Групи // Ред на елемент и циклична група

$(G,\cdot)$ $\quad k\in\mathbb{N}$

$a^k = \underbrace{a\cdot a\cdot\ldots\cdot a}_{k}$

$a^{-k} = \underbrace{a^{-1}\cdot a^{-1}\ldots\cdot a^{-1}}_{k}$

$a^0 = e$

$a^k\cdot a^s = a^{k+s}$

$(a^k)^s = a^{k\cdot s}$

$(\mathbb{Z},+)$ $\quad k\in\mathbb{N}$

$x\in\mathbb{Z}$

$k(x) = \underbrace{x+\ldots+x}_{k}$

$-k(x) = \underbrace{(-x)+(-x)+\ldots+(-x)}_{k}$

$0(x)=0$

$k(x)+s(x)=(k+s)(x)$

$s(k(x))=(sk)(x)$

$G\neq\emptyset$ $\quad (G,\cdot)$

0) $\forall a,b\in G \to a\cdot b\in G$

1) $(ab)c=a(bc),\ \forall a,b,c\in G$

2) $\exists e:\ a\cdot e=e\cdot a=a\ \forall a$

3) $\forall a\in G,\ \exists b:\ ab=ba=e$

$\emptyset\neq H\subset G$ $\quad (G,\circ)$

$(H,\cdot)$ $\quad H<G$

Тв $\emptyset\neq H\subset G$

$H<G \iff \begin{cases} ab\in H,\ \forall a,b\in H \\ a^{-1}\in H,\ \forall a\in H\end{cases}$

$H<G,\ a\in H \Rightarrow a^k\in H,\ \forall k\in\mathbb{Z}$

$\langle a\rangle = \{a^k \mid k\in\mathbb{Z}\}$ циклична подгрупа породена от $\underline{a}$

$\langle a\rangle\subset H<G$

$(\mathbb{Z},+)$

$\langle x\rangle = \{k(x) \mid k\in\mathbb{Z}\}$

G е циклична, когато $\exists\, a \in G$! $G = \langle a \rangle$
$G = \{a^k \mid k \in \mathbb{Z}\}$

Опр. $a \in G$
$|a| = r = ord(a)$, когато r е min. естествено число за което $a^r = e$

Ако такова не съществува $\Rightarrow o(a) = |a| = \infty$

Опр. $(L, +)$
$|x| = t$ когато t е min. ест. число за което $t(x) = 0 = \underbrace{x + \dots + x}_{t}$

T Нека $(G, \cdot)$, $a \in G$!
$|a| = k \neq \infty$
1) $a^s = e \Leftrightarrow k \mid s$
2) $a^s = a^t \Leftrightarrow s \equiv t \pmod{k}$

Тв $(G, \cdot)$ $a \in G$, $|a| = k \neq \infty$
Тогава $|a^s| = \frac{k}{(s,k)}$ $\quad k_1 = \frac{k}{(k,s)}$

Д-во Нека $|a^s| = t$ $\quad$ НОД$(s,k) = d$
$s = s_1 d$
$k = k_1 d$
$(a^s)^t = a^{st} = e \Rightarrow k \mid st$
$\Rightarrow k_1 d \mid s_1 d t \Rightarrow k_1 \mid s_1 t \Rightarrow k_1 \mid t$
$(a^s)^{k_1} = a^{s_1 d k_1} = a^{s_1 k} = e \Rightarrow t \mid k_1 \Rightarrow k_1 = t$

I / $(G,\cdot)$ и $a\in G$ и $|a|=k\neq\infty$
тогава $|\langle a\rangle|=k$

Д-во / $a^s=a^t \Leftrightarrow s\equiv t \pmod{k}$
$\bar{s}=\{s+kl \mid k\in\mathbb{Z}\}$
$p\in\bar{s} \Rightarrow a^p=a^s$
$\Rightarrow \langle a\rangle=\{a^p \mid p\in\mathbb{Z}\}=\{a^0,a^1,\dots,a^{k-1}\}$

