MAT 425 Notes

Max Chien

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Introduction

This document contains notes taken for the class MAT 425: Integration Theory and Hilbert Spaces at Princeton University, taken in the Spring 2025 semester. These notes are primarily based on lectures by Professor Jacob Shapiro. Other references used in these notes include *Real Analysis* by Elias Stein and Rami Shakarchi, *Real and Complex Analysis* by Walter Rudin, *Real Analysis (2nd Edition)* by Halsey Royden, *The Elements of Integration and Lebesgue Measure* by Robert Bartle, *Measure Theory* by Paul Halmos, and *Real Analysis: Modern Techinques and Their Applications* by Gerald Folland. Since these notes were primarily taken live, they may contains typos or errors.

Chapter 1

Introductory Measure Theory

1.1 Motivations

The formal study of measure theory is motivated historically by the insufficiency of the Riemann integral as a complete tool for describing integration. Considering some bounded function $f : [a, b] \to \mathbb{R}$, there are many desirable properties that we might expect from an integral.

1. We might ask that the integral produces the average value of the function f on $[a, b]$, as
$$\overline{f} = \frac{1}{b-a} \int_a^b f$$
2. Geometrically, we can interpret the integral as the signed area between the graph of f and the x-axis:
$$A = \int_a^b f$$
3. We also think of integrals as the continuous generalization of summation.

Recall that the Riemann integral of f over $[a, b]$ is defined by considering, for fixed $N \in \mathbb{N}$, the upper and lower sums L_N, U_N defined by

$$L_N(f) = \frac{b-a}{N} \sum_{j=0}^{N-1} \inf \left\{ f(x) : x \in a + [n, n+1] \frac{b-a}{N} \right\}$$
$$U_N(f) = \frac{b-a}{N} \sum_{j=0}^{N-1} \sup \left\{ f(x) : x \in a + [n, n+1] \frac{b-a}{N} \right\}$$

We say that f is Riemann integrable with integral $I = \int_a^b f \in \mathbb{R}$ if $\lim L_N, \lim U_N$ both exist and are equal to I.

From our previous studies, Lebesgue's criterion gave a convenient characterization of Riemann integrable functions.

Definition 1.1

A set $S \subseteq \mathbb{R}$ has **measure zero** if for any $\varepsilon > 0$ there exists a collection $\{U_n\}_{n\in\mathbb{N}}$ of open intervals such that $S \subseteq \bigcup U_n$ and $\sum |U_n| < \varepsilon$, where $|U_n|$ is the length of U_n.

Example 1.1

The Cantor set $\mathcal{C}$ has measure zero. This is a consequence of the fact that at each iterative step in the construction of the Cantor set, we have a collection of open intervals covering the Cantor set, and the total length at step k is given by $\left(\frac{2}{3}\right)^k \to 0$.

Theorem 1.1: Lebesgue's Theorem

A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if the set of discontinuities of f has measure zero.

In particular, continuous functions are always Riemann integrable. The indicator function $\chi_{\mathcal{C}}$ of the Cantor set is Riemann integrable, since its discontinuities are of measure zero. However, $\chi_{\mathbb{Q}}$ (restricted to some compact interval) is not, since it is discontinuous at *every* point (this is precisely Dirichlet's function).

One can define a Riemann integral for unbounded functions or on unbounded domains by considering appropriate limits of Riemann integrals on compact intervals.

Example 1.2

The improper integral $\int_0^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x$ is computed as

$$\int_{[0,1]} \frac{1}{\sqrt{x}}\,\mathrm{d}x = \lim_{n\to\infty} \int_{[\frac{1}{n},1]} \frac{1}{\sqrt{x}}\,\mathrm{d}x = \lim_{n\to\infty} 2\sqrt{x}|_{\frac{1}{n}}^{1} = \lim_{n\to\infty} \left[2 - \frac{2}{\sqrt{n}}\right] = 2$$

This method may be naturally extended to functions with a finite number of "integrable" discontinuities, or sometimes countable discontinuities. However, the following example shows that it fails in the general case.

Example 1.3

Let $\{\eta_n\}_{n\in\mathbb{N}}$ be an enumeration of the set $(0, 1) \cap \mathbb{Q}$. Define $f_n : [0, 1] \to \mathbb{R}$ by

$$f_n : x \mapsto \begin{cases} \frac{1}{\sqrt{x-\eta_n}} : & x > \eta_n \\ 0 : & x \leq \eta_n \end{cases}$$

Then define

$$f(x) \coloneqq \sum_{n=1}^{\infty} 2^{-n} f_n(x)$$

By density, f is unbounded in every open subinterval of $[0,1]$. As a result, there is no limit of intervals increasing to $[0,1]$ which we could use to define the integral of f over $[0,1]$, in the sense used in the previous example.

To try to figure out a way around this, note that our work in the previous example shows that

$$\int_{[0,1]} f_n = 2\sqrt{1-\eta_n}$$

Now, consider the (unjustified) interchange of the integral and sum:

$$\int_{[0,1]} f = \int_{[0,1]} \sum_{n=1}^{\infty} 2^{-n} f_n \longrightarrow \sum_{n=1}^{\infty} 2^{-n} \int_{[0,1]} f_n = \sum_{n=1}^{\infty} 2^{-n} 2\sqrt{1-\eta_n} < \infty$$

As the above example demonstrates, an important question in analysis is which operations respect the limiting process. In particular, we know that uniform convergence respects the limit:

Theorem 1.2

Let $f_n : [a,b] \to \mathbb{R}$ be a sequence of bounded Riemann integrable functions which converge uniformly to f. Then f is Riemann integrable and $\lim \int_{[a,b]} f_n = \int_{[a,b]} f$.

However, it is desirable to us to apply this interchange under weaker hypotheses than uniform convergence, so that we can develop a more powerful and general theory of integration.

Example 1.4

Consider again the enumeration $\{\eta_n\}_{n\in\mathbb{N}}$ of $(0,1)\cap\mathbb{Q}$. Define

$$f_n \coloneqq \chi_{\{\eta_j : j\in[1,n]\}}$$

In words, $f_n(x) = 1$ if $x = \eta_j$ for some $j \leq n$ and 0 otherwise. $\int_{[0,1]} f_n = 0$ for all n, so we would like to assign the value 0 to $\int_{[0,1]} \lim f$. However, observe that f_n converges pointwise to Dirichlet's function, which is not Riemann integrable.

The development of the Lebesgue integral, which solves many issues with the Riemann integral, will be accomplished by first discussing the general theory of measure and integration, and following the construction of the Lebesgue measure and integral.

1.2 Abstract Measure Theory

The development of a measure space structure on a set is accomplished by defining a collection of "measurable" subsets, not unlike a topology, which satisfies particular structural constraints.

Definition 1.2

Let X be a set, and consider a collection of subsets $\mathcal{M} \subseteq \mathcal{P}(X)$. We say that $\mathcal{M}$ is a **σ-algebra** on X if

1. $X \in \mathcal{M}$,
2. If $A \in \mathcal{M}$ then $X \setminus A \in \mathcal{M}$,
3. If $\{A_n\}_{n\in\mathbb{N}}$ is a countable collection of elements of $\mathcal{M}$, then $\bigcup A_n \in \mathcal{M}$.

If $\mathcal{M}$ is a σ-algebra on X, then $(X, \mathcal{M})$ is called a **measurable space**. An element of $\mathcal{M}$ is called a **measurable set**. If the σ-algebra on X is understood by context, then $\mathrm{Meas}(X)$ denotes the set of measurable subsets of X (that it, it denotes the implied σ-algebra).

Notice that while a topology is required to be closed under arbitrary unions, a σ-algebra is only required to be closed under countable unions. Moreover, the following follows directly from the axioms of σ-algebras:

Proposition 1.3

$\varnothing \in \mathrm{Meas}(X)$ and $\mathrm{Meas}(X)$ is clsoed under countable intersections.

For comparison, recall the following definition of a topology:

Definition 1.3

Let X be a set, and consider a collection of subsets $\mathcal{T} \subseteq \mathcal{P}(X)$. We say that $\mathcal{T}$ is a **topology** on X if

1. $X, \varnothing \in \mathcal{T}$,
2. $\bigcap_{n=1}^{N} V_n \in \mathcal{T}$ whenever each $V_n \in \mathcal{T}$,
3. $\bigcup_{\alpha\in A} V_\alpha \in \mathcal{T}$ whenever $V_\alpha \in \mathcal{T}$ for an arbitrary indexing set A.

By direct comparison, a topology is not automatically a σ-algebra, since it may not be closed under complements.

Again in analogy to topology, recall that continuous functions are the morphisms of topological spaces. Thus, we can ask which functions can be considered to be the morphisms of measurable spaces. Indeed, just as continuous functions are topologically characterized by preserving open sets under preimages, we define measurable space morphisms similarly:

Definition 1.4

A function $f : X \to Y$ for measurable spaces X, Y is said to be a **measurable function** if $f^{-1}(A) \in \mathrm{Meas}(X)$ whenever $A \in \mathrm{Meas}(Y)$.

It follows immediately that the composition of measurable functions is measurable.

As with topologies, any set automatically comes equipped with two σ-algebras: the power set $\mathcal{P}(X)$ and $\{\varnothing, X\}$. These are the largest and smallest σ-algebras on X, respectively.

Example 1.5

Let $X = \{1, 2, 3, 4\}$. Then the following is a nontrivial σ-algebra:

$$\mathcal{M} = \{\varnothing, X, \{1, 2\}, \{3, 4\}\}$$

Generalizing the above, for any $A \subseteq X$, the σ-algebra $\{\varnothing, X, A, X \setminus A\}$ is the smallest σ-algebra containing A.

Remark 1.1

The arbitrary intersection of σ-algebras on a common set is again a σ-algebra, but not necessarily unions.

