MAT 335 Notes

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Fall 2024

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Introduction

This document contains notes taken for the class MAT 335: Complex Analysis at Princeton University, taken in the Fall 2024 semester. These notes are primarily based on lectures by Professor Assaf Naor. Other references used in these notes include *Complex Analysis* by Elias Stein and Rami Shakarchi, *Complex Analysis* by Lars Ahlfors, *Visual Complex Analysis* by Tristan Needham, and *Real and Complex Analysis* by Walter Rudin. Since these notes were primarily taken live, they may contains typos or errors.

Chapter 1

Preliminaries

1.1 The Complex Number System

The set of complex numbers, denoted $\mathbb{C}$ is identified with ordered pairs $(x, y) \in \mathbb{R}^2$. We may alternately write this as $x + iy$, where the symbol i is currently undefined.

For a given complex number $z = x + iy$, $x = \mathrm{Re}(z)$ is called the **real part** of z, $y = \mathrm{Im}(z)$ is called the **imaginary part**, $|z| = \sqrt{x^2 + y^2}$ is the **modulus** of z, and the **argument** of z, $\theta = \arg(z)$, is the angle between (x, y) and the x-axis, defined up to integer multiples of 2π.

Definition 1.1

Let $\theta \in \mathbb{R}$. We define

$$e^{i\theta} = \cos\theta + i\sin\theta = (\cos\theta, \sin\theta)$$

One can observe using the identity $\cos^2 + \sin^2 = 1$ that $e^{i\theta}$ lies on the unit circle. Moreover, if $r = |z|$, then elementary geometry shows that we have $z = re^{i\theta}$ using the definition above.

Proposition 1.2

For any $z \in \mathbb{C}$, $|\mathrm{Re}(z)| \leq |z|$ and $|\mathrm{Im}(z)| \leq |z|$.

Proof. $|z| = \sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2}$. □

One of the distinguishing features of $\mathbb{C}$ from the real plane $\mathbb{R}^2$ is the algebraic structure present on $\mathbb{C}$.

Definition 1.3

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then we define addition and multiplication on $\mathbb{C}$ by

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$
$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)$$

Taking $i = (0, 1)$, then we observe that $i^2 = -1 + 0i = -1$. Thus we recover the basic identity $i^2 = -1$.

Proposition 1.4

Addition and multiplication over $\mathbb{C}$ are commutative and associative. Moreover, multiplication distributes over addition.

Proof. Commutative and associativity of addition is inherited from $\mathbb{R}$. $\square$

Using the definition of $e^{i\theta}$, we can reinterpret complex multiplication in a much more pleasant manner than the definition above.

Proposition 1.5

If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then

$$z_1 z_2 = (r_1 r_2) e^{i(\theta_1 + \theta_2)}$$

Proof. We have proved commutativity. From here, we apply trig identities. $\square$

Thus multiplication results in multiplication of lengths and addition of arguments.

Proposition 1.6

For $z_1, z_2 \in \mathbb{C}$, the **triangle inequality** holds:

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

Proof. Choose r, θ such that $z_1 + z_2 = re^{i\theta}$. Then

$$|z_1 + z_2| = r = (z_1 + z_2)e^{-i\theta} = z_1 e^{-i\theta} + z_2 e^{-i\theta} = \mathrm{Re}(z_1 e^{-i\theta} + z_2 e^{-i\theta})$$

Now note that $\mathrm{Re}(z + w) = \mathrm{Re}(z) + \mathrm{Re}(w)$. So

$$\mathrm{Re}(z_1 e^{-i\theta} + z_2 e^{-i\theta}) = \mathrm{Re}(z_1 e^{-i\theta}) + \mathrm{Re}(z_2 e^{-i\theta}) \leq |z_1 e^{-i\theta}| + |z_2 e^{-i\theta}| = |z_1| + |z_2| \quad \square$$

Corollary 1.7

The reverse triangle inequality also holds:

$$||z| - |w|| \leq |z - w|$$

Definition 1.8

Let $z = x + iy \in \mathbb{C}$. Then the **complex conjugate** of z is defined as

$$\overline{z} = x - iy$$

Geometrically, this is reflection over the x axis.

Proposition 1.9

For $z \in \mathbb{C}$, $z\overline{z} = |z|^2$.

