MAT 425 Notes

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Fall 2025

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Introduction

This document contains notes taken for the class MAT 425: Integration Theory and Hilbert Spaces at Princeton University, taken in the Spring 2025 semester. These notes are primarily based on lectures by Professor Jacob Shapiro. Other references used in these notes include *Real Analysis* by Elias Stein and Rami Shakarchi, *Real and Complex Analysis* by Walter Rudin, *Real Analysis (2nd Edition)* by Halsey Royden, *The Elements of Integration and Lebesgue Measure* by Robert Bartle, *Measure Theory* by Paul Halmos, and *Real Analysis: Modern Techinques and Their Applications* by Gerald Folland. Since these notes were primarily taken live, they may contains typos or errors.

Chapter 1

1.1 Motivations

The formal study of measure theory is motivated historically by the insufficiency of the Riemann integral as a complete tool for describing integration. Considering some bounded function $f : [a, b] \to \mathbb{R}$, there are many desirable properties that we might expect from an integral.

1. We might ask that the integral produces the average value of the function f on $[a, b]$, as
$$\overline{f} = \frac{1}{b-a} \int_a^b f$$
2. Geometrically, we can interpret the integral as the signed area between the graph of f and the x-axis:
$$A = \int_a^b f$$
3. We also think of integrals as the continuous generalization of summation.

Recall that the Riemann integral of f over $[a, b]$ is defined by considering, for fixed $N \in \mathbb{N}$, the upper and lower sums L_N, U_N defined by

$$L_N(f) = \frac{b-a}{N} \sum_{j=0}^{N-1} \inf \left\{ f(x) : x \in a + [n, n+1] \frac{b-a}{N} \right\}$$
$$U_N(f) = \frac{b-a}{N} \sum_{j=0}^{N-1} \sup \left\{ f(x) : x \in a + [n, n+1] \frac{b-a}{N} \right\}$$

We say that f is Riemann integrable with integral $I = \int_a^b f \in \mathbb{R}$ if $\lim L_N, \lim U_N$ both exist and are equal to I.

From our previous studies, Lebesgue's criterion gave a convenient characterization of Riemann integrable functions.

Definition 1.1

A set $S \subseteq \mathbb{R}$ has **measure zero** if for any $\varepsilon > 0$ there exists a collection $\{U_n\}_{n\in\mathbb{N}}$ of open intervals such that $S \subseteq \bigcup U_n$ and $\sum |U_n| < \varepsilon$, where $|U_n|$ is the length of U_n.

Example 1.1

The Cantor set $\mathcal{C}$ has measure zero. This is a consequence of the fact that at each iterative step in the construction of the Cantor set, we have a collection of open intervals covering the Cantor set, and the total length at step k is given by $\left(\frac{2}{3}\right)^k \to 0$.

Theorem 1.1: Lebesgue's Theorem

A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if the set of discontinuities of f has measure zero.

In particular, continuous functions are always Riemann integrable. The indicator function $\chi_{\mathcal{C}}$ of the Cantor set is Riemann integrable, since its discontinuities are of measure zero. However, $\chi_{\mathbb{Q}}$ (restricted to some compact interval) is not, since it is discontinuous at *every* point (this is precisely Dirichlet's function).

One can define a Riemann integral for unbounded functions or on unbounded domains by considering appropriate limits of Riemann integrals on compact intervals.

Example 1.2

The improper integral $\int_0^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x$ is computed as

$$\int_{[0,1]} \frac{1}{\sqrt{x}}\,\mathrm{d}x = \lim_{n\to\infty} \int_{[\frac{1}{n},1]} \frac{1}{\sqrt{x}}\,\mathrm{d}x = \lim_{n\to\infty} 2\sqrt{x}|_{\frac{1}{n}}^{1} = \lim_{n\to\infty} \left[2 - \frac{2}{\sqrt{n}}\right] = 2$$

This method may be naturally extended to functions with a finite number of "integrable" discontinuities, or sometimes countable discontinuities. However, the following example shows that it fails in the general case.

Example 1.3

Let $\{\eta_n\}_{n\in\mathbb{N}}$ be an enumeration of the set $(0, 1) \cap \mathbb{Q}$. Define $f_n : [0, 1] \to \mathbb{R}$ by

$$f_n : x \mapsto \begin{cases} \frac{1}{\sqrt{x-\eta_n}} : & x > \eta_n \\ 0 : & x \leq \eta_n \end{cases}$$

Then define

$$f(x) := \sum_{n=1}^{\infty} 2^{-n} f_n(x)$$

By density, f is unbounded in every open subinterval of $[0,1]$. As a result, there is no limit of intervals increasing to $[0,1]$ which we could use to define the integral of f over $[0,1]$, in the sense used in the previous example.

To try to figure out a way around this, note that our work in the previous example shows that

$$\int_{[0,1]} f_n = 2\sqrt{1-\eta_n}$$

Now, consider the (unjustified) interchange of the integral and sum:

$$\int_{[0,1]} f = \int_{[0,1]} \sum_{n=1}^{\infty} 2^{-n} f_n \longrightarrow \sum_{n=1}^{\infty} 2^{-n} \int_{[0,1]} f_n = \sum_{n=1}^{\infty} 2^{-n} 2\sqrt{1-\eta_n} < \infty$$

As the above example demonstrates, an important question in analysis is which operations respect the limiting process. In particular, we know that uniform convergence respects the limit:

Theorem 1.2

Let $f_n : [a,b] \to \mathbb{R}$ be a sequence of bounded Riemann integrable functions which converge uniformly to f. Then f is Riemann integrable and $\lim \int_{[a,b]} f_n = \int_{[a,b]} f$.

