Линейни оператори. Ядро и образ на линеен оператор.[3]

Задача. Докажете, че изображението $\varphi : \begin{cases} M_2(F) \to M_2(F) \\ A \mapsto A + A^t \end{cases}$ е линеен оператор в пространството $M_2(F)$ на всички квадратни матрици от втори ред. Намерете матрицата на φ в базиса $E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ на $M_2(F)$. Определете $\mathrm{Ker}\varphi$ и $\mathrm{Im}\varphi$.

Нека $A, B \in M_2(F)$ и нека $\lambda \in F$. Тогава

$$\begin{aligned}\varphi(A+B) &= (A+B) + (A+B)^t = A + B + A^t + B^t = (A + A^t) + (B + B^t) = \varphi(A) + \varphi(B) \\ \varphi(\lambda A) &= \lambda A + (\lambda A)^t = \lambda A + \lambda A^t = \lambda(A + A^t) = \lambda\varphi(A).\end{aligned}$$

Имаме

$$\begin{aligned}\varphi(E_{11}) &= E_{11} + E_{11}^t = E_{11} + E_{11} = 2E_{11} + 0E_{12} + 0E_{21} + 0E_{22} \\ \varphi(E_{12}) &= E_{12} + E_{12}^t = E_{12} + E_{21} = 0E_{11} + 1E_{12} + 1E_{21} + 0E_{22} \\ \varphi(E_{21}) &= E_{21} + E_{21}^t = E_{21} + E_{12} = 0E_{11} + 1E_{12} + 1E_{21} + 0E_{22} \\ \varphi(E_{22}) &= E_{22} + E_{22}^t = E_{22} + E_{22} = 0E_{11} + 0E_{12} + 0E_{21} + 2E_{22}\end{aligned}$$

и следователно φ има матрица

$$A = \begin{array}{cccc} \varphi(E_{11}) & \varphi(E_{12}) & \varphi(E_{21}) & \varphi(E_{22}) \\ \downarrow & \downarrow & \downarrow & \downarrow \end{array} \\ \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

в базиса E_{11}, E_{12}, E_{21}, E_{22}.

[3]**Дефиниция.** Нека V е линейно пространство над полето F и нека $\varphi : V \to V$. Ще казваме, че φ е линеен оператор във V и ще бележим $\varphi \in \mathrm{Hom}V$, ако
1) $\varphi(\boldsymbol{a}_1 + \boldsymbol{a}_2) = \varphi(\boldsymbol{a}_1) + \varphi(\boldsymbol{a}_2)$ за всяко $\boldsymbol{a}_1, \boldsymbol{a}_2 \in V$;
2) $\varphi(\lambda\boldsymbol{a}) = \lambda\varphi(\boldsymbol{a})$ за всяко $\lambda \in F$ и всяко $\boldsymbol{a} \in V$.

Забележка. Изображението $\varphi : V \to V$ е линеен оператор тогава и само тогава, когато $\varphi(\lambda_1\boldsymbol{a}_1 + \cdots + \lambda_k\boldsymbol{a}_k) = \lambda_1\varphi(\boldsymbol{a}_1) + \cdots + \lambda_k\varphi(\boldsymbol{a}_k)$ за всяко $\lambda_1, \ldots, \lambda_k \in F$ и всяко $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \in V$ (т.е. когато образът на произволна линейна комбинация на вектори от V е същата линейна комбинация от образите им).

Нека $\dim V = n$, $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ е фиксиран базис на V и нека $\varphi \in \mathrm{Hom}V$. Нека

$$\begin{aligned}\varphi(\boldsymbol{e}_1) &= a_{11}\boldsymbol{e}_1 + a_{21}\boldsymbol{e}_2 + \cdots + a_{n1}\boldsymbol{e}_n \\ \varphi(\boldsymbol{e}_2) &= a_{12}\boldsymbol{e}_1 + a_{22}\boldsymbol{e}_2 + \cdots + a_{n2}\boldsymbol{e}_n \\ &\vdots \\ \varphi(\boldsymbol{e}_n) &= a_{1n}\boldsymbol{e}_1 + a_{2n}\boldsymbol{e}_2 + \cdots + a_{nn}\boldsymbol{e}_n\end{aligned}$$

Дефиниция. Квадратната матрица $A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}$, чиито стълбове са съставени от координатите на векторите $\varphi(\boldsymbol{e}_1)$, $\varphi(\boldsymbol{e}_2)$, …, $\varphi(\boldsymbol{e}_n)$ спрямо базиса $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$, се нарича матрица на линейния оператор φ в базиса $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$.

