MAT 425 Notes

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Contents

Introduction

This document contains notes taken for the class MAT 425: Integration Theory and Hilbert Spaces at Princeton University, taken in the Spring 2025 semester. These notes are primarily based on lectures by Professor Jacob Shapiro. Other references used in these notes include *Real Analysis* by Elias Stein and Rami Shakarchi, *Real and Complex Analysis* by Walter Rudin, *Real Analysis (2nd Edition)* by Halsey Royden, *The Elements of Integration and Lebesgue Measure* by Robert Bartle, *Measure Theory* by Paul Halmos, and *Real Analysis: Modern Techinques and Their Applications* by Gerald Folland. Since these notes were primarily taken live, they may contains typos or errors.

Chapter 1

Introductory Measure Theory

1.1 Motivations

The formal study of measure theory is motivated historically by the insufficiency of the Riemann integral as a complete tool for describing integration. Considering some bounded function $f : [a, b] \to \mathbb{R}$, there are many desirable properties that we might expect from an integral.

1. We might ask that the integral produces the average value of the function f on $[a, b]$, as
$$\overline{f} = \frac{1}{b-a} \int_a^b f$$
2. Geometrically, we can interpret the integral as the signed area between the graph of f and the x-axis:
$$A = \int_a^b f$$
3. We also think of integrals as the continuous generalization of summation.

Recall that the Riemann integral of f over $[a, b]$ is defined by considering, for fixed $N \in \mathbb{N}$, the upper and lower sums L_N, U_N defined by

$$L_N(f) = \frac{b-a}{N} \sum_{j=0}^{N-1} \inf \left\{ f(x) : x \in a + [n, n+1] \frac{b-a}{N} \right\}$$
$$U_N(f) = \frac{b-a}{N} \sum_{j=0}^{N-1} \sup \left\{ f(x) : x \in a + [n, n+1] \frac{b-a}{N} \right\}$$

We say that f is Riemann integrable with integral $I = \int_a^b f \in \mathbb{R}$ if $\lim L_N, \lim U_N$ both exist and are equal to I.

From our previous studies, Lebesgue's criterion gave a convenient characterization of Riemann integrable functions.

Definition 1.1

A set $S \subseteq \mathbb{R}$ has **measure zero** if for any $\varepsilon > 0$ there exists a collection $\{U_n\}_{n\in\mathbb{N}}$ of open intervals such that $S \subseteq \bigcup U_n$ and $\sum|U_n| < \varepsilon$, where $|U_n|$ is the length of U_n.

Example 1.1

The Cantor set $\mathcal{C}$ has measure zero. This is a consequence of the fact that at each iterative step in the construction of the Cantor set, we have a collection of open intervals covering the Cantor set, and the total length at step k is given by $\left(\frac{2}{3}\right)^k \to 0$.

Theorem 1.1: Lebesgue's Theorem

A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if the set of discontinuities of f has measure zero.

In particular, continuous functions are always Riemann integrable. The indicator function $\chi_{\mathcal{C}}$ of the Cantor set is Riemann integrable, since its discontinuities are of measure zero. However, $\chi_{\mathbb{Q}}$ (restricted to some compact interval) is not, since it is discontinuous at *every* point (this is precisely Dirichlet's function).

One can define a Riemann integral for unbounded functions or on unbounded domains by considering appropriate limits of Riemann integrals on compact intervals.

Example 1.2

The improper integral $\int_0^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x$ is computed as

$$\int_{[0,1]} \frac{1}{\sqrt{x}}\,\mathrm{d}x = \lim_{n\to\infty} \int_{[\frac{1}{n},1]} \frac{1}{\sqrt{x}}\,\mathrm{d}x = \lim_{n\to\infty} 2\sqrt{x}|_{\frac{1}{n}}^{1} = \lim_{n\to\infty} \left[2 - \frac{2}{\sqrt{n}}\right] = 2$$

This method may be naturally extended to functions with a finite number of "integrable" discontinuities, or sometimes countable discontinuities. However, the following example shows that it fails in the general case.

Example 1.3

Let $\{\eta_n\}_{n\in\mathbb{N}}$ be an enumeration of the set $(0, 1) \cap \mathbb{Q}$. Define $f_n : [0, 1] \to \mathbb{R}$ by

$$f_n : x \mapsto \begin{cases} \frac{1}{\sqrt{x-\eta_n}} : & x > \eta_n \\ 0 : & x \leq \eta_n \end{cases}$$

Then define

$$f(x) := \sum_{n=1}^{\infty} 2^{-n} f_n(x)$$

By density, f is unbounded in every open subinterval of $[0,1]$. As a result, there is no limit of intervals increasing to $[0,1]$ which we could use to define the integral of f over $[0,1]$, in the sense used in the previous example.

To try to figure out a way around this, note that our work in the previous example shows that

$$\int_{[0,1]} f_n = 2\sqrt{1-\eta_n}$$

Now, consider the (unjustified) interchange of the integral and sum:

$$\int_{[0,1]} f = \int_{[0,1]} \sum_{n=1}^{\infty} 2^{-n} f_n \longrightarrow \sum_{n=1}^{\infty} 2^{-n} \int_{[0,1]} f_n = \sum_{n=1}^{\infty} 2^{-n} 2\sqrt{1-\eta_n} < \infty$$

As the above example demonstrates, an important question in analysis is which operations respect the limiting process. In particular, we know that uniform convergence respects the limit:

Theorem 1.2

Let $f_n : [a,b] \to \mathbb{R}$ be a sequence of bounded Riemann integrable functions which converge uniformly to f. Then f is Riemann integrable and $\lim \int_{[a,b]} f_n = \int_{[a,b]} f$.

However, it is desirable to us to apply this interchange under weaker hypotheses than uniform convergence, so that we can develop a more powerful and general theory of integration.

