Задача. Нека $\boldsymbol{a}_1 = (1,1,1)$, $\boldsymbol{a}_2 = (1,1,-1)$, $\boldsymbol{b} = (1,2,3)$ са вектори от F^3, $U = l(\boldsymbol{a}_1, \boldsymbol{a}_2)$, $V = l(\boldsymbol{b})$. В стандартния базис $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ да се напише матрицата на такъв линеен оператор φ, за който $\mathrm{Ker}\varphi = U$ и $\mathrm{Im}\varphi = V$.

Решение. Имаме
$$\begin{array}{c} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \end{array}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & -2 \end{pmatrix}$$
и значи $\boldsymbol{a}_1, \boldsymbol{a}_2$ са линейно независими и базис на U. Полагаме $\boldsymbol{a}_3 = (0,1,0)$ и тогава $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ е базис на F^3. Тогава, ако $\varphi \in \mathrm{Hom}F^3$ и $\varphi(\boldsymbol{a}_1) = \varphi(\boldsymbol{a}_2) = \boldsymbol{0}$, $\varphi(\boldsymbol{a}_3) = \boldsymbol{b}$, то $\mathrm{Ker}\varphi = U$ и $\mathrm{Im}\varphi = V$. Действително, нека $\boldsymbol{v} \in F^3$ и $\boldsymbol{v} = \lambda_1 \boldsymbol{a}_1 + \lambda_2 \boldsymbol{a}_2 + \lambda_3 \boldsymbol{a}_3$, $\lambda_1, \lambda_2, \lambda_3 \in F$. Тогава
$$\boldsymbol{v} \in \mathrm{Ker}\varphi \iff \varphi(\boldsymbol{v}) = \lambda_1 \varphi(\boldsymbol{a}_1) + \lambda_2 \varphi(\boldsymbol{a}_2) + \lambda_3 \varphi(\boldsymbol{a}_3) = \lambda_3 \boldsymbol{b} = \boldsymbol{0} \iff \lambda_3 = 0$$
и значи $\mathrm{Ker}\varphi = U$.
$$\mathrm{Im}\varphi = l(\varphi(\boldsymbol{a}_1), \varphi(\boldsymbol{a}_2), \varphi(\boldsymbol{a}_3)) = l(\boldsymbol{b}) = V.$$
Имаме
$$\begin{array}{lcccccl} \varphi(\boldsymbol{a}_1) = \varphi(\boldsymbol{e}_1) & + & \varphi(\boldsymbol{e}_2) & + & \varphi(\boldsymbol{e}_3) & = & (0,0,0) \\ \varphi(\boldsymbol{a}_2) = \varphi(\boldsymbol{e}_1) & + & \varphi(\boldsymbol{e}_2) & - & \varphi(\boldsymbol{e}_3) & = & (0,0,0) \\ \varphi(\boldsymbol{a}_3) = & & \varphi(\boldsymbol{e}_2) & & & = & (1,2,3) \end{array}$$
$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & (0 & 0 & 0) \\ 1 & 1 & -1 & (0 & 0 & 0) \\ 0 & 1 & 0 & (1 & 2 & 3) \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & (0 & 0 & 0) \\ 0 & 0 & 1 & (0 & 0 & 0) \\ 0 & 1 & 0 & (1 & 2 & 3) \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & (-1 & -2 & -3) \\ 0 & 0 & 1 & (0 & 0 & 0) \\ 0 & 1 & 0 & (1 & 2 & 3) \end{array}\right)$$
и следователно
$$\begin{array}{lcl} \varphi(\boldsymbol{e}_1) & = & (-1,-2,-3) \\ \varphi(\boldsymbol{e}_2) & = & (1,2,3) \\ \varphi(\boldsymbol{e}_3) & = & (0,0,0) \end{array}$$
и следователно матрицата на φ в базиса $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ е $A = \begin{pmatrix} -1 & 1 & 0 \\ -2 & 2 & 0 \\ -3 & 3 & 0 \end{pmatrix}$.

Задача. В линейното пространство $F^4[x]$ на полиномите с коефициенти от F от степен, ненадминаваща 3, е дадено изображението $\tau : F^4[x] \to F^4[x]$, действащо по правилото $\tau(f(x)) = f(x+1)$.

а) Да се докаже, че τ е обратим линеен оператор и да се намери неговият обратен.

б) Да се намери матрицата на линейния оператор τ спрямо базиса $1, x, x^2, x^3$ на $F^4[x]$.

Решение. а) Нека $f, g \in F^4[x]$, $\lambda \in F$ Тогава
$$\begin{array}{ccccccc} \tau((f+g)(x)) & = & (f+g)(x+1) & = & f(x+1) + g(x+1) & = & \tau(f(x)) + \tau(g(x)) \\ \tau((\lambda f)(x) & = & (\lambda f)(x+1) & = & \lambda f(x+1) & = & \lambda \tau(f(x)) \end{array}$$
и следователно τ е линеен оператор. Нека $\tau' : F^4[x] \to F^4[x]$, действащо по правилото $\tau'(f(x)) = f(x-1)$. Тогава
$$\begin{array}{ccccccccc} \tau'\tau(f(x)) & = & \tau'(f(x+1)) & = & f((x-1)+1) & = & f(x) & = & \varepsilon(f(x)) \\ \tau\tau'(f(x)) & = & \tau(f(x-1)) & = & f((x+1)-1) & = & f(x) & = & \varepsilon(f(x)) \end{array}.$$
Следователно $\tau'\tau = \tau\tau' = \varepsilon$ и значи τ е обратим и $\tau^{-1} = \tau'$.

