Задача. Нека $U = l(\mathbf{a}_1, \mathbf{a}_2)$, където $\mathbf{a}_1 = (2,1,0,1)$, $\mathbf{a}_2 = (-3,-1,1,1)$, а

$$W : \left| \begin{array}{rcrcrcrcl} 7x_1 & - & x_2 & - & 3x_3 & - & x_4 & = & 0 \\ 4x_1 & + & 3x_2 & - & x_3 & - & 2x_4 & = & 0 \\ -3x_1 & - & x_2 & + & x_3 & + & x_4 & = & 0 \end{array} \right.$$

Да се намерят базиси на $U+W$ и $U \cap W$.
Решение. Първо намираме ФСР на W.

$$\begin{pmatrix} 7 & -1 & -3 & -1 \\ 4 & 3 & -1 & -2 \\ -3 & -1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 4 & -2 & -2 & 0 \\ -2 & 1 & 1 & 0 \\ -3 & -1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ -5 & 0 & 2 & 1 \end{pmatrix}$$

Полагаме $x_1 = p$, $x_3 = q$ и тогава $x_2 = 2p - q$, $x_4 = 5p - 2q$. Следователно

$$W = \{(p, 2p-q, q, 5p-2q) \mid p, q \in F\}$$

$$\left. \begin{array}{ll} p=1, q=0: & \mathbf{c}_1 = (1,2,0,5) \\ p=0, q=1: & \mathbf{c}_2 = (0,-1,1,-2) \end{array} \right\} \text{ФСР, т.е. базис на } W$$

Тогава $U + W = l(\mathbf{a}_1, \mathbf{a}_2) + l(\mathbf{c}_1, \mathbf{c}_2) = l(\mathbf{a}_1, \mathbf{a}_2, \mathbf{c}_1, \mathbf{c}_2)$.

$$\begin{array}{c} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{c}_1 \\ \mathbf{c}_2 \end{array} \begin{pmatrix} 2 & 1 & 0 & 1 \\ -3 & -1 & 1 & 1 \\ 1 & 2 & 0 & 5 \\ 0 & -1 & 1 & -2 \end{pmatrix} \sim \begin{pmatrix} 0 & -3 & 0 & -9 \\ 0 & 5 & 1 & 16 \\ 1 & 2 & 0 & 5 \\ 0 & -1 & 1 & -2 \end{pmatrix} \begin{array}{l} \mid : -3 \\ \\ \\ \\ \end{array} \sim \begin{pmatrix} 0 & 1 & 0 & 3 \\ 0 & 6 & 0 & 18 \\ 1 & 2 & 0 & 5 \\ 0 & -1 & 1 & -2 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Следователно векторите $\mathbf{f}_1 = (0,1,0,3)$, $\mathbf{f}_2 = (1,0,0,-1)$, $\mathbf{f}_3 = (0,0,1,1)$ са базис на $U+W$.

Разглеждаме хомогенната система с коефициенти координатите на $\mathbf{a}_1$, $\mathbf{a}_2$ и ѝ намираме ФСР.

$$\begin{pmatrix} 2 & 1 & 0 & 1 \\ -3 & -1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & 0 & 1 \\ -5 & -2 & 1 & 0 \end{pmatrix}$$

Полагаме $x_1 = p$, $x_2 = q$ и тогава $x_3 = 5p + 2q$, $x_4 = -2p - q$, така че множеството от решенията на разглежданата хомогенна система е

$$\{p, q, 5p+2q, -2p-q \mid p, q \in F\}.$$

$$\left. \begin{array}{ll} p=1, q=0: & \mathbf{b}_1 = (1,0,5,-2) \\ p=0, q=1: & \mathbf{b}_2 = (0,1,2,-1) \end{array} \right\} \text{ФСР.}$$

и тагова

$$U : \left| \begin{array}{rcrcrcrcl} x_1 & & & + & 5x_3 & - & 2x_4 & = & 0 \\ & & x_2 & + & 2x_3 & - & x_4 & = & 0 \end{array} \right. .$$

Оттук

$$U \cap W : \left| \begin{array}{rcrcrcrcl} x_1 & & & + & 5x_3 & - & 2x_4 & = & 0 \\ & & x_2 & + & 2x_3 & - & x_4 & = & 0 \\ 7x_1 & - & x_2 & - & 3x_3 & - & x_4 & = & 0 \\ 4x_1 & + & 3x_2 & - & x_3 & - & 2x_4 & = & 0 \\ -3x_1 & - & x_2 & + & x_3 & + & x_4 & = & 0 \end{array} \right.$$

$$\begin{pmatrix} 1 & 0 & 5 & -2 \\ 0 & 1 & 2 & -1 \\ 7 & -1 & -3 & -1 \\ 4 & 3 & -1 & -2 \\ -3 & -1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 5 & -2 \\ 0 & 1 & 2 & -1 \\ 0 & -1 & -38 & 13 \\ 0 & 3 & -21 & 6 \\ 0 & -1 & 16 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 5 & -2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -36 & 12 \\ 0 & 0 & -27 & 9 \\ 0 & 0 & 18 & -6 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 5 & -2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Полагаме $x_3 = p$ и тогава $x_4 = 3p$, $x_2 = p$, $x_1 = p$, така че

$$U \cap W = \{(p,p,p,3p) \mid p \in F\}.$$

$$p = 1: \quad \mathbf{d} = (1,1,1,3) \text{ — ФСР, т.е. базис на } U \cap W.$$

Детерминанти

$$\begin{array}{ccc|cc} + & + & + & & \\ a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \\ - & - & - & & \end{array} =$$

$$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}.$$

Задача. Да се реши системата (чрез формули на Крамер)

$$\left|\begin{array}{ccccccc} x_1 & + & 2x_2 & + & x_3 & = & 4 \\ 2x_1 & + & 3x_2 & + & 3x_3 & = & 8 \\ x_1 & + & 3x_2 & - & 2x_3 & = & 2 \end{array}\right.$$

