MAT 425 Notes

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Introduction

This document contains notes taken for the class MAT 425: Integration Theory and Hilbert Spaces at Princeton University, taken in the Spring 2025 semester. These notes are primarily based on lectures by Professor Jacob Shapiro. Other references used in these notes include *Real Analysis* by Elias Stein and Rami Shakarchi, *Real and Complex Analysis* by Walter Rudin, *Real Analysis (2nd Edition)* by Halsey Royden, *The Elements of Integration and Lebesgue Measure* by Robert Bartle, *Measure Theory* by Paul Halmos, and *Real Analysis: Modern Techinques and Their Applications* by Gerald Folland. Since these notes were primarily taken live, they may contains typos or errors.

Chapter 1

1.1 Motivations

The formal study of measure theory is motivated historically by the insufficiency of the Riemann integral as a complete tool for describing integration. Considering some bounded function $f : [a, b] \to \mathbb{R}$, there are many desirable properties that we might expect from an integral.

1. We might ask that the integral produces the average value of the function f on $[a, b]$, as
$$\overline{f} = \frac{1}{b-a} \int_a^b f$$
2. Geometrically, we can interpret the integral as the signed area between the graph of f and the x-axis:
$$A = \int_a^b f$$
3. We also think of integrals as the continuous generalization of summation.

Recall that the Riemann integral of f over $[a, b]$ is defined by considering, for fixed $N \in \mathbb{N}$, the upper and lower sums L_N, U_N defined by

$$L_N(f) = \frac{b-a}{N} \sum_{j=0}^{N-1} \inf \left\{ f(x) : x \in a + [n, n+1] \frac{b-a}{N} \right\}$$
$$U_N(f) = \frac{b-a}{N} \sum_{j=0}^{N-1} \sup \left\{ f(x) : x \in a + [n, n+1] \frac{b-a}{N} \right\}$$

We say that f is Riemann integrable with integral $I = \int_a^b f \in \mathbb{R}$ if $\lim L_N, \lim U_N$ both exist and are equal to I.

From our previous studies, Lebesgue's criterion gave a convenient characterization of Riemann integrable functions.

Definition 1.1

A set $S \subseteq \mathbb{R}$ has **measure zero** if for any $\varepsilon > 0$ there exists a collection $\{U_n\}_{n\in\mathbb{N}}$ of open intervals such that $S \subseteq \bigcup U_n$ and $\sum |U_n| < \varepsilon$, where $|U_n|$ is the length of U_n.

Example 1.1

The Cantor set $\mathcal{C}$ has measure zero. This is a consequence of the fact that at each iterative step in the construction of the Cantor set, we have a collection of open intervals covering the Cantor set, and the total length at step k is given by $\left(\frac{2}{3}\right)^k \to 0$.

Theorem 1.1: Lebesgue's Theorem

A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if the set of discontinuities of f has measure zero.

In particular, continuous functions are always Riemann integrable. The indicator function $\chi_{\mathcal{C}}$ of the Cantor set is Riemann integrable, since its discontinuities are of measure zero. However, $\chi_{\mathbb{Q}}$ (restricted to some compact interval) is not, since it is discontinuous at *every* point (this is precisely Dirichlet's function).

One can define a Riemann integral for unbounded functions or on unbounded domains by considering appropriate limits of Riemann integrals on compact intervals.

Example 1.2

The improper integral $\int_0^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x$ is computed as

$$\int_{[0,1]} \frac{1}{\sqrt{x}}\,\mathrm{d}x = \lim_{n\to\infty} \int_{[\frac{1}{n},1]} \frac{1}{\sqrt{x}}\,\mathrm{d}x = \lim_{n\to\infty} 2\sqrt{x}|_{\frac{1}{n}}^{1} = \lim_{n\to\infty} \left[2 - \frac{2}{\sqrt{n}}\right] = 2$$

This method may be naturally extended to functions with a finite number of "integrable" discontinuities, or sometimes countable discontinuities. However, the following example shows that it fails in the general case.

Example 1.3

Let $\{\eta_n\}_{n\in\mathbb{N}}$ be an enumeration of the set $(0, 1) \cap \mathbb{Q}$. Define $f_n : [0, 1] \to \mathbb{R}$ by

$$f_n : x \mapsto \begin{cases} \frac{1}{\sqrt{x-\eta_n}} : & x > \eta_n \\ 0 : & x \leq \eta_n \end{cases}$$

Then define

$$f(x) := \sum_{n=1}^{\infty} 2^{-n} f_n(x)$$

By density, f is unbounded in every open subinterval of $[0,1]$. As a result, there is no limit of intervals increasing to $[0,1]$ which we could use to define the integral of f over $[0,1]$, in the sense used in the previous example.