$0\le p\neq t<k$
Допускаме че $a^p=a^t \Rightarrow k\mid(p-t)$ против.
$\Rightarrow$ елементите $a^0,a^1,\dots,a^{k-1}$ са различни помежду си
$\langle a\rangle=\{a^0,a^1,\dots,a^{k-1}\}$
$|\langle a\rangle|=k$

когато $|a|=\infty$
$a^s=a^t \Leftrightarrow s=t$
за $s\neq t$ $a^s\neq a^t$
всички са различни помежду си
$\{a^s \mid s\in\mathbb{Z}\}$
$\Rightarrow |\langle a\rangle|=\infty$

Т/ Всяка подгрупа на циклична група е циклична.

$G = \langle a \rangle \qquad H < G$

Нека $a^s \in H$ е такъв че s е min естествено число за което $a^s \in H$

$a^s \in H \Rightarrow \langle a^s \rangle \subset H$

Нека $a^d \in H$ (произволен)

$d = sq + r, \quad 0 \le r < s$

$\underbrace{a^d}_{\in H} = a^{sq+r} = \underbrace{(a^s)^q}_{\in H} \cdot a^r \Rightarrow a^r = a^d \cdot a^{-sq} \in H$

$a^r \in H$ и от s - min $\Rightarrow r = 0$

$\Rightarrow d = sq \Rightarrow a^d = (a^s)^q \in \langle a^s \rangle$

$\Rightarrow H \subset \langle a^s \rangle \Rightarrow H = \langle a^s \rangle$

Тв/ Нека $G = \langle a \rangle$

$|\langle a \rangle| = k$

$H < G$

$\Rightarrow H = \langle a^d \rangle$ $\quad d$ - min естеств. число

$\Rightarrow d \mid k$ и $|H| = \frac{k}{d}$

$$C_n = \{1, w_1, w_1^2, \dots, w_1^{n-1}\} = \{x \in \mathbb{C} \mid x^n = 1\} = \langle w_1 \rangle$$

$$w_1 = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$$

$C_{12} = \langle w_1 \rangle$

$\langle w_1^2 \rangle = \{1, w_1^2, w^4, w^6, w^8, w^{10}\} = C_6 = \langle w^{10} \rangle$

$|w_1^2| = \frac{12}{(12,2)} = 6$

$\langle w_1^3 \rangle = \{1, w^3, w^6, w^9\} = C_4 = \langle w^9 \rangle$

$|w^3| = \frac{12}{(12,3)} = 4$

$\langle w^4 \rangle = \{1, w^4, w^8\} = C_3 = \langle w^8 \rangle = \{1, w^8, w^4\}$

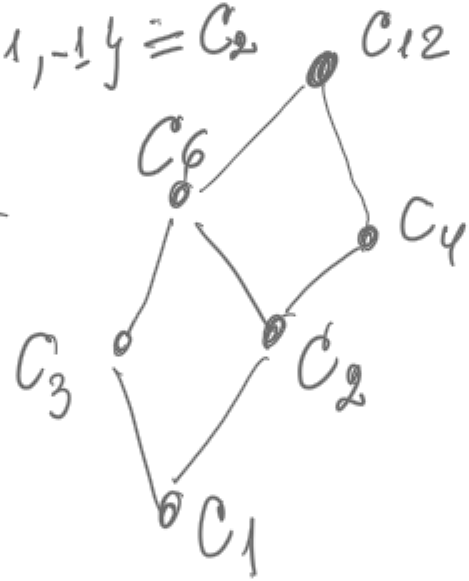
$|w^5| = \frac{12}{(12,5)} = 12$

$\langle w^5 \rangle = C_{12} = \langle w_1 \rangle = \langle w_1^7 \rangle = \langle w^{11} \rangle$