б) Имаме
$$\begin{array}{ccccccccccc} \tau(1) & = & & & 1 & & & & & & \\ \tau(x) & = & x+1 & = & 1 & + & x & & & & \\ \tau(x^2) & = & (x+1)^2 & = & 1 & + & 2x & + & x^2 & & \\ \tau(x^3) & = & (x+1)^3 & = & 1 & + & 3x & + & 3x^2 & +x^3 & \end{array}$$
и следователно матрицата на φ в базиса $1, x, x^2, x^3$ е $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

Задача. Нека V е крайномерно линейно пространство над F. Нека $\varphi_1, \varphi_2, \psi \in \mathrm{HomV}$ са такива, че

$$\varphi_1\varphi_2 + \psi = \varepsilon$$

и

$$\psi(\boldsymbol{v}) = \boldsymbol{v} \iff \boldsymbol{v} = \boldsymbol{0}.$$

Да се докаже, че операторите φ_1 и φ_2 са обратими.

Решение. Нека $\dim V = n$, $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ е базис на V и φ_1, φ_2, ψ имат съответно матрици A, B, C в този базис. Тогава $AB + C = E$. От $\psi(\boldsymbol{v}) = \boldsymbol{v} \iff \boldsymbol{v} = \boldsymbol{0}$ следва, че 1 не е собствена стойност на ψ, т.е. $f_C(1) = \det(C - E) \neq 0$. Сега от $C - E = -AB$ следва $0 \neq \det(C - E) = (-1)^n \det(AB) = (-1)^n \det A \det B$. Следователно $\det A \neq 0$, $\det B \neq 0$ и значи φ_1, φ_2 са обратими.

Задача. Нека $\varphi : V \to V$ е линеен оператор в крайномерно пространство V, такъв че $\varphi^2 = 4\varepsilon$. Да се докаже, че:
а) φ е обратим линеен оператор;

Решение. Имаме $\left(\frac{1}{4}\varphi\right)\varphi = \varphi\left(\frac{1}{4}\varphi\right) = \frac{1}{4}\varphi^2 = \frac{1}{4}.4\varepsilon = \varepsilon$ и следователно φ е обратим и $\varphi^{-1} = \frac{1}{4}\varphi$.

б) ако λ е собствена стойност на φ, то $\lambda = \pm 2$.

Решение. Нека $\boldsymbol{v}$ е собствен вектор, съответстващ на λ. Тогава $\varphi(\boldsymbol{v}) = \lambda\boldsymbol{v}$ и $4\boldsymbol{v} = \varphi^2(\boldsymbol{v}) = \varphi(\varphi(\boldsymbol{v})) = \varphi(\lambda\boldsymbol{v}) = \lambda\varphi(\boldsymbol{v}) = \lambda^2\boldsymbol{v}$. Следователно $(\lambda^2 - 4)\boldsymbol{v} = \boldsymbol{0}$ и тъй като $\boldsymbol{v} \neq \boldsymbol{0}$, то $\lambda^2 = 4$, т.е. $\lambda = \pm 2$.

в) $\mathrm{Ker}(\varphi + 2\varepsilon) \cap \mathrm{Ker}(\varphi - 2\varepsilon) = \{\boldsymbol{0}\}$.

Решение. Нека $\boldsymbol{v} \in \mathrm{Ker}(\varphi + 2\varepsilon) \cap \mathrm{Ker}(\varphi - 2\varepsilon)$. Тогава

$$\begin{aligned}(\varphi + 2\varepsilon)(\boldsymbol{v}) &= \varphi(\boldsymbol{v}) + 2\boldsymbol{v} = \boldsymbol{0}, \text{ т.е. } \varphi(\boldsymbol{v}) = -2\boldsymbol{v},\\ (\varphi - 2\varepsilon)(\boldsymbol{v}) &= \varphi(\boldsymbol{v}) - 2\boldsymbol{v} = \boldsymbol{0}, \text{ т.е. } \varphi(\boldsymbol{v}) = 2\boldsymbol{v}\end{aligned}$$

и следователно $\boldsymbol{v} = \boldsymbol{0}$.

г) $\mathrm{Im}(\varphi + 2\varepsilon) = \mathrm{Im}((\varphi + 2\varepsilon)^2)$.

Решение. Нека $\boldsymbol{v} \in V$. Тогава

$$(\varphi + 2\varepsilon)^2(\boldsymbol{v}) = ((\varphi + 2\varepsilon)(\varphi(\boldsymbol{v}) + 2\boldsymbol{v}) = \varphi(\varphi(\boldsymbol{v}) + 2\boldsymbol{v}) + 2(\varphi(\boldsymbol{v}) + 2\boldsymbol{v}) = 4\boldsymbol{v} + 2\varphi(\boldsymbol{v}) + 2\varphi(\boldsymbol{v}) + 4\boldsymbol{v} = (4(\varphi + 2\varepsilon))(\boldsymbol{v}).$$

Тогава за $\boldsymbol{u} \in V$ имаме

$$\begin{aligned}\boldsymbol{u} \in \mathrm{Im}((\varphi + 2\varepsilon)^2) &\iff \boldsymbol{u} = (\varphi + 2\varepsilon)^2(\boldsymbol{v}) \text{ за някое } \boldsymbol{v} \in V\\ &\iff \boldsymbol{u} = 4(\varphi + 2\varepsilon)(\boldsymbol{v}) \text{ за някое } \boldsymbol{v} \in V\\ &\iff \frac{1}{4}\boldsymbol{u} \in \mathrm{Im}(\varphi + 2\varepsilon)\\ &\iff \boldsymbol{u} \in \mathrm{Im}(\varphi + 2\varepsilon)\end{aligned}$$

и следователно $\mathrm{Im}((\varphi + 2\varepsilon)^2) = \mathrm{Im}(\varphi + 2\varepsilon)$.

д) $\mathrm{Ker}(\varphi - 2\varepsilon) = \mathrm{Im}(\varphi + 2\varepsilon)$.

