and say that g is the **logarithm** of f with respect to Ω, z_0, c_0.

This allows us to define a branch of the logarithm without respect to a certain function, by picking a logarithm with respect to the identity $z \mapsto z$.

Corollary 3.16

Let $\Omega \subseteq \mathbb{C}$ be simply connected and suppose $0 \notin \Omega$, $1 \in \Omega$. Let $\mathbb{D}_r(1) \subseteq \Omega$ for some $0 < r < 1$. Then there exists $F : \Omega \to \mathbb{C}$ holomorphic such that

$$e^{F(z)} = z$$

for all $z \in \Omega$ and for $1 - r < t < 1 + r$, $F(t) = \log t$.

The first conclusion says that F is a function which matches our intution for what the logarithm should do, and the second says that it also coincides with the real version of the logarithm.

Proof. We take $F = \log_{\Omega,1,0}(f)$, where $f(z) = z$ is the identity. Note that this coincides with the real logarithm since for any $1 - r < t < 1 + r$, we have

$$F(t) = \int_1^t \frac{1}{s}\,\mathrm{d}s = \log t \qquad \square$$

Definition 3.12

The function F produced in Corollary 3.16 is denoted $F(z) = \log_\Omega z$.

Intuitively, Corollary 3.16 says we can define a logarithm so long as it is not possible to wrap around the origin, so that the argument of a function is actually well-defined.

Definition 3.13

The **principal branch** of the logarithm is the function $\mathrm{Log} = \log_\Omega$ where $\Omega = \mathbb{C} \setminus (-\infty, 0]$ is the slit complex plane.

Example 3.4

Let us verify that Log satisfies the property

$$\mathrm{Log}\, re^{i\theta} = \log r + i\theta$$

Let $z = re^{i\theta} \in \mathbb{C} \setminus (-\infty, 0]$. Then integrating first along the real axis and then along an arc, we have

$$\mathrm{Log}\, z = \int_1^r \frac{\mathrm{d}s}{s} + \int_0^\theta \frac{1}{e^{it}} ie^{it}\,\mathrm{d}t = \ln r + i\theta$$

Thus the principal branch of the logarithm preserves some of the properties of the real logarithm. However, it does not conserve all of these properties. For instance, let $z_1 = z_2 = e^{2\pi i/3}$. Then

$$\mathrm{Log}\, z_1 = \mathrm{Log}\, z_2 = \frac{2\pi i}{3}$$

But $z_1 z_2 = e^{4\pi i/3} = e^{-2\pi i/3}$ so

$$\mathrm{Log}\, z_1 z_2 = -\frac{2\pi i}{3} \neq \mathrm{Log}\, z_1 + \mathrm{Log}\, z_2$$

Definition 3.14

For $\alpha \in \mathbb{C}$, $z \notin (-\infty, 0]$, we define

$$z^\alpha := e^{\alpha\, \mathrm{Log}\, z}$$

Chapter 4

Conformal Mappings

In this chapter, we investigate conformal mappings, which is a class of geometric maps. Our study will lead us to the Riemann mapping theorem.

4.1 Conformal Equivalence

Definition 4.1

Let $U, V \subseteq \mathbb{C}$ be open. $f : U \to V$ is called a **conformal equivalence** of U, V if f is bijective, holomorphic, and $f^{-1} : V \to U$ is holomorphic (meaning f is **biholomorphic**). If such an f exists, then U, V are said to be conformally equivalent, sometimes denoted $U \cong V$.

The importance of conformal mappings is that we may treat conformally equivalent subsets as being *essentially* equivalent in the realm of complex variables. That is, if there is some holomorphic function $h : U \to \mathbb{C}$, then this corresponds uniquely to the holomorphic function $h \circ f^{-1} : V \to \mathbb{C}$. This is similar to to the treatment of isomorphic algebraic structures as equivalent. Although conformal equivalence is slightly weaker than these isomorphisms, it turns out that many questions about open sets may be answered by understanding conformally equivalent sets.

Let us investigate which sets may be conformally equivalent. Suppose U, V are conformally equivalent. Suppose that V is simply connected. Then let γ_0, γ_1 be curves in U which share endpoints. Then $f(\gamma_0), f(\gamma_1)$ are curves in V which share endpoints. So there exists a homotopy $H : [a, b] \times [0, 1] \to V$ between $f(\gamma_0), f(\gamma_1)$. Then $f^{-1} \circ H$ is a homotopy between γ_0, γ_1. So if V is simply connected, so is U, and vice versa. In other words, sets can only be conformally equivalent if both are simply connected or neither is.

Example 4.1

$\mathbb{D}$ and $\mathbb{D} \setminus \{0\}$ are not conformally equivalent.

Example 4.2

Suppose that $U = \mathbb{C}$ and $V = \mathbb{D}$. Suppose there were some holomorphic function $f : \mathbb{C} \to \mathbb{D}$. Then f is entire, and it maps into $\mathbb{D}$ so it is bounded. Thus f is constant and is not a conformal mapping. So $\mathbb{C}$ is not conformally equivalent to $\mathbb{D}$. In particular, $\mathbb{C}$ is not conformally equivalent to any bounded set.

Thus we have seen that the property of being simply connected is preserved under conformal mappings, and that $\mathbb{C}$ is not conformally equivalent to bounded subsets. We could continue trying to find more limitations; however, the Riemann mapping theorem shows that these are in fact the only two important restrictions. We state it for now without proof as motivation for the following work:

Theorem: Riemann Mapping Theorem

Let $U, V \subseteq \mathbb{C}$ be simply connected, with $\varnothing \neq U, V \neq \mathbb{C}$. Then U, V are conformally equivalent.

The following theorem shows that every bijective holomorphic function is biholomorphic, so the condition that f^{-1} is holomorphic may be dropped from the definition of conformal equivalences.

Theorem 4.1

Let $U, V \subseteq \mathbb{C}$ be open. If $f : U \to V$ is holomorphic and injective, then $f' \neq 0$ on U, and $f^{-1} : f(U) \to U$ is holomorphic.

We note that $f(U)$ is open by the Open Mapping theorem.

The intuition is that by power series expansion, vanishing points of the derivative correspond to double roots of f. These must be isolated by analytic continuation, so by perturbing slightly, $f - z_0 - w$ must have distinct roots for w sufficiently small.

Proof. Suppose there exists $z_0 \in U$ such that $f'(z_0) = 0$. f is injective so it is nonconstant. Then we can locally write

$$f(z) = f(z_0) + \sum_{n=m}^{\infty} a_n (z - z_0)^n = f(z_0) + (z - z_0)^m a_m + (z - z_0)^{m+1} h(z)$$

where $m \geq 2$ and h is holomorphic. Suppose δ is small and $w \in \partial \mathbb{D}_{\delta/2}$. Let

$$F(z) = a(z - z_0)^m - w$$
$$G(z) = (z - z_0)^{m+1} h(z)$$

h is bounded on $\mathbb{D}_\delta$, say by M, so (again assuming δ is small) given $z \in \partial \mathbb{D}_\delta$ we have

$$|F(z)| \geq \frac{1}{2}\delta - |a_m|\delta^m = \delta^{m+1}\left(\frac{1}{2\delta^m} - \frac{|a_m|}{\delta}\right)$$
$$|G(z)| \leq \delta^{m+1} M$$

Thus for δ sufficiently small, we will have $|F(z)| > |G(z)|$ on $\partial\mathbb{D}_\delta$. Applying Rouche's Theorem, F has m roots so

$$F + G = f - f(z_0) - w$$

also has m roots in $\mathbb{D}_\delta$. But this means that $f = f(z_0) + w$ for $m \geq 2$ points in U. This is only up to multiplicity. However, $f'(z) = 0$ wherever f has a double root, and f' cannot be zero arbitrary close to z_0 or else f would be constant by analytic continuation. So assuming δ is small enough, $f = f(z_0) + w$ at at least two distinct points, contradicting injectivity. So f' is nonvanishing.

Since $f' \neq 0$, the inverse function theorem tells us f^{-1} is continuously differentiable in the real sense. The Jacobian of f is a composition of rotations and dilations, so its inverse (the Jacobian of f^{-1}) is as well. Thus f^{-1} satisfies Cauchy-Riemann and is holomorphic. In particular,

$$\left(f^{-1}(\omega)\right)' = \frac{1}{f'(f^{-1}(\omega))}$$

□

Example 4.3

Let $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ be the upper half plane. Define $F : \mathbb{H} \to \mathbb{C}$ by

$$F(z) = \frac{i - z}{i + z}$$

The denominator does not vanish since $z \in \mathbb{H}$. The absolute value of the numerator is the distance between z, i, and the absolute value of the numerator is the distance between $z, -i$. So the denominator is larger in absolute value and thus $|F(z)| \leq 1$. Therefore F maps $\mathbb{H}$ into $\mathbb{D}$. F is holomorphic since the denominator does not vanish. Define $G : \mathbb{D} \to \mathbb{H}$ by

$$G(w) = i\frac{1 - w}{1 + w}$$

We have

$$\operatorname{Im}\left(G(u + iv)\right) = \frac{1 - u^2 - v^2}{(1 + u)^2 + v^2} > 0$$

on $\mathbb{D}$ so G indeed maps into $\mathbb{H}$. We then calculate that

$$F(G(w)) = w, \qquad G(F(z)) = z$$

so F is bijective. Therefore $\mathbb{H}, \mathbb{D}$ are conformally equivalent.

The types of conformal mappings encountered in Example 4.3 are common conformal mappings.

Definition 4.2

A function of the form

$$f(z) = \frac{az + b}{cz + d}$$

for $a, b, c, d \in \mathbb{C}$ is called a **fractional linear transformation** or a **Mobius mapping**.

Example 4.4

Let $\mathbb{H}$ be the upper half plane, and define the open strip $V = \{u + iv : u \in \mathbb{R}, v \in (0, \pi)\}$. Let $\Omega := \mathbb{C} \setminus i[0, -\infty)$ be the complex plane slit on the negative imaginary axis. Consider $f(z) = \log_\Omega z$. Ω may be written as the sector $\{re^{i\theta} : r > 0, -\frac{\pi}{2} < \theta < \frac{3\pi}{2}\}$. For $z \in \Omega$,

$$\log z = \log r + i\theta$$

For $z \in \mathbb{H}$, $0 < \theta < \pi$, so the imaginary part is between 0 and π. Thus $f(\mathbb{H}) \subseteq V$. By construction, $\log_\Omega$ has an inverse given by the exponential map, so it is bijective. Thus $\mathbb{H}, V$ are conformally equivalent.

One consequence of the Riemann mapping theorem is that a simply connected, nonempty proper subset of $\mathbb{C}$ is conformally equivalent to $\mathbb{D}$. Thus, we derive results about $\mathbb{D}$, which may be generalized to statements about other sets.

Theorem 4.2: Schwarz Lemma

Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic with $f(0) = 0$. Then:

1. $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$.
2. If $|f(z_0)| = |z_0|$ for some nonzero $z_0 \in \mathbb{D}$, then f is a rotation $x \mapsto e^{i\theta}x$.
3. $|f'(0)| \leq 1$.
4. If $|f'(0)| = 1$, then f is a rotation.

Proof. Expand f as a power series on $\mathbb{D}$ as

$$f(z) = a_1 z + a_2 z^2 + \ldots = z(a_1 + a_2 z + \ldots) = zg(z)$$

where g is holomorphic and converges on $\mathbb{D}$. Fix $0 < r < 1$, and let $z \in \partial\mathbb{D}_r$. Then

$$|g(z)| = \frac{|f(z)|}{|z|} = \frac{|f(z)|}{r} < \frac{1}{r}$$

By the Maximum Modulus principle, $g(z) < \frac{1}{r}$ for all $z \in \mathbb{D}_r$. Therefore

$$|f(z)| \leq \frac{1}{r}|z|$$

for $z \in \mathbb{D}_r$. As $r \to 1$ this becomes $|f(z)| \leq |z|$. If we have equality, then g attains a maximum at z_0. By the strong form of the maximum modulus principle, g attains its maximum on the interior of $\mathbb{D}$ so it is constant, say equal to c. At z_0, $|g(z_0)| = 1$ so $f(z) = cz$ for $|c| = 1$. Thus f is a rotation.

We write the difference quotient:

$$f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z} = \lim_{z\to 0} g(z) \le 1$$

If we have equality, then $|g(0)| = 1$ so g attains its maximum and by the logic above, f is a rotation. $\square$

Definition 4.3

Let $\Omega \subseteq \mathbb{C}$ be open. A conformal equivalence from Ω to itself is called a **conformal automorphism** of Ω. The set of all conformal automorphisms of Ω dentoed $\mathrm{Aut}(\Omega)$.

Note that the automorphism groups (they are indeed groups under composition) for conformally equivalent sets are isomorphic. Indeed, suppose U, V are conformally equivalent adn $F : U \to V$ is a conformal equivalence. Then for $\psi \in \mathrm{Aut}(U)$, $F \circ \psi \circ F^{-1}$ is an automorphism of V, and similarly in the other direction.

As a result of the above fact and the Riemann mapping theorem, it will be of interest to us to be able to compute $\mathrm{Aut}(\mathbb{D})$, which we will do now.

1. $\mathrm{id}_{\mathbb{D}} \in \mathrm{Aut}(\mathbb{D})$, as the identity is in any automorphism group.
2. The rotations $r_\theta : z \mapsto e^{i\theta} z$ for $\theta \in \mathbb{R}$ form a one-parameter family of automorphisms.
3. We showed in homework that for $\alpha \in \mathbb{D}$, the **Blaschke factor**

$$\psi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha} z}$$

   maps $\mathbb{D}$ into itself. It is also holomorphic and is its own inverse, so the Blaschke factors form another family of automorphisms.
4. Of course, we may compose rotations and Blaschke factors at will.

It turns out that this is a complete classification of $\mathrm{Aut}(\mathbb{D})$.

Theorem 4.3

Let $f \in \mathrm{Aut}(\mathbb{D})$. Then there exists some $\theta \in \mathbb{R}, \alpha \in \mathbb{D}$ such that

$$f(z) = e^{i\theta} \frac{\alpha - z}{1 - \overline{\alpha} z}$$

Proof. Let $f : \mathbb{D} \to \mathbb{D}$ be an element of $\mathrm{Aut}(\mathbb{D})$. Then there exists $\alpha \in \mathbb{D}$ such that $f(\alpha) = 0$. Define

$$g = f \circ \psi_\alpha \in \mathrm{Aut}(\mathbb{D})$$

Then $g(0) = 0$. By the Schwarz Lemma,

$$|g(z)| \le |z|$$

for all $z \in \mathbb{D}$. Since $g \in \mathrm{Aut}(\mathbb{D})$, $g^{-1}(0) = 0$ and another application shows that

$$\left|g^{-1}(w)\right| \leq |w|$$

Then for $w = g(z)$,

$$|z| \leq |g(z)| \leq |z|$$

So $|g(z)| = |z|$ for all $z \in \mathbb{D}$. By the Schwarz Lemma, g is a rotation $z \mapsto e^{i\theta}z$. Sp

$$f(z) = f(\psi_\alpha(\psi_\alpha(z))) = g(\psi_\alpha(z)) = e^{i\theta}\frac{\alpha - z}{1 - \overline{\alpha}z}$$ □

Corollary 4.4

The only automorphisms of $\mathbb{D}$ which preserve the origin are the rotations.

Proof. If the origin is preserved then $\alpha = 0$ and the Blaschke factor drops out. □

Remark 4.1

$\mathrm{Aut}(\mathbb{D})$ acts transitively on $\mathbb{D}$, meaning that for any $\alpha, \beta \in \mathbb{D}$, there exists $f \in \mathrm{Aut}(\mathbb{D})$ (for instance $f = \psi_\beta \circ \psi_\alpha$) such that $f(\alpha) = \beta$. In other words, if $O_\alpha = \{f(\alpha) : f \in \mathrm{Aut}(\mathbb{D})\}$ denotes the orbit of α under the action of $\mathrm{Aut}(\mathbb{D})$ on $\mathbb{D}$, then $O_\alpha = \mathbb{D}$ and the group action is transitive. By extension, the Riemann mapping theorem $\mathrm{Aut}(\Omega)$ acts transitively on Ω for any simply connected Ω.

Example 4.5

Consider $\mathrm{Aut}(\mathbb{H})$, where $\mathbb{H}$ is the upper half plane $\mathbb{H} = \{z : \mathrm{Im}(z) > 0\}$. We showed previously that

$$F(z) = \frac{i - z}{i + z}$$

is a conformal equivalence from $\mathbb{H}$ to $\mathbb{D}$, and

$$F^{-1}(w) = i\frac{1 - w}{1 + w}$$

So for any $f \in \mathrm{Aut}(\mathbb{H})$, $F \circ f \circ F^{-1} \in \mathrm{Aut}(\mathbb{D})$, and we may write

$$f(z) = F^{-1}\left(e^{i\theta}\frac{\alpha - F(z)}{1 - \overline{\alpha}F(z)}\right)$$

for appropriate $\theta \in \mathbb{R}, \alpha \in \mathbb{D}$. As a result, every conformal automorphism of $\mathbb{H}$ is a Mobius mapping of the form

$$f(z) = \frac{az + b}{cz + d}$$

where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. If we associate this with a matrix of determinant 1

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \iff f_M(z) = \frac{az + b}{cz + d}$$

then computation shows that

$$f_{M_1} \circ f_{M_2} = f_{M_1 M_2}$$

Thus $\mathrm{Aut}(\mathbb{H})$ is closely related to $\mathrm{SL}_2(\mathbb{R})$, with the exception that $f_M = f_{-M}$, so we need to quotient by this relation, and $\mathrm{Aut}(\mathbb{H}) \cong \mathrm{PSL}_2(\mathbb{R}) := {}^{\mathrm{SL}_2(\mathbb{R})}\!/\!_{\{I,-I\}}$. Since $\mathrm{Aut}(\mathbb{D}) \cong \mathrm{Aut}(\mathbb{H})$, we see that the automorphism group for any nontrivial simply connected subset of $\mathbb{C}$ is isomorphic to $\mathrm{PSL}_2(\mathbb{R})$.

4.2 The Riemann Mapping Theorem

Having now previewed some consequences of the Riemann mapping theorem, we now work to prove the theorem. We first restate the theorem here:

Theorem: Riemann Mapping Theorem

Let $\Omega \subseteq \mathbb{C}$ be simply connected with $\Omega \neq \mathbb{C}, \varnothing$. Let $z_0 \in \Omega$. Then there is a unique conformal equivalence $F : \Omega \to \mathbb{D}$ such that $F(z_0) = 0$, $F'(z_0) \in (0, \infty)$.