Definition 1.5

Let $f : X \to Y$, where X is an arbitrary set and Y is a measurable space. Then the σ-algebra $\sigma(f)$ **generated** by f is

$$\sigma(f) \coloneqq \left\{f^{-1}(A) : A \in \mathrm{Meas}(Y)\right\}$$

Essentially, $\sigma(f)$ is generated by pulling back the measurable structure of Y through f. It is straightforward to verify that $\sigma(f)$ is actually a σ-algebra, and it follows that $\sigma(f)$ is the smallest σ-algebra on X such that f is measurable. In other words, if $\mathcal{M}$ is a σ-algebra on X, then f is measurable with respect to $(X, \mathcal{M}), Y$ if and only if $\sigma(f) \subseteq \mathcal{M}$.

We can generalize the construction of "smallest σ-algebra" type constructions to find the smallest σ-algebra containing a certain collection of subsets. It is somewhat nonobvious that such an algebra exists or is unique.

Theorem 1.4

Let $\mathcal{F} \subseteq \mathcal{P}(X)$. Then there exists a unique minimal σ-algebra $\sigma(\mathcal{F})$ on X such that $\mathcal{F} \subseteq \sigma(\mathcal{F})$.

Proof. Let Ω be the set of collection of all σ-algebras on X which contain $\mathcal{F}$. Ω is nonempty since $\mathcal{P}(X) \subseteq \Omega$. Define

$$\sigma(\mathcal{F}) = \bigcap_{\mathcal{M} \in \Omega} \mathcal{M}$$

Since the arbitrary intersection of σ-algebras is a σ-algebra, $\sigma(\mathcal{F})$ is indeed a σ-algebra. Moreover, by construction $\sigma(\mathcal{F})$ is contained in any element of Ω, and it is thus minimal. $\square$

As we remarked above, a topology is not in general a σ-algebra. The two notions are linked by considering the Borel σ-algebra, which is generated by the open sets on a space.

Definition 1.6

Let X be a topological space with topology $\mathcal{T}$. Then the **Borel σ-algebra** on X is given by $\mathscr{B}(X) = \sigma(\mathcal{T})$.

Note that since σ-algebras are closed under complements, by definition the closed sets on X are in $\mathscr{B}(X)$. It is also the case that countable intersections of open sets and countable unions of closed sets are contained in $\mathscr{B}(X)$, when this is not necessarily true in $\mathcal{T}$. Elements of a Borel σ-algebra are called **Borel sets**. In general, when we refer to topological spaces without specifying a σ-algebra, the Borel algebra is implicitly taken.

Under Hausdorff's terminology, sets which are the countable union of closed sets are denoted F_σ sets. Analogously, sets which are the countable intersection of open sets are denoted G_δ sets.

To make more precise the connection between topologies and measurable spaces through the Borel σ-algebra, we make the following claim:

Proposition 1.5

Let $f : X \to Y$ be a mapping between topological spaces such that $f^{-1}(V) \in \mathscr{B}(X)$ for any open set $V \subseteq Y$. Then f is measurable with respect to $\mathscr{B}(X), \mathscr{B}(Y)$.

Proof. Define the collection

$$\mathcal{M} = \left\{A \in \mathcal{P}(Y) : f^{-1}(A) \in \mathscr{B}(X)\right\}$$

It can be verified that $\mathcal{M}$ is a σ-algebra on Y. Then, by assumption the open sets in Y are contained in $\mathcal{M}$. Moreover, by definition $\mathscr{B}(Y)$ is the smallest σ-algebra containing the open sets. Therefore we have $\mathrm{Open}(Y) \subseteq \mathscr{B}(Y) \subseteq \mathcal{M}$. Since $\mathscr{B}(Y)$ is contained in $\mathcal{M}$ it follows by definition that f is measurable with respect to $\mathscr{B}(X), \mathscr{B}(Y)$. $\square$

Note that the above proposition implies that any continuous mapping between topological spaces is measurable with respect to their Borel algebras. We prove the following statement, which will aid our understanding of complex measurable functions:

Proposition 1.6

Let X be a measurable space and Y a topological space. Let $u, v : X \to \mathbb{R}$ be measurable and $\varphi : \mathbb{R}^2 \to Y$ be continuous. Then $h : X \to Y$ defined by

$$h(x) = \varphi(u(x), v(x))$$

is measurable with respect to $\mathrm{Meas}(X), \mathscr{B}(Y)$.

Proof. From the previous proposition, φ is measurable with respect to $\mathscr{B}(\mathbb{R})$ and $\mathscr{B}(Y)$. Let $f : X \to \mathbb{R}^2$ be $x \mapsto (u(x), v(x))$. Then $h = \varphi \circ f$, and the composition of measurable functions is measurable. So it suffices to show f is measurable with respect to $\text{Meas}(X), \mathscr{B}(\mathbb{R})$.

Take some rectangle $R = I_1 \times I_2$ for intervals I_1, I_2. Then $f^{-1}(\mathbb{R}) = u^{-1}(I_1) \cap v^1(I_2)$. $f^{-1}(\mathbb{R})$ is then a measurable set since both u, v are measurable functions. Now, let $V \in \text{Open}(\mathbb{R}^2)$. Then V can be written as the countable union of rectangles. So we have

$$f^{-1}(V) = f^{-1}\left(\bigcup_{n=1}^{\infty} R_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(R_n) \in \text{Meas}(X)$$

From the previous proposition it follows that f is measurable. $\square$

We can now use this fact to produce measurable functions from other measurable functions.

Theorem 1.7

Let X be a measurable space. Then:

1. If $u, v : X \to \mathbb{R}$ are measurable, then so is $u + iv : X \to \mathbb{C}$.
2. If $f : X \to \mathbb{C}$ is measurable, then so are $\text{Re}(f), \text{Im}(f), |f|$.
3. If $f, g : X \to \mathbb{C}$ are measurable then $f + g$ and fg are both measurable.
4. If $A \in \text{Meas}(X)$ then $\chi_A : X \to \mathbb{R}$ is measurable as well.
5. If $f : X \to \mathbb{C}$ is measurable then there exists $\alpha : X \to \mathbb{C}$ measurable such that $f = \alpha|f|$.

It is often of interest to us to work in the extended real line, so that we can consider functions or measures which assign infinite values to some points or sets. This is also helpful since the extended real line is compact.

Definition 1.7

The **extended real line** is denoted $[-\infty, \infty]$ or $\overline{\mathbb{R}}$, and consists of the set $\mathbb{R} \cup \{\pm\infty\}$, together with the topology that contains open sets in $\mathbb{R}$ and countable unions of sets of the form $(a, \infty]$ and $[-\infty, a)$.

Theorem 1.8

Let $f : X \to \overline{\mathbb{R}}$ with X a measurable space. If

$$f^{-1}((a, \infty]) \in \text{Meas}(X)$$

for all $a \in \mathbb{R}$, then f is measurable.

Proof. The point is to show that any open set in $\overline{\mathbb{R}}$ may be constructed countably from sets of the form $(a, \infty]$.

First we consider sets of the form $[-\infty, a)$. Let $\{a_n\} \to a$ be a sequence of points with $a_n < a$ for all a_n. Then

$$[-\infty, a) = \bigcup_{n=1}^{\infty} [-\infty, a_n] = \bigcup_{n=1}^{\infty} (a_n, \infty]^c$$

so $f^{-1}([-\infty, a)) \in \text{Meas}(X)$. We can similarly write

$$(a, b) = [-\infty, b) \cap (a, \infty]$$

so that $f^{-1}((a, b)) \in \text{Meas}(X)$ as well. Now it follows that any open set in the topology on $\overline{\mathbb{R}}$ has a preimage in $\text{Meas}(X)$, so it follows that f is measurable with respect to the Borel algebra on $\overline{\mathbb{R}}$. □

Theorem 1.9

Let $f_n : X \to \overline{\mathbb{R}}$ be a sequence of measurable functions. Then the functions $\sup f_n, \limsup f_n, \inf f_n, \liminf f_n$, which are defined pointwise, are all measurable.

Proof. By the previous theorem, it suffices to check that $(\sup f_n)^{-1}((a, \infty])$ is measurable for all $a \in \mathbb{R}$, which we will do by expressing these sets as countable unions of preimages through the individual f_n.

We claim that

$$(\sup f_n)^{-1}((a, \infty]) = \bigcup_{n=1}^{\infty} f_n^{-1}((a, \infty])$$

While this is not true in general, it holds for the half-open infinite intervals. We show double inclusion.

($\subseteq$) Let $x \in (\sup f_n)^{-1}((a, \infty])$. Then $\sup f_n(x) > a$. Thus there exists n such that $f_n(x) > \sup f_n - \varepsilon$ for ε sufficiently small that $\sup f_n - \varepsilon > a$. So $x \in f_n^{-1}(a, \infty]$.

($\supseteq$) Similarly, if $x \in \bigcup_{n=1}^{\infty} f_n^{-1}((a, \infty])$, then there exists n with $f_n(x) > a$, which then implies that $\sup f_n(x) > a$ as well.

By hypothesis, $f_n^{-1}((a, \infty]) \in \text{Meas}(X)$ for all n. Thus $\sup f_n$ is measurable. Of course this is true for inf as well.

To show that lim sup is measurable as well, we simply use the representation of lim sup as

$$\limsup a_n = \inf_{n \geq 1} \left(\sup_{m \geq n} a_m \right)$$

Thus $\limsup f_n$ and $\liminf f_n$ are both measurable as well. □

Corollary 1.10

If $\lim f_n$ exists and each $f_n : X \to \overline{\mathbb{R}}$ is measurable, then so is $\lim f_n$.

