AN ERROR ANALYSIS PRIMER FOR PHYS 252

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1. Does There Even Have to be Error?

Seeing as how one of the goals of this class it to learn to write at an eschalon worthy of publication, the topic of reporting experimentally measured values in proper scientific notation must be discussed. Any measured valued needs to be reported $(a \pm \delta a) \times 10^k$ units for $a, \delta a \in \mathbb{R}$ (the real numbers) and $k \in \mathbb{Z}$ (the integers). One sees that, in the proper scientific notation one must determine what the measurement uncertainty δa must be. The determination of δa will be the focus of this document.

First, I would like to examine the question if there even has to be error, that is if $\delta a = 0$, namely the error is *EXACTLY* equal to zero. There are two general schools of thought on this prospect: one ontological and one epistemological. Both of these schools of thoughts are philosophical disciplines concerning what can exist any why-ontology, and the study of knowledge-epistemology. In particular ontology applied to this proposition queries if an exact value of some physical quantity a even exists at all. On the other hand, the epistemological question is concerned with how one determines truths from falsehoods, and if one adopts an epistemology of empiricism-where one believes that truth can be determined by objective measurement of reality-the epistemic question boils down to what can be measured and how well. One then has the conditional flow chart of Fig. 1.1, where first the ontological question must be decided, and next the epistemic. I will argue that it is possible, however incredibly unlikely, for

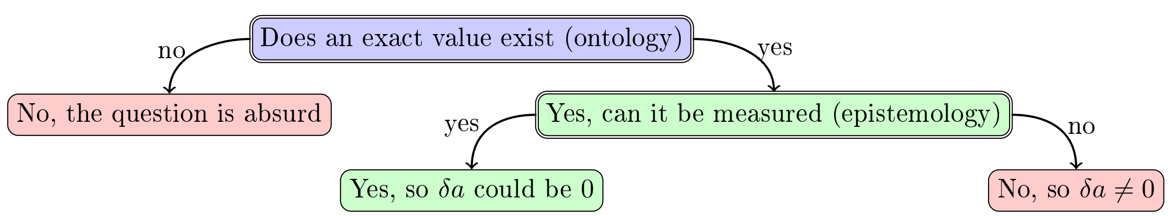

Figure 1.1. Flowchart of philosophical possibilities of measuring an exact value. First an exact value must exist-the ontological concern-and next it must be able to be measured-the epistemological concern.

a physical quantity to exist with an exact value. Next, given this ontological existence, it could also be possible, albeit almost impossibly difficult, to measure it with exact certainty. However, the impediments for this occurring are enormous, and without a known theory of quantum gravity nothing can be said for certain.

1.1. Epistemic Concerns.

1.1. Epistemic Concerns. Assume that an exact value for a quantity to be measured, a, does exist: is it possible to measure it exactly? To examine this question, I will, via reducto ad absurdum, discuss the difficulties of

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performing an exact measurement. First and foremost, and of the most pertinent to the experiments which will be performed in PHYS 252 this semester, is the accuracy and precision limitations of the laboratory equipment. Each

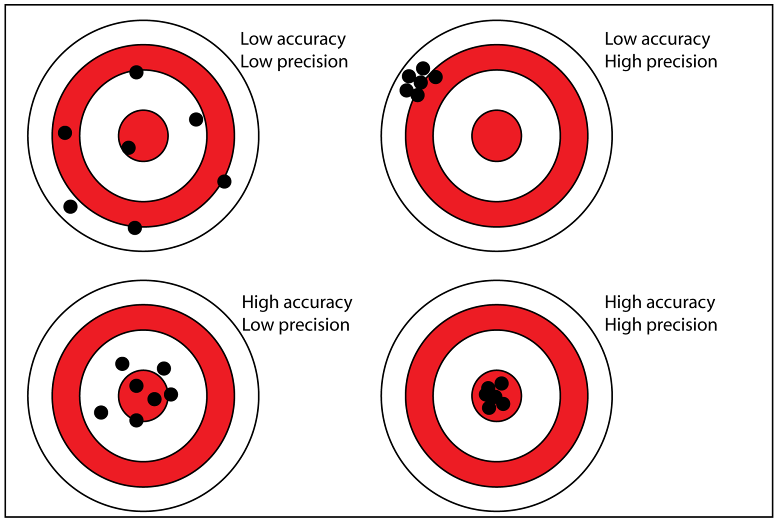

FIGURE 1.2. A review of accuracy and precision. Accuracy is the difference between the accepted and measured values, whereas precision is the variance of the measurements. Image taken from http://cdn.antarcticglaciers.org/wp-content/uploads/2013/11/precision_accuracy.png

piece of lab equipment has a maximum precision it can measure, albeit by the millimeter markings on a meter stick or the number of bits in an analog to digital converter (ADC). In fact, many manual measuring devices have a second scale which allows tenths of a unit to be estimated well; calipers, sphereometer, and spectroscopes are such instruments which will be used this semester. Once a measuring device has been properly set up to ensure that it is operating properly at its maximum precision, it must be calibrated against a known value to ensure accuracy.

I would like to return to the ADC for a moment: if one accepts that the quantity $a \in \mathbb{R}$, which could possibly require an unaccountably infinite amount of information to store (if a is irrational), one sees that being modeled by a fixed-bit number in the ADC data is certainly lost. Assume now that $a \in \mathbb{F}$, where $\mathbb{F}$ is a number system[1]for which every element can be represented by a finite number of bits. It is possible that such an $a \in \mathbb{F}$ could be measured exactly with the proper equipment.