We first prove uniqueness before proceeding to the main part of the proof:

Proof of uniqueness. Suppose $F, G : \Omega \to \mathbb{D}$ are conformal equivalences with $F(z_0) = G(z_0) = 0$ and $F'(z_0), G'(z_0) \in (0, \infty)$. Then let $H = F \circ G^{-1} : \mathbb{D} \to \mathbb{D}$. We know

$$H(0) = 0$$

and $H \in \mathrm{Aut}(\mathbb{D})$). By Corollary 4.4, H is a rotation of the form

$$H(z) = e^{i\theta} z$$

Thus

$$e^{i\theta} = H'(z) = \frac{F'(G^{-1}(z))}{G'(G^{-1}(z))}$$

So by assumption,

$$e^{i\theta} = H'(0) = \frac{F'(z_0)}{G'(z_0)} > 0$$

which implies that $e^{i\theta} = 1$. So $F = G$. $\square$

Having proved uniqueness, we now proceed to showing existence. The proof proceeds as follows:

1. Let $\mathcal{F}$ be the family of all injective holomorphic functions $F : \Omega \to \mathbb{D}$ satisfying $F(z_0) = 0$. We will show that $\mathcal{F}$ is nonempty.

2. We will then show that there exists $F \in \mathcal{F}$ for which $|F'(z_0)|$ is maximal.

3. We will show that this choice of F is onto. From there, we can compose F with a rotation r_θ so that $(r_\theta \circ F)'(z_0) \in (0, \infty)$.

We proceed by first assuming Step 1 and proving Step 2.

Definition 4.4

Let $\Omega \subseteq \mathbb{C}$ be open and $\mathcal{F}$ a family of functions $f : \Omega \to \mathbb{C}$. Then $\mathcal{F}$ is called a **normal family** if for any sequence of functions $\{f_n\} \subseteq \mathcal{F}$, there exists a subsequence $\{f_{n_k}\}$ which uniformly converges on compact subsets of Ω. Note that it is not required that the limit function is in $\mathcal{F}$.

Definition 4.5

Let $\Omega \subseteq \mathbb{C}$ be open and $\mathcal{F}$ a family of functions $f : \Omega \to \mathbb{C}$. Then $\mathcal{F}$ is said to be **uniformly bounded** on compact subsets if for every $K \subseteq \Omega$ there exists $M > 0$ such that

$$|f(z)| \leq M$$

for every $f \in \mathcal{F}$, $z \in K$.

In the case of real variables, a uniformly bounded family need not be normal. For instance, consider $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$, where $f_n(x) = \sin nx$. Then $\mathcal{F}$ is uniformly bounded on compact subsets of $(0, 1)$, but there is no uniformly convergent subsequence (or even convergent).

This kind of counterexample does not occur in the case of holomorphic functions. To prove this, we will make use of the Arzela-Ascoli theorem, which is a theorem that is proved in the general setting of compact metric spaces.

Definition 4.6

Let $\Omega \subseteq \mathbb{C}$ be open and $\mathcal{F}$ a family of functions $f : \Omega \to \mathbb{C}$. $\mathcal{F}$ is said to be **equicontinuous** on compact subsets if for every $K \subseteq \Omega$ compact and every $\varepsilon > 0$ there exists $\delta > 0$ such that if $w, z \in K$ and $|w - z| < \delta$ then

$$|f(w) - f(z)| < \varepsilon$$

for all $f \in \mathcal{F}$.

Theorem 4.5: Arzela-Ascoli Theorem

Let $\Omega \subseteq \mathbb{C}$ be open. If $\mathcal{F}$ is a family of functions (not necessarily holomorphic) on Ω which is equicontinuous on compact subsets and uniformly bounded on compact subsets, then $\mathcal{F}$ is a normal family.

Proof. Let $\{f_n\} \subseteq \mathcal{F}$ be a sequence of functions on Ω. Fix some dense subset $\{\omega_j\}_{j=1}^\infty \subseteq \Omega$ (for instance, the set of points with rational coordinates in Ω). Consider the sequence

$$\{f_n(\omega_1)\}_{n=1}^\infty$$

Since $\mathcal{F}$ is uniformly bounded on the compact singleton $\{\omega_1\}$, this sequence is bounded. So there exists a convergent subsequence

$$\{f_{n_k}(\omega_1)\}_{k=1}^{\infty} = \{f_{n,1}(\omega_1)\}_{n=1}^{\infty}$$

Now consider the sequence

$$\{f_{n,1}(\omega_2)\}_{n=1}^{\infty}$$

By the same logic, this sequence is bounded and has a convergent subsequence

$$\{f_{n,2}(\omega_2)\}_{n=1}^{\infty}$$

We continue this recursive refinement process, so that for each $k \in \mathbb{N}$ there exists a subsequence $\{f_{n,k}\}_{n=1}^{\infty}$ such that $\{f_{n,k}(\omega_j)\}_{n=1}^{\infty}$ converges for $j \leq k$.

Now define $g_n : \Omega \to \mathbb{C}$ by $g_n = f_{n,n}$. $\{g_n\}$ is a subsequence of $\{f_n\}$, and for every $j \in \mathbb{N}$, $\{g_n(\omega_j)\}$ converges, say to $g(\omega_j)$. Let $K \subseteq \Omega$ be compact. By compactness there exists $r > 0$ such that

$$K_r = \bigcup_{z \in K} \overline{\mathbb{D}_r}(z) \subseteq \Omega$$

is compact. Let $\varepsilon > 0$ $\{f_n\}$ is equicontinuous on compact subsets so $\{g_n\}$ is as well, so there exists $\delta > 0$ (let us assume $\delta < r$) such that whenever $z, z' \in K_r$ and $|z - z'| < \delta$, then for all $n \in \mathbb{N}$,

$$|g_n(z) - g_n(z')| < \frac{\varepsilon}{3}$$

Observe that $\{\mathbb{D}_\delta(\omega_j)\}_{j=1}^{\infty}$ is an open cover for K since ω_j is dense. Thus we can pick a finite subcover $\{\mathbb{D}_\delta(\omega_j)\}_{j=1}^{J}$. The sequences $\{g_n(\omega_1)\}, \ldots, \{g_n(\omega_J)\}$ all converge, so we may pick N large enough that for any $m, n \geq N$ and $1 \leq j \leq J$,

$$|g_n(\omega_j) - g_m(\omega_j)| < \frac{\varepsilon}{3}$$

Take $z \in K$. Then $z \in \mathbb{D}_\delta(\omega_j)$ for some $1 \leq j \leq J$. For $n, m \geq N$,

$$|g_n(z) - g_m(z)| \leq |g_n(z) - g_n(\omega_j)| + |g_n(\omega_j) - g_m(\omega_j)| + |g_m(\omega_j) - g_m(z)| < \varepsilon$$

Thus $\{g_n\}$ converges pointwise on K to some function g. Then taking the limit as $m \to \infty$,

$$|g_n(z) - g(z)| \leq |g_n(z) - g_m(z)| < \varepsilon$$

for all $z \in K, n \geq N$. Thus $g_n \rightrightarrows g$ on K. □

Theorem 4.6: Montel's Theorem

Let $\Omega \subseteq \mathbb{C}$ be open. Then any family of holomorphic functions on Ω which is uniformly bounded on compact subsets is a normal family.

Proof. Let $\mathcal{F}$ be a family of holomorphic functions on Ω. By Arzela-Ascoli, we just need to show that $\mathcal{F}$ is equicontinuous on compact subsets. Let $K \subseteq \Omega$ be compact. Then $d(K, \mathbb{C} \setminus \Omega) > 0$ so we may pick $\delta > 0$ such that $\overline{\mathbb{D}_{2\delta}}(z) \subseteq \Omega$ for all $z \in K$. It follows that

$$K_{2\delta} := \bigcup_{z \in K} \overline{\mathbb{D}_{2\delta}}(z)$$

is compact. By assumption, there exists $M > 0$ such that

$$|f(z)| \leq M$$

for all $z \in K_{2\delta}, f \in \mathcal{F}$. Let $z, w \in K$ with $|z - w| < \frac{\delta}{2}$, so that $z, w \in \mathbb{D}_\delta(z)$. By the Cauchy integral formula,

$$f(z) = \frac{1}{2\pi i} \int_{\partial \mathbb{D}_\delta(z)} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta$$

and

$$f(w) = \frac{1}{2\pi i} \int_{\partial \mathbb{D}_\delta(z)} \frac{f(\zeta)}{\zeta - w} \, \mathrm{d}\zeta$$

Thus

$$\begin{aligned} |f(z) - f(w)| &\leq \frac{1}{2\pi} \left| \int_{\partial \mathbb{D}_\delta(z)} \frac{f(\zeta)(w - z)}{(\zeta - z)(\zeta - w)} \, \mathrm{d}\zeta \right| \\ &\leq \frac{1}{2\pi} 2\pi\delta \frac{|w - z| M}{\delta \cdot \frac{\delta}{2}} \\ &= \frac{2M}{\delta} |w - z| \end{aligned}$$

Recalling that δ is a constant here, we are done. $\square$

Theorem 4.7

Let $\Omega \subseteq \mathbb{C}$ be open and connected, and $\{f_n : \Omega \to \mathbb{C}\}_{n=1}^\infty$ a sequence of injective holomorphic functions which converges uniformly to some $f : \Omega \to \mathbb{C}$ on compact subsets. Then f is either injective or constant.

Proof. Suppose for contradiction that f is nonconstant and there exists $z_1, z_2 \in \Omega$ distinct such that $f(z_1) = f(z_2)$. Denote $g_n(z) = f_n(z) - f_n(z_1)$ and $g(z) = f(z) - f(z_1)$. By design, $g(z_1) = g(z_2) = 0$. Each f_n is injective, so z_1 is the only zero of g_n.

We know $g_n \rightrightarrows g$ on compact subsets. $g(z_2) = 0$, so by uniqueness of analytic continuation there exists $r > 0$ such that $0 \notin g(\overline{\mathbb{D}_r}(z_2) \setminus \{z_2\})$ (since f is nonconstant, it cannot be 0 everywhere). Let us assume that $r < |z_1 - z_2|/2$. By the argument principle,

$$\frac{1}{2\pi i} \int_{\partial \mathbb{D}_r(z_2)} \frac{g'(\zeta)}{g(\zeta)} \, \mathrm{d}\zeta = k \geq 1 \tag{1}$$

where k is the multiplicity of the zero of g at z_2. However, also by the argument principle,

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}_r(z_2)}\frac{g_n'(\zeta)}{g_n(\zeta)}\,\mathrm{d}\zeta = 0 \tag{2}$$

Thus if we show that the sequence of integrals in (2) converges to the integral in (1), we will arrive at a contradiction. To see this, calculate

$$\left|\frac{g_n'(\zeta)}{g_n(\zeta)} - \frac{g'(\zeta)}{g(\zeta)}\right| = \frac{|g(\zeta)g_n'(\zeta) - g_n(\zeta)g'(\zeta)|}{|g_n(\zeta)||g(\zeta)|}$$

By continuity, $g(\zeta) \geq \varepsilon > 0$ for some $\varepsilon > 0$ and all $\zeta \in \partial\mathbb{D}_r(z_2)$. By uniform convergence, we can also guarantee that $|g_n(\zeta)| \geq \varepsilon/2$ for $n \geq N$. Thus for large n, $|g_n(\zeta)g(\zeta)| \geq \varepsilon^2/2$. Since $g_n \rightrightarrows g$ on compact subsets, $g_n' \rightrightarrows g'$ on compact subsets. Thus

$$|g(\zeta)g_n'(\zeta) - g_n(\zeta)g'(\zeta)| \rightrightarrows 0$$

on compact subsets. Uniform convergence of the integrand implies that the integrals converge, so

$$0 = \frac{1}{2\pi i}\int_{\partial\mathbb{D}_r(z_2)}\frac{g_n'(\zeta)}{g_n(\zeta)}\,\mathrm{d}\zeta \longrightarrow \frac{1}{2\pi i}\int_{\partial\mathbb{D}_r(z_2)}\frac{g'(\zeta)}{g(\zeta)}\,\mathrm{d}\zeta \geq 1$$

contradiction. □

We now have all the tools necessary to prove the Riemann mapping theorem.

Theorem 4.8: Riemann Mapping Theorem

Let $\Omega \subseteq \mathbb{C}$ be simply connected with $\Omega \neq \mathbb{C}, \varnothing$. Let $z_0 \in \Omega$. Then there is a unique conformal equivalence $F : \Omega \to \mathbb{D}$ such that $F(z_0) = 0$, $F'(z_0) \in (0, \infty)$.

Proof. Uniqueness was proved previously.

Step 1: We claim that Ω is conformally equivalent to some open subset of $\mathbb{D}$ containing 0.

Fix $\alpha \in \mathbb{C} \setminus \Omega$. Write $\psi(z) = z - \alpha$. ψ is holomorphic and nonvanishing on Ω. Ω is simply connected, so by our work on logarithms, there exists a holomorphic function $f : \Omega \to \mathbb{C}$ such that $e^f = \psi$.

Claim 4.1

f is injective.

Proof. If $f(z_1) = f(z_2)$ then

$$z_1 - \alpha = e^{f(z_1)} = e^{f(z_2)} = z_2 - \alpha$$

and $z_1 = z_2$. ■

Ω is nonempty so fix some $\omega \in \Omega$.

Claim 4.2

There exists $r > 0$ such that

$$\mathbb{D}_r(f(\omega) + 2\pi i) \cap f(\Omega) = \varnothing$$

Proof. If not, then for each $n \in \mathbb{N}$ there exists $z_n \in f(\Omega)$ such that

$$|f(z_n) - (f(\omega) + 2\pi i)| < \frac{1}{n}$$

so

$$\lim_{n\to\infty} f(z_n) = f(\omega) + 2\pi i$$

so by continuity,

$$\lim_{n\to\infty} z_n - \alpha = \lim_{n\to\infty} e^{f(z_n)} = e^{f(\omega+2\pi i)} = e^{f(\omega)} = \omega - \alpha$$

and

$$f(\omega) + 2\pi i = \lim_{n\to\infty} f(z_n) = f(\omega)$$

contradiction. ∎

Denote

$$F(z) = \frac{1}{f(z) - (f(\omega) + 2\pi i)}$$

F is injective since f is. It is also bounded by $\frac{1}{r}$. Shift F once again and normalize, giving

$$G(z) = \frac{1}{\frac{2}{r} + \frac{1}{2\pi}}\left(F(z) - F(\omega)\right) = \frac{1}{\frac{2}{r} + \frac{1}{2\pi}}\left(F(z) - \frac{i}{2\pi}\right)$$

G is still holomorphic and injective. By construction, $G(\omega) = 0$ so $0 \in G(\Omega)$. Also,

$$|G(z)| < \frac{1}{\frac{2}{r} + \frac{1}{2\pi}}\left(\frac{2}{r} + \frac{1}{2\pi}\right) = 1$$

so $G(\Omega) \subseteq \mathbb{D}$. Thus Ω is conformally equivalent to the open set $G(\Omega) \subseteq \mathbb{D}$ which contains 0.

This proves Step 1. From this point, we will assume that $\Omega \subseteq \mathbb{D}$ and $0 \in \Omega$.

Step 2: Let $\mathcal{F}$ be the set of all functions $f : \Omega \to \mathbb{D}$ which are holomorphic, injective, and satisfy $f(0) = 0$. By Step 1 $\mathcal{F}$ is nonempty. Let

$$s = \sup_{f\in\mathcal{F}} |f'(0)|$$

Claim 4.3

s is finite and is attained.

Proof. First observe that $\mathcal{F}$ is uniformly bounded on $\mathbb{D}$, since every function maps into $\mathbb{D}$ and thus a uniform bound of $M = 1$ suffices. Moreover, $s \geq 1$ since $\mathrm{id} \in \mathcal{F}$.

To show that $s < \infty$, $0 \in \Omega$ so we may pick $\overline{\mathbb{D}_\varepsilon} \subseteq \Omega$. By Cauchy's Inequality, for any $f \in \mathcal{F}$,

$$|f'(0)| \leq \frac{1}{\varepsilon} \sup_{\zeta \in \partial \mathbb{D}_\varepsilon} |f(\zeta)| \leq \frac{1}{\varepsilon}$$

Thus s is finite. Then we may pick $\{f_n\} \subseteq \mathcal{F}$ such that $|f_n'(0)|$ tends to s. Since $\mathcal{F}$ is uniformly bounded, by Montel's Theorem, $\mathcal{F}$ is normal so there exists a subsequence $\{f_{n_k}\}$ which converges to some f uniformly on compact subsets.

We want to show that $f \in \mathcal{F}$. f is holomorphic since it is the uniform limit of holomorphic funcitons. Also,

$$f(0) = \lim_{k \to \infty} f_{n_k}(0) = 0$$

Notice that since this is a uniform limit, we have

$$|f'(0)| = \lim_{k \to \infty} \left|f_{n_k}'(0)\right| = s \neq 0$$

Thus f is nonconstant.

By Theorem 4.7, f is nonconstant and the uniform limit of injective holomorphic functions, so f is injective. Also, f maps Ω into $\mathbb{D}$, since for each $z \in \Omega$,

$$|f(z)| = \lim_{k \to \infty} |f_{n_k}(z)| \leq 1$$

So f certainly maps Ω into $\overline{\mathbb{D}}$. f is nonconstant, so by the maximum modulus principle, f does not attain a maximum on Ω, so this inequality is actually strict and f takes Ω into $\mathbb{D}$.

So $f \in \mathcal{F}$ and therefore s is attained. ∎

Step 3: Consider the function f from above. We want to show that $f(\Omega) = \mathbb{D}$. Suppose for contradiction that there exists $\alpha \in \mathbb{D} \setminus f(\Omega)$. Define the Blaschke factor

$$\psi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha} z}$$

Then let $U = \psi_\alpha \circ f(\Omega)$. Then $\alpha \notin f(\Omega)$ so $0 \notin U$, and U is simply connected since it is conformally equivalent to Ω. Thus we may define a holomorphic logarithm $\log_U$. Let

$$g(\omega) = e^{\frac{1}{2} \log_U(\omega)}$$

Then g is holomorphic and

$$g(\omega)^2 = e^{\log_U(\omega)} = \omega$$

So g takes U into $\mathbb{D}$. Define $F : \Omega \to \mathbb{D}$ by

$$F = \psi_{g(\alpha)} \circ g \circ \psi_\alpha \circ f$$

We have

$$F(0) = \psi_{g(\alpha)} \circ g(\psi_\alpha \circ f(0)) = \psi_{g(\alpha)}(g(\alpha)) = 0$$

Clearly F is holomorphic. If $g(\omega_1) = g(\omega_2)$, then

$$\omega_1 = g(\omega_1)^2 = g(\omega_2)^2 = \omega_2$$

so F is the composition of injective functions and thus injective. Thus $F \in \mathcal{F}$. Denote $h : \mathbb{D} \to \mathbb{D}$ defined by $h(z) = z^2$, so that $h \circ g = \text{id}$. Define $\Phi : \mathbb{D} \to \mathbb{D}$ by

$$\Phi = \psi_\alpha \circ h \circ \psi_{g(\alpha)}$$

Then

$$\begin{aligned}
\Phi \circ F &= \psi_\alpha \circ h \circ \psi_{g(\alpha)} \circ \psi_{g(\alpha)} \circ g \circ \psi_\alpha \circ f \\
&= \psi_\alpha \circ h \circ g \circ \psi_\alpha \circ f \\
&= \psi_\alpha \circ \psi_\alpha \circ f \\
&= f
\end{aligned}$$

We have

$$\Phi(z) = \psi_\alpha([\psi_{g(\alpha)}]^2)$$

By the Schwarz Lemma, $|\Phi'(0)| \leq 1$, and if $|\Phi'(0)| = 1$ then Φ is a rotation. But Φ is not one to one since

$$\Phi\left(\psi_{g(\alpha)}\left(\frac{1}{2}\right)\right) = \psi_\alpha\left(\frac{1}{4}\right) = \psi_\alpha\left(\left(-\frac{1}{2}\right)^2\right) = \Phi\left(\psi_{g(\alpha)}\left(-\frac{1}{2}\right)\right)$$

$\psi_{g(\alpha)}$ is injective so we conclude that $\psi_{g(\alpha)}\left(\frac{1}{2}\right) \neq \psi_{g(\alpha)}\left(-\frac{1}{2}\right)$. Thus Φ is not injective and in particular it is not a rotation. Thus $|\Phi'(0)| < 1$. By the chain rule,

$$|f'(0)| = |\Phi'(F(0))||F'(0)| = |\Phi'(0)||F'(0)| < |F'(0)|$$

But $F \in \mathcal{F}$, so this contradicts the maximality of $|f'(0)|$. Thus our choice of α was invalid, and $f(\Omega) = \mathbb{D}$. Thus f is a conformal mapping between Ω and $\mathbb{D}$. Then if $f'(0) = e^{i\theta}r$, we have

$$\left(e^{-i\theta}f\right)'(0) = r \in (0, \infty)$$

so $e^{-i\theta}f$ is our desired conformal mapping and we are done. $\square$

Chapter 5

Entire Functions

As Liouville's Theorem showed us previously, functions defined everywhere on $\mathbb{C}$ must satisfy relatively strict conditions in order to be holomorphic everywhere. This is in large part due to the local-global nature of holomorphicity. Thus we will investigate the possible behaviors of entire functions; namely, where they can vanish, their behavior at infinity, and the factorization of entire functions in terms of their zeroes.

