д) (циркуланта)

$$\Delta_n = \begin{vmatrix} a_0 & a_1 & a_2 & \dots & a_{n-2} & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \dots & a_{n-3} & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \dots & a_{n-4} & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_2 & a_3 & a_4 & \dots & a_0 & a_1 \\ a_1 & a_2 & a_3 & \dots & a_{n-1} & a_0 \end{vmatrix}$$

Решение. Нека $\omega_k = \cos\dfrac{2k\pi}{n} + i\sin\dfrac{2k\pi}{n}$, $0 \le k \le n-1$, са n-тите корени на единицата (т.е. решенията на уравнението $z^n = 1$). Да означим

$$\begin{array}{rcrcrcrcccrcr} f_0(x) & = & a_0 & + & a_1x & + & a_2x^2 & + & \cdots & + & a_{n-2}x^{n-2} & + & a_{n-1}x^{n-1} \\ f_1(x) & = & a_{n-1} & + & a_0x & + & a_1x^2 & + & \cdots & + & a_{n-3}x^{n-2} & + & a_{n-2}x^{n-1} \\ f_2(x) & = & a_{n-2} & + & a_{n-1}x & + & a_0x^2 & + & \cdots & + & a_{n-4}x^{n-3} & + & a_{n-3}x^{n-1} \\ \vdots & & & & & & & & & & & & \\ f_{n-1}(x) & = & a_1 & + & a_2x & + & a_3x^2 & + & \cdots & + & a_{n-1}x^{n-2} & + & a_0x^{n-1} \end{array}$$

Тогава

$$\begin{aligned} f_1(\omega_k) &= a_{n-1} + a_0\omega_k + a_1\omega_k^2 + \cdots + a_{n-3}\omega_k^{n-2} + a_{n-2}\omega_k^{n-1} \\ &= \omega_k(a_0 + a_1\omega_k + a_2\omega_k^2 + \cdots + a_{n-2}\omega_k^{n-2} + a_{n-1}\omega_k^{n-1}) \\ &= \omega_k f_0(\omega_k) \end{aligned}$$

за $0 \le k \le n-1$. Аналогично

$$\begin{aligned} f_2(\omega_k) &= \omega_k f_1(\omega_k) \\ f_3(\omega_k) &= \omega_k f_2(\omega_k) \\ &\vdots \\ f_{n-1}(\omega_k) &= \omega_k f_{n-2}(\omega_k) \end{aligned}$$

и следователно

$$\left.\begin{aligned} f_1(\omega_k) &= \omega_k f_0(\omega_k) \\ f_2(\omega_k) &= \omega_k^2 f_0(\omega_k) \\ &\vdots \\ f_{n-1}(\omega_k) &= \omega_k^{n-1} f_0(\omega_k) \end{aligned}\right. \qquad (*)$$

за $0 \le k \le n-1$. Тогава

$$\Delta_n W(\omega_0, \omega_1, \dots, \omega_{n-1}) =$$

$$\begin{vmatrix} a_0 & a_1 & a_2 & \dots & a_{n-2} & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \dots & a_{n-3} & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \dots & a_{n-4} & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_2 & a_3 & a_4 & \dots & a_0 & a_1 \\ a_1 & a_2 & a_3 & \dots & a_{n-1} & a_0 \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ \omega_0 & \omega_1 & \omega_2 & \dots & \omega_{n-2} & \omega_{n-1} \\ \omega_0^2 & \omega_1^2 & \omega_2^2 & \dots & \omega_{n-2}^2 & \omega_{n-1}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \omega_0^{n-2} & \omega_1^{n-2} & \omega_2^{n-2} & \dots & \omega_{n-2}^{n-2} & \omega_{n-1}^{n-2} \\ \omega_0^{n-1} & \omega_1^{n-1} & \omega_2^{n-1} & \dots & \omega_{n-2}^{n-1} & \omega_{n-1}^{n-1} \end{vmatrix} =$$

