GEO 441 Notes

Max Chien

Fall 2025

Contents

Introduction

This document contains notes taken for the class GEO 441: Computational Geophysics at Princeton University, taken in the Spring 2025 semester. These notes are primarily based on lectures by Professor Jeroen Tromp. This class covers finite-difference, finite-element, and spectral methods for numerical solutions to the wave and heat equations. Since these notes were primarily taken live, they may contains typos or errors.

Chapter 1

Continuum Mechanics and the Equations of Motion

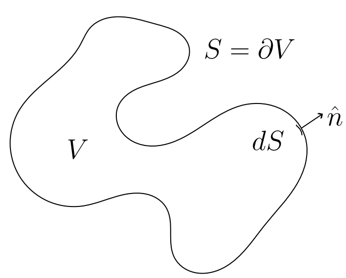
In this class, we will primarily focus on the wave and heat equations, which are important in the study of geophysics, and more broadly, continuum mechanics. As such, we will begin with an introduction to basic continuum mechanics to better understand the role of the differential equations we study.

Continuum mechanics are primarily governed by four conservation laws:

1. Conservation of mass,
2. Conservation of linear momentum,
3. Conservation of angular momentum,
4. Conservation of energy.

The wave and heat equations arise as a result of (2) and (4), respectively, but in actual applications it is often the case that coupled systems of conservation laws must be solved.

1.1 Conservation of Mass



We consider a "comoving volume" V. By "comoving volume", one can imagine a bag of some fluid mass deposited in a river, which can be deformed as it moves, but nevertheless maintains a constant mass throughout. We also denote the surface of V by $S = \partial V$, and for small surface elements $\mathrm{d}S$ we denote the unit outward normal vector by $\hat{n}$.[1]

We also adopt the Einstein summation convention, in which repeated indices that are not otherwise used are implied to be summed over:

$$\vec{u} = u^i e_i$$

If we consider a change of basis to some new basis $\{e'_1, e'_2\}$, this can then be written as

$$\vec{u} = u^{i'} e'_i$$

where $u^{i'}$ denotes the ith component of $\vec{u}$ in the new basis.

While $\vec{u}$ is invariant under change of basis, the components are of course not. The way that they transform under change of basis is given by the change of basis matrix $\boldsymbol{\lambda}$, and this relationship is expressed under Einstein summation notation by

$$u^i = \lambda^i_{i'} u^{i'}$$
$$e_i = \lambda^i_{i'} e'_i$$

The reverse transformation may be denoted by $\boldsymbol{\Lambda}$. The fact that they are inverses may be expressed by the equation

$$\lambda^i_{i'} \Lambda^{i'}_j = \delta^i_j$$

where δ^i_j is the Kronecker delta (in coordinates, the RHS is the identity matrix). This then allows us to express the reverse relationships for change of basis:

$$u^{i'} = \Lambda^{i'}_i u^i$$
$$e'_i = \Lambda^{i'}_i e_i$$

Now, to formalize the notion of the mass of V, we first consider the mass density, considered as a function $\rho(\vec{x}, t)$ of both space and time (with respect to some coordinate system). For an infinitesimal volume element $\mathrm{d}V$, the mass of the volume is given by $\rho\,\mathrm{d}V$. Notice that the dimensions of mass density is

$$[\rho] = \frac{\mathrm{kg}}{\mathrm{m}^3}$$

so that the dimensions of mass are

$$[\rho]\,[\mathrm{d}V] = \mathrm{kg}$$

More generally, the mass of V is given by integrating against mass density,

$$M = \int_V \rho\,\mathrm{d}V$$

[1] In this course we adopt the convention that a vector is denoted by $\vec{v}$, a unit vector by $\hat{v}$, and the ith component of a vector by v_i or v^i. (The distinction is the distinction between covariant and contravariant indices, but is not necessary for this course). Moreover, we denote the standard basis vectors in the x and y directions by $e_x = \hat{x}$ and $e_y = \hat{y}$, respectively.

In Cartesian coordinates this is

$$M = \int_V \rho(x, y, z, t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

Notice that the integrand is time dependent. Moreover, we allow V to deform over time as well, so that this equation might be more appropriately written as

$$M(t) = \int_{V(t)} \rho(x, y, z, t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

Then the conservation of mass law is expressed as the ODE

$$0 = \frac{\mathrm{d}M}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \rho\,\mathrm{d}V$$

If V is constant (that is, if we allow for no deformation), then Feynman's trick give us

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \int_V \frac{\partial \rho}{\partial t}\,\mathrm{d}V$$

However, because V is time-dependent, this fails to hold. Instead, we first appeal to the single-dimensional case by considering Leibniz's rule, which handles integration with time-dependent limits and integrand of the form

$$I(t) = \int_{a(t)}^{b(t)} f(x, t)\,\mathrm{d}x$$

In this case, by considering I as the area under the curve, it is clear that (at least for continuous a, b) the value $\frac{\mathrm{d}I}{\mathrm{d}t}$ must take into account both the values $\frac{\partial f}{\partial t}|_{[a,b]}$, but also the area which is added or removed by the change in a, b.