I have now assumed, for the sake of argument that the laws of physics permit a measuring instrument to measure a, tacitly assumed up to this point to be a single particle, exactly. The next constraint comes from statistical ensembles of particles-assume there are N particles. In particular, assume that N is "large" namely $N \gtrsim N_A$, Avogadro's Number. Per the previous discussion, under the assumptions at hand it is theoretically possible to exactly measure the position and momentum of all N particles, but with current technology this is effectively an impossible task. Thus, to classify the states of such an ensemble, like moles of an ideal gas, one uses thermodynamic variables such as temperature T, entropy S, pressure P, and volume V which have inherent fluctuations (and hence uncertainty) as part of the theory. In fact, even if one could measure the exact position

[1]For the mathematically inclined this object is known as an (algebraic) *field*: $\mathbb{R}$ and $\mathbb{C}$ are examples of fields, however many fields with finite elements exist. Computing physical theories under finite fields is a topic of current research.

and momentum of all N particles one could not completely hermetically isolate the system from the rest of the universe so there would be (albeit small) uncertainties from inevitable temperature interactions possibly caused by external vibrations (phonons), radiation (photons, neutrinos, gravitational waves, etc . . .), or even quantum vacuum interactions! However, one could take the entire (observable) universe to the system at hand so, possibly, a could be measured exactly.

All this is well and good, and it would certainly be incredibly difficult to exactly measure every particles in an ensemble but not strictly impossible.. But wait! We have assumed that position and momentum could be measured exactly simultaneously: this contradicts the Heisenberg uncertainly principle that

$$\sigma_{\hat{\mathbf{X}}}\sigma_{\hat{\mathbf{P}}} \geq \frac{h}{2} \tag{1.1}$$

where, without going into too much quantum mechanics $\sigma_{\hat{\mathbf{X}}}$ is the uncertainty (standard deviation) of the measured (observed) position and $\sigma_{\hat{\mathbf{P}}}$ is the uncertainty (standard deviation) of the measured (observed) momentum. Thus, for any statistical ensemble at least, one must have error of at least $\frac{h}{2}$. If one makes the assumption that the universe is an ensemble of particles then Eq. 1.1 implies an exact value of position and momentum cannot be simultaneously known. We have not yet contradicted our assumption that a in fact, *HAS*, a definite value-an ontological assumption-but Eq. 1.1 says that one cannot know these quantities-an epistemic concern. In fact, the Heisenberg uncertainty principle can be generalized to most observable quantities[2].

So this is it, right? Given Eq. 1.1 one must conclude that it is epistemologically impossible to make measurements with exact precision. Not quite: one can work hard to prepare quantum system in a special state known as an eigenstate, for which a particular measurement will deterministically yield an exact value, which circumvents the Heisenberg uncertainty relations[3]. Thus, given this, it is possible to measure a exactly.

One must go down a single level deeper to the realm of quantum field theory. At this level of formalism, our final answer is revealed through the well documented phenomena of vacuum fluctuations. The seeming "empty" quantum vacuum, $|\Omega\rangle$, is in fact not! First, $|\Omega\rangle$ has non-zero energy density! Next, although, on average, a measurement of the quantum vacuum will yield a result with no particles, any individual measurement may not. One can actually extract this energy from the vacuum to do useful work, which manifests as an (attractive) Casimir force; in fact, at nanoscales overcoming this vacuum stickiness is a major issue in nanoengineering. No one knows[4] the eigenstates of $|\Omega\rangle$ so Eq. 1.1 applies at the level of the standard model and there is fundamental uncertainty for any measured non-gravitational quantity. Remember, that the standard model does not include gravity!

[2]For the interested reader, given quantum observable operators $\hat{A}, \hat{B}$ the Heisenberg uncertainty relationship between them is given to be $\sigma_{\hat{A}}\sigma_{\hat{B}} \geq \frac{1}{2}\left|\left[\hat{A}, \hat{B}\right]\right|$

[3]A propos the previous footnote, this enumerates the case that $\left[\hat{A}, \hat{B}\right] = 0$

[4]This is one of the things which a solution of the "Yang-Mills Mass Gap and Existence Problem" would elucidate: see [The18] for more information

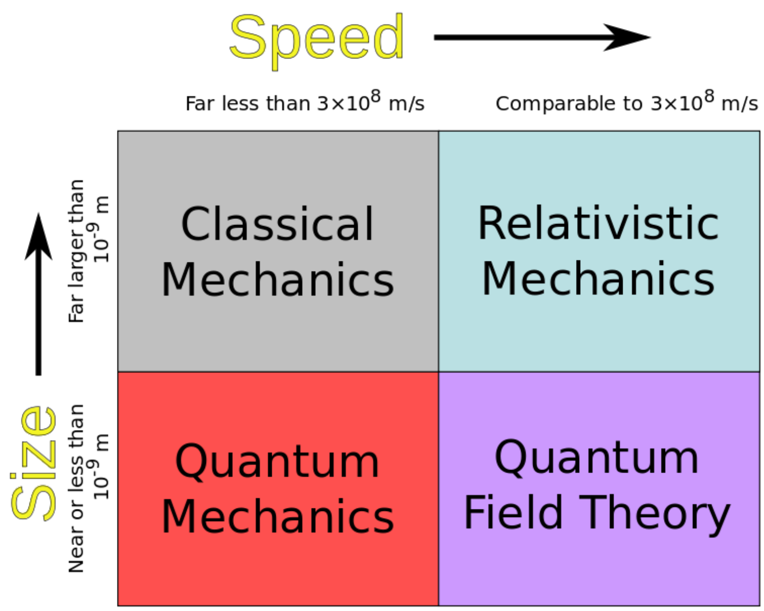

FIGURE 1.3. A hierarchy of physics theories. For "slow" and "big" things (see axis scales) classical mechanics, including thermodynamics and statistical mechanics, dominates. As things become "smaller" but still "slow" (regular) quantum mechanics dominates. As the speed of objects approaches c, the speed of light, but they are still "large" relativistic mechanics, certainly general and sometime special relativity dominates - we will be learning some special relativity (SR) this semester. Finally, for "fast" and "small" things the field of quantum field theory (QFT) dominates. This theory cannot incorporate gravitational effects in a (renormalizable) consistent fashion, thus the consummation of the marrage of big and small eludes us. Such a theory which would be called a theory of quantum gravity (not shown in the figure). Image taken from https://en.wikipedia.org/wiki/Classical_mechanics#/media/File:Physicsdomains.svg.