Нека $\boldsymbol{v} \in V$ и $\boldsymbol{v} = \alpha_1\boldsymbol{e}_1 + \alpha_2\boldsymbol{e}_2 + \cdots + \alpha_n\boldsymbol{e}_n$, $\varphi(\boldsymbol{v}) = \beta_1\boldsymbol{e}_1 + \beta_2\boldsymbol{e}_2 + \cdots + \beta_n\boldsymbol{e}_n$. Да означим с α и β следните матрици стълбове $\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$, $\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$.

Твърдение. В сила е матричното равенство $\beta = A\alpha$, т.е. $\boxed{\varphi(\boldsymbol{v})_{\boldsymbol{e}} = A_{\boldsymbol{e}}\boldsymbol{v}_{\boldsymbol{e}}}$.

Дефиниция. Под *ядро* на линейния оператор φ ще разбираме множеството

$$\mathrm{Ker}\,\varphi = \{\boldsymbol{v} \in V \mid \varphi(\boldsymbol{v}) = \boldsymbol{0}\},$$

а под *образ* на φ ще разбираме множеството

$$\mathrm{Im}\,\varphi = \{\boldsymbol{u} \in V \mid \exists\boldsymbol{v} \in V : \varphi(\boldsymbol{v}) = \boldsymbol{u}\}.$$

Имаме

$$\text{Ker}\varphi = \{A \in M_2(F) \mid \varphi(A) = A + A^t = \mathbf{0}\} = \{A \in M_2(F) \mid A^t = -A\} = \left\{ \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \mid a \in F \right\}$$

т.е. $\text{Ker}\varphi$ е пространството T от всички антисиметрични матрици от ред 2 с базис $E_{12} - E_{21}$.

Нека $A \in M_2(F)$, тогава $\varphi(A) = A + A^t \in \text{Im}\varphi$ и $(\varphi(A))^t = (A + A^t)^t = A^t + A = \varphi(A)$, т.е. $\varphi(A) \in S$ — пространството от всички симетрични матрици от ред 2, следователно $\text{Im}\varphi \subseteq S$. Обратно, нека $B \in S$ (т.е. $B^t = B$) и да разгледаме матрицата $A = \dfrac{1}{2}B$. Имаме

$$\varphi(A) = A + A^t = \frac{1}{2}(B + B^t) = \frac{2B}{2} = B,$$

т.е. $B \in \text{Im}\varphi$ и значи

$$\text{Im}\varphi = S = \left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} \mid a, b, c \in F \right\}$$

с базис E_{11}, $E_{12} + E_{21}$, E_{22}.

Задача. Докажете, че изображението $\delta : \begin{cases} F^4[x] \to F^4[x] \\ f \mapsto f' \end{cases}$ е линеен оператор (оператор на диференцирането) в пространството $F^4[x]$ от полиномите от степен, ненадминаваща 3. Намерете матрицата на δ в базиса $1, x, x^2, x^3$ на $F^4[x]$. Определете $\text{Ker}\delta$ и $\text{Im}\delta$

Решение. Нека $f, g \in F^4[x]$ и $\lambda \in F$. Тогава

$$\begin{aligned} \delta(f + g) &= (f + g)' = f' + g' = \delta(f) + \delta(g) \\ \delta(\lambda f) &= (\lambda f)' = \lambda f' = \lambda \delta(f). \end{aligned}$$

