Multivariable Calculus

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Contents

1 Vectors

1.1 Definition

Vectors are defined as mathematical quantities with both direction and magnitude.

1.2 Notation

$$
\begin{aligned}
\vec{v}&: \quad \text{A vector}\\
\hat{u}&: \quad \text{Unit vector (length 1)}\\
\hat{i},\, \hat{j},\, \hat{k}&: \quad \text{Unit vectors in } x,\, y,\, z \text{ directions, respectively}\\
(a_1, a_2)&: \quad \text{Point with coordinates } (a_1, a_2)\\
\langle a_1, a_2\rangle&: \quad \text{Vector given by } a_1\hat{i} + a_2\hat{j}\\
\overrightarrow{PQ}&: \quad \text{Vector between points P and Q}\\
\vec{P} = \overrightarrow{OP} = \boldsymbol{P}&: \quad \text{Origin vector (origin as tail)}\\
|\vec{A}| = \sqrt{a_1^2 + a_2^2}&: \quad \text{Magnitude or length of } \vec{A}
\end{aligned}
$$

1.3 Basic Operations

Let $\vec{A} = \langle a_1, a_2\rangle$, $\vec{B} = \langle b_1, b_2\rangle$, $c =$ constant. Then:

$$
\begin{aligned}
c\vec{A} &= \langle ca_1, ca_2\rangle\\
\vec{A} + \vec{B} &= \langle a_1 + b_1, a_2 + b_2\rangle\\
\vec{A} - \vec{B} = \vec{A} + (-\vec{B}) &= \langle a_1 - b_1, a_2 - b_2\rangle
\end{aligned}
$$

1.4 Dot Product

Let $\vec{A} = \langle a_1, a_2, a_3\rangle$, $\vec{B} = \langle b_1, b_2, b_3\rangle$. Then

$$
\begin{aligned}
\vec{A}\cdot\vec{B} &= a_1b_1 + a_2b_2 + a_3b_3 & \vec{A}\cdot\vec{B} &= |\vec{A}||\vec{B}|\cos\theta\\
\vec{A}\cdot\vec{B} &= \sum_{i=1}^{n} a_ib_i & \vec{A}\cdot\vec{A} &= |\vec{A}|^2\\
& & \vec{A}\perp\vec{B} &\iff \vec{A}\cdot\vec{B} = 0
\end{aligned}
$$

1.5 Cross Product

Let $\vec{A} = \langle a_1, a_2, a_3\rangle$, $\vec{B} = \langle b_1, b_2, b_3\rangle$. Then

$$
\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3\\ b_2 & b_3 \end{vmatrix}\hat{i} + \begin{vmatrix} a_1 & a_3\\ b_1 & b_3 \end{vmatrix}\hat{j} + \begin{vmatrix} a_1 & a_2\\ b_1 & b_2 \end{vmatrix}\hat{k}
$$

$$
\vec{A}\perp(\vec{A}\times\vec{B})\perp\vec{B} \text{ (direction given by right hand rule)}
$$

$$
\begin{aligned}
|\vec{A}\times\vec{B}| &= |\vec{A}||\vec{B}|\sin\theta & \vec{A}\times\vec{A} &= \vec{0}\\
\vec{A}\times\vec{B} &= -\vec{B}\times\vec{A} & \vec{A}\times(\vec{B}\times\vec{C}) &\neq (\vec{A}\times\vec{B})\times\vec{C}
\end{aligned}
$$