$$\begin{vmatrix} f_0(\omega_0) & f_0(\omega_1) & f_0(\omega_2) & \dots & f_0(\omega_{n-2}) & f_0(\omega_{n-1}) \\ f_1(\omega_0) & f_1(\omega_1) & f_1(\omega_2) & \dots & f_1(\omega_{n-2}) & f_1(\omega_{n-1}) \\ f_2(\omega_0) & f_2(\omega_1) & f_2(\omega_2) & \dots & f_2(\omega_{n-2}) & f_2(\omega_{n-1}) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{n-2}(\omega_0) & f_{n-2}(\omega_1) & f_{n-2}(\omega_2) & \dots & f_{n-2}(\omega_{n-2}) & f_{n-2}(\omega_{n-1}) \\ f_{n-1}(\omega_0) & f_{n-1}(\omega_1) & f_{n-1}(\omega_2) & \dots & f_{n-1}(\omega_{n-2}) & f_{n-1}(\omega_{n-1}) \end{vmatrix} \overset{(*)}{=}$$

$$\begin{vmatrix} f_0(\omega_0) & f_0(\omega_1) & f_0(\omega_2) & \dots & f_0(\omega_{n-2}) & f_0(\omega_{n-1}) \\ \omega_0 f_0(\omega_0) & \omega_1 f_0(\omega_1) & \omega_2 f_0(\omega_2) & \dots & \omega_{n-2} f_0(\omega_{n-2}) & \omega_{n-1} f_0(\omega_{n-1}) \\ \omega_0^2 f_0(\omega_0) & \omega_1^2 f_0(\omega_1) & \omega_2^2 f_0(\omega_2) & \dots & \omega_{n-2}^2 f_0(\omega_{n-2}) & \omega_{n-1}^2 f_0(\omega_{n-1}) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \omega_0^{n-2} f_0(\omega_0) & \omega_1^{n-2} f_0(\omega_1) & \omega_2^{n-2} f_0(\omega_2) & \dots & \omega_{n-2}^{n-2} f_0(\omega_{n-2}) & \omega_{n-1}^{n-2} f_0(\omega_{n-1}) \\ \omega_0^{n-1} f_0(\omega_0) & \omega_1^{n-1} f_0(\omega_1) & \omega_2^{n-1} f_0(\omega_2) & \dots & \omega_{n-2}^{n-1} f_0(\omega_{n-2}) & \omega_{n-1}^{n-1} f_0(\omega_{n-1}) \end{vmatrix} =$$

$$
f_0(\omega_0)f_0(\omega_1)\dots f_0(\omega_{n-1})\begin{vmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ \omega_0 & \omega_1 & \omega_2 & \dots & \omega_{n-2} & \omega_{n-1} \\ \omega_0^2 & \omega_1^2 & \omega_2^2 & \dots & \omega_{n-2}^2 & \omega_{n-1}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \omega_0^{n-2} & \omega_1^{n-2} & \omega_2^{n-2} & \dots & \omega_{n-2}^{n-2} & \omega_{n-1}^{n-2} \\ \omega_0^{n-1} & \omega_1^{n-1} & \omega_2^{n-1} & \dots & \omega_{n-2}^{n-1} & \omega_{n-1}^{n-1} \end{vmatrix} =
$$

$$
f_0(\omega_0)f_0(\omega_1)\dots f_0(\omega_{n-1})W(\omega_0,\omega_1,\dots,\omega_{n-1}).
$$

Следователно

$$
\Delta_n W(\omega_0,\omega_1,\dots,\omega_{n-1}) = f_0(\omega_0)f_0(\omega_1)\dots f_0(\omega_{n-1})W(\omega_0,\omega_1,\dots,\omega_{n-1})
$$

и тъй като $W(\omega_0,\omega_1,\dots,\omega_{n-1}) \neq 0$ ($\omega_i \neq \omega_j$ при $i \neq j$, $0 \leq i,j \leq n-1$), то

$$
\Delta_n = f_0(\omega_0)f_0(\omega_1)\dots f_0(\omega_{n-1}).
$$

Матрични уравнения. Обратна матрица

Задача. Да се реши матричното уравнение

а) $\begin{pmatrix} 2 & 1 & -1 \\ 1 & -1 & 3 \\ 3 & 2 & 1 \end{pmatrix} X = \begin{pmatrix} 2 & 6 & 5 \\ 3 & 0 & 8 \\ 6 & 10 & 13 \end{pmatrix}$

