Introduction to Real Analysis (MAT 215)

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Contents

1

Real Numbers

1.1 Sets

A set is defined by the elements in it. We write $x \in A$ if the object x is an element contained in A, and $x \notin A$ if it is not in A.

Definition 1.1.1. A set A is a **subset** of a set B if $\forall x \in A, x \in B$.

Definition 1.1.2. The **union** of two sets A, B is given by $A \cup B = \{x : x \in A \text{ or } x \in B\}$.

Definition 1.1.3. The **intersection** of two sets A, B is given by $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

Definition 1.1.4. Two sets A, B are equal if and only if $x \in A \iff x \in B$.

Definition 1.1.5. Two sets A, B are **disjoint** if $A \cap B = \emptyset$.

Remark. The set of natural numbers, integers, rationals, real numbers, and complex numbers are denoted $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$, respectively.

Definition 1.1.6. A set A is called an **inductive set** if

(i) $1 \in A$

(ii) $x \in A \implies x + 1 \in A$

Remark. If a set $S \subseteq \mathbb{N}$ is an inductive set, then $S = \mathbb{N}$.

1.2 Properties of the Real Numbers

Definition 1.2.1. A number $b \in \mathbb{R}$ is called an **upper bound** for a set $A \subseteq \mathbb{R}$ if $\forall a \in A, a \leq b$.

Definition 1.2.2. A set $A \subseteq \mathbb{R}$ is **bounded above** if $\exists b \in \mathbb{R}$ such that b is an upper bound for A.

Definition 1.2.3. A number $s \in \mathbb{R}$ is called the **supremum** or **least upper bound** for a set $A \subseteq \mathbb{R}$ if

(i) s is an upper bound for A

(ii) if b is any upper bound for A, then $s \leq b$.

Definition 1.2.4. A number $m \in \mathbb{R}$ is called the **maximum** of a set $A \subseteq \mathbb{R}$ if m is an upper bound for A and $m \in A$.

The terms **lower bound**, **bounded below**, **infimum**, **greatest lower bound**, and **minimum** are defined similarly.

Axiom of Completeness Every nonempty set $A \subseteq \mathbb{R}$ that is bounded above has a supremum.

Lemma 1.2.1. An alternate definition of the **supremum** says that for $s \in \mathbb{R}$ that is an upper bound for $A \subseteq \mathbb{R}$, $s = \sup A$ if and only if $\forall \varepsilon > 0, \exists a \in A$ s.t. $s - \varepsilon < a$.

Theorem 1.2.2 (Nested Interval Property). Suppose for each $n \in \mathbb{N}$ there is an associated interval $I_n = [a_n, b_n]$, and suppose $I_n \supseteq I_{n+1}$ such that $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ Then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof. Let $A = a_n : n \in \mathbb{N}$. Then A is nonempty and bounded above by any b_n, so we may use the Axiom of Completeness to set $x = \sup A$. For an arbitrary $I_n = [a_n, b_n]$, since x is an upper bound of A, $a_n \leq x$, and since all b_n are upper bounds of A and $x = \sup A$, $x \leq b_n$, so $\forall n \in \mathbb{N}, x \in I_n \implies x \in \bigcap_{n=1}^{\infty} I_n$, so the intersection is nonempty. □

Theorem 1.2.3 (Archimedean Property). This theorem has two parts:

(i) $\forall x \in \mathbb{R}, \exists n \in \mathbb{N}$ s.t. $n > x$.

(ii) $\forall y > 0, \exists n \in \mathbb{N}$ s.t. $1/n < y$.

Proof. (i) Suppose not. Then $\exists x \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}, n \leq x$. Then $\mathbb{N}$ is nonempty and bounded above, so we may set $x = \sup \mathbb{N}$. By Lemma 1.2.1, $\exists n \in \mathbb{N}$ s.t. $x - n < 1$. But then $x < n + 1$ and $n + 1 \in \mathbb{N}$, so x is not an upper bound for $\mathbb{N}$ and we have a contradiction.

(ii) Use (i) to select $n \in \mathbb{N}$ s.t. $n > 1/y$. Then $y > 1/n$. □

1.3 Cardinality

Definition 1.3.1. A **relation** $R : X \to Y$ is called a **function** if $\forall x \in X$, $y, z \in Y$ if $(x, y) \in R$ and $(x, z) \in R$, then $y = z$, and $\forall x \in X \exists y \in Y$, $(x, y) \in R$.

Definition 1.3.2. A function $f : A \to B$ is called **one-to-one** or **1-1** if $\forall a_1, a_2 \in A$, $f(a_1) = f(a_2) \implies a_1 = a_2$.

Definition 1.3.3. A function $f : A \to B$ is called **onto** if $\forall b \in B$, $\exists a \in A$ s.t. $f(a) = b$.

Definition 1.3.4. Two sets A and B have the same **cardinality** or are called **equicardinal**, denoted $A \sim B$, if $\exists f : A \to B$ such that f is 1-1 and onto.

Definition 1.3.5. A set A is **countably infinite** if $A \sim \mathbb{N}$.

Definition 1.3.6. A set A is **countable** if it is countably infinite or finite.

Remark. Some authors use "countable" to denote a set $A \sim \mathbb{N}$, and do not have a term similar to the definition of "countable" presented here.

Definition 1.3.7. A set A is **uncountable** if it is not countable.

Theorem 1.3.1. $\mathbb{Q}$ is countably infinite.

Proof. Ordering the rationals in a 2x2 grid and taking a diagonal path results in a 1-1, onto function. □

Theorem 1.3.2. $\mathbb{R}$ is uncountable.

Proof. Proved by Cantor's diagonalization argument. □

Theorem 1.3.3. If $A \subseteq B$ and B is countably infinite, then A is countable.

Theorem 1.3.4. The countable union of countable sets is countable.

Definition 1.3.8. Given a set A, the **power set** of A is $\mathcal{P}(A) = \{B : B \subseteq A\}$.

Theorem 1.3.5 (Cantor's Theorem). Given any set A, there does not exist an onto function $f : A \to \mathcal{P}(A)$.

2

Sequences and Series

2.1 Sequences

Definition 2.1.1. A **sequence** is a function whose domain is $\mathbb{N}$.

Definition 2.1.2. A sequence (a_n) **converges** to a real number a if $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $|a_n - a| < \varepsilon$. This is denoted $(a_n) \to a$ or $\lim a_n = a$.

Definition 2.1.3. A sequence (a_n) is called a **convergent sequence** if $\exists a \in \mathbb{R}$ s.t. $(a_n) \to a$.

Definition 2.1.4. A sequence (a_n) is called a **divergent sequence** if it does not converge to any $a \in \mathbb{R}$.

Definition 2.1.5. A sequence (a_n) is said to **eventually** possess a property P if $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, a_n possesses P.

Definition 2.1.6. Given $a \in \mathbb{R}, \varepsilon > 0$, the ε**-neighborhood** of a is $V_\varepsilon(a) = \{x \in \mathbb{R} : |x - a| < \varepsilon\}$.

