Полиноми на една променлива

от училище $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$

$a_0 + a_1 x + \dots + a_k x^k$

x?

$a_s x^s$ - едночлени

$a_s x^s + b_s x^s = (a_s + b_s) x^s$

$a_s x^s + b_k x^k$

$(a_s x^s) \cdot (b_k x^k) = a_s b_k x^{s+k}$

$f(x) + g(x) = g(x) + f(x)$

$(f(x) \cdot g(x)) = g(x) \cdot f(x)$

$f(g+h) = fg + fh$ и т.н

модел с финитни редици

Нека A - комутативен пр. с 1

$(a_0, a_1, a_2, \dots, a_k, 0, 0, \dots)$

финитна редица

безкрайна редица с ел. от A в която само краен брой са $\neq 0$

$(a_0, a_1, \dots \dots a_k, \dots)$

$\exists N: \quad a_{N+i} = 0, \ i \geq 0$

$2 + 3x^2 - 4x^4$

$\updownarrow$

$(2, 0, 3, 0, -4, 0, 0, \dots)$

A – комутативен пр. 1

$B = \{ \alpha = (a_0, a_1, \dots, \ ) \mid$ финитна редица $\}$

$\alpha = (a_0, a_1, \dots \ ) \in B$ $\beta = (b_0, b_1, \dots \ ) \in B$

$\alpha + \beta = (a_0 + b_0, a_1 + b_1, a_2 + b_2 \dots \ )$

(1) $\alpha + \beta$ също е финитна

$N_1: \ a_{N_1+i} = 0, \ \forall i \in N$ | $N = \max\{N_1, N_2\}$

$N_2: \ b_{N_2+i} = 0 \ \ \forall i \in N$ | $a_{N+i} + b_{N+i} = 0, \forall i$

$\longrightarrow \alpha + \beta$ е финитна

(2) $\alpha + \beta = \beta + \alpha$; (3) $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$

(4) $\theta = (0, 0, 0, \dots \ ) \in B$

$\alpha + \theta = \alpha, \ \forall \alpha$

(5) $\alpha + \underbrace{(-a_0, -a_1, -a_2, \dots \ )}_{= -\alpha \in B} = \theta$

$a_k x^k + b_k x^k = (a_k + b_k) x^k$

"+" е бинарна операция в B

$B = \{ (a_0, a_1, \dots, a_n, \dots) \mid$ финитни редици $\}$

$\alpha = (a_0, a_1, \dots, a_n, \dots)$, $\beta = (b_0, b_1, \dots, b_s, \dots)$

$\alpha \cdot \beta = \gamma = (c_0, c_1, c_2, \dots)$

$$\underline{\underline{c_k}} = a_0 b_k + a_1 b_{k-1} + \dots + a_k b_0 = \sum_{i=0}^{k} a_i b_{k-i} = \sum_{i+j=k} a_i b_j$$

$a_k x^k \cdot b_s x^s = a_k b_s x^{k+s}$

$x^5 \to ?$

$a_0 x^0 \cdot b_5 x^5 + a_1 x^1 b_4 x^4 + a_2 x^2 b_3 x^3 \dots a_5 x^5 b_0$

① γ - финитна редица

Нека $N : a_{N+i} = 0, \forall i \geq 0$; $M : b_{M+i} = 0, \forall i \geq 0$

$$c_{M+N+1+i} = \underbrace{a_0 \underbrace{b_{M+N+1+i}}_{=0} + \dots + a_{N+i} \underbrace{b_{M+1}}_{=0}}_{=0} + \underbrace{a_{N+i+1}}_{=0} b_N + \dots + \underbrace{a_{N+M+i+1}}_{=0} b_0$$

$c_{M+N+1+i} = 0, \forall i \geq 0 \Rightarrow \gamma$ е финитна

② $\alpha\beta = \beta\alpha$ $\quad \alpha\beta = \gamma = (c_0, c_1, \dots)$, $\beta\alpha = \delta = (d_0, d_1, \dots)$

$c_k = a_0 b_k + a_1 b_{k-1} + \dots + a_k b_0 = b_0 a_k + b_1 a_{k-1} + \dots + b_k a_0 = d_k$

Тв/ Ако $\alpha = (a_0, a_1, \dots)$ $\beta = (b_0, b_1, \dots)$ $\gamma = (c_0, c_1, \dots)$

$\alpha, \beta, \gamma \in B$, то гаь $(\alpha\beta)\gamma = \alpha(\beta\gamma)$

Д-во $\alpha\beta = (u_0, u_1, \dots)$ $u_s = \sum_{i=0}^{s} a_i b_{s-i} = \sum_{i+j=s} a_i b_j$

