MAT 345 Notes

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Contents

Introduction

This document contains notes taken for the class MAT 345: Algebra I at Princeton University, taken in the Fall 2024 semester. These notes are primarily based on lectures and lecture notes by Professor Jakub Witaszek. Other references used in these notes include *Algebra* by Michael Artin, *Abstract Algebra* by David Dummit and Richard Foote, *Contemporary Abstract Algebra* by Joseph Gallian, and *A Book of Abstract Algebra* by Charles Pinter. Since these notes were primarily taken live, they may contains typos or errors.

Chapter 1

Elementary Number Theory

This course will study algebraic structures, primarily groups, rings, and fields. These objects serve as abstractions of objects which we are familiar with performing algebra over, such as $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$. As such, we will begin with a brief survey of algebraic operations over these familiar objects, before progressing to their abstracted counterparts.

1.1 The Euclidean Algorithm

The most important theorem of the structure of the integers is the following:

Theorem 1.1: Fundamental Theorem of Arithmetic

Let $n \in \mathbb{N}$. Then there is a unique representation of n as a product of powers of primes (up to ordering), as

$$n = p_1^{\alpha_1} \cdot \ldots \cdot p_k^{\alpha_k}$$

Another important operation to abstract is that of division. This requires phrasing it in terms that are easily generalized to other objects:

Theorem 1.2: Division Algorithm

Let $n, d \in \mathbb{Z}$ with $d > 0$. Then there exist unique $q, r \in \mathbb{Z}$ such that

$$n = qd + r$$

and

$$0 \leq r < d$$

Proof. **Existence:** Define

$$S = \{n - dx | x \in \mathbb{Z}, n - dx \geq 0\}$$

Let $r = \min S$ and let $q \in \mathbb{Z}$ be the corresponding value such that $n - qd = r$. Suppose that $r \geq d$. Then

$$n - (q+1)d = n - qd - d = r - d \geq 0$$

so $r - d \in S$, contradicting $r = \min S$. So $0 \leq r < d$. Thus we have shown existence.

Uniqueness: Let $n = qd + r = q'd + r'$. Then

$$d(q - q') + r - r' = 0$$

so $d|r - r'$. But we also have $-d < r - r' < d$, so $r - r' = 0$ and thus $r = r'$. It follows that $q = q'$. □

We call d the divisor, q the quotient, and r the remainder. Explicitly, we have

$$n = \left\lfloor \frac{n}{d} \right\rfloor d + (n \bmod d)$$

The proof of the Fundamental Theorem of Arithmetic requires the proof of some other lemmas:

Definition 1.3

Let $a, b \in \mathbb{Z}$. We write $a|b$ if there exists $c \in \mathbb{Z}$ such that $ac = b$.

Lemma 1.4: Euclid's lemma

Let p be prime and $a, b \in \mathbb{Z}$. If $p|ab$, then $p|a$ or $p|b$.

This, in turn, relies on another identity.

Definition 1.5

Let $a, b \in \mathbb{N}$. Then define $\gcd(a, b)$ to be a common divisor which divides any other common divisor.

We should note that we have not shown that $\gcd(a, b)$ exists and is unique. However, consideration of the extended Euclidean algorithm shows both of these, and moreover that $\gcd(a, b)$ is the largest common divisor of a and b.

Proposition 1.6: Bezout's Identity

Let $a, b \in \mathbb{Z}$ be nonzero. Then there exist $k, l \in \mathbb{Z}$ such that

$$ka + lb = \gcd(a, b)$$

Example 1.7

if $a = 9$ and $b = 24$, then

$$3 \cdot 9 + (-1) \cdot 24 = 3 = \gcd(9, 24)$$

Bezout's Identity follows from the **extended Euclidean Algorithm**.

Proof of Euclid's Lemma. If $p|a$, then we are done. So suppose it doesn't. Then $\gcd(p, a) = 1$. By Bezout's identity, there exist $k, l \in \mathbb{Z}$ such that

$$kp + la = 1$$

So $kpb + lab = b$. p divides the left hand side since it is in the product, and divides the right hand side since it divides ab. □

1.2 Modular Arithmetic

Definition 1.8

Let $a, b \in \mathbb{Z}$, and let $n > 0$ be an integer. Then a is **congruent** to b modulo n (denoted $a \equiv b \pmod{n}$) if

$$n|a - b$$

It follows that congruence modulo n is an equivalence relation for any n, dividing the integers into n classes based on their remainders after dividing by n.

We may equivalently define this congruence as follows:

Proposition 1.9

$a \equiv b \pmod{n}$ if and only if $a \bmod n = b \bmod n$ (where $a \bmod n$ represents the remainder of a when divided by n.)

A convenient example of modular arithmetic is the use of a 12-hour clock system, where the hour hand resets after each multiple of 12. We may similarly visualize modular arithmetic for any n as movement around a circle with n distinct positions.