1.6 Equation of Planes

$$\left.\begin{aligned}\vec{N} &= \langle a,b,c\rangle \\ \vec{P_1} &= \langle x_0,y_0,z_0\rangle \\ \vec{P} &= \langle x,y,z\rangle\end{aligned}\right\} \implies \begin{cases}\vec{P}_1P \cdot \vec{N} = 0 \\ \vec{P}\cdot\vec{N} = \vec{P_1}\cdot\vec{N} \\ a(x-x_0)+b(y-y_0)+c(z-z_0)=0\end{cases}$$

$$P_1, P_2, P_3 \text{ in plane} \implies \vec{N} = \overrightarrow{P_1P_2} \times \overrightarrow{P_2P_3}$$

$$\text{intercepts } (a,0,0),(0,b,0),(0,0,c) \implies \frac{x}{a}+\frac{b}{y}+\frac{c}{z}=1$$

$$ax+by+cz=d \implies \vec{N} = \langle a,b,c\rangle$$

1.7 Applications

$$\text{Component of } \vec{A} \text{ in direction of } \hat{u} : \vec{A}_{\hat{u}} = \vec{A}\cdot\hat{u}$$

$$\text{Area of parallelogram with sides } \vec{A} \text{ and } \vec{B} : A = \det(\vec{A},\vec{B}) = |\vec{A}\times\vec{B}|$$

$$\text{Volume of parallelepiped with sides } \vec{A},\ \vec{B},\vec{C} : V = \det(\vec{A},\vec{B},\vec{C}) = \vec{A}\cdot(\vec{B}\times\vec{C})$$

$$\text{Distance from point P to plane} : d = \frac{|\overrightarrow{PQ}\cdot\vec{N}|}{|\vec{N}|}$$

2 Matrices

2.1 Definition

An $m \times n$ matrix has m rows and n columns.

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} = 3 \times 4 \text{ matrix}$$

2.2 Notation

Given matrix A,

$$\begin{aligned} a_{ij} &= \text{entry at row } i\text{, column j} \\ (a_{ij}) &= \text{matrix composed of } a_{ij} \text{ at each entry} \\ A = B \iff & \text{corresponding entries equal} \\ A^T &= \text{transpose of } A \\ A^{-1} &= \text{inverse of} A \\ \det(A) &= |A| \\ I_n &= \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} = n \times n \text{ identity matrix} \end{aligned}$$

2.3 Basic Operations

$$\begin{aligned} cA &= (ca_{ij}) \\ A+B &= (a_{ij}+b_{ij}) \\ A-B &= (a_{ij}-b_{ij}) \\ A^T &= (a_{ji}) \\ &= \text{ switch rows and columns} \end{aligned}$$

2.4 Properties

$$\begin{aligned} A(B+C) = AB+AC, (A+B)C &= AC+BC \\ (AB)C &= A(BC) \\ AB &\neq BA \text{ (generally, if defined)} \\ \det(AB) &= \det(A)\det(B) \\ I_m A = AI_n &= A \text{ (for } m \times n \ A) \\ AA^{-1} &= A^{-1}A = I \end{aligned}$$

2.5 Matrix Multiplication

$$\underset{m\times n}{A} \cdot \underset{n\times p}{B} = \underset{m\times p}{C}$$

$$C_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$C_{ij} = \text{ dot product of } i\text{-th row, } j\text{-th column}$$

2.6 Determinant

Laplace expansion along first row:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\det(A) = \text{dot product of entries and cofactors along row}$$

2.7 Inverse Matrices

For 2×2 A,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \implies A^{-1} - \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

For square A, $\det(A) \neq 0$:

$$\underset{\text{Matrix}}{\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}} \implies \underset{\text{Minors}}{\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2.1} & a_{2,2} & a_{2.3} \\ a_{3,1} & a_{3.2} & a_{3.3} \end{bmatrix}} \implies \underset{\text{Cofactors}}{\begin{bmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2.3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{bmatrix}} \implies \underset{\text{Transpose of Cofactors}}{A^{-1} = \frac{1}{\det(A)} C^T}$$

Where $a_{i,j}$ = determinant of matrix with i-th row, j-th column deleted and $C_{i,j} = \pm a_{i,j}$ according to checkerboard pattern:

$$\text{Sign of cofactor} = \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

2.8 Linear Systems

Let $A = n \times n$ square matrix, $X = n \times 1$ column matrix, $B = n \times 1$ column matrix. $AX = B$ is a linear system of equations.