Theorem 1.1: Leibniz's Rule

Let $f(x, t)$ be jointly continuous with $\frac{\partial}{\partial t} f(x, t)$ also jointly continuous in some region given by $a(t) \leq x \leq b(t)$, $t_0 \leq t \leq t_1$. If a, b are both continuously differentiable, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{a(t)}^{b(t)} f(x, t)\,\mathrm{d}x\right) = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}(x, t)\,\mathrm{d}x + f(b(t), t)\frac{\mathrm{d}b}{\mathrm{d}t}(t) - f(a(t), t)\frac{\mathrm{d}a}{\mathrm{d}t}(t)$$

This can be derived using the limit formulation of the derivative by writing

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\int_{a(t+\Delta t)}^{b(t+\Delta t)} f(x, t+\Delta t)\,\mathrm{d}x - \int_{a(t)}^{b(t)} f(x, t)\,\mathrm{d}x\right]$$

As a first order approximation for the change in area if the integration limits are constant, Feynman's rule holds and we have

$$\int_{a(t)}^{b(t)} \frac{1}{\Delta t}\lim_{\Delta t \to 0}\left[f(x, t+\Delta t) - f(x)\right]\mathrm{d}x + O((\Delta t)^2) = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}(x, t)\,\mathrm{d}x$$

At the upper limit, f is also near constant, so the change in area is approximated to first order by

$$f(b(t),t)\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[b(t+\Delta t)-b(t)\right]=f(b(t),t)\frac{\mathrm{d}b}{\mathrm{d}t}(t)$$

The lower limit is similar with a negative sign. Combining the three approximations, we get

$$\frac{\mathrm{d}I}{\mathrm{d}t}=\int_{a(t)}^{b(t)}\frac{\partial f}{\partial t}(x,t)\,\mathrm{d}x+f(b(t),t)\frac{\mathrm{d}b}{\mathrm{d}t}(t)-f(a(t),t)\frac{\mathrm{d}a}{\mathrm{d}t}(t)$$

Now, we return to the case of our comoving volume. Taking inspiration from Leibniz's rule, the main term that we have to adjust in the 2-dimensional case is the change in boundary area. This is approximated by considering the volume over which a surface element moves within an infinitesimal time interval.

For a given surface element $\mathrm{d}S(t)$, we consider both the associated normal $\hat{n}(t)$ and the velocity vector $\vec{v}$. Then the component of the velocity of $\mathrm{d}S(t)$ in the normal direction is given by

$$\vec{v}\cdot\hat{n}(t)=v^i(t)n^i(t)$$

Note that, as usual we also define the length of u by

$$\|\vec{u}\|^2=(u^i)^2$$

Now, the flux of mass through $\mathrm{d}S(t)$ in the period $[t,t+\Delta t]$ is then

$$\rho|_{\mathrm{d}S(t)}\vec{v}\cdot\hat{n}$$

Then we can now include the correct error term to calculate $\frac{\mathrm{d}M}{\mathrm{d}t}$:

$$\frac{\mathrm{d}M}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)}\rho\,\mathrm{d}V=\int_{V(t)}\frac{\partial\rho}{\partial t}\,\mathrm{d}V+\int_{S(t)}\rho\vec{v}\cdot\hat{n}\,\mathrm{d}S$$

(where S is equipped with the outward-facing orientation). Lastly, we can replace the second term with an integral over $V(t)$ using the divergence theorem:

$$\int_S\vec{u}\cdot\hat{n}\,\mathrm{d}S=\int_V\nabla\cdot\vec{u}\,\mathrm{d}V$$

We combine the integrals:

$$\frac{\mathrm{d}M}{\mathrm{d}t}=\int_{V(t)}\left[\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\vec{v})\right]\mathrm{d}V$$

Note that the divergence is taken against $\rho\vec{v}$, since this is the quantity which is dotted against $\hat{n}$.

Because the integral must be zero for all possible V, the integrand is identically zero. Thus we express the conservation of mass law for a comoving volume (also known as the **continuity equation**) by

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\vec{v})=0$$

We can expand this using summation notation as

$$\partial_t \rho + v^i \nabla_i \rho + \rho \nabla_i v^i = 0$$

The first two terms $\partial_t \rho + v^i \nabla_i \rho$ is known as the **material derivative**

$$D_t \rho = \partial_t \rho + \vec{v} \cdot \nabla \rho$$

where the first term is the local change in density, and the second is the advection term (which is the directional derivative of the density in the direction of velocity). In other words, the rate of change of local mass along a path is given by the pointwise rate of change together with the change given by the motion of the path against the gradient. We then rephrase the continuity equation as

$$D_t \rho + \rho \nabla \cdot \vec{v} = 0$$

or equivalently

$$\frac{1}{\rho} D_t \rho = -\nabla \cdot \vec{v}$$

This essentially says that the relative change in density along a path is the negative of the velocity divergence. This makes sense because when divergence is positive, mass is moving away and density decreases, while density increases with velocity divergence is negative. In particular, if the mass is incompressible, $\nabla \cdot \vec{v} = 0$, so that density is constant along any path. In this case, we don't need to worry about conservation of mass.