To try to figure out a way around this, note that our work in the previous example shows that

$$\int_{[0,1]} f_n = 2\sqrt{1-\eta_n}$$

Now, consider the (unjustified) interchange of the integral and sum:

$$\int_{[0,1]} f = \int_{[0,1]} \sum_{n=1}^{\infty} 2^{-n} f_n \longrightarrow \sum_{n=1}^{\infty} 2^{-n} \int_{[0,1]} f_n = \sum_{n=1}^{\infty} 2^{-n} 2\sqrt{1-\eta_n} < \infty$$

As the above example demonstrates, an important question in analysis is which operations respect the limiting process. In particular, we know that uniform convergence respects the limit:

Theorem 1.2

Let $f_n : [a,b] \to \mathbb{R}$ be a sequence of bounded Riemann integrable functions which converge uniformly to f. Then f is Riemann integrable and $\lim \int_{[a,b]} f_n = \int_{[a,b]} f$.

However, it is desirable to us to apply this interchange under weaker hypotheses than uniform convergence, so that we can develop a more powerful and general theory of integration.

Example 1.4

Consider again the enumeration $\{\eta_n\}_{n\in\mathbb{N}}$ of $(0,1) \cap \mathbb{Q}$. Define

$$f_n \coloneqq \chi_{\{\eta_j : j \in [1,n]\}}$$

In words, $f_n(x) = 1$ if $x = \eta_j$ for some $j \leq n$ and 0 otherwise. $\int_{[0,1]} f_n = 0$ for all n, so we would like to assign the value 0 to $\int_{[0,1]} \lim f$. However, observe that f_n converges pointwise to Dirichlet's function, which is not Riemann integrable.

The development of the Lebesgue integral, which solves many issues with the Riemann integral, will be accomplished by first discussing the general theory of measure and integration, and following the construction of the Lebesgue measure and integral.

1.2 Abstract Measure Theory

The development of a measure space structure on a set is accomplished by defining a collection of "measurable" subsets, not unlike a topology, which satisfies particular structural constraints.

Definition 1.2

Let X be a set, and consider a collection of subsets $\mathcal{M} \subseteq \mathcal{P}(X)$. We say that $\mathcal{M}$ is a **σ-algebra** on X if

1. $X \in \mathcal{M}$,
2. If $A \in \mathcal{M}$ then $X \setminus A \in \mathcal{M}$,
3. If $\{A_n\}_{n\in\mathbb{N}}$ is a countable collection of elements of $\mathcal{M}$, then $\bigcup A_n \in \mathcal{M}$.

If $\mathcal{M}$ is a σ-algebra on X, then $(X, \mathcal{M})$ is called a **measurable space**. An element of $\mathcal{M}$ is called a **measurable set**. If the σ-algebra on X is understood by context, then $\mathrm{Meas}(X)$ denotes the set of measurable subsets of X (that it, it denotes the implied σ-algebra).

Notice that while a topology is required to be closed under arbitrary unions, a σ-algebra is only required to be closed under countable unions. Moreover, the following follows directly from the axioms of σ-algebras:

Proposition 1.3

$\varnothing \in \mathrm{Meas}(X)$ and $\mathrm{Meas}(X)$ is clsoed under countable intersections.

For comparison, recall the following definition of a topology:

Definition 1.3

Let X be a set, and consider a collection of subsets $\mathcal{T} \subseteq \mathcal{P}(X)$. We say that $\mathcal{T}$ is a **topology** on X if

1. $X, \varnothing \in \mathcal{T}$,
2. $\bigcap_{n=1}^{N} V_n \in \mathcal{T}$ whenever each $V_n \in \mathcal{T}$,
3. $\bigcup_{\alpha\in A} V_\alpha \in \mathcal{T}$ whenever $V_\alpha \in \mathcal{T}$ for an arbitrary indexing set A.

By direct comparison, a topology is not automatically a σ-algebra, since it may not be closed under complements.

Again in analogy to topology, recall that continuous functions are the morphisms of topological spaces. Thus, we can ask which functions can be considered to be the morphisms of measurable spaces. Indeed, just as continuous functions are topologically characterized by preserving open sets under preimages, we define measurable space morphisms similarly:

Definition 1.4

A function $f : X \to Y$ for measurable spaces X, Y is said to be a **measurable function** if $f^{-1}(A) \in \mathrm{Meas}(X)$ whenever $A \in \mathrm{Meas}(Y)$.