Решение. Имаме

$$\Delta = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 1 & 3 & -2 \end{vmatrix} = 2 \neq 0.$$

Следователно системата има единствено решение

$$x_1 = \frac{\Delta_1}{\Delta}, \qquad x_2 = \frac{\Delta_2}{\Delta}, \qquad x_3 = \frac{\Delta_3}{\Delta},$$

където

$$\Delta_1 = \begin{vmatrix} 4 & 2 & 1 \\ 8 & 3 & 3 \\ 2 & 3 & -2 \end{vmatrix} = 2$$

$$\Delta_2 = \begin{vmatrix} 1 & 4 & 1 \\ 2 & 8 & 3 \\ 1 & 2 & -2 \end{vmatrix} = 2$$

$$\Delta_3 = \begin{vmatrix} 1 & 2 & 4 \\ 2 & 3 & 8 \\ 1 & 3 & 2 \end{vmatrix} = 2$$

В сила са равенствата

$$\det A = \sum_{k=1}^{n} (-1)^{p+k} a_{pk}\Delta_{pk} = (-1)^{p+1}a_{p1}\Delta_{p1} + (-1)^{p+2}a_{p2}\Delta_{p2} + \cdots + (-1)^{p+n}a_{pn}\Delta_{pn} \tag{1}$$

$$\det A = \sum_{k=1}^{n} (-1)^{k+q} a_{kq}\Delta_{kq} = (-1)^{1+q}a_{1q}\Delta_{1q} + (-1)^{2+q}a_{2q}\Delta_{2q} + \cdots + (-1)^{n+q}a_{nq}\Delta_{nq} \tag{2}$$

за $1 \leq p, q \leq n$. Равенства (1) и (2) се наричат съответно развитие на $\det A$ по p-ти ред и развитие на $\det A$ по q-ти стълб.

Задача. Да се пресметне детерминантата

а)

$$\Delta = \begin{vmatrix} 1 & -1 & 0 & 2 \\ 3 & 1 & -2 & 4 \\ 2 & 3 & -1 & 0 \\ 5 & 6 & 2 & 7 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 & 2 \\ 0 & 4 & -2 & -2 \\ 0 & 5 & -1 & -4 \\ 0 & 11 & 2 & -3 \end{vmatrix} =$$

$$-2\begin{vmatrix} 1 & -1 & 0 & 2 \\ 0 & -2 & 1 & 1 \\ 0 & 5 & -1 & -4 \\ 0 & 11 & 2 & -3 \end{vmatrix} = 2\begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & -1 & 5 & -4 \\ 0 & 2 & 11 & -3 \end{vmatrix} = 2\begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 15 & -5 \end{vmatrix} =$$

$$2.3.5\begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 3 & -1 \end{vmatrix} = 30\begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 2 \end{vmatrix} = 60$$

б)

$$\Delta = \begin{vmatrix} 1 & 1 & 1 & 0 & 2 \\ 2 & -1 & 3 & 4 & 1 \\ -1 & 0 & 1 & 5 & -2 \\ 3 & 4 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 0 & 2 \\ 3 & 0 & 4 & 4 & 3 \\ -1 & 0 & 1 & 5 & -2 \\ -1 & 0 & -3 & 2 & -5 \\ 1 & 0 & 1 & 2 & 3 \end{vmatrix} = 1.(-1)^{1+2}\begin{vmatrix} 3 & 4 & 4 & 3 \\ -1 & 1 & 5 & -2 \\ -1 & -3 & 2 & -5 \\ 1 & 1 & 2 & 3 \end{vmatrix}$$

$$= -\begin{vmatrix} 0 & 1 & -2 & -6 \\ 0 & 2 & 7 & 1 \\ 0 & -2 & 4 & -2 \\ 1 & 1 & 2 & 3 \end{vmatrix} = -(-1)^{1+4}2\begin{vmatrix} 1 & -2 & -6 \\ 2 & 7 & 1 \\ -1 & 2 & -1 \end{vmatrix} = 2\begin{vmatrix} 1 & -2 & -6 \\ 0 & 11 & 13 \\ 0 & 0 & -7 \end{vmatrix} = -154.$$

в)

$$\Delta = \begin{vmatrix} 3 & 0 & -2 & 1 \\ 2 & 1 & 0 & 2 \\ -2 & 6 & 1 & 4 \\ 1 & -1 & 2 & -2 \end{vmatrix} = \begin{vmatrix} 3 & 0 & -2 & 1 \\ 3 & 0 & 2 & 0 \\ 4 & 0 & 13 & -8 \\ 1 & -1 & 2 & -2 \end{vmatrix} =$$

$$-1(-1)^{4+2}\begin{vmatrix} 3 & -2 & 1 \\ 3 & 2 & 0 \\ 4 & 13 & -8 \end{vmatrix} = -\begin{vmatrix} 3 & -2 & 1 \\ 3 & 2 & 0 \\ 28 & -3 & 0 \end{vmatrix} = -1(-1)^{1+3}(-9-56) = 65$$