1.2 Conservation of Linear Momentum

Linear momentum is given by the product of mass with velocity. In continuum mechanics this is given by $\rho \vec{v} \, \mathrm{d}V$. Thus the total momentum of a volume is simply

$$p = \int_{V(t)} \rho \vec{v} \, \mathrm{d}V$$

The statement of conservation of linear momentum is essentially that the only way to change linear momentum is to apply (external) forces to our volume. This is basically Newton's second law, written as $\vec{F} = \dot{p}$. One can consider a body force $\vec{f}$ which pulls on small volume elements $\mathrm{d}V$. We can also consider forces $\vec{t}$ which act only on the surface of the volume. Thus we write

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{V(t)} \rho \vec{v} \, \mathrm{d}V = \int_{V(t)} f \, \mathrm{d}V + \int_{S(t)} \vec{t} \, \mathrm{d}S$$

We can differentiate the left hand side the same way as we did in the conservation mass equation:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{V(t)} \rho \vec{v} \, \mathrm{d}V = \int_{V(t)} \partial_t (\rho \vec{v}) \, \mathrm{d}V + \int_{S(t)} (\rho \vec{v}) \hat{n} \cdot \vec{v} \, \mathrm{d}S$$

To conceptualize the surface-acting forces, we consider the **stress tensor**, which is a rank 2 tensor (or matrix) $\mathbf{T}$ such that $\mathbf{T} \cdot \vec{n}$ gives the traction force on $\mathrm{d}S$, if the unit outward normal of $\mathrm{d}S$ is $\hat{n}$. In indices, this is

$$t_i = T_{ij} \hat{n}_j \, \mathrm{d}S$$

(Note that in general $T_{ij}\hat{n}_j \neq \hat{n}_j T_{ji}$, but this is true if $\mathbf{T}$ is a symmetric tensor). Then the right hand side of our equation is writen as

$$\int_{V(t)} \vec{f}\,\mathrm{d}V + \int_{S(t)} \mathbf{T}\cdot\hat{n}\,\mathrm{d}S$$

Once again we use the divergence theorem to convert these to volume integrals, so that our equation is given by

$$\int_{V(t)} \left[\partial_t(\rho\vec{v}) + \nabla\cdot(\rho\vec{v}\otimes\vec{v})\right]\mathrm{d}V = \int_{V(t)} \left[\vec{f} + \nabla\cdot\mathbf{T}\right]\mathrm{d}V$$

Note that both integrals are vector quantities. In components, the integrand on the left can be given by

$$\partial_t \rho v^i + \nabla_j \cdot \left(\rho v^i v^j\right)$$

(By convention, the divergence theorem is written in indices as $\int_S u^i \hat{n}_i\,\mathrm{d}S = \int_V \nabla_i u^i\,\mathrm{d}V$).

Similarly, the divergence of $\mathbf{T}$ is given by contracting the gradient against the last index of $\mathbf{T}$, so that the integrand on the right is given in indices by

$$f^i + \nabla_j \cdot T^{ij}$$

Equating the integrands again, the conservation of linear momentum law is thus given by

$$\partial_t(\rho\vec{v}) + \nabla\cdot(\rho\vec{v}\otimes\vec{v}) = \nabla\cdot\mathbf{T} + \vec{f}$$

Some equivalent formulations are

$$\partial_t(\rho\vec{v}) = \nabla\cdot(\mathbf{T} - \rho\vec{v}\otimes\vec{v}) + \vec{f}$$
$$\partial_t(\rho\vec{v}) + \nabla\cdot(\rho\vec{v}\otimes\vec{v} - \mathbf{T}) = \vec{f}$$

The last formulation is the Eulerian form, which expresses the conservation law as the pointwise time derivative of a quantity plus its flux being equated to the source term.

Expressing this with the chain rule gives

$$(\partial_t\rho)\vec{v} + \rho\partial_t\vec{v} + \nabla\cdot(\rho\vec{v})\vec{v} + \rho\vec{v}\nabla\cdot\vec{v} = \nabla\cdot\mathbf{T} + \vec{f}$$

The first and third term are zero by conservation of mass. Thus this is equivalent to

$$\rho\left(\partial_t\vec{v} + \vec{v}\nabla\cdot\vec{v}\right) = \nabla\cdot\mathbf{T} + \vec{f}$$

The parenthetical term is again the material derivative, this time of velocity, so this is

$$\rho D_t\vec{v} = \nabla\cdot\mathbf{T} + \vec{f}$$

As formulated, the coupling of the conservation of mass and momentum laws gives four scalar equations. Even if body forces are given, this leaves as unknowns the mass density, velocity, and stress tensor. Thus we need constitutive relationships, which express some of these (particularly the stress tensor) in terms of the others in order to solve these. This makes sense given that the actual results will depend on material properties, which are specified in the stress tensor but nowhere else.