Example 1.4

Consider again the enumeration $\{\eta_n\}_{n\in\mathbb{N}}$ of $(0,1) \cap \mathbb{Q}$. Define

$$f_n \coloneqq \chi_{\{\eta_j : j \in [1,n]\}}$$

In words, $f_n(x) = 1$ if $x = \eta_j$ for some $j \le n$ and 0 otherwise. $\int_{[0,1]} f_n = 0$ for all n, so we would like to assign the value 0 to $\int_{[0,1]} \lim f$. However, observe that f_n converges pointwise to Dirichlet's function, which is not Riemann integrable.

The development of the Lebesgue integral, which solves many issues with the Riemann integral, will be accomplished by first discussing the general theory of measure and integration, and following the construction of the Lebesgue measure and integral.

1.2 Abstract Measure Theory

The development of a measure space structure on a set is accomplished by defining a collection of "measurable" subsets, not unlike a topology, which satisfies particular structural constraints.

Definition 1.2

Let X be a set, and consider a collection of subsets $\mathcal{M} \subseteq \mathcal{P}(X)$. We say that $\mathcal{M}$ is a **σ-algebra** on X if

1. $X \in \mathcal{M}$,
2. If $A \in \mathcal{M}$ then $X \setminus A \in \mathcal{M}$,
3. If $\{A_n\}_{n\in\mathbb{N}}$ is a countable collection of elements of $\mathcal{M}$, then $\bigcup A_n \in \mathcal{M}$.

If $\mathcal{M}$ is a σ-algebra on X, then $(X, \mathcal{M})$ is called a **measurable space**. An element of $\mathcal{M}$ is called a **measurable set**. If the σ-algebra on X is understood by context, then $\mathrm{Meas}(X)$ denotes the set of measurable subsets of X (that it, it denotes the implied σ-algebra).

Notice that while a topology is required to be closed under arbitrary unions, a σ-algebra is only required to be closed under countable unions. Moreover, the following follows directly from the axioms of σ-algebras:

Proposition 1.3

$\varnothing \in \mathrm{Meas}(X)$ and $\mathrm{Meas}(X)$ is clsoed under countable intersections.

For comparison, recall the following definition of a topology:

Definition 1.3

Let X be a set, and consider a collection of subsets $\mathcal{T} \subseteq \mathcal{P}(X)$. We say that $\mathcal{T}$ is a **topology** on X if

1. $X, \varnothing \in \mathcal{T}$,
2. $\bigcap_{n=1}^{N} V_n \in \mathcal{T}$ whenever each $V_n \in \mathcal{T}$,
3. $\bigcup_{\alpha\in A} V_\alpha \in \mathcal{T}$ whenever $V_\alpha \in \mathcal{T}$ for an arbitrary indexing set A.

By direct comparison, a topology is not automatically a σ-algebra, since it may not be closed under complements.

Again in analogy to topology, recall that continuous functions are the morphisms of topological spaces. Thus, we can ask which functions can be considered to be the morphisms of measurable spaces. Indeed, just as continuous functions are topologically characterized by preserving open sets under preimages, we define measurable space morphisms similarly:

Definition 1.4

A function $f : X \to Y$ for measurable spaces X, Y is said to be a **measurable function** if $f^{-1}(A) \in \mathrm{Meas}(X)$ whenever $A \in \mathrm{Meas}(Y)$.

It follows immediately that the composition of measurable functions is measurable.

As with topologies, any set automatically comes equipped with two σ-algebras: the power set $\mathcal{P}(X)$ and $\{\varnothing, X\}$. These are the largest and smallest σ-algebras on X, respectively.

Example 1.5

Let $X = \{1, 2, 3, 4\}$. Then the following is a nontrivial σ-algebra:

$$\mathcal{M} = \{\varnothing, X, \{1, 2\}, \{3, 4\}\}$$

Generalizing the above, for any $A \subseteq X$, the σ-algebra $\{\varnothing, X, A, X \setminus A\}$ is the smallest σ-algebra containing A.

Remark 1.1

The arbitrary intersection of σ-algebras on a common set is again a σ-algebra, but not necessarily unions.

Definition 1.5

Let $f : X \to Y$, where X is an arbitrary set and Y is a measurable space. Then the σ-algebra $\sigma(f)$ **generated** by f is

$$\sigma(f) \coloneqq \left\{ f^{-1}(A) : A \in \text{Meas}(Y) \right\}$$

Essentially, $\sigma(f)$ is generated by pulling back the measurable structure of Y through f. It is straightforward to verify that $\sigma(f)$ is actually a σ-algebra, and it follows that $\sigma(f)$ is the smallest σ-algebra on X such that f is measurable. In other words, if $\mathcal{M}$ is a σ-algebra on X, then f is measurable with respect to $(X, \mathcal{M}), Y$ if and only if $\sigma(f) \subseteq \mathcal{M}$.

We can generalize the construction of "smallest σ-algebra" type constructions to find the smallest σ-algebra containing a certain collection of subsets. It is somewhat nonobvious that such an algebra exists or is unique.

Theorem 1.4

Let $\mathcal{F} \subseteq \mathcal{P}(X)$. Then there exists a unique minimal σ-algebra $\sigma(\mathcal{F})$ on X such that $\mathcal{F} \subseteq \sigma(\mathcal{F})$.

Proof. Let Ω be the set of collection of all σ-algebras on X which contain $\mathcal{F}$. Ω is nonempty since $\mathcal{P}(X) \subseteq \Omega$. Define

$$\sigma(\mathcal{F}) = \bigcap_{\mathcal{M} \in \Omega} \mathcal{M}$$

Since the arbitrary intersection of σ-algebras is a σ-algebra, $\sigma(\mathcal{F})$ is indeed a σ-algebra. Moreover, by construction $\sigma(\mathcal{F})$ is contained in any element of Ω, and it is thus minimal. $\square$

As we remarked above, a topology is not in general a σ-algebra. The two notions are linked by considering the Borel σ-algebra, which is generated by the open sets on a space.

Definition 1.6

Let X be a topological space with topology $\mathcal{T}$. Then the **Borel σ-algebra** on X is given by $\mathscr{B}(X) = \sigma(\mathcal{T})$.