г)

$$\Delta = \begin{vmatrix} 2 & 0 & 1 & -1 & 1 \\ -1 & 1 & 2 & 2 & 0 \\ 3 & 3 & 1 & 0 & 2 \\ 1 & 2 & 1 & 3 & -2 \\ 2 & 1 & 2 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 1 & -1 & 1 \\ 3 & 1 & 4 & 0 & 2 \\ 3 & 3 & 1 & 0 & 2 \\ 7 & 2 & 4 & 0 & 1 \\ 4 & 1 & 3 & 0 & 4 \end{vmatrix} = -1(-1)^{1+4}\begin{vmatrix} 3 & 1 & 4 & 2 \\ 3 & 3 & 1 & 2 \\ 7 & 2 & 4 & 1 \\ 4 & 1 & 3 & 4 \end{vmatrix} =$$

$$\begin{vmatrix} 0 & 1 & 0 & 0 \\ -6 & 3 & -11 & -4 \\ 1 & 2 & -4 & -3 \\ 1 & 1 & -1 & 2 \end{vmatrix} = 1.(-1)^{1+2}(-1)\begin{vmatrix} 6 & 11 & 4 \\ 1 & -4 & -3 \\ 1 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 0 & 17 & -8 \\ 0 & -3 & -5 \\ 1 & -1 & 2 \end{vmatrix} =$$

$$1.(-1)^{3+1}(-17.5-8.3) = -85-24 = -109.$$

д)

$$\Delta = \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & -7 & 7 & 7 & 2 \\ 2 & 3 & 7 & 10 & 13 \\ 3 & 5 & 11 & 16 & 21 \\ 1 & 4 & 5 & 3 & 10 \end{vmatrix} = 52$$

е)

$$\Delta_{n+1} = \begin{vmatrix} a_0 & b_1 & b_2 & \dots & b_{n-1} & b_n \\ c_1 & a_1 & 0 & \dots & 0 & 0 \\ c_2 & 0 & a_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n-1} & 0 & 0 & \dots & a_{n-1} & 0 \\ c_n & 0 & 0 & \dots & 0 & a_n \end{vmatrix}$$

1 сл.) Нека $a_1, \dots, a_n$ са различни от нула. Тогава

$$\Delta_{n+1} = \begin{vmatrix} a_0 & b_1 & b_2 & \dots & b_{n-1} & b_n \\ c_1 & a_1 & 0 & \dots & 0 & 0 \\ c_2 & 0 & a_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n-1} & 0 & 0 & \dots & a_{n-1} & 0 \\ c_n & 0 & 0 & \dots & 0 & a_n \end{vmatrix} = \begin{vmatrix} a_0 - \sum_{i=1}^n \frac{c_i b_i}{a_i} & b_1 & b_2 & \dots & b_{n-1} & b_n \\ 0 & a_1 & 0 & \dots & 0 & 0 \\ 0 & 0 & a_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_{n-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & a_n \end{vmatrix} = \left(a_0 - \sum_{i=1}^n \frac{c_i b_i}{a_i} \right) a_1 a_2 \dots a_n =$$

$$a_0 a_1 \dots a_n - (c_1 b_1) a_2 \dots a_n - a_1 (c_2 b_2) a_3 \dots a_n - \dots - a_1 \dots a_{n-1} (c_n b_n).$$

2 сл.) Нека $a_i = 0$ за някое i, $1 \le i \le n$. Развиваме Δ_{n+1} по $(i+1)$-ви ред и получаваме

$$\Delta_{n+1} = \begin{vmatrix} a_0 & b_1 & b_2 & \dots & b_{i-1} & b_i & b_{i+1} & \dots & b_{n-1} & b_n \\ c_1 & a_1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ c_2 & 0 & a_2 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{i-1} & 0 & 0 & \dots & a_{i-1} & 0 & 0 & \dots & 0 & 0 \\ c_i & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ c_{i+1} & 0 & 0 & \dots & 0 & 0 & a_{i+1} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n-1} & 0 & 0 & \dots & 0 & 0 & 0 & \dots & a_{n-1} & 0 \\ c_n & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & a_n \end{vmatrix} =$$

$$(-1)^{i+1+1} c_i \begin{vmatrix} b_1 & b_2 & \dots & b_{i-1} & b_i & b_{i+1} & \dots & b_{n-1} & b_n \\ a_1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & a_2 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & a_{i-1} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & a_{i+1} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & a_{n-1} & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & a_n \end{vmatrix} =$$

$$(-1)^{i+2}c_ib_i(-1)^{1+i}\begin{vmatrix} a_1 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & a_2 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & a_{i-1} & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & a_{i+1} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & a_{n-1} & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & a_n \end{vmatrix} =$$

$$c_i(-1)^{i+2}b_i(-1)^{1+i}a_1a_2\dots a_{i-1}a_{i+1}\dots a_n = -a_1\dots a_{i-1}(c_ib_i)a_{i+1}\dots a_n.$$