Решение. Нека $\boldsymbol{e}_1, \dots, \boldsymbol{e}_k$ е базис на $\mathrm{Ker}(\varphi + 2\varepsilon)$. Допълваме $\boldsymbol{e}_1, \dots, \boldsymbol{e}_k$ до базис $\boldsymbol{e}_1, \dots, \boldsymbol{e}_k, \boldsymbol{e}_{k+1}, \dots, \boldsymbol{e}_n$ на V. Тогава $(\varphi + 2\varepsilon)(\boldsymbol{e}_{k+1}), \dots, (\varphi + 2\varepsilon)(\boldsymbol{e}_n)$ е базис на $\mathrm{Im}(\varphi + 2\varepsilon)$. Тъй като $(\varphi - 2\varepsilon)(\varphi + 2\varepsilon) = \boldsymbol{0}$, то $(\varphi + 2\varepsilon)(\boldsymbol{e}_{k+1}), \dots, (\varphi + 2\varepsilon)(\boldsymbol{e}_n) \in \mathrm{Ker}(\varphi - 2\varepsilon)$ и значи $\mathrm{Im}(\varphi + 2\varepsilon) \leq \mathrm{Ker}(\varphi - 2\varepsilon)$. Следователно $r(\varphi + 2\varepsilon) \leq d(\varphi - 2\varepsilon)$. От друга страна, съгласно подточка в), $d(\varphi + 2\varepsilon) + d(\varphi - 2\varepsilon) \leq n$ и значи $d(\varphi - 2\varepsilon) \leq n - d(\varphi + 2\varepsilon) = r(\varphi + 2\varepsilon)$. Оттук $r(\varphi + 2\varepsilon) = d(\varphi - 2\varepsilon)$ и следователно $\mathrm{Ker}(\varphi - 2\varepsilon) = \mathrm{Im}(\varphi + 2\varepsilon)$.

Евклидови пространства.

Задача. Проверете, че векторите $\boldsymbol{a}_1 = (1, -3, -1, 1)$, $\boldsymbol{a}_2 = (5, 0, 1, -4)$ от $\mathbb{R}^4$ са ортогонални и да се допълни системата до ортогонален базис на $\mathbb{R}^4$.

Решение. $(\boldsymbol{a}_1, \boldsymbol{a}_2) = 1.5 - 3.0 - 1.1 + 1.(-4) = 0$, следователно $\boldsymbol{a}_1, \boldsymbol{a}_2$ са ортогонални. Търсим всички вектори $\boldsymbol{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$, такива че $(\boldsymbol{a}_1, \boldsymbol{x}) = (\boldsymbol{a}_2, \boldsymbol{x}) = 0$, т.е.

$$\left| \begin{array}{lcccccccc} x_1 & - & 3x_2 & - & x_3 & + & x_4 & = & 0 \\ 5x_1 & & & + & x_3 & - & 4x_4 & = & 0 \end{array} \right. .$$

Полагаме $x_1 = p$, $x_4 = q$, тогава $x_3 = -5p + 4q$, $x_2 = \dfrac{p + 5p - 4q + q}{3} = \dfrac{6p - 3q}{3} = 2p - q$ и

$$\{(p, 2p - q, -5p + 4q, q) \mid p, q \in \mathbb{R}\}.$$

При $p = 1$, $q = 0$ получаваме $\boldsymbol{a}_3 = (1, 2, -5, 0)$ — ортогонален на $\boldsymbol{a}_1$ и $\boldsymbol{a}_2$. Търсим вектор $\boldsymbol{a}_4$ във вида $\boldsymbol{a}_4 = (p, 2p - q, -5p + 4q, q)$, така че

$$0 = (\boldsymbol{a}_3, \boldsymbol{a}_4) = p + 4p - 2q + 25p - 20q = 30p - 22q,$$

т.е. $p = \dfrac{11}{15} q$. Следователно при $q = 15$ имаме $p = 11$ и $\boldsymbol{a}_4 = (11, 7, 5, 15)$ — ортогонален на $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$.

Задача. Докажете, че векторите $\mathbf{a}_1 = (2, 1, 1, -1)$, $\mathbf{a}_2 = (5, 0, 3, -1)$, $\mathbf{a}_3 = (-3, -10, 3, 8)$, $\mathbf{a}_4 = (-1, -1, 1, 5)$ са базис на $\mathbb{R}^4$. Да се ортогонализират тези вектори по метода на Грам-Шмид и да се намери съответния ортонормиран базис.

Решение.

$$\begin{vmatrix} 2 & 1 & 1 & -1 \\ 5 & 0 & 3 & -1 \\ -3 & -10 & 3 & 8 \\ -1 & -1 & 1 & 5 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 1 & -1 \\ 5 & 0 & 3 & -1 \\ 17 & 0 & 13 & -2 \\ 1 & 0 & 2 & 4 \end{vmatrix} = 1(-1)^{1+2} \begin{vmatrix} 5 & 3 & -1 \\ 17 & 13 & -2 \\ 1 & 2 & 4 \end{vmatrix} =$$

$$- \begin{vmatrix} 5 & 3 & -1 \\ 7 & 7 & 0 \\ 21 & 14 & 0 \end{vmatrix} = -1.(-1)(-1)^{1+3} 49 \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} = -49 \neq 0$$

Следователно векторите $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$, $\boldsymbol{a}_4$ са линейно независими и значи образуват базис на $\mathbb{R}^4$.