Proof. If the limit exists then it is equal to both the lim sup and lim inf. □

Corollary 1.11

If $f, g : X \to \overline{\mathbb{R}}$ are measurable then so is $\max\{f, g\}$ and $\min\{f, g\}$.

Proof. Define $f_1 = f$ and $f_n = g$ for all $n \geq 2$. □

The following theorem, which is a direct result of the above, is useful for considering an arbitrary function in terms of two nonnegative functions, which are easier to work with.

Proposition 1.12

For any $f : X \to \overline{\mathbb{R}}$, we can decompose it into positive and negative parts as $f = f^+ - f^-$, with

$$f^+ := \max\{f, 0\}$$
$$f^- := -\min\{f, 0\}$$

If f is measurable then so are f^+, f^-.

Proof. Based on the previous theorems, we just need to show that the constant zero function is measurable. But this is clear since the preimage of any subset of $\overline{\mathbb{R}}$ will be all of X if the subset contains 0, and $\varnothing$ otherwise. □

1.3 Measures and Integration

Our next goal is to define integration of measurable functions. To do so, we will first consider simple functions, which will be the smallest building blocks that we define an integral on.

Definition 1.8

A function $s : X \to \mathbb{C}$ is a **simple function** if it has finite image. A simple nonnegative function is a simple function $s : X \to [0, \infty)$.

Because a simple function s assumes only finitely many values, we can always express it as the weighted sum of characteristic functions:

$$s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$$

where the α_i are the values in the image, and the A_i are their preimages.

Proposition 1.13

A simple function expressed as

$$s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$$

is measurable if and only if each A_i is measurable.

Proposition 1.14

Products and sums of simple functions are simple.

Proof. Clearly there are only finitely many values in the image. □

The utility of simple functions is that we may use them to approximate arbitrary measurable functions. Thus, so long as our integral operator interchanges with limits, we will be free to define integrals solely over simple functions.

Theorem 1.15

Let $f : X \to [0, \infty]$ be measurable. Then there exists a sequence of simple nonnegative measurable functions $s_n : X \to [0, \infty)$ such that:

- $0 \leq s_1 \leq s_2 \leq \ldots \leq f$.
- $s_n \to f$ pointwise.

Proof. We first provide an approximation for the identity, and then compose this with our function f. This approximation is made easier since we only need a pointwise limit. Thus we can consider a step function which both has finer steps (in order to approach the identity), and approximates the identity on a larger range (so that at there are always finite points in the range). Thus, we define

$$\varphi_n(t) = \begin{cases} 2^{-n}\lfloor 2^n t \rfloor, & 0 \leq t < n \\ n, & t \geq n \end{cases}$$

φ_n is simple since it has $\sim 2^{-n}$ values in its image. Additionally its preimages are half-open intervals so φ_n is measurable.

Now, we need to show that $\varphi_n \leq \varphi_{n+1}$ and φ_n converges to the identity pointwise. To show this, we prove that $t - 2^{-n} < \varphi_n(t) \leq t$ for all t. Then it is clear that as $n \to \infty$, φ_n approaches the identity.

Now, the conclusion to the proof is to set $s_n \coloneqq \varphi_n \circ f$. s_n is simple since we factor through the simple function φ_n, and it is measurable as the composition of measurable functions. □

Now, we have established the technical background to define integration of simple functions. To do this, we essentially just assign each possible preimage of the simple functions a weight (which must be additive). Such a weight is a generalization of the notions of area, volume, mass, and so on, and is called a measure. We make two slightly different definitions for real and complex measures.

Definition 1.9

A **complex measure** on X is a function $\mu : \mathrm{Meas}(X) \to \mathbb{C}$ which is countably additive, meaning that whenever $\{A_n\}$ is a countable sequence of pairwise disjoint measurable sets, we have

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

Definition 1.10

A **nonnegative measure** on X is a map $\mu : \mathrm{Meas}(X) \to [0, \infty]$ which is countably additive, and such that there is at least one set $A \in \mathrm{Meas}(X)$ with finite measure. A **measure space** is a triple $(X, \mathcal{M}, \mu)$ where $(X, \mathcal{M})$ is a mesaurable space and μ is a measure on $(X, \mathcal{M})$.

Note it follows that $\mu(\varnothing) = 0$, which would not be true in the nonnegative case if we did not require the existence of a finite measure set.

Proposition 1.16

If μ is a nonnegative meausre on X, then:

1. $\mu(\varnothing) = 0$.
2. μ is finitely additive.
3. If $A \subseteq B$ then $\mu(A) \leq \mu(B)$.
4. If $A_1 \subseteq A_2 \subseteq \dots$ is a countable sequence of measurable sets, then
$$\lim \mu(A_n) = \mu\left(\bigcup A_n\right)$$
5. If $A_1 \supseteq A_2 \supseteq \dots$ is a countable sequence of measurable sets and at least one A_n has finite measure, then
$$\lim \mu(A_n) = \mu\left(\bigcap A_n\right)$$

Roughly speaking, (4) and (5) tell us that measures may be approximated from either inside or outside.

Proof. 1. Take A measurable with finite measure, and consider the sequence $A_1 = A$, $A_n = \varnothing$ For $n \geq 2$. Then

$$\infty > \mu(A) = \mu\left(\bigcup A_n\right) = \sum \mu(A_n) = \mu(A) + \sum \mu(\varnothing)$$

which implies that we must have $\mu(\varnothing) = 0$.

2. Follows from countable additivity now that we know $\mu(\varnothing) = 0$.

3. We write $B = A \sqcup (B \setminus A)$ and apply (2).

4. Define $B_1 = A_1$, $B_2 = A_2 \setminus A_1$, and $B_n = A_n \setminus A_{n-1}$ for $n \geq 2$. Then apply countable additivity.

5. EXERCISE □

The most important example of a nonnegative measure is the Lebesgue measure. Because it is harder to define, we start by defining a few simpler measures.

Definition 1.11

Let X be a measurable space with $\mathrm{Meas}(X) = \mathcal{P}(X)$. The **counting measure** is defined as $c : \mathrm{Meas}(X) \to [0, \infty]$ such that $c(A)$ is the cardinality of A (possibly infinite).

Definition 1.12

Let X be a measurable space with $\mathrm{Meas}(X) = \{\varnothing, X, \{x_0\}, X \setminus \{x_0\}\}$ for some distinguished point x_0. Then the **Dirac delta measure** at x_0 is defined by

$$S \mapsto \begin{cases} 1, & x_0 \in S \\ 0, & x_0 \notin S \end{cases}$$

We can now define the integral of a positive function against a measure. We will do so by first defining the integral of simple functions, then passing to the limit.

Definition 1.13

Let $\mu : \mathrm{Meas}(X) \to [0, \infty]$ be a nonnegative measure. Let $s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$ be a simple measurable function, and let $E \in \mathrm{Meas}(X)$. Then we define the **Lebesgue integral** of s over E with respect to μ to be

$$\int_E s \, \mathrm{d}\mu := \sum_{i=1}^{n} \alpha_i \mu(A_i \cap E)$$

By convention, if $\alpha_i = 0$ on a set of infinte measure, the entire term is considered to be zero.

Definition 1.14

Let $f : X \to [0,\infty]$ be measurable, and let $\mu : \mathrm{Meas}(X) \to [0,\infty]$ be a nonnegative measure. Let $E \in \mathrm{Meas}(X)$. Then the **Lebesgue integral** of f over E with respect to μ is

$$\int_E f \,\mathrm{d}\mu := \sup_{0 \le s \le f} \int_E s \,\mathrm{d}\mu$$

where the supremum is taken over all nonnegative simple measurable functions which satisfy $0 \le s \le f$.

Note that the second definition agrees with the first since the supremum is attained by f.

Example 1.6

Set $X = \mathbb{N}$, $\mathrm{Meas}(X) = \mathcal{P}(X)$, and c to be the counting measure on X. Then

$$\int_A f \,\mathrm{d}c = \sum_{x \in A} f(x)$$

when $A \subseteq \mathbb{N}$. This is clear for finite A but requires limit theorems for countable A. Thus we have represented the sum as an integral against the counting measurable, meaning that our integral theorems will apply to sums as well.

1.4 Limit Theorems

We now turn to the question of interchanging the limit operator and integral, which is a major motivation for the definition of the integral in this way. We begin first with a few elementary properties.

Proposition 1.17

Let $0 \le f \le g$ be nonnegative measurable functions. Then:

1. $\int f \le \int g$.
2. If $A \subseteq B$ then $\int_A f \le \int_B f$.
3. If $0 \le c < \infty$, then $\int cf = c \int f$.
4. If $f \equiv 0$ then $\int_E f = 0$ for any measurable E, even if E has infinite measure.
5. If E is measurable with $\mu(E) = 0$, then $\int_E f = 0$.
6. For E measurable, $\int_E f = \int \chi_E f$.

Theorem 1.18

Let $s, t \geq 0$ be nonnegative simple functions and μ a measure. Define

$$\varphi_s(E) = \int_E s \, \mathrm{d}\mu$$

Then φ_s is a measure, and $\varphi_{s+t} = \varphi_s + \varphi_t$.

Proof. Let $E = \bigsqcup E_i$ be the disjoint countable union of some E_i. By definition,

$$\varphi_s(E) = \sum_{i=1}^{n} \alpha_i \mu(E \cap A_i) = \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{\infty} \mu(E_j \cap A_i)$$

Because s is simple we can interchange the finite sum:

$$\sum_{i=1}^{n} \alpha_i \sum_{j=1}^{\infty} \mu(E_j \cap A_i) = \sum_{j=1}^{\infty} \sum_{i=1}^{n} \alpha_i \mu\left(E_j \cap E_i\right) = \sum_{j=1}^{\infty} \varphi_s(E_j)$$

Thus φ_s is a measure. Linearity follows since we are only adding two simple functions, and so there are at most finitely many sets to work with. $\square$

Example 1.7

To give an example of a sequence where the limit and integral cannot be interchanged, define $f_n = n\chi_{(0,1/n)}$. Then $\int f_n = 1$ for all n, but the pointwise limit is 0 everywhere.

