DATA605 Homework 13

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```
library(calculus)
library(Deriv)
```

Assignment

This week, we'll work out some Taylor Series expansions of popular functions.

1. $f(x) = (1 - x)$
2. $f(x) = e^x$
3. $f(x) = ln(1 + x)$
4. $f(x) = x^{(1/2)}$

For each function, only consider its valid ranges as indicated in the notes when you are computing the Taylor Series expansion. Please submit your assignment as an R-Markdown document.

Response

For each function, I'll start work out the expansion using the base formula for Taylor Series.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

I'll work out the first four terms, centered on zero.

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3$$

Function 1

$$f(x) = (1 - x)$$

This first function is a bit unique, in that the function is already linear. So, a Taylor Series expansion does not serve a functional purpose, in that there is no need to approximate!

Still, we can work through the expansion. The first derivative of $f(x) = (1 - x)$ with respect to x is simply -1. That means that all further derivatives are simply zero. So, our Taylor Series looks as follows.

$$f(x) = 1 - 0 + -1(x-c) + \frac{0}{2!}(x-c)^2 + \frac{0}{3!}(x-c)^3 + \ldots$$

Beyond the first two terms, all are zero, so we just have the following.

$$f(x) = 1 + -1(x-c)$$

Which we can simplify to $1 - x + c$. If we center this on $c = 0$, then we are left with $1 - x$, which is our original function. It makes sense that the Taylor Series of a linear function $f(x)$ is simply $f(x)$. Taylor Series are a linear approximation of non-linear functions. So if we apply the approach to an already linear function, we get an exact match.

Let's check our result against the `taylor` function from the `calculus` package.

```
f1_taylor <- taylor('1-x', var = 'x', order = 4)
print(f1_taylor$f)
```

```
## [1] "(1) * 1 + (-1) * x^1"
```

It's the same!

Function 2

$$f(x) = e^x$$

Figure 8.8.1 in the text gives us the simplified expansion for e^x as follows.

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

Let's work to this same spot using the base Taylor Series formula. First, we need to evaluate $f(x)$ at $c = 0$ for the first term, and e^0 is just 1.

Second, we need the first, second and third derivatives of e^x. For this function, however, the answer is simple: it's always e^x. In fact, that is the part of what defines e^x: it represents the constant that, when used as a base in an exponential function, has a derivative that is proportional to itself by a factor of 1. In other words, e^x is the unique function that equals its own derivative. So, for all terms in our Taylor Series, the function that we will evaluate is just e^x.

Let's plug these values in to our Taylor Series formula with $c = 0$.

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \frac{f'''(c)}{3!}(x - c)^3$$

$$f(x) = 1 + e^0(x - 0) + \frac{e^0}{2!}(x - 0)^2 + \frac{e^0}{3!}(x - 0)^3$$

$$f(x) = 1 + 1(x) + \frac{1}{2!}(x)^2 + \frac{1}{3!}(x)^3$$

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

With that, we've landed back at what we had in the text. Let's run a final check with `calculus`.

```
f2_taylor <- taylor('exp(x)', 'x', order = 4)
print(f2_taylor$f)
```

```
## [1] "(1) * 1 + (1) * x^1 + (0.5) * x^2 + (0.166666666666667) * x^3 + (0.0416666666666667) * x
^4"
```

It appears to match, but let's check these coefficients.

```
print(1/factorial(2))
```

```
## [1] 0.5
```

```
print(1/factorial(3))
```

```
## [1] 0.1666667
```

```
print(1/factorial(4))
```

```
## [1] 0.04166667
```

Looks good!

Function 3

$$f(x) = ln(1+x)$$

Figure 8.8.1 gives us the expansion for $ln(x)$ as follows:

$$(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \ldots$$

I would presume $ln(1+x)$ follows a similar pattern, but let's work it out.

First, we find that $f(0) = ln(1) = 0$, so our first term drops off. Next, we find derivatives.