Полагаме $\boldsymbol{b}_1 = \boldsymbol{a}_1 = (2, 1, 1, -1)$. Търсим $\boldsymbol{b}_2$ във вида $\boldsymbol{b}_2 = \boldsymbol{a}_2 + \lambda \boldsymbol{b}_1$, така че

$$0 = (\boldsymbol{b}_2, \boldsymbol{b}_1) = (\boldsymbol{a}_2 + \lambda \boldsymbol{b}_1, \boldsymbol{b}_1) = (\boldsymbol{a}_2, \boldsymbol{b}_1) + \lambda (\boldsymbol{b}_1, \boldsymbol{b}_1),$$

оттук

$$\lambda = -\frac{(\boldsymbol{a}_2, \boldsymbol{b}_1)}{(\boldsymbol{b}_1, \boldsymbol{b}_1)} = -\frac{10 + 0 + 3 + 1}{7} = -2.$$

Следователно

$$\boldsymbol{b}_2 = (5, 0, 3, -1) - 2(2, 1, 1, -1) = (1, -2, 1, 1).$$

Търсим $\boldsymbol{b}_3$ във вида $\boldsymbol{b}_3 = \boldsymbol{a}_3 + \lambda_1 \boldsymbol{b}_1 + \lambda_2 \boldsymbol{b}_2$, така че

$$\begin{array}{ccccccc} 0 & = & (\boldsymbol{b}_3, \boldsymbol{b}_1) & = & (\boldsymbol{a}_3 + \lambda_1 \boldsymbol{b}_1 + \lambda_2 \boldsymbol{b}_2, \boldsymbol{b}_1) & = & (\boldsymbol{a}_3, \boldsymbol{b}_1) + \lambda_1 (\boldsymbol{b}_1, \boldsymbol{b}_1) + \lambda_2 \overbrace{(\boldsymbol{b}_2, \boldsymbol{b}_1)}^{=0} \\ 0 & = & (\boldsymbol{b}_3, \boldsymbol{b}_2) & = & (\boldsymbol{a}_3 + \lambda_1 \boldsymbol{b}_1 + \lambda_2 \boldsymbol{b}_2, \boldsymbol{b}_2) & = & (\boldsymbol{a}_3, \boldsymbol{b}_2) + \lambda_1 \underbrace{(\boldsymbol{b}_1, \boldsymbol{b}_2)}_{=0} + \lambda_2 (\boldsymbol{b}_2, \boldsymbol{b}_2), \end{array}$$

т.е.

$$\begin{array}{rcl} \lambda_1 & = & -\dfrac{(\boldsymbol{a}_3, \boldsymbol{b}_1)}{(\boldsymbol{b}_1, \boldsymbol{b}_1)} = -\dfrac{-6 - 10 + 3 - 8}{7} = 3 \\ \lambda_2 & = & -\dfrac{(\boldsymbol{a}_3, \boldsymbol{b}_2)}{(\boldsymbol{b}_2, \boldsymbol{b}_2)} = -\dfrac{-3 + 20 + 3 + 8}{7} = -4. \end{array}$$

Следователно

$$\boldsymbol{b}_3 = (-3,-10,3,8) + 3(2,1,1,-1) - 4(1,-2,1,1) = (-1,1,2,1).$$

Търсим $\boldsymbol{b}_4 = \boldsymbol{a}_4 + \mu_1\boldsymbol{b}_1 + \mu_2\boldsymbol{b}_2 + \mu_3\boldsymbol{b}_3$, където

$$\begin{aligned}
\mu_1 &= -\frac{(\boldsymbol{a}_4,\boldsymbol{b}_1)}{(\boldsymbol{b}_1,\boldsymbol{b}_1)} = -\frac{-2-1+1-5}{7} = 1\\
\mu_2 &= -\frac{(\boldsymbol{a}_4,\boldsymbol{b}_2)}{(\boldsymbol{b}_2,\boldsymbol{b}_2)} = -\frac{-1+2+1+5}{7} = -1\\
\mu_3 &= -\frac{(\boldsymbol{a}_4,\boldsymbol{b}_3)}{(\boldsymbol{b}_3,\boldsymbol{b}_3)} = -\frac{1-1+2+5}{7} = -1
\end{aligned}$$

Следователно

$$\boldsymbol{b}_4 = (-1,-1,1,5) + (2,1,1,-1) - (1,-2,1,1) - (-1,1,2,1) = (1,1,-1,2).$$

Векторите $\boldsymbol{b}_1,\boldsymbol{b}_2,\boldsymbol{b}_3,\boldsymbol{b}_4$ са ортогонален базис на $\mathbb{R}^4$. След нормирането на тези вектори, получаваме търсения ортонормиран базис на $\mathbb{R}^4$, а именно

$$\begin{aligned}
\boldsymbol{e}_1 &= \frac{\boldsymbol{b}_1}{|\boldsymbol{b}_1|} = \frac{1}{\sqrt{7}}(2,1,1,-1),\\
\boldsymbol{e}_2 &= \frac{\boldsymbol{b}_2}{|\boldsymbol{b}_2|} = \frac{1}{\sqrt{7}}(1,-2,1,1),\\
\boldsymbol{e}_3 &= \frac{\boldsymbol{b}_3}{|\boldsymbol{b}_3|} = \frac{1}{\sqrt{7}}(-1,1,2,1),\\
\boldsymbol{e}_4 &= \frac{\boldsymbol{b}_4}{|\boldsymbol{b}_4|} = \frac{1}{\sqrt{7}}(1,1,-1,2)
\end{aligned}$$

Задача. Да се построи по метода на Грам-Шмид ортогонален базис на $U = l(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4)$, където $\mathbf{a}_1 = (1,-2,2,2)$, $\mathbf{a}_2 = (-1,9,-5,-5)$, $\mathbf{a}_3 = (1,5,-1,-1)$, $\mathbf{a}_4 = (1,12,-3,-5)$.