Theorem 2.1.1. A sequence (a_n) converges to $a \in \mathbb{R}$ if and only if $\forall \varepsilon > 0$, (a_n) is eventually in $V_\varepsilon(a)$.

Theorem 2.1.2. If a sequence (a_n) has a limit, the limit is unique.

Definition 2.1.7. A sequence (a_n) is **bounded** if $\exists M > 0$ s.t. $\forall n \in \mathbb{N}, |a_n| < M$.

Theorem 2.1.3. Every convergent sequence is bounded.

Proof. Suppose $(x_n) \to l$. Then $\exists N$ s.t. $\forall n \geq N, |x_n - l| < 1 \implies |x_n| < |l| + 1$. Then (x_n) is bounded by $M = \max\{|x_1|, |x_2|, |x_3|, \ldots |x_{N-1}|, |l| + 1\}$. □

Theorem 2.1.4 (Algebraic Limit Theorem for Sequences). Let $\lim a_n = a, \lim b_n = b, c \in \mathbb{R}$. Then

(i) $\lim(ca_n) = ca$

(ii) $\lim(a_n + b_n) = a + b$

(iii) $\lim(a_n b_n) = ab$

(iv) $\lim(a_n / b_n) = a/b, b \neq 0$

Proof. (i) Let $\varepsilon > 0$ be given. Then $\exists N$ such that $\forall n \geq N$ we have $|a_n - a| < \varepsilon/|c|$. So $|ca_n - ca| = |c||a_n - a| < |c|\varepsilon/|c| = \varepsilon$. So $(ca_n) \to ca$.

(ii) Let $\varepsilon > 0$ be given. Then $\exists N_1, N_2$ such that $\forall n \geq N_1$ we have $|a_n - a| < \varepsilon/2$ and $\forall n \geq N_2$ we have $|b_n - b| < \varepsilon/2$. Then for $n \geq N = \max\{N_1, N_2\}$, $|a_n + b_n - (a+b)| \leq |a_n - a| + |b_n - b| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

(iii) (b_n) converges, so it is bounded. Suppose $\forall n$, $|b_n| \leq M$ for some $M > 0$. Let $\varepsilon > 0$ be given. Then $\exists N_1, N_2$ such that $\forall n \geq N_1$, $|a_n - a| < \varepsilon/2M$, and $\forall n \geq N_2$, $|b_n - b| < \varepsilon/2|a|$. Then $\forall n \geq N = \max\{N_1, N_2\}$, $|a_n b_n - ab| = |a_n b_n - ab_n + ab_n - ab| \leq |a_n - a||b_n| + |a||b_n - b| < \varepsilon/2M(M) + |a|\varepsilon/2|a| = \varepsilon/2 + \varepsilon/2 = \varepsilon$. So $(a_n b_n) \to ab$.

(iv) It suffices to show $(b_n) \to b \implies (1/b_n) \to 1/b$. Let $\varepsilon > 0$ be given. Then $\exists N_1$ such that $\forall n \geq N_1$, $|b_n - b| < |b|/2 \implies |b_n| > |b|/2$. Then choose N_2 such that $\forall n \geq N_2$, $|b_n - b| < \varepsilon|b|^2/2$. So for $n \geq N = \max\{N_1, N_2\}$, we have $|\frac{1}{b_n} - \frac{1}{b}| = |\frac{b-b_n}{b_n b}| < \varepsilon|b|^2/2|\frac{1}{b_n b}| < \varepsilon|b|^2/2|\frac{1}{|b|b/2}| = \varepsilon|b|^2/|b|^2 = \varepsilon$. So $(1/b_n) \to 1/b$ and $(a_n/b_n) \to a/b$ by part (iii).

□

Theorem 2.1.5 (Order Limit Theorem). Let $\lim a_n = a, \lim b_n = b$. Then

(i) If $\forall n \in \mathbb{N}$, $a_n \geq 0$, then $a \geq 0$

[(ii) If $\forall n \in \mathbb{N}$, $a_n \leq b_n$ then $a \leq b$

(iii) If $\forall n \in \mathbb{N}$, $a_n \geq c$ for $c \in \mathbb{R}$,$a \geq c$.

Theorem 2.1.6 (Squeeze Theorem). If $\forall n \in \mathbb{N}$, $x_n \leq y_n \leq z_n$ and $\lim x_n = \lim z_n = l$, then $\lim y_n = l$.

Definition 2.1.8. A sequence is **increasing** if $\forall n \in \mathbb{N}$, $a_{n+1} \geq a_n$.

A **decreasing** sequence is defined analogously. The terms **strictly increasing** and **strictly decreasing** are defined as above using strict inequalities.

Definition 2.1.9. A sequence is **monotone** if it is increasing or decreasing.

Theorem 2.1.7 (Monotone Convergence Theorem). A bounded monotone sequence converges.

Proof. Suppose (a_n) is monotonically increasing (consider $(-a_n)$ if not. Let $s = \sup\{a_n : n \in \mathbb{N}\}$ (this set is bounded). Then by Lemma 1.2.1, for all $\varepsilon > 0$ we have N such that $l - a_N < \varepsilon$. This holds for $n \geq N$ because (a_n) is increasing. So $(a_n) \to s$. □

Definition 2.1.10. Given a sequence (a_n) and a strictly increasing sequence (n_k), then the sequence (a_{n_k}) is called a **subsequence** of (a_n).

Theorem 2.1.8. Subsequences of a convergent sequence converge to the same limit.

Proof. Let $(a_n) \to l$. Then (a_n) is eventually in every ε-neighborhood of l, so any subsequence is also eventually in every ε-neighborhood of l, so any subsequence also converges to l. □

Theorem 2.1.9 (Bolzano-Weierstrass Theorem). Every bounded sequence has a convergent subsequence.

Proof. Let (x_n) be bounded by M. We begin with an interval $I_1 = [-M, M]$. For each n, we bisect I_n and choose I_{n+1} as one of these bisections such that I_{n+1} contains an infinite number of terms of (x_n). If we choose (y_{n_k}) in I_k such that (n_k) is strictly increasing, then we have a subsequence that converges to x_0, where $x_0 \in \bigcap_{\mathbb{N}} I_n$ is guaranteed by the Nested Interval Property. □

Definition 2.1.11. A sequence (a_n) is called a **Cauchy sequence** if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall m, n \geq N$, $|a_m - a_n| < \varepsilon$.

Theorem 2.1.10. Cauchy sequences are bounded.

Proof. Let $\varepsilon = 1$. So $\exists N$ such that $\forall m \geq N$, $|a_m - a_N| < 1$. So $\forall m \geq N, |a_m| < |a_N| + 1$. Let $M = \max\{|a_1|, |a_2|, \dots |a_{N-1}|, |a_N| + 1\}$. Then M bounds (a_n). $\square$

Theorem 2.1.11 (Cauchy Criterion for Sequences). A sequence is convergent if and only if it is Cauchy.

Proof. ($\implies$) Trivial by definitions.

