DATA605 Homework 7

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```
library(tidyverse)
```

Question 1

Let $X_1, X_2, \ldots, X_n$ be n mutually independent random variables, each of which is uniformly distributed on the integers from 1 to k. Let Y denote the minimum of the X_i's. Find the distribution of Y.

Response

Our goal is to find a function that describes the distribution of Ys, i.e. the probability mass function $P(Y = y)$. We can start by defining the cumulative distribution of Y, i.e. $F(Y) = P(Y \leq y)$. Because Y represents the minimum of each X_i, the probability that Y will be less than or equal to some value y is the same as the probability that at least one of the random variables X is less than or equal to y. In other words, its the compliment of the joint probability that all variables X are greater than y.

$$P(Y \leq y) = 1 - P(X_1 > y) \times P(X_2 > y) \times \ \ldots \ \times P(X_n > y)$$

Each individual variable X is uniformly distributed, so $P(X_i > y) = (k - y)/k$, and the joint probability can be expressed as $P(Y > y) = [(k - y)/k]^n$, and our CDF can be expressed as $P(Y \leq y) = 1 - [(k - y)/k]^n$.

Because all variables X are discrete, we can define the probability that Y is a specific value y as the difference between the CDFs given y and $y - 1$. In other words:

$$P(Y = y) = P(Y \leq y) - P(Y \leq y - 1)$$

If we substitute our CDF definitions above, we get the following.

$$P(Y = y) = 1 - (\frac{k - y}{k})^n - [1 - (\frac{k - (y - 1)}{k})^n]$$

We can rearrange this and simplify as follows.

$$P(Y = y) = (\frac{k - y + 1}{k})^n - (\frac{k - y}{k})^n$$

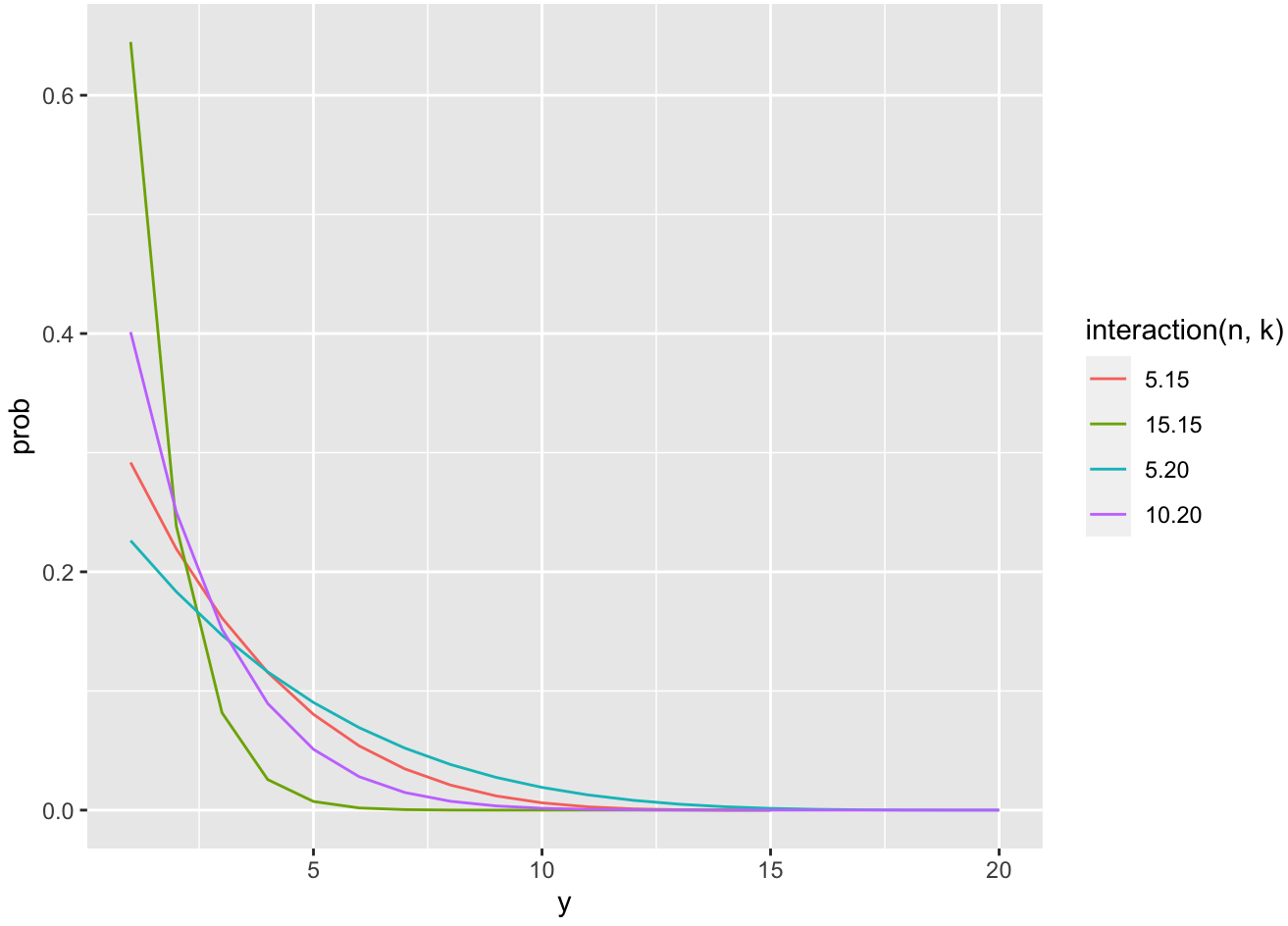
We can put together a function to test this distribution for certain values of n and k, given a certain value y. We generate plots for different combinations of n and k to observe the varied distributions.