5.1 Jensen's Formula

We first briefly prove a lemma that makes the following proof a bit more concise:

Lemma 5.1: Mean Value Property

If f is holomorphic on $\mathbb{D}_R(z_0)$ and $0 < r < R$, then

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, \mathrm{d}\theta$$

Proof. By Cauchy's integral formula,

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial \mathbb{D}_r} \frac{f(z)}{z}\, \mathrm{d}z = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} rie^{i\theta}\, \mathrm{d}\theta = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta})\, \mathrm{d}\theta \qquad \square$$

The first important result in our study of entire function is Jensen's formula, which (roughly speaking) says that the average logarithmic value of a function over a circle is related to the number of zeroes in the circle.

Theorem 5.2: Jensen's Formula

Let $R > 0$. Let $\Omega \subseteq \mathbb{C}$ be open with $\overline{\mathbb{D}_R} \subseteq \Omega$. Let $f : \Omega \to \mathbb{C}$ be holomorphic with $f(0) \neq 0$, and suppose f does not vanish on $\partial\mathbb{D}_R$ and the zeroes of f in $\mathbb{D}_R$ (with multiplicity) are $z_1, \ldots, z_N$. Then

$$\frac{1}{2\pi}\int_0^{2\pi} \ln\left|f(Re^{i\theta})\right| \, \mathrm{d}\theta = \ln|f(0)| + \sum_{k=1}^{N} \ln\left(\frac{R}{|z_k|}\right)$$

We remark that by appropriately scaling f, the constant term drops out. In this form, Jensen's formula says that the size of f on $\partial\mathbb{D}_R$ is completely determined by the location of the zeroes in $\mathbb{D}_R$.

Proof. Define $g : \Omega \setminus \{z_1, \ldots, z_N\} \to \mathbb{C}$ by

$$g(z) = \frac{f(z)}{(z - z_1)\cdots(z - z_N)}$$

g has removable singularities at $z_1, \ldots, z_N$, and it is holomorphic on the rest of Ω. Thus we may extend g to all of Ω so that it is still holomorphic. Also, g has no zeroes in $\overline{\mathbb{D}_R}$, and it is continuous, so there exists $R' > R$ such that $\mathbb{D}_{R'} \subseteq \Omega$ and g has no zeroes on $\mathbb{D}_{R'}$. $\mathbb{D}_{R'}$ is simply connected and g does not vanish, so there exists a logarithm $h : \mathbb{D}_{R'} \to \mathbb{C}$ such that

$$e^{h(z)} = g(z)$$

so

$$|g(z)| = \left|e^{h(z)}\right| = e^{\mathrm{Re}(h(z))} \implies \mathrm{Re}(h(z)) = \log|g(z)|$$

Taking real parts of the mean value property,

$$\log|g(z)| = \mathrm{Re}(h(0)) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left(h(Re^{i\theta})\right) \mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} \log\left|g(re^{i\theta})\right| \mathrm{d}\theta \qquad (*)$$

We calculate that

$$|g(0)| = \frac{|f(0)|}{|-z_1|\cdots|-z_N|} = \frac{|f(0)|}{|z_1|\cdots|z_N|}$$

so

$$\log|g(0)| = \log|f(0)| - \sum_{k=1}^{N} \log|z_k|$$

By our integral formula, this gives

$$\begin{aligned}
\log|f(0)| - \sum_{k=1}^{N} \log|z_k| &= \frac{1}{2\pi}\int_0^{2\pi} \log\left(\frac{\left|f(Re^{i\theta})\right|}{|Re^{i\theta} - z_1|\cdots|Re^{i\theta} - z_N|}\right) \mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left(\log\left|f(Re^{i\theta})\right|\right) \mathrm{d}\theta - \sum_{k=1}^{N} \frac{1}{2\pi}\int_0^{2\pi} \log\left|Re^{i\theta} - z_k\right| \mathrm{d}\theta
\end{aligned}$$

So

$$\frac{1}{2\pi}\int_0^{2\pi}\log\left|f(Re^{i\theta})\right|\mathrm{d}\theta = \log|f(0)| + \sum_{k=1}^{N}\frac{1}{2\pi}\int_0^{2\pi}\log\frac{\left|Re^{i\theta}-z_k\right|}{|z_k|}\,\mathrm{d}\theta$$

Then we just need to show that for each k,

$$\frac{1}{2\pi}\int_0^{2\pi}\log\left|Re^{i\theta}-z_k\right|\mathrm{d}\theta = \log R$$

This can be shown by direct computation; we will instead show this in a roundabout way. Define $\psi_k : \mathbb{D} \to \mathbb{C}$ by

$$\psi_k(z) = 1 - \frac{z_k}{R}z$$

ψ_k has no zeroes in $\overline{\mathbb{D}}$. Notice that the formula $(*)$ holds for any function with no zeroes in $\mathbb{D}_R$. So we can apply it to ψ_k on $\mathbb{D}$ to get

$$\begin{aligned} 0 = \log|\psi_k(0)| &= \frac{1}{2\pi}\int_0^{2\pi}\log\left|\psi_k(e^{i\theta})\right|\mathrm{d}\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi}\log\left|1-\frac{z_k}{R}e^{i\theta}\right|\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi}\log\frac{\left|Re^{-i\theta}-z_k\right|}{|Re^{-i\theta}|}\,\mathrm{d}\theta \end{aligned}$$

so

$$\log R = \frac{1}{2\pi}\int_0^{2\pi}\log\left|Re^{-i\theta}-z_k\right|\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi}\log\left|Re^{i\theta}-z_k\right|\mathrm{d}\theta$$

and we are done. □

Definition 5.1

Let $\Omega \subseteq \mathbb{C}$ open with $\overline{\mathbb{D}_R} \subseteq \Omega$, and $f : \Omega \to \mathbb{C}$ holomorphic on Ω with a finite number of zeroes in $\mathbb{D}_R$. For $0 < r < R$ we denote by $n_f(r)$ the number of zeroes of f in $\mathbb{D}_r$.

The below lemma shows that the value of the sum in Jensen's formula is simply a result of the magnitudes of each zero of f.

Lemma

In the setting of Jensen's formula,

$$\sum_{k=1}^{N}\ln\left(\frac{R}{|z_k|}\right) = \int_0^R \frac{n_f(r)}{r}\,\mathrm{d}r$$

Proof. For $k = 1, \ldots, N$, define

$$\eta_k(r) = \begin{cases} 1, & r > |z_k| \\ 0, & r \leq |z_k| \end{cases}$$

Then

$$n_f(r) = \sum_{k=1}^{N} \eta_k(r)$$

So

$$\int_0^R \frac{n_f(r)}{r} = \sum_{k=1}^{N} \int_0^R \frac{\eta_k(r)}{r} \, \mathrm{d}r = \sum_{k=1}^{N} \int_{|z_k|}^R \frac{1}{r} \, \mathrm{d}r = \sum_{k=1}^{N} \ln\left(\frac{R}{|z_k|}\right) \qquad \square$$

We can use this lemma to rewrite Jensen's formula:

Corollary 5.3

In the setting of Jensen's formula,

$$\frac{1}{2\pi} \int_0^{2\pi} \ln\left|f(Re^{i\theta})\right| \mathrm{d}\theta = \ln|f(0)| + \int_0^R \frac{n_f(r)}{r} \, \mathrm{d}r$$

Definition 5.2

Let $f : \mathbb{C} \to \mathbb{C}$ be entire. We say that f has **order of growth** at most $\rho \geq 0$ if there are $A, B > 0$ such that

$$|f(z)| \leq Ae^{B|z|^\rho}$$

for all $z \in \mathbb{C}$. We define the **growth rate** of f to be

$$\rho_f := \inf\{\rho \geq 0 : f \text{ has order of growth at most } \rho\}$$

Example 5.1

Let $f(z) = e^{z^2}$. Then

$$|f(z)| = \left|\sum_{n=0}^{\infty} \frac{z^{2n}}{n!}\right| \leq \sum_{n=0}^{\infty} \frac{|z|^{2n}}{n!} = e^{|z|^2}$$

So f has order of growth at most ρ. By substituting $x \in \mathbb{R}_{>0}$ we see that this is also the infimum, so $\rho_f = 2$.

Example 5.2

Consider

$$f(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{(2n)!}$$

(Intuitively we may think of f as $\cos(z^{\frac{1}{2}})$, although this is not strictly meaningful).

Then

$$|f(z)| \leq \sum_{n=0}^{\infty} \frac{|z|^n}{(2n)!} = \sum_{n=0}^{\infty} \frac{\sqrt{|z|}^{2n}}{(2n)!} \leq \sum_{n=0}^{\infty} \frac{\sqrt{|z|}^n}{n!} = e^{\sqrt{|z|}}$$

So $\rho_f \leq \frac{1}{2}$. We can check that this is sharp by substituting some $z = -a$ for $a > 0$, from which it follows that

$$f(z) = e^{\sqrt{a}}$$

Remark 5.1

If f has order of growth at most ρ, then $z^m f(z)$ has order of growth at most ρ for any $m \in \mathbb{N}$. To see this, we have

$$|z|^m |f(z)| \leq A|z|^m e^{B|z|^\rho} \leq A' e^{(B+1)|z^\rho|}$$

Theorem 5.4

Let $f : \mathbb{C} \to \mathbb{C}$ be entire and not identically zero. Suppose f has order of growth at most ρ. Then:

1. There exists $c > 0$ such that $n_f(r) \leq cr^\rho$ for all $r \geq 1$.
2. If $z_1, z_2, \ldots$ are the zeroes of f in $\mathbb{C} \setminus \{0\}$ with multiplicity. Then for any $s > \rho$,

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|^s} < \infty$$

We remark that since f is not identically zero, there are at most countably many zeroes of f. Moreover, if there are a countably infinite number of zeroes, then we must have $|z_n| \to \infty$ since we cannot have a bounded sequence of zeroes.

Proof. We just need to prove the theorem when $f(0) \neq 0$. If $f(0) = 0$, then by local description of zeroes, $f(z) = z^m g(z)$ with g entire, and $g(0) \neq 0$. By the remark, f, g have the same order of growth. Also, the zeroes of g are precisely $z_1, z_2, \ldots$, and $n_f(r) = n_g(r) + m$, so the conclusion holds for f if it holds for g.

Thus assume $f(0) \neq 0$, and we can normalize it so that $f(0) = 1$. Fix $r \geq 1$. Then by Jensen's formula for $R = 2r$,

$$\frac{1}{2\pi} \int_0^{2\pi} \ln\left|f(2re^{i\theta})\right| \mathrm{d}\theta = \int_0^{2r} \frac{n_f(t)}{t} \,\mathrm{d}t \geq \int_r^{2r} \frac{n_f(t)}{t} \,\mathrm{d}t \geq \int_r^{2r} \frac{n_f(r)}{t} \,\mathrm{d}t = n_f(r) \ln 2$$

Since f has growth rate at most ρ, pick $A, B > 0$ such that $|f(z)| \leq Ae^{B|z|^\rho}$. Then

$$\begin{aligned}
n_f(r) &\leq \frac{1}{2\pi \ln 2} \int_0^{2\pi} \ln\left|f(2re^{i\theta})\right| \mathrm{d}\theta \\
&\leq \frac{1}{2\pi \ln 2} \int_0^{2\pi} \ln\left(Ae^{B(2r)^\rho}\right) \mathrm{d}\theta \\
&\leq \frac{1}{\ln 2}\left(\ln A + B2^\rho r^\rho\right) \leq Cr^\rho
\end{aligned}$$

where

$$C = \frac{\ln A + B2^\rho}{\ln 2}$$

For part 2, we have

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|^s} = \sum_{n:|z_n|<1} \frac{1}{|z_n|^s} + \sum_{n:|z_n|\geq 1} \frac{1}{|z_n|^2}$$

The first sum has at most finitely many terms, so we only care about the second sum. This is

$$\sum_{n:|z_n|\geq 1} \frac{1}{|z_n|^s} = \sum_{j=0}^{\infty} \sum_{n:2^j \leq |z_n| < 2^{j+1}} \frac{1}{|z_n|^s} \leq \sum_{j=0}^{\infty} \sum_{n:2^j \leq |z_n| < 2^{j+1}} \frac{1}{2^{js}} \leq \sum_{j=0}^{\infty} \frac{n_f(2^{j+1})}{2^{js}}$$

By part 1,

$$\sum_{j=0}^{\infty} \frac{n_f(2^{j+1})}{2^{js}} \leq \sum_{j=0}^{\infty} \frac{C(2^{j+1})^\rho}{2^{js}} \leq \sum_{j=0}^{\infty} C2^\rho \frac{1}{(2^{s-\rho})^j}$$

which converges since $s > \rho$ so $2^{s-\rho} > 1$. □

Example 5.3

Let

$$f(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{(2n)!}$$

Suppose $f(z) = 0$. If $z = re^{i\theta}$, then write $w = \sqrt{r}e^{i\frac{\theta}{2}}$. Then $z = w^2$ and $0 = f(w^2) = \cos w$. Thus z is a zero of f only if w is a zero of cos. Thus the zeroes are

$$z_n = \left(\pi\left(n + \frac{1}{2}\right)\right)^2$$

and each has multiplicity 1. Then $|z_n| \sim n^2$ so

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|^s} \sim \sum_{n=1}^{\infty} \frac{1}{n^{2s}}$$

which converges for $s > \frac{1}{2} = \rho_f$.

5.2 The Weierstrass Product Formula

As we remarked above, if f is entire and not identically zero, with $z_1, z_2, \ldots$ the zeroes of f, then $|z_n| \to \infty$. We want to show the converse, which is that if $|z_n| \to \infty$ for some sequence $\{z_n\}$, then there exists an entire function which vanishes at $z_1, z_2, \ldots$ and nowhere else. To do this, we first need to study infinite products.

Proposition 5.5

Let $\Omega \subseteq \mathbb{C}$ be an open set and let $\{F_n : \Omega \to \mathbb{C}\}$ be a sequence of holomorphic functions. Suppose $\{c_n\} \subseteq (0, \infty)$ is a sequence of real numbers such that $\sum_{n=1}^{\infty} c_n$ converges. Suppose that

$$|F_n(z) - 1| \leq c_n$$

for all $z \in \Omega$. Then:

1. The sequence of products

$$\{G_N(z)\} = \left\{\prod_{n=1}^{N} F_n(z)\right\}$$

   converges uniformly on Ω to some F. We define

$$\prod_{n=1}^{\infty} F_n(z) = F(z)$$

2. If $F_n(z) \neq 0$ for every $z \in \Omega$ and $n \in \mathbb{N}$ then F does not vanish on Ω. Also,

$$\frac{F'(z)}{F(z)} = \sum_{n=1}^{\infty} \frac{F_n'(z)}{F_n(z)}$$

   for all z.

Proof. Denote $a_n(z) = F_n(z) - 1$, so that $|a_n(z)| \leq c_n$ for all z. By assumption,

$$\sum_{n=1}^{\infty} c_n < \infty$$

so $\lim c_n = 0$. Pick $N_0 \in \mathbb{N}$ such that for $n \geq N_0$, $c_n < \frac{1}{2}$. Let $N > N_0$. Then

$$\prod_{n=1}^{N} (F_n(z)) = \prod_{n=1}^{N} (1 + a_n(z)) = \left(\prod_{n=1}^{N_0} (1 + a_n(z))\right) \prod_{n=N_0+1}^{N} (1 + a_n(z))$$

The factors in the second product are all nonzero by our choice of N_0. Thus

$$\left(\prod_{n=1}^{N_0} (1 + a_n(z))\right) \prod_{n=N_0+1}^{N} (1 + a_n(z)) = \left(\prod_{n=1}^{N_0} (1 + a_n(z))\right) \exp\left(\sum_{n=N_0+1}^{N} b_n(z)\right)$$

where $b_n(z) = \log(1 + a_n(z))$. Note that log is the principal branch, which is obtained by the fact that $1 + a_n(z)$ does not vanish in the second product. We know that

$$b_n(z) = \log(1 + a_n(z)) = \sum_{k=1}^{\infty} (-1)^k \frac{(a_n(z))^k}{k}$$

so

$$|b_n(z)| \leq \sum_{k=1}^{\infty} \frac{|a_n(z)|^k}{k} \leq \sum_{k=1}^{\infty} \frac{c_n^k}{k} = c_n \sum_{k=1}^{\infty} \frac{c_n^{k-1}}{k} \leq c_n \sum_{k=1}^{\infty} c_n \leq c_n^{k-1} \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{k-1} = 2c_n$$

So

$$b(z) = \sum_{n=N_0+1}^{\infty} b_n(z)$$

converges uniformly and thus b is holomorphic. So

$$F(z) = \lim_{N \to \infty} \prod_{n=1}^{N} F_n(z) = \left(\prod_{n=1}^{N_0} F_n(z)\right) e^{b(z)}$$

From here, part 2 follows since the exponential never vanishes, and thus if none of the F_n vanishes, F cannot vanish either. Lastly, define

$$G_N(z) = \prod_{n=1}^{N} F_n(z)$$

Since $G \rightrightarrows F$ on $\mathbb{C}$, it converges uniformly on compact subsets, and thus

$$G_N'(z) \to F'(z)$$

Also, since F_n does not vanish, then for any $K \subseteq \mathbb{C}$ compact there exists $\delta > 0$ such that $|F(z)| \geq \delta$. Thus

$$\frac{G_N'}{G_N} \rightrightarrows \frac{F'}{F}$$

on compact subsets. But we have

$$\frac{G_N'}{G_N} = \sum_{n=1}^{N} \frac{F_n'}{F_n}$$

whic converges uniformly to the infinite sum. ☐

Theorem 5.6: Weierstrass Product Formula

Let $\{z_n\} \subseteq \mathbb{C}$ be such that $|z_n| \to \infty$. Then there exists an entire function f which vanishes at $\{z_n\}$ and nowhere else, with the multiplicities of the z_k. Moreover, if f_1, f_2 are two functions which satisfy the conclusion, then there exists an entire function g such that

$$f_2 = e^g f_1$$

Proof. For uniqueness, let f_1, f_2 be two functions which satisfy the conclusion. Then $\frac{f_2}{f_1} : \mathbb{C} \setminus \{a_1, a_2, \ldots\} \to \mathbb{C}$. The singularities are all removable since f_1, f_2 have identical order zeroes. So we may extend this holomorphically to all of $\mathbb{C}$. This function also does not vanish. Thus we may take an entire logarithm $g : \mathbb{C} \to \mathbb{C}$ such that

$$\frac{f_2}{f_1} = e^g$$

We want to form an infinite product of the type $\prod_n \left(1 - \frac{z}{a_n}\right)$; however, convergence is not guaranteed in this form. Instead, we adjust this product using the following factors:

Definition 5.3

For each $k \in \mathbb{N}$, we define the **Weierstrass canonical factor** of degree k to be

$$E_k(z) = (1-z)e^{z + \frac{z^2}{2} + \ldots + \frac{z^k}{k}}$$

with

$$E_0(z) = (1-z)$$

Notice that a Weierstrass canonical factor is zero if and only if $z = 1$. Also, for $|z| < 1$, for the principal branch of the logarithm,

$$\operatorname{Log}(1-z) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(-z)^n}{n} = -\sum_{n=1}^{\infty} \frac{z^n}{n}$$

So on $\mathbb{D}$,

$$E_k(z) = e^{\operatorname{Log}(1-z) + \left(z + \ldots + \frac{z^k}{k}\right)} = \exp\left(-\sum_{n=k+1}^{\infty} \frac{z^n}{n}\right)$$

Lemma

There exists $c > 0$ such that for all k and all $|z| \le \frac{1}{2}$, $|1 - E_k(z)| \le c|z|^{k+1}$.