($\impliedby$) Let (a_n) be Cauchy. Then it is bounded. Apply the Bolzano-Weierstrass Theorem to produce (a_{n_k}) convergent. Let $x = \lim a_{n_k}$. So $\exists N_1$ such that $\forall m, n \geq N_1$ we have $|a_m - a_n| < \varepsilon/2$. Choose $n_k > N_1$ such that $\forall n_{k'} \geq n_k$, $|a_{n_{k'}} - x| < \varepsilon/2$. So $\forall n \geq n_k$ we have $|a_n - x| = |a_n - a_{n_k} + a_{n_k} - x| \leq |a_n - a_{n_k}| + |a_{n_k} - x| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. So $(a_n) \to x$. $\square$

2.2 Series

Definition 2.2.1. Given a sequence (b_n), an **infinite series** is an expression of the form $\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + \dots$.

Definition 2.2.2. Given a series $\sum b_n$, the corresponding sequence of **partial sums** is given by $s_k = \sum_{n=1}^{k} b_n$.

Definition 2.2.3. A series $\sum b_n$ is a **convergent series** if and only if (s_k) converges. If so, then we say $\sum b_n$ converges to B, where $B = \lim s_k$.

Theorem 2.2.1. The harmonic series $\sum 1/n$ diverges.

Theorem 2.2.2. The series $\sum 1/n^p$ converges if and only if $p > 1$.

Theorem 2.2.3 (Algebraic Limit Theorem for Series). Let $\sum a_n = a, \sum b_n = b, c \in \mathbb{R}$. Then

(i) $\sum (ca_n) = ca$

(ii) $\sum (a_n + b_n) = a + b$

Proof. This follows directly from the Algebraic Limit Theorem for Series by considering the sequence of partial sums. $\square$

Remark. It is not true that $\sum a_n b_n = ab$.

Theorem 2.2.4. If $\sum a_n$ converges, then $(a_n) \to 0$.

Theorem 2.2.5 (Comparison Test). Let $(a_k), (b_k)$ be sequences satisfying $\forall k \in \mathbb{N}, 0 \leq a_k \leq b_k$. Then

(i) $\sum b_k$ converges $\implies \sum a_k$ converges.

(ii) $\sum a_k$ diverges $\implies \sum b_k$ diverges.

Theorem 2.2.6 (Geometric Series). $\sum ar^k = \frac{a}{1-r}$ if and only if $|r| < 1$.

Theorem 2.2.7 (Absolute Convergence Test). If $\sum |a_n|$ converges then $\sum a_n$ converges.

Theorem 2.2.8 (Alternating Series Test). Suppose (a_n) is decreasing and $(a_n) \to 0$. Then $\sum (-1)^n a_n$ converges.

Definition 2.2.4. If $\sum |a_n|$ converges, then $\sum a_n$ is **absolutely convergent**. If $\sum a_n$ converges but $\sum |a_n|$ diverges, then $\sum a_n$ is **conditionally convergent.**

Definition 2.2.5. A series $\sum b_k$ is called a **rearrangement** of $\sum a_k$ if there exists a 1-1 function $f : \mathbb{N} \to \mathbb{N}$ such that $\forall k \in \mathbb{N}$, $b_{f(k)} = a_k$.

Theorem 2.2.9. If a series converges absolutely, any rearrangement converges to the same limit.

Proof. Let $\varepsilon > 0$ be given. (s_k) is Cauchy, so $\exists N$ such that $\forall m \geq n \geq N$, $|s_m - s_n| = |a_{n+1} + a_{n+2} + \ldots + a_m| \leq \sum_{i=n}^{m} |a_i| < \varepsilon$. Then consider a rearrangement $\sum a'_n$, with partial sums (s'_n). Then choose p such that $\forall 1 \leq m \leq N$, $f(m) \leq p$. So $\forall k \geq p$, $|s'_k - s_k|$ cancels all terms with index $j \leq N$, so $|s'_k - s_k| \leq \sum_{i=n}^{m} |a_i| < \varepsilon$. So (s'_k) converges to the same limit as (s_k). $\square$

Theorem 2.2.10 (Ratio Test). Given (a_n) with $a_n \neq 0$, if $\lim \left|\frac{a_{n+1}}{a_n}\right| = r < 1$, then $\sum a_n$ converges absolutely.

Proof. Let (a_n) be given with $a_n \neq 0$. Suppose $\lim \left|\frac{a_{n+1}}{a_n}\right| = r < 1$. Let $\varepsilon >= (1-r)/2 > 0$ be given. Then $\exists N$ such that $\forall n \geq N$, $| \left|\frac{a_{n+1}}{a_n}\right| - r| < \varepsilon$. So $\left|\frac{a_{n+1}}{a_n}\right| < r + \varepsilon$. So $|a_{n+1}| < (r+\varepsilon)|a_n|$. So $\forall m \geq 0$, $|a_{N+m}| < (r+\varepsilon)^m |a_N|$. $\sum (r+\varepsilon)^m$ converges by the geometric series (because $r + \varepsilon < 1$), and $|a_N|$ is constant, so $\sum (r+\varepsilon)^m |a_N|$ converges. $|a_{N+m}| < (r+\varepsilon)^m |a_N|$, so $\sum |a_{N+m}| = \sum |a_n|$ converges by comparison. So $\sum a_n$ converges absolutely. $\square$

3

Point Set Topology

3.1 Open and Closed Sets

Definition 3.1.1. A set $O \subseteq \mathbb{R}$ is **open** if $\forall a \in O\ \exists \varepsilon > 0$ s.t. $V_\varepsilon(a) = (a - \varepsilon, a + \varepsilon) \subseteq O$.

Definition 3.1.2. Given a set $E \subseteq \mathbb{R}$, the **interior** of E is $E^o = \{x \in E : \exists V_\varepsilon(x) \subseteq E\}$.

Theorem 3.1.1. For any set $E \subseteq \mathbb{R}$, E^o is open and the largest open set contained within E.

Theorem 3.1.2. A set $E \subseteq \mathbb{R}$ is open if and only if $E^o = E$.

Theorem 3.1.3. The union of an arbitrary collection of open sets is open.

Proof. Let $\mathcal{U}$ be an arbitrary collection of open sets. Then let $U = \bigcup \mathcal{U}$. Choose some $x \in U$. Then $x \in O$ for some O in $\mathcal{U}$. Since O is open, $\exists V_\varepsilon(x) \subseteq O \subseteq U$. So U is open. □

Theorem 3.1.4. The intersection of a finite collection of open sets is open.

Proof. Let $O_1, O_2, O_3, \ldots O_n$ be a finite collection of open sets. Let $U = \bigcap_{i=1}^n O_i$. Then choose some $x \in U$. $x \in O_1$, so $\exists \varepsilon_1$ such that $V_{\varepsilon_1}(x) \subseteq O_1$. Repeat with every O_k to obtain $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots \varepsilon_n$. Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots \varepsilon_n\}$. Then for each k, $V_\varepsilon(x) \subseteq V_{\varepsilon_k}(x) \subseteq O_k$, so $V_\varepsilon(x)$ is in each O_k, so $V_\varepsilon(x) \subseteq U$. So U is open. □

Definition 3.1.3. A point x is a **limit point** of a set A if $\forall \varepsilon > 0, \exists y \in V_\varepsilon(x) \cap A : y \neq x$.