	$\det(A) \neq 0$	$\det(A) = 0$
$AX = 0$ (homogeneous)	$X = 0$ is only solution	line through origin perpendicular to each row of A
$AX = B$ (nonhomogeneous)	$X = A^{-1}B$ is only solution	either 0 or infinitely many solutions

3 Parametric Curves

3.1 Definition

A parametric curve $C = \vec{r}(t)$ is the set of values of $\vec{r}(t)$ within a given interval of t (trajectory of moving point).

3.2 Equation of a Line

Line containing $\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$ parallel to $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \implies \vec{r}(t) = \begin{bmatrix} x_0 + at \\ y_0 + bt \\ z_0 + ct \end{bmatrix} = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$

3.3 Derived Quantities

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle \qquad \text{Speed} = |\vec{v}| = \left|\frac{ds}{dt}\right| = \left|\frac{d\vec{r}}{dt}\right|$$

$$\vec{v}(t) = \frac{d}{dt}\vec{r} = \langle x'(t), y'(t), z'(t) \rangle \qquad \hat{T} = \frac{\vec{v}}{|\vec{v}|} = \text{dir}(\vec{v})$$

$$\vec{a}(t) = \frac{d^2}{dt^2}\vec{r} = \langle x''(t), y''(t), z''(t) \rangle \qquad \frac{d\vec{r}}{dt} = \vec{v} = \hat{T}\frac{ds}{dt}$$

3.4 Parametric Vector Differentiation

$$\frac{d}{dt}(\vec{u} \cdot \vec{v}) = \frac{d\vec{u}}{dt} \cdot \vec{v} + \vec{u} \cdot \frac{d\vec{v}}{dt}$$

$$\frac{d}{dt}(\vec{u} \times \vec{v}) = \frac{d\vec{u}}{dt} \times \vec{v} + \vec{u} \times \frac{d\vec{v}}{dt}$$

4 Partial Derivatives

4.1 Definition

Given a function $f(x, y)$,

$$f_x = \frac{\partial}{\partial x} f = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

4.2 Approximation Formulae

$$\Delta f \approx f_x \Delta x + f_y \Delta y: \text{ tangent plane approximation}$$

$$z - z_0 = f_x(x - x_0) + f_y(y - y_0): \text{ tangent plane}$$

4.3 Gradient

$$
\begin{aligned}
\nabla f &= \langle f_x, f_y \rangle \\
\nabla f &\perp (S \coloneqq f(x,y) = c) \\
\mathrm{dir}(\nabla f) &= \mathrm{dir}(\text{steepest increase}) \\
\left.\frac{df}{ds}\right|_{\hat{u}} &= \nabla f \cdot \hat{u}
\end{aligned}
$$

4.4 Optimization

Critical points of f occur when $\nabla f = \vec{0}$, extrema lie at either critical points or along boundary.

4.5 Second Derivative Test

Let $A = f_{xx}, B = f_{xy} = f_{yx}, C = f_{yy}$. Then

$$
AC - B^2 \implies \begin{cases} > 0 & : \begin{cases} A < 0 & : \text{local max} \\ A > 0 & : \text{local min} \end{cases} \\ = 0 & : \text{inconclusive} \\ < 0 & : \text{saddle point} \end{cases}
$$

4.6 Total Differentials, Chain Rule

$$
\begin{aligned}
df &= f_x dx + f_y dy \\
\frac{\partial f}{\partial u} &= f_x \frac{\partial x}{\partial u} + f_y \frac{\partial y}{\partial u}
\end{aligned}
$$

4.7 Lagrange Multipliers

To optimize $f(x,y,z)$ given a constraint $g(x,y,z) = c$, solve the system of equations

$$
\nabla f = \lambda \nabla g \implies \begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \\ g(x,y,z) = c \end{cases}
$$