Имаме

$$\begin{aligned} \delta(1) &= 1' = 0 = 0.1 + 0x + 0x^2 + 0x^3 \\ \delta(x) &= x' = 1 = 1.1 + 0x + 0x^2 + 0x^3 \\ \delta(x^2) &= (x^2)' = 2x = 0.1 + 2x + 0x^2 + 0x^3 \\ \delta(x^3) &= (x^3)' = 3x^2 = 0.1 + 0x + 3x^2 + 0x^3 \end{aligned}$$

и следователно δ има матрица

$$\begin{array}{cc} & \begin{array}{cccc} \delta(1) & \delta(x) & \delta(x^2) & \delta(x^3) \\ \downarrow & \downarrow & \downarrow & \downarrow \end{array} \\ A = & \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

в базиса $1, x, x^2, x^3$.

Имаме

$$\text{Ker}\delta = \{f \in F^4[x] \mid \delta(f) = f' = 0\} = F.$$

Нека $f = a_0x^3 + a_1x^2 + a_2x + a_3 \in F^4[x]$, тогава $\delta(f) = 3a_0x^2 + 2a_1x + a_2 \in F^3[x]$, т.е. $\text{Im}\delta \subseteq F^3[x]$.

Обратно, нека $g = b_0x^2 + b_1x + b_2 \in F^3[x]$ и да разгледаме полинома $f = \dfrac{b_0}{3}x^3 + \dfrac{b_1}{2}x^2 + b_2x \in F^4[x]$, имаме $\delta(f) = f' = g$, т.е. $g \in \text{Im}\delta$ и значи $F^3[x] \subseteq \text{Im}\delta$. Окончателно $\text{Im}\delta = F^3[x]$ с базис $1, x, x^2$.

Задача. В тримерното линейно пространство V с базис $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ е зададен линеен оператор φ, действащ по правилото

$$\varphi(\alpha_1\boldsymbol{e}_1 + \alpha_2\boldsymbol{e}_2 + \alpha_3\boldsymbol{e}_3) = (\alpha_1 - 2\alpha_2 + 2\alpha_3)\boldsymbol{e}_1 + (2\alpha_1 - 4\alpha_2 + 4\alpha_3)\boldsymbol{e}_2 + (-\alpha_1 + 2\alpha_2 - 2\alpha_3)\boldsymbol{e}_3 \text{ за всяко } \alpha_1, \alpha_2, \alpha_3 \in F.$$

а) Намерете матрицата на φ в базиса $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ и координатите на $\varphi(\boldsymbol{v})$ в този базис, ако $\boldsymbol{v} = 3\boldsymbol{e}_1 + \boldsymbol{e}_2$.

б) Определете $\text{Ker}\varphi$ и $\text{Im}\varphi$.

Решение. а)

$$\begin{array}{llllcrcrcr} \alpha_1 = 1 & \alpha_2 = 0 & \alpha_3 = 0: & \varphi(\boldsymbol{e}_1) & = & \boldsymbol{e}_1 & + & 2\boldsymbol{e}_2 & - & \boldsymbol{e}_3 \\ \alpha_1 = 0 & \alpha_2 = 1 & \alpha_3 = 0: & \varphi(\boldsymbol{e}_2) & = & -2\boldsymbol{e}_1 & - & 4\boldsymbol{e}_2 & + & 2\boldsymbol{e}_3 \\ \alpha_1 = 0 & \alpha_2 = 0 & \alpha_3 = 1: & \varphi(\boldsymbol{e}_3) & = & 2\boldsymbol{e}_1 & + & 4\boldsymbol{e}_2 & - & 2\boldsymbol{e}_3 \end{array}$$

Следователно φ има матрица и

$$\begin{array}{cccc} & \varphi(\boldsymbol{e}_1) & \varphi(\boldsymbol{e}_3) & \varphi(\boldsymbol{e}_3) \\ & \downarrow & \downarrow & \downarrow \\ A = & \multicolumn{3}{c}{\begin{pmatrix} 1 & -2 & 2 \\ 2 & -4 & 4 \\ -1 & 2 & -2 \end{pmatrix}} \end{array}$$

в базиса $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$. Тогава

$$\varphi(\boldsymbol{v})_{\boldsymbol{e}} = A_{\boldsymbol{e}}\boldsymbol{v}_{\boldsymbol{e}} = \begin{pmatrix} 1 & -2 & 2 \\ 2 & -4 & 4 \\ -1 & 2 & -2 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix},$$

т.е. $\varphi(\boldsymbol{v}) = \boldsymbol{e}_1 + 2\boldsymbol{e}_2 - \boldsymbol{e}_3$.