Finally, even though a consistent theory of quantum gravity does not currently exist, it is a reasonable expectation of this theory that, if one is found it reduce to a quantum field theory. Although many uncontrolled infinities abound attempting to quantum gravitational calculations, calculations in specialized cases can be crunched, and some properties of a quantum gravitational theory can be inferenced without knowing the theory at large. It is largely accepted, then, that the quantum gravitational vacuum fluctuations would occur at what is known at the *Planck length*

$$\begin{aligned} \ell_p &= \sqrt{\frac{\hbar G}{c^3}} \\ &\simeq (1.616229 \pm 0.000038) \times 10^{-35}\mathrm{m} \end{aligned} \tag{1.2}$$

In fact, if one accepts that such a vacuum gravitational fluctuation must exist in the context of a QFT, that a gravitational Heisenberg-type uncertainty relationship can be written

$$\sigma_g \sigma_L \geq \frac{\ell_p}{2\sqrt{\pi}} \tag{1.3}$$

where σ_g represents the size of the gravitational fluctuation[5], and σ_L denotes an uncertainty in the measuring apparatus [Pad85]. Thus, there seems to be some fundamental uncertainty σ_L associated with any measurement a.

1.2. **Ontological Concerns.** So, where does the argument of the previous section leave us concerning the (existence) ontological status of some value a one wants to measure exactly? I suppose one cannot strictly rule out an exact value of a existing, but it must be very very difficult - if not downright impossible - to measure exactly if it does, see Fig. 1.4. . First, one must have the value a contain no more information than exists in the observable

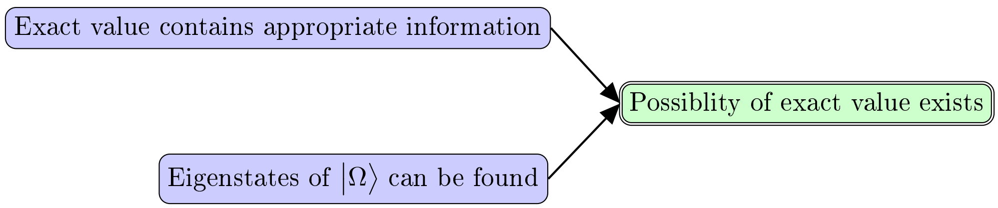

FIGURE 1.4. The possibility of having no error, namely $\delta a = 0$, cannot be ruled out: however if it does exist it must be difficult. If it happens to be that the two major hurdles on the left of the diagram are overcome it is possible that a may, in fact, have an exact value. However, given current science, the probability that this value is exactly measured - an epistemological concern - is negligibly small.

universe, which scientists put at $\sim 10^{120}$ bits . This means that $a \in \mathbb{F}$, a number system with finite elements, not the usual number systems of $\mathbb{R}$ and $\mathbb{C}$ used in physics today, which I would argue are used mostly for computational convenience not metaphysical significance [6]. Next, one must first know about, and then prepare the gravitational quantum vacuum $|\Omega\rangle$ in an eigenstate, which could be measured exactly. It is quite possible that a theory of quantum gravity could be discovered in the future - but not necessarily a guarantee as many physicists believe: see Gödel's Incompleteness Theorem for more information - where properties of $|\Omega\rangle$ could be elucidated, but preparing a system in this eigenstate may not be possible.

I concede that given the two necessary conditions stated above, it is *possible,* albeit exceedingly unlike considering Occam's Razor, that there exists some physical quantities which could be measured exactly with no error. However, the epistemic hurdles to measuring such a quantity are enormous. To tie this back into our PHYS 252 class, *the main hurdles which will be encountered in this lab are the precision limitations of lab equipment.* This is the main take away from this section! For the intrepid few who read the remainder of my treatise on this subject, I thank you! However, this is not information which you are expected to know for this class!

[5]Where, for the interested reader, g denotes the gravitational metric from general relativity (GR)

[6]For the mathematically interested reader, I am asserting that the *Axiom of Choice* does not hold in physical reality. The Axiom of Choice is an extension of the usual Zermelo-Frankel set axioms we all learn some variant or another of in high school, is in general is accepted by mathematicians and physicists, and the existence of continuums such as $\mathbb{R}$ and $\mathbb{C}$ is predicated upon this axiom. I have been convinced by Prof. Roger Penrose in that this is false in physical reality. If this were not the case, such absurd things as [SB07, PG15] or would be possible.

2. Categories of Error

Now that I have established that error is something which we will all have to accept in the PHYS 252 laboratory section, how can we categorize the the different types of error?. In this class I want you to know three different types of error: systematic error, statistical (random) error, and fitting error. Before continuing, please examine Fig. 2.1

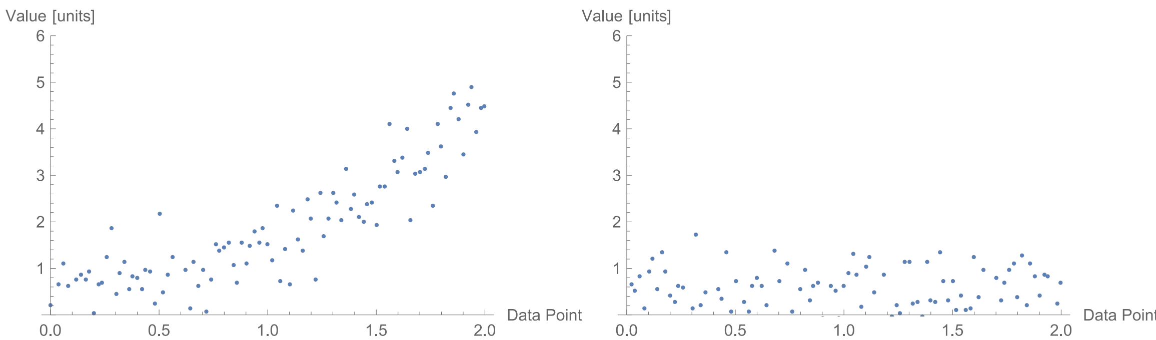

Figure 2.1. Which of these graphs contains systematic error? Which of these graphs contains random error? Why?