Theorem 3.1.5. A point x is a limit point of a set A if and only if $\exists (a_n)$ with $\forall a_n, a_n \in A, a_n \neq x$, and $(a_n) \to x$.

Proof. ($\Longrightarrow$). If x is a limit point, then every $V_\varepsilon(x)$ contains some $x_0 \neq x$ satisfying $|x_0 - x| < \varepsilon$. It is easy to see that any decreasing sequence of ε tending to 0 will produce a sequence converging to x.

($\Longleftarrow$). By the definition of convergence, every $V_\varepsilon(x)$ contains some x_0, which by assumption is in A and $x_0 \neq x$, so x is a limit point. □

Theorem 3.1.6. A set $F \subseteq \mathbb{R}$ is closed if and only if every Cauchy sequence in F tends to an element of F.

Definition 3.1.4. A point $a \in A$ is an **isolated point** of A if it is not a limit point of A.

Remark. A limit point need not be in A. An isolated point is always in A.

Definition 3.1.5. Given a set $F \subseteq \mathbb{R}$, let L be the set of limit points of F. F is **closed** if $L \subseteq F$.

Definition 3.1.6. Given a set $F \subseteq \mathbb{R}$, the **closure** of F is $\overline{F} = F \cup L$.

Theorem 3.1.7. A set $F \subseteq \mathbb{R}$ is closed if and only if $F = \overline{F}$.

Theorem 3.1.8. For any $A \subseteq \mathbb{R}$, $\overline{A}$ is closed and the smallest closed set containing A.

Theorem 3.1.9. The union of a finite collection of closed sets is closed.

Theorem 3.1.10. The intersection of an arbitrary collection of closed sets is closed.

Theorem 3.1.11. A set O is open if and only if O^c is closed. A set F is closed if and only if F^c is open.

3.2 Compact Sets

Definition 3.2.1. Given $A \subseteq \mathbb{R}$, a **cover** for A is a collection of sets $\{E_\lambda : \lambda \in \Lambda\}$ such that $A \subseteq \bigcup_{\lambda \in \Lambda} E_\lambda$.

Definition 3.2.2. A cover C is called an **open cover** if every set in C is open.

Definition 3.2.3. If C is a cover for A, then D is a **subcover** of C if D is a subcollection of C and is also a cover of A.

Definition 3.2.4. A set $K \subseteq \mathbb{R}$ is **compact** if every open cover of K has a finite subcover.

Theorem 3.2.1. Closed intervals $I = [a, b] \subseteq \mathbb{R}$ are compact.

Proof. Let $\mathcal{U}$ be an open covering of I. Assume for contradiction it has no finite subcover. Bisect $I = I_1 = [a_1, b_1]$ into two intervals $[a_1, c_1]$ and $[c_1, b_1]$. At least one of those intervals has no finite subcover (if they both did, the total subcover would be finite which is a contradiction). Let I_2 be this interval. Continue this process to obtain a nested sequence of closed intervals. By the nested interval property, there exists some $x_0 \in \bigcap_{\mathbb{N}} I_n$. Since $x_0 \in I$, there exists some $U_0 \in \mathcal{U}$ with $x_0 \in U_0$. Since U_0 is open, there is some $V_\varepsilon(x_0) \in U_0$. Since each I_n has length $(b-a)2^{n-1}$, there is some N for which $n \geq N \implies I_n \subseteq V_\varepsilon(x)$. So I_N is covered by U_0 which is a finite subcover of $\mathcal{U}$. This is a contradiction. So every open covering of I has a finite subcover, and I is compact. □

Theorem 3.2.2. Closed subsets of compact sets are compact.

Proof. Let $F \subseteq K$, where F is closed and K is compact. Then let $\mathcal{U}$ be an arbitrary open cover of F. Adding F^c (which is open) to $\mathcal{U}$ gives an open cover of K. Since K is compact, there is some finite subcover of this collection that covers K. Call this subcover $\mathcal{C}$. Then $\mathcal{C} \setminus F^c$ still covers F, and is a subcollection of $\mathcal{U}$. So $\mathcal{C} \setminus F^c$ is the desired finite subcover. □

Theorem 3.2.3 (Nested Compact Set Property). If $K_1 \subseteq K_2 \subseteq K_3 \subseteq \dots$ is a nested sequence of nonempty compact sets, then the intersection $\bigcap_{n=1}^{\infty} K_n$ is nonempty.

Proof. Construct a sequence (x_n) such that $x_n \in K_n$ for each n. Then (x_n) is contained in K_1, so it converges to a limit in K_1. Similarly, (x_n) (excluding the first term) is contained in K_2, and this is true for all K_n, so $x = \lim x_n$ is in $\bigcap_{n=1}^{\infty} K_n$. □

Theorem 3.2.4 (Heine-Borel Theorem). If $K \subseteq \mathbb{R}$, then all of the following three statements are equivalent:

(i) Every sequence in K has a subsequence that converges to a limit in K.

(ii) K is closed and bounded.

(iii) Every open cover of K has a finite subcover.

Proof. ((i) $\implies$ (ii)) Assume for contradiction that K is not bounded. Then we may construct a sequence of terms (x_n) that is increasing with $(x_n) \to \infty$. But this sequence has no convergent subsequence, so this is a contradiction and K must be bounded. Let (y_n) be an arbitrary convergent sequence in K. Then it has a convergent subsequence $(y_{n_k}) \to y, y \in K$. Since subsequences converge to the same limit, $(y_n) \to y$ for some $y \in K$, so K is closed.

((ii) $\implies$ (i)) Let (x_n) be an arbitrary sequence in K. K is bounded, so (x_n) is bounded. Then by the Bolzano-Weierstrass Theorem, (x_n) has a convergent subsequence (x_{n_k}). Since this is a convergent sequence in K, which is closed, $(x_{n_k}) \to x$ for some $x \in K$.

((ii) $\implies$ (iii)) If K is bounded, then there exists some closed interval I with $K \subseteq I$. I is compact. So K is a closed subset of a compact set and is therefore compact.

((iii) $\implies$ (ii)) Construct an open cover for K by defining $O_x = (x-1, x+1)$. Then $\{O_x : x \in K\}$ has a finite subcover $\{O_{x_1}, O_{x_2}, O_{x_3}, \ldots O_{x_n}\}$. Since K is contained in a finite union of bounded sets, K is bounded. Suppose for contradiction that K is not closed. Then let (y_n) be a Cauchy sequence in K with $y = \lim y_n$, but $y \notin K$. So every $x \in K$ satisfies $|x-y| > 0$. Construct an open cover for K by defining $O_x = (x - |x-y|/2, x + |x-y|/2)$ for each $x \in K$. Since we assume (iii), there is a finite subcover of K given by $\{O_{x_1}, O_{x_2}, O_{x_3}, \ldots O_{x_n}\}$. But let $\varepsilon_0 = \min\{|x_i - y|/2 : 1 \leq i \leq n$. Since $(y_n) \to y$, there exists $y_n \in K$ with $|y_n - y| < \varepsilon_0$. But this y_n is not in any O_{x_n}, so the subcover does not actually cover K, contradiction. So K is closed. $\square$

Remark. The equivalence between statements (ii) and (iii) above is true only for compact sets in Euclidean space $\mathbb{R}^n$. It is *not* true for general closed metric spaces.