Proof. We choose $c = 2e$. From above, we have

$$E_k(z) = e^w$$

where

$$w = -\sum_{n=k+1}^{\infty} \frac{z^n}{n}$$

Then

$$|w| \le \sum_{n=k+1}^{\infty} \frac{|z|^n}{n} = |z|^{k+1} \sum_{n=k+1}^{\infty} \frac{|z|^{n-k-1}}{n} \le |z|^{k+1} \sum_{n=k+1}^{\infty} \left(\frac{1}{2}\right)^{n-k-1} = 2|z|^{k+1}$$

so

$$|1 - E_k(z)| = |1 - e^w| = \left|1 - \sum_{n=0}^{\infty} \frac{w^n}{n!}\right| = \left|-\sum_{n=1}^{\infty} \frac{w^n}{n!}\right| \le \sum_{n=1}^{\infty} \frac{|w|^n}{n!} \le |w| \sum_{n=1}^{\infty} \frac{|w|^{n-1}}{n!}$$
$$\le 2|z|^{k+1} \sum_{n=1}^{\infty} \frac{1}{n!} \le 2e|z|^{k+1} = c|z|^{k+1} \qquad \blacksquare$$

Suppose we want f to vanish of order ℓ at 0 and $a_1, a_2, \in \mathbb{C} \setminus \{0\}$ are the other zeroes. We will check that

$$f(z) = z^{\ell} \prod_{n=1}^{\infty} E_n\left(\frac{z}{a_n}\right)$$

works. At 0 as well as the various a_n, f vanishes and has the prescribed orders, since $E_n\left(\frac{z}{a_n}\right) = 0$ if and only if $z = a_n$. It also does not vanish anywhere else (assuming we actually have convergence).

To show convergence, pick $R > 0$ and fix $z \in \mathbb{D}_R$. Then

$$\prod_{n=1}^{N} E_n\left(\frac{z}{a_n}\right) = \left(\prod_{n \le N : |a_n| \le 2R} E_n\left(\frac{z}{a_n}\right)\right)\left(\prod_{n \le N : |a_n| > 2R} E_n\left(\frac{z}{a_n}\right)\right)$$

For any R, the left product has a bounded number of factors as $N \to \infty$. Thus it is a finite product and we can ignore it. If $|a_n| \ge 2R$, then $\left|\frac{z}{a_n}\right| \le \frac{1}{2}$. So we can apply our lemma to see that

$$\left|1 - E_n\left(\frac{z}{a_n}\right)\right| \le 2e\left|\frac{z}{a_n}\right|^{n+1} \le \frac{e}{2^n}$$

So from Proposition 5.5,

$$\prod_{n : |a_n| > 2R} E_n\left(\frac{z}{a_n}\right)$$

converges uniformly on $\mathbb{D}$, and therefore the infinite product is nonvanishing and holomorphic on $\mathbb{D}_R$. R was arbitrary, so f is entire and we are done. $\square$

5.3 Hadamard's Factorization Theorem

Having proved the Weierstrass product theorem, we continue to a refinement of the factorization, due to Hadamard, concerning functions with restricted order of growth.

Theorem: Hadamard Factorization Theorem

Let $f : \mathbb{C} \to \mathbb{C}$ be entire with order of growth ρ_0. Let $k = \lfloor \rho_0 \rfloor$. If f has a zero of order ℓ at 0 and its zeroes on $\mathbb{C} \setminus \{0\}$ are $a_1, a_2, \ldots$, then

$$f(z) = e^{p(z)} z^\ell \prod_{n=1}^{\infty} E_k\left(\frac{z}{a_n}\right)$$

where p is a polynomial of degree $\leq k$. Notice that E_k is fixed and does not grow with n.

Before proving the theorem, we first demonstrate an example.

Example 5.4

Let $f(z) = \sin z = \frac{e^{iz} - e^{-iz}}{2}$. We can calculate that it has growth rate $\rho = 1$. Moreover, the zeroes of sin are $\{\pi n : n \in \mathbb{Z}\}$, each of order 1. So by Hadamard's theorem, we know

$$\begin{aligned}
\sin z &= z e^{az+b} \prod_{n \in \mathbb{Z}, n \neq 0}^{\infty} E_1\left(\frac{z}{\pi n}\right) \\
&= z e^{az+b} \prod_{n=1}^{\infty} E_1\left(\frac{z}{\pi n}\right) E_1\left(\frac{z}{-\pi n}\right) \\
&= z e^{az+b} \prod_{n=1}^{\infty} \left(\left(1 - \frac{z}{\pi n}\right) e^{\frac{z}{\pi n}} \left(1 + \frac{z}{\pi n}\right) e^{-\frac{z}{\pi n}}\right) \\
&= z e^{az+b} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\pi^2 n^2}\right)
\end{aligned}$$

To solve for a, b, we know that

$$\frac{\sin z}{z} \xrightarrow{z \to 0} 1$$

But this is equal to

$$e^{az+b} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\pi^2 n^2}\right) \xrightarrow{z \to 0} e^b$$

so $b = 0$. Also, by symmetry we know that for $z \neq 0$,

$$e^{-az} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\pi^2 n^2}\right) = \frac{\sin(-z)}{(-z)} = \frac{\sin z}{z} = e^{az} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\pi^2 n^2}\right)$$

so $e^{-az} = e^{az}$ for all $z \neq 0$. Thus $a = 0$. So

$$\sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\pi^2 n^2}\right)$$

In particular, letting $f = \sin$, $F_0(z) = z$, and $F_n = 1 - \frac{z^2}{\pi^2 n^2}$, we know from Proposition 5.5 that

$$\cot z = \frac{F'(z)}{F(z)} = \frac{F_0'(z)}{z} + \sum_{n=1}^{\infty} \frac{F_n'(z)}{F_n(z)} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{\frac{-2z}{\pi^2 n^2}}{1 - \frac{z^2}{\pi^2 n^2}} = \frac{1}{z} - 2z \sum_{n=1}^{\infty} \frac{1}{\pi^2 n^2 - z^2}$$

In the following, we denote:

$$E(z) = z^\ell \prod_{n=1}^{\infty} E_k\left(\frac{z}{a_n}\right)$$

Lemma 5.7

E is entire.

Proof. For $z \in \mathbb{D}_R$,

$$\prod_{n=1}^{N} E_k\left(\frac{z}{a_n}\right) = \left(\prod_{n \le N : |a_n| \le 2R} E_k\left(\frac{z}{a_n}\right)\right)\left(\prod_{n \le N : |a_n| > 2R} E_k\left(\frac{z}{a_n}\right)\right)$$

As before, the first product has a bounded number of terms as $N \to \infty$, so we ignore it. For the second product, if $|a_n| > 2R$ then $\left|\frac{z}{a_n}\right| < \frac{1}{2}$ so

$$\left|1 - E_k\left(\frac{z}{a_n}\right)\right| \le 2e\frac{|z|^{k+1}}{|a_n|^{k+1}} \le \left(2eR^{k+1}\right)\frac{1}{|a_n|^{k+1}}$$

Since $k = \lfloor \rho \rfloor$, $k + 1 > r$, and by Theorem 5.4,

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^{k+1}} < \infty$$

Thus by Proposition 5.5, E converges and is holomorphic on $\mathbb{D}_R$ arbitrary, so it is entire. $\square$

Observe that E has the same zeroes as f with multiplicity. So as before, $\frac{f}{E}$ has only removable singularities and is nonzero, and thus we can take a logarithm. Thus $\frac{f}{E} = e^g$ for some entire g. Thus the only claim that we need to prove is that g is a polynomial with degree at most k.

Lemma 5.8

Fix some $\rho_0 < s < k + 1$. Then there are radii $0 < r_1 < r_2 < \dots$ with $\lim_{m \to \infty} r_m =$

∞ and a constant $C > 0$ such that

$$\left| \prod_{n=1}^{\infty} E_k\left(\frac{z}{a_n}\right) \right| \geq e^{-C|z|^s}$$

for all m and $z \in \partial\mathbb{D}_{r_m}$.

Intuitively, the above says that we can find certain sets where we can bound the product from below. We cannot hope to achieve a global bound since the product is zero at a_k, so we instead pick radii in such a way that z is bounded away from the zeroes.

Proof. We make three key claims in the proof.

Claim 1: If $|z| \leq \frac{1}{2}$ then

$$|E_k(z)| \geq e^{-2|z|^{k+1}}$$

To see this, we are in the radius of convergence of the principal branch of the logarithm:

$$\mathrm{Log}(1-z) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(-z)^n}{n} = -\sum_{n=1}^{\infty} \frac{z^n}{n}$$

so

$$E_k(z) = (1-z)e^{z+\frac{z^2}{2}+\ldots+\frac{z^k}{k}} = e^w$$

where

$$w = -\sum_{n=k+1}^{\infty} \frac{z^n}{n}$$

Now,

$$|w| \leq \sum_{n=k+1}^{\infty} \frac{|z|^n}{n} \leq |z|^{k+1} \sum_{n=k+1}^{\infty} |z|^{n-k-1} \leq |z|^{k+1} \sum_{n=k+1}^{\infty} \left(\frac{1}{2}\right)^{n-k-1} = 2|z|^{k+1}$$

so

$$E_k(z) = |e^w| = e^{\mathrm{Re}(w)} \geq e^{-|w|} \geq e^{-2|z|^{k+1}}$$

Claim 2: If $|z| \geq \frac{1}{2}$ then

$$|E_k(z)| \geq |1-z|e^{-2^k|z|^k}$$

To prove this claim, we have

$$|E_k(z)| = |1-z|\left|e^{z+\frac{z^2}{2}+\ldots+\frac{z^k}{k}}\right| \geq |1-z|e^{-\left|z+\frac{z^2}{2}+\ldots+\frac{z^k}{k}\right|}$$

$$\geq |1-z|e^{-\left(|z|+\frac{|z|^2}{2}+\ldots+\frac{|z|^k}{k}\right)} \geq |1-z|e^{-(|z|+|z|^2+\ldots+|z|^k)} = |1-z|e^{-|z|^k\left(\frac{1}{|z|^{k-1}}+\frac{1}{|z|^{k-2}}+\ldots+1\right)}$$

$$\geq |1-z|e^{-|z|^k\left(2^{k-1}+2^{k-2}+\ldots+1\right)} \geq |1-z|e^{-2^k|z|^k}$$

Claim 3: There exists $C > 0$ such that if $|z| \geq 1$ and

$$z \in \mathbb{C} \setminus \left(\bigcup_{n=1}^{\infty} \mathbb{D}_{\frac{1}{|a_n|^{k+1}}}(a_n) \right)$$

then

$$\left| \prod_{n=1}^{\infty} E_k\left(\frac{z}{a_n}\right) \right| \geq e^{-C|z|^s}$$

In other words, we bound the product away from some disks around the zeroes. Also, the centers of the disks tend to infinity and their radii tend to zero.

Take some such z. Then

$$\prod_{n=1}^{\infty} \left| E_k\left(\frac{z}{a_n}\right) \right| = \prod_{n:|a_n|<2|z|} \left| E_k\left(\frac{z}{a_n}\right) \right| \prod_{n:|a_n|\geq 2|z|} \left| E_k\left(\frac{z}{a_n}\right) \right|$$

For the second factor, $\left|\frac{z}{a_n}\right| < \frac{1}{2}$ so by Claim 1,

$$\prod_{n:|a_n|\geq 2|z|} \left| E_k\left(\frac{z}{a_n}\right) \right| \geq \prod_{n:|a_n|\geq 2|z|} e^{-2\frac{|z|^{k+1}}{|a_n|^{k+1}}} = \exp\left(-2|z|^{k+1} \sum_{n:|a_n|\geq 2|z|} \frac{1}{|a_n|^{k+1}} \right)$$
$$= \exp\left(-2|z|^{k+1} \sum_{n:|a_n|\geq 2|z|} \frac{1}{|a_n|^s |a_n|^{k+1-s}} \right) \geq \exp\left(-2|z|^{k+1} \sum_{n:|a_n|\geq 2|z|} \frac{1}{|a_n|^s (2|z|)^{k+1-s}} \right)$$
$$= \exp\left(-2^{s-k} \sum_{|a_n|\geq 2|z|} \frac{1}{|a_n|^s} |z|^s \right) \geq \exp\left(-\left(2^{s-k} \sum_{n=1}^{\infty} \frac{1}{|a_n|^s} \right) |z|^s \right)$$

We know from Theorem 5.4 that $\sum \frac{1}{|a_n|^s}$ converges. So we let

$$0 < C_1 = 2^{s-k} \sum_{n=1}^{\infty} \frac{1}{|a_n|^s} < \infty$$

so

$$\prod_{n:|a_n|\geq 2|z|} \left| E_k\left(\frac{z}{a_n}\right) \right| \geq e^{-C_1|z|^s}$$

For the first factor, since $a_n \to \infty$, the product and sum are always finite. We apply Claim 2 to get

$$\prod_{n:|a_n|<2|z|} \left| E_k\left(\frac{z}{a_n}\right) \right| \geq \prod_{n:|a_n|<2|z|} \left| 1 - \frac{z}{a_n} \right| e^{-2^k \frac{|z|^k}{|a_n|^k}}$$
$$= \left(\prod_{n:|a_n|<2|z|} \left| 1 - \frac{z}{a_n} \right| \right) \exp\left(-2^k |z|^k \sum_{n:|a_n|<2|z|} \frac{1}{|a_n|^k} \right)$$

We will proceed by bounding these two factors separately. For the second,

$$\exp\left(-2^k|z|^k \sum_{n:|a_n|<2|z|} \frac{1}{|a_n|^k}\right) = \exp\left(-2^k|z|^k \sum_{n:|a_n|<2|z|} \frac{|a_n|^{s-k}}{|a_n|^s}\right)$$
$$\geq \exp\left(-2^k|z|^k \sum_{n:|a_n|<2|z|} \frac{(2|z|)^{s-k}}{|a_n|^s}\right) \geq \exp\left(-2^s|z|^s \sum_{n=1}^{\infty} \frac{1}{|a_n|^s}\right) = e^{-C_2|z|^s}$$

where

$$C_2 = 2^s \sum_{n=1}^{\infty} \frac{1}{|a_n|^s}$$

For the final factor, we use the assumption on z to get

$$\prod_{n:|a_n|<2|z|} \left|1 - \frac{z}{a_n}\right| = \prod_{n:|a_n|<2|z|} \left|\frac{z-a_n}{a_n}\right| \geq \prod_{n:|a_n|<2|z|} \frac{1}{|a_n|^{k+2}}$$
$$\geq \prod_{n:|a_n|<2|z|} \frac{1}{(2|z|)^{k+2}} = \left(\frac{1}{(2|z|)^{k+2}}\right)^{n_f(2|z|)-\ell}$$

where $n_f(r)$ is the number of zeroes of f in $\mathbb{D}_r$ as we previously defined, and ℓ is the order of the zero at 0. Fix some ρ such that $\rho_0 < \rho < s$. Then Theorem 5.4 showed that there exists $c > 0$ such that $n_f(r) \leq cr^\rho$ when $r \geq 1$. So since $|z| \geq \frac{1}{2}$,

$$\left(\frac{1}{(2|z|)^{k+2}}\right)^{n_f(2|z|)-\ell} \geq \frac{1}{2|z|}^{(k+2)c(2|z|)^\rho} = e^{-c(k+2)2^\rho|z|^\rho \ln(2|z|)} \geq e^{-C_3|z|^s}$$

where C_3 is some constant that we get since $s > \rho$ so $|z|^\rho \ln(2|z|)$ is $O(|z|^s)$. Let $C = C_1 + C_2 + C_3$ and we have proved Claim 3.

At this point we have bounded our function away from some disks. Thus all that remains is to show that there exist radii tending to infinity that do not intersect the disks. By Theorem 5.4,

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^{k+1}} < \infty$$

so we may choose $N \in \mathbb{N}$ such that

$$\sum_{n=N}^{\infty} \frac{1}{|a_n|^{k+1}} < \frac{1}{2}$$

Fix

$$m > 2\max\left\{1, |a_1|, \ldots, |a_N|, \frac{1}{|a_1|^{k+1}}, \ldots, \frac{1}{|a_N|^{k+1}}\right\}$$

We want to show that there is $m \leq r \leq m+1$ such that

$$\partial\mathbb{D}_r \cap \left(\bigcup_{n=1}^{\infty} \mathbb{D}_{\frac{1}{|a_n|^{k+1}}}(a_n)\right) = \varnothing$$

This will immediately imply the theorem as we can continue picking r as m increases.