To do this, we consider stress and strain. Fix some origin point and let $\vec{x}$ denote the starting point of some particle. Let $\vec{r}(\vec{x}, t)$ denote the position of particle $\vec{x}$ at time t. By definition $\vec{r}(\vec{x}, 0) = \vec{x}$. Define $\vec{s}(\vec{x}, t) = \vec{r}(\vec{x}, t) - \vec{x}$ to be the displacement vector. Suppose we consider two initially neighboring particles $\vec{x}, \vec{x} + \mathrm{d}\vec{x}$. As time progresses, their displacement becomes $\mathrm{d}\vec{r} = \vec{r}(\vec{x} + \mathrm{d}\vec{x}, t) - \vec{r}(\vec{x}, t)$. We take the first order Taylor expansion:

$$\mathrm{d}\vec{r} \approx \vec{r}(\vec{x}, t) + \mathrm{d}\vec{x} \cdot \nabla \vec{r}(\vec{x}, t) - \vec{r}(\vec{x}, t) = \mathrm{d}x^i \nabla_i \vec{r} = \mathrm{d}\vec{x} \cdot \nabla \vec{r}$$

We can express this as a tensor by

$$\nabla_j r^i \,\mathrm{d}x^j = F^i_j \,\mathrm{d}x^j$$

where $F^i_j = \nabla_j r^i$, or equivalently $\mathbf{F} = (\nabla \vec{r})^T$. The tensor $\mathbf{F}$ is known as the **deformation gradient**. Recalling that $\vec{r} = \vec{x} + \vec{s}$, we have

$$F = [\nabla(\vec{x} + \vec{s})]^T = [\nabla \vec{x} + \nabla s]^T$$

Since $\nabla \vec{x}$ is taken against $\vec{x}$ itself, its matrix formulation is just the identity:

$$\mathbf{I} = \nabla \vec{x} = \hat{x} \otimes \hat{x} + \hat{y} \otimes \hat{y} + \hat{z} \otimes \hat{z} = (\delta_{ij})$$

In summary, we can write

$$\mathbf{F} = \mathbf{I} + (\nabla \vec{s})^T$$

which is the identity plus the transpose of the displacement gradient. Physically, the displacement gradient represents the separation or convergence of material, or equivalently the deviation from uniform motion. Noting that

$$\mathrm{d}\vec{r} = \mathbf{F} \cdot \mathrm{d}\vec{x}$$

we have

$$\mathrm{d}\vec{r} = \left[\mathbf{I} + (\nabla \vec{s})^T\right] \mathrm{d}\vec{x} = \mathrm{d}\vec{x} + (\nabla \vec{s})^T \cdot \mathrm{d}\vec{x}$$

In general the tensor may not be symmetric; however we can always decompose a matrix into its symmetric and antisymmetric parts as

$$\mathbf{A} = \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^T\right) + \frac{1}{2}\left(\mathbf{A} - \mathbf{A}^T\right)$$

In particular, we can write F as

$$\mathbf{F} = \mathbf{I} + \boldsymbol{\varepsilon} + \boldsymbol{\omega}$$

where $\boldsymbol{\varepsilon}, \boldsymbol{\omega}$ are the symmetric and antisymmetric parts of $(\nabla \vec{s})^T$, respectively. $\boldsymbol{\varepsilon}$ is called the **strain** and $\boldsymbol{\omega}$ the **vorticity**. In other words, $\boldsymbol{\varepsilon}$ denotes the linear deviation from uniform displacement, or the linear deformation, and $\boldsymbol{\omega}$ denotes the twisting component.

Note that this implies the following:

$$\mathrm{tr}(\boldsymbol{\omega}) = 0$$
$$\mathrm{tr}(\boldsymbol{\varepsilon}) = \mathrm{tr}(\nabla \vec{s}) = \nabla \cdot \vec{s}$$

So $\mathrm{tr}(\boldsymbol{\varepsilon})$ can be seen to measure the local density or volume change.

It is shown in homework that we can calculate

$$\boldsymbol{\omega} \cdot \mathrm{d}\vec{x} = \frac{1}{2}\left(\nabla \times \vec{s}\right) \times \mathrm{d}\vec{x}$$

Thus we have

$$\mathrm{d}\vec{r} = \mathbf{F} \cdot \mathrm{d}\vec{x} = (\mathbf{I} + \boldsymbol{\varepsilon} + \boldsymbol{\omega}) = \mathrm{d}\vec{x} + \boldsymbol{\varepsilon} \cdot \mathrm{d}\vec{x} + \boldsymbol{\omega} \cdot \mathrm{d}\vec{x}$$

Which essentially says that final change in position differs by original change position by some linear strain component $\boldsymbol{\varepsilon} \cdot \mathrm{d}\vec{x}$, and rotationally by $\boldsymbol{\omega} \cdot \mathrm{d}\vec{x}$.

