① Упражнение 16 за 1, 2 и 3 група

Зад. 1 Пресметнете числата:

а) $A = \sin\left(2\operatorname{arctg}\frac{2}{3} - \operatorname{arccotg}\frac{12}{5}\right)$ ;

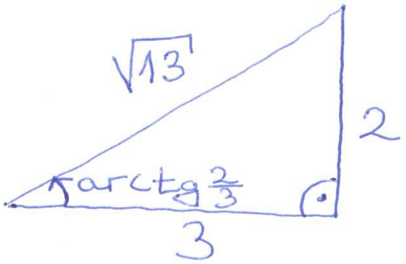
б) $B = \operatorname{arctg}2 + \operatorname{arctg}3$.

Решение: а) $A = \sin\left(2\operatorname{arctg}\frac{2}{3}\right)\cos\left(\operatorname{arccotg}\frac{12}{5}\right) -$

$- \sin\left(\operatorname{arccotg}\frac{12}{5}\right)\cos\left(2\operatorname{arctg}\frac{2}{3}\right) =$

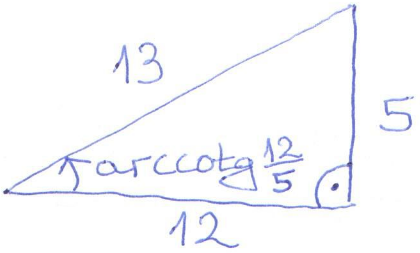
$= 2\sin\left(\operatorname{arctg}\frac{2}{3}\right)\cos\left(\operatorname{arctg}\frac{2}{3}\right)\cos\left(\operatorname{arccotg}\frac{12}{5}\right) -$

$- \sin\left(\operatorname{arccotg}\frac{12}{5}\right)\left(\cos^2\left(\operatorname{arctg}\frac{2}{3}\right) - \sin^2\left(\operatorname{arctg}\frac{2}{3}\right)\right).$




$\sin\left(\operatorname{arctg}\frac{2}{3}\right) = \frac{2}{\sqrt{13}}$

$\cos\left(\operatorname{arctg}\frac{2}{3}\right) = \frac{3}{\sqrt{13}}$



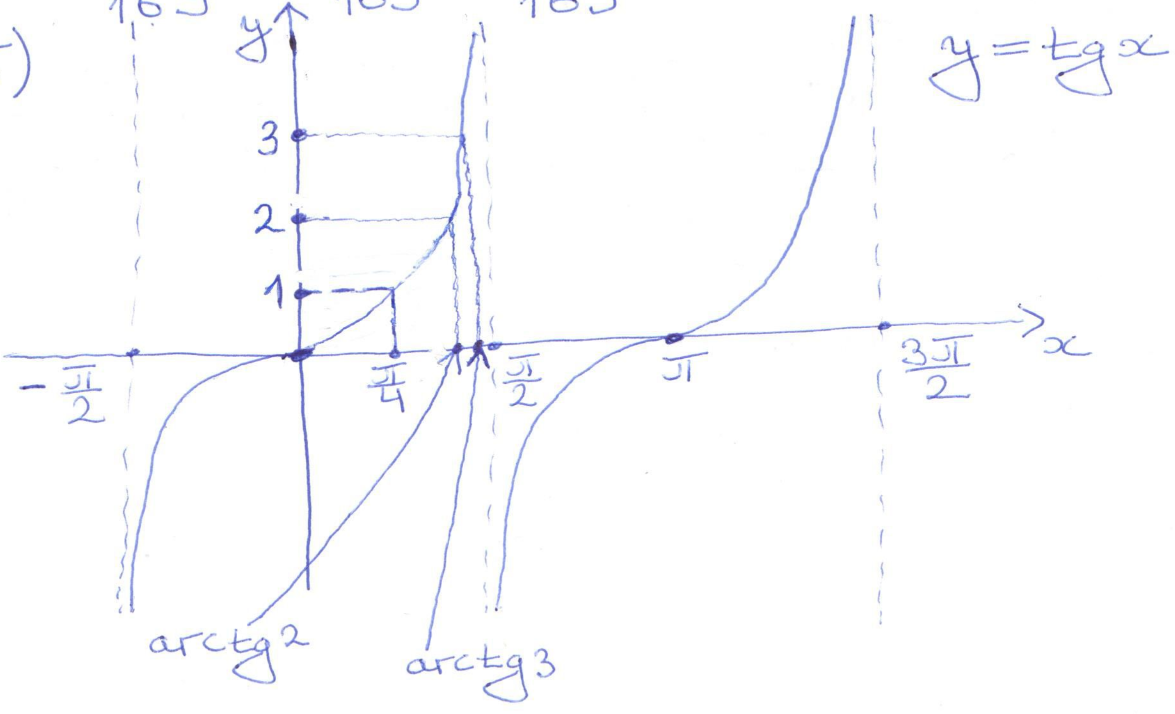

$\sin\left(\operatorname{arccotg}\frac{12}{5}\right) = \frac{5}{13}$

$\cos\left(\operatorname{arccotg}\frac{12}{5}\right) = \frac{12}{13}$

Сл. $A = 2 \cdot \frac{2}{\sqrt{13}} \cdot \frac{3}{\sqrt{13}} \cdot \frac{12}{13} - \frac{5}{13} \cdot \left(\frac{9}{13} - \frac{4}{13}\right) =$