ж)

$$\Delta_n = \begin{vmatrix} 5 & 2 & 2 & \dots & 2 \\ 2 & 3^2+2 & 2 & \dots & 2 \\ 2 & 2 & 3^3+2 & \dots & 2 \\ \vdots & \vdots & \vdots & & \vdots \\ 2 & 2 & 2 & \dots & 3^n+2 \end{vmatrix} = \begin{vmatrix} 5 & 2 & 2 & \dots & 2 \\ -3 & 3^2 & 0 & \dots & 0 \\ -3 & 0 & 3^3 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -3 & 0 & 0 & \dots & 3^n \end{vmatrix} =$$

$$\begin{vmatrix} 5+2\left(\frac{1}{3}+\frac{1}{3^2}+\dots+\frac{1}{3^{n-1}}\right) & 2 & 2 & \dots & 2 \\ 0 & 3^2 & 0 & \dots & 0 \\ 0 & 0 & 3^3 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 3^n \end{vmatrix} = \left[5+2.\frac{1}{3}\frac{\left(\frac{1}{3}\right)^{n-1}-1}{\frac{1}{3}-1}\right].3^{2+3+\dots+n} =$$

$$\left[5+1-\frac{1}{3^{n-1}}\right].3^{2+3+\dots+n} = (6.3^{n-1}-1).3^{2+3+\dots+n-n+1} =$$

$$(6.3^{n-1}-1).3^{1+2+\dots+n-1} = (2.3^n-1).3^{\frac{(n-1)n}{2}}$$

з)

$$\Delta_n = \begin{vmatrix} 3 & 7 & 7 & \dots & 7 & 7 \\ 7 & 3 & 7 & \dots & 7 & 7 \\ 7 & 7 & 3 & \dots & 7 & 7 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 7 & 7 & 7 & \dots & 3 & 7 \\ 7 & 7 & 7 & \dots & 7 & 3 \end{vmatrix}$$

Първи начин:

$$\Delta_n = \begin{vmatrix} 3 & 7 & 7 & \dots & 7 & 7 \\ 7 & 3 & 7 & \dots & 7 & 7 \\ 7 & 7 & 3 & \dots & 7 & 7 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 7 & 7 & 7 & \dots & 3 & 7 \\ 7 & 7 & 7 & \dots & 7 & 3 \end{vmatrix} = \begin{vmatrix} 3 & 7 & 7 & \dots & 7 & 7 \\ 4 & -4 & 0 & \dots & 0 & 0 \\ 4 & 0 & -4 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 4 & 0 & 0 & \dots & -4 & 0 \\ 4 & 0 & 0 & \dots & 0 & -4 \end{vmatrix} =$$

$$\begin{vmatrix} 3+(n-1)7 & 7 & 7 & \dots & 7 & 7 \\ 0 & -4 & 0 & \dots & 0 & 0 \\ 0 & 0 & -4 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -4 & 0 \\ 0 & 0 & 0 & \dots & 0 & -4 \end{vmatrix} = (7n-4).(-1)^{n-1}4^{n-1}.$$

Втори начин:

$$\Delta_n = \begin{vmatrix} 3 & 7 & 7 & \dots & 7 & 7 \\ 7 & 3 & 7 & \dots & 7 & 7 \\ 7 & 7 & 3 & \dots & 7 & 7 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 7 & 7 & 7 & \dots & 3 & 7 \\ 7 & 7 & 7 & \dots & 7 & 3 \end{vmatrix} =$$

$$\begin{vmatrix} 3+(n-1)7 & 3+(n-1)7 & 3+(n-1)7 & \dots & 3+(n-1)7 & 3+(n-1)7 \\ 7 & 3 & 7 & \dots & 7 & 7 \\ 7 & 7 & 3 & \dots & 7 & 7 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 7 & 7 & 7 & \dots & 3 & 7 \\ 7 & 7 & 7 & \dots & 7 & 3 \end{vmatrix} =$$

$$(7n-4)\begin{vmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 7 & 3 & 7 & \dots & 7 & 7 \\ 7 & 7 & 3 & \dots & 7 & 7 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 7 & 7 & 7 & \dots & 3 & 7 \\ 7 & 7 & 7 & \dots & 7 & 3 \end{vmatrix} =$$

$$(7n-4)\begin{vmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & -4 & 0 & \dots & 0 & 0 \\ 0 & 0 & -4 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -4 & 0 \\ 0 & 0 & 0 & \dots & 0 & -4 \end{vmatrix} = (7n-4).(-1)^{n-1}4^{n-1}$$

и)

$$\Delta_n = \begin{vmatrix} 0 & \dots & 0 & 0 & a_{1n} \\ 0 & \dots & 0 & a_{2,n-1} & a_{2n} \\ 0 & \dots & a_{3,n-2} & a_{3,n-1} & a_{3n} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{n1} & \dots & a_{n,n-2} & a_{n,n-1} & a_{nn} \end{vmatrix} = (-1)^{n-1+n-2+\dots+2+1}\begin{vmatrix} a_{1n} & 0 & 0 & \dots & 0 \\ a_{2n} & a_{2,n-1} & 0 & \dots & 0 \\ a_{3n} & a_{3,n-1} & a_{3,n-2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{nn} & a_{n,n-1} & a_{n,n-2} & \dots & a_{n1} \end{vmatrix} =$$

$$(-1)^{\frac{n(n-1)}{2}} a_{1n}a_{2,n-1}\dots a_{n1}.$$

к)

$$\Delta_n = \begin{vmatrix} 1 & 1 & 1 & 1 & \dots & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & \dots & 3^2 & 0 & 2 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 3^{n-2} & 0 & 0 & \dots & 0 & 0 & 2 \\ 3^{n-1} & 0 & 0 & 0 & \dots & 0 & 0 & 2 \end{vmatrix} =$$

$$\begin{vmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1-2\left(\frac{1}{3}+\frac{1}{3^2}+\cdots+\frac{1}{3^{n-1}}\right) \\ 0 & 0 & 0 & 0 & \dots & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & \dots & 3^2 & 0 & 2 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 3^{n-2} & 0 & 0 & \dots & 0 & 0 & 2 \\ 3^{n-1} & 0 & 0 & 0 & \dots & 0 & 0 & 2 \end{vmatrix} =$$

$$(-1)^{\frac{n(n-1)}{2}}\left[1-2.\frac{1}{3}\frac{\left(\frac{1}{3}\right)^{n-1}-1}{\frac{1}{3}-1}\right].3^{1+2+\cdots+n-1}=(-1)^{\frac{n(n-1)}{2}}\left[1-1+\frac{1}{3^{n-1}}\right].3^{1+2+\cdots+n-1}=$$

$$(-1)^{\frac{n(n-1)}{2}}.3^{1+2+\cdots+n-2}=(-1)^{\frac{n(n-1)}{2}}.3^{\frac{(n-1)(n-2)}{2}}.$$