2.1. **Systematic Error.** The first type of error I would like to discuss is that of *systematic error*, namely error which one knows about and can "subtract out" in the final calculation. Before I go any further, I will pontificate one of my "pet peeves" when it comes to lab reports: *"HUMAN ERROR" IS PROFANE.* I will discuss this more later. There are many sources of systematic error, and, upon staring upon an aberrant data set, attempting to infer what phenomenology might be contributing to systematic error is one of the more difficult challenges to do in physics. I direct the reader to [Cho12] where, for a period of time from 2011-2012 the OPERA neutrino detector in Italy seemed to measure superluminal neutrinos. It took the scientific community approximately a year to discern that this 0.002% discrepancy was caused by a fiber optic cable not quite exactly mating with a detector - I blame a graduate student. The global scientific community did not accept an explanation of "human error," the exact mechanisms of the error which caused a superluminal measurement were precisely explained. I, also, expect you to make reasonable inferences on the sources of your error.

The biggest source of systematic error which will be encountered in PHYS 252 lab is that of calibration. Many measurement apparati do not measure absolute scales, they can only measure relative differences. Given this, one must give the apparatus a known value - and possibly an extrapolation scheme - so it can equate what actual physical quantity it measures - usually a voltage binned by an ADC - to some physical quantity of interest: this is done in the gas expansion lab this semester, for example. If one improperly calibrates an apparatus, it is easy to fix ex post facto by transforming the data to the proper calibration factor. This being said, however; *please* properly calibrate your lab equipment for all our sakes.

Systematic error can be more subtle: in experiments where many trials are conducted clear trends in the data as a function of trial number can be indicative of systematic error. One example of this which you might be familiar with is the friction-putty lab conducted for ASU freshman mechanics. In this experiment, students measure both μ_s and μ_k by pulling a block of putty along a lab table attached to a force meter. As more and more trials are conducted, however, one notices an increasing trend in the experimentally measured values of μ_s and μ_k. What might cause this? Well, as I have verified, the more trials which are conducted the stickier the table got and the higher these friction coefficients rose. What could one do to mitigate this error? Well, two things: first one could attempt a curve fit to the data and try to subtract off the trend after the fact (the difficult option) or just clean the table between trials (the easy option)! So, in a lab report, a student should write something akin to "we detected a systematic error in the direction of increasing μ_k/μ_s caused by the table becoming sticky from subsequent trials. This error was corrected by cleaning the table carefully between trials" versus simply stating "our μs were off due to human error." Your error analysis sections should be written in the style of the former, not the latter.

Common sources of systematic error are difficulty in reading a measurement apparatus (the steam occlusion of the heat lab, and the parallax errors through optical scopes are examples) and a system not being exactly closed (heat, sound, gas particles, etc. . . escaping from a system) are two examples which will be seen this semester. This is certainly not an exhaustive list: I look forward to seeing how you cleverly infer mechanisms of systematic error for the experiments this semester in your lab reports!

2.2. **Statistical Error.** As I argued in Section 1, even if it is ontologically possible for a value to exist without error, it is virtually impossible. Thus, for all intents and purposes in this class, any measurement you take will have some intrinsic statistical or random error which you cannot do anything about. Imagine, for a moment a situation as in Fig 2.2. If one has an exact value of $\frac{1}{2}$ units, but the bins are graduated in terms of 0.05 units, one

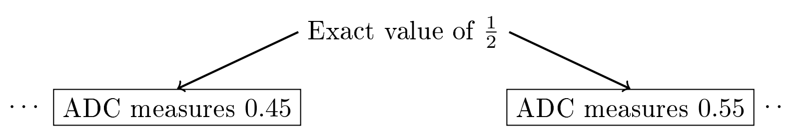

FIGURE 2.2. An example of random error. For an ADC with a precision of 0.05 units, how will a value of $\frac{1}{2}$ units, taken to be exact, be binned? There is a probability that it will end up in either the 0.45 units bin or the 0.50 units bin, but a priori, one does not know which one.

does not know a priori which bin will be triggered by a measurement. Now, one could play the game we argued in Section 1 that, perhaps, if we had more information we would be able to predict which bin would trigger with more certainty. However, this information is exactly what we are trying to measure with the apparatus at hand, so one does not have this information: it is an absurdity, then to speculate on it! Thus, our lack of knowledge of the system - our current epistemological situation to use the parlance of Section 1 - is our limiting factor and is the

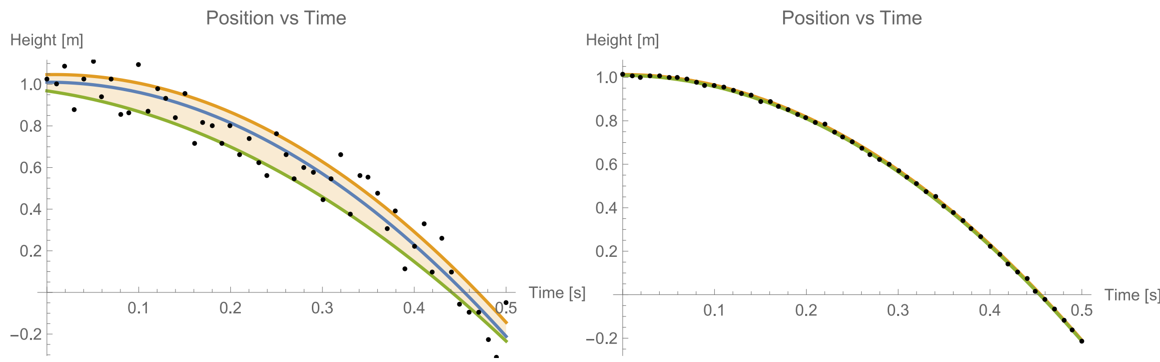

FIGURE 2.3. Example of fitting error. Here two trials of measuring g from a falling object are performed. In the first trial, there is significant error in the measurements leading to multiple curves which Mathematica predicts could fit the data; in the second trial measurement errors are minimal and all three curves coincide. The blue curve is the exact trajectory, the orange curve is given by adding all fitting errors to the exact parameters, and the green curve is given by subtracting all fitting errors from the exact parameters. Here the exact $g = 9.77\,\mathrm{m/s^2}$, $v_0 = 0.00\,\mathrm{m/s}$ and $x_0 = 1.01\,\mathrm{m}$

source of our statistical error. However, it happens to occur that exact values do not even exist, there will always be some statistical error.