$= \frac{144}{169} - \frac{25}{169} = \frac{119}{169}$. Отг. на а): $A = \frac{119}{169}$.

б)



② Имаме, че $\operatorname{arctg} 2 \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ и $\operatorname{arctg} 3 \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$, сл. $B \in \left(\frac{\pi}{2}, \pi\right)$.

Освен това $\operatorname{tg} B = \operatorname{tg}(\operatorname{arctg} 2 + \operatorname{arctg} 3) =$

$$= \frac{\operatorname{tg}(\operatorname{arctg} 2) + \operatorname{tg}(\operatorname{arctg} 3)}{1 - \operatorname{tg}(\operatorname{arctg} 2) . \operatorname{tg}(\operatorname{arctg} 3)} = \frac{2+3}{1-2.3} = \frac{5}{-5} = -1.$$

$$\left.\begin{array}{l} B \in \left(\frac{\pi}{2}, \pi\right) \\ \operatorname{tg} B = -1 \end{array}\right\} \Rightarrow B = \frac{3\pi}{4}.$$ Отг. на б): $B = \frac{3\pi}{4}$.

Зад. 2 Док. че $\sin(2 \arccos x) = 2x\sqrt{1-x^2}$, $x \in [-1, 1]$.

Решение: Нека $\arccos x = \alpha$, т.е. $\alpha \in [0, \pi]$ и $\cos \alpha = x$.

Тогава $\sin(2 \arccos x) = \sin 2\alpha =$

$$= 2 \sin \alpha \cos \alpha \underset{\substack{\uparrow \\ \alpha \in [0,\pi] \Rightarrow \sin\alpha \geq 0}}{=} 2\sqrt{1-\cos^2 \alpha}\cos \alpha = 2x\sqrt{1-x^2}.$$

Зад. 3 Док. че $\operatorname{arctg} \frac{x-1}{x+1} = \begin{cases} \operatorname{arctg} x - \frac{\pi}{4}, \text{ ако } x \in (-1, +\infty) \\ \operatorname{arctg} x + \frac{3\pi}{4}, \text{ ако } x \in (-\infty, -1) \end{cases}$.

Решение:

Нека $\operatorname{arctg} x = \alpha$, т.е. $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus \left\{-\frac{\pi}{4}\right\}$ и $\operatorname{tg} \alpha = x$ ($\alpha \neq -\frac{\pi}{4}$, защото $x \neq -1$).

Тогава $\operatorname{arctg} \frac{x-1}{x+1} = \operatorname{arctg} \frac{\operatorname{tg}\alpha - \operatorname{tg}\frac{\pi}{4}}{1 + \operatorname{tg}\alpha . \operatorname{tg}\frac{\pi}{4}} =$

$$= \operatorname{arctg}\left[\operatorname{tg}\left(\alpha - \frac{\pi}{4}\right)\right] = \begin{cases} \alpha - \frac{\pi}{4}, \text{ ако } \alpha \in \left(-\frac{\pi}{4}, \frac{\pi}{2}\right) \\ \left(\alpha - \frac{\pi}{4}\right) + \pi, \text{ ако } \alpha \in \left(-\frac{\pi}{2}, -\frac{\pi}{4}\right) \end{cases} =$$

$$= \begin{cases} \operatorname{arctg} x - \frac{\pi}{4}, \text{ ако } x \in (-1, +\infty) \\ \operatorname{arctg} x + \frac{3\pi}{4}, \text{ ако } x \in (-\infty, -1) \end{cases}.$$

③ Зад. 4 Док. че $\operatorname{arctg} x + \operatorname{arctg} y = \operatorname{arctg}\frac{x+y}{1-xy}$, ако $xy<1$.

Решение: Нека $\operatorname{arctg} x = \alpha$ и $\operatorname{arctg} y = \beta$, т.е. $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$; $\operatorname{tg}\alpha = x$ и $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$; $\operatorname{tg}\beta = y$.