4.8 Constrained Partial Derivatives

When $f(x,y,z)$ is subject to the constraint $g(x,y,z) = c$,

$$
\begin{aligned}
f_x &= \text{formal partial (all treated independent)} \\
\left(\frac{\partial f}{\partial x}\right)_y = f_x + f_z \frac{\partial z}{\partial x} &= \text{partial with } y \text{ independent, } z \text{ dependent}
\end{aligned}
$$

5 Vector Fields

5.1 Definition

A vector field $\vec{F}$ is associated with a vector valued function $\vec{F}(x,y,z)$.

5.2 Conservative Fields

$$\vec{F} \text{ is conservative } \iff \begin{cases} \vec{F} = \nabla f \text{ for some function } f(x,y,z) \\ \oint_C \vec{F}\cdot d\vec{r} = 0 \text{ for all closed curves } C \\ \int_C \vec{F}\cdot d\vec{r} = 0 \text{ is path independent} \\ \text{curl}(\vec{F}) = 0 \text{ on a simply connected region} \end{cases}$$

5.3 Potential Functions

If $\vec{F}$ is conservative, then to find a function f representing its potential, use:
Method 1:

$$f(x_1,y_1,z_1) = \int_{(a,b,c)}^{(x_1,y_1,z_1)} \vec{F}\cdot d\vec{r} = \int_0^{x_1} Pdx\Big|_{\substack{y=0\\z=0}} + \int_0^{y_1} Qdy\Big|_{\substack{x=x_1\\z=0}} + \int_0^{z_1} Rdz\Big|_{\substack{x=x_1\\y=y_1}}$$

Method 2:

$$f_x = P \implies f = \int Pdx + g(y,z) \implies f_y = \frac{\partial}{\partial x}\int Pdx + \frac{\partial}{\partial y}g(y,z) = Q\ldots$$

5.4 Curl

2D Curl (scalar valued):

$$\text{curl}(\vec{F}) = \nabla\times\vec{F} = N_x - M_y$$

3D Curl (vector valued):

$$\text{curl}(\vec{F}) = \nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix}$$

$$\begin{aligned} \text{dir}(\nabla\times\vec{F}) &= \text{ main axis of rotation} \\ |\nabla\times\vec{F}| &= \text{ magnitude of rotation about axis} \\ \omega(\hat{n}) &= \frac{1}{2}|\nabla\times\vec{F}|\cdot\hat{n} \end{aligned}$$

5.5 Divergence

$$\text{div}(\vec{F}) = \nabla\cdot\vec{F} = P_x + Q_y + R_z$$

5.6 Del Notation

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}. \frac{\partial}{\partial z} \right\rangle \qquad \text{div}(\vec{F}) = \nabla\cdot\vec{F} = \frac{\partial}{\partial x}P + \frac{\partial}{\partial y}Q + \frac{\partial}{\partial z}R$$

$$\text{grad}(f) = \nabla f = \left\langle \frac{\partial}{\partial x}f, \frac{\partial}{\partial y}f. \frac{\partial}{\partial z}f \right\rangle \qquad \text{curl}(\vec{F}) = \nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

6 Line Integrals

6.1 Definition

$$\int_C f(x,y,z)ds = \text{ integral over curve C}$$

6.2 Scalar Line Integrals

$$\int_C f(x,y,z)ds = \int_C f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$
$$\int_C f(x,y,z)dx = \int_C f(x(t),y(t),z(t))x'(t)dt$$

6.3 Vector Line Integrals

$$\text{Work} = \int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\hat{T}ds = \int_C Pdx + Qdy + Rdz$$
$$\text{Flux} = \int_C \vec{F}\cdot\hat{n}ds = \int_C -Ndx + Mdy \quad (\text{in 2D, } \hat{n} = -\langle dy, -dx\rangle)$$