б) Имаме $\boldsymbol{x} \in \mathrm{Ker}\varphi \iff \varphi(\boldsymbol{x})_{\boldsymbol{e}} = A_{\boldsymbol{e}}\boldsymbol{x}_{\boldsymbol{e}} = \mathbf{0}$. Следователно

$$\mathrm{Ker}\varphi : \left| \begin{array}{rcrcrcl} x_1 & - & 2x_2 & + & 2x_3 & = & 0 \\ 2x_1 & - & 4x_2 & + & 4x_3 & = & 0 \\ -x_1 & + & 2x_2 & - & 2x_3 & = & 0 \end{array} \right.$$

Полагаме $x_2 = p$, $x_3 = q$, тогава $x_1 = 2p - 2q$ и

$$\mathrm{Ker}\varphi = \{(2p - 2q, p, q) \mid p, q \in F\}.$$

$$\left.\begin{array}{ll} p = 1, q = 0: & \boldsymbol{v}_1 = (2, 1, 0) \\ p = 0, q = 1: & \boldsymbol{v}_2 = (-2, 0, 1) \end{array}\right\} \text{ФСР, т.е. базис на Ker}\varphi.$$

Имаме $\mathrm{Im}\varphi = l(\varphi(\boldsymbol{e}_1), \varphi(\boldsymbol{e}_2), \varphi(\boldsymbol{e}_3))$.

$$\begin{array}{c} \varphi(\boldsymbol{e}_1) \\ \varphi(\boldsymbol{e}_2) \\ \varphi(\boldsymbol{e}_3) \end{array} \begin{pmatrix} 1 & 2 & -1 \\ -2 & -4 & 2 \\ 2 & 4 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Следователно $\varphi(\boldsymbol{e}_1) = (1, 2, -1)$ е базис на $\mathrm{Im}\varphi$.

Задача. Нека $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ е базис на линейното пространство $\mathbb{V}$. Нека

$$\begin{array}{rcrcrcr} \mathbf{a}_1 & = & \mathbf{e}_1 & + & \boldsymbol{e}_2 & & \\ \mathbf{a}_2 & = & -\mathbf{e}_1 & + & \mathbf{e}_2 & + & 3\,\mathbf{e}_3 \\ \mathbf{a}_3 & = & \mathbf{e}_1 & & & + & \mathbf{e}_3 \end{array} \quad \text{и} \quad \begin{array}{rcrcrcr} \mathbf{b}_1 & = & \mathbf{a}_1 & + & 2\mathbf{a}_2 & - & \mathbf{a}_3 \\ \mathbf{b}_2 & = & \mathbf{a}_1 & - & \mathbf{a}_2 & & \\ \mathbf{b}_3 & = & -\mathbf{a}_1 & + & 2\mathbf{a}_2 & + & 2\mathbf{a}_3 \end{array}$$

са други два базиса на $\mathbb{V}$. Намерете матриците на прехода[4] $T = T_{\mathbf{e}\to\mathbf{a}}$, $U = U_{\mathbf{a}\to\mathbf{b}}$, $Q = Q_{\mathbf{e}\to\mathbf{b}}$, както и координатите на вектора $\mathbf{v} = 2\mathbf{e}_1 - 3\mathbf{e}_3$ в другите два базиса.