2.3. **Fitting Error.** The final category of error which will be discussed in this course is that of fitting error. This error appears when one is is using some sort of computer algebra system to fit a curve to data points. Usually, when one is performing an experiment one is testing a theory, so ahead of time, one will usually have an idea of what general equation type the data should conform to. However, one does not know the exact value for the constants in the equation. Some general equation types are listed in Table. 1. The point of fitting, then, is to

Equation Class	Equation Form	Constants
Linear	$ax + b$	a-slope, b-intercept
Dampened Oscillation	$Ae^{-\beta t}\cos(\omega t + \phi) + x_0$	A-amplitude, β-decay constant, ω-angular frequency, ϕ-phase, x_0-offset
Logarithmic	$A\ln\left(\frac{t}{\tau}\right) + x_0$	A-amplitude, τ-time constant, x_0-offset
Gaussian	$Ae^{-(x-\mu)^2/\left(2\sigma^2\right)} + x_0$	A-amplitude, μ-mean, σ-standard deviation, x_0-offset

TABLE 1. Some general equation types which commonly occur in physical systems. Curve fitting determines estimates for the constants based on a data set, however there is uncertainty in this estimate, hence fitting error.

make inferences on what the value of the constants are; however, there is always some uncertainty of the exact value of a constant. This error can be interpreted as there being many different possible curves which could possibly fit the data. Lets consider a concrete example: an object of mass m is dropped from a height h, and would one like to measure what the acceleration of gravity g, is: the data from two trials of this experiment are shown in Fig. 2.3.

The parameters from the fitting equation of $-\left(\frac{1}{2}\right)at^2 + bt + c$ are shown in Table. 2. One sees that there are multiple curves which could reasonably fit the data within error. In the first trial the measurements were poor, so there is a bit of fitting error and the different curves can be explicitly seen, however in the second trial, the measurements were much more precise, so all three curves virtually coincide.

	Exact Trajectory	Trial 1	Trial 2
g m/s^2	9.77	8.64 ± 1.61	9.78 ± 0.07
v_0 m/s	0	$(-3.01 \pm 4.18) \times 10^{-1}$	$(-0.01 \pm 1.81) \times 10^{-2}$
x_0 m	1.01	1.04 ± 0.05	1.01 ± 0.01

TABLE 2. Table of parameters from the exact, trial 1 and trial 2 of this simulated experiment to measure g.

Quite often times computer algebra systems will give fitting errors of $\mathcal{O}\left(10^{-15}\right)$ to $\mathcal{O}\left(10^{-17}\right)$ or so. All this signifies is that the fitting algorithm cannot discriminate different possible fits within its current precision, it does not mean that your data is precise to within this range for there can still be notable statistical error, see Fig. 2.4. . For example if one were measuring length and found error of $\mathcal{O}\left(10^{-15}\right)$, it means you know the length to within

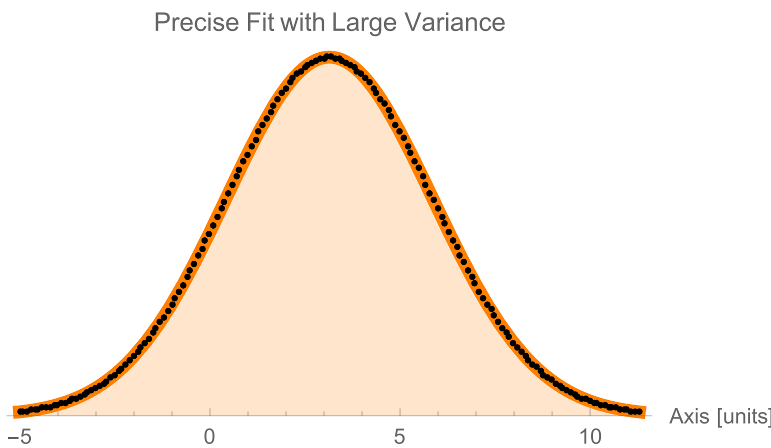

FIGURE 2.4. An example of an almost exact fit of a data set with considerable statistical variance (error). The data set shown is the normal (Gaussian) distribution of $\mathrm{n}\left(\pi, e^2\right)$ - mean π and variance e^2 - which would coorespond to an experimenter reporting the value measured by this experiment as $\pi \pm e$. A relative error calculation yeilds $\frac{e}{\pi} \simeq 86\%$. However, the best fit parameters for this data set are $\mu = 3.1416$, $\sigma = 2.7183$ but $\delta\mu = 1.4363 \times 10^{-13}, \delta\sigma = 1.438 \times 10^{-13}$ which is hardly indicitave of 86% relative error! The small $\delta\mu, \delta\sigma$ are most likely determined by limits of the numerical precision of the fitting calculation.

a Fermi, about the radius of a proton. I highly doubt this as seeing as it took LIGO multiple years to achieve this precision in order to detect gravitational waves. So, when you see fitting errors of this order, it simply means the fitting error is negligible: you need to look for other sources of error in your experiment, like the precision of the ruler, the ADC you are using, or at worst statistical variance.