Lemma

Let $a, b, c, d \in \mathbb{Z}$ and let $n \in \mathbb{N}$. Suppose that

$$\begin{cases} a \equiv c \pmod{n} \\ b \equiv d \pmod{n} \end{cases}$$

Then

$$\begin{cases} a + b \equiv c + d \pmod{n} \\ ab \equiv cd \pmod{n} \end{cases}$$

Essentially, the above lemma says that we may replace any number by another number which is equivalent modulo n (for addition and multiplication).

Example 1.10

We have

$$7 \cdot 22 \equiv 1 \cdot 4 \equiv 4 \pmod{6}$$

Similarly,

$$(5 + 12)8 + 13 \equiv (5 + 5)1 + 6 \equiv 3 \cdot 1 + 6 \equiv 2 \pmod{7}$$

Theorem 1.11

Let p be prime and let $k \in \mathbb{Z}$, and suppose p does not divide k. Then

$$k \bmod p, 2k \bmod p, \ldots, (p-1)k \bmod p$$

is a permutation of

$$1, 2, \ldots, p - 1$$

Proof. Suppose that not all of these values are different, such that there exist $1 \leq n_1, n_2 \leq p - 1$ but $n_1 k \bmod p = n_2 k \bmod p$. But this means that $(n_2 - n_1)k \bmod p = 0$, so p divides $(n_2 - n_1)k$. It doesn't divide k, so it divides $n_2 - n_1$. But $-p < n_2 - n_1 < p$. The only number in this range which p divides is 0, so $n_1 = n_2$.

Thus the list

$$k \bmod p, \ldots, (p-1)k \bmod p$$

is a list of $p-1$ distinct numbers between 1 and $p-1$. So each number occurs at least once, and we have just shown that they are distinct, so each number occurs exactly once. □

One interpretation of this is that if you repeatedly take k steps around a circle with p positions, then if p does not divide k, we will not repeat spaces until we have covered all of them.

Corollary 1.12

Let p be prime and $a \in \mathbb{Z}$ such that p does not divide a. Then there exists $b \in \mathbb{Z}$ such that

$$ab \equiv 1 \pmod{p}$$

For any b which satisfies the above, we call b a **multiplicative inverse** of a.

Proof. By Theorem 1.11, there exists some n with $1 \leq n \leq p-1$ such that $nk \bmod p = 1$ □

Note that multiplicative inverses found this way are *not* unique. Thus it is improper to write an expression of the form $\frac{1}{a} \pmod{p}$.

Remark

A multiplicative inverse may be found using the extended Euclidean algorithm.

Theorem 1.13: Fermat's Little Theorem

Let p be prime and $a \in \mathbb{Z}$ such that p does not divide a. Then

$$a^{p-1} \equiv 1 \pmod{p}$$

Example 1.14

With $a = 2, p = 7$ we have

$$\begin{aligned}
2^0 = 1 &\equiv 1 \pmod{7} \\
2^1 = 2 &\equiv 2 \pmod{7} \\
2^2 = 4 &\equiv 4 \pmod{7} \\
2^3 = 8 &\equiv 1 \pmod{7} \\
2^4 = 16 &\equiv 2 \pmod{7} \\
2^5 = 32 &\equiv 4 \pmod{7} \\
2^6 = 64 &\equiv 1 \pmod{7}
\end{aligned}$$

Note that $7 - 1 = 6$ is not the first b with $a^b \equiv 1 \pmod{p}$. However, the remainders do occur in cycles, and the period of this cycle divides $p - 1$.

Lemma

Suppose n does not divide k. If

$$ak \equiv bk \pmod{n}$$

then

$$a \equiv b \pmod{n}$$

Proof. We have $n|(a-b)k$, so by Euclid's Lemma $n|a-b$. Thus $a \equiv b \pmod{n}$. □

Proof of Fermat's Little Theorem. Take the product

$$a \cdot 2a \cdot \ldots \cdot (p-1)a \equiv a^{p-1}(p-1)! \pmod{p}$$

(Note that this is a simple equality). But Theorem 1.11 tells us that modulo p, these factors are a rearrangement of $1, \ldots, p-1$. So we have

$$(p-1)! \equiv a \cdot 2 \ldots \cdot (p-1) \pmod{p}$$

Combining these two congruences and applying the Lemma, we have

$$a^{p-1} \equiv 1 \pmod{p}$$ □

1.3 Fields

We recall the definition of a field:

Definition 1.15

A field is a nonempty set F together with two operations $+ : F \times F \to F$ and $\cdot : F \times F \to F$ as well as distinct elements $0 \neq 1 \in F$ such that

- $+$ and $\cdot$ are commutative.
- $+$ and $\cdot$ are associative.
- 0 is an additive identity and 1 a multiplicative identity.
- Additive inverses exist (denoted $-\alpha$).
- Multiplicative inverses exists for any $\alpha \neq 0$ (denoted α^{-1}).
- $\cdot$ distributes over $+$.

Some familiar examples of fields are $\mathbb{Q}, \mathbb{R}, \mathbb{C}$. A nonexample is $\mathbb{Z}$ (which does not have multiplicative inverses.)