л)

$$\Delta_{n+1}=\begin{vmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} & a_n \\ -x & x & 0 & \dots & 0 & 0 \\ 0 & -x & x & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & x & 0 \\ 0 & 0 & 0 & \dots & -x & x \end{vmatrix} = \begin{vmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} & \sum_{i=0}^n a_i \\ -x & x & 0 & \dots & 0 & 0 \\ 0 & -x & x & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & x & 0 \\ 0 & 0 & 0 & \dots & -x & 0 \end{vmatrix} =$$

$$=\left(\sum_{i=0}^n a_i\right).(-1)^{1+n+1}(-1)^n x^n = x^n(a_0+a_1+\cdots+a_n).$$

м) $$\Delta_{n+1}=\begin{vmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} & a_n \\ -y_1 & x_1 & 0 & \dots & 0 & 0 \\ 0 & -y_2 & x_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & x_{n-1} & 0 \\ 0 & 0 & 0 & \dots & -y_n & x_n \end{vmatrix}$$

Решение. Развиваме детерминантата по първи ред.

$$\to\begin{vmatrix} a_0 & a_1 & a_2 & \dots & a_{i-2} & a_{i-1} & a_i & a_{i+1} & a_{i+2} & \dots & a_{n-1} & a_n \\ -y_1 & x_1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & -y_2 & x_2 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & x_{i-2} & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & -y_{i-1} & x_{i-1} & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & -y_i & x_i & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & -y_{i+1} & x_{i+1} & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & -y_{i+2} & x_{i+2} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & x_{n-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & -y_n & x_n \end{vmatrix} = \sum_{i=0}^n a_i(-1)^{1+i+1}\Delta_{1,i+1},$$

където

$$\Delta_{1,i+1} = \begin{vmatrix} -y_1 & x_1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & -y_2 & x_2 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & x_{i-2} & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & -y_{i-1} & x_{i-1} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & -y_i & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & x_{i+1} & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & -y_{i+2} & x_{i+2} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & x_{n-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & -y_n & x_n \end{vmatrix} =^{1}$$

$$(-y_1)(-y_2)\dots(-y_{i-1})(-y_i)x_{i+1}x_{i+2}\dots x_n = (-1)^i y_1 y_2 \dots y_i x_{i+1}\dots x_n.$$

Следователно $\Delta_{n+1} = \sum_{i=0}^{n}(-1)^{i+2}a_i(-1)^i y_1\dots y_i x_{i+1}\dots x_n = a_0x_1\dots x_n + a_1y_1x_2\dots x_n + a_2y_1y_2x_3\dots x_n + \dots + a_ny_1y_2\dots y_n$

н)

$$\Delta_n = \begin{vmatrix} a_1+b_1 & a_1+b_2 & \dots & a_1+b_n \\ a_2+b_1 & a_2+b_2 & \dots & a_2+b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n+b_1 & a_n+b_2 & \dots & a_n+b_n \end{vmatrix} = \begin{vmatrix} a_1 & a_1+b_2 & \dots & a_1+b_n \\ a_2 & a_2+b_2 & \dots & a_2+b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_n+b_2 & \dots & a_n+b_n \end{vmatrix} + \begin{vmatrix} b_1 & a_1+b_2 & \dots & a_1+b_n \\ b_1 & a_2+b_2 & \dots & a_2+b_n \\ \vdots & \vdots & \ddots & \vdots \\ b_1 & a_n+b_2 & \dots & a_n+b_n \end{vmatrix} =$$

$$\begin{vmatrix} a_1 & b_2 & \dots & b_n \\ a_2 & b_2 & \dots & b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n & b_2 & \dots & b_n \end{vmatrix} + b_1\begin{vmatrix} 1 & a_1 & \dots & a_1 \\ 1 & a_2 & \dots & a_2 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & \dots & a_n \end{vmatrix} = \begin{cases} 0, & n > 2 \\ a_1 + b_1, & n = 1 \\ b_1(a_1-a_2) + b_1(a_2-a_1) = (a_1-a_2)(b_2-b_1), & n = 2. \end{cases}$$

о)

$$\Delta_{n+1} = \begin{vmatrix} 0 & 1 & 1 & 1 & \dots & 1 & 1 \\ x_1 & a_1 & 0 & 0 & \dots & 0 & 0 \\ x_2 & x_2 & a_2 & 0 & \dots & 0 & 0 \\ x_3 & x_3 & x_3 & a_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{n-1} & x_{n-1} & x_{n-1} & x_{n-1} & \dots & a_{n-1} & 0 \\ x_n & x_n & x_n & x_n & \dots & x_n & a_n \end{vmatrix} =$$