Note that since σ-algebras are closed under complements, by definition the closed sets on X are in $\mathscr{B}(X)$. It is also the case that countable intersections of open sets and countable unions of closed sets are contained in $\mathscr{B}(X)$, when this is not necessarily true in $\mathcal{T}$. Elements of a Borel σ-algebra are called **Borel sets**. In general, when we refer to topological spaces without specifying a σ-algebra, the Borel algebra is implicitly taken.

Under Hausdorff's terminology, sets which are the countable union of closed sets are denoted F_σ sets. Analogously, sets which are the countable intersection of open sets are denoted G_δ sets.

To make more precise the connection between topologies and measurable spaces through the Borel σ-algebra, we make the following claim:

Proposition 1.5

Let $f : X \to Y$ be a mapping between topological spaces such that $f^{-1}(V) \in \mathscr{B}(X)$ for any open set $V \subseteq Y$. Then f is measurable with respect to $\mathscr{B}(X), \mathscr{B}(Y)$.

Proof. Define the collection

$$\mathcal{M} = \left\{A \in \mathcal{P}(Y) : f^{-1}(A) \in \mathscr{B}(X)\right\}$$

It can be verified that $\mathcal{M}$ is a σ-algebra on Y. Then, by assumption the open sets in Y are contained in $\mathcal{M}$. Moreover, by definition $\mathscr{B}(Y)$ is the smallest σ-algebra containing the open sets. Therefore we have $\mathrm{Open}(Y) \subseteq \mathscr{B}(Y) \subseteq \mathcal{M}$. Since $\mathscr{B}(Y)$ is contained in $\mathcal{M}$ it follows by definition that f is measurable with respect to $\mathscr{B}(X), \mathscr{B}(Y)$. $\square$

Note that the above proposition implies that any continuous mapping between topological spaces is measurable with respect to their Borel algebras. We prove the following statement, which will aid our understanding of complex measurable functions:

Proposition 1.6

Let X be a measurable space and Y a topological space. Let $u, v : X \to \mathbb{R}$ be measurable and $\varphi : \mathbb{R}^2 \to Y$ be continuous. Then $h : X \to Y$ defined by

$$h(x) = \varphi(u(x), v(x))$$

is measurable with respect to $\mathrm{Meas}(X), \mathscr{B}(Y)$.

Proof. From the previous proposition, φ is measurable with respect to $\mathscr{B}(\mathbb{R})$ and $\mathscr{B}(Y)$. Let $f : X \to \mathbb{R}^2$ be $x \mapsto (u(x), v(x))$. Then $h = \varphi \circ f$, and the composition of measurable functions is measurable. So it suffices to show f is measurable with respect to $\text{Meas}(X), \mathscr{B}(\mathbb{R})$.

Take some rectangle $R = I_1 \times I_2$ for intervals I_1, I_2. Then $f^{-1}(\mathbb{R}) = u^{-1}(I_1) \cap v^1(I_2)$. $f^{-1}(\mathbb{R})$ is then a measurable set since both u, v are measurable functions. Now, let $V \in \text{Open}(\mathbb{R}^2)$. Then V can be written as the countable union of rectangles. So we have

$$f^{-1}(V) = f^{-1}\left(\bigcup_{n=1}^{\infty} R_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(R_n) \in \text{Meas}(X)$$

From the previous proposition it follows that f is measurable. $\square$

We can now use this fact to produce measurable functions from other measurable functions.

Theorem 1.7

Let X be a measurable space. Then:

1. If $u, v : X \to \mathbb{R}$ are measurable, then so is $u + iv : X \to \mathbb{C}$.
2. If $f : X \to \mathbb{C}$ is measurable, then so are $\text{Re}(f), \text{Im}(f), |f|$.
3. If $f, g : X \to \mathbb{C}$ are measurable then $f + g$ and fg are both measurable.
4. If $A \in \text{Meas}(X)$ then $\chi_A : X \to \mathbb{R}$ is measurable as well.
5. If $f : X \to \mathbb{C}$ is measurable then there exists $\alpha : X \to \mathbb{C}$ measurable such that $f = \alpha|f|$.

It is often of interest to us to work in the extended real line, so that we can consider functions or measures which assign infinite values to some points or sets. This is also helpful since the extended real line is compact.

Definition 1.7

The **extended real line** is denoted $[-\infty, \infty]$ or $\overline{\mathbb{R}}$, and consists of the set $\mathbb{R} \cup \{\pm\infty\}$, together with the topology that contains open sets in $\mathbb{R}$ and countable unions of sets of the form $(a, \infty]$ and $[-\infty, a)$.

Theorem 1.8

Let $f : X \to \overline{\mathbb{R}}$ with X a measurable space. If

$$f^{-1}((a, \infty]) \in \text{Meas}(X)$$

for all $a \in \mathbb{R}$, then f is measurable.

Proof. The point is to show that any open set in $\overline{\mathbb{R}}$ may be constructed countably from sets of the form $(a, \infty]$.

First we consider sets of the form $[-\infty, a)$. Let $\{a_n\} \to a$ be a sequence of points with $a_n < a$ for all a_n. Then

$$[-\infty, a) = \bigcup_{n=1}^{\infty} [-\infty, a_n] = \bigcup_{n=1}^{\infty} (a_n, \infty]^c$$

so $f^{-1}([-\infty, a)) \in \mathrm{Meas}(X)$. We can similarly write

$$(a, b) = [-\infty, b) \cap (a, \infty]$$

so that $f^{-1}((a, b)) \in \mathrm{Meas}(X)$ as well. Now it follows that any open set in the topology on $\overline{\mathbb{R}}$ has a preimage in $\mathrm{Meas}(X)$, so it follows that f is measurable with respect to the Borel algebra on $\overline{\mathbb{R}}$. □

Theorem 1.9

Let $f_n : X \to \overline{\mathbb{R}}$ be a sequence of measurable functions. Then the functions $\sup f_n, \limsup f_n, \inf f_n, \liminf f_n$, which are defined pointwise, are all measurable.

Proof. By the previous theorem, it suffices to check that $(\sup f_n)^{-1}((a, \infty])$ is measurable for all $a \in \mathbb{R}$, which we will do by expressing these sets as countable unions of preimages through the individual f_n.