Definition 1.16

Let p be prime. Then we define $\mathbb{F}_p = \{\ldots, \overline{-2}, \overline{-1}, \overline{0}, \overline{1}, \overline{2}, \ldots\}$, where the elements $\overline{k}$ are defined such that

$$\overline{a} = \overline{b} \iff a \equiv b \pmod{p}$$

We define

$$\overline{a} + \overline{b} = \overline{a+b}$$

and

$$\overline{a} \cdot \overline{b} = \overline{ab}$$

Example 1.17

With $p = 6$, we have

$$\overline{2} \cdot \overline{3} = \overline{6} = \overline{1}$$

Equivalently, since we identify numbers congruent modulo p, Theorem 1.11 we can simply write

$$\mathbb{F}_p = \{\overline{0}, \ldots, \overline{p-1}\}$$

all of which are distinct. Moreover, Corollary 1.12 assures us of the existence of multiplicative inverses. The remaining axioms are simpler to check, but this demonstrates that $\mathbb{F}_p$ is in fact a field.

Definition 1.18

The set $\mathbb{Z}/n\mathbb{Z}$ is defined similarly to $\mathbb{F}_p$ (where n is not necessarily prime), with only the operation of addition defined.

We can use this to prove the following theorem:

Theorem 1.19

Let p be prime with $p \equiv 1 \pmod{4}$. Then $p = x^2 + y^2$ for some $x, y \in \mathbb{Z}$.

We can check the first few cases by hand:

$$\begin{aligned} 5 &= 1^2 + 2^2 \\ 13 &= 2^2 + 3^2 \\ 17 &= 1^2 + 4^2 \\ 29 &= 2^2 + 5^2 \end{aligned}$$

For the cases $p \geq 37$, we will develop a bit more theory.

Proof. □

Definition 1.20

$a \in \mathbb{F}_p$ is called a **quadratic residue** if $a = x^2$ for some $x \in \mathbb{F}_p$.

Equivalently:

Definition 1.21

$a \in \mathbb{Z}$ is a quadratic residue mod p if $a \equiv x^2 \pmod{p}$ for some $x \in \mathbb{Z}$.

Example 1.22

With $p = 5$, we have

$$\begin{cases} 0^2 \equiv 0 \\ 1^2 \equiv 1 \\ 2^2 \equiv 4 \\ 3^2 \equiv 4 \\ 4^2 \equiv 1 \end{cases} \pmod{5}$$

so the quadratic residues are $0, 1, 4$ (note that 0 is always a quadratic residue.)

The necessary result is as follows:

Lemma

-1 (or $p-1$) is a quadratic residue mod p if and only if $p \equiv 1 \pmod{4}$.

Proof. Skipped. □

We can now return to the previous proof.

Proof of Theorem 1.19. **Claim 1:** There exists $x, y \in \mathbb{Z}$ with $0 < x, y < p$ and

$$x^2 + y^2 \equiv 0 \pmod{p}$$

To show this, by the Lemma we have that -1 is a quadratic residue, so there exists $a \in \mathbb{Z}$ with

$$a^2 \equiv -1 \pmod{p}$$

or

$$1^2 + a^2 \equiv 0 \pmod{p}$$

Now let $x = 1$, $y = a \bmod p$. Claim 1 is proved.

Claim 2: There exist $x, y \in \mathbb{Z}$ with $x^2 + y^2 < 2p$ and $x^2 + y^2 \equiv 0 \pmod{p}$.

To show this, apply Claim 1 to produce x, y with $x^2 + y^2 \equiv 0 \pmod{p}$. Then let S be the set

$$S = \{(x_0, y_0), \ldots, (x_{p-1}, y_{p-1})\} \subseteq \mathbb{Z}^2$$

where

$$(x_i, y_i) = (ix \bmod p, iy \bmod p)$$

This set may be seen as the set of integer multiples of the point (x, y), modulo p.

Now, we claim that there exists $0 \leq i < j \leq p - 1$ such that

$$d((x_i, y_i), (x_j, y_j)) < \sqrt{2p}$$

To show this, we draw circles of radius

$$\frac{\sqrt{2p}}{2}$$

around. If the claim is false then the circles do not overlap. All the circles are subsetes of

$$\left[-\frac{\sqrt{2p}}{2}, p + \frac{\sqrt{2p}}{2}\right]^2$$

If they do not overlap, then the total area is less than that of the square. But

$$1.57 \approx \frac{\pi}{2}p^2 = p\pi(\frac{\sqrt{2p}}{2})^2 \leq (p + \sqrt{2p})^2 = p(1 + \sqrt{\frac{2}{p}}) \leq p(1 + \sqrt{\frac{2}{37}})^2 \approx 1.51$$

We checked the lower cases, so the claim is proved. Then pick

$$(x', y') = (|x_j - x_i|, |y_j - y_i|)$$

We then show that p divides $(x')^2 + (y')^2$, but also this number is less than $2p$, so it is p. $\square$

Definitions