```
Y_pmf <- function(y, n, k) {
  prob <- ((k - y + 1) / k)^n - ((k - y) / k)^n
  return(prob)
}

n_values <- c(5,10,5,15)
k_values <- c(20,20,15,15)
results <- data.frame()

for (i in 1:4) {
  n <- n_values[i]
  k <- k_values[i]
  for (y in 1:k) {
    prob <- Y_pmf(y, n, k)
    result <- list(y = y, n = n, k = k, prob = prob)
    results <- rbind(results, result)
  }
}

results %>%
  ggplot(aes(y, prob, color = interaction(n, k))) +
  geom_line()
```



Question 2

Your organization owns a copier (future lawyers, etc.) or MRI (future doctors). This machine has a manufacturer's expected lifetime of 10 years. This means that we expect one failure every ten years. (Include the probability statements and R Code for each part.)

Part A

What is the probability that the machine will fail after 8 years? Provide also the expected value and standard deviation. Model as a geometric. *Hint: the probability is equivalent to not failing during the first 8 years.*

Response

Here, we can define the machine failing as a "success", and each year as a separate "trial". In this case, our probability of success per trial (i.e. the likelihood of failure each year) is 1/10. If we are looking for the probability that the machine fails after 8 years, then we are looking for $P(X > 8)$.

We can use the CDF of the geometric distribution to find $P(X \leq 8)$, then take the complement (i.e. $1 - P(X \leq 8)$) to find $P(X > 8)$. The CDF is defined as $F(x) = P(X \leq x) = 1 - (1 - p)^{x+1}$, so we simply need to plug in our values for p and x (i.e., 1/10 and 8, respectively).

The expected value and standard deviation of the geometric distribution are defined as $1/p$ and $\sqrt{(1 - p)/p^2}$.

```
# Parameters
p <- 1/10
x <- 8

# Calculations
probability <- 1 - (1 - (1 - p)^(x+1) )
expected_value <- 1/p
standard_deviation <- sqrt( (1-p) / p^2 )

# Results
cat("Probability of failure after 8 years:", probability, "\n")
```

```
## Probability of failure after 8 years: 0.3874205
```

```
cat("Expected value:", expected_value, "\n")
```

```
## Expected value: 10
```

```
cat("Standard deviation:", standard_deviation, "\n")
```

```
## Standard deviation: 9.486833
```

We'll check our results with the `pgeom` function.