$(\alpha\beta)\gamma = (v_0, v_1, \dots)$

$$v_n = \sum_{i+s=n} u_i c_s = \sum_{i+s=n} \Big(\sum_{k+l=i} a_k b_l\Big) c_s = \sum_{i+s=n} \sum_{k+l=i} a_k b_l c_s =$$

$$= \sum_{k+l+s=n} a_k b_l c_s$$

$\beta\gamma = (q_0, q_1, \dots)$ $q_t = \sum_{i+j=t} b_i c_j$

$\alpha(\beta\gamma) = (p_0, p_1, \dots)$

$$p_n = \sum_{k+t=n} a_k q_t = \sum_{k+t=n} a_k \Big(\sum_{i+j=t} b_i c_j\Big) = \sum_{k+t=n} \sum_{i+j=t} a_k b_i c_j =$$

$$= \sum_{k+i+j=n} a_k b_i c_j$$

$\Longrightarrow$ $\alpha(\beta\gamma) = (\alpha\beta)\gamma$

Св-во $\alpha, \beta, \gamma \in B \Rightarrow \alpha(\beta+\gamma) = \alpha\beta + \alpha\gamma$

$\alpha(\beta+\gamma) = (p_0, p_1, p_2, \ldots)$

$p_n = \sum_{i+j=n} a_i(b_j + c_j) = \sum_{i+j=n} a_i b_j + \sum_{i+j=n} a_i c_j \Rightarrow \alpha(\beta+\gamma) = \alpha\beta + \alpha\gamma$

$\sum_{i+j=n} a_i b_j \to$ № n от $\alpha\beta$

$\sum_{i+j=n} a_i c_j \to$ № n от $\alpha\gamma$

Св-во 1 е единица на A $\quad e = 1 = (1, 0, 0, 0, \ldots)$

$e\alpha = \alpha e = \alpha, \forall \alpha \qquad e\alpha = (c_0, c_1, c_2, \ldots)$

$c_k = e_0 a_k + \underbrace{e_1}_{=0} a_{k-1} + \ldots + \underbrace{e_k}_{=0} a_0 = a_k \Rightarrow e\alpha = \alpha$

Тв/ Ако A е комутативен пръстен с единица тогава мн-вото B от всички финитни редици с ед. от A ($B = \{(a_0, \ldots, a_n, \ldots) \mid$ финитни ред$\}$) е комутативен пръстен с единица.

B - пръстен на полиномите с коеф. от A

$B = A[X]$

Връзка финитни редици с полиноми $\boxed{B = A[X]}$

(1) $B = A[X]$ $\quad A_0 = \{(a_0, 0, 0, \dots) \mid a_0 \in A\} \subset B$

$\varphi: A \to A_0 \subset A[X]$

$\varphi(a) = (a, 0, 0, 0, \dots)$

$\varphi(a+b) = (a+b, 0, 0, \dots)$
$= (a, 0, 0, \dots) + (b, 0, 0, \dots)$
$= \varphi(a) + \varphi(b)$

$\varphi(ab) = (ab, 0, 0, 0, \dots) =$
$= (a, 0, 0, \dots) \cdot (b, 0, 0, \dots)$

φ е биекция
$\Rightarrow A \cong A_0 \trianglelefteq B$

(2) $(a_0, 0, 0, 0, \dots) \cdot (b_0, b_1, b_2, \dots) = (a_0 b_0, a_0 b_1, \dots)$

(3) Нека да бележим. $(0, 1, 0, 0, \dots) = X$

$(0, \underset{=}{1}, 0, 0, \dots)(0, \underset{=}{1}, 0, 0, \dots) = X \cdot X = (0, 0, 1, 0, 0, \dots)$

$X^k = (\underbrace{0, 0, \dots 0}_{k}, \underset{\uparrow \text{№} = k}{1}, 0, 0, \dots)$

(4) Тогава $\alpha = (a_0, a_1, \dots, a_N, 0, 0, \dots) =$
$= (a_0, 0, 0 \dots) + (0, a_1, 0, 0 \dots 0) + \dots + (0, 0, \dots, a_N, 0, \dots)$
$= \boxed{a_0 1 + a_1 X + \dots + a_N X^N}$

a_0 – свободен коеф.

Опр. Нека $\alpha = (a_0, a_1, \dots\dots) \in A[X]$

α има степен n, ако $a_n \neq 0$ и $a_{n+i} = 0$, $\forall i > 0$

$\deg \alpha = n$

$\deg(0,0,0,\dots) = -\infty < n$ $(n \geq 0)$

$\alpha \in A_0$

$\deg \alpha = 0$ или $\deg \alpha = -\infty$

Опр. $\deg(a_0 + a_1 x + \dots + a_n x^n) = n$ ако $a_n \neq 0$

най-високата степен на „x" която има ненулев коеф.