Тогава $\operatorname{arctg}\frac{x+y}{1-xy} = \operatorname{arctg}\frac{\operatorname{tg}\alpha + \operatorname{tg}\beta}{1-\operatorname{tg}\alpha\operatorname{tg}\beta} =$

$$= \operatorname{arctg}\left[\operatorname{tg}(\alpha+\beta)\right] \underset{(\alpha+\beta)\in\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)}{=} \alpha+\beta = \operatorname{arctg} x + \operatorname{arctg} y.$$

Остана да обосновем защо $(\alpha+\beta)\in\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$:

$$\underset{\text{по условие}}{xy < 1} \Longrightarrow \operatorname{tg}\alpha . \operatorname{tg}\beta < 1 \underset{\alpha, \beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \Rightarrow \cos\alpha > 0, \cos\beta > 0}{\Longrightarrow} \sin\alpha\sin\beta < \cos\alpha\cos\beta$$

$$\Longrightarrow \cos\alpha\cos\beta - \sin\alpha\sin\beta > 0 \Longrightarrow$$

$$\Longrightarrow \cos(\alpha+\beta) > 0 \underset{\alpha, \beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \Rightarrow (\alpha+\beta) \in (-\pi, \pi)}{\Longrightarrow} (\alpha+\beta) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Следващите две граници се използват често.

Зад. 5 (!) Док. че: а) $\lim\limits_{x\to 0}\frac{\arcsin x}{x} = 1$;

б) $\lim\limits_{x\to 0}\frac{\operatorname{arctg} x}{x} = 1$.

Решение:

а) Нека $\arcsin x = \alpha$, т.е. $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\}$ и $\sin\alpha = x$ ($\alpha \neq 0$, защото $x \neq 0$).

$x \to 0 \Longrightarrow \alpha \to 0$. Имаме, че

$$\lim_{x\to 0}\frac{\arcsin x}{x} = \lim_{\alpha\to 0}\frac{\alpha}{\sin\alpha} = 1.$$

б) Нека $\operatorname{arctg} x = \alpha$, т.е. $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus \{0\}$ и $\operatorname{tg}\alpha = x$ ($\alpha \neq 0$, защото $x \neq 0$).

$x \to 0 \Longrightarrow \alpha \to 0$. Имаме, че

$$\lim_{x\to 0}\frac{\operatorname{arctg} x}{x} = \lim_{\alpha\to 0}\frac{\alpha}{\operatorname{tg}\alpha} = 1.$$

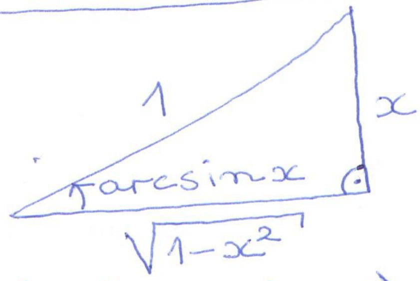
Зад. 6 Пресметнете границата

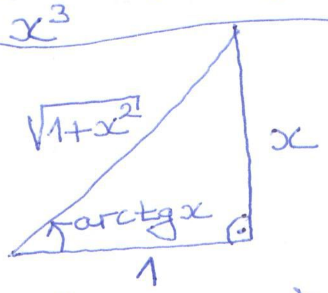
$$L=\lim_{x\to 0}\frac{\arcsin x-\operatorname{arctg} x}{x^3}.$$

Решение: $\left[\frac{0}{0}\right]$.

$$L=\lim_{x\to 0}\frac{\arcsin x-\operatorname{arctg} x}{\sin(\arcsin x-\operatorname{arctg} x)}\cdot\lim_{x\to 0}\frac{\sin(\arcsin x-\operatorname{arctg} x)}{x^3}=$$

$$=1\cdot\lim_{x\to 0}\frac{\sin(\arcsin x)\cos(\operatorname{arctg} x)-\sin(\operatorname{arctg} x)\cos(\arcsin x)}{x^3}$$




$\sin(\arcsin x)=x$

$\cos(\arcsin x)=\sqrt{1-x^2}$

$\sin(\operatorname{arctg} x)=\dfrac{x}{\sqrt{1+x^2}}$

$\cos(\operatorname{arctg} x)=\dfrac{1}{\sqrt{1+x^2}}$

$$\text{Сл. } L=\lim_{x\to 0}\frac{\cancel{x}\cdot\frac{1}{\sqrt{1+x^2}}-\frac{\cancel{x}}{\sqrt{1+x^2}}\cdot\sqrt{1-x^2}}{x^{\cancel{3}\,2}}=$$

$$=\lim_{x\to 0}\frac{1}{\sqrt{1+x^2}}\cdot\lim_{x\to 0}\frac{1-\sqrt{1-x^2}}{x^2}=$$

$$=1\cdot\lim_{x\to 0}\frac{\cancel{x^2}}{\cancel{x^2}\left(1+\sqrt{1-x^2}\right)}=\frac{1}{2}.$$

Отг. на зад. 6: $L=\frac{1}{2}$.