2.4. **Tying it All Together: A Laboratory Example.** Now, let us revisit Fig. 2.1 and look at it in the lens of these three types of errors. First, the left graph contains both systematic and statistical error whereas the right plot has only statistical error. I think at this point, it would be helpful for me to reveal precisely how I generated both of those plots. For both I generated a data set about an exact value of $\frac{1}{2}$ augmented by normal (Gaussian) variate error of mean 0 and variance $\frac{1}{2}$ but for the left plot I added a systematic error term of x^2.

One can now imagine the following situation: you are writing your lab report, and while performing calculations you notice an increasing trend in your data, however the theoretical prediction indicates that the value should be constant. What should you do to fix this? Well, you notice that the general trend of the data is increasing, and it does not look like it is increasing in a linear fashion, so you try the next best thing, a quadratic fit. So, you fit the curve ax^2+bx+c to the data set - Fig. 2.5. Your computer algebra system comes back with the fitting parameters

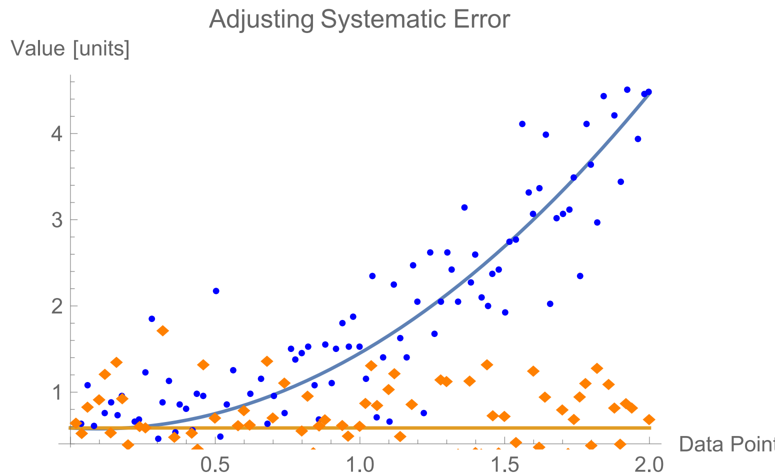

FIGURE 2.5. Example of fixing systematic error via statistical methods. By fitting a quadratic curve to the data set with systematic error and taking the constant term, one can get an estimate of the data with only statistical error. The data set and fit with systematic error is shown in blue and the "raw" data set without this error and fit are shown in orange.

shown in Table. 3. Because one expects the final value to be constant, one estimates what the measured value

	a units	b units	c units
Adjusted Systematic Error	1.20715 ± 0.15434	$(-3.96389 \pm 3.19006) \times 10^{-1}$	$(5.64687 \pm 1.38046) \times 10^{-1}$
Only Statistical Error	0	0	$(4.87767 \pm 0.54338) \times 10^{-1}$

TABLE 3. Best fit parameters table of statistically adjusting systematic error ex post facto. One sees that the "exact" value of $\frac{1}{2}$ is within error of both calculations, however by not including systematic error, the uncertainty in c, given by δc is approximately reduced by a third.

should be by taking c, which is the constant term of the fitting equation: all other terms very and hence must be attributed to systematic error of some sort. Thus, one can, vis à vis purely statistical methods attempt to recover what the measurement should be ex post facto, but the fitting error in this case has approximately tripled. I hope, now, that the reader sees that it is much easier to just "clean the table" between trials then perform a statistical analysis by which to correct systematic error.

3. Error Propagation

Now that we have established that we will have to deal with some sort of error in this course, now that we have discussed the major categories of this error, we can finally ask the question of how this error propagates. One does not typically directly measure the quantity you are interested in, call this f: in general f is a function of many other variables, call these[7] $x_1, \cdots, x_n$. Thus f can be stated $f = f(\{x_k\})$. Now each x_k has some uncertainty

[7]A notation note: quite often I will write the finite set $x_1, x_2,, \cdots, x_n$ in the shorthand of $\{x_k\}_{k=1}^n$ often written $\{x_k\}$ where it is assumed I am summing over all appropriate values of k. Additionally, note that there are n total elements and writing an index of k is akin to select any one of the n variables to think about a propos some example.

δx_k, so one would like to calculate how all of these errors $\{\delta x_k\}$ contribute to the uncertainty in f denoted δf. It is possible that one δx_k contributes greatly to the error δf and another one, δx_ℓ contributes very little. How does one answer this question, so, the final value of f can be reported in scientific notation $(f \pm \delta f) \times 10^k$ units?

3.1. **An Instructive Example.** Consider the following problem

Problem. Let X be a square with length $\ell = (3.72 \pm 0.34)$ m and width $\varpi = (3.14 \pm 0.76)$ m. What is the uncertainty in the area $A(\ell, \varpi) = \ell \cdot \varpi$?

Solution. Our goal is to construct some upper bound on what the error δA is given the errors $\delta\ell = 0.34\,\mathrm{m}$ and $\delta\varpi = 0.76\,\mathrm{m}$. Given the simplicity of the problem, one could attempt to do this geometrically. First, fix the width and look at the two related rectangles of $A(\ell + \delta\ell, \varpi)$ and $A(\ell - \delta\ell, \varpi)$ as shown in Fig. 3.1. Thus one has

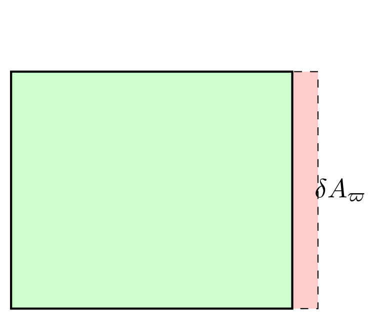

FIGURE 3.1. A geometric interpretation of the effect of error on a rectangle while holding one side constant. The additional errors in area while holding ϖ, ℓ constant are $\delta A_\varpi, \delta A_\ell$ respectively.