We show this by contradiction. If no such r exists, then for all $r \in [m, m+1]$, then there is $z_r \in \partial\mathbb{D}_r$ such that

$$z_r \in \mathbb{D}_{\frac{1}{|a_n|^{k+1}}}(a_n)$$

for some $n \in \mathbb{N}$. This means that

$$|z_r - a_n| < \frac{1}{|a_n|^{k+1}} \tag{$*$}$$

Observe that we must have $n \geq N$, because otherwise n appears in the list of numbers that m is greater than. Rearranging $(*)$, we get

$$\frac{m}{2} > |a_n| \geq |z| - \frac{1}{|a_n|^{k+1}} = r - \frac{1}{|a|^{k+1}} > r - \frac{m}{2}$$

which would imply $m > r$. So $n \geq N$. We also know from $(*)$ that

$$|a_n| - \frac{1}{|a_n|^{k+1}} < |z| = r < |a_n| + \frac{1}{|a_n|^{k+1}}$$

Since this works for all r, we get

$$[m, m+1] \subseteq \bigcup_{n=N+1}^{\infty} \left(|a_n| - \frac{1}{|a_n|^{k+1}}, |a_n| + \frac{1}{|a_n|^{k+1}} \right)$$

We are essentially done by measure theory as the length on the right is strictly less than 1 by our assumption on N. To prove this without measure theory, we proceed as follows. Since $[m.m+1]$ is compact, a finite union suffices as a cover, so we get

$$[m, m+1] \subseteq \bigcup_{n=N=1}^{M} \left(|a_n| - \frac{1}{|a_n|^{k+1}}, |a_n| + \frac{1}{|a_n|^{k+1}} \right)$$

Using the indicator function 1_A, we could rewrite this as

$$1_{[m,m+1]} \leq \sum_{n=N+1}^{M} 1_{\left(|a_n| - \frac{1}{|a_n|^{k+1}}, |a_n| + \frac{1}{|a_n|^{k+1}}\right)}$$

So

$$1 = \int_0^\infty 1_{[m,m+1]}(t)\,\mathrm{d}t \leq \int_0^\infty \sum_{n=N+1}^{M} 1_{\left(|a_n| - \frac{1}{|a_n|^{k+1}}, |a_n| + \frac{1}{|a_n|^{k+1}}\right)}(t)\,\mathrm{d}t \leq \sum_{n=N=1}^{\infty} \frac{2}{|a_n|^{k+1}} < 1$$

So we get a contradiction and such an r exists. $\square$

Now, we continue with some basic estimates. We have

$$\left|\frac{f(z)}{E(z)}\right| = \left|e^{g(z)}\right| = e^{\operatorname{Re} g(z)}$$

Recall also that by our definition of growth rate, for any $\rho_0 < \rho < s < k+1$ we have

$$|f(z)| \leq Ae^{B|z|^\rho}$$

for constants $A, B > 0$. Thus when z lies in one of the radii, we have

$$\mathrm{Re}(g(z)) = \ln\left(\left|\frac{f(z)}{E(z)}\right|\right) \leq \ln\left(\frac{Ae^{B|z|^\rho}}{e^{-c|z|^s}}\right) = \ln\left(Ae^{B|z|^\rho + c|z|^s}\right) \leq C'|z|^s$$

for some $C' > 0$.

Lemma 5.9

If g is entire and

$$\mathrm{Re}(g(z)) \leq Cr_m^s$$

for $C > 0$, $|z| = r_m$ and $r_m \to \infty$, then g is a polynomial with degree at most s.

Proof. Write g as a power series about the origin:

$$g(z) = \sum_{n=0}^{\infty} b_n z^n$$

Fix r to be one of the r_m. Notice that

$$\frac{1}{2\pi}\int_0^{2\pi} g(re^{i\theta})e^{-in\theta}\,\mathrm{d}\theta = \begin{cases} b_n r^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

This is true by the Cauchy integral formula, since if $n \geq 0$ then

$$b_n = \frac{1}{2\pi i}\int_{\partial\mathbb{D}_r} \frac{g(z)}{z^{n+1}}\,\mathrm{d}z = \frac{1}{2\pi i}\int_0^{2\pi} \frac{g(re^{i\theta})}{r^{n+1}e^{i(n+1)\theta}} rie^{i\theta}\,\mathrm{d}\theta = \frac{1}{r^n 2\pi}\int_0^{2\pi} g(re^{i\theta})e^{-in\theta}\,\mathrm{d}\theta$$

If $n < 0$ then the integral is zero since it is a closed curve. Also, for $n > 0$,

$$\frac{1}{2\pi}\int_0^{2\pi} \overline{g(re^{i\theta})}e^{-in\theta} = \overline{\frac{1}{2\pi}\int_0^{2\pi} g(re^{i\theta})e^{in\theta}\,\mathrm{d}\theta} = 0$$

Adding these two results, we have

$$b_n r^n = \frac{1}{\pi}\int_0^{2\pi} \mathrm{Re}(g(re^{i\theta}))e^{-in\theta}\,\mathrm{d}\theta$$

we can add or subtract a constant in the integrand since this amounts to adding a function with a primitive, so

$$b_n = \frac{1}{\pi r^n}\int_0^{2\pi} \left(\mathrm{Re}(g(re^{i\theta}) - Cr^s)e^{-in\theta}\right)\mathrm{d}\theta$$

We can now directly integrate, using the mean value property for one of the terms to conclude that

$$\begin{aligned} |b_n| &\leq \frac{1}{\pi r^n} \int_0^{2\pi} \left| \operatorname{Re}(g(re^{i\theta}) - Cr_m^s) \right| \mathrm{d}\theta \\ &= \frac{1}{\pi r^n} \int_0^{2\pi} \left(Cr^s - \operatorname{Re}(g(re^{i\theta})) \right) \mathrm{d}\theta \\ &= \frac{2C}{r^{n-s}} - \frac{2\operatorname{Re}(b_0)}{r_m^n} \end{aligned}$$

Then if $n > s$, we have

$$|b_n| \leq \frac{2C}{r^{n-s}} - \frac{2\operatorname{Re}(b_0)}{r_m^n} \xrightarrow{m\to\infty} 0$$

This concludes the proof. $\square$

Chapter 6

Special Functions

Having developed the general theory of complex functions, we will now apply our results to specific functions which arise in both complex analysis and other fields.

6.1 The Gamma Function

The Gamma function serves as an analytic generalization of the factorial function. Although it is not the only analytic function which agrees with the factorial function (analytic continuation does not apply as the factorial is defined on $\mathbb{N}$, which has no limit point), it nevertheless posses many important properties which cause it to appear in many applications.

Definition 6.1

Denote the **gamma function** $\Gamma : \{s \in \mathbb{C} : \operatorname{Re}(s) > 0\} \to \mathbb{C}$ by

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} \, \mathrm{d}t$$

In particular, notice that when $\operatorname{Re}(s) = \sigma > 0$,

$$\left| e^{-t} t^{s-1} \right| = e^{-t} t^{\sigma - 1}$$

so the integral converges absolutely.

Proposition 6.1

Γ is holomorphic on $\{s \in \mathbb{C} : \operatorname{Re}(s) > 0\}$.

Proof. For $n \in \mathbb{N}$, define

$$F_n(s) = \int_{\frac{1}{n}}^n e^{-t} t^{s-1} \, \mathrm{d}t$$

The integrand is continuous and holomorphic in s, so F_n is holomorphic (in fact, it is entire). It is enough, then, to prove that $F_n \rightrightarrows \Gamma$ on compact subsets of the right half plane.

Every compact subset K of the right half plane is contained in a strip $\mathrm{Re}(K) \subseteq [\delta, M]$ for some $0 < \delta < M < \infty$. So we just need to show uniform convergence on every strip.

For $s \in K$, denote $\sigma = \mathrm{Re}(s)$. Then

$$\begin{aligned}
|\Gamma(s) - F_n(s)| &= \left| \int_0^{\frac{1}{n}} t^{s-1}e^{-t}\,\mathrm{d}t + \int_n^{\infty} t^{s-1}e^{-t}\,\mathrm{d}t \right| \\
&\leq \int_0^{\frac{1}{n}} \left|t^{s-1}\right| \underbrace{\left|e^{-t}\right|}_{\leq 1} \mathrm{d}t + \int_n^{\infty} \left|t^{s-1}\right|\left|e^{-t}\right| \mathrm{d}t \\
&\leq \int_0^{\frac{1}{n}} t^{\sigma-1}\,\mathrm{d}t + \int_n^{\infty} t^{\sigma-1}e^{-t}\,\mathrm{d}t \\
&\leq \int_0^{\frac{1}{n}} t^{\delta-1}\,\mathrm{d}t + \int_n^{\infty} t^{M-1}e^{-t}\,\mathrm{d}t
\end{aligned}$$

The first integral evaluates directly to $\frac{1}{\delta n^{\delta}}$. For the second integral we have

$$\int_n^{\infty} t^{M-1}e^{-t}\,\mathrm{d}t = \int_n^{\infty} \left(t^{M-1}e^{-\frac{t}{2}}\right) e^{-\frac{t}{2}}\,\mathrm{d}t \leq \sup\{t^{M-1}e^{-\frac{t}{2}} : t \geq 1\} \int_n^{\infty} e^{-\frac{t}{2}}\,\mathrm{d}t$$

This supremum is some constant $C(M)$, and so this evaluates to

$$C(M)2e^{-\frac{n}{2}}$$

So

$$|\Gamma(s) - F_n(s)| \leq \frac{1}{\delta n^{\delta}} + \frac{2C(M)}{e^{\frac{n}{2}}} \rightrightarrows 0$$ □

Remark 6.1

The above proof shows that

$$\int_1^{\infty} t^{s-1}e^{-t}\,\mathrm{d}t$$

is an entire function in s, since the only part where $\mathrm{Re}(s) > 0$ was required was in the $\int_0^{\frac{1}{n}}$ term, which does not exist when integrating from 1 (or any $\delta > 0$).

Now that we have shown that Γ is holomorphic on the right half plane, it is of interest to us to extend Γ to as much of the entire plane as possible. This becomes a theme throughout all of our study of these special functions, since analytic continuation guarantees we do not lose any generality when performing this extension.

Lemma 6.2

If $\mathrm{Re}(s) > 0$, then $\Gamma(s+1) = s\Gamma(s)$.

Proof. Notice that by analytic continuation, it suffices to check this for $s \in (0, \infty)$. (This is because $\Gamma(s+1) - s\Gamma(s)$ will be zero on the real axis, which implies it is zero everywhere).

Since $t^s e^{-t} \to 0$ as $t \to 0$ and $t \to \infty$,

$$0 = \int_0^\infty \left(t^s e^{-t}\right)' \,\mathrm{d}t = \int_0^\infty (st^{s-1}e^{-t} - t^s e^{-t}) \,\mathrm{d}t = s\Gamma(s) - \Gamma(s+1) \qquad \square$$

In particular, since $\Gamma(1) = 1$, we see that $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$.

Theorem 6.3

There exists a unique holomorphic function, which we still denote by Γ, on $\mathbb{C} \setminus \{0, -1, -2, \ldots\}$ that coincides with our original definition of Γ on $\{\mathrm{Re}(s) > 0\}$. Moreover, this function has simple poles at $\{0, -1, -2, \ldots\}$ and its residues are given by

$$\mathrm{res}_{-n}(\Gamma) = \frac{(-1)^n}{n!}$$

Proof. For each $m \in \mathbb{N}$ we define $F_m : \{s \in \mathbb{C} : \mathrm{Re}(s) > -m\} \to \mathbb{C}$ by

$$F_m(s) = \frac{\Gamma(s+m)}{(s+m-1)(s+m-2)\cdots s}$$

We want to show that F_m is holomorphic on $\{\mathrm{Re}(s) > -m\} \setminus \{0, \ldots, -m+1\}$. To do this, notice that we have simple poles at $s = -n$ for $n \in \{0, \ldots, m-1\}$, and

$$\mathrm{res}_{-n}(F_m) = \lim_{s \to -n} (s+n) F_m(s) = \lim_{s \to -n} (s+n) \frac{\Gamma(s+m)}{(s+m-1)\cdots s}$$
$$= \frac{\Gamma(m-n)}{(m-n-1)\cdots 1 \cdot (-1) \cdot (-2) \cdots (-n)} = \frac{(m-n-1)!}{(m-n-1)! \cdot (-1)^n \cdot n!} = \frac{(-1)^n}{n!}$$

Observe also that if $\mathrm{Re}(s) > -m$, then $F_{m+1}(s) = F_m(s)$. This is because

$$F_{m+1}(s) = \frac{\Gamma(s+m+1)}{(s+m)(s+m-1)\cdots s} = \frac{(s+m)\Gamma(s+m)}{(s+m)\cdot s} = \frac{\Gamma(s+m)}{(s+m-1)\cdots s} = F_m(s)$$

This for $s \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$, we simply define $\Gamma(s) = F_m(s)$ for any $m \in \mathbb{N}$ such that $\mathrm{Re}(s) > -m$, which will thus be independent of our choice of m. $\square$

We also present a nonalgebraic proof which will be instructive in our proofs of extending other functions.

Alternate Proof. When $\mathrm{Re}(s) > 0$, we know that

$$\Gamma(s) = \int_0^1 t^{s-1} e^{-t} \,\mathrm{d}t + \int_1^\infty t^{s-1} e^{-t} \,\mathrm{d}t$$

We remarked earlier that the second integral is entire. So we just need to show that the first integral may be extended to the whole plane, minus the poles. Still assuming that $\mathrm{Re}(s) > 0$, we have

$$\int_0^1 t^{s-1}e^{-t}\,\mathrm{d}t = \int_0^1 t^{s-1}\left(\sum_{n=0}^{\infty}\frac{(-1)^n t^n}{n!}\right)\mathrm{d}t$$

This is an integral on a bounded interval of an absolutely convergent series, so we may perform the interchange:

$$\int_0^1 t^{s-1}\left(\sum_{n=0}^{\infty}\frac{(-1)^n t^n}{n!}\right)\mathrm{d}t = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^1 t^{n+s-1}\,\mathrm{d}t = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!(n+s)}$$

So we have shown that for $\mathrm{Re}(s) > 0$, we may alternatively write

$$\Gamma(s) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!(n+s)} + \int_1^{\infty} t^{s-1}e^{-t}\,\mathrm{d}t$$

From here, the extension is clear: we simply allow s to be any complex number, save the nonpositive integers, which will cause one of the denominators in the series to vanish. This also makes the residue calculation trivial. To check that this extension converges, fix $R > 0$, $N > 2R$, and take $s \in \mathbb{D}_R \setminus \{0, -1, -2, \ldots\}$. Then

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(n+s)} = \sum_{n=0}^{N}\frac{(-1)^n}{n!(n+s)} + \sum_{n=N+1}^{\infty}\frac{(-1)^n}{n!(n+s)}$$

We just need to check the infinite sum. We have

$$\sum_{n=N+1}^{\infty}\frac{(-1)^n}{n!(n+s)} = \lim_{M\to\infty}\sum_{n=N+1}^{M}\frac{(-1)^n}{n!(n+s)}$$

The finite sum is holomorphic on $\mathbb{D}_R \setminus \{0, -1, -2, \ldots\}$, so we just need to show that the convergence is uniform there. In particular, we will demonstrate absolute convergence. We have

$$\left|\frac{(-1)^n}{n!(n+s)}\right| \le \frac{1}{n!(n-|s|)} \le \frac{1}{n!(2R-R)} \le \frac{1}{n!R}$$

and the bound is uniform, so we are done. Thus Γ may be extended to $\mathbb{C}\setminus\{0, -1, -2, \ldots\}$. □

Having shown the existence of the gamma function, we now work to derive identities that will be useful to us, and also will be applied in our study of later functions.

Theorem 6.4: Euler's Reflection Formula

For all $s \in \mathbb{C} \setminus \mathbb{Z}$,

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$

Proof. By analytic continuation, it is enough to prove this for any set with a limit point, in particular the real interval $(0,1)$. So using our integral formula, for $s \in (0,1)$ we have

$$\Gamma(s)\Gamma(1-s) = \int_0^\infty t^{s-1}e^{-t}\Gamma(1-s)\,\mathrm{d}t = \int_0^\infty t^{s-1}e^{-t}\left(\int_0^\infty u^{-s}e^{-u}\,\mathrm{d}u\right)\mathrm{d}t$$

For fixed t, we write $u = vt$ and make the substitution:

$$\begin{aligned}\int_0^\infty t^{s-1}e^{-t}\left(\int_0^\infty u^{-s}e^{-u}\,\mathrm{d}u\right)\mathrm{d}t &= \int_0^\infty t^{s-1}e^{-t}\left(\int_0^\infty e^{-vt}(vt)^{-s}t\,\mathrm{d}v\right)\mathrm{d}t \\ &= \int_0^\infty \left(\int_0^\infty e^{-t(v+1)}v^{-s}\,\mathrm{d}v\right)\mathrm{d}t\end{aligned}$$

The integrand is positive and decays exponentially, so the integral converges absolutely and by Fubini's Theorem we may interchange the integrals. Then

$$\int_0^\infty \left(\int_0^\infty e^{-t(v+1)}v^{-s}\,\mathrm{d}v\right)\mathrm{d}t = \int_0^\infty \left(\int_0^\infty e^{-(v+1)t}\,\mathrm{d}t\right)v^{-s}\,\mathrm{d}v = \int_0^\infty \frac{v^{-s}}{v+1}\,\mathrm{d}v$$

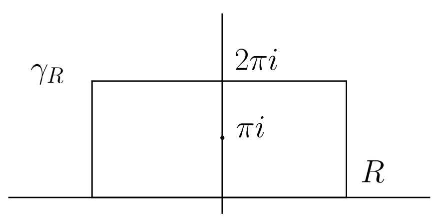
Claim: For $0 < a < 1$,

$$\int_0^\infty \frac{v^{a-1}}{1+v}\,\mathrm{d}v = \frac{\pi}{\sin(\pi a)}$$

We apply a change of variables:

$$\int_0^\infty \frac{v^{a-1}}{1+v}\,\mathrm{d}v = \int_{-\infty}^\infty \frac{e^{ax}}{1+e^x}\,\mathrm{d}x$$

Now, taking $f(z) = \frac{e^{az}}{1+e^z}$, we evaluate this integral over the contour:



Then we observe that the only pole lies at πi, and the residue is given by

$$\operatorname{res}_{\pi i}(f) = \lim_{z\to\pi i}(z-\pi i)\frac{e^{az}}{1-e^{z-\pi i}} = -e^{a\pi i}$$

so

$$\int_{\gamma_R} f(z)\,\mathrm{d}z = 2\pi i\operatorname{res}_{\pi i}(f) = -2\pi i e^{a\pi i}$$

To show that the vertical sides go to zero, we have

$$\left| i\int_0^{2\pi} \frac{e^{a(R+it)}}{1+e^{R+it}}\,\mathrm{d}t\right| \le 2\pi\frac{e^{aR}}{e^{R-1}} \rightrightarrows 0$$

For the bottom side, we have

$$\int_{-R}^{R} f(z)\,\mathrm{d}z \xrightarrow{R\to\infty} I$$

and for the top

$$\int_{-R}^{R} f(t+2\pi i)\,\mathrm{d}t = \int_{-R}^{R} \frac{e^{a(t+2\pi i)}}{1+e^{t+2\pi i}}\,\mathrm{d}t \xrightarrow{R\to\infty} e^{2\pi a i} I$$

so

$$I - e^{2\pi a i} I = -2\pi i e^{a\pi i}$$

which recovers

$$I = \frac{\pi}{\sin(\pi a)}$$

From here, we see that

$$\Gamma(s)\Gamma(1-s) = \int_0^\infty \frac{v^{-s}}{v+1}\,\mathrm{d}v = \frac{\pi}{\sin(\pi(1-s))} = \frac{\pi}{\sin \pi s}$$

□

Note also that this gives

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Theorem 6.5

The function $\frac{1}{\Gamma(s)}$ is an entire function with simple zeroes at $\{0,-1,-2,\ldots\}$ and

$$\left|\frac{1}{F(s)}\right| \le C_1 e^{C_2 |s| \ln(1+|s|)}$$

where $C_1, C_2 > 0$, so that the order of growth is at most 1.