$\langle w^6 \rangle = \{1, w^6\} = \{1, -1\} = C_2$

$|w^8| = \frac{12}{(12,8)} = 3$

$\langle 1 \rangle = \{1\} = C_1$



Как изглеждат подгрупи на C_8; C_{20}; C_{24}

Изоморфизъм при групи

$(G, *)$, $(L, \circ)$ групи

изобр. $\varphi : G \to L$

което $\underline{\varphi(a * b) = \varphi(a) \circ \varphi(b)}$ –

и което е биекция

φ е $\boxed{\text{изоморфизъм}}$

$G \cong L$ $\quad$ $G \cong L$

I/ Нека $G=\langle a\rangle$. Тогава

- ако $|G|=\infty \Rightarrow G\cong\mathbb{Z}$
- ако $|G|=n \Rightarrow G\cong\mathbb{Z}_n\ (\cong C_n)$

Д-во

I сл. Ако $|G|=\infty$ $(G,\cdot)$ $G=\langle a\rangle \Rightarrow |a|=\infty$

съставяме $\varphi:(\mathbb{Z},+)\to(G,\cdot)$ $\varphi(k)=a^k$

Ако $k\neq s \Rightarrow a^k\neq a^s$ (защото ако ѓри. те са равни $\Rightarrow a^{k-s}=e \Rightarrow |a|\neq\infty \Rightarrow$?!)

$\langle a\rangle=\{a^x \mid x\in\mathbb{Z}\}=\{\varphi(k)\mid k\in\mathbb{Z}\}\Rightarrow \varphi$ - биекция

и $\varphi(k+s)=a^{k+s}=a^k\cdot a^s=\varphi(k)\cdot\varphi(s)$

$\Rightarrow \varphi$ е изоморфизъм

$\Rightarrow G\cong\mathbb{Z}$

II сл. $|G| = n = |a|$; $\langle a \rangle = G$, $(G, \cdot)$

$|a| = n \Rightarrow \langle a \rangle = \{e, a, a^2, \dots, a^{n-1}\}$

$\varphi : G \to \mathbb{Z}_n = \{\bar{0}, \bar{1}, \dots, \overline{n-1}\}$ $\quad e = a^0$

$\varphi(a^k) = \bar{k}$, $k = 0, 1, \dots, n-1$

φ – биекция $\quad (0 \le k \ne s < n \Rightarrow a^k \ne a^s$ и $(\bar{k} \ne \bar{s}$ в $\mathbb{Z}_n)$

$\varphi(a^k \cdot a^s) = \varphi(a^{k+s}) = \varphi(a^{nq+r}) = \varphi((a^n)^q a^r) = \varphi(a^r) = \bar{r} = \bar{k} + \bar{s} = \varphi(a^k) + \varphi(a^s)$

разделяне $k+s$ на n: $k+s = nq + r$; $0 \le r < n$

От $k+s = nq + r \Rightarrow \bar{k} + \bar{s} = \bar{r}$

$\Rightarrow \varphi$ е изоморфизъм

$\Rightarrow G \cong \mathbb{Z}_n$

С точност до изоморф. съществува единствена циклична група от ред n

Симетрична група

$M \neq \emptyset$ $\quad S(M) = \{ \varphi : M \to M \mid \varphi - \text{биекция} \}$

$\varphi \circ \psi (x) = \varphi(\psi(x)) \quad x \in M$ $\quad (\varphi \circ \psi$ „след" композиция$)$

$\varphi \circ \psi \in S(M)$

- $(\varphi \circ \psi) \circ \tau = \varphi \circ (\psi \circ \tau)$
- $\varphi \circ id = id \circ \varphi = \varphi$
- $\varphi \circ \varphi^{-1} = \varphi^{-1} \circ \varphi = id$

$id : M \to M$
$id(x) = x, \ \forall x \in M$

когато $|M| > 2$ $S(M)$ не е Абелева

когато $|M| = n$ $\quad M = \{1, 2, \dots, n\}$ тогава $S(M) = S_n$

S_n - симетрична група от степен n

(*) $\varphi = \begin{pmatrix} 1 & 2 & \dots & n \\ i_1 & i_2 & \dots & i_n \end{pmatrix}$