these additional areas $\delta A_\varpi, \delta A_\ell$: is it possible to find an analytic solution for these based on only on the function $A(\ell, \varpi)$? The answer is yes! Remember, from multivariate calculus that

$$\left.\frac{\partial A}{\partial \ell}\right|_{(\ell,\varpi)} = \lim_{\delta\ell \to 0} \frac{A(\ell + \delta\ell, \varpi) - A(\ell, \varpi)}{\delta\ell} \tag{3.1}$$

now, if one relaxes Eq. 3.1 so one does not take $\delta\ell \to 0$ but lets $\delta\ell$ be "small", usually written as $\delta\ell \ll 1$, then one deduces

$$\begin{aligned} \left.\frac{\partial A}{\partial \ell}\right|_{(\ell,\varpi)} \delta\ell &= A(\ell + \delta\ell, \varpi) - A(\ell, \varpi) \\ &= \delta A_\varpi \end{aligned} \tag{3.2}$$

where one uses the tautological fact that $A(\ell + \delta\ell, \varpi) - A(\ell, \varpi)$ is exactly the red shaded region δA_ϖ in Fig. 3.1, expressed algebraically. Finally, for plus or minus $(\pm)$ bounds, one only is concerned with the absolute value

$$\|\delta A_\varpi\| = \left\| \left.\frac{\partial A}{\partial \ell}\right|_{(\ell,\varpi)} \delta\ell \right\| \tag{3.3}$$

A similar calculation can be performed by taking $\ell \leftrightarrow \varpi$. Now, the total differential δA can be calculated as

$$\delta A = \left.\frac{\partial A}{\partial \ell}\right|_{(\ell,\varpi)} \delta\ell + \left.\frac{\partial A}{\partial \varpi}\right|_{(\ell,\varpi)} \delta\varpi \tag{3.4}$$

and the absolute value of δA can be calculated via the Pythagorean Theorem as

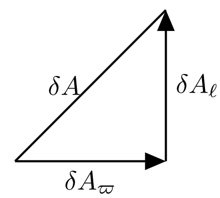

FIGURE 3.2. Diagram of error contributions δA_ℓ and δA_ϖ contributing to δA

$$\|\delta A\| = \sqrt{\left(\left.\frac{\partial A}{\partial \ell}\right|_{(\ell,\varpi)} \delta\ell\right)^2 + \left(\left.\frac{\partial A}{\partial \varpi}\right|_{(\ell,\varpi)} \delta\varpi\right)^2} \tag{3.5}$$

Thus, having derived Eq. 3.5 one can solve the problem

$$\begin{aligned}
\delta A &= \sqrt{\left(\left.\frac{\partial A}{\partial \ell}\right|_{(\ell,\varpi)} \delta\ell\right)^2 + \left(\left.\frac{\partial A}{\partial \varpi}\right|_{(\ell,\varpi)} \delta\varpi\right)^2} \\
&= \sqrt{(\varpi \cdot \delta\ell)^2 + (\ell \cdot \delta\varpi)^2} \\
&= \sqrt{(3.14\,\mathrm{m} \cdot 0.34\,\mathrm{m})^2 + (3.72\,\mathrm{m} \cdot 0.76\,\mathrm{m})^2} \\
&= 4.61\,\mathrm{m}^2
\end{aligned}$$

So, finally, the area of the rectangle can be reported as $(1.17 \pm 0.46) \times 10^1\,\mathrm{m}^2$ in scientific notation.

Remark. Note how I carefully did the previous example in 4 lines: this is *EXACTLY* how I want you to do calculations. The first line is the general equation from the theory, the next line is *symbolically* plugging in for expressions in the first line. The third line is plugging in numbers, *with significant figures and units,* into the calculation. The final line is the overall numerical answer with *significant figures and units!*

3.2. **The General Error Propagation Equation.** Having seen the derivation of the previous subsection, I will now state the general formula to be utilized for error propagation. Assume one wants to measure a quantity $f\left(\{x_k\}_{k=1}^n\right)$, which is a function of n independent variables x_k, each with associated uncertainty δx_k. The absolute error in f, denoted as δf is given by the equation

$$\delta f = \sqrt{\sum_{k=1}^{n} \left(\left.\frac{\partial f}{\partial x_k}\right|_{\mathbf{x}} \delta x_k\right)^2} \tag{3.6}$$

I think it would be helpful to do another, but more complicated example problem:

Problem. Measure the mass of the Earth

Solution. The first man to determine the mass of the Earth was British scientist Henry Cavendish in 1798 which he did by measuring a value for Newton's gravitational constant G utilizing a torsion system consisting of lead masses. Starting with Newton's (Universal) Law of Gravitation one has

$$\mathbf{F}_g = \frac{GM_\oplus m}{r_\oplus^2}\hat{\mathbf{r}} \tag{3.7}$$

where $M_\oplus, r_\oplus$ are mass and radius of the earth respectively. If one considers a test particle of mass m at the surface of the earth, one can write[8]

$$g = \frac{GM_\oplus}{r_\oplus^2} \tag{3.8}$$

reaping

$$M_\oplus = \frac{gr_\oplus^2}{G} \tag{3.9}$$

so as a function $M_\oplus(g, r_\oplus, G)$ expresses the mass of the earth. One looks up[9], $g = (9.78033 \pm 0.04892)\ \mathrm{m/s^2}$, $r_\oplus = (6.378137 \pm 0.255126) \times 10^6\,\mathrm{m}$, $G = (6.67408 \pm 0.00467) \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$and calculates

$$\begin{aligned} M_\oplus &= \frac{(9.78033\,\mathrm{m/s^2}) \cdot (6.37813 \times 10^6\,\mathrm{m})^2}{(6.67408 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}})} \\ &= 5.96141 \times 10^{24}\,\mathrm{kg} \end{aligned}$$