We now prove our first limit theorem:

Theorem 1.19: Monotone Convergence Theorem

Let $0 \leq f_n \nearrow f \leq \infty$ be a sequence of nonnegative measurable functions. Then f is measurable and

$$\int f_n \to \int f$$

Proof. First note that the sequence $\int f_n$ is monotone increasing, so it has a limit (in the extended reals). Thus we have

$$L = \lim \int f_n \leq \int f$$

Pick a simple function $s \leq f$ and $\varepsilon < 1$. We want to show that $L \geq \varepsilon \int s$, which will then prove the result by taking $\varepsilon \to 1$ and $s \to f$.

For each n, define

$$E_n = \{x : f_n(x) \geq \varepsilon s(x)\}$$

For any point $x \in X$, we have $f_n(x) \to f(x) > \varepsilon s(x)$, so

$$\bigcup E_n = X$$

Then for each n we have

$$\int_{E_n} \varepsilon s \leq \int_{E_n} f_n \leq \int_X f_n \to L$$

We also have

$$\int_{E_n} \varepsilon s \to \int_X \varepsilon s$$

so

$$\int \varepsilon s \leq L$$

for all $\varepsilon < 1, s \leq f$. Thus

$$\int f \leq L$$

so we have both inequalities and thus

$$\int f = L = \lim \int f_n$$

□

Corollary 1.20

If f, g are nonnegative and measurable then $\int f + g = \int f + \int g$.

Proof. Take two sequences of simple functions $s_i \nearrow f$ and $t_i \nearrow g$. The monotone convergence theorem gives the result. □

Corollary 1.21

If $f_n \geq 0$ is a sequence of nonnegative measurable functions then

$$\int \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \int f_n(x)$$

Proof. Combine the monotone convergence theorem with the previous corollary. □

Corollary 1.22

If a_{ij} is a sequence of nonnegative numbers then

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$$

Proof. We write one of the sums as an integral with the counting measure. □

Lemma 1.23: Fatou's Lemma

Let $f_n \geq 0$ be a sequence of nonnegative measurable functions. Then

$$\int \liminf f_n \leq \liminf \int f_n$$

Proof. Define $g_n(x) = \inf_{m \geq n} f_m(x)$. Then by definition, $g_n \nearrow \liminf f_n$. Also $\int g_n \leq \int f_n$ for each n. So by monotone convergence we have

$$\int \liminf f_n = \lim \int g_n = \liminf \int g_n \leq \liminf \int f_n \qquad \square$$

Having established limit theorems for nonnegative functions, we now make our definition of arbitrary integrals.

Definition 1.15

Let $f : X \to \overline{\mathbb{R}}$ be a measurable function. Writing $f = f^+ - f^-$, we define

$$\int f = \int f^+ - \int f^-$$

For a complex measurable function $F = u + iv : X \to \mathbb{C}$, we define

$$\int F = \int u + i \int v$$

Clearly this definition agrees with our previous one. However, there is a slight subtlety, which is that our definition may end up with an expression like $\infty - i\infty$. As such, we restrict this definition to those f which make the integral absolutely convergent (meaning $\int |f| < \infty$).

Proposition 1.24

For f measurable,

$$\left| \int f \right| \leq \int |f|$$

Proof. For f real valued, we write

$$\left| \int f \right| = \left| \int f^+ - \int f^- \right| \leq \left| \int f^+ \right| + \left| \int f^- \right| = \int f^+ + \int f^- = \int |f| \qquad \square$$

A similar proof shows the result for complex functions.

Our new integral inherits the properties we have shown for integrals of nonnegative functions, assuming the limits are finite. To capture this we make the following classification:

Definition 1.16

Let μ be a measure on X. Then we define the L^1 **space** to be

$$L^1(\mu) = \left\{ f : X \to \mathbb{C} : \int |f| \, \mathrm{d}\mu < \infty \right\}$$

Theorem 1.25: Dominated Convergence Theorem

If $f_n \to f$ and there exists $g \in L^1$ such that $|f_n| \leq g$, then:

- $f_n \in L^1$,
- $\lim \int |f - f_n| = 0$ (equivalently, $f_n \to f$ in L^1),
- $\lim \int f_n = \int f$ (weak convergence)

Proof. First note that we have

$$|f_n| \leq g \longrightarrow |f| \leq g$$

so $f_n, f \in L^1$. Moreover, we have

$$|f_n - f| \leq 2g$$

so the differences are in L^1 as well. Moreover, we have $2g - |f_n - f| \geq 0$. Thus we can apply Fatou's lemma:

$$\begin{aligned} \int 2g = \int \lim \left(2g - |f - f_n|\right) = \int \liminf \left(2g - |f - f_n|\right) \\ \leq \liminf \int \left(2g - |f - f_n|\right) = \int 2g + \liminf \int -|f - f_n| \end{aligned}$$

Because $\int 2g < \infty$, we can subtract it from both sides to see that

$$0 \leq \liminf \left(- \int |f - f_n| \right) \implies \limsup \int |f - f_n| \leq 0$$

Since the RHS is nonnegative we conclude that $\lim \int |f - f_n|$ exists and is equal to zero. To demonstrate weak convergence, we have

$$\left| \int f_n - \int f \right| = \left| \int f_n - f \right| \leq \int |f_n - f| \to 0$$

□

Example 1.8

Consider $f_n = n\chi_{(0,1/n^2)}$. These functions are bounded by $g(x) = \frac{1}{\sqrt{x}} \in L^1$. Moreover, we have

$$\lim \int f_n = \lim \frac{1}{n} = 0 = \int 0 = \int \lim f_n$$

Chapter 2

The Lebesgue Measure

To this point we have defined integrals in a way that allows us to interchange them with limit operators in various settings. We have also defined an appropriate σ-algebra, $\mathscr{B}(\mathbb{R})$, on $\mathbb{R}$, which we can use to work with this integral. Now we have to define a measure on $\mathscr{B}(\mathbb{R})$ that extends the Riemann integral. To make this definition we will essentially present an existence and uniqueness proof.

More precisely we show that there exists a unique positive, *translation invariant* meausre $\lambda : \mathscr{B}(\mathbb{R}) \to [0, \infty]$ such that $\lambda([0, 1]) = 1$.

In this search it will also turn out that the measurable sets under λ is larger than the Borel algebra.

Definition 2.1

For a set $S \subseteq \mathbb{R}$ and $x \in \mathbb{R}$, we define **translation** by

$$S + x \coloneqq \{s + x : s \in S\}$$

A measure μ is called **translation invariant** if $\mu(S) = \mu(S + x)$ for all x, S.

Our work will involve first developing theorems about how to construct measures out of more primitive objects. Applying this to $\mathbb{R}$ with some geometric intuition will give us the Lebesgue measure.

2.1 Premeasures and Outer Measures

Consider some nonempty set X, and let $\rho : E \to [0, \infty]$ be a map which is initially defined on some subset E of $\mathcal{P}(X)$, with $\rho(\varnothing) = 0$. We do not assume that E is a σ-algebra; however it will generate the σ-algebra that is used by the final measure.

Definition 2.2

If X is nonempty, an **outer measure** on X is a map $\varphi : \mathcal{P}(X) \to [0, \infty]$ such that

1. $\varphi(\varnothing) = 0$,
2. $\varphi(A) \leq \varphi(B)$ whenever $A \subseteq B$,
3. $\varphi\left(\bigcup A_n\right) \leq \sum \varphi(A_n)$ for any countable collection of sets A_n.

Note that an outer measure is not a measure.

We now define an outer measure $\varphi_\rho : \mathcal{P}(X) \to [0, \infty]$ using the data from ρ.

Proposition 2.1

Let X be nonempty, $\rho : E \to [0, \infty]$ for $E \subseteq \mathcal{P}(X)$ containing $\{\varnothing, X\}$, and $\rho(\varnothing) = 0$. Then the function $\varphi_\rho : \mathcal{P}(X) \to [0, \infty]$ defined by

$$\varphi_\rho(A) \coloneqq \inf\left\{\sum \rho(E_n) : \{E_n\}_{n\in\mathbb{N}} \subseteq E, A \subseteq \bigcup E_n\right\}$$

is an outer measure. Here the infimum is over all countable covers of A with elements of E. If no such cover exists then by definition the infimum is ∞.

Proof. It is clear that $\varphi_\rho(\varnothing) = 0$ since we can take the cover to be $E_n = \varnothing$. To show monotonicity, if $A \subseteq B$ then any cover of B covers A, so $\varphi_\rho(A) \leq \varphi_\rho(B)$ (this still holds when one or both sets admit no covers).

If $A = \bigcup A_n$, then for any $\varepsilon > 0$ we can pick covers $\{E_{n,i}\}_i$ for each n such that

$$\sum_{i=1}^{\infty} \rho(E_{n,i}) \geq \varphi_\rho(A_n) - \frac{\varepsilon}{2^n}$$

Then the collection $\{E_{n,i}\}_{n,i}$ is a countable cover of A, and we have

$$\sum_{n,i} \rho(E_{n,i}) = \sum_n \sum_i \rho(E_{n,i}) = \sum_n \left(\varphi(A_n) - \frac{\varepsilon}{2^n}\right) = \sum_n \varphi_\rho(A_n) - \varepsilon$$

Taking $\varepsilon \to 0$ and taking the infimum, we conclude that

$$\varphi_\rho\left(\bigcup_n A_n\right) \leq \sum_n \varphi_\rho(A_n)$$

□

Example 2.1

Taking E to be the set of intervals and letting $\rho((a, b)) = b - a$, we generate the Lebesgue outer measure.