Решение.

$$
\begin{aligned}
&\left(\begin{array}{ccc|ccc} 2 & 1 & -1 & 2 & 6 & 5 \\ 1 & -1 & 3 & 3 & 0 & 8 \\ 3 & 2 & 1 & 6 & 10 & 13 \end{array}\right)
&\sim &\left(\begin{array}{ccc|ccc} 1 & -1 & 3 & 3 & 0 & 8 \\ 2 & 1 & -1 & 2 & 6 & 5 \\ 3 & 2 & 1 & 6 & 10 & 13 \end{array}\right) \begin{array}{l} (-2)\ (-3) \\ + \\ + \end{array} \\
& &\sim &\left(\begin{array}{ccc|ccc} 1 & -1 & 3 & 3 & 0 & 8 \\ 0 & 3 & -7 & -4 & 6 & -11 \\ 0 & 5 & -8 & -3 & 10 & -11 \end{array}\right) \begin{array}{l} \\ (-2) \\ + \quad |.(-1) \end{array} \\
& &\sim &\left(\begin{array}{ccc|ccc} 1 & -1 & 3 & 3 & 0 & 8 \\ 0 & 1 & -6 & -5 & 2 & -11 \\ 0 & 3 & -7 & -4 & 6 & -11 \end{array}\right) \begin{array}{l} \\ (-3) \\ + \end{array} \\
& &\sim &\left(\begin{array}{ccc|ccc} 1 & -1 & 3 & 3 & 0 & 8 \\ 0 & 1 & -6 & -5 & 2 & -11 \\ 0 & 0 & 11 & 11 & 0 & 22 \end{array}\right) \begin{array}{l} \\ \\ |:11 \end{array} \\
& &\sim &\left(\begin{array}{ccc|ccc} 1 & -1 & 3 & 3 & 0 & 8 \\ 0 & 1 & -6 & -5 & 2 & -11 \\ 0 & 0 & 1 & 1 & 0 & 2 \end{array}\right) \begin{array}{l} + \\ + \\ 6 \quad (-3) \end{array} \\
& &\sim &\left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & 0 & 2 \end{array}\right) \begin{array}{l} + \\ \\ \\ \end{array} \\
& &\sim &\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & 0 & 2 \end{array}\right)
\end{aligned}
$$

Следователно $X = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix}$.

б) $\begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 1 & 1 & 2 \end{pmatrix} X = \begin{pmatrix} 4 & 2 & 2 \\ 1 & -1 & 5 \\ 5 & 1 & 7 \end{pmatrix}$

Решение.

$$\left(\begin{array}{ccc|ccc}1 & 2 & -1 & 4 & 2 & 2\\ 0 & -1 & 3 & 1 & -1 & 5\\ 1 & 1 & 2 & 5 & 1 & 7\end{array}\right) \sim \left(\begin{array}{ccc|ccc}1 & 2 & -1 & 4 & 2 & 2\\ 0 & -1 & 3 & 1 & -1 & 5\\ 0 & -1 & 3 & 1 & -1 & 5\end{array}\right) \sim \left(\begin{array}{ccc|ccc}1 & 2 & -1 & 4 & 2 & 2\\ 0 & -1 & 3 & 1 & -1 & 5\\ 0 & 0 & 0 & 0 & 0 & 0\end{array}\right)$$

Нека $X = \begin{pmatrix} x_1 & y_1 & z_1\\ x_2 & y_2 & z_2\\ x_3 & y_3 & z_3\end{pmatrix}$. Тогава $\begin{pmatrix}1 & 2 & -1\\ 0 & -1 & 3\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} x_1 & y_1 & z_1\\ x_2 & y_2 & z_2\\ x_3 & y_3 & z_3\end{pmatrix} = \begin{pmatrix}4 & 2 & 2\\ 1 & -1 & 5\\ 0 & 0 & 0\end{pmatrix} \iff$