$i_1, \dots, i_n$ - пермутация на $1, 2, \dots, n$

1) $|S_n| = n!$ (броя на различ. запис (*))

2) $n > 2$, S_n не е абелева

S_3

$\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix} = id$

$\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix} = \varphi_1$

$\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix} = \varphi_2$

$\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix} = \varphi_3$

$\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix} = \varphi_4$

$\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix} = \varphi_5$

$$\varphi_1 \circ \varphi_3 = \underset{\varphi_1}{\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}} \circ \underset{\varphi_3}{\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}} = \begin{pmatrix}1&2&3\\ \cancel{1}&\cancel{3}&\cancel{2}\\2&1&3\end{pmatrix} = \begin{pmatrix}1&2&3\\2&1&3\end{pmatrix} = \varphi_2$$

$$\varphi_3 \circ \varphi_1 = \underset{\varphi_3}{\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}} \circ \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix} = \begin{pmatrix}1&2&3\\ \cancel{2}&\cancel{3}&\cancel{1}\\3&2&1\end{pmatrix} = \begin{pmatrix}1&2&3\\3&2&1\end{pmatrix} = \varphi_5$$

$$\varphi_1 \circ \varphi_2 = \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix} \circ \begin{pmatrix}1&2&3\\2&1&3\end{pmatrix} = \begin{pmatrix}1&2&3\\ \cancel{2}&\cancel{1}&\cancel{3}\\3&2&1\end{pmatrix} = \begin{pmatrix}1&2&3\\3&2&1\end{pmatrix} = \varphi_5$$

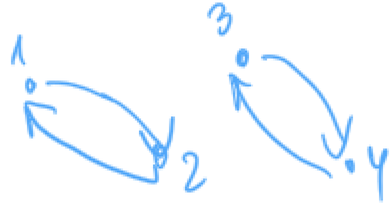
φ_1

φ_2

$\circ$	id	φ_1	φ_2	φ_3	φ_4	φ_5
id	id	φ_1	φ_2	φ_3	φ_4	φ_5
φ_1	φ_1	φ_4	φ_5	φ_2	id	
φ_2	φ_2		id			
φ_3	φ_3	φ_5		id		
φ_4	φ_4					
φ_5	φ_5					

S_4 не всеки елем. от S_n е цикъл

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$



Опр. $\varphi \in S_n$

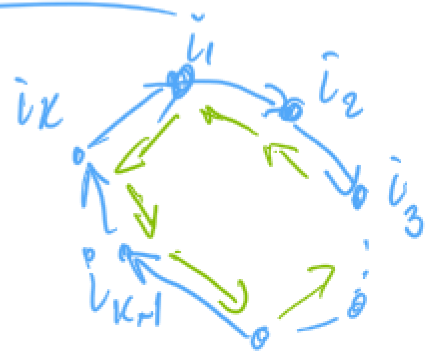
φ е цикъл с дължина k, когато

$$\boxed{\varphi = (i_1, i_2, \dots, i_k)}$$

$\varphi(i_1)=i_2$; $\varphi(i_2)=i_3$; $\dots$; $\varphi(i_{k-1})=i_k$

$\varphi(i_k)=i_1$ и $\varphi(j)=j$ за $j \notin \{i_1, \dots, i_k\}$

$(1,2,3)=(2,3,1)=(3,1,2)$



Св-во $1 < t \le k$

$(i_1, i_2, \dots, i_k) =$

$(i_t, i_{t+1}, \dots, i_k, i_1, \dots, i_{t-1})$

може да се запише по k различни начина

Св-во

$(i_1, \dots, i_k)^{-1} =$

$= (i_k, i_{k-1}, \dots, i_1)$

S_n

опр.// $\varphi=(i_1,\dots,i_k)$; $\psi=(j_1,\dots,j_s)$

φ и ψ са независими цикли, когато

$\{i_1,\dots,i_k\}\cap\{j_1,\dots,j_s\}=\emptyset$ &

Тв.// Ако $\varphi=(i_1,\dots,i_k)$, $\psi(j_1\dots j_s)\in S_n$ са независими цикли $\Rightarrow$ $\varphi\circ\psi=\psi\circ\varphi$