We claim that

$$(\sup f_n)^{-1}((a, \infty]) = \bigcup_{n=1}^{\infty} f_n^{-1}((a, \infty])$$

While this is not true in general, it holds for the half-open infinite intervals. We show double inclusion.

($\subseteq$) Let $x \in (\sup f_n)^{-1}((a, \infty])$. Then $\sup f_n(x) > a$. Thus there exists n such that $f_n(x) > \sup f_n - \varepsilon$ for ε sufficiently small that $\sup f_n - \varepsilon > a$. So $x \in f_n^{-1}(a, \infty]$.

($\supseteq$) Similarly, if $x \in \bigcup_{n=1}^{\infty} f_n^{-1}((a, \infty])$, then there exists n with $f_n(x) > a$, which then implies that $\sup f_n(x) > a$ as well.

By hypothesis, $f_n^{-1}((a, \infty]) \in \mathrm{Meas}(X)$ for all n. Thus $\sup f_n$ is measurable. Of course this is true for inf as well.

To show that lim sup is measurable as well, we simply use the representation of lim sup as

$$\limsup a_n = \inf_{n \geq 1} \left(\sup_{m \geq n} a_m \right)$$

Thus $\limsup f_n$ and $\liminf f_n$ are both measurable as well. □

Corollary 1.10

If $\lim f_n$ exists and each $f_n : X \to \overline{\mathbb{R}}$ is measurable, then so is $\lim f_n$.

Proof. If the limit exists then it is equal to both the lim sup and lim inf. $\square$

Corollary 1.11

If $f, g : X \to \overline{\mathbb{R}}$ are measurable then so is $\max\{f, g\}$ and $\min\{f, g\}$.

Proof. Define $f_1 = f$ and $f_n = g$ for all $n \geq 2$. $\square$

The following theorem, which is a direct result of the above, is useful for considering an arbitrary function in terms of two nonnegative functions, which are easier to work with.

Proposition 1.12

For any $f : X \to \overline{\mathbb{R}}$, we can decompose it into positive and negative parts as $f = f^+ - f^-$, with

$$f^+ := \max\{f, 0\}$$
$$f^- := -\min\{f, 0\}$$

If f is measurable then so are f^+, f^-.

Proof. Based on the previous theorems, we just need to show that the constant zero function is measurable. But this is clear since the preimage of any subset of $\overline{\mathbb{R}}$ will be all of X if the subset contains 0, and $\varnothing$ otherwise. $\square$

1.3 Measures and Integration

Our next goal is to define integration of measurable functions. To do so, we will first consider simple functions, which will be the smallest building blocks that we define an integral on.

Definition 1.8

A function $s : X \to \mathbb{C}$ is a **simple function** if it has finite image. A simple nonnegative function is a simple function $s : X \to [0, \infty)$.

Because a simple function s assumes only finitely many values, we can always express it as the weighted sum of characteristic functions:

$$s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$$

where the α_i are the values in the image, and the A_i are their preimages.

Proposition 1.13

A simple function expressed as

$$s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$$

is measurable if and only if each A_i is measurable.

Proposition 1.14

Products and sums of simple functions are simple.

Proof. Clearly there are only finitely many values in the image. □

The utility of simple functions is that we may use them to approximate arbitrary measurable functions. Thus, so long as our integral operator interchanges with limits, we will be free to define integrals solely over simple functions.

Theorem 1.15

Let $f : X \to [0, \infty]$ be measurable. Then there exists a sequence of simple nonnegative measurable functions $s_n : X \to [0, \infty)$ such that:

- $0 \leq s_1 \leq s_2 \leq \ldots \leq f$.
- $s_n \to f$ pointwise.

Proof. We first provide an approximation for the identity, and then compose this with our function f. This approximation is made easier since we only need a pointwise limit. Thus we can consider a step function which both has finer steps (in order to approach the identity), and approximates the identity on a larger range (so that at there are always finite points in the range). Thus, we define

$$\varphi_n(t) = \begin{cases} 2^{-n}\lfloor 2^n t \rfloor, & 0 \leq t < n \\ n, & t \geq n \end{cases}$$

φ_n is simple since it has $\sim 2^{-n}$ values in its image. Additionally its preimages are half-open intervals so φ_n is measurable.

Now, we need to show that $\varphi_n \leq \varphi_{n+1}$ and φ_n converges to the identity pointwise. To show this, we prove that $t - 2^{-n} < \varphi_n(t) \leq t$ for all t. Then it is clear that as $n \to \infty$, φ_n approaches the identity.

Now, the conclusion to the proof is to set $s_n := \varphi_n \circ f$. s_n is simple since we factor through the simple function φ_n, and it is measurable as the composition of measurable functions. □

Now, we have established the technical background to define integration of simple functions. To do this, we essentially just assign each possible preimage of the simple functions a weight (which must be additive). Such a weight is a generalization of the notions of area, volume, mass, and so on, and is called a measure. We make two slightly different definitions for real and complex measures.

Definition 1.9

A **complex measure** on X is a function $\mu : \text{Meas}(X) \to \mathbb{C}$ which is countably additive, meaning that whenever $\{A_n\}$ is a countable sequence of pairwise disjoint measurable sets, we have

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

Definition 1.10

A **nonnegative measure** on X is a map $\mu : \text{Meas}(X) \to [0, \infty]$ which is countably additive, and such that there is at least one set $A \in \text{Meas}(X)$ with finite measure. A **measure space** is a triple $(X, \mathcal{M}, \mu)$ where $(X, \mathcal{M})$ is a mesaurable space and μ is a measure on $(X, \mathcal{M})$.

Note it follows that $\mu(\varnothing) = 0$, which would not be true in the nonnegative case if we did not require the existence of a finite measure set.