6.4 Fundamental Theorem of Line Integrals

$$\vec{F} = \nabla f \implies \int_{P_0}^{P_1} \nabla f\cdot d\vec{r} = f(P_1) - f(P_0) \iff \oint_C \nabla f\cdot d\vec{r} = 0$$

7 Double Integrals, Triple Integrals

7.1 Definition

$$\iint_R f(x,y)dA = \text{ integral over planar region R}$$
$$\iiint_D f(x,y,z)dV = \text{ integral over domain in space D}$$

7.2 Iterated Integrals

$$\begin{aligned}\iint_R f(x,y)dA &= \iint_R f dxdy = \int_{y_0}^{y_1}\int_{x_0(y)}^{x_1(y)} f dxdy\\ &= \iint_R f dydx = \int_{x_0}^{x_1}\int_{y_0(x)}^{y_1(x)} f dydx\end{aligned}$$

7.3 Polar, Cylindrical, Spherical Coordinates

Polar:

$$\begin{array}{cc} x = r\cos\theta & r = \sqrt{x^2+y^2} \\ y = r\sin\theta & \theta = \tan^{-1}\left(\frac{y}{x}\right) \end{array} \quad dA = rdrd\theta$$

Cylindrical:

$$\begin{array}{cc} x = r\cos\theta & r = \sqrt{x^2+y^2} \\ y = r\sin\theta & \theta = \tan^{-1}\left(\frac{y}{x}\right) \\ z = z & z = z \end{array} \quad dV = dzrdrd\theta$$

Spherical:

$$\begin{array}{cc} x = \rho\sin\phi\cos\theta & \rho = \sqrt{x^2+y^2+z^2} \\ y = \rho\sin\phi\sin\theta & \theta = \tan^{-1}\left(\frac{y}{x}\right) \\ z = \cos\phi & \phi = \tan^-1\left(\frac{r}{z}\right) \end{array} \quad dV = \rho^2\sin\phi d\rho d\phi d\theta$$

7.4 Change of Variables, Jacobian

$$\begin{cases} x \\ y \\ z \end{cases} \rightarrow \begin{cases} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{cases} \implies dxdydz = \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| dudvdw$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_w & y_w & z_w \end{vmatrix}, \quad \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{vmatrix}$$

7.5 Applications

$$\text{Area} = \iint_R dA$$

$$\text{Volume} = \iiint_V dV$$

$$\text{Mass} = \iint_R \delta dA = \iiint_V \delta dV$$

$$\text{Average} = \frac{1}{A} \iint_R f dA = \frac{1}{V} \iiint_V f dV$$

$$\text{Weighted Average} = \frac{1}{M} \iint_R f \delta dA = \frac{1}{M} \iiint_V f \delta dV$$

$$x_{CM} = \frac{1}{M} \iiint_V x \delta dV = \frac{1}{V} \iiint_V x dV$$

$$F_{gz} = Gm \iiint_M \sin\phi \cos\phi \, \delta \, d\rho d\phi d\theta$$

$$I = \iint_R r^2 \delta dA = \iiint_V r^2 \delta dV$$

8 Surface Integrals

8.1 Definition

$$\iint_S f(x,y,z) = \lim_{\Delta S \to 0} \sum_i f(x_i^*, y_i^*, z_i^*) \Delta S$$

8.2 Scalar Surface Integrals

Suppose $z = g(x,y)$. Then

$$\iint_S f(x,y,z) dS = \iint_R f(x,y,g(x,y)) \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1} \, dxdy$$

If S is parameterized by $\vec{r}(u,v)$, then

$$\iint_S f(x,y,z) dS = \iint_{S'} f(\vec{r}(u,v)) \left| \vec{r}_u \times \vec{r}_v \right| dudv$$

8.3 Surface Flux

$$\text{Flux} = \iint_S \vec{F} \cdot \hat{n} dS = \iint_S \vec{F} \cdot d\vec{S}$$

Evaluating flux requires an orientation (choice of set of $\hat{n}$). For closed S, $\hat{n}$ conventionally points outward.