$$\iff \left|\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & 4\\ & - & x_2 & + & 3x_3 & = & 1\end{array}\right., \quad \left|\begin{array}{rcrcrcr} y_1 & + & 2y_2 & - & y_3 & = & 2\\ & - & y_2 & + & 3y_3 & = & -1\end{array}\right., \quad \left|\begin{array}{rcrcrcr} z_1 & + & 2z_2 & - & z_3 & = & 2\\ & - & z_2 & + & 3z_3 & = & 5\end{array}\right. \iff$$

$$\left|\begin{array}{lcc} x_1 & = & 4-2(-1+3p)+p = 6-5p\\ x_2 & = & -1+3p\\ x_3 & = & p\end{array}\right., \quad \left|\begin{array}{lcc} y_1 & = & 2-2(1+3q)+q = -5q\\ y_2 & = & 1+3q\\ y_3 & = & q\end{array}\right., \quad \left|\begin{array}{lcc} z_1 & = & 2-2(-5+3r)+r = 12-5r\\ z_2 & = & -5+3r\\ z_3 & = & r\end{array}\right.$$

Следователно $X = \begin{pmatrix} 6-5p & -5q & 12-5r\\ -1+3p & 1+3q & -5+3r\\ p & q & r\end{pmatrix}$ за произволни числа $p, q, r \in F$.

в) $X\begin{pmatrix}1 & 2 & 1\\ 2 & -2 & 3\\ -3 & 1 & 2\end{pmatrix} = \begin{pmatrix}5 & -1 & 13\\ 5 & -2 & 7\\ -3 & 8 & 7\end{pmatrix}$

Решение. Дадено е уравнението $XA = B$. Като транспонираме това равенство, получаваме $A^tX^t = B^t$. Затова $(A^t \mid B^t) \overset{\text{по ред.}}{\sim} (E \mid X^t)$ и $X = (X^t)^t$.

$$\left(\begin{array}{ccc|ccc}1 & 2 & -3 & 5 & 5 & -3\\ 2 & -2 & 1 & -1 & -2 & 8\\ 1 & 3 & 2 & 13 & 7 & 7\end{array}\right) \sim \left(\begin{array}{ccc|ccc}1 & 2 & -3 & 5 & 5 & -3\\ 0 & -6 & 7 & -11 & -12 & 14\\ 0 & 1 & 5 & 8 & 2 & 10\end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc}1 & 2 & -3 & 5 & 5 & -3\\ 0 & 1 & 5 & 8 & 2 & 10\\ 0 & 0 & 37 & 37 & 0 & 74\end{array}\right) \;\; | : 37$$

$$\sim \left(\begin{array}{ccc|ccc}1 & 2 & -3 & 5 & 5 & -3\\ 0 & 1 & 5 & 8 & 2 & 10\\ 0 & 0 & 1 & 1 & 0 & 2\end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc}1 & 2 & 0 & 8 & 5 & 3\\ 0 & 1 & 0 & 3 & 2 & 0\\ 0 & 0 & 1 & 1 & 0 & 2\end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc}1 & 0 & 0 & 2 & 1 & 3\\ 0 & 1 & 0 & 3 & 2 & 0\\ 0 & 0 & 1 & 1 & 0 & 2\end{array}\right)$$

Следователно $X = (X^t)^t = \begin{pmatrix}2 & 3 & 1\\ 1 & 2 & 0\\ 3 & 0 & 2\end{pmatrix}$.

г) $XA = A + 3X$, където $A = \begin{pmatrix}2 & -3 & -3\\ 1 & 7 & 6\\ 1 & 2 & 2\end{pmatrix}$.

Решение. Имаме $XA - 3X = A$, $XA + X.(-3E) = A$ и следователно $X(A - 3E) = A$, т.е.

$$X\begin{pmatrix}-1 & -3 & -3\\ 1 & 4 & 6\\ 1 & 2 & -1\end{pmatrix} = \begin{pmatrix}2 & -3 & -3\\ 1 & 7 & 6\\ 1 & 2 & 2\end{pmatrix}.$$

$$\left(\begin{array}{ccc|ccc} -1 & 1 & 1 & 2 & 1 & 1 \\ -3 & 4 & 2 & -3 & 7 & 2 \\ -3 & 6 & -1 & -3 & 6 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & -1 & -1 & -2 & -1 & -1 \\ 0 & 1 & -1 & -9 & 4 & -1 \\ 0 & 3 & -4 & -9 & 3 & -1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -1 & -1 & -2 & -1 & -1 \\ 0 & 1 & -1 & -9 & 4 & -1 \\ 0 & 0 & 1 & -18 & 9 & -2 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & -20 & 8 & -3 \\ 0 & 1 & 0 & -27 & 13 & -3 \\ 0 & 0 & 1 & -18 & 9 & -2 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -47 & 21 & -6 \\ 0 & 1 & 0 & -27 & 13 & -3 \\ 0 & 0 & 1 & -18 & 9 & -2 \end{array}\right).$$