Definition 1.10

For $z \neq 0$, define

$$\frac{1}{z} = \frac{\overline{z}}{|z|^2}$$

The above proposition and definition show that

$$z \cdot \frac{1}{z} = 1$$

Definition 1.11

A **sequence** of complex numbers $\{z_n\}_{n=1}^{\infty}$ **converges** to $z \in \mathbb{C}$ (written $\lim_{n\to\infty} z_n = z$) if

$$\begin{cases} \lim_{n\to\infty} \mathrm{Re}(z_n) = \mathrm{Re}(z) \\ \lim_{n\to\infty} \mathrm{Im}(z_n) = \mathrm{Im}(z) \end{cases}$$

We similarly define the limit of a complex function $\lim_{z\to a} f(z)$.

Definition 1.12

A **Cauchy sequence** is a sequence $(z_n) \subseteq \mathbb{C}$ such that $(\mathrm{Re}(z_n))$ and $(\mathrm{Im}(z_n))$ are both Cauchy.

Proposition 1.13

A Cauchy sequence is convergent.

Proof. Follows from completeness of $\mathbb{R}$. $\square$

1.2 Topology of $\mathbb{C}$

The topological nature of $\mathbb{C}$ should not be unfamiliar to the reader, since it is essentially the same as that of $\mathbb{R}^2$, rephrased slightly using complex variables.

Definition 1.14

Let $r > 0$ and $z_0 \in \mathbb{C}$. Then the **open disk** of radius ε about z is the set

$$\mathbb{D}_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$$

and the **closed disk** as

$$\overline{\mathbb{D}}_r(z_0) = \{z \in \mathbb{C} : |z - z_0| \leq r\}$$

We also specify $\mathbb{D}_r = \mathbb{D}_r(0)$ and $\mathbb{D} = \mathbb{D}_1$.

Definition 1.15

An **interior point** $z_0 \in \Omega$ of a subset $\Omega \subseteq \mathbb{C}$ is a point such that there exists $r > 0$ where $\mathbb{D}_r(z_0) \subseteq \Omega$.

Definition 1.16

The set of interior point in Ω is the **interior** of Ω, denoted $\operatorname{int}\Omega$.

Definition 1.17

An **open set** in $\mathbb{C}$ is a subset $\Omega \subseteq \mathbb{C}$ such that for any $z_0 \in \Omega$ there exists $\varepsilon > 0$ such that $D_\varepsilon(z_0) \subseteq \Omega$.

It is immediate that Ω is open if and only if $\operatorname{int}\Omega = \Omega$.

Definition 1.18

Let $\Omega \in \mathbb{C}$ and let $z \in \mathbb{C}$. z is a **limit point** of Ω if there exists a sequence of points $\{z_n\}_{n=1}^{\infty} \subseteq \Omega$ such that $z_n \neq z$ for each n and $\lim z_n = z$.

We can equivalently define a limit point as a point z such that $\mathbb{D}_r(z) \setminus \{z\} \cap \Omega \neq \varnothing$ for each $r > 0$

Definition 1.19

$A \subseteq \mathbb{C}$ is a **closed set** if $\mathbb{C} \setminus A$ is open.

Proposition 1.20

A is closed if and only if it contains all its limit points.

Proof. ($\implies$) Suppose not. Then pick z which is a limit point of A that is not in A. Then there is no disk around z entirely contained in $\mathbb{C} \setminus A$. Thus A is not closed.

($\impliedby$) Suppose A is not closed. Then there exists $z \notin A$ such that each $\mathbb{D}_r(z) \setminus \{z\}$ intersects A. Then z is a limit point of A. □

Definition 1.21

The **closure** of $\Omega \subseteq \mathbb{C}$, denoted $\overline{\Omega}$, is the union of Ω with its limit points.

Definition 1.22

The **boundary** of $\Omega \subseteq \mathbb{C}$, denoted $\partial\Omega$, is defined as $\overline{\Omega} \setminus \operatorname{int}\Omega$.

Definition 1.23

$\Omega \subseteq \mathbb{C}$ is **bounded** if there exists $M > 0$ such that $|z| < M$ for each $z \in \Omega$ (or equivalently, $\Omega \subseteq \mathbb{D}_M$).

Definition 1.24

Let $\Omega \subseteq \mathbb{C}$ be bounded. Then the **diameter** of Ω is defined as

$$\operatorname{diam}\Omega = \sup_{z,w\in\Omega} |z - w|$$

The following definition, as in the real case, is critical:

Definition 1.25

$\Omega \subseteq \mathbb{C}$ is **compact** if it is closed and bounded.

Theorem 1.26: Bolzano-Weierstrass Theorem

Let $\Omega \subseteq \mathbb{C}$. Then the following conditions are equivalent:

1. Ω is compact.
2. Each sequence $\{z_n\}_{n=1}^{\infty} \subseteq \Omega$ has a subsequence $\{z_{n_k}\}_{k=1}^{\infty}$ which converges to some $z \in \Omega$.