Proof. By the reflection formula, for $s \notin \mathbb{Z}$, Γ is not zero, and

$$\frac{1}{\Gamma(s)} = \Gamma(1-s)\frac{\sin(\pi s)}{\pi}$$

At $s \in \mathbb{N}$, $\Gamma(1-s)$ has a simple pole, and $\sin(\pi s)$ has a simple zero. So they cancel and thus the singularities at $s \in \mathbb{N}$ are removable. Also, at the poles of $\Gamma(s)$, $\frac{1}{\Gamma(s)}$ has a zero. So it has simple zeroes at $\{0,-1,-2,\ldots\}$ and nowhere else. We plug in the identity

$$\Gamma(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+s)} + \int_1^\infty t^{s-1} e^{-t}\,\mathrm{d}t$$

to get

$$\frac{1}{\Gamma(s)} = \Gamma(1-s)\frac{\sin \pi s}{\pi} = \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1-s)}\right)\frac{\sin \pi s}{\pi} + \left(\int_1^\infty e^{-t} t^{-s}\,\mathrm{d}t\right)\frac{\sin \pi s}{\pi}$$

By bounding the Taylor series, we get

$$\left|\frac{\sin \pi s}{\pi}\right| \leq e^{-s}$$

Let $\sigma = \mathrm{Re}(s)$. Then

$$\left|\int_1^\infty e^{-t}t^{-s}\,\mathrm{d}t\right| \leq \int_1^\infty e^{-t}t^{-\sigma}\,\mathrm{d}t \leq \int_1^\infty e^{-t}t^{|s|}\,\mathrm{d}t$$

Pick n such that $|s| \leq n \leq |s|+1$. Then

$$\int_1^\infty e^{-t}t^{|s|}\,\mathrm{d}t \leq \int_1^\infty e^{-t}t^{n}\,\mathrm{d}t = \Gamma(n+1) = n! \leq n^n = e^{n\ln n} \leq e^{(|s|+1)\ln(|s|+1)}$$

To bound the sum, consider separately the cases when s is close to an integer, and when it is not. If $|n+1-s| \geq \frac{1}{2}$ for all $n \in \mathbb{N} \cup \{0\}$, then

$$\left|\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1-s)}\right| \leq \sum_{n=0}^{\infty} \frac{2}{n!} = 2e < \infty$$

Otherwise, if there is $k \in \mathbb{N} \cup \{0\}$ such that $|k+1-s| < \frac{1}{2}$, then only one such integer exists. So we can pull out this term and

$$\left|\sum_{\substack{n=0\\n\neq k}}^{\infty} \frac{(-1)^n}{n!(n+1-s)}\right| < 2e < \infty$$

The last term can be handled by multiplying with the sin factor. Since sin has a zero and the sum term has a pole, they cancel out and the product is bounded:

$$\left|\frac{(-1)^k}{k!(k+1-s)}\frac{\sin \pi s}{\pi}\right| \leq C$$

for some $C > 0$. Thus we have shown the bound. $\square$

The following theorem is a well-known result called the product formula for $\frac{1}{\Gamma}$. It involves the **Euler-Mascheroni constant** $\gamma \approx 0.57721$, which appears in many applications but is not well understood.

Theorem 6.6

$$\frac{1}{\Gamma(s)} = e^{\gamma s} s \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-\frac{s}{n}}$$

where

$$\gamma = \lim_{N\to\infty} \left(\left(\sum_{n=1}^{N} \frac{1}{n}\right) - \ln(N+1)\right)$$

Proof. Since the nonzero zeroes of $\frac{1}{\Gamma}$ are $-\mathbb{N}$, and all the zeroes are simple, Hadamard's factorization theorem tells us that

$$\frac{1}{\Gamma(s)} = e^p s \prod_{n=1}^{\infty} E_1\left(\frac{s}{-n}\right)$$

where p is a polynomial of degree no more than the growth rate of $\frac{1}{\Gamma}$. But we just showed that the growth rate is at most 1, so we may write $p = as + b$. Then expanding E_1, we have

$$e^p s \prod_{n=1}^{\infty} E_1\left(\frac{s}{-n}\right) = e^{as+b} s \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-\frac{s}{n}}$$

By our identities for Γ,

$$\frac{1}{\Gamma(s+1)} = \frac{1}{s\Gamma(s)} = e^{as+b} \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-\frac{s}{n}}$$

As $s \to 0$, the RHS approaches e^b, and the LHS is $\Gamma(1) = 1$, so $e^b = 1$. Plugging in $s = 1$,

$$e^{-a} = \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right) e^{-\frac{1}{n}} = \lim_{N\to\infty} \prod_{n=1}^{N} \frac{n+1}{n} e^{-\frac{1}{n}}$$
$$= \lim_{N\to\infty} (N+1) e^{-\sum_{n=1}^{N} \frac{1}{n}} = e^{\lim_{N\to\infty} (\ln(N+1)) - \sum_{n=1}^{N} \frac{1}{n}}$$

This proves that $a = \gamma$ and that the limit for γ exists. □

6.2 The Θ Function

In order to prove the prime number theorem, we briefly introduce the Θ function and prove a functional equation. The function we study is part of a far more general class of functions, given by a two-parameter function known as the Jacobi theta function.

Definition 6.2

For $t > 0$, we define the **theta function** by

$$\Theta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t} = 1 + 2 \sum_{=1}^{\infty} e^{-\pi n^2 t}$$

The following functional equation is satisfied by Θ:

Theorem 6.7

For $t > 0$,

$$\Theta\left(\frac{1}{t}\right) = \sqrt{t}\Theta(t)$$

Proof. Fix $t > 0$. Define the entire function

$$f(z) = e^{-\pi t z^2}$$

so that $\Theta(t)$ is the sum of the values of $f(n)$ for $n \in \mathbb{Z}$. Denote

$$g(z) = \frac{f(z)}{e^{2\pi i z} - 1}$$

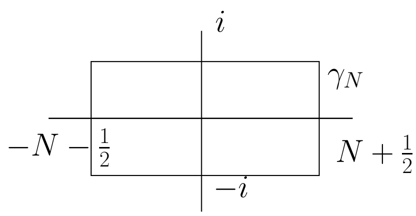
g is holomorphic on $\mathbb{C}\setminus\mathbb{Z}$ and has simple poles at the integers. We can calculate the residues using L'Hopital's rule:

$$\operatorname{res}_n(g) = \lim_{z \to n} (z - n) g(z) = \frac{f(n)}{2\pi i}$$

So

$$\Theta(t) = \sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} 2\pi i \operatorname{res}_n(g) = \lim_{N \to \infty} \int_{\gamma_N} g(z)\,\mathrm{d}z$$

where γ_N is some closed curve containg the integers in $[-N, N]$. Explicitly, consider



We want to show the vertical segments vanish as $N \to \infty$. For the right hand side, which we denote as $\gamma_{N,R}$ this is

$$\left|\int_{\gamma_{N,R}} g(z)\,\mathrm{d}z\right| \le \int_{-1}^{1} \left|g\left(N + \frac{1}{2} + is\right)\right| \mathrm{d}s \le \int_{-1}^{1} \frac{\left|e^{-\pi t\left(N+\frac{1}{2}+is\right)^2}\right|}{\left|e^{2\pi i\left(N+\frac{1}{2}+is\right)} - 1\right|}\,\mathrm{d}s$$

$$= \int_{-1}^{1} \frac{e^{-\pi t\left(N+\frac{1}{2}\right)^2 + \pi t s^2}}{e^{-2\pi s} + 1}\,\mathrm{d}s \le 2e^{\pi t} e^{-\pi t\left(N+\frac{1}{2}\right)^2} \xrightarrow{N \to \infty} 0$$

The left side contour is similar. For the top and bottom contours, they will approach

$$\int_{-\infty}^{\infty} g(s - i)\,\mathrm{d}s - \int_{-\infty}^{\infty} g(s + i)\,\mathrm{d}s$$

This limit exists since f exhibits quadratic decay. Then we have

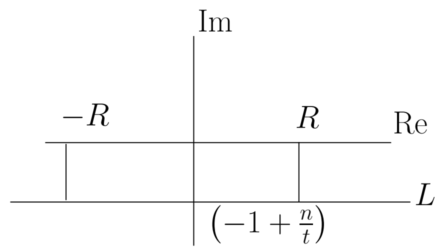
$$\int_{-\infty}^{\infty} g(s - i)\,\mathrm{d}s = \int_{-\infty}^{\infty} \frac{e^{-\pi t(s-i)^2}}{e^{2\pi i(s-i)} - 1}\,\mathrm{d}s$$

$$= \int_{-\infty}^{\infty} \frac{e^{-\pi t(s-i)^2}}{1 - e^{-2\pi i(s-i)}} \cdot e^{-2\pi i(s-i)}\,\mathrm{d}s = \int_{-\infty}^{\infty} e^{-\pi t(s-i)^2 - 2\pi i(s-i)} \left(\sum_{n=1}^{\infty} e^{-2\pi i(s-i)n}\right) \mathrm{d}s$$

Notice that the ratio of the geometric series has absolute value $e^{-2\pi} < 1$. So we may interchange the sum and integral:

$$\begin{aligned}\sum_{n=1}^{\infty}\int_{-\infty}^{\infty} e^{-\pi t(s-i)^2-2\pi i(s-i)n}\,\mathrm{d}s &= \sum_{n=1}^{\infty} e^{-\pi\frac{n^2}{t}}\int_{-\infty}^{\infty} e^{-\pi ts\left(s-i+\frac{n}{t}i\right)^2}\,\mathrm{d}s \\ &= \sum_{n=1}^{\infty} e^{-\frac{\pi n^2}{t}}\int_{L} e^{-\pi tz^2}\,\mathrm{d}z\end{aligned}$$

where L is the infinite horizontal line segment which intersects the y axis at $\left(-1+\frac{n}{t}\right)$. Then we can find this by looking at the contour:



As $R \to \infty$, the vertical line segments again cancel out, and the function we integrate is entire, so the horizontal segments are equal. Thus

$$\sum_{n=1}^{\infty} e^{-\frac{\pi n^2}{t}}\int_{L} e^{-\pi tz^2}\,\mathrm{d}z = 2\sum_{n=1}^{\infty} e^{-\frac{\pi n^2}{t}}\int_{0}^{\infty} e^{-\pi tx^2}\,\mathrm{d}x$$

Making the change of variables $x = \frac{\sqrt{y}}{\sqrt{\pi t}}$, this becomes

$$2\sum_{n=1}^{\infty} e^{-\frac{\pi n^2}{t}}\int_{0}^{\infty} e^{-y}\frac{1}{2\sqrt{y}\sqrt{\pi t}}\,\mathrm{d}y = \sum_{n=1}^{\infty} e^{-\frac{\pi n^2}{t}}\frac{1}{\sqrt{\pi t}}\underbrace{\Gamma\left(\frac{1}{2}\right)}_{=\frac{1}{2}} = \frac{1}{\sqrt{t}}\sum_{n=1}^{\infty} e^{-\frac{\pi n^2}{t}}$$

One shows by the same computation that

$$-\int_{-\infty\infty} g(s+i)\,\mathrm{d}s = \frac{1}{\sqrt{t}}\sum_{n=-\infty}^{0} e^{-\frac{\pi n^2}{t}}$$

so

$$\Theta(t) = \frac{1}{\sqrt{t}}\sum_{n=-\infty}^{\infty} e^{-\frac{\pi n^2}{t}} = \frac{1}{\sqrt{t}}\Theta\left(\frac{1}{t}\right)$$

□

6.3 The Riemann Zeta Function

Definition 6.3

For $s \in \mathbb{C}$ such that $\operatorname{Re}(s) > 1$, we define the **zeta function**

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

We previously proved that ζ is holomorphic on $\{s \in \mathbb{C} : \operatorname{Re}(s) > 1\}$.

Theorem 6.8

If $\operatorname{Re}(s) > 1$, then

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \int_0^{\infty} u^{\frac{s}{2}-1} \frac{\Theta(u) - 1}{2} \, \mathrm{d}u$$

Proof. We first verify that the integral actually converges. Recall that we wrote

$$\Theta(u) = 1 + 2 \sum_{n=1}^{\infty} e^{-\pi n^2 u}$$

so for $u \geq 1$,

$$\frac{\Theta(u) - 1}{2} = \sum_{n=1}^{\infty} e^{-\pi n^2 u} \leq \sum_{n=1}^{\infty} e^{-\pi u} = \frac{e^{-\pi n u}}{1 - e^{-\pi u}} \leq \frac{e^{-\pi u}}{1 - e^{\pi}}$$

This decays faster than $u^{\frac{s}{2}}$ at infinity. For $u \leq 1$,

$$\Theta(u) = \frac{1}{\sqrt{u}} \Theta\left(\frac{1}{u}\right) \sim \frac{1}{\sqrt{u}}$$

and

$$\left| u^{\frac{s}{2}-1} \right| = u^{\frac{\operatorname{Re}(s)}{2}-1}$$

so near 0, the exponent of the integrand is strictly greater than -1, so the integral absolutely converges. Now, we compute the right hand side:

$$\int_0^{\infty} u^{\frac{s}{2}-1} \frac{\Theta(u) - 1}{2} \, \mathrm{d}u = \int_0^{\infty} \left(u^{\frac{s}{2}-1} \sum_{n=1}^{\infty} e^{-\pi n^2 u} \right) \mathrm{d}u$$

As before, we have absolute convergence of the sum, so

$$\int_0^{\infty} \left(u^{\frac{s}{2}-1} \sum_{n=1}^{\infty} e^{-\pi n^2 u} \right) \mathrm{d}u = \sum_{n=1}^{\infty} \int_0^{\infty} u^{\frac{s}{2}-1} e^{-\pi n^2 u} \, \mathrm{d}u$$

For each integral in the sum, apply the change of variables $t = \pi n^2 u$. Then this becomes

$$\sum_{n=1}^{\infty} \int_0^{\infty} \left(\frac{t}{\pi n^2}\right)^{\frac{s}{2}-1} e^{-t} \frac{1}{\pi n^2} \, \mathrm{d}t = \pi^{-\frac{s}{2}} \sum_{n=1}^{\infty} \frac{1}{n^s} \int_0^{\infty} t^{\frac{s}{2}-1} e^{-t} \, \mathrm{d}t = \pi^{-\frac{s}{2}} \zeta(s) \Gamma\left(\frac{s}{2}\right) \qquad \square$$

For convenience, we denote the left hand side of the above identity by the following shorthand:

Definition 6.4

The **xi function** is defined for $\mathrm{Re}(s) > 1$ by

$$\xi(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

Theorem 6.9

ξ is holomorphic on $\{s \in \mathbb{C} : \mathrm{Re}(s) > 1\}$ and admits an analytic continuation to $\mathbb{C} \setminus \{0, 1\}$ which has simple poles at $0, 1$, and satisfies the identity

$$\xi(s) = \xi(1 - s)$$

In other words, ξ is symmetric about the axis $\mathrm{Re}(z) = \frac{1}{2}$, which will be an important axis in the study of the zeta function.

Proof. For $u > 0$, denote

$$\Psi(u) = \frac{\Theta(u) - 1}{2}$$

From Theorem 6.8

$$\xi(s) = \int_0^{\infty} u^{\frac{s}{2}-1} \Psi(u) \, \mathrm{d}u$$

We also know

$$\Theta(u) = \frac{1}{\sqrt{u}} \Theta\left(\frac{1}{u}\right)$$

So

$$\Psi(u) = \frac{\frac{1}{\sqrt{u}} \Theta\left(\frac{1}{u}\right) - 1}{2} = \frac{1}{\sqrt{u}} \Psi\left(\frac{1}{u}\right) + \frac{1}{2\sqrt{u}} - \frac{1}{2}$$

As in the previous proof, we break the integral into two parts:

$$\xi(s) = \int_0^1 u^{\frac{s}{2}-1} \Psi(u) \, \mathrm{d}u + \int_1^{\infty} u^{\frac{s}{2}-1} \Psi(u) \, \mathrm{d}u$$

As before, the second integral is entire. Then we apply the substitution $u \mapsto \frac{1}{x}$ to write the first integral as

$$\begin{aligned} \int_0^1 u^{\frac{s}{2}-1} \Psi(u) \, \mathrm{d}u &= \int_0^1 u^{\frac{s}{2}-1} \left(\frac{1}{\sqrt{u}} \Psi\left(\frac{1}{u}\right) + \frac{1}{2\sqrt{u}} - \frac{1}{2}\right) \mathrm{d}u \\ &= \frac{1}{s-1} - \frac{1}{s} + \int_1^{\infty} \left(\frac{1}{x}\right)^{\frac{s}{2}-1} \sqrt{x} \Psi(x) \frac{1}{x^2} \, \mathrm{d}x \end{aligned}$$

Combining our two integrals, we have

$$\begin{aligned}\xi(s) &= \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty \left(\frac{1}{x}\right)^{\frac{s}{2}-1} \sqrt{x}\Psi(x)\frac{1}{x^2}\,\mathrm{d}x + \int_1^\infty u^{\frac{s}{2}-1}\Psi(u)\,\mathrm{d}u \\ &= \frac{1}{s-1} + \frac{1}{s} + \int_1^\infty \left(x^{-\frac{s}{2}-1} + x^{\frac{s}{2}-1}\right)\Psi(x)\,\mathrm{d}x\end{aligned}$$

The integral converges everywhere because of the rapid decay of Ψ, so this formula holds not only on $\{\mathrm{Re}(s) > 1\}$ but also on all of $\mathbb{C}$, except for $0, 1$. It is also clear from this equation that the poles are $0, 1$, and that replacing s with $1-s$ gives the same value, so that $\xi(s) = \xi(1-s)$. ☐

This allows us to prove the existence of the analytic continuation of the zeta function.