Th// A - комутативен пръстен с 1 и без делители на 0

$\alpha, \beta \in A[X]$

а) $\deg(\alpha + \beta) \leq \max\{\deg \alpha; \deg \beta\}$

б) $\deg(\alpha \cdot \beta) = \deg \alpha + \deg \beta$

$\deg \alpha = n$, $\deg \beta = k$

$s = \max\{n, k\}$ $\quad a_n \neq 0$, $b_k \neq 0$

$\alpha\beta = $

$c_s = \sum_{i+j=s} a_i b_j$

$a_n b_k \neq 0$

$c_{s+i} = \sum_{p+q=s+i} a_p b_q = 0$

$i > 0$

Д-во $\deg(\alpha\cdot\beta) = \deg\alpha + \deg\beta$

$\deg\alpha = n$ $\qquad \deg\beta = k$

$\alpha\cdot\beta = \gamma$

$$c_{n+k+i} = \sum_{p+q=n+k+i} a_p b_q = 0$$

$$c_{n+k} = \sum_{p+q=n+k} a_p b_q = a_n \cdot b_k \neq 0$$

Сл. Ако в A няма делители на 0 $\Rightarrow$ в $A[X]$ също няма делители на 0

I/ (Th за деление с частно и остатък)
А - област на цялост, $f, g \in A[X]$ и $g \neq 0$ и старшия коеф. на g е обратим елем. в А
Тогава $\exists\ q, r \in A[X]$: $f = qg + r$ и $\deg r < \deg g$
q - частно, r - остатък

Д-во // 1 сл. Нека $\deg g = 0$ $g = (b_0, 0, 0, \ldots)$, $b_0 \neq 0$
$\exists$: b_0 - обратим в А $\Rightarrow \exists\ b_0^{-1} \in A$
$f = (b_0^{-1} f) \cdot g + 0$ $q = b_0^{-1} f$, $r = 0$

2 сл. $\deg g = m > 0$ $\deg f = n$ Индукция по n
База $n < m \Rightarrow f = 0 \cdot g + f$ $\deg f < \deg g$ $q = 0$; $r = f$
ИП Нека допуснем, че е вярно за всички полиноми от ст $< n$
$n \geq m$ $x^{n-m} g = (0, \ldots, 0, b_0, b_1, \ldots, b_m, 0 \ldots)$ $N = n$
$f_1 = f - a_n b_m^{-1} x^{n-m} g \Rightarrow \deg f < n$
ст n, ст. коеф a_n ст n, ст коеф a_n

$\mathbb{Z}_5$ $\quad f = \bar{2} + \bar{4}x + \bar{3}x^2 + \bar{2}x^3 + x^4 \qquad g = \bar{2} + \bar{4}x + \bar{3}x^2$

$$\begin{array}{l|l}
\bar{1}x^4 + \bar{2}x^3 + \bar{3}x^2 + \bar{4}x + \bar{2} & \bar{3}x^2 + \bar{4}x + \bar{2} = g \\
\cline{2-2}
-\;x^4 + \bar{3}x^3 + \bar{4}x^2 & \boxed{\bar{2}x^2 + \bar{3}x + \bar{4}} = q(x) \\
\hline
\quad 4x^3 + \bar{4}x^2 + \bar{4}x + \bar{2} & \\
-\;4x^3 + \bar{2}x^2 + \bar{1}x & \\
\hline
\quad \bar{2}x^2 + \bar{3}x + \bar{2} & \\
-\;\bar{2}x^2 + \bar{1}x + 3 & \\
\hline
\quad \bar{2}x + \bar{4} = r(x) &
\end{array}$$

$$\Rightarrow f = (\bar{2}x^2 + \bar{3}x + \bar{4})\,g + (\bar{2}x + \bar{4})$$

$\deg(q_2 - q_1) = -\infty \Rightarrow q_2 = q_1$ și $r_2 = r_1 \Rightarrow$ egalitate

Правило на Хорнер

Нека $\deg f = n$

$f = a_0x^n + a_1x + \dots + a_n$

Делим на $x - \beta = g$

$q = b_0x^{n-1} + b_1x^{n-2} + \dots + b_{n-1} = q$

$r = b_n$ $\quad$ $f = qg + r$

Д - во

	a_0	a_1	a_2	$\dots$	a_{n-1}	a_n
β	b_0	b_1	b_2	$\dots$	b_{n-1}	b_n

$b_0 = a_0$

$b_s = \beta b_{s-1} + a_s$, $\quad s = 1, 2, \dots n$

$(b_0x^{n-1} + b_1x^{n-2} + \dots + b_{n-1})(x - \beta) + b_n =$

$\underbrace{b_0}_{=a_0}x^n + \underbrace{(b_1 - \beta b_0)}_{=a_1}x^{n-1} + \underbrace{(b_2 - \beta b_1)}_{=a_2}x^{n-2} + \dots + \underbrace{(b_{n-1} - \beta b_{n-2})}_{a_{n-1}}x + \underbrace{(b_n - \beta b_{n-1})}_{=a_n}$

$\mathbb{Z}_5$ $\quad f = x^4 + \bar{2}x^3 + \bar{3}x^2 + \bar{4}x + \bar{2}$

$x - \bar{1} = x + \bar{4}$

	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{2}$
1	1	3	1	0	2
2	1	4	1	1	4
$\bar{4}$	1	1	2	2	0

$$f = \overset{=(x+4)}{(x-\bar{1})}\left(x^3 + 3x^2 + x\right) + \bar{2}$$

$$f = (x-2)\left(x^3 + \bar{4}x^2 + x + \bar{1}\right) + \bar{4}$$

$$f = (x-4)\left(x^3 + x^2 + \bar{2}x + \bar{2}\right) + \bar{0}$$