So far we have placed no assumptions on ρ. In order to get outer measures and measures which we can work with nicely, it is helpful to impose a few conditions. To see this, we examine some possible difficulties with pathological ρ. For instance, if ρ itself is not countably additive, then φ_ρ could fail to coincide with ρ on E.

Definition 2.3

If φ is an outer measure on X, a set $A \subseteq X$ is called **φ-measurable** if for all $Q \in \mathcal{P}(X)$,

$$\varphi(Q) = \varphi(Q \cap A) + \varphi(Q \cap A^c)$$

The set of φ-measurable sets is denoted $\mathscr{A}_\varphi$.

Essentially, a φ-measurable set splits with respect to measure. It is not a priori obvious that nonmeasurable sets should exist under this definition, but we will see later that they do. Note that we always have

$$\varphi(Q) \leq \varphi(Q \cap A) + \varphi(Q \cap A^c)$$

by countable subadditivity of φ. Thus in general we can check φ-measurability just by verifying that

$$\varphi(Q) \geq \varphi(Q \cap A) + \varphi(Q \cap A^c)$$

Moreover, when $\varphi(Q) = \infty$ this is automatically true.

A natural question is then to ask whether φ_ρ-measurable sets form a σ-algebra. The answer to this question is yes; moreover the restriction theorem that we prove shows that φ_ρ is also a measure when restricted to these sets.

Theorem 2.2: Caratheodory's Restriction Theorem

Let X be a nonempty set and φ an outer measure on X. Then $\mathscr{A}_\varphi$ is a σ-algebra, and $\mu_\varphi := \varphi|_{\mathscr{A}_\varphi}$ is a measure.

Proof. Take $\varnothing$. By the remark above, it suffices to show that for any $Q \in X$,

$$\varphi(Q) \geq \varphi(Q \cap \varnothing) + \varphi(Q \cap \varnothing^c)$$

But this is clear since the right hand side is just

$$\varphi(\varnothing) + \varnothing(Q) = \varnothing(Q)$$

It is also obvious that $\mathscr{A}_\varphi$ is closed under complements since the definition treats A, A^c symmetrically.

To show closure under countable unions, we first show finite unions. For $A, B \in \mathscr{A}_\varphi$, and pick $Q \in \mathcal{P}(X)$ with $\varphi(Q) < \infty$ (recall from above that we can assume finite outer measure). Then

$$\begin{aligned} \varphi(Q) &= \varphi(Q \cap A) + \varphi(Q \cap A^c) \\ &= \varphi(Q \cap A \cap B) + \varphi(Q \cap A \cap B^c) + \varphi(Q \cap A^c \cap B) + \varphi(Q \cap A^c \cap B^c) \end{aligned}$$

We have the identity

$$A \cup B = (A \cap B) \cup (A \cap B^c) \cup (A^c \cap B)$$

Since φ is an outer measure, it follows that

$$\varphi(A \cup B) \leq \varphi(A \cap B) + \varphi(A \cap B^c) + \varphi(A^c \cap B)$$

take a sequence $\{A_n\} \subseteq \mathscr{A}_\varphi$. Intersecting with Q on both sides, we have

$$\varphi(Q) \geq \varphi(Q \cap (A \cup B)) + \varphi(Q \cap A^c \cap B^c) = \varphi(Q \cap (A \cup B)) + \varphi(Q \cap (A \cup B)^c)$$

Now we extend this to countable unions $\bigcup A_n$. It suffices to assume that the A_n are pairwise disjoint by picking

$$A_n' = A_n \setminus \left(\bigcup_{m=1}^{n-1} A_m \right)$$

Then A_n' are in $\mathscr{A}_\varphi$ by our work showing that complements and finite unions were closed. Now take Q with $\varphi(Q) < \infty$. Then for any N, $\bigcup^N A_n \in \mathscr{A}_\varphi$. Therefore we can write

$$\begin{aligned}
\varphi(Q) &= \varphi\left(Q \cap \left(\bigcup_{n=1}^N A_n \right) \right) + \varphi\left(Q \cap \left(\bigcup_{n=1}^N A_n \right)^c \right) \\
&\geq \varphi\left(Q \cap \left(\bigcup^N A_n \right) \cap A_n \right) + \varphi\left(Q \cap \left(\bigcup^N A_n \right) \cap A_N^c \right) \\
&= \varphi\left(Q \cap (A)_N\right) + \varphi\left(Q \cap \left(\bigcup^{N-1} A_n \right) \right) \\
&\vdots \\
&\geq \sum_{n=1}^N \varphi(Q \cap A_n) + \varphi\left(Q \cap \left(\bigcup^N A_n \right)^c \right)
\end{aligned}$$

Now, we have $\bigcup^\infty A_n \supseteq \bigcup^N A_n$, so we have

$$\varphi\left(Q \cap \left(\bigcup^N A_n \right)^c \right) \geq \varphi\left(Q \cap \left(\bigcup^\infty A_n \right)^c \right)$$

Taking $N \to \infty$, we have

$$\varphi(Q) \geq \sum_{n=1}^\infty \varphi(Q \cap A_n) + \varphi\left(Q \cap \left(\bigcup^\infty \varphi(A_n) \right)^c \right)$$

Since φ is countably subadditive,

$$\sum_{n=1}^\infty \varphi(Q \cap A_n) \geq \varphi\left(Q \cap \left(\bigcup^\infty A_n \right) \right)$$

Thus

$$\varphi(Q) \geq \varphi\left(Q \cap \left(\bigcup^{\infty} A_n\right)\right) + \varphi\left(Q \cap \left(\bigcup^{\infty} \varphi(A_n)\right)^c\right)$$

showing that $\bigcup^{\infty} A_n \in \mathscr{A}_\varphi$. Thus $\mathscr{A}_\varphi$ is a σ-algebra.

We know there is a set with finite measure since $\mu_\varphi(\varnothing) = 0$. To demonstrate finite additivity, pick $A, B \in \mathscr{A}_\varphi$ disjoint. Then

$$\mu_\varphi(A \cup B) = \varphi(A \cup B) = \varphi((A \cup B) \cap A) + \varphi((A \cup B) \cap A^c) = \varphi(A) + \varphi(B) = \mu_\varphi(A) + \mu_\varphi(B)$$

The proof for countable additivity is the same. □

It is also worth noting that the measure produced by the restriction theorem has the property of being a "complete" measure.

Definition 2.4

A measure $\mu : \text{Meas}(X) \to [0, \infty]$ is said to be **complete** if for any $M \subseteq N$ with N measurable and $\mu(N) = 0$, M is measurable.

In short, any subset of a measure zero set is measurable. (We already know that any such measurable set has measure zero, but a priori it is not clear that such sets are measurable in the first place).

Proposition 2.3

Given X and φ an outer measure on X, the measure μ_φ as defined in Caratheodory's Restriction Theorem is complete.

Proof. Pick $B \in \mathscr{A}_\varphi$ with $\mu_\varphi(B) = 0$, and take $A \subseteq B$. Take some $Q \subseteq X$ with $\varphi(Q) < \infty$. Then we have

$$Q \cap A \subseteq Q \cap B \subseteq B \implies \varphi(Q \cap A) \leq \varphi(B) = \mu_\varphi(B) = 0$$

Also we have

$$Q \cap A^c \subseteq Q \implies \varphi(Q \cap A^c) \leq \varphi(Q)$$

so

$$\varphi(Q) \geq \varphi(Q \cap A) + \varphi(Q \cap A^c)$$

and thus $A \in \mathscr{A}_\varphi$. □

Proposition 2.4

Given any measure $\mu : \mathcal{M} \to [0, \infty]$, there exists a unique complete measure $\overline{\mu} : \overline{\mathcal{M}} \to [0, \infty]$, wehre $\overline{\mathcal{M}} \supseteq \mathcal{M}$ is another σ-algebra on X and $\overline{\mu}|_{\mathcal{M}} = \mu$.

We have thus illustrated a method to pass from a primitive function $\rho : E \to [0, \infty]$ to a full measure μ_{φ_ρ} on $\mathscr{A}_{\varphi_\rho}$. To that end it is worth investigating the relationship between $\sigma(E)$ and $\mathscr{A}_{\varphi_\rho}$. In order to properly do this it is best to impose additional conditions on ρ.

Definition 2.5

An **algebra** on a set X is a collection $\mathscr{A} \subseteq \mathcal{P}(X)$ which contains X, is closed under complements, and closed under finite unions.

Note the only difference between a σ-algebra and an algebra is we require σ-algebras to be closed under countable unions as well.

Definition 2.6

Let $\mathscr{A} \subseteq \mathcal{P}(X)$ be an algebra. Then a function $\rho : \mathscr{A} \to [0, \infty]$ is called a **premeasure** if:

1. $\rho(\varnothing) = 0$,
2. If $\{A_n\} \subseteq \mathscr{A}$ is a countable collection of pairwise disjoint sets, and if in addition $\bigcup A_n \in \mathscr{A}$, then
$$\rho\left(\bigcup A_n\right) = \sum \rho(A_n).$$

Proposition 2.5

If $\rho : \mathscr{A} \to [0, \infty]$ is a premeasure, then for $A \subseteq B$ with $A, B \in \mathscr{A}$, $\rho(A) \leq \rho(B)$.

Proof. $B \setminus A \in \mathscr{A}$, so by countable additivity

$$\rho(B) = \rho(A) + \rho(B \setminus A) \geq \rho(A)$$ □

Adding the condition that ρ is a premeasure ensures that our extended constructions are proper extensions, in the sense that they are consistent with our original data.