Решение.

$$\begin{array}{c} \boldsymbol{a}_1\\ \boldsymbol{a}_2\\ \boldsymbol{a}_3\\ \boldsymbol{a}_4 \end{array}
\begin{pmatrix} 1 & -2 & 2 & 2\\ -1 & 9 & -5 & -5\\ 1 & 5 & -1 & -1\\ 1 & 12 & -3 & -5 \end{pmatrix}
\sim
\begin{pmatrix} 1 & -2 & 2 & 2\\ 0 & 7 & -3 & -3\\ 0 & 7 & -3 & -3\\ 0 & 14 & -5 & -7 \end{pmatrix}
\sim
\begin{pmatrix} 1 & -2 & 2 & 2\\ 0 & 7 & -3 & -3\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & -1 \end{pmatrix}$$

Следователно векторите $\boldsymbol{a}_1,\boldsymbol{a}_2,\boldsymbol{a}_4$ са базис на U. Полагаме

$$\boldsymbol{b}_1 = \boldsymbol{a}_1 = (1,-2,2,2).$$

Търсим $\boldsymbol{b}_2 = \boldsymbol{a}_2 + \lambda\boldsymbol{b}_1$, където

$$\lambda = -\frac{(\boldsymbol{a}_2,\boldsymbol{b}_1)}{(\boldsymbol{b}_1,\boldsymbol{b}_1)} = -\frac{-1-18-10-10}{13} = 3.$$

Следователно

$$\boldsymbol{b}_2 = (-1,9,-5,-5) + 3(1,-2,2,2) = (2,3,1,1).$$

Търсим $\boldsymbol{b}_3 = \boldsymbol{a}_4 + \lambda_1\boldsymbol{b}_1 + \lambda_2\boldsymbol{b}_2$, където

$$\begin{aligned}
\lambda_1 &= -\frac{(\boldsymbol{a}_4,\boldsymbol{b}_1)}{(\boldsymbol{b}_1,\boldsymbol{b}_1)} = -\frac{1-24-6-10}{13} = 3;\\
\lambda_2 &= -\frac{(\boldsymbol{a}_4,\boldsymbol{b}_2)}{(\boldsymbol{b}_2,\boldsymbol{b}_2)} = -\frac{2+36-3-5}{15} = -2.
\end{aligned}$$

Следователно

$$\boldsymbol{b}_3 = (1,12,-3,-5) + 3(1,-2,2,2) - 2(2,3,1,1) = (0,0,1,-1).$$

и $\boldsymbol{b}_1,\boldsymbol{b}_2,\boldsymbol{b}_3$ са ортогонален базис на U.

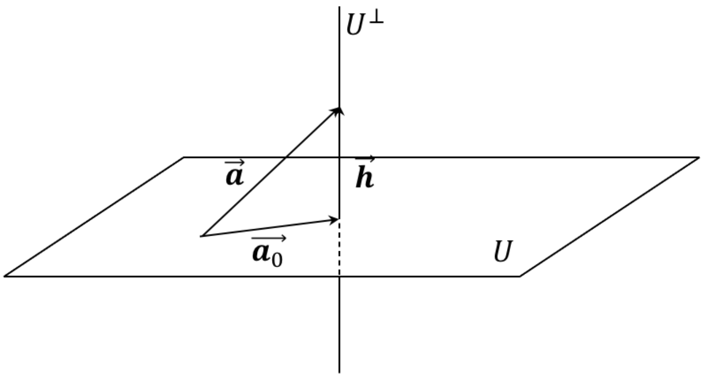
Дефиниция 1. Под ортогонално допълнение на подпространството U на V ще разбираме множеството

$$U^{\perp} = \{\boldsymbol{a} \in V \mid (\boldsymbol{a}, \boldsymbol{u}) = 0 \text{ за всяко } \boldsymbol{u} \in U\}.$$

Твърдение 1. Нека V е крайномерно евклидово пространство и $U \leq V$. Тогава

$$V = U \oplus U^{\perp}.$$

Следствие 1. Нека V е крайномерно евклидово пространство, $U \leq V$ и $\boldsymbol{a} \in V$. Тогава съществуват единствени вектори $\boldsymbol{a}_0 \in U$, $\boldsymbol{h} \in U^{\perp}$, такива че $\boldsymbol{a} = \boldsymbol{a}_0 + \boldsymbol{h}$ ($\boldsymbol{a}_0$ наричаме ортогонална проекция на $\boldsymbol{a}$ върху U, а $\boldsymbol{h}$ — перпендикуляр, спуснат от $\boldsymbol{a}$ към U).



Задача. Нека $U = l(\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3)$, където $\boldsymbol{a}_1 = (1, 1, 1, 1)$, $\boldsymbol{a}_2 = (1, 2, 2, -1)$, $\boldsymbol{a}_3 = (1, 0, 0, 3)$ и нека $\boldsymbol{a} = (4, -1, -3, 4)$. Да се намерят оротогоналната проекция $\boldsymbol{a}_0$ на $\boldsymbol{a}$ върху U и перпендикулярът $\boldsymbol{h}$ от $\boldsymbol{a}$ към U.

Решение. Първо намираме ортогонален базис на U.

$$\begin{array}{c} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \boldsymbol{a}_3 \end{array} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & -1 \\ 1 & 0 & 0 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & -1 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Следователно $\boldsymbol{a}_1, \boldsymbol{a}_2$ са базис на U.

Полагаме $\boldsymbol{b}_1 = \boldsymbol{a}_1 = (1, 1, 1, 1)$. Търсим $\boldsymbol{b}_2 = \boldsymbol{a}_2 + \lambda \boldsymbol{b}_1$, където $\lambda = -\dfrac{(\boldsymbol{a}_2, \boldsymbol{b}_1)}{(\boldsymbol{b}_1, \boldsymbol{b}_1)} = -\dfrac{1+2+2-1}{4} = -1$. Следователно $\boldsymbol{b}_2 = (1, 2, 2, -1) - (1, 1, 1, 1) = (0, 1, 1, -2)$ и $\boldsymbol{b}_1, \boldsymbol{b}_2$ е ортогонален базис на U.