Corollary 6.10

ζ has an analytic continuation to $\mathbb{C} \setminus \{1\}$ with a simple pole at 1, and it satisfies the identity

$$\zeta(s) = \pi^{s-\frac{1}{2}} \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \zeta(1-s)$$

Proof. From the definition of ξ,

$$\zeta(s) = \pi^{\frac{s}{2}} \frac{\xi(s)}{\Gamma\left(\frac{s}{2}\right)}$$

ξ can be extended everywhere except $0, 1$. Also, $\frac{1}{\Gamma}$ is entire, so ζ is certainly holomorphic on $\mathbb{C} \setminus \{0, 1\}$. But we also know $\frac{1}{\Gamma}$ has a simple zero at the nonpositive integers, so it has a simple zero at zero which cancels with the simple zero of ξ at 0. This means that ζ may be continued to 0 as well. The identity follows from the identity

$$\xi(s) = \xi(1-s)$$ ☐

We now prove the existence of a sequence of function swhich will aid us in proving the prime number theorem.

Proposition 6.11

There exists a sequence of entire functions $\delta_n : \mathbb{C} \to \mathbb{C}$ such that

$$|\delta_n(s)| \leq \frac{|s|}{n^{\mathrm{Re}(s)+1}}$$

and

$$|\delta_n(s)| \leq \frac{2}{n^{\mathrm{Re}(s)}}$$

such that

$$\sum_{n=1}^{N-1} \frac{1}{n^s} - \int_1^N \frac{1}{x^s}\,\mathrm{d}s = \sum_{n=1}^{N-1} \delta_n(s)$$

Proof. Define

$$\delta_n(s) = \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s} \right) \mathrm{d}x$$

Then summing $\delta_n(s)$ gives the identity desired. To show the bounds, we have

$$|\delta_n(s)| \leq \left| \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s} \,\mathrm{d}x \right) \right| \leq \int_n^{n+1} \left(\frac{1}{n^{\mathrm{Re}(s)}} + \frac{1}{x^{\mathrm{Re}(s)}} \right) \mathrm{d}x \leq \frac{2}{n^{\mathrm{Re}(s)}}$$

To demonstrate the first bound, we apply the mean value theorem to each $x \in [n, n+1]$ to produce $y \in [n, x]$ such that

$$\frac{1}{n^s} - \frac{1}{x^s} = -\frac{s(n-x)}{y^{s+1}} \leq \frac{s}{y^s}$$

which gives

$$\left| \frac{1}{n^s} - \frac{1}{x^s} \right| = \frac{|s|}{y^{\mathrm{Re}(s)+1}} \leq \frac{|s|}{n^{\mathrm{Re}(s)+1}}$$

□

This allows us to bound the analytic continuation on a subset of $\mathbb{C}$.

Corollary 6.12

$\zeta(s) - \frac{1}{s-1}$ has an analytic continuation to $\{s \in \mathbb{C} : \mathrm{Re}(s) > 0\} \setminus \{1\}$.

Proof. Suppose $\mathrm{Re}(s) > 1$. Then for all N,

$$\sum_{n=1}^{N-1} \frac{1}{n^s} - \frac{1}{s-1} \left(1 - \frac{1}{N^{s-1}} \right) = \sum_{n=1}^{N-1} \delta_n(s)$$

As $N \to \infty$, this tends to the equation

$$\zeta(s) - \frac{1}{s-1} = \sum_{n=1}^{\infty} \delta_n(s)$$

To check that this converges, we use the bound

$$|\delta_n(s)| \leq \frac{2}{n^{\mathrm{Re}(s)}}$$

together with the fact that $\mathrm{Re}(s) > 1$ to conclude convergence. We now denote

$$H(s) = \sum_{n=1}^{\infty} \delta_n(s)$$

so that

$$\zeta(s) = \frac{1}{s-1} + H(s)$$

Using the bound

$$|\delta_n(s)| \leq \frac{|s|}{n^{\mathrm{Re}(s)+1}}$$

we have uniform convergence on compact subsets, so H is holomorphic on $\{s \in \mathbb{C} : \mathrm{Re}(s) > 0\}$. Then this is an analytic continuation of ζ to $\{s \in \mathbb{C} : \mathrm{Re}(s) > 0\} \setminus \{1\}$. $\square$

We take a brief moment to prove the following technical lemma, to be used later.

Lemma 6.13

For $0 < \varepsilon < \frac{1}{2}$ and $\varepsilon \leq \sigma_0 \leq 1$, there exists $C = C_{\varepsilon,\sigma_0} > 0$ such that

1. $|\zeta(\sigma + it)| \leq C|t|^{1-\sigma_0+\varepsilon}$ when $\sigma \geq \sigma_0$ and $|t| \geq \frac{1}{2}$.
2. $|\zeta'(\sigma + it)| \leq C|t|^{2\varepsilon}$ for $\sigma \geq 1, |t| \geq 1$.

Proof. From the previous proof, we have

$$\zeta(s) = \frac{1}{s-1} + H(s)$$

so for $s = \sigma + it$, with $|t| \geq \frac{1}{2}$ and $\sigma > 0$,

$$|\zeta(s)| \leq \frac{1}{\sqrt{(\sigma-1)^2 + t^2}} + \sum_{n=1}^{\infty} |\delta_n(s)|$$

For any $\eta \in (0,1)$, we know

$$|\delta_n(s)| = |\delta_n(s)|^{\eta} |\delta_n(s)|^{1-\eta} \leq \left(\frac{|s|}{n^{\sigma+1}}\right)^{\eta} \left(\frac{2}{n^{\sigma}}\right)^{1-\eta} = \frac{2^{1-\eta}|s|^{\eta}}{n^{\eta+\sigma}} \leq \frac{2|s|^{\eta}}{n^{\sigma_0+\eta}}$$

Choosing $\eta = 1 - \sigma_0 + \varepsilon$, we get

$$|\delta_n(s)| \leq \frac{2|s|^{1-\sigma_0+\varepsilon}}{n^{1+\varepsilon}}$$

so

$$|\zeta(s)| \leq \frac{1}{\sqrt{(\sigma-1)^2 + t^2}} + \sum_{n=1}^{\infty} |\delta_n(s)| \leq 2 + 2|s|^{1-\sigma_0+\varepsilon} \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}}}_{=C_{\varepsilon}}$$

When $\sigma \leq 3|t|$, this gives us what we want. Otherwise, $\sigma > 3|t| > \frac{3}{2}$, so we are in the region of convergence fo the series, so

$$|\zeta(s)| \leq \sum_{n=1}^{\infty} \frac{1}{n^{\sigma}} \leq \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$$

which converges.

The second inequality follows from the first by Cauchy's integral formula. If $|t| \geq 1$, then we can draw a circle of radius ε around s. Then

$$\zeta'(s) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}_\varepsilon(s)} \frac{\zeta(w)}{(w-s)^2}\,\mathrm{d}w \leq \frac{1}{2\pi i\varepsilon^2}\int_0^{2\pi} \left|\zeta(s+\varepsilon e^{i\theta})\right|\left|(\varepsilon i e^{i\theta})\right|\mathrm{d}\theta$$

On the boundary of the disk, we apply the first inequality for $\sigma_0 = 1-\varepsilon$:

$$\begin{aligned}\frac{1}{2\pi i\varepsilon}\int_0^{2\pi} \left|\zeta(s+\varepsilon e^{i\theta})\right|\mathrm{d}\theta &\leq \frac{1}{2\pi\varepsilon}\int_0^{2\pi} \left|\zeta\left(\sigma+\varepsilon\cos\theta + i\left(t+\varepsilon\sin\theta\right)\right)\right|\mathrm{d}\theta \\ &\leq C'|t+\varepsilon\sin\theta|^{2\varepsilon} \leq C'|2t|^{2\varepsilon}\end{aligned}$$

□

6.4 The Prime Number Theorem

Having studied the gamma, theta, and zeta functions, and established functional equations and bounds for them, we now progerss to proving the prime number theorem. This theorem was proved independently by Hadamard and de la Vallee Poussin in 1896. The theorem concerns the prime counting function π. Denoting the set of primes as $\mathbb{P}$, for $x > 0$ we define

$$\pi(x) = |\{p \in \mathbb{P} : p \leq x\}|$$

The theorem asserts that π satisfies the asymptotic relation

$$\pi(x) \sim \frac{x}{\ln x}$$

The proof of the theorem involves a connection between prime numbers and the zeta function, given by the following identity:

Theorem 6.14: Euler's Product Formula

For $\mathrm{Re}(s) > 1$,

$$\zeta(s) = \prod_{p\in\mathbb{P}} \frac{1}{1-p^{-s}}$$

Proof. Observe that

$$\sum_{p\in\mathbb{P}} p^{-s}$$

converges absolutely and uniformly on compact subsets of $\{\mathrm{Re}(s) > 1\}$, since it is dominated by n^{-s}. Therefore, we know from our study of infinite products that

$$\prod_{p\in\mathbb{P}} (1-p^{-s})$$

is defined, holomorphic, and nonvanishing on $\{\mathrm{Re}(s) > 1\}$. Thus

$$\frac{1}{\prod_{p\in\mathbb{P}}(1-p^{-s})} = \prod_{p\in\mathbb{P}} \frac{1}{1-p^{-s}}$$

is also holomorphic on $\{\mathrm{Re}(s) > 1\}$. So by analytic continuation, we just need to show that this product is equal to ζ on some set with a limit point. We will take the real axis $s > 1$.

Observe that

$$\sum_{n=1}^{N} \frac{1}{n^s} \leq \prod_{p \leq N} \left(1 + \frac{1}{p^s} + \ldots + \frac{1}{p^{sN}}\right)$$

To see this, since the terms are nonnegative and greater than 1, we just need to show that each term in the sum appears in the expanded form of the product. By prime factorization, for each $n \leq N$, we can write $n = p_1^{k_1} \cdots p_\ell^{k_\ell}$ where each $p_j \leq N$. Thus

$$\frac{1}{n^s} = \frac{1}{p_1^{sk_1}} \cdots \frac{1}{p_\ell^{sk_\ell}}$$

This appears in the expansion of the product, so the inequality holds. We can weaken this bound to

$$\sum_{n=1}^{N} \frac{1}{n^s} \leq \prod_{p \leq N} \left(1 + \frac{1}{p^s} + \ldots + \frac{1}{p^{sN}}\right) \leq \prod_{p \leq N} \sum_{k=0}^{\infty} \frac{1}{p^{ks}} = \prod_{p \leq N} \frac{1}{1 - p^{-s}}$$

As $N \to \infty$, we get

$$\zeta(s) \leq \prod_{p \in \mathbb{P}} \frac{1}{1 - p^{-s}}$$

To show the reverse inequality, let N, M be integers. We claim that

$$\prod_{p \leq N} \left(1 + \frac{1}{p^s} + \ldots + \frac{1}{p^{sM}}\right) \leq \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s)$$

The expansion of the product gives many terms of the form $\frac{1}{n^s}$ for some n. By uniqueness of prime factorization, none show up in the sum more than once, so the inequality holds. Then as $M \to \infty$, we get

$$\prod_{p \leq N} \frac{1}{1 - p^{-s}} \leq \zeta(s)$$

Taking $N \to \infty$, we recover the correct inequality and therefore the product formula. □

Corollary 6.15

If $\mathrm{Re}(s) > 1$, then $\zeta(s) \neq 0$.

Proof. None of the terms in the infinite product are zero, and it converges uniformly, so $\zeta(s) \neq 0$. □

Now, we develop estimates for ζ which will allow us to prove the prime number theorem.

Definition 6.5

The **von Mangoldt** function $\Lambda : \mathbb{N} \to \mathbb{R}$ is defined by

$$\Lambda(n) = \begin{cases} \ln p, & n = p^m \text{ for } m \in \mathbb{N}, p \in \mathbb{P} \\ 0 \end{cases}$$

Lemma 6.16

For $\sigma > 1$ and $s = \sigma + it$, then

$$\operatorname{Log} \zeta(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s \ln n} = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s \ln n}$$

so

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s}$$

Proof. First note that the logarithm is the principal baanch of the logarithm, which is well defined and holomorphic here since ζ does not vanish on the simply connected region $\mathrm{Re}(s) > 1$. Also note that

$$\sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s \ln n} \leq \sum_{n=2}^{\infty} \frac{1}{n^s} = \zeta(s) - 1$$

and ζ converges absolutely, so the series is holomorphic. So by analytic continuation we just need to prove the identity for s real. By the product formula for ζ, we have

$$\begin{aligned} \operatorname{Log} \zeta(s) &= \lim_{N\to\infty} \operatorname{Log} \prod_{p \leq N} \frac{1}{1 - p^{-s}} = \lim_{N\to\infty} \sum_{p\leq N} - \ln(1 - p^{-s}) = \lim_{N\to\infty} \sum_{p\leq N} \sum_{m=1}^{\infty} \frac{1}{p^{sm} m} \\ &= \lim_{N\to\infty} \sum_{\substack{n=p^m: \\ p\leq N \\ m\in\mathbb{N}}} \frac{\ln p}{n^s \ln n} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s \ln n} = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s \ln n} \end{aligned}$$

Since we have absolute convegence, the value $\frac{\zeta'(s)}{\zeta(s)}$ may be found by differentiating the series term-by-term, which recovers the second formula. $\square$

Lemma 6.17

If $\sigma > 1$ and t is real, then

$$\log\left|\zeta^3(\sigma)\zeta^4(\sigma + it)\zeta(\sigma + 2it)\right| \geq 0$$

Proof. We have (note that the branch cut doesn't matter since the real part of the logarithm is the same for all branches)

$$\begin{aligned}\log\left|\zeta^3(\sigma)\zeta^4(\sigma+it)\zeta(\sigma+2it)\right| &= 3\log|\zeta(\sigma)| + 4\log|\zeta(\sigma+it)| + \log|\zeta(\sigma+2it)| \\ &= 3\operatorname{Re}\log\zeta(\sigma) + 4\operatorname{Re}\log\zeta(\sigma+it) + \operatorname{Re}\log(\zeta(\sigma+2it))\end{aligned}$$

By the previous lemma, this becomes

$$\sum_{n=1}^{\infty} c_n n^{-\sigma}(3 + 4\cos\theta_n + \cos 2\theta_n)$$

Observe that

$$\begin{aligned}0 \le 2(1+\cos\theta)^2 = 2(1+2\cos\theta+\cos^2\theta) = 3+4\cos\theta+2\cos^2\theta - 1 \\ = 3+4\cos\theta+\cos^2\theta-(1-\cos^2\theta) = 3+4\cos\theta+\cos^2\theta-\sin^2\theta = 3+4\cos\theta+\cos 2\theta\end{aligned}$$

which proves the lemma. $\square$

Corollary 6.18

For $s = \sigma + it$,

$$\left|\zeta^3(\sigma)\zeta^4(s)\zeta(\sigma+2it)\right| \ge 1$$

Theorem 6.19

For $\varepsilon > 0$, there exists $c_\varepsilon > 0$ such that for any $s = \sigma + it$, with $\sigma \ge 1, |t| \ge 1$,

$$\frac{1}{|\zeta(s)|} \le c_\varepsilon |t|^\varepsilon$$

Proof. Referring to Lemma 6.13, for small ε and $\sigma_0 = 1$, there exists C_ε such that

$$\begin{aligned}|\zeta(s)| &\le C_\varepsilon |t|^{1-\sigma_0+\varepsilon} \\ |\zeta'(s)| &\le C_\varepsilon |t|^\varepsilon\end{aligned}$$

By Corollary 6.12, we have

$$\zeta(s) = \frac{1}{1-s} + H(s)$$

where

$$H(s) = \sum_{n=1}^{\infty} \delta_n(s)$$

and $H(s) \ge 0$ when $s \ge 0$.

By Corollary 6.18, we have

$$|\zeta(s)| \ge |\zeta(\sigma)|^{-\frac{3}{4}} |\zeta(\sigma+2it)|^{-\frac{1}{4}} \ge C_\varepsilon |\zeta(\sigma)|^{-\frac{3}{4}} |t|^{-\frac{\varepsilon}{4}} \ge C_\varepsilon (\sigma-1)^{\frac{3}{4}} |t|^{-\frac{\varepsilon}{4}}$$

From here, we have two cases, which will depend on a constant A which we pick later:

- If $\sigma - 1 \geq A|t|^{-5\varepsilon}$, then
$$|\zeta(s)| \geq C_\varepsilon A^{\frac{3}{4}} |t|^{-4\varepsilon}$$

- If $\sigma - 1 < A|t|^{-5\varepsilon}$, then choose $\sigma' > \sigma$ such that
$$\sigma' - 1 = A|t|^{-5\varepsilon}$$

  Then applying the first case to σ', and applying the mean value theorem to the bound on ζ',
$$\begin{aligned} |\zeta(\sigma + it)| &\geq |\zeta(\sigma' + it)| - |\zeta(\sigma' + it) - |\zeta(\sigma + it)|| \\ \geq C_\varepsilon A^{\frac{3}{4}} |t|^{-4\varepsilon} - (\sigma' - \sigma)|\zeta'(\sigma'' + it)| &\geq C_\varepsilon A^{\frac{3}{4}} |t|^{-4\varepsilon} - C'(\sigma' - 1)|t|^{\varepsilon} \\ = C_\varepsilon a^{\frac{3}{4}} |t|^{-4\varepsilon} C'(\sigma' - 1)A|t|^{-4\varepsilon} &= |t|^{-4\varepsilon} \left(C_\varepsilon A^{\frac{3}{4}} - C'A \right) \end{aligned}$$

  where $\sigma < \sigma'' < \sigma'$, and C' is a constant produced by applying Lemma 6.13 again. By picking
$$A = \left(\frac{C_\varepsilon}{2C'} \right)^4$$
  we conclude. $\square$

Theorem 6.20

The only zeroes of the ζ function in $\mathbb{C} \setminus \{s \in \mathbb{C} : 0 \leq \mathrm{Re}(s) < 1\}$ are the simple zeroes at $-2, -4, -6, \ldots$.

The zeroes at $-2, -4, -6, \ldots$ are called the **trivial zeroes** and the set $\{s \in \mathbb{C} : 0 \leq \mathrm{Re}(s) < 1\}$ is called the **critical strip**.

Proof. We have already proved that ζ does not vanish on $\{\mathrm{Re}(s) > 1\}$. When $\mathrm{Re}(s) < 0$, we have the following functional equation:
$$\zeta(s) = \pi^{s - \frac{1}{2}} \Gamma\left(1 - \frac{s}{2}\right) \Gamma\left(\frac{s}{2}\right) \zeta(1 - s)$$

$\frac{1}{\Gamma}$ is entire, so Γ does not vanish. $\pi^{s-\frac{1}{2}}$ is an exponential so it never vanishes. $\zeta(1 - s) \neq 0$ since $\mathrm{Re}(1 - s) > 1$. Then $\frac{1}{\Gamma}$ has simple zeroes at the negative integers, and thus $\zeta(s) = 0$ only at $-2, -4, -6, \ldots$.

Thus we just need to show that ζ is nonzero when $\mathrm{Re}(s) = 1$. Suppose that $\zeta(1 + it) = 0$ for some $t \in \mathbb{R} \setminus \{0\}$. We can draw a small disk around this point of radius $\varepsilon > 0$. Then ζ will be holomorphic and thus bounded, so for every $1 < \sigma < 1 + \varepsilon$,
$$|\zeta(\sigma + it)| \leq A(\sigma - 1)$$

At 1, ζ has a simple pole so
$$|\zeta(\sigma)| \leq \frac{B}{\sigma - 1}$$

We also have

$$|\zeta(\sigma + 2it)| \leq C$$

Then we can plug this into Corollary 6.18 to get

$$1 \geq \left(\frac{B}{\sigma - 1}\right)^3 (A(\sigma - 1))^4 C = B^3 A^4 C(\sigma - 1)$$

As $\sigma \to 1$ the RHS tends to zero, contradiction. So no zero exists. $\square$

We introduce the following auxiliary function to ease our proof of the prime number theorem.