Proposition 1.16

If μ is a nonnegative meausre on X, then:

1. $\mu(\varnothing) = 0$.
2. μ is finitely additive.
3. If $A \subseteq B$ then $\mu(A) \leq \mu(B)$.
4. If $A_1 \subseteq A_2 \subseteq \ldots$ is a countable sequence of measurable sets, then
$$\lim \mu(A_n) = \mu\left(\bigcup A_n\right)$$
5. If $A_1 \supseteq A_2 \supseteq \ldots$ is a countable sequence of measurable sets and at least one A_n has finite measure, then
$$\lim \mu(A_n) = \mu\left(\bigcap A_n\right)$$

Roughly speaking, (4) and (5) tell us that measures may be approximated from either inside or outside.

Proof. 1. Take A measurable with finite measure, and consider the sequence $A_1 = A$, $A_n = \varnothing$ For $n \geq 2$. Then

$$\infty > \mu(A) = \mu\left(\bigcup A_n\right) = \sum \mu(A_n) = \mu(A) + \sum \mu(\varnothing)$$

which implies that we must have $\mu(\varnothing) = 0$.

2. Follows from countable additivity now that we know $\mu(\varnothing) = 0$.

3. We write $B = A \sqcup (B \setminus A)$ and apply (2).

4. Define $B_1 = A_1$, $B_2 = A_2 \setminus A_1$, and $B_n = A_n \setminus A_{n-1}$ for $n \geq 2$. Then apply countable additivity.

5. EXERCISE □

The most important example of a nonnegative measure is the Lebesgue measure. Because it is harder to define, we start by defining a few simpler measures.

Definition 1.11

Let X be a measurable space with $\text{Meas}(X) = \mathcal{P}(X)$. The **counting measure** is defined as $c : \text{Meas}(X) \to [0, \infty]$ such that $c(A)$ is the cardinality of A (possibly infinite).

Definition 1.12

Let X be a measurable space with $\text{Meas}(X) = \{\varnothing, X, \{x_0\}, X \setminus \{x_0\}\}$ for some distinguished point x_0. Then the **Dirac delta measure** at x_0 is defined by

$$S \mapsto \begin{cases} 1, & x_0 \in S \\ 0, & x_0 \notin S \end{cases}$$

We can now define the integral of a positive function against a measure. We will do so by first defining the integral of simple functions, then passing to the limit.

Definition 1.13

Let $\mu : \text{Meas}(X) \to [0, \infty]$ be a nonnegative measure. Let $s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$ be a simple measurable function, and let $E \in \text{Meas}(X)$. Then we define the **Lebesgue integral** of s over E with respect to μ to be

$$\int_E s \, \mathrm{d}\mu := \sum_{i=1}^{n} \alpha_i \mu(A_i \cap E)$$

By convention, if $\alpha_i = 0$ on a set of infinte measure, the entire term is considered to be zero.

Definition 1.14

Let $f : X \to [0,\infty]$ be measurable, and let $\mu : \text{Meas}(X) \to [0,\infty]$ be a nonnegative measure. Let $E \in \text{Meas}(X)$. Then the **Lebesgue integral** of f over E with respect to μ is

$$\int_E f \,\mathrm{d}\mu := \sup_{0 \le s \le f} \int_E s \,\mathrm{d}\mu$$

where the supremum is taken over all nonnegative simple measurable functions which satisfy $0 \le s \le f$.

Note that the second definition agrees with the first since the supremum is attained by f.

Example 1.6

Set $X = \mathbb{N}$, $\text{Meas}(X) = \mathcal{P}(X)$, and c to be the counting measure on X. Then

$$\int_A f \,\mathrm{d}c = \sum_{x \in A} f(x)$$

when $A \subseteq \mathbb{N}$. This is clear for finite A but requires limit theorems for countable A. Thus we have represented the sum as an integral against the counting measurable, meaning that our integral theorems will apply to sums as well.

1.4 Limit Theorems

We now turn to the question of interchanging the limit operator and integral, which is a major motivation for the definition of the integral in this way. We begin first with a few elementary properties.

Proposition 1.17

Let $0 \le f \le g$ be nonnegative measurable functions. Then:

1. $\int f \le \int g$.
2. If $A \subseteq B$ then $\int_A f \le \int_B f$.
3. If $0 \le c < \infty$, then $\int cf = c \int f$.
4. If $f \equiv 0$ then $\int_E f = 0$ for any measurable E, even if E has infinite measure.
5. If E is measurable with $\mu(E) = 0$, then $\int_E f = 0$.
6. For E measurable, $\int_E f = \int \chi_E f$.

Theorem 1.18

Let $s, t \geq 0$ be nonnegative simple functions and μ a measure. Define

$$\varphi_s(E) = \int_E s \, \mathrm{d}\mu$$

Then φ_s is a measure, and $\varphi_{s+t} = \varphi_s + \varphi_t$.

Proof. Let $E = \bigsqcup E_i$ be the disjoint countable union of some E_i. By definition,

$$\varphi_s(E) = \sum_{i=1}^{n} \alpha_i \mu(E \cap A_i) = \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{\infty} \mu(E_j \cap A_i)$$

Because s is simple we can interchange the finite sum:

$$\sum_{i=1}^{n} \alpha_i \sum_{j=1}^{\infty} \mu(E_j \cap A_i) = \sum_{j=1}^{\infty} \sum_{i=1}^{n} \alpha_i \mu\left(E_j \cap E_i\right) = \sum_{j=1}^{\infty} \varphi_s(E_j)$$

Thus φ_s is a measure. Linearity follows since we are only adding two simple functions, and so there are at most finitely many sets to work with. □

Example 1.7

To give an example of a sequence where the limit and integral cannot be interchanged, define $f_n = n\chi_{(0,1/n)}$. Then $\int f_n = 1$ for all n, but the pointwise limit is 0 everywhere.

We now prove our first limit theorem:

Theorem 1.19: Monotone Convergence Theorem

Let $0 \leq f_n \nearrow f \leq \infty$ be a sequence of nonnegative measurable functions. Then f is measurable and

$$\int f_n \to \int f$$

Proof. First note that the sequence $\int f_n$ is monotone increasing, so it has a limit (in the extended reals). Thus we have

$$L = \lim \int f_n \leq \int f$$

Pick a simple function $s \leq f$ and $\varepsilon < 1$. We want to show that $L \geq \varepsilon \int s$, which will then prove the result by taking $\varepsilon \to 1$ and $s \to f$.