Следователно $X = (X^t)^t = \begin{pmatrix} -47 & -27 & -18 \\ 21 & 13 & 9 \\ -6 & -3 & -2 \end{pmatrix}$.

д) $\begin{pmatrix} 3 & 4 & 3 \\ 1 & 0 & 5 \\ 1 & 1 & 2 \end{pmatrix} X = \begin{pmatrix} 2 & 1 & -2 \\ 3 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

Решение.

$$\left(\begin{array}{ccc|ccc} 3 & 4 & 3 & 2 & 1 & -2 \\ 1 & 0 & 5 & 3 & 0 & 1 \\ 1 & 1 & 2 & 1 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 5 & 3 & 0 & 1 \\ 3 & 4 & 3 & 2 & 1 & -2 \\ 1 & 1 & 2 & 1 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 5 & 3 & 0 & 1 \\ 0 & 4 & -12 & -7 & 1 & -5 \\ 0 & 1 & -3 & -2 & 1 & 0 \end{array}\right) \sim$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 5 & 3 & 0 & 1 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -3 & -5 \end{array}\right)$$

и следователно уравнението няма решение.

е) $\begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 5 & -1 & -3 \end{pmatrix} X \begin{pmatrix} 3 & 0 & -1 \\ 0 & 2 & -3 \\ 1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 13 & 6 & 0 \\ 13 & 12 & -9 \\ 1 & -44 & 59 \end{pmatrix}$.

Решение. Полагаме $X \begin{pmatrix} 3 & 0 & -1 \\ 0 & 2 & -3 \\ 1 & 2 & 0 \end{pmatrix} = Y$.

$$
\left(\begin{array}{ccc|ccc} 2 & 0 & 1 & 13 & 6 & 0 \\ 1 & 1 & 1 & 13 & 12 & -9 \\ 5 & -1 & -3 & 1 & -44 & 59 \end{array}\right)
\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 13 & 12 & -9 \\ 2 & 0 & 1 & 13 & 6 & 0 \\ 5 & -1 & -3 & 1 & -44 & 59 \end{array}\right)
\begin{array}{l} (-2)\ (-5) \\ + \\ + \end{array}
$$

$$
\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 13 & 12 & -9 \\ 0 & -2 & -1 & -13 & -18 & 18 \\ 0 & -6 & -8 & -64 & -104 & 104 \end{array}\right)
\begin{array}{l} \\ (-3)\ \ |\,.-1 \\ + \end{array}
$$

$$
\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 13 & 12 & -9 \\ 0 & 2 & 1 & 13 & 18 & -18 \\ 0 & 0 & -5 & -25 & -50 & 50 \end{array}\right)
\begin{array}{l} \\ \\ |:-5 \end{array}
$$

$$
\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 13 & 12 & -9 \\ 0 & 2 & 1 & 13 & 18 & -18 \\ 0 & 0 & 1 & 5 & 10 & -10 \end{array}\right)
\begin{array}{l} + \\ + \\ (-1)\ (-1) \end{array}
$$

$$
\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 8 & 2 & 1 \\ 0 & 2 & 0 & 8 & 8 & -8 \\ 0 & 0 & 1 & 5 & 10 & -10 \end{array}\right)
\begin{array}{l} \\ |:2 \\ \\ \end{array}
$$

$$
\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 8 & 2 & 1 \\ 0 & 1 & 0 & 4 & 4 & -4 \\ 0 & 0 & 1 & 5 & 10 & -10 \end{array}\right)
\begin{array}{l} + \\ (-1) \\ \\ \end{array}
$$

$$
\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -2 & 5 \\ 0 & 1 & 0 & 4 & 4 & -4 \\ 0 & 0 & 1 & 5 & 10 & -10 \end{array}\right).
$$