We can treat $\mathbb{C}$ similarly to $\mathbb{R}^2$ to prove this.

Proof. ($1 \implies 2$) If Ω is compact, then $\{z_n\}_{n=1}^\infty \subseteq \mathbb{C}$ may be written as $\{x_n + iy_n\}_{n=1}^\infty \subseteq \mathbb{C}$. Since Ω is bounded, there exists $M > 0$ such that $|z| < M$ for all $z \in \Omega$. In particular $\sqrt{x_n^2 + y_n^2} = |z_n| < M$. So the real sequences $\{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty$ are bounded. Apply the real version of Bolzano-Weierstrass, there exists a convergent subsequence $\{x_{n_k}\}$. Then consider the sequence $\{y_{n_k}\}$. This is also bounded, so we apply Bolzano-Weierstrass again to produce $\{y_{n_{k_i}}\}$ convergent. Then the sequence $\{z_{n_{k_i}}\}$ is a convergent subsequence. If $z = z_n$ for some n, then $z \in \Omega$; otherwise it is a limit point. Since Ω is closed it contains its limit points so $z \in \Omega$.

($2 \implies 1$) Suppose each sequence has a convergent subsequence. Let z be a limit point and let $\{z_n\} \subseteq \Omega \setminus \{z\}$ be a sequence converging to z. Then there exists a subsequence $\{z_{n_k}\}$ which converges to $z' \in \Omega$. But subsequences converge to the same value as the original sequence, so $z = z' \in \Omega$. So Ω is closed. If Ω is not bounded, then we may take $\{z_n\}$ such that $|z_n| \geq n$. Then we have some convergent subsequence $\{z_{n_k}\}$ by assumption. But this is impossible, so Ω is bounded. □

Definition 1.27

An **open cover** of a set $\Omega \subseteq \mathbb{C}$ is a collection $\mathcal{O}$ of open sets such that each $z \in \Omega$ is contained in some $O \in \mathcal{O}$. A **subcover** of $\mathcal{O}$ is a subcollection which is still a cover.

Theorem 1.28: Heine-Borel Theorem

A set $\Omega \subseteq \mathbb{C}$ is compact if and only if every open cover has a finite subcover.

Proof. □

Proposition 1.29: Nested Compact Set Property

Suppose that $\Omega_1 \supseteq \Omega_2 \supseteq \dots$ is a nested sequence of compact, nonempty subsets of $\mathbb{C}$. Then

$$\bigcap_{n=1}^{\infty} \Omega_n \neq \varnothing$$

Moreover, if $\lim_{n\to\infty} \operatorname{diam} \Omega_n = 0$, then there is a unique point $z \in \mathbb{C}$ such that $z \in \Omega_n$ for all n.

Proof. Choose $z_n \in \Omega_n$ for each n. Then the sequence of points $\{z_n\} \subseteq \Omega_1$, and Ω_1 is compact, so there exists a convergent subsequence $\{z_{n_k}\}$ tending to $z \in \Omega_1$. Then for arbitrary Ω_n, there exists a subsequence $\{z_{n_{k+k_0}}\} \subseteq \Omega_n$ for sufficiently large k_0, which converges to z and we see that $z \in \Omega_n$. So the intersection is nonempty.

To show uniqueness, take $z, w \in \bigcap_{n=1}^\infty \Omega_n$. Then

$$|z - w| \leq \operatorname{diam} \Omega_n$$

for each n, but $\operatorname{diam} \Omega_n \to 0$ so $|z - w| = 0$ and thus $z = w$. □

Remark

With the assumption that $\operatorname{diam}\Omega_n \to 0$, we need not take subsequences as $\{z_n\}$ itself is Cauchy. To see this, pick $\varepsilon > 0$ and let N be such that $\operatorname{diam}\Omega_n < \varepsilon$ for any $n \geq N$. Then for any $n, m \geq N$, $z_n, z_m \in \Omega_N$ and thus $|z_n - z_m| \leq \operatorname{diam}\Omega_N < \varepsilon$.

Definition 1.30

A set $\Omega \subseteq \mathbb{C}$ is **connected** if there are no disjoint sets $A, B \subseteq \Omega$ such that $\Omega = A \sqcup B$ such that $A \cap \overline{B} = \varnothing$ and $\overline{A} \cap B = \varnothing$.

The above general definition can be simplified for nicer sets:

Proposition 1.31

If Ω is open, then it is connected if and only if it cannot be written as the union of disjoint open sets. Similarly if F is closed then it is connected if and only if it is not the union of disjoint closed sets.