What is the uncertainty? One applies Eq. 3.6 to find

$$\begin{aligned} \delta M_\oplus &= \sqrt{\left(\frac{\partial M_\oplus}{\partial g}\bigg|_{(g,r_\oplus,G)} \delta g\right)^2 + \left(\frac{\partial M_\oplus}{\partial r_\oplus}\bigg|_{(g,r_\oplus,G)} \delta r_\oplus\right)^2 + \left(\frac{\partial M_\oplus}{\partial G}\bigg|_{(g,r_\oplus,G)} \delta G\right)^2} \\ &= \sqrt{\left(\frac{r_\oplus^2}{G}\delta g\right)^2 + \left(\frac{2gr_\oplus}{G}\delta r_\oplus\right)^2 + \left(-2\frac{gr_\oplus^2}{G}\delta G\right)^2} \\ &= \sqrt{\left(\frac{(6.37813 \times 10^6\,\mathrm{m})^2}{(6.67408 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}})} \cdot (0.00467\,\mathrm{m^3\,kg^{-1}\,s^{-2}})\right)^2 + \left(\frac{2 \cdot (9.78033\,\mathrm{m/s^2}) \cdot (6.37813 \times 10^6\,\mathrm{m})}{(6.67408 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}})} \cdot (0.25512\,\mathrm{m})\right)^2} \\ &\quad \overline{\cdots + \left(-2\frac{(9.78033\,\mathrm{m/s^2}) \cdot (6.37813 \times 10^6\,\mathrm{m})^2}{(6.67408 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}})^2} \cdot (0.00467\,\mathrm{m^3\,kg^{-1}\,s^{-2}})\right)^2} \\ &= 5.57278 \times 10^{22}\,\mathrm{kg} \end{aligned}$$

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[8]For those interested in general relativity (GR), g is the surface gravity of the earth. In fact, for $\mathbf{F}_g = m\mathbf{a}_g = \frac{GM_\oplus m}{r_\oplus^2}\hat{\mathbf{r}}$ the fact that the inertial mass m (on the left hand side) is exactly equal to the gravitational mass m on the right hand side and cancel exactly, is an example of the *weak equivalence principle* of GR.

[9]Here I have utilized the equatorial values of the independent variables, noting that the earth is not exactly a sphere, but is an oblique spheroid (ellipsoid).

so one can report the mass of the earth, $M_\oplus$, in scientific notation as $(5.96141 \pm 0.05573) \times 10^{24}$ kg.

How does one determine what the precision uncertainties, δx_k, are? In general there are two ways: the first is from the precision of the measuring apparatus and the second is from statistical analysis. *If at all possible, all δx_k should be determined by the precision of the measuring apparatus.* Only if this first method is not available should the δx_k be determined statistically, either through fit uncertainty of by a variance calculation.

The best way to determine the precision of an apparatus is to look up the technical specifications on the manufacturer's website, which will give an exact value determined by the actual engineers who build the detection apparatus. If this is not available, one can use $\pm$ the smallest unit of measurement on the device: for example on a meter stick this is millimeters. There are some devices with slide rule like bars which allow estimation to one tenth the smallest unit of precision, many calipers, spectroscopes and sphereometers have this. As for statistical methods, one simply calculates the standard deviation or variance on the data set or uses uncertainty in fit parameters as a δx_k.

3.3. **Absolute Versus Relative Error.** The last topic I would like to discuss about error propagation is the difference between absolute and relative error. Absolute error is when an error bar with units is given: the usual scientific notation of $x \pm \delta x$ is an example of this. Because δx is to be added or subtracted from the reported value of x it must have the same units as x. This gives, independent of the value of x what the uncertainty is. On the other hand one has relative error which is usually reported as a percentage calculated by $\left|\frac{\delta x}{x}\right|$: note that this quantity is dimensionless. In terms of utility, relative error is inferior to absolute error because the uncertainty is dependent upon the final value for x. *In this class, unless otherwise stated, all error values must be reported with absolute error!*

4. Conclusion

So, what have we all covered here in this primer on error? First, although it might be possible for there to ontologically exist physical quantities which can be measured without error, we will not find any of these in this laboratory class; thus *you will always have to deal with measurement uncertainty* for any measurement you perform. Most of this uncertainty will come from precision limitations on the laboratory equipment being used.

There are three categories of error which you will have to deal with: systematic error, statistical (random) error, and fitting error. Systematic errors can be corrected by properly calibrating lab equipment, using good laboratory technique, and a posteriori statistical adjustments if necessary. Determining which physical phenomena cause systematic error of various forms is one of the most difficult problems to solve in the laboratory. *In my class the phrase "HUMAN ERROR" is profane: do not use it!* Next, for a multitude of complicated reasons, some of which will we be studying this semester, there will always be statistical error which will appear in measurements. Finally, especially for data sets with a sizable variance, if a computer algebra system is utilized to fit experimental data to a curve then there will be errors in the best fit parameters corresponding to multiple curves which could be fit to the data. Fit error can be artificially low - $\left(\mathcal{O}\left(10^{-15}\right)-\mathcal{O}\left(10^{-17}\right)\right)$ - so it this is the case consider using the statistical variance for error propagation instead.

Finally, one is usually interested in reporting a function $f\left(\{x_i\}\right)$, of measured quantities $\{x_i\}$ with errors $\{\delta x_i\}$, so one needs a way to calculate how measurement errors propagate to an absolute error $\delta f\left(\{x_i\},\{\delta x_i\}\right)$. A method of differentials is used to derive Eq. 3.6,

$$\delta f=\sqrt{\sum_{k=1}^{n}\left(\left.\frac{\partial f}{\partial x_k}\right|_{\mathbf{x}}\delta x_k\right)^2}$$

which has been restated for the reader's convenience. One must somehow determine what the $\{\delta x_i\}$ are: the best ways to do this are by looking up technical specifications of laboratory equipment from the equipment manufacturer, using the maximum precision based on examining the equipment its self, using the variance from statistics on experimental data sets, and finally, if all else fails, looking at fit error if appropriate.

I now believe that all PHYS 252 students have a basic primer of common knowledge regarding error and error propagation appropriate for this class. This is just an introduction, however: as you pursue your careers in physics and make great discoveries hopefully you won't have as much error in your error and less variation in your uncertainty!

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