Based on this physical interpretation, it is clear that applying a stress force on the exterior of a body should impart a strain on the interior. If we relate the two, this can help us reduce the dimensionality of our PDE. One possible assumption is **Hooke's law**, which postulates that this is a linear relationship.

In other words, for each component T_{ij} of the stress tensor, there should be coefficients $a_{ij}, \ldots, f_{ij}$ such that

$$T_{ij} = a_{ij}\varepsilon_{11} + b_{ij}\varepsilon_{12} + \ldots + f_{ij}\varepsilon_{33}$$

(Note that there are only six degrees of freedom since $\boldsymbol{\varepsilon}$ is symmetric). These coefficients can be collected in a fourth-order tensor called the **elastic tensor**. This is summarized as

$$T_{ij} = c_{ijkl}\varepsilon_{kl}$$

Returning to the conservation of momentum law and substitute this relationship, we have

$$\rho\partial_t^2\vec{s} = \nabla \cdot (\mathbf{c} : \boldsymbol{\varepsilon}) + \vec{f}$$

or in components,

$$\rho\partial_t^2 s^i = \nabla_j \cdot \left(c^{ijkl}\varepsilon_{kl}\right) + f^i$$

A priori, we have not really reduced the dimensionality, since $\mathbf{c}$ has 81 components. However, conservation of angular momentum forces the stress tensor to be symmetric, and conservation of energy gives symmetricity across the first two and last two indices. This reduces the number of independent components to 21.

We now investigate further the importance of Hooke's law in developing the wave equation. Consider the one-dimensional case of Hooke's law, which can be imagined by a spring of length L and spring constant k. If it is given an initial displacement $\Delta\ell$, then Hooke's law says that the spring force is given by $F = k\Delta L$.

By graphing the displacement at $t = 0$ against the position along the spring x, the displacement linearly increases from 0 to $\Delta\ell$. In other words,

$$s(x) = \frac{\Delta\ell}{L}x$$

so the strain is given by

$$\varepsilon = \frac{\mathrm{d}}{\mathrm{d}x}s = \frac{\Delta\ell}{L}$$

which is therefore constant along the spring. Then the stress is linear in ε, so that

$$\sigma = \mu\varepsilon = \mu\frac{\Delta\ell}{L} = \frac{\mu}{kL}F$$

Relating this back to stress forces, the force against a unit area is the stress force:

$$\sigma = \frac{F}{A}$$

so that

$$\sigma = \frac{\mu A}{kL}\sigma$$

or

$$\mu = \frac{kL}{A}$$

1.3 The 1D Wave Equation

Here we develop the one-dimensional wave equation PDE as a consequence of conservation of linear momentum. Imagine a horizontal string of length L, and suppose that the string experiences perpendicular displacement given by $\vec{s}(x,t) = s(x,t)\hat{y}$ in the vertical direction. The gradient of $\vec{s}$ is given by

$$\nabla\vec{s} = \hat{x} \otimes \hat{y}\partial_x s$$

The strain is the symmetric part, which is therefore given by

$$\boldsymbol{\varepsilon} = \frac{1}{2}\partial_x s\left(\hat{x} \otimes \hat{y} + \hat{y} \otimes \hat{x}\right)$$

We apply Hooke's law to linearly relate stress and strain:

$$\mathbf{T} = 2\mu\boldsymbol{\varepsilon} = T_{xy}\left(\hat{x} \otimes \hat{y} + \hat{y} \otimes \hat{x}\right)$$

(The factor of 2 is conventional). It is thus clear that $\mathbf{T}$ has to be symmetric in the 1D case under Hooke's law, so that

$$T_{xy} = \mu\partial_x s$$

To calculate the divergence of $\mathbf{T}$, we have

$$\nabla \cdot \mathbf{T} = \partial_x\left(\mu\partial_x s\right)\hat{y}$$

Since the acceleration is also vertical, it is given by

$$\rho\partial_t^2\vec{s} = \rho\partial_t^2 s\hat{y}$$

Plugging this into the conservation of momentum equation, we get

$$\rho\partial_t^2 s = \partial_x\left(\mu\partial_x s\right)$$

Note that a priori we allow the **shear modulus** μ to vary over the string. However, if it is constant, then we can conclude

$$\partial_t^2 s = \beta^2 \partial_x^2 s$$
$$\beta = \sqrt{\frac{\mu}{\rho}}$$

where β is the shear wave speed.

Let us now consider the propagation of sound waves through fluids. In a fluid, the traction must be perpendicular to the surface, so that

$$\vec{t} \sim -p\hat{n}\,\mathrm{d}S$$

where p is the pressure. For an isotropic fluid, the forces are the same in all directions and only governed by pressure, so that the stress tensor can be written as

$$\mathbf{T} = -p\mathbf{I}$$

In a fluid, the pressure fluctuations are thus governed by the strain

$$\operatorname{tr} \boldsymbol{\varepsilon} = \nabla \cdot \vec{s}$$

This is expressed as

$$p = -\kappa \nabla \cdot \vec{s}$$

where κ is the **bulk modulus** or incompressibility of the fluid.