It follows immediately that the composition of measurable functions is measurable.

As with topologies, any set automatically comes equipped with two σ-algebras: the power set $\mathcal{P}(X)$ and $\{\varnothing, X\}$. These are the largest and smallest σ-algebras on X, respectively.

Example 1.5

Let $X = \{1, 2, 3, 4\}$. Then the following is a nontrivial σ-algebra:

$$\mathcal{M} = \{\varnothing, X, \{1, 2\}, \{3, 4\}\}$$

Generalizing the above, for any $A \subseteq X$, the σ-algebra $\{\varnothing, X, A, X \setminus A\}$ is the smallest σ-algebra containing A.

Remark 1.1

The arbitrary intersection of σ-algebras on a common set is again a σ-algebra, but not necessarily unions.

Definition 1.5

Let $f : X \to Y$, where X is an arbitrary set and Y is a measurable space. Then the σ-algebra $\sigma(f)$ **generated** by f is

$$\sigma(f) := \left\{ f^{-1}(A) : A \in \mathrm{Meas}(Y) \right\}$$

Essentially, $\sigma(f)$ is generated by pulling back the measurable structure of Y through f. It is straightforward to verify that $\sigma(f)$ is actually a σ-algebra, and it follows that $\sigma(f)$ is the smallest σ-algebra on X such that f is measurable. In other words, if $\mathcal{M}$ is a σ-algebra on X, then f is measurable with respect to $(X, \mathcal{M}), Y$ if and only if $\sigma(f) \subseteq \mathcal{M}$.

We can generalize the construction of "smallest σ-algebra" type constructions to find the smallest σ-algebra containing a certain collection of subsets. It is somewhat nonobvious that such an algebra exists or is unique.

Theorem 1.4

Let $\mathcal{F} \subseteq \mathcal{P}(X)$. Then there exists a unique minimal σ-algebra $\sigma(\mathcal{F})$ on X such that $\mathcal{F} \subseteq \sigma(\mathcal{F})$.

Proof. Let Ω be the set of collection of all σ-algebras on X which contain $\mathcal{F}$. Ω is nonempty since $\mathcal{P}(X) \subseteq \Omega$. Define

$$\sigma(\mathcal{F}) = \bigcap_{\mathcal{M} \in \Omega} \mathcal{M}$$

Since the arbitrary intersection of σ-algebras is a σ-algebra, $\sigma(\mathcal{F})$ is indeed a σ-algebra. Moreover, by construction $\sigma(\mathcal{F})$ is contained in any element of Ω, and it is thus minimal. $\square$

As we remarked above, a topology is not in general a σ-algebra. The two notions are linked by considering the Borel σ-algebra, which is generated by the open sets on a space.

Definition 1.6

Let X be a topological space with topology $\mathcal{T}$. Then the **Borel σ-algebra** on X is given by $\mathcal{B}(X) = \sigma(\mathcal{T})$.

Note that since σ-algebras are closed under complements, by definition the closed sets on X are in $\mathcal{B}(X)$. It is also the case that countable intersections of open sets and countable unions of closed sets are contained in $\mathcal{B}(X)$, when this is not necessarily true in $\mathcal{T}$. Elements of a Borel σ-algebra are called **Borel sets**. In general, when we refer to topological spaces without specifying a σ-algebra, the Borel algebra is implicitly taken.

Under Hausdorff's terminology, sets which are the countable union of closed sets are denoted F_σ sets. Analogously, sets which are the countable intersection of open sets are denoted G_δ sets.

To make more precise the connection between topologies and measurable spaces through the Borel σ-algebra, we make the following claim:

Proposition 1.5

Let $f : X \to Y$ be a mapping between topological spaces such that $f^{-1}(V) \in \mathcal{B}(X)$ for any open set $V \subseteq Y$. Then f is measurable with respect to $\mathcal{B}(X), \mathcal{B}(Y)$.