Proposition 2.6

Suppose $\rho : \mathscr{A} \to [0, \infty]$ is a premeasure. Then $\varphi_\rho|_{\mathscr{A}} = \rho$. Moreover, $\mathscr{A} \subseteq \mathscr{A}_{\varphi_\rho}$.

Proof. Note that clearly $\varphi_\rho|_{\mathscr{A}} \leq \rho$, since any set in $\mathscr{A}$ is a cover for itself.

For the reverse inequality, pick $Q \in \mathscr{A}$ and a countable cover $\{E_n\} \subseteq \mathscr{A}$. Note that by monotonicity we can assume the E_n are disjoint (call the disjoint parts $\{F_n\}$), and we can also assume that $Q = \sqcup F_n$; that is the F_n do not overcover Q. Then we have

$$\rho(Q) = \rho\left(\sqcup_{n=1}^{\infty} F_n\right) = \sum_{n=1}^{\infty} \rho(F_n) \leq \sum_{n=1}^{\infty} \rho(E_n)$$

So $\rho \leq \varphi_\rho|_{\mathscr{A}}$ as well and thus $\varphi_\rho|_{\mathscr{A}} = \rho$.

To show that $\mathscr{A} \subseteq \mathscr{A}_{\varphi_\rho}$, pick $A \in \mathscr{A}$ and $Q \subseteq X$. Then by the definition of φ_ρ, we can pick a countable cover $\{E_n\} \subseteq \mathscr{A}$ with

$$\varphi_\rho(Q) \geq \sum_{n=1}^{\infty} \rho(E_n) - \varepsilon$$

Since each $E_n \in \mathscr{A}$ and $A \in \mathscr{A}$, we have $\rho(E_n) = \rho(E_n \cap A) + \rho(E_n \cap A^c)$. Thus we have

$$\varphi_\rho(Q) + \varepsilon \geq \sum_{n-1}^{\infty} \rho(E_n) = \sum_{n=1}^{\infty} \rho(E_n \cap A) + \rho(E_n \cap A^c) = \varphi_\rho(E_n \cap A) + \varphi_\rho(E_n \cap A^c)$$

Since $Q \cap A \subseteq \bigcup E_n \cap A$, by countable subadditivity of φ we have

$$\varphi_\rho(Q \cap A) \leq \sum_{n=1}^{\infty} \varphi_\rho(E_n \cap A)$$

and similarly for the A^c term. Thus we have

$$\varphi_\rho(Q) + \varepsilon \geq \varphi_\rho(Q \cap A) + \varphi_\rho(Q \cap A^c)$$

Taking $\varepsilon \to 0$, we see that A is φ_ρ-measurable and thus $\mathscr{A} \subseteq \mathscr{A}_{\varphi_\rho}$. ☐

Thus, since we can now properly use premeasures to build outer measures, we can apply the restriction theorem to actually extend premeasures.

Theorem 2.7: Caratheodory's Extension Theorem

Let $\rho : \mathscr{A} \to [0, \infty]$ be a premeasure and φ_ρ, $\mathscr{A}_{\varphi_\rho}$, μ_{φ_ρ} be as defined above. Then:

1. $\sigma(\mathscr{A}) \subseteq \mathscr{A}_{\varphi_\rho}$;
2. If $\nu : \sigma(\mathscr{A}) \to [0, \infty]$ is any other measure such that $\nu|_{\mathscr{A}} = \rho$, then $\nu \leq \mu_{\varphi_\rho}$ on $\sigma(\mathscr{A})$, and moreover for any $E \in \sigma(\mathscr{A})$ with $\mu_{\varphi_\rho}(E) < \infty$, $\nu(E) = \mu_{\varphi_\rho}(E)$;
3. If X is **σ-finite** with respect to ρ, meaning that there is a countable collection $\{A_n\} \subseteq \mathscr{A}$ with $\rho(A_n) < \infty$ and $X = \bigcup A_n$, then μ_{φ_ρ} is the unique extension of ρ to $\sigma(\mathscr{A})$.

Proof.

1. This is clear since $\mathscr{A} \subseteq \mathscr{A}_{\varphi_\rho}$ with $\mathscr{A}_{\varphi_\rho}$ a σ-algebra. Then by definition $\sigma(\mathscr{A})$ is the smallest σ-algebra containing $\mathscr{A}$ so $\sigma(\mathscr{A}) \subseteq \mathscr{A}_{\varphi_\rho}$.
2. Let $E \in \sigma(\mathscr{A})$ and pick a cover $\{E_n\} \subseteq \mathscr{A}$. Then we have subadditivity (note the E_n are not necessarily disjoint):

$$\nu(E) \leq \sum_{n=1}^{\infty} \nu(E_n) = \sum_{n=1}^{\infty} \rho(E_n)$$

Taking the infimum, we have $\nu(E) \leq \mu_{\varphi_\rho}(E)$.

Note that in general, by approximation from inside, if we pick a sequence of sets $\{E_n\} \subseteq \mathscr{A}$, then

$$\nu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \nu(E_n) = \lim_{n\to\infty} \mu_{\varphi_\rho}(E_n)$$

Suppose $\mu_{\varphi_\rho}(E) < \infty$. By the infimum property, for $\varepsilon > 0$ we can pick a cover $\{E_n\} \subseteq \mathscr{A}$ such that

$$\mu_{\varphi_\rho}\left(\bigcup_{n=1}^{\infty} E_n\right) < \mu_{\varphi_\rho}(E) + \varepsilon \implies \mu_{\varphi_\rho}\left(\left(\bigcup_{n=1}^{\infty} E_n\right) \setminus E\right) < \varepsilon$$

Then we have

$$\mu_{\varphi_\rho}(E) \leq \mu_{\varphi_\rho}\left(\bigcup_{n=1}^{\infty} E_n\right) = \nu\left(\bigcup_{n=1}^{\infty} E_n\right)$$

$$= \nu\left(\left(\bigcup_{n=1}^{\infty} E_n\right) \cap E\right) + \nu\left(\left(\bigcup_{n=1}^{\infty}\right) \cap E^c\right) = \nu(E) + \nu\left(\left(\bigcup_{n=1}^{\infty} E_n\right) \setminus E\right)$$

$$\leq \nu(E) + \mu_{\varphi_\rho}\left(\left(\bigcup_{n=1}^{\infty} E_n\right) \setminus E\right) \leq \nu(E) + \varepsilon$$

3. If X is σ-finite with respect to ρ, then pick a pairwise disjoint countable collection $\{A_n\} \subseteq \mathscr{A}$ with $\rho(A_n) < \infty$ and $X = \bigcup A_n$. Then we have

$$\mu(E) = \sum_{n=1}^{\infty} \mu(E \cap A_n) = \sum_{n=1}^{\infty} \nu(E \cap A_n) = \nu(E) \qquad \square$$

2.2 The Lebesgue Premeasure

Having now developed a theory of how to define measures on simpler sets, we apply this to construct the Lebesgue measure.

To do this, we first need to consider the simpler sets on which we will define a premeasure. For the case of the Levesgue measure, we will take the collection of half open intervals:

$$\mathscr{A}_0 \coloneqq \{\varnothing\} \cup \{(a, b] : a \in [-\infty, \infty), a < b\} \cup \{(a, \infty) : a \in [-\infty, \infty)\} \subseteq \mathcal{P}(\mathbb{R})$$

This is the set of all intervals which are open on the left and closed on the right, with appropriate consideration of infinite endpoints.

Definition 2.7

An **elementary family** on X is a collection $\mathscr{F} \subseteq \mathcal{P}(X)$ such that

1. $\varnothing \in \mathscr{F}$,
2. $\mathscr{F}$ is closed under finite intersections,

3. For any $E \in \mathscr{F}$, $X \setminus E$ is a finite disjoint union of elements of $\mathscr{F}$.

Proposition 2.8

$\mathscr{A}_0$ is an elementary family on $\mathbb{R}$.

Proof. 1 is true by definition.

For intersections, in general we have

$$(a, b] \cap (a', b'] = \begin{cases} \varnothing \\ (a', b'], & a < a' < b' < b \\ \vdots \end{cases}$$

Here we do not show all of the cases but it is true in general. Complements are similar:

$$\begin{aligned} \varnothing^c &= \mathbb{R} \\ (a, b]^c &= (-\infty, a] \cup (b, \infty) \\ &\vdots \end{aligned}$$

□

Proposition 2.9

If $\mathcal{E}$ is an elementary family then the collection $\mathscr{A}$ of finite disjoint unions of elements of $\mathcal{E}$ is an algebra.

Proposition 2.10

$\sigma(\mathscr{A}) = \mathscr{B}(\mathbb{R})$, where $\mathscr{A}$ is the algebra given by the collection of finite disjoint unions of elements of the half open intervals $\mathscr{A}_0$.

Proof. We can write half open intervals as countable intersections of open intervals:

$$(a, b] = \bigcap_{n=1}^{\infty} \left(a, b + \frac{1}{n}\right)$$

so $\sigma(\mathscr{A}_0) \subseteq \mathscr{B}(\mathbb{R})$. In the other direction, we have

$$(a, b) = \bigcup_{n=1}^{\infty} \left(a, b - \frac{1}{n}\right)$$

Here we have handwaved some of the cases away but it is nevertheless true that $\mathscr{B}(\mathbb{R}) \subseteq \sigma(\mathscr{A})$ as well. □

Now, we have developed an suitable algebra to define a premeasure on. (Note that we would have liked to simply define our premeasure on intervals. However this is not an algebra and so not sufficient to define a premeasure on. Nevertheless we can define essentially the same premeasure on disjoint unions of intervals; with some extra work to verify that it is well-defined.)