```
probability <- 1 - pgeom(x, prob = p)
cat("Probability of failure after 8 years:", probability, "\n")
```

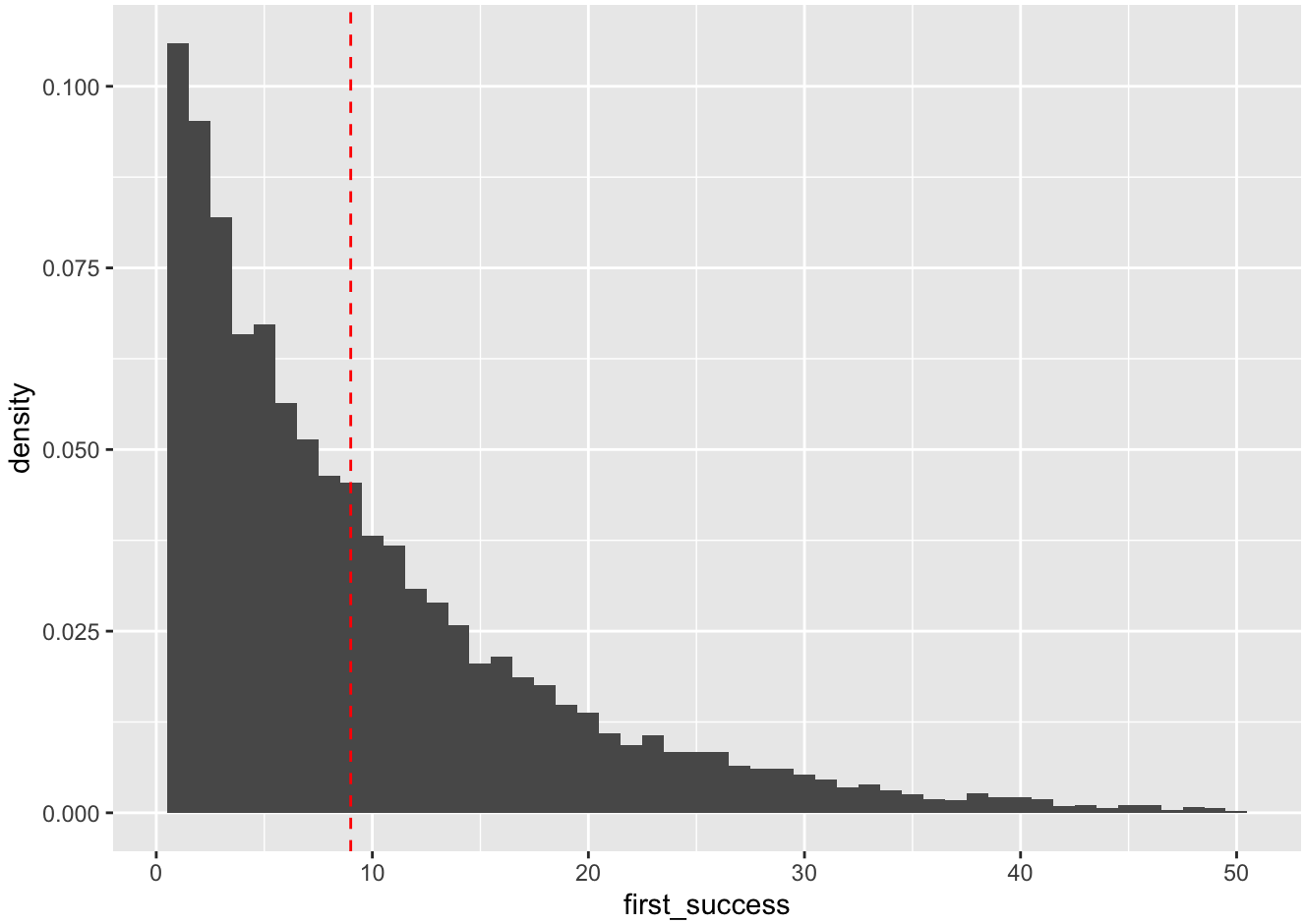
```
## Probability of failure after 8 years: 0.3874205
```

Our results match! However, this results appears somewhat lower than I would expect, given the expected value is greater than our x. As a sanity check, I'll run a quick simulation, in which we randomly sample a 1 or 0 (representing success or failure) using $p = 0.1$, then extract the index of the first appearance of a 1. This gives us an empirical measure of how many trials are required before we realize a success (i.e. how many years before a machine failure occurs). We'll iterate this simulation 100,000 times and aggregate results.

```
n_sims <- 10000
n_samples <- 50
p <- 1/10
results <- data.frame()

for (sim in 1:n_sims) {
  samples <- sample(0:1, size = n_samples, replace = TRUE, prob = c(1-p, p))
  first_success <- min(which(samples == 1))
  result <- list(sim = sim, first_success = first_success)
  results <- rbind(results, result)
}

results %>%
  ggplot(aes(first_success, ..density..)) +
  geom_histogram(bins = 50) +
  geom_vline(aes(xintercept = 9), linetype="dashed", color="red")
```



To my surprise, the plot conforms to our earlier results. Surprising, but this helps convince me!

Part B

What is the probability that the machine will fail after 8 years?. Provide also the expected value and standard deviation. Model as an exponential.

Response

Again, we can define the machine failing as a "success". We don't have discrete trials when modeling as an exponential process, but instead use our probability of success as a rate of success. This rate (i.e. λ) is 1/10. If we are looking for the probability that the machine fails after 8 years, then we are looking for $P(X > 8)$.

The CDF of the exponential distribution is defined as $F(x) = P(X \leq x) = 1 - e^{-\lambda x}$, so we simply need to plug in our values for λ and x (i.e., 1/10 and 8, respectively). Again, we'll take the compliment (i.e. $1 - P(X \leq 8)$) to find $P(X > 8)$.

The expected value and standard deviation of the geometric distribution are both defined as $1/\lambda$.