Under isotropy, the stress is completely governed by the shear modulus and bulk modulus, which reduces from 81 to 2 parameters. Hooke's law can be written as

$$\mathbf{T} = \kappa \operatorname{tr}(\boldsymbol{\varepsilon})\mathbf{I} + 2\mu \mathbf{d}$$

where $\mathbf{d}$ is the deviatoric strain tensor, which is essentially the traceless part of the strain: $\mathbf{d} = \boldsymbol{\varepsilon} - \frac{1}{3}\operatorname{tr}(\boldsymbol{\varepsilon})\mathbf{I}$.

Using these relations, we have

$$\rho \partial_t^2 \vec{s} = -\nabla p$$

Or alternately,

$$\partial_t^2 \vec{s} = -\frac{1}{\rho}$$
$$\partial_t^2 \nabla \cdot \vec{s} = -\nabla \cdot \left(\frac{1}{\rho \nabla p}\right)$$
$$\frac{1}{\kappa}\partial_t^2 p = \nabla \cdot \left(\frac{1}{\rho}\nabla p\right)$$

Under constant density assupmtions, we have the **acoustic wave equation**

$$\partial_t^2 p = c^2 \partial_x^2 p$$

where

$$c = \sqrt{\frac{\kappa}{\rho}}$$

is the sound wave speed.

1.4 Conservation of Angular Momentum

Since a small quantity of linear momentum can be calculated as $\rho\vec{v}\,\mathrm{d}V$, a small quantity of angular momentum is given by $\vec{r}\times\rho\vec{v}\,\mathrm{d}V$. The total angular momentum of a comoving volume is therefore

$$\int_{V(t)} \vec{r}\times\rho\vec{v}\,\mathrm{d}V$$

As with linear momentum, we can express the changes in angular momentum as a sum of body torques and surface torques:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \vec{r}\times\rho\vec{v}\,\mathrm{d}V = \int_{V(t)} \vec{r}\times\vec{f}\,\mathrm{d}V + \int_{S(t)} \vec{r}\times\vec{t}\,\mathrm{d}S$$

We proceed on the left side as before, differentiating and applying the divergence theorem:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \vec{r}\times\rho\vec{v}\,\mathrm{d}V = \int_{V(t)} \left[\partial_t\left(\vec{r}\times\rho\vec{v}\right) + \nabla\cdot\left(\rho\vec{r}\times\vec{v}\otimes\vec{v}\right)\right]\mathrm{d}V$$

On the right, we have

$$\int_{V(t)} \vec{r}\times\vec{f}\,\mathrm{d}V + \int_{S(t)} \vec{r}\times\vec{t}\,\mathrm{d}S = \int_{V(t)} \left[\vec{r}\times\vec{f} + \nabla\cdot\left(\vec{r}\times\mathbf{T}\right)\right]\mathrm{d}V$$

(Note that $\vec{r}\times\mathbf{T}$ should be interpreted as taking the cross product against the first index of $\mathbf{T}$.) On the left hand side, there is a $\partial_t\rho + \nabla\cdot(\rho\vec{v})$ term, which is zero by conservation of lienar momentum. Thus the final expression of conservation of angular momentum is

$$\rho D_t(\vec{r}\times\vec{v}) - \nabla\cdot(\vec{r}\times\mathbf{T}) = \vec{r}\times\vec{f}$$

For the material derivative of angular velocity, we note that $D_t\vec{r} = \vec{v}$, so that we can pull the $\vec{r}\times$ outside:

$$\vec{r}\times\left(\rho D_t\vec{v} - \nabla\cdot\mathbf{T} - \vec{f}\right) - \boldsymbol{\epsilon}:\mathbf{T} = \vec{0}$$

where $\boldsymbol{\varepsilon}$ is the rank three alternating tensor. By conservation of linear momentum,

$$\rho D_t\vec{v} = \nabla\cdot\mathbf{T} + \vec{f}$$

so that the entire first term vanishes. Thus we conclude that

$$\boldsymbol{\epsilon}:\mathbf{T} = \vec{0}$$

In other words, this tells us that

$$\epsilon_{ijk}T_{kj} = 0$$

for all i, which implies that $T_{23} = T_{32}$ and similarly $T_{13} = T_{31}, T_{12} = T_{21}$. Therefore under conservation of angular momentum and linear momentum, the stress tensor has to be symmetric. Thus the elastic tensor c_{ijkl} is symmetric in i,j, as well as k,l since ε is also symmetric. Thus we have 6 independent components in i,j and 6 in k,ℓ, so there are 36 independent components.

1.5 Conservation of Energy

Our final conservation law is conservation of energy, which leads to the heat equation.