Нека $\boldsymbol{a} = \boldsymbol{a}_0 + \boldsymbol{h}$, $\boldsymbol{a}_0 \in U$, $\boldsymbol{h} \in U^{\perp}$. Нека $\boldsymbol{a}_0 = \lambda_1 \boldsymbol{b}_1 + \lambda_2 \boldsymbol{b}_2$. Координатите λ_1, λ_2 на $\boldsymbol{a}_0$ ще определим от

$$\begin{aligned} 0 = (\boldsymbol{h}, \boldsymbol{b}_1) = (\boldsymbol{a} - \lambda_1 \boldsymbol{b}_1 - \lambda_2 \boldsymbol{b}_2, \boldsymbol{b}_1) = (\boldsymbol{a}, \boldsymbol{b}_1) - \lambda_1 (\boldsymbol{b}_1, \boldsymbol{b}_1) - \lambda_2 \overbrace{(\boldsymbol{b}_2, \boldsymbol{b}_1)}^{=0}, \\ 0 = (\boldsymbol{h}, \boldsymbol{b}_2) = (\boldsymbol{a} - \lambda_1 \boldsymbol{b}_1 - \lambda_2 \boldsymbol{b}_2, \boldsymbol{b}_2) = (\boldsymbol{a}, \boldsymbol{b}_2) - \lambda_1 \underbrace{(\boldsymbol{b}_1, \boldsymbol{b}_2)}_{=0} - \lambda_2 (\boldsymbol{b}_2, \boldsymbol{b}_2), \end{aligned}$$

следователно

$$\begin{aligned} \lambda_1 &= \frac{(\boldsymbol{a}, \boldsymbol{b}_1)}{(\boldsymbol{b}_1, \boldsymbol{b}_1)} = \frac{4-1-3+4}{4} = 1 \\ \lambda_2 &= \frac{(\boldsymbol{a}, \boldsymbol{b}_2)}{(\boldsymbol{b}_2, \boldsymbol{b}_2)} = \frac{0-1-3-8}{6} = -2 \end{aligned}$$

Следователно

$$\begin{array}{rcl} \boldsymbol{a}_0 & = & 1.\boldsymbol{b}_1 - 2.\boldsymbol{b}_2 = (1,1,1,1) - 2(0,1,1,-2) = (1,-1,-1,5) \\ \boldsymbol{h} & = & \boldsymbol{a} - \boldsymbol{a}_0 = (4,-1,-3,4) - (1,-1,-1,5) = (3,0,-2,-1). \end{array}$$

Втори начин. Както по-горе проверяваме, че $\boldsymbol{a}_1, \boldsymbol{a}_2$ са базис на U и нека $\boldsymbol{a} = \boldsymbol{a}_0 + \boldsymbol{h}$, $\boldsymbol{a}_0 \in U$, $\boldsymbol{h} \in U^\perp$. Нека $\boldsymbol{a}_0 = \beta_1\boldsymbol{a}_1 + \beta_2\boldsymbol{a}_2$. Като умножим последователно това равенство скаларно с $\boldsymbol{a}_1, \boldsymbol{a}_2$ и вземем предвид, че $(\boldsymbol{a}_0, \boldsymbol{a}_i) = (\boldsymbol{a} - \boldsymbol{h}, \boldsymbol{a}_i) = (\boldsymbol{a}, \boldsymbol{a}_i) - (\boldsymbol{h}, \boldsymbol{a}_i) = (\boldsymbol{a}, \boldsymbol{a}_i)$, $i = 1, 2$, за координатите β_1, β_2 получаваме

$$\left| \begin{array}{rcl} \beta_1(\boldsymbol{a}_1, \boldsymbol{a}_1) + \beta_2(\boldsymbol{a}_2, \boldsymbol{a}_1) & = & (\boldsymbol{a}, \boldsymbol{a}_1) \\ \beta_1(\boldsymbol{a}_1, \boldsymbol{a}_2) + \beta_2(\boldsymbol{a}_2, \boldsymbol{a}_2) & = & (\boldsymbol{a}, \boldsymbol{a}_2) \end{array} \right.$$

Тъй като $(\boldsymbol{a}_1, \boldsymbol{a}_1) = 4$, $(\boldsymbol{a}_1, \boldsymbol{a}_2) = (\boldsymbol{a}_2, \boldsymbol{a}_1) = 4$, $(\boldsymbol{a}_2, \boldsymbol{a}_2) = 10$, $(\boldsymbol{a}, \boldsymbol{a}_1) = 4$, $(\boldsymbol{a}, \boldsymbol{a}_2) = -8$, системата е

$$\left| \begin{array}{rcrcr} 4\beta_1 & + & 4\beta_2 & = & 4 \\ 4\beta_1 & + & 10\beta_2 & = & -8 \end{array} \right. ,$$

откъдето $\beta_1 = 3$, $\beta_2 = -2$ и значи $\boldsymbol{a}_0 = 3\boldsymbol{a}_1 - 2\boldsymbol{a}_2 = 3(1,1,1,1) - 2(1,2,2,-1) = (1,-1,-1,5)$ и $\boldsymbol{h} = \boldsymbol{a} - \boldsymbol{a}_0 = (3,0,-2,-1)$.

Дефиниция. Нека U е подпространство на крайномерното евклидово пространство V и нека $\boldsymbol{a} \in V$. Ъгъл между вектор $\boldsymbol{a}$ и подпространството U се нарича ъгълът между $\boldsymbol{a}$ и ортогоналната проекция $\boldsymbol{a}_0$ на $\boldsymbol{a}$ върху U.