3.3 Perfect Sets

Definition 3.3.1. A set $P \subseteq \mathbb{R}$ is called **perfect** if it is closed and contains no isolated points.

Theorem 3.3.1. The Cantor set is perfect.

Proof. Since the Cantor set is an intersection of finite unions of closed intervals, it is closed. Every point in the Cantor set is a limit point. So the Cantor set is perfect. $\square$

Theorem 3.3.2. A nonempty perfect set is uncountable.

Proof. If P is perfect and nonempty, it cannot be finite, since then it would only have isolated points. Suppose it is countable. Then $P = \{x_1, x_2, x_3, \ldots\}$. Create a nested sequence of compact sets K_n in P such that $x_1 \notin K_2, x_2 \notin K_3, \ldots$, with each K_n nonempty. We do this by letting I_1 be a closed interval containing x_1 in its interior. Since x_1 is not isolated, $\exists y_2 \neq x_1$ in I_0. Construct $I_2 \subseteq I_1$, a closed interval centered around y_2 satisfying $x_1 \notin I_2$. Continue this process inductively. Then $I_{n+1} \subseteq I_n$, $x_n \notin I_{n+1}$, and $I_n \cap P \neq \emptyset$ because $y_n \in I_n$ and $y_n \in P$. let $K_n = I_n \cap P$. By the Nested Compact Set Property, the intersection $\bigcap_{n=1}^{\infty} K_n$ is nonempty, but $x \in \bigcap_{n=1}^{\infty} K_n \subseteq P$ is not in the list $\{x_1, x_2, x_3 \ldots\}$ by construction. So P is uncountable. $\square$

3.4 Connected Sets

Definition 3.4.1. Two nonemtpy sets $A, B \subseteq \mathbb{R}$ are **separated** if $\overline{A} \cap B$ and $A \cap \overline{B}$ are both empty.

Definition 3.4.2. A set $E \subseteq \mathbb{R}$ is **disconnected** if it can be written as $E = A \cup B$, where A and B are nonempty separated sets. E is **connected** if it is not disconnected.

Theorem 3.4.1. A set $E \subseteq \mathbb{R}$ is connected if and only if, for all nonempty disjoint A and B satisfying $E = A \cup B$, there exists a convergent sequence $(x_n) \to x$ with (x_n) contained in either A or B, and x in the other.

Theorem 3.4.2. A set $E \subseteq \mathbb{R}$ is connected if and only if whenever $a < c < b$ with $a, b \in E$, then $c \in E$.

4 Functional Limits and Continuity

4.1 Functional Limits

Definition 4.1.1. Let $f : A \to \mathbb{R}$, and let c be a limit point of A. Then the **functional limit** $\lim_{x\to c} f(x) = L$ means that $\forall\varepsilon > 0$, $\exists\delta > 0$ s.t. $0 < |x - c| < \delta, x \in A$ implies $|f(x) - L| < \varepsilon$

Theorem 4.1.1. Given $f : A \to \mathbb{R}$ and c a limit point of A, $\lim_{x\to c} f(x) = L$ if and only if for all sequences $(x_n) \subseteq A$ satisfying $x_n \neq c$ and $(x_n) \to c$, $(f(x_n)) \to L$.

Theorem 4.1.2 (Algebraic Limit Theorem for Functional Limits). Let f and g be functions defined on $A \subseteq \mathbb{R}$ and assume $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$ for some limit point c of A. Then

(i) $\lim_{x\to c} kf(x) = kL$ for $k \in \mathbb{R}$

(ii) $\lim_{x\to c}[f(x) + g(x)] = L + M$

(iii) $\lim_{x\to c}[f(x)g(x)] = LM$

(iv) $\lim_{x\to c}[f(x)/g(x)] = L/M$ if $M \neq 0$

Theorem 4.1.3. Let $f : A \to \mathbb{R}$ with $A \subseteq \mathbb{R}$, and let c be a limit point of A. if there exist two sequences (x_n) and (y_n) in A with $x_n \neq c$ and $y_n \neq c$ and $\lim x_n = \lim y_n = c$ but $\lim f(x_n) \neq \lim f(y_n)$, then $\lim_{x\to c} f(x)$ does not exist.

4.2 Continuity

Definition 4.2.1. A function $f : A \to \mathbb{R}$ is **continuous** at a point $c \in A$ if $\forall\varepsilon > 0 \exists\delta > 0$ such that $|x - c| < \delta$ and $x \in A$ implies $|f(x) - f(c)| < \varepsilon$.

If f is continuous at every point in A, then f is continuous on A.

Remark. If c is a limit point of A, then the above definition is equivalent to the statement that $\lim_{x\to c} f(x) = f(c)$. If c is isolated, then the limit is undefined, but the definition is still valid. It follows from the definition that f is continuous at every isolated point of A.

Theorem 4.2.1. Let $f : A \to \mathbb{R}$ and let $c \in A$. Then f is continuous at c if and only if for all $(x_n) \to c$ contained in A, $(f(x_n)) \to f(c)$.

Theorem 4.2.2. Let $F : A \to \mathbb{R}$ and $c \in A$ be a limit point of A. If there exists a sequence $(x_n) \subseteq A$ with $(x_n) \to c$ but $f(x_n)$ does not converge to $f(c)$, then f is discontinuous at c.

Definition 4.2.2. Let $f : M \to N$. Then the **preimage** of a set $V \subseteq N$ under f is $f^{pre}(V) = \{x \in M : f(x) \in V\}$.

Theorem 4.2.3. Let $f : M \to N$. Then f is continuous on M if and only if the preimage of each closed set in N is closed in M.

Theorem 4.2.4. Let $f : M \to N$. Then f is continuous on M if and only if the preimage of each open set in N is open in M.

Theorem 4.2.5 (Algebraic Continuity Theorem). Let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ be continuous at $c \in A$. Then

(i) $kf(x)$ is continuous at c for $k \in \mathbb{R}$

(ii) $f(x) + g(x)$ is continuous at c

(iii) $f(x)g(x)$ is continuous at c

(iv) $f(x)/g(x)$ is continuous at c, provided it is defined.

Theorem 4.2.6. Let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$, with the range of f contained in B. If f is continuous at $c \in A$ and g is continuous at $f(c) \in B$, then $g \circ f$ is continuous at c.

Theorem 4.2.7. Let $f : A \to \mathbb{R}$ be continuous on A. If $K \subseteq A$ is compact, then $f(K)$ is compact.