$$\begin{vmatrix} 1 & 1 & 1 & 1 & \dots & 1 & 1 \\ x_1 & a_1 & 0 & 0 & \dots & 0 & 0 \\ x_2 & x_2 & a_2 & 0 & \dots & 0 & 0 \\ x_3 & x_3 & x_3 & a_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{n-1} & x_{n-1} & x_{n-1} & x_{n-1} & \dots & a_{n-1} & 0 \\ x_n & x_n & x_n & x_n & \dots & x_n & a_n \end{vmatrix} + \begin{vmatrix} -1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & a_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & x_2 & a_2 & 0 & \dots & 0 & 0 \\ 0 & x_3 & x_3 & a_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & x_{n-1} & x_{n-1} & x_{n-1} & \dots & a_{n-1} & 0 \\ 0 & x_n & x_n & x_n & \dots & x_n & a_n \end{vmatrix} =$$

$$\begin{vmatrix} 1 & 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & a_1-x_1 & -x_1 & -x_1 & \dots & -x_1 & -x_1 \\ 0 & 0 & a_2-x_2 & -x_2 & \dots & -x_2 & -x_2 \\ 0 & 0 & 0 & a_3-x_3 & \dots & -x_3 & -x_3 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & a_{n-1}-x_{n-1} & -x_{n-1} \\ 0 & 0 & 0 & 0 & \dots & 0 & a_n-x_n \end{vmatrix} + (-1)(-1)^{1+1}a_1a_2\dots a_n$$

$$= \prod_{i=1}^{n}(a_i - x_i) - a_1a_2\dots a_n$$

[1] Развиваме последователно i пъти по първи стълб

п)

$$\Delta_n = \begin{vmatrix} x & a & a & \dots & a & a \\ -a & x & a & \dots & a & a \\ -a & -a & x & \dots & a & a \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -a & -a & -a & \dots & x & a \\ -a & -a & -a & \dots & -a & x \end{vmatrix} = \begin{vmatrix} x & a & a & \dots & a & a \\ -a & x & a & \dots & a & a \\ -a & -a & x & \dots & a & a \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -a & -a & -a & \dots & x & a \\ -a & -a & -a & \dots & a & a \end{vmatrix} + \begin{vmatrix} x & a & a & \dots & a & 0 \\ -a & x & a & \dots & a & 0 \\ -a & -a & x & \dots & a & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -a & -a & -a & \dots & x & 0 \\ -a & -a & -a & \dots & -a & x-a \end{vmatrix} =$$

$$\begin{vmatrix} x+a & 2a & 2a & \dots & 2a & a \\ 0 & x+a & 2a & \dots & 2a & a \\ 0 & 0 & x+a & \dots & 2a & a \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & x+a & a \\ 0 & 0 & 0 & \dots & 0 & a \end{vmatrix} + (x-a)(-1)^{n+n}\Delta_{n-1} = a(x+a)^{n-1} + (x-a)\Delta_{n-1},$$

т.е. $\Delta_n = a(x+a)^{n-1} + (x-a)\Delta_{n-1}$.

Транспонираме детерминантата и получаваме

$$\Delta_n = \begin{vmatrix} x & -a & -a & \dots & -a & -a \\ a & x & -a & \dots & -a & -a \\ a & a & x & \dots & -a & -a \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a & a & a & \dots & x & -a \\ a & a & a & \dots & a & x \end{vmatrix} = -a(x-a)^{n-1} + (x+a)\Delta_{n-1}.$$

Тогава

$$-\left|\begin{array}{rcl} \Delta_n & = & a(x+a)^{n-1} + (x-a)\Delta_{n-1} \\ \Delta_n & = & -a(x-a)^{n-1} + (x+a)\Delta_{n-1} \end{array}\right., \qquad \text{откъдето} \quad \Delta_{n-1} = \frac{a(x+a)^{n-1} + a(x-a)^{n-1}}{2a}.$$

Следователно

$$\Delta_n = \frac{(x+a)^n + (x-a)^n}{2}.$$

р)

$$\Delta_n = \begin{vmatrix} 1-b_1 & b_2 & 0 & 0 & \dots & 0 & 0 \\ -1 & 1-b_2 & b_3 & 0 & \dots & 0 & 0 \\ 0 & -1 & 1-b_3 & b_4 & \dots & 0 & 0 \\ 0 & 0 & -1 & 1-b_4 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1-b_{n-1} & b_n \\ 0 & 0 & 0 & 0 & \dots & -1 & 1-b_n \end{vmatrix} =$$

$$\begin{vmatrix} 1 & b_2 & 0 & 0 & \dots & 0 & 0 \\ -1 & 1-b_2 & b_3 & 0 & \dots & 0 & 0 \\ 0 & -1 & 1-b_3 & b_4 & \dots & 0 & 0 \\ 0 & 0 & -1 & 1-b_4 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1-b_{n-1} & b_n \\ 0 & 0 & 0 & 0 & \dots & -1 & 1-b_n \end{vmatrix} + \begin{vmatrix} -b_1 & b_2 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1-b_2 & b_3 & 0 & \dots & 0 & 0 \\ 0 & -1 & 1-b_3 & b_4 & \dots & 0 & 0 \\ 0 & 0 & -1 & 1-b_4 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1-b_{n-1} & b_n \\ 0 & 0 & 0 & 0 & \dots & -1 & 1-b_n \end{vmatrix}$$

$$\begin{vmatrix} 1 & b_2 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & b_3 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & b_4 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & b_n \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{vmatrix} + (-1)^{1+1}(-b_1)\Delta_{n-1}$$

Следователно $\Delta_n = 1 - b_1\Delta_{n-1}$ Оттук

$$\begin{array}{lcl} \Delta_n & = & 1 - b_1\Delta_{n-1} \\ \Delta_{n-1} & = & 1 - b_2\Delta_{n-2} \\ \Delta_{n-2} & = & 1 - b_3\Delta_{n-3} \\ \vdots & & \\ \Delta_2 & = & 1 - b_{n-1}\Delta_1 \\ \Delta_1 & = & 1 - b_n \end{array}$$