Следователно $X \begin{pmatrix} 3 & 0 & -1 \\ 0 & 2 & -3 \\ 1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 4 & -2 & 5 \\ 4 & 4 & -4 \\ 5 & 10 & -10 \end{pmatrix}$. Оттук

$$
\left(\begin{array}{ccc|ccc} 3 & 0 & 1 & 4 & 4 & 5 \\ 0 & 2 & 2 & -2 & 4 & 10 \\ -1 & -3 & 0 & 5 & -4 & -10 \end{array}\right)
\begin{array}{l} \\ |:2 \\ |\,.-1 \end{array}
\sim \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & -5 & 4 & 10 \\ 0 & 1 & 1 & -1 & 2 & 5 \\ 3 & 0 & 1 & 4 & 4 & 5 \end{array}\right)
\begin{array}{l} (-3) \\ \\ + \end{array}
$$

$$
\sim \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & -5 & 4 & 10 \\ 0 & 1 & 1 & -1 & 2 & 5 \\ 0 & -9 & 1 & 19 & -8 & -25 \end{array}\right)
\begin{array}{l} \\ 9 \\ + \end{array}
$$

$$
\sim \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & -5 & 4 & 10 \\ 0 & 1 & 1 & -1 & 2 & 5 \\ 0 & 0 & 10 & 10 & 10 & 20 \end{array}\right)
\begin{array}{l} \\ \\ |:10 \end{array}
$$

$$
\sim \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & -5 & 4 & 10 \\ 0 & 1 & 1 & -1 & 2 & 5 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{array}\right)
\begin{array}{l} \\ + \\ (-1) \end{array}
$$

$$
\sim \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & -5 & 4 & 10 \\ 0 & 1 & 0 & -2 & 1 & 3 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{array}\right)
\begin{array}{l} + \\ (-3) \\ \\ \end{array}
$$

$$
\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & -2 & 1 & 3 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{array}\right).
$$

Следователно $X = (X^t)^t = \begin{pmatrix} 1 & -2 & 1 \\ 1 & 1 & 1 \\ 1 & 3 & 2 \end{pmatrix}$.

$\square$

Нека $A \in M_n(F)$. Матрицата A е обратима тогава и само тогава, когато $\det A \neq 0$ (т.е. когато A е неособена). При това, ако $\det A \neq 0$, то $A^{-1} = \dfrac{1}{\det A}\begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \ddots & A_{nn} \end{pmatrix}$, където A_{ij} е адюнгираното количество на елемента a_{ij}, $1 \leq i, j \leq n$.

Задача. Да се намери A^{-1}, където

а) $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ и $\det A = ad - bc \neq 0$.

Решение.

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

б) $A = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$

Решение. $\det A = -1 + 1 + 1 + 1 + 1 + 1 = 4 \neq 0$ и

$$A^{-1} = \frac{1}{4}\begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$$

Сега ще намерим A^{-1} като решение на матричното уравнение $AX = E$.

$$\left(\begin{array}{ccc|ccc} -1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} |\,.-1 \\ + \\ + \end{array} \sim \left(\begin{array}{ccc|ccc} 1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \end{array}\right) \begin{array}{l} \\ |\,:2 \\ |\,:2 \end{array} \sim$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \end{array}\right) \begin{array}{l} +\ + \\ \\ \\ \end{array} \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \end{array}\right).$$

Следователно $A^{-1} = \dfrac{1}{2}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

в) $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$

Решение.