$\varphi\circ\psi(i_t)=\varphi(\psi(i_t))=\varphi(i_t)=i_{t+1}$

$\psi\circ\varphi(i_t)=\psi(\varphi(i_t))=\psi(i_{t+1})=i_{t+1}$

$\varphi\circ\psi(j_l)=\varphi(\psi(j_l))=\varphi(j_{l+1})=j_{l+1}$

$\psi\circ\varphi(j_l)=\psi(\varphi(j_l))=\psi(j_l)=j_{l+1}$

Ако $p\notin\{i_1,\dots,i_k\}\cup\{j_1,\dots,j_s\}$

$\varphi\circ\psi(p)=\varphi(p)=p$

$\psi\circ\varphi(p)=\psi(p)=p$

I) Всеки елемент φ от S_n ($\varphi \neq id$) може да се представи като произведение на независими цикли и това представяне е единствено с точност до реда на множителите

Д-во) $\varphi \in S_n$, $\varphi \neq id$ $\quad M_\varphi = \{t \mid \varphi(t) \neq t\} \subseteq \{1, 2, \dots, n\}$

Индукция по $m_\varphi = |M_\varphi|$

- $m_\varphi = 1$??? (не е възможно) $\quad \varphi(t) \neq t \Rightarrow$
  $\Rightarrow \varphi(t) = u \neq t \Rightarrow \varphi(u) \neq u$
- $m_\varphi = 2$ и $M_\varphi = \{t, u\}$ и $\varphi(s) = s$ за $s \neq t, u$
  $\varphi(t) = u$ и $\varphi(u) = t$
  $\Rightarrow \varphi = (u, t)$
- Нека $m_\varphi > 2$ и да допуснем че е док. тв. за стойността за които $2 \leq m_\psi < m_\varphi$

$i_1 \in M_\varphi : \varphi(i_1) \neq i_1 \quad i_2 = \varphi(i_1)$; $i_3 = \varphi(i_2)$, ... $i_t = \varphi(i_{t-1})$

$i_1, i_2, i_3, \dots, i_t, i_{t+1}, \dots$ има повторения някъде

Нека $i_t = i_s \rightarrow \varphi(i_{t-1}) = \varphi(i_{s-1})$

$\Rightarrow i_{t-1} = i_{s-1}$

Първото число което се повтаря в редицата $i_1 i_2 \ldots$ е i_1

$i_1, \ldots, i_k, i_1, i_2, \ldots, i_k, i_1 \ldots i_k, \ldots$

различни

Разглеждаме $\psi = (i_1, \ldots, i_k)$

$\varphi_1 = \psi^{-1} \circ \varphi$ $\quad \varphi_1(i_1) = i_1 \quad \varphi_1(i_2) = i_2, \ldots \varphi_1(i_k) = i_k$

Ако $j \notin \{i_1, \ldots i_k\}$ $\Rightarrow \varphi(j) \notin \{i_1, \ldots, i_k\}$ $\quad \varphi_1(j) = \psi^{-1}(\varphi(j)) = \varphi(j)$

$\Rightarrow \quad M_{\varphi_1} = M_\varphi \setminus \{i_1, \ldots, i_k\}, \quad m_{\varphi_1} = m_\varphi - k$

Прилагаме инд. за $\varphi_1 = \tau_1 \circ \tau_2 \circ \ldots \circ \tau_s$ независими