[4]Нека

$$\boldsymbol{e}_1,\ \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n \tag{3}$$
$$\boldsymbol{f}_1,\ \boldsymbol{f}_2, \ldots, \boldsymbol{f}_n \tag{4}$$

са два базиса на крайномерното линейно пространство V. Нека

$$\begin{array}{rcl} \boldsymbol{f}_1 & = & \tau_{11}\boldsymbol{e}_1 + \tau_{21}\boldsymbol{e}_2 + \cdots + \tau_{n1}\boldsymbol{e}_n \\ \boldsymbol{f}_2 & = & \tau_{12}\boldsymbol{e}_1 + \tau_{22}\boldsymbol{e}_2 + \cdots + \tau_{n2}\boldsymbol{e}_n \\ \vdots & & \\ \boldsymbol{f}_n & = & \tau_{1n}\boldsymbol{e}_1 + \tau_{2n}\boldsymbol{e}_2 + \cdots + \tau_{nn}\boldsymbol{e}_n \end{array}$$

Дефиниция. Квадратната матрица

$$T = \begin{pmatrix} \tau_{11} & \tau_{12} & \dots & \tau_{1n} \\ \tau_{21} & \tau_{22} & \dots & \tau_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tau_{n1} & \tau_{n2} & \dots & \tau_{nn} \end{pmatrix}$$

чиито стълбове са съставени от координатите на векторите от втория базис спрямо първия базис, се нарича матрица на прехода от базиса (3) към базиса (4).

Решение.

$$T = T_{\boldsymbol{e}\to\boldsymbol{a}} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}, \qquad U = U_{\boldsymbol{a}\to\boldsymbol{b}} = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 2 \\ -1 & 0 & 2 \end{pmatrix}.$$

$$\begin{tikzcd} & \mathbf{e} \arrow[dl, "T"'] \arrow[dr, "Q"] & \\ \mathbf{a} \arrow[rr, "U"] & & \mathbf{b} \end{tikzcd}$$

В сила е равенството $TU = Q$.

Действително, нека $\boldsymbol{v}$ е произволен вектор от V. Тогава $\boldsymbol{v_e} = T_{\boldsymbol{e}\to\boldsymbol{a}}\boldsymbol{v_a} = T_{\boldsymbol{e}\to\boldsymbol{a}}U_{\boldsymbol{a}\to\boldsymbol{b}}\boldsymbol{v_b}$ и $\boldsymbol{v_e} = Q_{\boldsymbol{e}\to\boldsymbol{b}}\boldsymbol{v_b}$ и следователно $TU\boldsymbol{v_b} = Q\boldsymbol{v_b}$, където $\boldsymbol{v_b}$ е стълбът от координатите на произволен вектор $\boldsymbol{v}$ в базиса $\boldsymbol{b}_1$, $\boldsymbol{b}_2$, $\boldsymbol{b}_3$. В частност, при $\boldsymbol{v} = \boldsymbol{b}_1$ имаме $\boldsymbol{v_b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ и от $TU\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = Q\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ следва, че първите стълбове на матриците TU и Q са равни. Аналогично при $\boldsymbol{v} = \boldsymbol{b}_2$ и $\boldsymbol{v} = \boldsymbol{b}_3$ се установява, че тези матрици имат още равни втори и трети стълб.

Следователно $Q = TU = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 2 \\ -1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} -2 & 2 & -1 \\ 3 & 0 & 1 \\ 5 & -3 & 8 \end{pmatrix}$.

$\boldsymbol{v_e} = T_{\boldsymbol{e}\to\boldsymbol{a}}\boldsymbol{v_a}$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & -3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 2 & -1 & -2 \\ 0 & 3 & 1 & -3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & -1 & -2 & 1 \\ 0 & 3 & 1 & -3 \end{array}\right) \ |\,.-1 \sim$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -5 & 0 \end{array}\right) \begin{array}{l} \\ \\ |\,:-5 \end{array} \sim \left(\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right),$$

т.е. $\boldsymbol{v} = \boldsymbol{a}_1 - \boldsymbol{a}_2$.