Proof. Let (b_n) be an arbitrary sequence in fK. Then let (a_k) be a sequence in K such that $f(a_n) = b_n$ for all n. Since K is compact, there exists a subsequence (a_{n_k}) that converges to some $p \in K$. By the continuity of f, $(a_{n_k}) \to p \implies (f(a_{n_k})) \to f(p) \in fK$. So (b_n) has a subsequence $b_{n_k} = f(a_{n_k})$ that converges to a limit in fK. □

Theorem 4.2.8 (Extreme Value Theorem). Let $f : K \to \mathbb{R}$ be continuous on $K \subseteq \mathbb{R}$ compact. Then $\exists x_0, x_1 \in K$ such that $f(x_0) \leq f(x) \leq f(x_1)$ for all $x \in K$.

4.3 Uniform Continuity

Definition 4.3.1. A function $f : A \to \mathbb{R}$ is **uniformly continuous** on A if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall x, y \in A$, $|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$.

Remark. Because there is no notion of uniform continuity at a point, δ is not allowed to depend on the point in A.

Theorem 4.3.1. Let $f : A \to \mathbb{R}$. Then f is not uniformly continuous on A if and only if there exists $\varepsilon > 0$ and two sequences (x_n) and (y_n) in A with $|x_n - y_n| \to 0$ but $|f(x_n) - f(y_n)| \geq \varepsilon$

Theorem 4.3.2. A function that is continuous on a compact set K is uniformly continuous on K.

Proof. Suppose f is continuous on K. Assume it is not uniformly continuous on K. Then there exists $\varepsilon > 0$ and two sequences (x_n) and (y_n) in A with $|x_n - y_n| \to 0$ but $|f(x_n) - f(y_n)| \geq \varepsilon$. Since K is compact, there exists a subsequence (x_{n_k}) that converges to some $x \in K$. Consider (y_{n_k}). Since $\lim(x_n - y_n) = 0$, we have $\lim y_{n_k} = \lim(y_{n_k} - x_{n_k} + x_{n_k}) = 0 + x$. So $(y_{n_k}) \to x$. By the continuity of f, $\lim f(x_{n_k}) = f(x) = \lim f(y_{n_k})$, so $\lim f(x_{n_k}) - f(y_{n_k}) = 0$. But this contradicts the statement that $|f(x_n) - f(y_n)| \geq \varepsilon$ for all n. So f must be uniformly continuous on K. □

Theorem 4.3.3. If $f : G \to \mathbb{R}$ is continuous and $E \subseteq G$ is connected, then $f(E)$ is connected.

Theorem 4.3.4 (Intermediate Value Theorem). Let $f : [a, b] \to \mathbb{R}$ be continuous. If L is a real number with $f(a) < L < f(b)$ or $f(b) < L < f(a)$, then $\exists c \in (a, b)$ such that $f(c) = L$.

5 Differentiation

5.1 Derivatives

Definition 5.1.1. Let $g : A \to \mathbb{R}$. Then the **derivative** of g at $c \in A$ is $g'(c) = \lim_{x\to c} \frac{g(x)-g(c)}{x-c}$, provided the limit exists.

Definition 5.1.2. $g : A \to \mathbb{R}$ is **differentiable** at $c \in A$ if the derivative of g at c exists. It is differentiable on A if it is differentiable at every $c \in A$.

Theorem 5.1.1. If $f : (a, b) \to \mathbb{R}$ is differentiable at $c \in (a, b)$, then there exists f_c^* continuous at c such that $f(x) - f(c) = (x - c)f_c^*(x)$ for any $x \in (a, b)$, with $f_c^*(c) = f'(c)$.

Proof. Let $f_c^*(x) = \frac{f(x)-f(c)}{x-c}$, $f_c^*(c) = f'(c)$. □

Theorem 5.1.2. If $g : A \to \mathbb{R}$ is differentiable at $c \in A$, then it is continuous at $c \in A$.

Proof. By Theorem 5.1.1, f_c^* exists and satisfies $f(x) - f(c) = (x - c)f_c^*(x)$. $\lim_{x\to c}(x - c)f_c^*(x) = 0$ so $\lim_{x\to c} f(x) = f(c)$ so f is continuous at c. □

Theorem 5.1.3. Let $f, g : A \to \mathbb{R}$ be differentiable at $c \in A$. Then

(i) $(f + g)'(c) = f'(c) + g'(c)$

(ii) $(kf)'(c) = kf'(c)$ for $k \in \mathbb{R}$

(iii) $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$

(iv) $(f/g)'(c) = (g(c)f'(c) - g'(c)f(c))/[(g(c))^2]$, provided $g(c) \neq 0$

Theorem 5.1.4 (Chain Rule). Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ such that $f(A) \subseteq B$ and $g \circ f$ is defined. If f is differentiable at $c \in A$ and g is differentiable at $f(c) \in B$ then $(g \circ f)$ is differentiable at c with $(g \circ f)'(c) = g'(f(c))f'(c)$.

Proof. Apply Theorem 5.1.1 to produce $f_c^*(x)$, $g_{f(c)}^*(x)$ satisfying $f(x) - f(c) = (x - c)f_c^*(x)$ and $g(y) - g[f(c)] = [y - f(c)]g_{f(c)}^*(y)$. Then we have $g[f(x)] - g[f(c)] = [f(x) - f(c)]g_{f(c)}^*[f(x)] = (x - c)[f_c^*(x)][g_{f(c)}^*(x)]$ and $\lim_{x\to c} g_{f(c)}^*(x) = g'[f(c)]$, so $\lim_{x\to c} \frac{g(f(x))-g(f(c))}{x-c} = g'[f(c)]f'(c)$. □

Theorem 5.1.5. Let $f : (a, b) \to \mathbb{R}$ satisfy $f'(c) > 0$ at some $c \in (a, b)$. Then $\exists\delta > 0$ s.t. for any $x \in V_\delta(c)$, $x > c \implies f(x) - f(c)$ and $x < c \implies f(x) < f(c)$, with a similar statement for $f'(c) < 0$.

Proof. By Theorem 5.1.1, we construct $f_c^*(x)$ with $f_c^*(c) = f'(c) > 0$. Since f_c^* is continuous, there is some $V_\delta(c)$ so $f_c^*(x) > 0$ on this interval, which leads to the conclusion. □

Definition 5.1.3. Let $f : A \to \mathbb{R}$. Let $a \in A$. f has a **local maximum** at a if there exists some $V_\delta(x)$ such that $f(x) \le f(a)$ for any $x \in V_\delta \cap A$. A **local minimum** is defined similarly.

Theorem 5.1.6. Let f be differentiable on (a, b). If f has a local extremum at $c \in (a, b)$, then $f'(c) = 0$.

Proof. By Theorem 5.1.5, if $f'(c) > 0$ or $f'(c) < 0$, then no $V_\delta(c)$ works. So $f'(c) = 0$. □

5.2 Mean Value Theorems

Theorem 5.2.1 (Rolle's Theorem). $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$ then $\exists c \in (a, b)$ such that $f'(c) = 0$.