For each n, define

$$E_n = \{x : f_n(x) \geq \varepsilon s(x)\}$$

For any point $x \in X$, we have $f_n(x) \to f(x) > \varepsilon s(x)$, so

$$\bigcup E_n = X$$

Then for each n we have

$$\int_{E_n} \varepsilon s \leq \int_{E_n} f_n \leq \int_X f_n \to L$$

We also have

$$\int_{E_n} \varepsilon s \to \int_X \varepsilon s$$

so

$$\int \varepsilon s \leq L$$

for all $\varepsilon < 1, s \leq f$. Thus

$$\int f \leq L$$

so we have both inequalities and thus

$$\int f = L = \lim \int f_n \qquad \square$$

Corollary 1.20

If f, g are nonnegative and measurable then $\int f + g = \int f + \int g$.

Proof. Take two sequences of simple functions $s_i \nearrow f$ and $t_i \nearrow g$. The monotone convergence theorem gives the result. $\square$

Corollary 1.21

If $f_n \geq 0$ is a sequence of nonnegative measurable functions then

$$\int \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \int f_n(x)$$

Proof. Combine the monotone convergence theorem with the previous corollary. $\square$

Corollary 1.22

If a_{ij} is a sequence of nonnegative numbers then

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$$

Proof. We write one of the sums as an integral with the counting measure. $\square$

Lemma 1.23: Fatou's Lemma

Let $f_n \geq 0$ be a sequence of nonnegative measurable functions. Then

$$\int \liminf f_n \leq \liminf \int f_n$$

Proof. Define $g_n(x) = \inf_{m \geq n} f_m(x)$. Then by definition, $g_n \nearrow \liminf f_n$. Also $\int g_n \leq \int f_n$ for each n. So by monotone convergence we have

$$\int \liminf f_n = \lim \int g_n = \liminf \int g_n \leq \liminf \int f_n \qquad \square$$

Having established limit theorems for nonnegative functions, we now make our definition of arbitrary integrals.

Definition 1.15

Let $f : X \to \overline{\mathbb{R}}$ be a measurable function. Writing $f = f^+ - f^-$, we define

$$\int f = \int f^+ - \int f^-$$

For a complex measurable function $F = u + iv : X \to \mathbb{C}$, we define

$$\int F = \int u + i \int v$$

Clearly this definition agrees with our previous one. However, there is a slight subtlety, which is that our definition may end up with an expression like $\infty - i\infty$. As such, we restrict this definition to those f which make the integral absolutely convergent (meaning $\int |f| < \infty$).

Proposition 1.24

For f measurable,

$$\left| \int f \right| \leq \int |f|$$

Proof. For f real valued, we write

$$\left| \int f \right| = \left| \int f^+ - \int f^- \right| \leq \left| \int f^+ \right| + \left| \int f^- \right| = \int f^+ + \int f^- = \int |f| \qquad \square$$

A similar proof shows the result for complex functions.

Our new integral inherits the properties we have shown for integrals of nonnegative functions, assuming the limits are finite. To capture this we make the following classification:

Definition 1.16

Let μ be a measure on X. Then we define the L^1 **space** to be

$$L^1(\mu) = \left\{ f : X \to \mathbb{C} : \int |f| \, \mathrm{d}\mu < \infty \right\}$$

Theorem 1.25: Dominated Convergence Theorem

If $f_n \to f$ and there exists $g \in L^1$ such that $|f_n| \le g$, then:

- $f_n \in L^1$,
- $\lim \int |f - f_n| = 0$ (equivalently, $f_n \to f$ in L^1),
- $\lim \int f_n = \int f$ (weak convergence)

Proof. First note that we have

$$|f_n| \le g \longrightarrow |f| \le g$$

so $f_n, f \in L^1$. Moreover, we have

$$|f_n - f| \le 2g$$

so the differences are in L^1 as well. Moreover, we have $2g - |f_n - f| \ge 0$. Thus we can apply Fatou's lemma:

$$\begin{aligned} \int 2g &= \int \lim \left(2g - |f - f_n|\right) = \int \liminf \left(2g - |f - f_n|\right) \\ &\le \liminf \int \left(2g - |f - f_n|\right) = \int 2g + \liminf \int -|f - f_n| \end{aligned}$$

Because $\int 2g < \infty$, we can subtract it from both sides to see that

$$0 \le \liminf \left(- \int |f - f_n| \right) \implies \limsup \int |f - f_n| \le 0$$

Since the RHS is nonnegative we conclude that $\lim \int |f - f_n|$ exists and is equal to zero. To demonstrate weak convergence, we have

$$\left| \int f_n - \int f \right| = \left| \int f_n - f \right| \le \int |f_n - f| \to 0$$

□

Example 1.8

Consider $f_n = n\chi_{(0,1/n^2)}$. These functions are bounded by $g(x) = \frac{1}{\sqrt{x}} \in L^1$. Moreover, we have

$$\lim \int f_n = \lim \frac{1}{n} = 0 = \int 0 = \int \lim f_n$$

Chapter 2

The Lebesgue Measure

To this point we have defined integrals in a way that allows us to interchange them with limit operators in various settings. We have also defined an appropriate σ-algebra, $\mathscr{B}(\mathbb{R})$, on $\mathbb{R}$, which we can use to work with this integral. Now we have to define a measure on $\mathscr{B}(\mathbb{R})$ that extends the Riemann integral. To make this definition we will essentially present an existence and uniqueness proof.

More precisely we show that there exists a unique positive, *translation invariant* meausre $\lambda : \mathscr{B}(\mathbb{R}) \to [0, \infty]$ such that $\lambda([0, 1]) = 1$.

In this search it will also turn out that the measurable sets under λ is larger than the Borel algebra.

Definition 2.1

For a set $S \subseteq \mathbb{R}$ and $x \in \mathbb{R}$, we define **translation** by

$$S + x \coloneqq \{s + x : s \in S\}$$

A measure μ is called **translation invariant** if $\mu(S) = \mu(S + x)$ for all x, S.