$\boldsymbol{v_a} = U_{\boldsymbol{a}\to\boldsymbol{b}}\boldsymbol{v_b}$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 2 & -1 & 2 & -1 \\ -1 & 0 & 2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -3 & 4 & -3 \\ 0 & 1 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 7 & 0 \end{array}\right) \begin{array}{l} \\ \\ |\,:7 \end{array} \sim$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right),$$

т.е. $\boldsymbol{v} = \boldsymbol{b}_2$.

Задача. Нека $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ е базис на линейното пространство $\mathbb{V}$. Нека

$$\begin{array}{rcl} \mathbf{a}_1 &=& \mathbf{e}_1 - \mathbf{e}_3 \\ \mathbf{a}_2 &=& 2\,\mathbf{e}_1 + \mathbf{e}_2 + 3\,\mathbf{e}_3 \\ \mathbf{a}_3 &=& -\,\mathbf{e}_1 + 2\,\mathbf{e}_2 \end{array} \quad \text{и} \quad \begin{array}{rcl} \mathbf{b}_1 &=& \mathbf{e}_1 - \mathbf{e}_2 + 5\,\mathbf{e}_3 \\ \mathbf{b}_2 &=& 4\,\mathbf{e}_1 + 7\,\mathbf{e}_2 + 9\,\mathbf{e}_3 \\ \mathbf{b}_3 &=& 2\,\mathbf{e}_1 + 3\,\mathbf{e}_2 + 2\,\mathbf{e}_3 \end{array}$$

са други два базиса на $\mathbb{V}$. Намерете матриците на прехода $T = T_{\mathbf{e}\to\mathbf{a}}$, $U = U_{\mathbf{a}\to\mathbf{b}}$, $Q = Q_{\mathbf{e}\to\mathbf{b}}$, както и координатите на вектора $\mathbf{v} = \mathbf{e}_1 + 5\mathbf{e}_2 + 2\mathbf{e}_3$ в базиса $\mathbf{b}_1$, $\boldsymbol{b}_2$, $\boldsymbol{b}_3$.

Решение.

$$T = T_{\boldsymbol{e}\to\boldsymbol{a}} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ -1 & 3 & 0 \end{pmatrix}, \qquad Q = Q_{\boldsymbol{e}\to\boldsymbol{b}} = \begin{pmatrix} 1 & 4 & 2 \\ -1 & 7 & 3 \\ 5 & 9 & 2 \end{pmatrix}.$$

Нека $\boldsymbol{v} \in V$ и нека $\boldsymbol{v} = \xi_1\boldsymbol{e}_1 + \xi_2\boldsymbol{e}_2 + \cdots + \xi_n\boldsymbol{e}_n$, $\boldsymbol{v} = \eta_1\boldsymbol{f}_1 + \eta_2\boldsymbol{f}_2 + \cdots + \eta_n\boldsymbol{f}_n$. Да означим с ξ и η следните матрици стълбове

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix}, \quad \eta = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_n \end{pmatrix}$$

Твърдение. В сила е матричното равенство

$$\xi = T\eta, \text{ т.е. } \boxed{\boldsymbol{v}_e = T_{e\to f}\boldsymbol{v}_f.}$$

Имаме $TU = Q$.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 4 & 2 \\ 0 & 1 & 2 & -1 & 7 & 3 \\ -1 & 3 & 0 & 5 & 9 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 4 & 2 \\ 0 & 1 & 2 & -1 & 7 & 3 \\ 0 & 5 & -1 & 6 & 13 & 4 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 4 & 2 \\ 0 & 1 & 2 & -1 & 7 & 3 \\ 0 & 0 & -11 & 11 & -22 & -11 \end{array}\right) \begin{array}{c} \\ \\ |:-11 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 4 & 2 \\ 0 & 1 & 2 & -1 & 7 & 3 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 6 & 3 \\ 0 & 1 & 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 0 & 1 \\ 0 & 1 & 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{array}\right).$$

Следователно $U = U_{\boldsymbol{a}\to\boldsymbol{b}} = \begin{pmatrix} -2 & 0 & 1 \\ 1 & 3 & 1 \\ -1 & 2 & 1 \end{pmatrix}$.