8.4 Calculating dS

$$x^2+y^2+z^2=a^2 \implies \begin{cases} \hat{n} &= \pm\frac{\langle x,y,z\rangle}{a} \\ dS &= a^2\sin\phi\, d\phi\, d\theta \end{cases}$$

$$x^2+y^2=a^2 \implies \begin{cases} \hat{n} &= \pm\frac{\langle x,y,0\rangle}{a} \\ dS &= a\, d\theta\, dz \end{cases}$$

$$z=a \implies \begin{cases} \hat{n} &= \pm\hat{k} \\ dS &= dxdy \end{cases}$$

$$z=z(x,y) \implies \hat{n}dS = \pm\langle -z_x, -z_y, 1\rangle dxdy$$

$$\left.\begin{array}{r} F(x,y,z)=c \\ z=z(x,y) \end{array}\right\} \implies \hat{n}dS = \pm\frac{\nabla F}{F_z}dxdy$$

$$\langle x,y,z\rangle = \vec{r}(u,v) \implies \hat{n}dS = \pm\left(\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\right)dudv$$

$$\vec{N}=\vec{N}(x,y,z) \implies \hat{n}dS = \frac{\vec{N}}{\vec{N}\cdot\hat{k}}dxdy$$

9 Integral Theorems

9.1 Theorem Relationships

	1D	2D	3D
Work	Fund. Theorem for Line Integrals	Green's Theorem (tangential form)	Stokes' Theorem
Flux		Green's Theorem (normal form)	Divergence Theorem

9.2 Green's Theorem

Statement (Tangential Form): If C is a positively oriented (counterclockwise) simple, closed, piecewise smooth curve in $\mathbb{R}^2$ enclosing a region R, and $\vec{F}$ is defined and differentiable on C and R, then

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R (N_x - M_y)dA = \iint_R \text{curl}(\vec{F})\cdot\hat{k}dA$$

Statement (Normal Form): If C is a positively oriented (counterclockwise) simple, closed, piecewise smooth curve in $\mathbb{R}^2$ enclosing a region R, and $\vec{F}$ is defined and differentiable on C and R, then

$$\oint_C \vec{F}\cdot\hat{n}ds = \iint_R (M_x + N_y)dA = \iint_R \text{div}(\vec{F})dA$$

Converse: If $\vec{F}$ is defined and differentiable on a simply connected region $R\subseteq\mathbb{R}^2$, then

$$\text{curl}(\vec{F})=0 \implies \vec{F} \text{ is conservative}$$

9.3 Stokes' Theorem

Statement: If C is a simple, closed, piecewise smooth curve in $\mathbb{R}^3$, and S is any surface with boundary C, and $\vec{F}$ is defined and differentiable on C and S, then

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_s (\nabla\times\vec{F})\cdot\hat{n}dS = \iint_S \text{curl}(\vec{F})\cdot\hat{n}dS$$

Converse: If $\vec{F}$ is defined and differentiable on a simply connected region $R\subseteq\mathbb{R}^3$, then

$$\text{curl}(\vec{F})=\vec{0} \implies \vec{F} \text{ is conservative}$$

Note: To choose a compatible orientation for C and S, use the right hand rule on C: the thumb points in the positive direction on C, index points into S, and middle finger points in the direction of $\hat{n}$.

9.4 Divergence Theorem

Statement: If S is a closed surface, oriented with $\hat{n}$ outward, S encloses a region D, and $\vec{F}$ is defined and differentiable everywhere in S and D, then

$$\oiint_S \vec{F}\cdot\hat{n}\,dS = \iiint_D (P_x+Q_y+R_Z)\,dV = \iiint_D \text{div}(\vec{F})\,dV$$