Задача. Да се намери ъгълът α, който векторът $\boldsymbol{a} = (1,2,3,\sqrt{2})$ сключва с подпространството $U = l(\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3)$ на V, където $\boldsymbol{a}_1 = (1,0,-1,1)$, $\boldsymbol{a}_2 = (3,-1,-2,1)$, $\boldsymbol{a}_3 = (-1,1,0,4)$.

Решение. Първо намираме ортогонален базис на U.

$$\begin{array}{c} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \boldsymbol{a}_3 \end{array} \begin{pmatrix} 1 & 0 & -1 & 1 \\ 3 & -1 & -2 & 1 \\ -1 & 1 & 0 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & -1 & 1 & -2 \\ 0 & 1 & -1 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & -1 & 1 & -2 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Следователно $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ са базис на U.

Полагаме $\boldsymbol{b}_1 = \boldsymbol{a}_1 = (1,0,-1,1)$. Търсим $\boldsymbol{b}_2 = \boldsymbol{a}_2 + \lambda\boldsymbol{b}_1$, където $\lambda = -\dfrac{(\boldsymbol{a}_2, \boldsymbol{b}_1)}{(\boldsymbol{b}_1, \boldsymbol{b}_1)} = -\dfrac{3+0+2+1}{3} = -2$. Следователно

$$\boldsymbol{b}_2 = (3,-1,-2,1) - 2(1,0,-1,1) = (1,-1,0,-1).$$

Търсим $\boldsymbol{b}_3$ във вида $\boldsymbol{b}_3 = \boldsymbol{a}_3 + \mu_1\boldsymbol{b}_1 + \mu_2\boldsymbol{b}_2$, където

$$\begin{array}{rcl} \mu_1 & = & -\dfrac{(\boldsymbol{a}_3, \boldsymbol{b}_1)}{(\boldsymbol{b}_1, \boldsymbol{b}_1)} = -\dfrac{-1-0+0+4}{3} = -1 \\ \mu_2 & = & -\dfrac{(\boldsymbol{a}_3, \boldsymbol{b}_2)}{(\boldsymbol{b}_2, \boldsymbol{b}_2)} = -\dfrac{-1-1+0-4}{3} = 2. \end{array}$$

Следователно

$$\boldsymbol{b}_3 = (-1,1,0,4) - (1,0,-1,1) + 2(1,-1,0,-1) = (0,-1,1,1)$$

и $\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3$ е ортогонален базис на U.

Нека $\boldsymbol{a} = \boldsymbol{a}_0 + \boldsymbol{h}$, $\boldsymbol{a}_0 \in U$, $\boldsymbol{h} \in U^\perp$. Нека $\boldsymbol{a}_0 = \lambda_1\boldsymbol{b}_1 + \lambda_2\boldsymbol{b}_2 + \lambda_3\boldsymbol{b}_3$. Координатите λ_1, λ_2, λ_3 на $\boldsymbol{a}_0$ ще определим от

$$\begin{array}{l} 0 = (\boldsymbol{h}, \boldsymbol{b}_1) = (\boldsymbol{a} - \lambda_1\boldsymbol{b}_1 - \lambda_2\boldsymbol{b}_2 - \lambda_3\boldsymbol{b}_3, \boldsymbol{b}_1) = (\boldsymbol{a}, \boldsymbol{b}_1) - \lambda_1(\boldsymbol{b}_1, \boldsymbol{b}_1) - \lambda_2\underbrace{(\boldsymbol{b}_2, \boldsymbol{b}_1)}_{=0} - \lambda_3\underbrace{(\boldsymbol{b}_3, \boldsymbol{b}_1)}_{=0}, \\ 0 = (\boldsymbol{h}, \boldsymbol{b}_2) = (\boldsymbol{a} - \lambda_1\boldsymbol{b}_1 - \lambda_2\boldsymbol{b}_2 - \lambda_3\boldsymbol{b}_3, \boldsymbol{b}_2) = (\boldsymbol{a}, \boldsymbol{b}_2) - \lambda_1\underbrace{(\boldsymbol{b}_1, \boldsymbol{b}_2)}_{=0} - \lambda_2(\boldsymbol{b}_2, \boldsymbol{b}_2) - \lambda_3\underbrace{(\boldsymbol{b}_3, \boldsymbol{b}_2)}_{=0}, \\ 0 = (\boldsymbol{h}, \boldsymbol{b}_3) = (\boldsymbol{a} - \lambda_1\boldsymbol{b}_1 - \lambda_2\boldsymbol{b}_2 - \lambda_3\boldsymbol{b}_3, \boldsymbol{b}_3) = (\boldsymbol{a}, \boldsymbol{b}_3) - \lambda_1\underbrace{(\boldsymbol{b}_1, \boldsymbol{b}_3)}_{=0} - \lambda_2\underbrace{(\boldsymbol{b}_2, \boldsymbol{b}_3)}_{=0} - \lambda_3(\boldsymbol{b}_3, \boldsymbol{b}_3), \end{array}$$

следователно

$$\lambda_1 = \frac{(\boldsymbol{a}, \boldsymbol{b}_1)}{(\boldsymbol{b}_1, \boldsymbol{b}_1)} = \frac{1 - 3 + \sqrt{2}}{3} = \frac{-2 + \sqrt{2}}{3}$$
$$\lambda_2 = \frac{(\boldsymbol{a}, \boldsymbol{b}_2)}{(\boldsymbol{b}_2, \boldsymbol{b}_2)} = \frac{1 - 2 - \sqrt{2}}{3} = \frac{-1 - \sqrt{2}}{3}$$
$$\lambda_3 = \frac{(\boldsymbol{a}, \boldsymbol{b}_3)}{(\boldsymbol{b}_3, \boldsymbol{b}_3)} = \frac{-2 + 3 + \sqrt{2}}{3} = \frac{1 + \sqrt{2}}{3}$$

Следователно

$$\boldsymbol{a}_0 = \frac{-2 + \sqrt{2}}{3}(1, 0, -1, 1) + \frac{-1 - \sqrt{2}}{3}(1, -1, 0, -1) + \frac{1 + \sqrt{2}}{3}(0, -1, 1, 1) = (-1, 0, 1, \sqrt{2}).$$

Оттук $\cos\alpha = \dfrac{(\boldsymbol{a}, \boldsymbol{a}_0)}{|\boldsymbol{a}||\boldsymbol{a}_0|} = \dfrac{4}{\sqrt{16}\sqrt{4}} = \dfrac{1}{2}$ и значи $\alpha = \dfrac{\pi}{3}$.