Theorem 5.2.2 (Mean Value Theorem). Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then $\exists c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Theorem 5.2.3. Let f, g be continuous on $[a, b]$ and differentiable on (a, b). Then $\exists c \in (a, b)$ such that $|f(b) - f(a)|g'(c) = g(b) - g(a)f'(c)$. If $g' \neq 0$ on $[a, b]$ the

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Theorem 5.2.4. Suppose f is differentiable on $[a, b]$. Then for any k satisfying $f'(a) < k < f'(b)$ or $f'(b) < k < f'(a)$, there exists $c \in (a, b)$ with $f'(c) = k$.

Proof. Use Theorem 5.1.1 to produce $f_a^*(x) = \frac{f(x)-f(a)}{x-a}$ and $f_b^*(x) = \frac{f(x)-f(b)}{x-b}$. Since each function is continuous, f_a^* takes on all values from $\frac{f(b)-f(a)}{b-a}$ to $f'(a)$ and $f_b^*(x)$ all values from $\frac{f(b)-f(a)}{b-a}$ to $f'(b)$. So for any k, one of the functions satisfies $f^*_{a \text{ or } b}(z) = k$. By the Mean Value Theorem, there exists $c \in (a, b)$ such that $f'(c) = f^*_{a \text{ or } b}(z) = k$ □

Theorem 5.2.5 (L'Hospital's Rule). Let f and g be differentiable on (a, b), with $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$, $g(x), g'(x) \neq 0$, and $\lim_{x\to c} f'(x)/g'(x) = L$. Then $\lim_{x\to c} f(x)/g(x) = L$. The result also holds if $c = \pm\infty$ or if $\lim_{x\to c} g = \infty$.

6 Sequences and Series of Functions

6.1 Sequences of Functions

Definition 6.1.1. Let (f_n) be a sequence of functions on $A \subseteq \mathbb{R}$. (f_n) **converges pointwise** on A to a function f (denoted $(f_n) \to f$) if, for any $x \in A$, $(f_n(x)) \to f(x)$.

Definition 6.1.2. Let (f_n) be a sequence of functions on $A \subseteq \mathbb{R}$. (f_n) **converges uniformly** on A to a function f (denoted $(f_n) \rightrightarrows f$) if, $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ such that $\forall n \geq N$, $\forall x \in A$, $|f_n(x) - f(x)| < \varepsilon$.

Theorem 6.1.1. Let (f_n) be a sequence of functions on $A \subseteq \mathbb{R}$. (f_n) converges uniformly if and only if $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ such that $\forall m, n \geq N$, $\forall x \in A$, $|f_n(x) - f_m(x)| < \varepsilon$.

Theorem 6.1.2. Let $(f_n) \rightrightarrows f$ on $A \subseteq \mathbb{R}$. If each f_n is continuous at some $c \in A$, then f is continuous at c.

Theorem 6.1.3. Let $(f_n) \to f$ on $[a, b]$. If each f_n is differentiable and $(f_n') \rightrightarrows g$ on $[a, b]$, then f is differentiable with $f' = g$.

Theorem 6.1.4. Let (f_n) be a sequence of differentiable functions on $[a, b]$. Suppose $(f_n') \rightrightarrows g$ on $[a, b]$ and $(f_n(c))$ is convergent at some $c \in [a, b]$. Then (f_n) converges uniformly on $[a, b]$, and the limit function $f = \lim f_n$ is differentiable with $f' = g$.

6.2 Series of Functions

Definition 6.2.1. Let f_n be defined on $A \subseteq \mathbb{R}$ for each n. Then the series $\sum f_n$ converges pointwise to f if the sequence $s_k = \sum_{i=1}^{k} f_i$ converges pointwise to f, and converges uniformly if (s_k) converges uniformly.

Theorem 6.2.1. Suppose each f_n is continuous on $A \subseteq \mathbb{R}$ and $\sum f_n$ converges uniformly to f on A. Then f is continuous on A.

Theorem 6.2.2. Suppose each f_n is differentiable on $[a, b]$ and $\sum f_n'$ converges uniformly to g on $[a, b]$. If $\sum f_n(c)$ converges for some $c \in [a, b]$, then $\sum f_n$ converges uniformly to f on $[a, b]$ and $f' = g$.

Theorem 6.2.3. $\sum f_n$ converges uniformly on $A \subseteq \mathbb{R}$ if and only if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\forall n > m \geq N$, $\forall x \in A$, $|\sum_{k=m+1}^{n} f_k(x)| < \varepsilon$.

Theorem 6.2.4. Given some (f_n) on $A \subseteq \mathbb{R}$, suppose some (M_n) satisfies $|f_n(x)| \leq M_n$ for each $x \in A$ and each $n \in \mathbb{N}$. If $\sum M_n$ converges, then $\sum f_n$ converges uniformly on A.

6.3 Power Series

Definition 6.3.1. A **power series** is a series of functions of the form $\sum_{n=0}^{\infty} a_n x^n$ for some constants $a_0, a_1, \ldots \in \mathbb{R}$.

Theorem 6.3.1. If a power series $\sum a_n x^n$ converges at some $x_0 \in \mathbb{R}$ then it converges absolutely for any x with $|x| < |x_0|$.

Corollary 6.3.1.1. The set of points that a power series converges on must be either $\{0\}$,$\mathbb{R}$, or some bounded interval of the form $[-R, R], (-R, R), [-R, R), (-R, R]$. The value of R is the **radius of convergence** of the power series. If the set is $\{0\}$ then the radius is 0 and if it is $\mathbb{R}$ then the radius is ∞.

Proof. If $\sum a_n x_0^n$ converges then $(a_n x_0^n) \to 0$ so $(a_n x_0^n)$ is bounded. Let $M \geq a_n x_0^n$ for all n. Then for $|x| < |x_0|$, we have $|a_n x_0^n| \leq M|x/x_0|^n$. Since $|x/x_0| < 1$, $\sum M|x/x_0|^n$ converges, so $\sum a_n x_0^n$ converges absolutely. □

Theorem 6.3.2. If a power series $\sum a_n x^n$ converges absolutely at some $x_0 > 0$ then it converges uniformly on the interval $[0, x_0]$.

Corollary 6.3.2.1. If a power series converges at some $x_0 > 0$ then it converges uniformly on any compact subset of $(-x_0, x_0]$.

Theorem 6.3.3 (Abel's Theorem). Suppose a power series converges at the point $x = R > 0$. Then the series converges uniformly on $[0, R]$ (moreover, it converges uniformly on $(-R, R]$. A similar result holds for $x = -R$.

Corollary 6.3.3.1. A power series converges uniformly on any compact subset of the interval it converges on.

Theorem 6.3.4. Power series are continuous everywhere they are defined.

Theorem 6.3.5. Power series are infinitely differentiable on the interior of their radius of convergence. Moreover, $(\sum a_n x^n)' = \sum n a_n x^{n-1}$.

6.4 Taylor Series

Definition 6.4.1. Let $f : I \to \mathbb{R}$ be infinitely differentiable on I. Then we say that $f \in C^\infty$ on I.