Our work will involve first developing theorems about how to construct measures out of more primitive objects. Applying this to $\mathbb{R}$ with some geometric intuition will give us the Lebesgue measure.

2.1 Premeasures

Consider some nonempty set X, and let $\rho : E \to [0, \infty]$ be a map which is initially defined on some subset E of $\mathcal{P}(X)$, with $\rho(\varnothing) = 0$. We do not assume that E is a σ-algebra; however it will generate the σ-algebra that is used by the final measure.

Definition 2.2

If X is nonempty, an **outer measure** on X is a map $\varphi : \mathcal{P}(X) \to [0, \infty]$ such that

1. $\varphi(\varnothing) = 0$,
2. $\varphi(A) \le \varphi(B)$ whenever $A \subseteq B$,
3. $\varphi\left(\bigcup A_n\right) \le \sum \varphi(A_n)$ for any countable collection of sets A_n.

Note that an outer measure is not a measure.

We now define an outer measure $\varphi_\rho : \mathcal{P}(X) \to [0, \infty]$ using the data from ρ.

Proposition 2.1

Let X be nonempty, $\rho : E \to [0, \infty]$ for $E \subseteq \mathcal{P}(X)$ containing $\{\varnothing, X\}$, and $\rho(\varnothing) = 0$. Then the function $\varphi_\rho : \mathcal{P}(X) \to [0, \infty]$ defined by

$$\varphi_\rho(A) \coloneqq \inf\left\{\sum \rho(E_n) : \{E_n\}_{n\in\mathbb{N}} \subseteq E, A \subseteq \bigcup E_n\right\}$$

is an outer measure. Here the infimum is over all countable covers of A with elements of E. If no such cover exists then by definition the infimum is ∞.

Proof. It is clear that $\varphi_\rho(\varnothing) = 0$ since we can take the cover to be $E_n = \varnothing$. To show monotonicity, if $A \subseteq B$ then any cover of B covers A, so $\varphi_\rho(A) \le \varphi_\rho(B)$ (this still holds when one or both sets admit no covers).

If $A = \bigcup A_n$, then for any $\varepsilon > 0$ we can pick covers $\{E_{n,i}\}_i$ for each n such that

$$\sum_{i=1}^{\infty} \rho(E_{n,i}) \ge \varphi_\rho(A_n) - \frac{\varepsilon}{2^n}$$

Then the collection $\{E_{n,i}\}_{n,i}$ is a countable cover of A, and we have

$$\sum_{n,i} \rho(E_{n,i}) = \sum_n \sum_i \rho(E_{n,i}) = \sum_n \left(\varphi(A_n) - \frac{\varepsilon}{2^n}\right) = \sum_n \varphi_\rho(A_n) - \varepsilon$$

Taking $\varepsilon \to 0$ and taking the infimum, we conclude that

$$\varphi_\rho\left(\bigcup_n A_n\right) \le \sum_n \varphi_\rho(A_n)$$

□

Example 2.1

Taking E to be the set of intervals and letting $\rho((a,b)) = b - a$, we generate the Lebesgue outer measure.

So far we have placed no assumptions on ρ. In order to get outer measures and measures which we can work with nicely, it is helpful to impose a few conditions. To see this, we examine some possible difficulties with pathological ρ. For instance, if ρ itself is not countably additive, then φ_ρ could fail to coincide with ρ on E.

Definition 2.3

If φ is an outer measure on X, a set $A \subseteq X$ is called **φ-measurable** if for all $Q \in \mathcal{P}(X)$,

$$\varphi(Q) = \varphi(Q \cap A) + \varphi(Q \cap A^c)$$

The set of φ-measurable sets is denoted $\mathscr{A}_\varphi$.

Essentially, a φ-measurable set splits with respect to measure. It is not a priori obvious that nonmeasurable sets should exist under this definition, but we will see later that they do. Note that we always have

$$\varphi(Q) \leq \varphi(Q \cap A) + \varphi(Q \cap A^c)$$

by countable subadditivity of φ. Thus in general we can check φ-measurability just by verifying that

$$\varphi(Q) \geq \varphi(Q \cap A) + \varphi(Q \cap A^c)$$

Moreover, when $\varphi(Q) = \infty$ this is automatically true.

A natural question is then to ask whether φ_ρ-measurable sets form a σ-algebra. The answer to this question is yes; moreover the restriction theorem that we prove shows that φ_ρ is also a measure when restricted to these sets.

Theorem 2.2: Caratheodory's Restriction Theorem

Let X be a nonempty set and φ an outer measure on X. Then $\mathscr{A}_\varphi$ is a σ-algebra, and $\mu_\varphi := \varphi|_{\mathscr{A}_\varphi}$ is a measure.

Proof. Take $\varnothing$. By the remark above, it suffices to show that for any $Q \in X$,

$$\varphi(Q) \geq \varphi(Q \cap \varnothing) + \varphi(Q \cap \varnothing^c)$$

But this is clear since the right hand side is just

$$\varphi(\varnothing) + \varnothing(Q) = \varnothing(Q)$$

It is also obvious that $\mathscr{A}_\varphi$ is closed under complements since the definition treats A, A^c symmetrically.