тук не участват $i_1 \ldots i_k$ цикли

$\psi^{-1} \circ \varphi = \tau_1 \ldots \tau_s \Rightarrow \varphi = \psi \circ \tau_1 \circ \ldots \tau_s$

независими

единственост

$\varphi = \tau_1 \circ \tau_2 \circ \dots \circ \tau_s$ ← независими цикли $\quad M_\varphi = M_{\tau_1} \cup M_{\tau_2} \cup \dots \cup M_{\tau_s}$

$\varphi = \psi_1 \circ \psi_2 \circ \dots \circ \psi_r$ ← независими $\quad M_\varphi = M_{\psi_1} \cup M_{\psi_2} \cup \dots \cup M_{\psi_r}$

$M_{\tau_1} \cup \dots \cup M_{\tau_s} = M_{\psi_1} \cup \dots \cup M_{\psi_r}$

$i_1 \in M_\varphi$ преномериране $\quad i_1 \in M_{\tau_1} \mid i_1 \in M_{\psi_1}$

$\tau_1(i_1) = \varphi(i_1) = \psi_1(i_1) = i_2$

$\tau_1 = (i_1, i_2, \underbrace{\varphi(i_2)}_{=i_3} \dots)$

$\psi_1 = (i_1, i_2, \underbrace{\varphi(i_2)}_{=i_3} \dots)$

двата цикъла са еднакви

$\tau_1 = \psi_1 \Rightarrow \tau_2 \circ \dots \circ \tau_s = \psi_2 \circ \dots \circ \psi_r$

и след краен бр. стъпки $\Rightarrow s = r$

$\Rightarrow \tau_p = \psi_p,\ p = 1, \dots, r$

$$\varphi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 3 & 7 & 5 & 9 & 1 & 12 & 11 & 6 & 10 & 8 & 2 & 4 \end{pmatrix} \in S_{12} =$$

$$= (1,3,5)(12,4,9,10,8,6)(7,11,2)$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \boxed{6} & 7 & 8 & 9 & 10 & 11 & 12 \\ 7 & 4 & 10 & 12 & 1 & \boxed{6} & 8 & 2 & 5 & 11 & 9 & 3 \end{pmatrix}$$

$$(3,10,11,9,5,1,7,8,2,4,12)\boxed{(6)}$$

$$\psi = \underbrace{(1,3,5,11)}_{\varphi_1} \circ \underbrace{(2,7,11,4)}_{\varphi_2} \circ \underbrace{(3,6,5,8)}_{\varphi_3} \circ \underbrace{(7,3,11,2,4,5,9)}_{\varphi_4}$$

$$\begin{array}{l}
1 \xrightarrow{\varphi_4} 1 \xrightarrow{\varphi_3} 1 \xrightarrow{\varphi_2} 1 \xrightarrow{\varphi_1} 3 \\
3 \longrightarrow 11 \xrightarrow{\varphi_3} 11 \longrightarrow 4 \longrightarrow 4 \\
4 \longrightarrow 5 \longrightarrow 8 \longrightarrow 8 \longrightarrow 8 \\
8 \longrightarrow 8 \longrightarrow 3 \longrightarrow 3 \longrightarrow 5 \\
5 \longrightarrow 9 \longrightarrow 9 \longrightarrow 9 \longrightarrow 9 \\
9 \longrightarrow 7 \longrightarrow 7 \xrightarrow{\varphi_2} 11 \longrightarrow 1
\end{array}
\qquad
\begin{array}{l}
2 \xrightarrow{\varphi_4} 4 \xrightarrow{\varphi_3} 4 \xrightarrow{\varphi_2} 2 \xrightarrow{\varphi_1} 2 \\
6 \to 6 \to 5 \to 5 \to 11 \\
11 \to 2 \to 2 \to 7 \to 7 \\
7 \to 3 \to 6 \to 6 \to 6
\end{array}$$

$$\psi = (1,3,4,8,5,9)(2)(6,11,7)$$