Definition 2.8

We define the **Lebesgue premeasure** $\rho : \mathscr{A} \to [0, \infty]$ by

$$\rho\left(\bigsqcup_{j=1}^{n}(a_j, b_j]\right) := \sum_{j=1}^{n} b_j - a_j$$
$$\rho(\varnothing) = 0$$

To check that it is well defined, note that alternate representations like $(0, 1] = (0, 1/2] \cup (1/2, 1]$ must "telescope" and hence since they are disjoint unions this is well defined.

Theorem 2.11

ρ is well defined, and it is a premeasure on $\mathscr{A}$.

Note that $\mathbb{R}$ is σ-finite with respect to ρ since it is the union of $[n, n+1]$ for $n \in \mathbb{Z}$.

Definition 2.9

The **Lebesgue measure** on $\mathbb{R}$ is the unique measure $\lambda = \mu_{\varphi_\rho}$ produced by applying Caratheodory's extension theorem to the Lebesgue premeasure ρ. Elements of the associated σ-algebra $\mathscr{A}_{\varphi_\rho}$ are called **Lebesgue measurable** subsets of $\mathbb{R}$.

Caratheodory's extension theorem assures us that λ is the unique extension of ρ to $\mathscr{B}(\mathbb{R})$. However, in some sense we need to remove the arbitrary choice of ρ from this construction for full generality of λ. This is accomplished by showing that λ satisfies another uniqueness condition.

Theorem 2.12

There exists a unique measure $\lambda : \mathscr{B}(\mathbb{R}) \to [0, \infty]$ such that $\lambda([0, 1]) = 1$ and λ is translation invariant.

Proof. Existence is satisfied by the Lebesgue measure we just showed. It is translation invariant since the premeasure it is defined on is, and it is not hard to verify that $[0, 1]$ has Lebesgue measure 1.

For uniqueness, pick another measure $\tilde{\lambda} : \mathscr{B}(\mathbb{R}) \to [0, \infty]$ obeying the hypotheses of the theorem. To show that $\lambda = \tilde{\lambda}$, we will need some regularity properties of Borel measures. We will delay the proofs of these properties, but essentially the statement is that measures on sufficiently nice Borel algebras may be approximated from the outside by open sets and from the inside from compact sets. More formally, for any $\mu : \mathscr{B}(\mathbb{R}) \to [0, \infty]$ and $S \in \mathscr{B}(\mathbb{R})$ we have

$$\mu(S) = \inf \{\mu(U) : S \subseteq U \in \text{Open}(\mathbb{R})\}$$
$$\mu(S) = \sup \{\mu(K) : S \supseteq K \in \text{Comapct}(\mathbb{R})\}$$

In particular it suffices to show that $\lambda, \tilde{\lambda}$ agree on open sets. Moreover since any open set in $\mathbb{R}$ is the countable disjoint union of open intervals, we can just show this for open intervals, and by translation invaraicne we just need to show that for $a > 0$,

$$\lambda((0, b)) = \tilde{\lambda}((0, b))$$

In essence, since we already know $\lambda([0, 1]) = 1$, we need to prove a scaling property of translation invariant measures.

To show that the endpoints of intervals will not be a problem, we show that singletons have measure zero in both $\lambda, \tilde{\lambda}$. If we pick N distinct points $x_1, \ldots, x_N$ in $[0, 1]$, then by translation invariance we have

$$N\lambda(\{0\}) = \lambda\left(\bigcup_{j=1}^{N} \{x_j\}\right) = \lambda([0, 1]) = 1$$

Taking $N \to \infty$, we conclude $\lambda(\{0\}) = 0$, and similarly for $\tilde{\lambda}$. This allows to treat open and closed intervals as the same in measure.

Claim: $\tilde{\lambda}\left(\left[0, \frac{1}{n}\right]\right) = \frac{1}{n}$. This is clear since $[0, 1]$ is formed of n translated copies:

$$1 = \lambda([0, 1]) = \lambda\left(\bigsqcup_{k=0}^{n-2} \left[\frac{k}{n}, \frac{k+1}{n}\right) \cup \left[\frac{n-1}{n}, 1\right]\right) = \sum_{k=0}^{n-1} \lambda\left(\left[\frac{k}{n}, \frac{k+1}{n}\right]\right) = n\lambda\left(\left[0, \frac{1}{n}\right]\right)$$

Similar work shows that $\lambda, \tilde{\lambda}$ obey a rational scaling factor:

$$\tilde{\lambda}\left(\left[0, \frac{m}{n}\right]\right) = \frac{m}{n}$$

To show that this is the case for real endpoints, we can use approximation from inside or outside for arbitrary measures to conclude that for $b > 0$ and $r_n \nearrow b$,

$$\lambda([0, b]) = \lim \lambda([0, r_n]) = \lim r_n = b$$

Thus we conclude that $\lambda = \tilde{\lambda}$ on the open intervals and hence all open sets, so that $\lambda = \tilde{\lambda}$ on $\mathscr{B}(\mathbb{R})$ (pending the regularity conditions for Borel measures). $\square$

2.3 Regularity of Borel Measures

Remark

At the moment the main proof provided here by Prof. Shapiro is incorrect. In part this is due to an incorrect theorem in Bogachev's book. The proof likely holds for metric spaces but may be difficult to repair for arbitrary topological spaces with the given assumptions.

In this section we discuss regularity properties connecting topological and measure spaces over arbitrary sets with sufficiently nice properties. These proofs are taken from Bogachev's *Measure Theory*.

Definition 2.10

Given a topological space X, a **Borel measure** is a measure on $\mathscr{B}(X)$.

Definition 2.11

Let $\mu : \mathscr{B}(X) \to [0, \infty]$ be a Borel measure on X. A set $A \in \mathscr{B}(X)$ is called **μ-outer regular** if

$$\mu(A) = \inf \{\mu(U) : A \subseteq U \in \operatorname{Open}(X)\}$$

and similarly it is **μ-inner regular** if either A is open or $\mu(A) < \infty$, and also

$$\mu(A) = \sup \{\mu(K) : A \supseteq K \in \operatorname{Compact}(X) \cap \mathscr{B}(X)\}$$

We say that μ is **regular** if every Borel set is μ-inner and μ-outer regular.

Definition 2.12

A topological space X is a **Hausdorff space** if for any $x, y \in X$ there exist open sets $x \in U$ and $y \in V$ with U, V disjoint.

Note that in a Hausdorff space, every compact set is closed and hence Borel, so there is no need to check measurability.

This leads us to the natural question of what assumptions must be placed on our topological space X such that every Borel measure is regular. The first is a requirement that essentially ensures the topological operations are compatible with the Borel algebra operations:

Definition 2.13

A topological space X is called **second countable** if there exists a countable basis for its topology.

Definition 2.14

A topological space X is called **σ-compact** if there exists a countable collection of compact sets $\{K_n\}_{n=1}^{\infty} \subseteq \text{Compact}(X)$ such that $X = \bigcup K_n$.

Definition 2.15

A measure $\mu : \mathscr{B}(X) \to [0, \infty]$ is **locally finite** if for any $x \in X$ there exists $U \ni x$ open with $\mu(U) < \infty$.

Proposition 2.13

If $\mu : \mathscr{B}(X) \to [0, \infty]$ is locally finite and X is Hausdorff, then $\mu(K) < \infty$ for any K compact.

Proof. (Note that since X is Hausdorff, K is closed and hence Borel, so it is actually measurable). For each $x \in K$ take $U_x \ni x$ of finite measure. Then $K \subseteq \bigcup_{x \in K} U_x$. Picking a finite subcover, we establish that $\mu(K) < \infty$. $\square$

Proposition 2.14

Let X be a second-countable σ-compact Hausdorff space and $\mu : \mathscr{B}(X) \to [0, \infty]$ a locally finite Borel measure on it. Then for any $\varepsilon > 0$ and $B \in \mathscr{B}(X)$ of finite measure, there exists $U_\varepsilon \in \text{Open}(X), K_\varepsilon \in \text{Compact}(X)$ with $K_\varepsilon \subseteq B \subseteq U_\varepsilon$ with

$$\mu(U_\varepsilon \setminus K_\varepsilon) < \varepsilon$$

Proof. We first substitute closed sets for compact. Define

$$\mathscr{A} = \{A \in \mathscr{B}(X) : \forall \varepsilon > 0 \exists U_\varepsilon \text{ open}, F_\varepsilon \text{ closed, s.t. } F_\varepsilon \subseteq A \subseteq U_\varepsilon, \mu(U_\varepsilon \setminus F_\varepsilon) < \varepsilon\}$$

We aim to show that the closed sets are contained in $\mathscr{A}$, which will show the result for closed sets since they generate the Borel algebra. Pick F closed. By second-countability, there exists a sequence of open sets $\{U_n\}$ such that $U_{n+1} \subseteq U_n$ and

$$\bigcap_{n=1}^{\infty} U_n$$

Now we apply σ-compactness to write $X = \bigcup_{n=1}^{\infty} K_n$ for compact K_n. Since X is Hausdorff and μ locally finite, each has finite measure so we can apply approximation from the outside for each K_n to write

$$\mu(F) = \lim \mu(U_n)$$

Since $\mu(U_n)$ is decreasing, we can pick $U_\varepsilon = U_n$ for some sufficiently large n. Taking $F_\varepsilon = F$, we establish the result.

Now we need to show that $\mathscr{A}$ is a σ-algebra. $\varnothing \in \mathscr{A}$ since it is clopen and has measure zero.

Pick $A \in \mathscr{A}$. Then taking $F_\varepsilon \subseteq U_\varepsilon$, we also have $U_\varepsilon^c \subseteq A^c \subseteq F_\varepsilon^c$. Also $\mu(F_\varepsilon^c \setminus U_\varepsilon^c) = \mu(U_\varepsilon \setminus F_\varepsilon) < \varepsilon$.