Definition 1.32

Suppose $\Omega \subseteq \mathbb{C}$ and $f : \Omega \to \mathbb{C}$. f is **continuous** at z_0 if for every $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $z \in \Omega$ and $|z - z_0| < \delta$, it follows that $|f(z) - f(z_0)| < \varepsilon$.

Proposition 1.33

f is continuous at z_0 if and only if for every $\{z_n\} \subseteq \Omega$ with $z_n \to z_0$, it follows that $f(z_n) \to f(z_0)$. We say that f is continuous on Ω if it is continuous at each point in Ω.

Definition 1.34

A **path** is a function $f : [0, 1] \to \mathbb{C}$. A continuous path is a continuous such function.

Definition 1.35

A set $\Omega \subseteq \mathbb{C}$ is **path connected** if for any $z, w \in \Omega$ there exists a continuous path with $f(0) = z$ and $f(1) = w$ with $f(t) \in \Omega$ for each $t \in [0, 1]$.

Proposition 1.36

Ω is path connected if and only if it is connected.

1.3 Functions on $\mathbb{C}$

We now turn our attention to functions which map complex numbers to complex numbers, the primary object of study in this course. Continuing from the definition of continuity from the previous section, we have the following:

Proposition 1.37

If f is continuous at z_0 then $|f|$ is continuous at z_0.

Proof. By the reverse triangle inequality we have $||f(z)| - |f(z_0)|| \leq |f(z) - f(z_0)|$. The conclusion follows. $\square$

Definition 1.38

f attains its maximum on $\Omega \subseteq \mathbb{C}$ if there exists $z_0 \in \Omega$ such that

$$|f(z)| \leq |f(z_0)|$$

for each $z \in \Omega$. The minimum case is analogous.

Theorem 1.39

Suppose that $\Omega \subseteq \mathbb{C}$ is compact and $f : \Omega \to \mathbb{C}$ is continuous. Then f attains is maximum (and minimum) on Ω.

Proof. First we show that f is bounded on Ω. If not, then we may take a sequence of points $\{z_n\} \subseteq \Omega$ such that $|f(z_n)| \to \infty$. Then $\{z_n\}$ contains a convergent subsequence $\{z_{n_k}\}$ tending to z. It follows that

$$|f(z_{n_k})| \to |f(z)|$$

But the left side diverges to ∞, contradiction. Thus $f(\Omega)$ is bounded.

Let $M = \sup |f|(\Omega)$. Then there exists a sequence $\{z_n\} \subseteq \Omega$ such that $|f(z_n)| \to M$. Then there exists a subsequence $\{z_{n_k}\}$ converging to $z \in \Omega$. By continuity we have

$$|f(z)| = \lim |f(z_{n_k})| = M$$

$\square$

We now make the most important definition of this course:

Definition 1.40

Let $\Omega \subseteq \mathbb{C}$ be open and let $z_0 \in \Omega$. Let $f : \Omega \to \mathbb{C}$. We say that f is **holomorphic** at z_0 (or complex differentiable) if the limit

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists. In this case, the limit is denoted $f'(z_0)$.

If f is holomorphic at every $z \in \Omega$, then we simply say it is holomorphic on Ω. If f is holomorphic on $\mathbb{C}$ it is said to be **entire**.

Note that the specification that Ω is open ensures that the difference quotient is actually defined (for sufficiently small h). Moreover, although this definition appears similar to the real analogue, the structure of the complex numbers means that it has far-reaching implications.

We will prove the following theorems in this class:

- (Cauchy's Theorem) If f is holomorphic on Ω, then it has derivatives of all orders.
- (Liouville's Theorem) If f is entire and bounded, then it is constant.
- (Prime Number Theorem) If $\pi(n)$ denotes the number of prime numbers less than or equal to n, then
$$\lim_{n \to \infty} \pi(n) \cdot \frac{\ln n}{n} = 1$$
- (Hardy-Ramanujan Theorem) Define $p(n)$ (the partition function) to be the number of ways to write $n = k_1 + k_2 + \ldots + k_n$ where $k_1 \geq k_2 \geq \ldots \geq k_n$ are all integers. For instance, $p(4) = 5$. Then
$$p(n) \sim \frac{1}{n\sqrt{48}} e^{\pi \sqrt{\frac{2}{3}} \cdot \sqrt{n}}$$

Example 1.41

The function $f(z) = z$ is holomorphic:

$$\frac{f(z+h) - f(z)}{h} = \frac{z + h - z}{h} = \frac{h}{h} = 1$$

so $z' = 1$.

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