To show closure under countable unions, we first show finite unions. For $A, B \in \mathscr{A}_\varphi$, and pick $Q \in \mathcal{P}(X)$ with $\varphi(Q) < \infty$ (recall from above that we can assume finite outer measure). Then

$$\begin{aligned}\varphi(Q) &= \varphi(Q \cap A) + \varphi(Q \cap A^c)\\ &= \varphi(Q \cap A \cap B) + \varphi(Q \cap A \cap B^c) + \varphi(Q \cap A^c \cap B) + \varphi(Q \cap A^c \cap B^c)\end{aligned}$$

We have the identity

$$A \cup B = (A \cap B) \cup (A \cap B^c) \cup (A^c \cap B)$$

Since φ is an outer measure, it follows that

$$\varphi(A \cup B) \leq \varphi(A \cap B) + \varphi(A \cap B^c) + \varphi(A^c \cap B)$$

take a sequence $\{A_n\} \subseteq \mathscr{A}_\varphi$. Intersecting with Q on both sides, we have

$$\varphi(Q) \geq \varphi(Q \cap (A \cup B)) + \varphi(Q \cap A^c \cap B^c) = \varphi(Q \cap (A \cup B)) + \varphi(Q \cap (A \cup B)^c)$$

Now we extend this to countable unions $\bigcup A_n$. It suffices to assume that the A_n are pairwise disjoint by picking

$$A'_n = A_n \setminus \left(\bigcup_{m=1}^{n-1} A_m \right)$$

Then A'_n are in $\mathscr{A}_\varphi$ by our work showing that complements and finite unions were closed. Now take Q with $\varphi(Q) < \infty$. Then for any N, $\bigcup^N A_n \in \mathscr{A}_\varphi$. Therefore we can write

$$\begin{aligned}
\varphi(Q) &= \varphi \left(Q \cap \left(\bigcup_{n=1}^{N} A_n \right) \right) + \varphi \left(Q \cap \left(\bigcup_{n=1}^{N} A_n \right)^c \right) \\
&\geq \varphi \left(Q \cap \left(\bigcup^{N} A_n \right) \cap A_n \right) + \varphi \left(Q \cap \left(\bigcup^{N} A_n \right) \cap A_N^c \right) \\
&= \varphi \left(Q \cap (A)_N \right) + \varphi \left(Q \cap \left(\bigcup^{N-1} A_n \right) \right) \\
&\vdots \\
&\geq \sum_{n=1}^{N} \varphi(Q \cap A_n) + \varphi \left(Q \cap \left(\bigcup^{N} A_n \right)^c \right)
\end{aligned}$$

Now, we have $\bigcup^\infty A_n \supseteq \bigcup^N A_n$, so we have

$$\varphi \left(Q \cap \left(\bigcup^{N} A_n \right)^c \right) \geq \varphi \left(Q \cap \left(\bigcup^{\infty} A_n \right)^c \right)$$

Taking $N \to \infty$, we have

$$\varphi(Q) \geq \sum_{n=1}^{\infty} \varphi(Q \cap A_n) + \varphi \left(Q \cap \left(\bigcup^{\infty} \varphi(A_n) \right)^c \right)$$

Since φ is countably subadditive,

$$\sum_{n=1}^{\infty} \varphi(Q \cap A_n) \geq \varphi \left(Q \cap \left(\bigcup^{\infty} A_n \right) \right)$$

Thus

$$\varphi(Q) \geq \varphi\left(Q \cap \left(\bigcup^{\infty} A_n\right)\right) + \varphi\left(Q \cap \left(\bigcup^{\infty} \varphi(A_n)\right)^c\right)$$

showing that $\bigcup^{\infty} A_n \in \mathscr{A}_\varphi$. Thus $\mathscr{A}_\varphi$ is a σ-algebra.

We know there is a set with finite measure since $\mu_\varphi(\varnothing) = 0$. To demonstrate finite additivity, pick $A, B \in \mathscr{A}_\varphi$ disjoint. Then

$$\mu_\varphi(A \cup B) = \varphi(A \cup B) = \varphi((A \cup B) \cap A) + \varphi((A \cup B) \cap A^c) = \varphi(A) + \varphi(B) = \mu_\varphi(A) + \mu_\varphi(B)$$

The proof for countable additivity is the same. □

We have thus illustrated a method to pass from a primitive function $\rho : E \to [0, \infty]$ to a full measure μ_{φ_ρ} on $\mathscr{A}_{\varphi_\rho}$. To that end it is worth investigating the relationship between $\sigma(E)$ and $\mathscr{A}_{\varphi_\rho}$. In order to properly do this it is best to impose additional conditions on ρ.

Definition 2.4

An **algebra** on a set X is a collection $\mathscr{A} \subseteq \mathcal{P}(X)$ which contains X, is closed under complements, and closed under finite unions.

Note the only difference between a σ-algebra and an algebra is we require σ-algebras to be closed under countable unions as well.

Definition 2.5

Let $\mathscr{A} \subseteq \mathcal{P}(X)$ be an algebra. Then a function $\rho : \mathscr{A} \to [0, \infty]$ is called a **premeasure** if:

1. $\rho(\varnothing) = 0$,
2. If $\{A_n\} \subseteq \mathscr{A}$ is a countable collection of pairwise disjoint sets, and if in addition $\bigcup A_n \in \mathscr{A}$, then
$$\rho\left(\bigcup A_n\right) = \sum \rho(A_n).$$

Adding the condition that ρ is a premeasure ensures that our extended constructions are proper extensions, in the sense that they are consistent with our original data.

Proposition 2.3

Suppose $\rho : \mathscr{A} \to [0, \infty]$ is a premeasure. Then $\varphi_\rho|_{\mathscr{A}} = \rho$. Moreover, $\mathscr{A} \subseteq \mathscr{A}_{\varphi_\rho}$.

Proof. □

Thus, since we can now properly use premeasures to build outer measures, we can apply the restriction theorem to actually extend premeasures.

Theorem 2.4: Caratheodory's Extension Theorem

Let $\rho : \mathscr{A} \to [0, \infty]$ be a premeasure and φ_ρ, $\mathscr{A}_{\varphi_\rho}$, μ_{φ_ρ} be as defined above. Then:

1. $\sigma(\mathscr{A}) \subseteq \mathscr{A}_{\varphi_\rho}$,
2. If $\nu : \mathscr{A}_{\varphi_\rho} \to [0, \infty]$ is any other measure such that $\nu|_{\mathscr{A}} = \rho$, then $\nu \leq \mu_{\varphi_\rho}$ on $\mathscr{A}_{\varphi_\rho}$,
3. If X is **σ-finite**, meaning that there is a countable collection $\{A_n\} \subseteq \mathscr{A}$ with $\rho(A_n) < \infty$ and $X = \bigcup A_n$, then μ_{φ_ρ} is the unique extension of ρ to $\sigma(\mathscr{A})$.

Definitions