Задача. Нека $U = l(\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3)$, където $\boldsymbol{a}_1 = (1, 2, 0, 1)$, $\boldsymbol{a}_2 = (3, 2, 1, 2)$, $\boldsymbol{a}_3 = (1, -2, 1, 0)$. Да се построи ортонормиран базис на $U^\perp$.

Решение. $\boldsymbol{x} = (x_1, x_2, x_3, x_4) \in U^\perp \iff (\boldsymbol{a}_i, \boldsymbol{x}) = 0,\ 1 \le i \le 3 \iff$

$$U^\perp : \left| \begin{array}{ccccccccl} x_1 & + & 2x_2 & & & + & x_4 & = 0 \\ 3x_1 & + & 2x_2 & + & x_3 & + & x_4 & = 0 \\ x_1 & - & 2x_2 & + & x_3 & & & = 0 \end{array} \right.$$

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 3 & 2 & 1 & 2 \\ 1 & -2 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & -4 & 1 & -1 \\ 0 & -4 & 1 & -1 \end{pmatrix}$$

Полагаме $x_2 = p$, $x_3 = q$, тогава $x_4 = -4p + q$, $x_1 = 2p - q$ и

$$U^\perp = \{(2p - q, p, q, -4p + q) \mid p, q \in \mathbb{R}\}.$$

$$\left. \begin{array}{ll} p = 0,\ q = 1: & \boldsymbol{c}_1 = (-1, 0, 1, 1) \\ p = 1,\ q = 0: & \boldsymbol{c}_2 = (2, 1, 0, -4) \end{array} \right\} \text{ФСР, т.е. базис на } U^\perp.$$

Полагаме $\boldsymbol{b}_1 = \boldsymbol{c}_1 = (-1, 0, 1, 1)$. Търсим $\boldsymbol{b}_2 = \boldsymbol{c}_2 + \lambda\boldsymbol{b}_1$, където $\lambda = -\dfrac{(\boldsymbol{c}_2, \boldsymbol{b}_1)}{(\boldsymbol{b}_1, \boldsymbol{b}_1)} = -\dfrac{-2 - 4}{3} = 2$. Тогава $\boldsymbol{b}_2 = (2, 1, 0, -4) + 2(-1, 0, 1, 1) = (0, 1, 2, -2)$. След нормиране на тези вектори, получаваме ортонормиран базис на $U^\perp$, а именно

$$\boldsymbol{e}_1 = \frac{\boldsymbol{b}_1}{|\boldsymbol{b}_1|} = \frac{1}{\sqrt{3}}(-1, 0, 1, 1), \quad \boldsymbol{e}_2 = \frac{\boldsymbol{b}_2}{|\boldsymbol{b}_2|} = \frac{1}{3}(0, 1, 2, -2).$$

Задача. Нека

$$U : \left| \begin{array}{ccccccccl} x_1 & + & x_2 & + & x_3 & + & x_4 & = 0 \\ 2x_1 & + & 3x_2 & - & x_3 & - & 2x_4 & = 0 \end{array} \right. .$$

Да се намери ортонормиран базис на $U^\perp$.

*Решение.*Означаваме $\boldsymbol{a}_1 = (1, 1, 1, 1)$, $\boldsymbol{a}_2 = (2, 3, -1, -2)$. Имаме $\boldsymbol{x} \in U \iff (\boldsymbol{a}_1, \boldsymbol{x}) = (\boldsymbol{a}_2, \boldsymbol{x}) = 0 \iff \boldsymbol{x} \in l(\boldsymbol{a}_1, \boldsymbol{a}_2)^\perp$, т.е. $U = l(\boldsymbol{a}_1, \boldsymbol{a}_2)^\perp$. Оттук $U^\perp = l(\boldsymbol{a}_1, \boldsymbol{a}_2)$.

Полагаме $\boldsymbol{b}_1 = \boldsymbol{a}_1 = (1, 1, 1, 1)$. Търсим $\boldsymbol{b}_2 = \boldsymbol{a}_2 + \lambda\boldsymbol{b}_1$, където $\lambda = -\dfrac{(\boldsymbol{a}_2, \boldsymbol{b}_1)}{(\boldsymbol{b}_1, \boldsymbol{b}_1)} = -\dfrac{2 + 3 - 1 - 2}{4} = -\dfrac{1}{2}$. Тогава $\boldsymbol{b}_2 = (2, 3, -1, -2) - \dfrac{1}{2}(1, 1, 1, 1) = \dfrac{1}{2}(3, 5, -3, -5)$. След нормиране на тези вектори, получаваме ортонормиран базис на $U^\perp$, а именно

$$\boldsymbol{e}_1 = \frac{\boldsymbol{b}_1}{|\boldsymbol{b}_1|} = \frac{1}{2}(1, 1, 1, 1), \quad \boldsymbol{e}_2 = \frac{\boldsymbol{b}_2}{|\boldsymbol{b}_2|} = \frac{1}{2\sqrt{17}}(3, 5, -3, -5).$$