```
# Parameters
lambda <- 1/10
x <- 8

# Calculations
probability <- 1 - (1 - exp(-lambda*x))
expected_value <- 1/lambda
standard_deviation <- sqrt( 1/lambda^2 )

# Results
cat("Probability of failure after 8 years:", probability, "\n")
```

```
## Probability of failure after 8 years: 0.449329
```

```
cat("Expected value:", expected_value, "\n")
```

```
## Expected value: 10
```

```
cat("Standard deviation:", standard_deviation, "\n")
```

```
## Standard deviation: 10
```

We'll check our results with the `pexp` function.

```
probability <- 1 - pexp(x, rate = lambda)
cat("Probability of failure after 8 years:", probability, "\n")
```

```
## Probability of failure after 8 years: 0.449329
```

Our results match!

Part C

What is the probability that the machine will fail after 8 years? Provide also the expected value and standard deviation. Model as a binomial. *Hint: 0 success in 8 years.*

Response

As with the first problem, we can define the machine failing as a “success”, and each year as a separate “trial”. In this case, our probability of success per trial (i.e. the likelihood of failure each year) is 1/10. If we are looking for the probability that the machine fails after 8 years, then we define our random variable as X~$B(n, p)$ with $n = 8$ and $p = 0.1$. Finally, we are looking for the probability that X remains zero across all 8 trials, i.e. $P(X = 0)$.

We can use the PMF of the binomial distribution to find $P(X = 0)$, defined as $\binom{n}{k}p^k(1-p)^{n-k}$. The expected value and standard deviation of are defined as $n * p$ and $\sqrt{n * p * (1-p)}$.

```
# Parameters
p <- 1/10
n <- 8
k <- 0

# Calculations
probability <- choose(n,k) * p^k * (1-p)^(n-k)
expected_value <- n*p
standard_deviation <- sqrt( n*p*(1-p) )

# Results
cat("Probability of failure after 8 years:", probability, "\n")
```

```
## Probability of failure after 8 years: 0.4304672
```

```
cat("Expected value:", expected_value, "\n")
```

```
## Expected value: 0.8
```

```
cat("Standard deviation:", standard_deviation, "\n")
```

```
## Standard deviation: 0.8485281
```

We’ll check our results with the `dbinom` function.

```
probability <- dbinom(0, n, p)
cat("Probability of failure after 8 years:", probability, "\n")
```

```
## Probability of failure after 8 years: 0.4304672
```

Our results match!

Part D

What is the probability that the machine will fail after 8 years? Provide also the expected value and standard deviation. Model as a Poisson.

Response

Typically, the Poisson distribution is used to estimate the probability of a certain number of occurrences happening over a fixed period. Here, we are looking specifically if the number of occurrences is zero over an 8-year period. If start with the same annual lambda as we had in Part B (1/10), we can convert this to an 8-year lambda, which we'll dub λ_8. We can then calculate the probability that X remains zero across the 8-year period, i.e. $P(X = 0)$ given λ_8.

The PMF of the Poisson distribution is defined as $P(X = x) = (\lambda^x e^{-\lambda})/x!$. The expected value and standard deviation are defined as $lambda$ and $\sqrt{\lambda}$.

```
# Parameters
lambda <- 1/10
lambda_8 <- lambda * 8
x <- 0

# Calculations
probability <- lambda_8^x * exp(-lambda_8) / factorial(x)
expected_value <- lambda
standard_deviation <- sqrt(lambda)
expected_value_8 <- lambda_8
standard_deviation_8 <- sqrt(lambda_8)

# Results
cat("Probability of failure after 8 years:", probability, "\n\n")
```

```
## Probability of failure after 8 years: 0.449329
```

```
cat("Expected number of failures for 1-year period:", expected_value, "\n")
```

```
## Expected number of failures for 1-year period: 0.1
```

```
cat("Standard deviation for 1-year period:", standard_deviation, "\n\n")
```

```
## Standard deviation for 1-year period: 0.3162278
```

```
cat("Expected number of failures for 8-year period:", expected_value_8, "\n")
```

```
## Expected number of failures for 8-year period: 0.8
```

```
cat("Standard deviation for 8-year period:", standard_deviation_8, "\n")
```

```
## Standard deviation for 8-year period: 0.8944272
```

Finally, we'll check out results with `dpois`.

```
probability <- dpois(x, lambda_8)
cat("Probability of failure after 8 years:", probability)
```

```
## Probability of failure after 8 years: 0.449329
```

Another match!