$$
\left(\begin{array}{rrrr|rrrr}
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0\\
1 & 1 & -1 & -1 & 0 & 1 & 0 & 0\\
1 & -1 & 1 & -1 & 0 & 0 & 1 & 0\\
1 & -1 & -1 & 1 & 0 & 0 & 0 & 1
\end{array}\right)
\sim
\left(\begin{array}{rrrr|rrrr}
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0\\
0 & 0 & -2 & -2 & -1 & 1 & 0 & 0\\
0 & -2 & 0 & -2 & -1 & 0 & 1 & 0\\
0 & -2 & -2 & 0 & -1 & 0 & 0 & 1
\end{array}\right)
\begin{array}{l} \\ |\,.-1 \\ |\,.-1 \\ \\ \end{array}
$$

$$
\sim
\left(\begin{array}{rrrr|rrrr}
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0\\
0 & 2 & 0 & 2 & 1 & 0 & -1 & 0\\
0 & 0 & 2 & 2 & 1 & -1 & 0 & 0\\
0 & -2 & -2 & 0 & -1 & 0 & 0 & 1
\end{array}\right)
\sim
\left(\begin{array}{rrrr|rrrr}
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0\\
0 & 2 & 0 & 2 & 1 & 0 & -1 & 0\\
0 & 0 & 2 & 2 & 1 & -1 & 0 & 0\\
0 & 0 & -2 & 2 & 0 & 0 & -1 & 1
\end{array}\right)
$$

$$
\sim
\left(\begin{array}{rrrr|rrrr}
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0\\
0 & 2 & 0 & 2 & 1 & 0 & -1 & 0\\
0 & 0 & 2 & 2 & 1 & -1 & 0 & 0\\
0 & 0 & 0 & 4 & 1 & -1 & -1 & 1
\end{array}\right)
\begin{array}{l} \\ |\,:2 \\ |\,:2 \\ |\,:4 \end{array}
$$

$$
\sim
\left(\begin{array}{rrrr|rrrr}
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0\\
0 & 1 & 0 & 1 & \frac{1}{2} & 0 & -\frac{1}{2} & 0\\
0 & 0 & 1 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 & 0\\
0 & 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4}
\end{array}\right)
\sim
\left(\begin{array}{rrrr|rrrr}
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4}\\
0 & 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4}\\
0 & 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4}
\end{array}\right)
$$

$$
\sim
\left(\begin{array}{rrrr|rrrr}
1 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4}\\
0 & 1 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4}\\
0 & 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4}\\
0 & 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4}
\end{array}\right)
$$

Следователно $A^{-1} = \dfrac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & 1 & -1\\ 1 & -1 & -1 & 1 \end{pmatrix} = \dfrac{1}{4}A.$

г) $A = \begin{pmatrix} 3 & -1 & -1 & \dots & -1\\ -1 & 3 & -1 & \dots & -1\\ -1 & -1 & 3 & \dots & -1\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ -1 & -1 & -1 & \dots & 3 \end{pmatrix}$

Решение.

$$
\left(\begin{array}{ccccc|ccccc}
3 & -1 & -1 & \dots & -1 & 1 & 0 & 0 & \dots & 0\\
-1 & 3 & -1 & \dots & -1 & 0 & 1 & 0 & \dots & 0\\
-1 & -1 & 3 & \dots & -1 & 0 & 0 & 1 & \dots & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\
-1 & -1 & -1 & \dots & 3 & 0 & 0 & 0 & \dots & 1
\end{array}\right) \sim
$$

$$
\left(\begin{array}{ccccc|ccccc}
4-n & 4-n & 4-n & \dots & 4-n & 1 & 1 & 1 & \dots & 1\\
-1 & 3 & -1 & \dots & -1 & 0 & 1 & 0 & \dots & 0\\
-1 & -1 & 3 & \dots & -1 & 0 & 0 & 1 & \dots & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\
-1 & -1 & -1 & \dots & 3 & 0 & 0 & 0 & \dots & 1
\end{array}\right)
\begin{array}{l} |:(4-n)\\ \\ \\ \\ \\ \end{array} \sim
$$

$$
\left(\begin{array}{ccccc|ccccc}
1 & 1 & 1 & \dots & 1 & \frac{1}{4-n} & \frac{1}{4-n} & \frac{1}{4-n} & \dots & \frac{1}{4-n}\\
-1 & 3 & -1 & \dots & -1 & 0 & 1 & 0 & \dots & 0\\
-1 & -1 & 3 & \dots & -1 & 0 & 0 & 1 & \dots & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\
-1 & -1 & -1 & \dots & 3 & 0 & 0 & 0 & \dots & 1
\end{array}\right) \sim
$$