The principle of conservation of energy essentially says that energy content is changed by work done. Considering again our comoving volume V. For any point particle with mass $\rho\,\mathrm{d}V$, the kinetic energy is $\frac{1}{2}\rho\,\mathrm{d}V\|\vec{v}\|^2$, so that the total kinetic energy is given by

$$KE = \frac{1}{2}\int_{V(t)} \rho\|\vec{v}\|^2\,\mathrm{d}V$$

To calculate the internal or potential energy term, simply consolidate all the internal energies into a term $\rho U\,\mathrm{d}V$, where U is the potential energy per unit mass. This gives

$$PE = \int_{V(t)} \rho U\,\mathrm{d}V$$

Therefore the total energy is expressed as

$$E = \int_{V(t)} \rho\left(\frac{1}{2}\|\vec{v}\|^2 + U\right)\mathrm{d}V$$

As with the previous conservation laws, we can relate the relate of change of the energy to the forces applied to our volume. In this case, work is calculated by considering the forces as

$$\int_{V(t)} \vec{v}\cdot\vec{f}\,\mathrm{d}V + \int_{S(t)} \vec{v}\cdot\vec{t}\,\mathrm{d}S$$

However, we also need to consider internal production of heat within the volume, for instance due to radioactivity. Similarly we need to consider heat fluxes out of the volume. These terms are given by

$$\int_{V(t)} h\,\mathrm{d}V - \int_{S(t)} \vec{H}\cdot\hat{n}\,\mathrm{d}S$$

where $\vec{H}$ is the heat flux out of V. Thus we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \rho\left(\frac{1}{2}\|\vec{v}\|^2 + U\right)\mathrm{d}V = \int_{V(t)} \vec{v}\cdot\vec{f}\,\mathrm{d}V + \int_{S(t)} \vec{v}\cdot(\mathbf{T}\cdot\hat{n})\,\mathrm{d}S + \int_{V(t)} h\,\mathrm{d}V - \int_{S(t)} \vec{H}\cdot\hat{n}\,\mathrm{d}S$$

Applying the same strategy as before, we have

$$\begin{aligned}\int_{V(t)} \rho D_t\left(\frac{1}{2}\|\vec{v}\|^2 + U\right)\mathrm{d}V &= \int_{V(t)} \left[\vec{v}\cdot\vec{f} + h + \nabla\cdot\left(\vec{v}\cdot\mathbf{T} - \vec{H}\right)\right]\mathrm{d}V \\ \implies \rho D_t\left(\frac{1}{2}\|\vec{v}\|^2 + U\right) + \nabla\cdot\left(\vec{H} - \vec{v}\cdot\mathbf{T}\right) &= h + \vec{v}\cdot\vec{f}\end{aligned}$$

Recall that one formulation of conservation of linear momentum was $\rho D_t\vec{v} - \nabla\cdot\mathbf{T} = \vec{f}$. Thus

$$\vec{v}\cdot\left(\rho D_t\vec{v} - \nabla\cdot\mathbf{T} - \vec{f}\right) = 0$$

Removing these terms from the equation, we get

$$\rho D_t U + \nabla \cdot \vec{H} = \mathbf{T} : \nabla \vec{v} + h$$

When $\mathbf{T}$ is symmetric, the contraction $\mathbf{T} : \nabla \vec{v}$ leaves only the symmetric part of $\nabla \vec{v}$ (the general principle is that a symmetric tensor contracted with an antisymmetric tensor gives zero). Thus we could replace $\nabla \vec{v}$ in the above with $\mathbf{D}$, where $\mathbf{D} = \frac{1}{2}\left[(\nabla \vec{v})^T + \nabla \vec{v}\right]$:

$$\rho D_t U + \nabla \cdot \vec{H} = \mathbf{T} : \mathbf{D} + h$$

In the case of waves propagating through elastic materials, particularly seismic waves, the rate of heat flux and heat production are negligible. Linearizing the wave equation, we have

$$\rho \partial_t U = \mathbf{T} : \nabla \partial_t \vec{s}$$

Applying Hooke's law, we write

$$\begin{aligned}\rho \partial_t U = (\mathbf{c} : \nabla \vec{s}) : \nabla \partial_t \vec{s} = c_{ijkl} \partial_k s_l \partial_i \partial_t s_j &= \frac{1}{2} \partial_t \left(\partial_i s_j c_{ijkl} \partial_k s_l \right) \\ &= \frac{1}{2} \partial_t \left(\varepsilon_{ij} c_{ijkl} \varepsilon_{kl} \right)\end{aligned}$$

Integrating against time, we have

$$\rho U = \rho_0 U_0 + \frac{1}{2} \varepsilon_{ij} c_{ijkl} \varepsilon_{kl} = \rho_0 U_0 + \frac{1}{2} \boldsymbol{\varepsilon} : \mathbf{c} : \boldsymbol{\varepsilon}$$