Оттук

$$\Delta_n = 1 - b_1(1 - b_2\Delta_{n-2}) = 1 - b_1 + b_1b_2(1 - b_3\Delta_{n-3}) = 1 - b_1 + b_1b_2 - b_1b_2b_3(1 - b_4\Delta_{n-4}) = \cdots =$$

$$1 - b_1 + b_1b_2 - b_1b_2b_3 + \cdots + (-1)^{n-1}b_1\ldots b_{n-1} + (-1)^n b_1\ldots b_n.$$

Задача 1. а) Да се намерят в алгебричен вид корените на уравнението

$$z^4 = 3.$$

Решение. Имаме $3 = 3(1 + i.0) = 3(\cos 0 + i \sin 0)$, откъдето (съгласно формулите на Моавър) решенията на уравнението са

$$z_k = \sqrt[4]{3}\left(\cos\frac{2k\pi}{4} + i\sin\frac{2k\pi}{4}\right), \text{ за } k = 0, 1, 2, 3$$

т.е.

$$\begin{aligned} z_0 &= \sqrt[4]{3}(\cos 0 + i\sin 0) = \sqrt[4]{3} \\ z_1 &= \sqrt[4]{3}(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}) = \sqrt[4]{3}i \\ z_2 &= \sqrt[4]{3}(\cos\pi + i\sin\pi) = -\sqrt[4]{3} \\ z_3 &= \sqrt[4]{3}(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}) = -\sqrt[4]{3}i \end{aligned}$$

б) Да се представят в тригонометричен вид решенията на уравнението

$$x^{129} - 9x^{86} + 40x^{43} - 32 = 0.$$

Решение. Полагайки $t = x^{43}$, даденото уравнение приема вида

$$t^3 - 9t^2 + 40t - 32 = 0. \tag{3}$$

Имаме

$$t^3 - 9t^2 + 40t - 32 = (t-1)(t^2 - 8t + 32) = (t-1)(t-(4+4i))(t-(4-4i)),$$

т.е. (3) има решения

$$t_1 = 1, \quad t_2 = 4 + 4i, \quad t_3 = 4 - 4i.$$

За $t_1 = 1$ имаме

$$x^{43} = 1 = 1(\cos 0 + i\sin 0),$$

откъдето

$$x_{1k} = \cos\frac{2k\pi}{43} + i\sin\frac{2k\pi}{43}, \quad k = 0, 1, \ldots, 42.$$

За $t_2 = 4 + 4i$ имаме

$$x^{43} = 4 + 4i = 4\sqrt{2}.(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$$

и значи

$$x_{2k} = \sqrt[86]{32}\left(\cos\frac{\frac{\pi}{4} + 2k\pi}{43} + i\sin\frac{\frac{\pi}{4} + 2k\pi}{43}\right), \quad k = 0, 1, \ldots, 42.$$

За $t_2 = 4 - 4i$ имаме

$$x^{43} = 4 - 4i = 4\sqrt{2}(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)),$$

откъдето

$$x_{3k} = \sqrt[86]{32}\left(\cos\frac{-\frac{\pi}{4} + 2k\pi}{43} + i\sin\frac{-\frac{\pi}{4} + 2k\pi}{43}\right), \quad k = 0, 1, \ldots, 42.$$

в) Да се представи в алгебричен вид комплексното число

$$\frac{(-1 + 3i\sqrt{3})^{175}}{(4 + 16i\sqrt{3})^{87}}$$

Решение.

$$\begin{aligned}
\frac{(-1+3\sqrt{3}i)^{175}}{(4+16\sqrt{3}i)^{87}} &= \frac{[(-1+3\sqrt{3}i)^2]^{87}}{(4+16\sqrt{3}i)^{87}}(-1+3\sqrt{3}i) = \left(\frac{-26-6\sqrt{3}i}{4+16\sqrt{3}i}\right)^{87}(-1+3\sqrt{3}i)\\
&= \frac{1}{2^{87}}\left(\frac{-13-3\sqrt{3}i}{1+4\sqrt{3}i}\right)^{87}(-1+3\sqrt{3}i)\\
&= \frac{1}{2^{87}}\left(\frac{(-13-3\sqrt{3}i)(1-4\sqrt{3}i)}{49}\right)^{87}(-1+3\sqrt{3}i)\\
&= \frac{1}{2^{87}}\left(\frac{-49+49\sqrt{3}i}{49}\right)^{87}(-1+3\sqrt{3}i)\\
&= \frac{1}{2^{87}}(-1+\sqrt{3}i)^{87}(-1+3\sqrt{3}i)\\
&= \frac{1}{2^{87}}\left(2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)\right)^{87}(-1+3\sqrt{3}i)\\
&= \frac{1}{2^{87}}.2^{87}(\cos 58\pi+i\sin 58\pi)(-1+3\sqrt{3}i)\\
&= -1+3\sqrt{3}i
\end{aligned}$$

Задача 2. Да се реши системата в зависимост от стойностите на параметъра λ:

$$\left|\begin{array}{rcrcrcrcr}
(-20-\lambda)x_1 & - & 2x_2 & + & x_3 & - & x_4 & = & -4\lambda\\
\lambda^2 x_1 & + & 2x_2 & - & x_3 & + & x_4 & = & 5\lambda+4\\
\lambda x_1 & + & 3x_2 & + & x_3 & + & 2x_4 & = & 3\lambda\\
20x_1 & - & x_2 & - & 2x_3 & - & x_4 & = & \lambda
\end{array}\right. .$$