Definition 6.4.2. Let $f \in C^\infty$ for some neighborhood of c. Then the **Taylor series** generated by f about c is the series $\sum_0^\infty f^{(n)}(c)(x-c)^n/n!$.

Theorem 6.4.1. Let f be differentiable $N+1$ times on $(-R, R)$. Let $S_N(x) = \sum_0^N f^{(n)}(0)x^n/n!$. Then for any $x \neq 0$ in $(-R, R)$, there is some $c(x)$ with $|c| < |x|$ such that $E_N(x) = f(x) - S_N(x) = f^{(N+1)}(c(x))x^{N+1}/(N+1)!$

Corollary 6.4.1.1. Let T be the Taylor series generated by f about 0. If $E_n(x) \to 0$ on some interval I then $T = f$ on that interval.

Corollary 6.4.1.2. Let $f \in C^\infty$ in some neighborhood I of c. Let T be the Taylor series generated by f about c. Then if there exists $M > 0$ such that $|f^{(n)}(x)| < M^n$ on I, $T = f$ on I.

6.5 Weierstrass Approximation Theorem

Definition 6.5.1. A function $\phi : [a, b] \to \mathbb{R}$ is **polygonal** if it is linear on a finite number of subintervals which cover $[a, b]$.

Theorem 6.5.1. For any continuous function f on $[a, b]$, for any $\varepsilon > 0$, there exists a polygonal function ϕ such that $|\phi - f| < \varepsilon$ on $[a, b]$.

Theorem 6.5.2. For any polygonal function ϕ on $[a, b]$, for any $\varepsilon > 0$, there exists a polynomial p such that $|\phi - p| < \varepsilon$ on $[a, b]$.

Theorem 6.5.3 (Weierstrass Approximation Theorem). For any continuous function f on $[a, b]$, for any $\varepsilon > 0$, there exists a polynomial p such that $|f - p| < \varepsilon$ on $[a, b]$.

7 Integration

7.1 Riemann Integral

Definition 7.1.1. Given an interval I, a **partition** P of I is a finite collection of points $x_n \in I$ such that $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$.

Definition 7.1.2. Given a partition P and partition P', P' is a **refinement** of P if $P \subseteq P'$.

Definition 7.1.3. Let P and Q be two partitions. Then the partition $J = P \cup Q$ is the **common refinement**, and J is a refinement of both P and Q.

Definition 7.1.4. Given a bounded function $f : [a, b] \to \mathbb{R}$ and a partition P of $[a, b]$, let $M_k = \sup\{f(x) : x_k \le x \le x_{k+1}\}$ and $m_k = \inf\{f(x) : x_k \le x \le x_{k+1}\}$. Then $U(f, P) = \sum_0^{k-1} M_k(x_{k+1} - x_k)$ the **upper sum** of f with respect to P, and $L(f, P)$ is the **lower sum**, defined with m_k.

Theorem 7.1.1. Given any f and partition P, $L(f, P) \le U(f, P)$.

Theorem 7.1.2. If P' is a refinement of P, then $U(f, P') \le U(f, P)$ and $L(f, P) \le L(f, P')$.

Theorem 7.1.3. If P and Q are two partitions, then $L(f, P) \le U(f, Q)$

Definition 7.1.5. Let f be bounded on $[a, b]$. Let $\mathcal{P}$ be the collection of all partitions of an interval $[a, b]$. Then the **upper integral** of f on $[a, b]$ is $\overline{\int_a^b} f = U(f) = \inf\{U(f, P) : P \in \mathcal{P}\}$. The **lower integral** is $\underline{\int_a^b} f = L(f) = \sup\{L(f, P) : P \in \mathcal{P}\}$.

Definition 7.1.6. Let f be a bounded function on $[a, b]$. If $U(f) = L(f)$ on $[a, b]$, then f is **Riemann integrable** on $[a, b]$, and we say $\int_a^b f = U(f) = L(f)$.

Theorem 7.1.4. A bounded function f is integrable on $[a, b]$ if and only if, for all $\varepsilon > 0$ there exists a partition P_ε such that $U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$.

Theorem 7.1.5. A continuous function $f : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$.

Theorem 7.1.6. A bounded function f with finite discontinuities on $[a, b]$ is integrable on $[a, b]$.

Definition 7.1.7. Let P be a partition of I. If $\{c_k\}$ is a collection of points such that $c_k \in [x_k, x_{k+1}]$ for all $0 \le k \le n - 1$, then $(P, \{c_k\})$ is called a **tagged partition**.

Definition 7.1.8. Let $(P, \{c_k\})$ be a tagged partition. Then given a bounded function f, the tagged sum of f with respect to $(P, \{c_k\})$ is $R(f, P, \{c_k\}) = \sum f(c_k)(x_{k+1} - x_k)$.

Definition 7.1.9. The **norm** of a partition P is $||P|| = \max\{x_{k+1} - x_k : 0 \le k \le n - 1\}$.

Theorem 7.1.7. A bounded function f is integrable with $\int f = A$ if and only if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for any partition P with $||P|| < \varepsilon$, we have $|R(f, P, \{c_k\}) - A| < \varepsilon$.

Theorem 7.1.8. If f is integrable on $[a, b]$ then it is integrable on any $I \subseteq [a, b]$.

Theorem 7.1.9. If f is integrable on $[c, b]$ for all $c \in (a, b)$, then f is integrable on $[a, b]$.

Theorem 7.1.10. If f is integrable on $[a, b]$ and integrable on $[b, c]$ then it is integrable on $[a, c]$. Moreover, $\int_a^c f = \int_a^b f + \int_b^c f$.

Definition 7.1.10. A set $A \subseteq \mathbb{R}$ has **measure zero** if for any $\varepsilon > 0$, there exists a cover of A by open intervals with total length less than ε.

Theorem 7.1.11. Given a bounded function f on $[a, b]$, define $D(f)$ to be the set of points at which f is continuous. f is integrable on $[a, b]$ if and only if $D(f)$ has measure 0.

Theorem 7.1.12. If f, g are integrable on $[a, b]$ then

- $f \pm g$ is integrable with $\int f \pm g = \int f \pm \int g$
- kf is integrable with $\int kf = k \int f$
- fg and f/g are integrable (assuming $g \neq 0$)
- $|f|$ is integrable and $|\int f| \leq \int |f|$
- If $m \leq f \leq M$ on $[a, b]$ then $(b - a)m \leq \int f \leq (b - a)M$

Theorem 7.1.13 (Fundamental Theorem of Calculus I). Let f be an integrable function on $[a, b]$. Suppose F satisfies $F = f'$ on $[a, b]$

Theorem 7.1.14 (Fundamental Theorem of Calculus II). Let f be an integrable function on $[a, b]$. Let $g(x) = \int_a^x f$. Then g is differentiable with $g' = f$.

Theorem 7.1.15. Let $f_n \rightrightarrows f$ on $[a, b]$, with each f_n integrable on $[a, b]$. Then f is integrable on $[a, b]$ with $\int_a^b f = \lim \int_a^b f_n$.

Index