1.6 The 1D Heat Equation

To develop the heat equation, consider a particle with zero initial velocity. Then conservation of energy gives

$$\rho \partial_t U + \nabla \cdot \vec{H} = h$$

In order to continue deriving this equation, we need assumptions on U. One possible assumption is **caloric equation of state**, which says that U may be expressed as a function $U(\theta)$ solely of temperature. We define the specific heat capacity at constant volume V to be

$$c_V = \frac{\mathrm{d}U}{\mathrm{d}\theta}$$

Fourier's law says that heat fluxes against the temperature gradient:

$$\vec{H} = -K \nabla \theta$$

(Under anisotropic conditions we may assume that there is cross-gradient heat flux; in this case we replace K with a tensor and contraction $\mathbf{K}\cdot$). Plugging this in, we arrive at the **heat equation** or diffusion equation

$$\rho c_V \partial_t \theta = \nabla \cdot (K \nabla \theta) + h$$

In one dimension this is

$$\rho c_v \partial_t \theta = \partial_x (K \partial_x \theta) + h$$

Chapter 2

Strong Methods for PDEs

Having introduced our model equations, we now turn to methods for numerically solving differential equations. We first begin with strong methods, which solve for solutions to the non-integrated form of the desired differential equation. Both of our model equations are linaer second order PDEs:

$$\begin{cases} \rho \partial_t^2 s = \partial_x \left(\mu \partial_x s \right) \\ \partial_t \theta = \partial_x \left(\alpha \partial_x \theta \right) + h \end{cases}$$

The most general form of a linear second order PDE is given by

$$A \partial_t^2 s + 2B \partial_t \partial_x s + c \partial_x^2 s + D \partial_t s + E \partial_x s + Fs + G = 0$$

Taking inspiration from conic sections we consider the discriminant of the second derivatives, given by $B^2 - AC$. When this quantity is positive the PDE is called **hyperbolic**; when it is negative the PDE is **elliptic**; and when it is zero it is **parabolic**.

2.1 Finite Difference Methods

The finite difference method is essentially rooted in the Taylor series. Essentially we can simulate the evolution of our system forward in time by simply linearizing and taking small steps forward in time:

$$\frac{\mathrm{d}f}{\mathrm{d}x}(x) = \frac{f(x + \Delta x) - f(x)}{\Delta x} + O(\Delta x)$$

This allows us to estimate the derivative of f at a point, so long as we know the values of f at points close to x. This is called the forward difference approximation. Of course we may approximate from below as well (backward difference approximation):

$$\frac{\mathrm{d}f}{\mathrm{d}x}(x) = \frac{f(x) - f(x - \Delta x)}{\Delta x} + O(\Delta x)$$

A third estimate combines the above approximations:

$$\frac{\mathrm{d}f}{\mathrm{d}x}(x) = \frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x} + O((\Delta x)^2)$$

This is called the centered-difference scheme, and it exhibits quadratic error, since approximating from both sides allows the linear term to cancel.

To calculate second derivatives, we can use the following first order approximation:

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = \frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{\Delta x^2} + O(\Delta x)$$

The finite difference method then uses a discretization of the relevant sample space. Essentially we define a finite grid of points that we will compute. Afterward, we discretize the time steps as well and progress it forward.

2.2 The 1D Wave Equation with Finite Difference

We can simplify the second order wave equation by depressing it to a first order system:

$$\rho(x)\partial_t^2 u(x,t) = \partial_x[\kappa(x)\partial_x u(x,t)]$$

becomes

$$\begin{cases} \rho(x)\partial_t v(x,t) = \partial_x T(x,t) \\ \partial_t T(x,t) = \kappa(x)\partial_x v(x,t) \end{cases}$$

where

$$\begin{cases} T(x,t) \coloneqq \kappa(x)\partial_x u(x,t) \\ v(x,t) = \partial_t u(x,t) \end{cases}$$

Discretizing the sample space, we approximate our finite differences as

$$\partial_t^2 u(x_i, t_n) \approx \frac{u(x_i, t_n + \Delta t) - 2u(x_i, t_n) + u(x_i, t_n - \Delta t)}{\Delta t^2} = \frac{1}{\Delta t^2}\left[u_i^{n+1} - 2u_i^n + u_i^{n-1}\right]$$

$$\partial_x^2 u(x_i, t_n) \approx \frac{1}{\Delta x^2}\left[u_{i+1}^n - 2u_i^n + u_{i-1}^n\right]$$

In the homogeneous case, our PDE is

$$\partial_t^2 u(x,t) = c^2 \partial_x^2 u(x,t)$$

Substituting in, we have

$$u_i^{n+1} = \frac{c^2 \Delta t^2}{\Delta x^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) + 2u_i^n - u_i^{n-1}$$

Now, it is important that we specify the behavior of the boundary conditions. Some options include the Dirichlet boundary conditions, which corresponds to a fixed boundary that satisfies $u(0,t) = 0$ for all t. On the other hand, we can pick the Neumann boundary conditions, which allow the boundary free movement. In other words, it expreiences no stress, so that $T(0,t) = 0$.

We also need to set initial conditions. It suffices to define $u(x,0)$ and $\partial_t u(x,0)$, which is just initial position and velocity.

Definitions