Решение. $$\left(\begin{array}{cccc|c}
-20-\lambda & -2 & 1 & -1 & -4\lambda\\
\lambda^2 & 2 & -1 & 1 & 5\lambda+4\\
\lambda & 3 & 1 & 2 & 3\lambda\\
20 & -1 & -2 & -1 & \lambda
\end{array}\right) \sim \left(\begin{array}{cccc|c}
-40-\lambda & -1 & 3 & 0 & -5\lambda\\
\lambda^2+20 & 1 & -3 & 0 & 6\lambda+4\\
\lambda+40 & 1 & -3 & 0 & 5\lambda\\
20 & -1 & -2 & -1 & \lambda
\end{array}\right) \sim$$

$$\left(\begin{array}{cccc|c}
0 & 0 & 0 & 0 & 0\\
\lambda^2-\lambda-20 & 0 & 0 & 0 & \lambda+4\\
\lambda+40 & 1 & -3 & 0 & 5\lambda\\
\lambda+60 & 0 & -5 & -1 & 6\lambda
\end{array}\right)$$

Да отбележим, че $\lambda^2-\lambda-20=(\lambda-5)(\lambda+4)$.

1. $\lambda=5$ — системата е несъвместима.

2. Нека $\lambda\neq 5$.

2.1 $\lambda=-4$. Системата е неопределена и общото ѝ решение зависи от два параметъра.

$$\{(p,-20-36p+3q,q,24+56p-5q)|p,q\in F\}.$$

2.1 $\lambda\neq-4$. Системата е неопределена и общото ѝ решение зависи от един параметър.

$$\left\{\left(\frac{1}{\lambda-5},5\lambda-\frac{\lambda+40}{\lambda-5}+3p,p,-6\lambda+\frac{\lambda+60}{\lambda-5}-5p\right)\mid p\in F\right\}.$$

Задача 3. Да се намери рангът на матрицата A в зависимост от стойностите на параметъра λ, където

$$A=\begin{pmatrix}
\lambda & \lambda & \lambda & \lambda & 8\\
\lambda & \lambda & \lambda & 7 & \lambda\\
\lambda & \lambda & 6 & \lambda & \lambda\\
\lambda & 5 & \lambda & \lambda & \lambda\\
1 & \lambda-4 & \lambda-5 & \lambda-6 & \lambda-7
\end{pmatrix}$$

Решение. $\begin{pmatrix} \lambda & \lambda & \lambda & \lambda & 8 \\ \lambda & \lambda & \lambda & 7 & \lambda \\ \lambda & \lambda & 6 & \lambda & \lambda \\ \lambda & 5 & \lambda & \lambda & \lambda \\ 1 & \lambda-4 & \lambda-5 & \lambda-6 & \lambda-7 \end{pmatrix} \sim \begin{pmatrix} \lambda & 0 & 0 & 0 & 8-\lambda \\ \lambda & 0 & 0 & 7-\lambda & 0 \\ \lambda & 0 & 6-\lambda & 0 & 0 \\ \lambda & 5-\lambda & 0 & 0 & 0 \\ 1 & \lambda-5 & \lambda-6 & \lambda-7 & \lambda-8 \end{pmatrix} \sim$

$$\begin{pmatrix} \lambda & 0 & 0 & 0 & 8-\lambda \\ \lambda & 0 & 0 & 7-\lambda & 0 \\ \lambda & 0 & 6-\lambda & 0 & 0 \\ \lambda & 5-\lambda & 0 & 0 & 0 \\ 1+4\lambda & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Следователно при $\lambda \notin \{-\frac{1}{4}, 5, 6, 7, 8\}$: $\operatorname{rank}(A) = 5$, а при $\lambda \in \{-\frac{1}{4}, 5, 6, 7, 8\}$: $\operatorname{rank}(A) = 4$.

Задача 4. Докажете, че множеството $\{x+y\sqrt{26} \mid x, y \in \mathbb{Q}\}$ е линейно пространство над полето на рационалните числа $\mathbb{Q}$ относно обичайните операции събиране на числа и умножение на число с рационално число. Намерете негов базис и определете размерността му. Спрямо намерения базис намерете координатите на $9 + 5\sqrt{26}$.

Упътване. Да означим $V = \{x + y\sqrt{26} \mid x, y \in \mathbb{Q}\}$. Забелязваме, че $V \subseteq \mathbb{R}$. V е линейно пространство над $\mathbb{Q}$, тъй като V е подпространство на $\mathbb{R}$, разгледано като линейно пространство над $\mathbb{Q}$.

Да означим $\mathbf{e}_1 = 1$ и $\mathbf{e}_2 = \sqrt{26}$. Очевидно $V = l(\mathbf{e}_1, \mathbf{e}_2)$. Нека $\lambda\mathbf{e}_1 + \mu\mathbf{e}_2 = \mathbf{0}$, $\lambda, \mu \in \mathbb{Q}$, т.е. $\lambda + \mu\sqrt{26} = 0$. Ако $\mu \neq 0$, то $\sqrt{26} = -\frac{\lambda}{\mu} \in \mathbb{Q}$, противоречие. Следователно $\mu = 0$, откъдето $\lambda = 0$ и значи $\mathbf{e}_1, \mathbf{e}_2$ са линейно независими. Така $\mathbf{e}_1, \mathbf{e}_2$ е базис на V и $\dim V = 2$.