For closure, we consider $\{A_j\}_{j=1}^\infty \subseteq \mathscr{A}$ and $\varepsilon > 0$. For each A_j pick $F_j \subseteq A_j \subseteq U_j$ with $\mu(U_j \setminus F_j) < \varepsilon 2^{-j}$. Taking

$$U = \bigcup U_j$$
$$Z_k := \bigcup_{j=1}^k F_j$$

We then have

$$Z_k \subseteq \bigcup A_j \subseteq U$$

for all k, and for sufficiently large k we also have $\mu(U \setminus Z_k) < \varepsilon$. To see this, again apply σ-compactness to work in compact sets. Then approximation from the outside gives us

$$0 = \mu\left(U \setminus \bigcup F_j\right) = \lim \mu(U \setminus Z_k) \implies \exists k \text{ s.t. } \mu(U \setminus Z_k) < \varepsilon$$

(when splitting into the σ-compact decomposition we use $\varepsilon 2^{-m}$ for our K_m).

Then since the closed sets generate $\mathscr{B}(X)$, we conclude $\mathscr{B}(X) \subseteq \mathscr{A}$.

We now need to strengthen the result to approximate with compact sets. Take B Borel of finite measure and pick $F_\varepsilon \subseteq B$ closed such that $\mu(B \setminus F_\varepsilon) < \varepsilon/2$.

Now write $X = \bigcup_{j=1}^\infty K_j$, and consider the collection $F_\varepsilon \cap K_j$. The intersection of closed and compact sets is compact, so we have a collection of compact sets approximating F_ε. Then by approximation from the inside there is a compact set $K_\varepsilon \subseteq F_\varepsilon$ with $\mu(F_\varepsilon \setminus K_\varepsilon) < \varepsilon/2$, proving the result. $\square$

We state the following without proof, which connects the criterion we just established with μ-regular sets. This is proved with monotonicity and approximation.

Theorem 2.15

Let X be a topological space and $\mu : \mathscr{B}(X) \to [0, \infty]$ a Borel measure. A set $A \in \mathscr{B}(X)$ of finite measure is μ-regular if and only if for all $\varepsilon > 0$ there exists U_ε open and $K_\varepsilon \in \mathscr{B}(X)$ compact with $K_\varepsilon \subseteq A \subseteq U_\varepsilon$ and $\mu(U_\varepsilon \setminus K_\varepsilon) < \varepsilon$.

Definition 2.16

A topological space X is called **locally compact** if for any $x \in X$ there exists $U \ni x$ open with $\overline{U}$ compact.

Theorem 2.16

Let X be locally compact, σ-compact, and Hausdorff with $\mu : \mathscr{B}(X) \to [0, \infty]$ a σ-finite and locally finite Borel measure. The:

1. For any $\varepsilon > 0$, $A \in \mathscr{B}(X)$, there exists U_ε open and F_ε closed with $\mu(U_\varepsilon \setminus F_\varepsilon) < \varepsilon$ with $F_\varepsilon \subseteq A \subseteq U_\varepsilon$.

2. μ is regular.

3. For any $A \in \mathscr{B}(X)$ there exists $F \in F_\sigma$ and $G \in G_\delta$ with $F \subseteq A \subseteq G$ and $\mu(G \setminus F) = 0$.

In particular, any Borel set may be written as the countable union of closed sets ($F \in F_\sigma$) and a measure zero set.

2.4 Product Measures

So far we have only defined a Lebesgue measure on $\mathbb{R}$, but we would like to extend this naturally to $\mathbb{R}^n$. Under sufficient assumptions we can ensure uniqueness of a Borel measure on $\mathbb{R}^n$; however showing existence may be done in two ways. One way to do so is to rerun the Lebesgue construction on $\mathbb{R}^n$, with rectangles in place of intervals. To do this more abstractly, we can define what it means to construct measures on product spaces more generally.

Definition 2.17

Let $\{X_\alpha\}_{\alpha \in A}$ be a collection of measurable spaces with associated σ-algebras $\{\mathcal{M}_\alpha\}_{\alpha \in A}$. Then the **product space** is the Cartesian product of the X_α:

$$X = \prod_{\alpha \in A} X_\alpha = \left\{ f : A \to \bigcup_{\alpha \in A} X_\alpha : f(\alpha) \in X_\alpha \right\}$$

For $\beta \in A$, let $\pi_\beta : X \to X_\beta$ denote the canonical projection map. We endow X with the **product σ-algebra**

$$\mathcal{M} \coloneqq \sigma\left(\left\{\pi_\alpha^{-1}(E_\alpha) : \alpha \in A, E_\alpha \in \mathcal{M}_\alpha\right\}\right) = \sigma\left(\{\pi_\alpha\}_{\alpha \in A}\right)$$

We notate this product as $\mathcal{M} = \bigotimes_{\alpha \in A} \mathcal{M}_\alpha$.

Proposition 2.17

If the index set is countable then $\mathcal{M}$ is generated by the "rectangular sets":

$$\mathcal{M} = \sigma\left(\left\{\prod_{\alpha \in A} E_\alpha : E_\alpha \in \mathcal{M}_\alpha\right\}\right)$$

Proof. Homework. $\square$

Having defined product spaces, we can now define the natural way to define a product measure. Here we will restrict ourselves to the case of finite products.

Definition 2.18

A **rectangular subset** of $\prod_{\alpha\in A} X_\alpha$ is a set of the form $\prod_{\alpha\in A} E_\alpha$ where $E_\beta \subseteq X_\beta$.

Proposition 2.18

Let $X = \prod_{j=1}^n X_j$ be a finite product space with associated nonnegative measures $\{\mu_j : \mathcal{M}_j \to [0,\infty]\}_{j=1}^n$. We write $\mathscr{A}_0$ to denote the measurable rectangular sets in X. Then $\mathscr{A}_0$ is an elementary family.

Proof. Clearly $\varnothing, X \in \mathscr{A}_0$. Intersections are closed since

$$\prod_{j=1}^n E_j \cap \prod_{k=1}^n F_k = \prod_{j=1}^n (E_j \cap F_j)$$

Also since our index set is finite, the complement of rectangular sets are finite disjoint unions of rectangular sets. $\square$

As a result it follows then that the collection $\mathscr{A}$ of finite disjoint unions of elements in $\mathscr{A}_0$ is an algebra. Note that because A is finite, $\mathcal{M} = \bigotimes_{j=1}^n \mathcal{M}_j = \sigma(\mathscr{A}_0)$. Since $\mathcal{M}$ is by definition the smallest σ-algebra containing $\mathscr{A}_0$ and $\mathscr{A}_0 \subseteq \mathscr{A} \subseteq \sigma(\mathscr{A})$, we conclude that $\sigma(\mathscr{A}) = \mathcal{M}$.

As in the case of the Lebesgue measure, we now define a premeasure on $\mathscr{A}$ by

$$\rho\left(\bigsqcup_{j=1}^m \prod_{k=1}^n E_{k,j}\right) = \sum_{j=1}^m \prod_{k=1}^n \mu_k(E_{k,j})$$

The fact that this is well defined essentially follows from additivity of each μ_j.

Proposition 2.19

ρ is a premeasure.

Proof. $\rho(\varnothing) = 0$ since each μ_j is a measure. Also it is essentially clear by definition that ρ is finitely additive. If a countable union $\bigcup_{n=1}^\infty A_n \in \mathscr{A}$, then since $\mathscr{A}$ is composed of finite unions, we must have $\bigcup_{n=1}^\infty A_n = \bigsqcup_{j=1}^k B_j$. Then we have

$$\bigcup_{n=1}^\infty A_n = \bigsqcup_{j=1}^k \prod_{\ell=1}^n F_{\ell,j} = \sum_{j=1}^k \prod_{\ell=1}^n \mu_\ell(F_{\ell,j})$$

Prof. Shapiro did not make it clear how to finish this argument. $\square$

Now that we have a premeasure, we can run the Caratheodory construction to generate a complete measure $\mu : \mathscr{A}_{\varphi_\rho} \to [0,\infty]$ where $\mu|_{\mathscr{A}} = \rho$ and $\mathscr{A}_{\varphi_\rho}$ is a σ-algebra containing $\sigma(\mathscr{A}) = \mathcal{M}$.

Thus we have to note that in general, the product σ-algebra is not the algebra on which the product measure is complete.

In particular we define the Lebesgue measure on $\mathbb{R}^n$ to be the n-fold product measure of the Lebesgue measure on $\mathbb{R}$. The domain of this completed measure is the set of Lebesgue measurable sets on $\mathbb{R}^n$. Note that this is strictly larger than simply the product σ-algebra:

$$\mathscr{L}_n \supsetneq \mathscr{L} \otimes \ldots \otimes \mathscr{L}$$

For instance $A \times B \subseteq \mathbb{R}^2$ is measurable when A is not measurable but B has measure zero.

2.5 Fubini-Tonelli

Here we develop results on iterated integrals with respect to measures. In multivariable analysis we noted that for sufficiently nice functions, the relation

$$\int_{X\times Y} f = \int_X \left(\int_Y f_x\right) = \int_Y \left(\int_X f_y\right)$$

holds. Proving this for arbitrary measures is particularly helpful since it allows us to extend these results to, say, double sums and combinations of sums and integrals. In the language of measure theory we would like to prove that given a sufficiently nice measurable function $f : X \times Y \to \mathbb{C}$, we have

$$\int_{X\times Y} f \,\mathrm{d}(\mu\times\nu) = \int_X \left(\int_Y f_x \,\mathrm{d}\nu\right)\mathrm{d}\mu = \int_Y \left(\int_X f_y \,\mathrm{d}\mu\right)\mathrm{d}\nu$$

The Fubini and Tonelli theorems provide two conditions for this to hold.

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