Proof. Define the collection

$$\mathcal{M} = \left\{ A \in \mathcal{P}(Y) : f^{-1}(A) \in \mathcal{B}(X) \right\}$$

It can be verified that $\mathcal{M}$ is a σ-algebra on Y. Then, by assumption the open sets in Y are contained in $\mathcal{M}$. Moreover, by definition $\mathcal{B}(Y)$ is the smallest σ-algebra containing the open sets. Therefore we have $\text{Open}(Y) \subseteq \mathcal{B}(Y) \subseteq \mathcal{M}$. Since $\mathcal{B}(Y)$ is contained in $\mathcal{M}$ it follows by definition that f is measurable with respect to $\mathcal{B}(X), \mathcal{B}(Y)$. $\square$

Note that the above proposition implies that any continuous mapping between topological spaces is measurable with respect to their Borel algebras. We prove the following statement, which will aid our understanding of complex measurable functions:

Proposition 1.6

Let X be a measurable space and Y a topological space. Let $u, v : X \to \mathbb{R}$ be measurable and $\varphi : \mathbb{R}^2 \to Y$ be continuous. Then $h : X \to Y$ defined by

$$h(x) = \varphi(u(x), v(x))$$

is measurable with respect to $\text{Meas}(X), \mathcal{B}(Y)$.

Proof. From the previous proposition, φ is measurable with respect to $\mathcal{B}(\mathbb{R})$ and $\mathcal{B}(Y)$. Let $f : X \to \mathbb{R}^2$ be $x \mapsto (u(x), v(x))$. Then $h = \varphi \circ f$, and the composition of measurable functions is measurable. So it suffices to show f is measurable with respect to $\text{Meas}(X), \mathcal{B}(\mathbb{R})$.

Take some rectangle $R = I_1 \times I_2$ for intervals I_1, I_2. Then $f^{-1}(\mathbb{R}) = u^{-1}(I_1) \cap v^1(I_2)$. $f^{-1}(\mathbb{R})$ is then a measurable set since both u, v are measurable functions. Now, let $V \in \text{Open}(\mathbb{R}^2)$. Then V can be written as the countable union of rectangles. So we have

$$f^{-1}(V) = f^{-1}\left(\bigcup_{n=1}^{\infty} R_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(R_n) \in \text{Meas}(X)$$

From the previous proposition it follows that f is measurable. ☐

We can now use this fact to produce measurable functions from other measurable functions.

Theorem 1.7

Let X be a measurable space. Then:

1. If $u, v : X \to \mathbb{R}$ are measurable, then so is $u + iv : X \to \mathbb{C}$.
2. If $f : X \to \mathbb{C}$ is measurable, then so are $\text{Re}(f), \text{Im}(f), |f|$.
3. If $f, g : X \to \mathbb{C}$ are measurable then $f + g$ and fg are both measurable.
4. If $A \in \text{Meas}(X)$ then $\chi_A : X \to \mathbb{R}$ is measurable as well.
5. If $f : X \to \mathbb{C}$ is measurable then there exists $\alpha : X \to \mathbb{C}$ measurable such that $f = \alpha|f|$.

It is often of interest to us to work in the extended real line, so that we can consider functions or measures which assign infinite values to some points or sets. This is also helpful since the extended real line is compact.

Definition 1.7

The **extended real line** is denoted $[-\infty, \infty]$ or $\overline{\mathbb{R}}$, and consists of the set $\mathbb{R} \cup \{\pm\infty\}$, together with the topology that contains open sets in $\mathbb{R}$ and countable unions of sets of the form $(a, \infty]$ and $[-\infty, a)$.

Theorem 1.8

Let $f : X \to \overline{\mathbb{R}}$ with X a measurable space. If

$$f^{-1}((a, \infty]) \in \text{Meas}(X)$$

for all $a \in \mathbb{R}$, then f is measurable.

Proof. The point is to show that any open set in $\overline{\mathbb{R}}$ may be constructed countably from sets of the form $[a, \infty)$.

First we consider sets of the form $[-\infty, a)$. Let $\{a_n\} \to a$ be a sequence of points with $a_n < a$ for all a_n. Then

$$[-\infty, a) = \bigcup_{n=1}^{\infty} [-\infty, a_n] = \bigcup_{n=1}^{\infty} (a_n, \infty]^c$$

so $f^{-1}([-\infty, a)) \in \text{Meas}(X)$. We can similarly write

$$(a, b) = [-\infty, b) \cap (a, \infty]$$

so that $f^{-1}((a, b)) \in \text{Meas}(X)$ as well. Now it follows that any open set in the topology on $\overline{\mathbb{R}}$ has a preimage in $\text{Meas}(X)$, so it follows that f is measurable with respect to the Borel algebra on $\overline{\mathbb{R}}$. $\square$

Theorem 1.9

Let $f_n : X \to \overline{\mathbb{R}}$ be a sequence of measurable functions. Consider the functions $\sup f_n, \limsup f_n$, which are defined pointwise, are both measurable.

Definitions