```
f <- function(x) log(1+x)
df <- Deriv(f)
df2 <- Deriv(df)
df3 <- Deriv(df2)
df4 <- Deriv(df3)

print(df)
```

```
## function (x)
## 1/(1 + x)
```

```
print(df2)
```

```
## function (x)
## -(1/(1 + x)^2)
```

```
print(df3)
```

```
## function (x)
## 2/(1 + x)^3
```

```
print(df4)
```

```
## function (x)
## -(6/(1 + x)^4)
```

```
print(df(0))
```

```
## [1] 1
```

```
print(df2(0))
```

```
## [1] -1
```

```
print(df3(0))
```

```
## [1] 2
```

```
print(df4(0))
```

```
## [1] -6
```

So, our coefficients are 1, -1, 2 and -6 for our second, third, fourth and fifth terms. Let's plug these in.

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \frac{f''''(c)}{4!}(x-c)^4$$

$$f(x) = 0 + 1(x-0) + \frac{-1}{2!}(x-0)^2 + \frac{2}{3!}(x-0)^3 + \frac{-6}{4!}(x-0)^4$$

$$f(x) = 0 + 1(x) + \frac{-1}{2!}(x)^2 + \frac{2}{3!}(x)^3 + \frac{-6}{4!}(x)^4$$

$$f(x) = 0 + x - \frac{x^2}{2!} + \frac{2x^3}{3!} - \frac{6x^4}{4!}$$

```
f3_taylor <- taylor('log(1+x)', 'x', order = 4)
print(f3_taylor$f)
```

```
## [1] "(1) * x^1 + (-0.5) * x^2 + (0.333333333333333) * x^3 + (-0.25) * x^4"
```

Looks good. We already confirmed that 1/2! equals 0.5 (duh), but I'll double check that those others.

```
2/factorial(3)
```

```
## [1] 0.3333333
```

```
6/factorial(4)
```

```
## [1] 0.25
```

We're good :)

Bringing it back to the Series for $ln(x)$, we see a similar positive-negative pattern and overall structure. But it seems that we've gotten rid of the $(x-1)$ format and introduced some other coefficients. Interesting!

Function 4

$$f(x) = x^{(1/2)}$$

For this function, we have x alone in the base, so centering on zero will result in the zeroing out of all terms. So, we'll center this Series on $c = 1$.

We can evaluate $f(1)$ as $1^{1/2} = 1$. Next, we can find the first three derivatives by applying the power rule.

$$f'(x) = \frac{1}{2}x^{-1/2}$$

$$f''(x) = -\frac{1}{4}x^{-3/2}$$

$$f'''(x) = \frac{3}{8}x^{-5/2}$$

We can see we've got some strange exponents to deal with. Luckily, we are centering on 1, and 1 to any power is just 1, which should clean things up.

Let's try and plug these in with $c = 1$.

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3$$

$$f(x) = 1 + \frac{1}{2}1^{-1/2}(x-1) + \frac{-\frac{1}{4}1^{-3/2}}{2!}(x-1)^2 + \frac{\frac{3}{8}1^{-5/2}}{3!}(x-1)^3$$

$$f(x) = 1 + \frac{1}{2}1(x-1) + \frac{-\frac{1}{4}1}{2!}(x-1)^2 + \frac{\frac{3}{8}1}{3!}(x-1)^3$$

$$f(x) = 1 + \frac{1}{2}(x-1) - \frac{\frac{1}{4}}{2!}(x-1)^2 + \frac{\frac{3}{8}}{3!}(x-1)^3$$

We can get some rough decimals to try and clean this up a bit.

```
(1/4)/factorial(2)
```

```
## [1] 0.125
```

```
(3/8)/factorial(3)
```

```
## [1] 0.0625
```

$$f(x) = 1 + 0.5(x-1) - 0.125(x-1)^2 + 0.0625(x-1)^3$$

Finally, let's check it. I don't believe we can center on any other value except 0 in the `taylor` function in `calculus`. Instead, we'll just check the coefficients using `pracma`.

```
f <- function(x) x^(1/2)
pracma::taylor(f, x0 = 1, n = 3)
```

```
## [1]  0.06250008 -0.31250025  0.93750025  0.31249992
```

Hm, they don't quite match. I'm not able to find any issues with my expansion, so I'll need to leave this as a point for further inquiry!