However, it is desirable to us to apply this interchange under weaker hypotheses than uniform convergence, so that we can develop a more powerful and general theory of integration.

Example 1.4

Consider again the enumeration $\{\eta_n\}_{n\in\mathbb{N}}$ of $(0,1)\cap\mathbb{Q}$. Define

$$f_n \coloneqq \chi_{\{\eta_j : j\in[1,n]\}}$$

In words, $f_n(x) = 1$ if $x = \eta_j$ for some $j \le n$ and 0 otherwise. $\int_{[0,1]} f_n = 0$ for all n, so we would like to assign the value 0 to $\int_{[0,1]} \lim f$. However, observe that f_n converges pointwise to Dirichlet's function, which is not Riemann integrable.

The development of the Lebesgue integral, which solves many issues with the Riemann integral, will be accomplished by first discussing the general theory of measure and integration, and following the construction of the Lebesgue measure and integral.

1.2 Abstract Measure Theory

The development of a measure space structure on a set is accomplished by defining a collection of "measurable" subsets, not unlike a topology, which satisfies particular structural constraints.

Definition 1.2

Let X be a set, and consider a collection of subsets $\mathcal{M} \subseteq \mathcal{P}(X)$. We say that $\mathcal{M}$ is a **σ-algebra** on X if

1. $X \in \mathcal{M}$,
2. If $A \in \mathcal{M}$ then $X \setminus A \in \mathcal{M}$,
3. If $\{A_n\}_{n\in\mathbb{N}}$ is a countable collection of elements of $\mathcal{M}$, then $\bigcup A_n \in \mathcal{M}$.

If $\mathcal{M}$ is a σ-algebra on X, then $(X, \mathcal{M})$ is called a **measure space**. An element of $\mathcal{M}$ is called a **measurable set**. If the σ-algebra on X is understood by context, then $\mathrm{Meas}(X)$ denotes the set of measurable subsets of X (that it, it denotes the implied σ-algebra).

Notice that while a topology is required to be closed under arbitrary unions, a σ-algebra is only required to be closed under countable unions. Moreover, the following follows directly from the axioms of σ-algebras:

Proposition 1.3

$\varnothing \in \mathrm{Meas}(X)$ and $\mathrm{Meas}(X)$ is clsoed under countable intersections.

For comparison, recall the following definition of a topology:

Definition 1.3

Let X be a set, and consider a collection of subsets $\mathcal{T} \subseteq \mathcal{P}(X)$. We say that $\mathcal{T}$ is a **topology** on X if

1. $X, \varnothing \in \mathcal{T}$,
2. $\bigcap_{n=1}^{N} V_n \in \mathcal{T}$ whenever each $V_n \in \mathcal{T}$,
3. $\bigcup_{\alpha\in A} V_\alpha \in \mathcal{T}$ whenever $V_\alpha \in \mathcal{T}$ for an arbitrary indexing set A.

By direct comparison, a topology is not automatically a σ-algebra, since it may not be closed under complements.

Again in analogy to topology, recall that continuous functions are the morphisms of topological spaces. Thus, we can ask which functions can be considered to be the morphisms of measure spaces. Indeed, just as continuous functions are topologically characterized by preserving open sets under preimages, we define measure space morphisms similarly:

Definition 1.4

A function $f : X \to Y$ for measure spaces X, Y is said to be a **measurable function** if $f^{-1}(A) \in \mathrm{Meas}(X)$ whenever $A \in \mathrm{Meas}(Y)$.

It follows immediately that the composition of measurable functions is measurable.

As with topologies, any set automatically comes equipped with two σ-algebras: the power set $\mathcal{P}(X)$ and $\{\varnothing, X\}$. These are the largest and smallest σ-algebras on X, respectively.

Example 1.5

Let $X = \{1, 2, 3, 4\}$. Then the following is a nontrivial σ-algebra:

$$\mathcal{M} = \{\varnothing, X, \{1, 2\}, \{3, 4\}\}$$

Generalizing the above, for any $A \subseteq X$, the σ-algebra $\{\varnothing, X, A, X \setminus A\}$ is the smallest σ-algebra containing A.

Remark 1.1

The arbitrary intersection of σ-algebras on a common set is again a σ-algebra, but not necessarily unions.

Definition 1.5

Let $f : X \to Y$, where X is an arbitrary set and Y is a measure space. Then the σ-algebra $\sigma(f)$ **generated** by f is

$$\sigma(f) \coloneqq \left\{f^{-1}(A) : A \in \operatorname{Meas}(Y)\right\}$$

It is straightforward to verify that $\sigma(f)$ is actually a σ-algebra, and it follows that $\sigma(f)$ is the smallest σ-algebra on X such that f is measurable. More generally, we have the following:

Theorem 1.4

Let $\mathcal{F} \subseteq \mathcal{P}(X)$. Then there exists a unique minimal σ-algebra $\mathcal{M}$ on X such that $\mathcal{F} \subseteq \mathcal{M}$.

Definitions