Definition 6.6

The **Chebyshev ψ-function** is defined for $x > 0$ as

$$\psi(x) = \sum_{n \leq x} \Lambda(n)$$

Notice that some summand $\Lambda(n)$ is nonzero if and only if $n = p^m \leq x$ for $m \in \mathbb{N}, p \in \mathbb{P}$. Then we have

$$m \leq \frac{\ln x}{\ln p} = \log_p x \iff m \leq \left\lfloor \frac{\ln x}{\ln p} \right\rfloor$$

so we can reindex the nonzero terms by the prime that they are a power of. For each $p \leq x$, there will be $\left\lfloor \frac{\ln x}{\ln p} \right\rfloor$ powers less than x, and $\Lambda(p^m) = \ln p$ for each, so

$$\psi(x) = \sum_{p \leq x} \left\lfloor \frac{\ln x}{\ln p} \right\rfloor \ln p$$

The next lemma shows that it suffices to prove an asymptotic relation for ψ rather than π directly.

Lemma 6.21

If $\psi(x) \sim x$ as $x \to \infty$, then $\pi(x) \sim \frac{x}{\ln x}$ as $x \to \infty$.

(Note that the converse is true as well, but we do not prove this as it is not needed here.)

Proof. We write

$$\psi(x) = \sum_{p \leq x} \left\lfloor \frac{\ln x}{\ln p} \right\rfloor \ln p \leq \sum_{p \leq x} \ln x = \pi(x) \ln x$$

For any $0 < \alpha < 1$,

$$\psi(x) = \sum_{p \leq x} \left\lfloor \frac{\ln x}{\ln p} \right\rfloor \ln p \geq \sum_{p \leq x} \ln p \geq \sum_{x^\alpha < p \leq x} \ln p \geq \sum_{x^\alpha < p \leq x} \alpha \ln x$$
$$\geq \alpha \ln x(\pi(x) - \pi(x^\alpha)) \geq \alpha \ln x(\pi(x) - x^\alpha)$$

We rearrange terms and combine our inequalities to get

$$\frac{\psi(x)}{x} \leq \pi(x)\frac{\ln x}{x} \leq \frac{1}{\alpha}\frac{\psi(x)}{x} + \frac{\ln x}{x^{1-\alpha}}$$

Thus, using the assumption $\psi(x) \sim x$,

$$1 \leq \liminf_{x\to\infty} \psi(x) \leq \liminf_{x\to\infty}\left(\pi(x)\frac{\ln x}{x}\right) \leq \limsup_{x\to\infty}\left(\pi(x)\frac{\ln x}{x}\right) \leq \frac{1}{\alpha}$$

Then taking $\alpha \to 1$ shows that

$$\lim_{x\to\infty} \pi(x)\frac{\ln x}{x} = 1$$

□

Definition 6.7

For $x > 1$, we define the **smoothing function** ψ_1 to be

$$\psi_1(x) = \int_1^x \psi(u)\,\mathrm{d}u$$

Using our definition of ψ, this is equal to

$$\int_1^x \left(\sum_{n\leq u} \Lambda(n)\right)\mathrm{d}u = \int_1^x \left(\sum_{n=1}^{x} \Lambda(n) f_n(u)\right)\mathrm{d}u$$

where

$$f_n(u) = \begin{cases} 1, & u \geq n \\ 0 \end{cases}$$

Then we can rearrange this as

$$\sum_{n\leq x} \Lambda(n) \int_1^x f_n(u)\,\mathrm{d}u = \sum_{n\leq x} \Lambda(n)(x-n)$$

We again reduce to a simpler asymptotic with the following theorem:

Lemma 6.22

If $\psi_1(x) \sim \frac{1}{2}x^2$ as $x \to \infty$, then $\psi(x) \sim x$ as $x \to \infty$.

Proof. Suppose $\psi_1(x) \sim \frac{1}{2}x^2$. Then for any $0 < \alpha < 1 < \beta$, we use the fact that ψ is nondecreasing to write the following averages:

$$\frac{1}{(1-\alpha)x}\int_{\alpha x}^{x} \psi(u)\,\mathrm{d}u \leq \psi(x) \leq \frac{1}{(\beta-1)x}\int_{x}^{\beta x} \psi(u)\,\mathrm{d}u$$

The first term is

$$\frac{\psi_1(x) - \psi_1(\alpha x)}{(1-\alpha)x}$$

and the second is

$$\frac{\psi_1(\beta x) - \psi - 1(x)}{(\beta - 1)x}$$

Then

$$\frac{\psi_1(x) - \psi_1(\alpha x)}{(1-\alpha)x^2} \leq \frac{\psi(x)}{x} \leq \frac{\psi_1(\beta x) - \psi_1(x)}{(\beta - 1)x^2}$$

So

$$\frac{1}{1-\alpha}\left[\frac{\psi_1(x)}{\frac{1}{2}x^2}\frac{1}{2} - \frac{\psi_1(\alpha x)}{\frac{1}{2}(\alpha x)^2}\frac{\alpha^2}{2}\right]\frac{\psi(x)}{x} \leq \frac{1}{\beta - 1}\left[\frac{\psi_1(\beta x)}{\frac{1}{2}(\beta x)^2}\frac{\beta^2}{2} - \frac{\psi_1(x)}{\frac{1}{2}x^2}\frac{1}{2}\right]$$

Using the asymptotic assumption, we have

$$\limsup_{x\to\infty} \frac{\psi(x)}{x} = \frac{\beta^2 - 1}{2(\beta - 1)} = \frac{\beta + 1}{2}$$

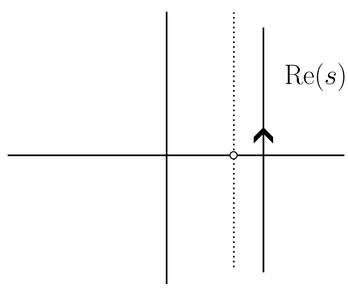
and similarly

$$\liminf_{x\to\infty} \frac{\psi(x)}{x} \geq \frac{\alpha + 1}{2}$$

Letting $\alpha, \beta \to 1$, $\psi(x) \sim x$. □

Thus our reduction work has simplified the problem to showing that the smoothing function obeys the asymptotic relation $\psi_1(x) \sim \frac{1}{2}x^2$. This will require the use of complex analysis tools.

We denote by $\mathrm{Re}(s) = c$ the following infinite curve, oriented so that the imaginary part ranges from $-\infty$ to ∞:

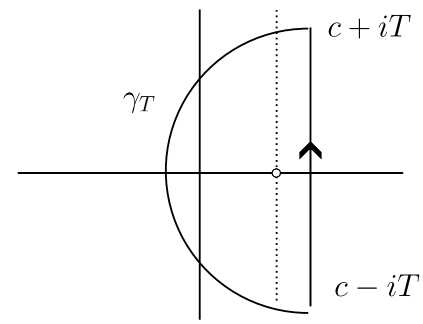


Lemma 6.23

For $a > 0$, $c > 1$

$$\frac{1}{2\pi i}\int_{\mathrm{Re}(s)=c} \frac{a^s}{s(s+1)}\,\mathrm{d}s = \begin{cases} 1 - \frac{1}{a}, & a > 1 \\ 0, & 0 < a \leq 1 \end{cases}$$

Proof. If $a > 1$, then denote $f(s) = \frac{a^s}{s(s+1)}$ and let $T > 1$. Then consider the curve γ_T given by:



The poles of f are $0, -1$, which lie in the interior of γ_T for sufficiently large T. Then by the residue theorem,

$$\frac{1}{2\pi i}\int_{\gamma_T} f(s)\,\mathrm{d}s = \mathrm{res}_0(f) + \mathrm{res}_1(f) = 1 - \frac{1}{a}$$

Then it suffices to show that the integral along the semicircular arc vanishes as $T \to \infty$. Denote this arc by C. Then for $s \in C$,

$$|f(s)| = \frac{a^{\mathrm{Re}(s)}}{|s||s+1|}$$

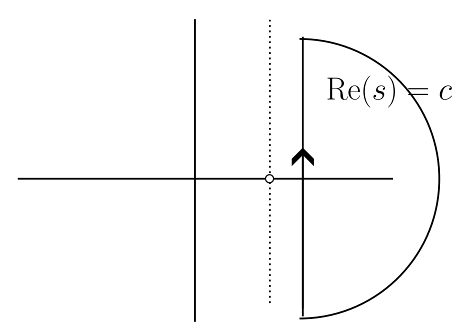
For $s \in \mathbb{C}$ we have $|s| \geq |s - c| - c = T - c$. Similarly $|s| \geq T - c - 1$.

$$|f(s)| \leq \frac{a^{\mathrm{Re}(s)}}{(T-c)(T-c-1)} \leq \frac{a^c}{(T-c)(T-c-1)}$$

Thus

$$\int_C f(s)\,\mathrm{d}s \leq \pi T \sup_{\zeta \in C} |f(\zeta)| \leq \pi T \frac{a^c}{(T-c)(T-c-1)} \xrightarrow{T\to\infty} 0$$

When $0 < a \leq 1$, we instead consider the right semicircular curve:



This curve contains no poles. A similar boudn shows that the integral over the arc is zero. So the integral in this case is zero. $\square$

Proposition 6.24

For $c, x \in \mathbb{R}$ with $c > 1, x > 0$,

$$\psi_1(x) = \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=c} \frac{x^{s+1}}{s(s+1)} \left(-\frac{\zeta'(s)}{\zeta(s)} \right) \mathrm{d}s$$

Proof. By Lemma 6.16, the integral is equal to

$$\frac{1}{\pi i} \int_{\mathrm{Re}(s)=c} \left(\frac{x^{s+1}}{s(s+1)} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \right) \mathrm{d}s$$

As we showedin Lemma 6.16, the series is absolutely convergent, so we rearrange this into

$$x \sum_{n=1}^{\infty} \Lambda(n) \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=c} \frac{\left(\frac{x}{n}\right)^s}{s(s+1)} \mathrm{d}s$$

By Lemma 6.23, the integrals are zero whenever $\frac{x}{n} > 1$. So only finitely many terms are nonzero:

$$x \sum_{n \le x} \Lambda(n) \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=c} \frac{\left(\frac{x}{n}\right)^s}{s(s+1)} \mathrm{d}s = x \sum_{n \le x} \Lambda(n) \left(1 - \frac{n}{x}\right) = \sum_{n \le x} \Lambda(n)(x-n) = \psi_1(x) \qquad \square$$

Now, consider the function

$$F(s) = \frac{x^{s+1}}{s(s+1)} \left(\frac{-\zeta'(s)}{\zeta(s)} \right)$$

Lemma 6.25

$\mathrm{res}_1 F = \frac{x^2}{2}$

Proof. Recall that we showed for $\mathrm{Re}(s) > 0$,

$$\zeta(s) = \frac{1}{s-1} + H(s)$$

where H is holomorphic on $\{\mathrm{Re}(s) > 0\}$. Then

$$\zeta'(s) = -\frac{1}{(s-1)^2} + H'(s)$$

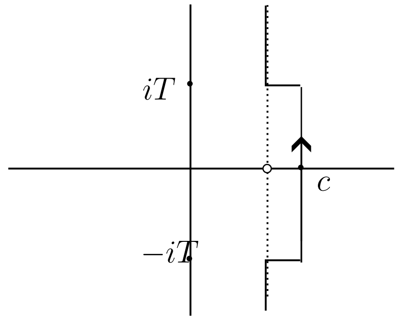
so

$$-\frac{\zeta'(s)}{\zeta(s)} = \frac{\frac{1}{(s-1)^2} - H'(s)}{\frac{1}{s-1} + H(s)} = \frac{1}{s-1} + \underbrace{\frac{(s-1)H'(s) - H(s)}{1 + (s-1)H(s)}}_{\text{holomorphic}}$$

Thus

$$\mathrm{res}_1 F = \frac{x^2}{2}$$

TODO: check this argument. $\square$

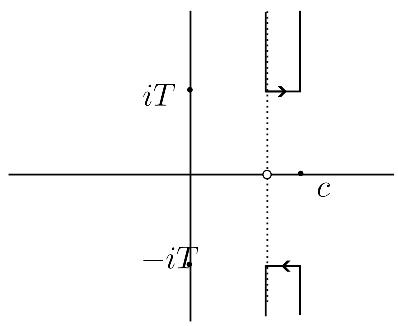


Let γ_T be the curve (set $c = 2$):

Lemma 6.26

$$\psi_1(x) = \frac{1}{2\pi i} \int_{\gamma_T} F(s)\,\mathrm{d}s$$

Proof. By cancellation, this is equal to the integral over the following curves:



Consider the upper curve (call it τ). The integral is given by

$$\int_\tau F(s)\,\mathrm{d}s = -i\int_T^\infty F(1+it)\,\mathrm{d}t + i\int_T^\infty F(2+it)\,\mathrm{d}t + \int_1^2 F(t+iT)\,\mathrm{d}t$$

We can pick $\varepsilon = \frac{1}{4}$ and apply Theorem 6.19 to get

$$|F(1+it)| \le \frac{x^2}{t^2}\frac{C_1(\varepsilon)|t|^\varepsilon}{C_2(\varepsilon)|t|^{-\varepsilon}} \le C_3(\varepsilon)\frac{x^2}{t^{2-2\varepsilon}}$$

so

$$\left|\int_T^\infty F(1+it)\,\mathrm{d}t\right| \le \int_T^\infty C_3(\varepsilon)\frac{x^2}{t^{2-2\varepsilon}}\,\mathrm{d}t = \frac{C(\varepsilon)}{1-2\varepsilon}x^2\frac{1}{T^{1-2\varepsilon}} \xrightarrow{T\to\infty} 0$$

The same bound works for

$$\left|\int_T^\infty F(2+it)\,\mathrm{d}t\right|$$

For the last integral, we bound by

$$|F(t+iT)| \leq \frac{x^{t+1}}{T^2} C(\varepsilon) M^{2\varepsilon} \overset{T\to\infty}{\longrightarrow} 0$$

and since the integral is from 1 to 2, it suffices to show that the integrand tends to 0. $\square$

Proposition 6.27

Fix $T > 1$. Then there exists $0 < \delta_T \leq \frac{1}{2}$ such that ζ has no zeroes in the set $\{\sigma + it : |t| \leq T, 1 - \delta_T \leq \sigma \leq T\}$.

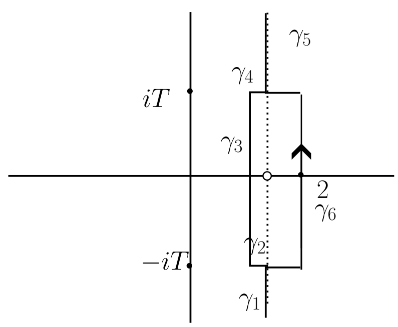
Proof. We apply a compactness argument. If no such δ_T exists then there is a sequence of zeroes with real part tending to 1. But we showed that there are no zeroes on the line $\mathrm{Re}(s) = 1$, so this is a contradiction. $\square$

We have now developed the ideas we need to conclude the proof of the prime number theorem.

Theorem 6.28

$\psi_1(x) \sim \frac{1}{2}x^2$.

Proof. Consider the curves:



Using the residue theorem,

$$\begin{aligned}\psi_1(x) - \frac{1}{2}x^2 &= \frac{1}{2\pi i}\int_{up+right+up+left+up} F(s)\,\mathrm{d}s - \frac{1}{2\pi i}\int_{box} F(s)\,\mathrm{d}s \\ &= \frac{1}{2\pi i}\left(\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} + \int_{\gamma_4} + \int_{\gamma_5}\right) F(s)\,\mathrm{d}s\end{aligned}$$

γ_1, γ_5 may be estimated the same way by symmetry. For $s = 1 + it$ and $t \geq T$, we have

$$|F(1+it)| = \frac{x^2}{t^2} C\sqrt{t} = C\frac{x^2}{t^{\frac{3}{2}}}$$

so

$$\left|\int_{\gamma_5} F(s)\,\mathrm{d}s\right| \le \int_T^\infty C\frac{x^2}{t^{\frac{3}{2}}}\,\mathrm{d}t = 2C\frac{x^2}{\sqrt{T}}$$

The same bound holds for γ_1. On γ_3, we have $s = 1 - \delta_T + it$ for $|t| \le T$. Then

$$|F(1-\delta_T+it)| \le \frac{x^{2-\delta_T}}{\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)} \underbrace{\max_{\zeta\in\gamma_3}\left|\frac{\zeta'(s)}{\zeta(s)}\right|}_{M_T} = 2M_T x^{2-\delta_T}$$

so

$$\left|\int_{\gamma_3} F(s)\,\mathrm{d}s\right| \le 4M_T x^{2-\delta_T}$$

On γ_4, we have $s = \sigma + it$ for $1 - \delta_T \le \sigma \le 1$.

$$|F(\sigma+it)| \le \frac{x^{\sigma+1}}{T^2} \underbrace{\max_{\zeta\in\gamma_4}\left|\frac{\zeta'(s)}{\zeta(s)}\right|}_{M_T'}$$

giving

$$\left|\int_{\gamma_4} F(s)\,\mathrm{d}s\right| \le \frac{M_T'}{T^2}x\int_{1-\delta_T}^1 e^{\sigma\ln x}\,\mathrm{d}\sigma = \frac{M_T'}{T^2}\frac{x^2}{\ln x}$$

The same bound holds for γ_2. So

$$\begin{aligned}\left|\psi_1(x) - \frac{x^2}{2}\right| &\le \frac{1}{2\pi}\left(\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} + \int_{\gamma_4} + \int_{\gamma_5}\right)|F(s)|\,\mathrm{d}s \\ &\le 4C\frac{x^2}{\sqrt{T}} + 4TM_T x^{2-\delta_T} + \frac{2M_T'}{T^2}\frac{x^2}{\ln x}\end{aligned}$$

Thus

$$\left|\frac{\psi_1(x)}{\frac{1}{2}x^2} - 1\right| = \frac{\left|\psi_1(x) - \frac{x^2}{2}\right|}{\frac{x^2}{2}} \le \frac{8C}{\sqrt{T}} + \frac{8TM_T}{x^{\delta_T}} + \frac{4M_T'}{T^2}\frac{1}{\ln x}$$

As $x \to \infty$ the x-dependent terms vanish and we are left with

$$\limsup_{x\to\infty}\left|\frac{\psi_1(x)}{\frac{1}{2}x^2} - 1\right| \le \frac{2C}{\sqrt{T}}$$

This is true for all T, so we conclude that

$$\lim_{x\to\infty}\left|\frac{\psi_1(x)}{\frac{1}{2}x^2} - 1\right| = 0$$

so $\psi_1(x) \sim \frac{1}{2}x^2$. $\square$

Definitions