$$
\left(\begin{array}{ccccc|ccccc}
1 & 1 & 1 & \dots & 1 & \frac{1}{4-n} & \frac{1}{4-n} & \frac{1}{4-n} & \dots & \frac{1}{4-n}\\
0 & 4 & 0 & \dots & 0 & \frac{1}{4-n} & \frac{5-n}{4-n} & \frac{1}{4-n} & \dots & \frac{1}{4-n}\\
0 & 0 & 4 & \dots & 0 & \frac{1}{4-n} & \frac{1}{4-n} & \frac{5-n}{4-n} & \dots & \frac{1}{4-n}\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\
0 & 0 & 0 & \dots & 4 & \frac{1}{4-n} & \frac{1}{4-n} & \frac{1}{4-n} & \dots & \frac{5-n}{4-n}
\end{array}\right)
\begin{array}{l} \\ |:4\\ |:4\\ \\ |:4 \end{array} \sim
$$

$$
\left(\begin{array}{ccccc|ccccc}
1 & 1 & 1 & \dots & 1 & \frac{1}{4-n} & \frac{1}{4-n} & \frac{1}{4-n} & \dots & \frac{1}{4-n}\\
0 & 1 & 0 & \dots & 0 & \frac{1}{4(4-n)} & \frac{5-n}{4(4-n)} & \frac{1}{4(4-n)} & \dots & \frac{1}{4(4-n)}\\
0 & 0 & 1 & \dots & 0 & \frac{1}{4(4-n)} & \frac{1}{4(4-n)} & \frac{5-n}{4(4-n)} & \dots & \frac{1}{4(4-n)}\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\
0 & 0 & 0 & \dots & 1 & \frac{1}{4(4-n)} & \frac{1}{4(4-n)} & \frac{1}{4(4-n)} & \dots & \frac{5-n}{4(4-n)}
\end{array}\right)
\begin{array}{l} \\ (-1)\\ (-1)\\ (-1)\\ (-1) \end{array} \sim
$$

$$
\left(\begin{array}{ccccc|ccccc}
1 & 0 & 0 & \dots & 0 & \frac{5-n}{4(4-n)} & \frac{1}{4(4-n)} & \frac{1}{4(4-n)} & \dots & \frac{1}{4(4-n)}\\
0 & 1 & 0 & \dots & 0 & \frac{1}{4(4-n)} & \frac{5-n}{4(4-n)} & \frac{1}{4(4-n)} & \dots & \frac{1}{4(4-n)}\\
0 & 0 & 1 & \dots & 0 & \frac{1}{4(4-n)} & \frac{1}{4(4-n)} & \frac{5-n}{4(4-n)} & \dots & \frac{1}{4(4-n)}\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\
0 & 0 & 0 & \dots & 1 & \frac{1}{4(4-n)} & \frac{1}{4(4-n)} & \frac{1}{4(4-n)} & \dots & \frac{5-n}{4(4-n)}
\end{array}\right)
$$

Следователно $A^{-1} = \dfrac{1}{4(n-1)} \begin{pmatrix} 5-n & 1 & 1 & \dots & 1\\ 1 & 5-n & 1 & \dots & 1\\ 1 & & 5-n & \dots & 1\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & 1 & 1 & \dots & 5-n \end{pmatrix}.$

Забележка. A е обратима тогава и само тогава, когато

$$
\det A = \begin{vmatrix} 3 & -1 & -1 & \dots & -1\\ -1 & 3 & -1 & \dots & -1\\ -1 & -1 & 3 & \dots & -1\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ -1 & -1 & -1 & \dots & 3 \end{vmatrix} = (4-n) \begin{vmatrix} 1 & 1 & 1 & \dots & 1\\ -1 & 3 & -1 & \dots & -1\\ -1 & -1 & 3 & \dots & -1\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ -1 & -1 & -1 & \dots & 3 \end{vmatrix} = (4-n) \begin{vmatrix} 1 & 1 & 1 & \dots & 1\\ 0 & 4 & 0 & \dots & 0\\ 0 & 0 & 4 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & 4 \end{vmatrix} = (4-n)4^{n-1} \neq 0,
$$

т.е. при $n \neq 4$.