$$\boldsymbol{v_e} = Q_{\boldsymbol{e}\to\boldsymbol{b}}\boldsymbol{v_b}.$$

$$\left(\begin{array}{ccc|c} 1 & 4 & 2 & 1 \\ -1 & 7 & 3 & 5 \\ 5 & 9 & 2 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 4 & 2 & 1 \\ 0 & 11 & 5 & 6 \\ 0 & -11 & -8 & -3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 4 & 2 & 1 \\ 0 & 11 & 5 & 6 \\ 0 & 0 & -3 & 3 \end{array}\right) \begin{array}{c} \\ \\ |:-3 \end{array} \sim$$

$$\left(\begin{array}{ccc|c} 1 & 4 & 2 & 1 \\ 0 & 11 & 5 & 6 \\ 0 & 0 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 4 & 0 & 3 \\ 0 & 11 & 0 & 11 \\ 0 & 0 & 1 & -1 \end{array}\right) \,|:11 \sim \left(\begin{array}{ccc|c} 1 & 4 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right) \sim$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right).$$

Следователно $\boldsymbol{v} = -\boldsymbol{b}_1 + \boldsymbol{b}_2 - \boldsymbol{b}_3$.

Задача. Нека $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ е базис на V и $\varphi \in \mathrm{Hom}V$ е такъв, че $\varphi(\boldsymbol{a}_i) = \boldsymbol{b}_i$, $i = 1,2,3$, където

$$\begin{array}{lcl} \boldsymbol{a}_1 &=& \boldsymbol{e}_2 + \boldsymbol{e}_3 \\ \boldsymbol{a}_2 &=& 2\boldsymbol{e}_1 + \boldsymbol{e}_3 \\ \boldsymbol{a}_3 &=& \boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3 \end{array} \qquad \begin{array}{lcl} \boldsymbol{b}_1 &=& \boldsymbol{e}_1 + 3\boldsymbol{e}_2 - 2\boldsymbol{e}_3 \\ \boldsymbol{b}_2 &=& \boldsymbol{e}_1 + \boldsymbol{e}_3 \\ \boldsymbol{b}_3 &=& 2\boldsymbol{e}_1 + 3\boldsymbol{e}_2 - \boldsymbol{e}_3 \end{array}$$

Намерете

а) матрицата на φ в базиса $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ и в базиса $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$. Намерете координатите на $\varphi(v)$ в тези два базиса, ако $\boldsymbol{v} = 7\boldsymbol{e}_1 + 4\boldsymbol{e}_2 + 6\boldsymbol{e}_3$;

б) базис на $\mathrm{Ker}\varphi$ и $\mathrm{Im}\varphi$.

Забележка. Да отбележим, че задачата е коректна, т.е. операторът е еднозначно определен чрез образите на три вектора $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$, само ако тези вектори също образуват базис на V. Тогава, ако $\boldsymbol{v} \in V$, $\boldsymbol{v} = \lambda_1\boldsymbol{a}_1 + \lambda_2\boldsymbol{a}_2 + \lambda_3\boldsymbol{a}_3$, то

$$\varphi(\boldsymbol{v}) = \lambda_1\varphi(\boldsymbol{a}_1) + \lambda_2\varphi(\boldsymbol{a}_2) + \lambda_3\varphi(\boldsymbol{a}_3) = \lambda_1\boldsymbol{b}_1 + \lambda_2\boldsymbol{b}_2 + \lambda_3\boldsymbol{b}_3.$$

Решение. а) $\begin{vmatrix} 0 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} \neq 0$, следователно $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ са линейно независими и значи са базис на V.

$$\begin{array}{lc|ccccccccccc} \varphi(\boldsymbol{a}_1) &=& & & \varphi(\boldsymbol{e}_2) &+& \varphi(\boldsymbol{e}_3) &=& \boldsymbol{b}_1 &=& (1, & 3, & -2)_{\boldsymbol{e}} \\ \varphi(\boldsymbol{a}_2) &=& 2\varphi(\boldsymbol{e}_1) & & &+& \varphi(\boldsymbol{e}_3) &=& \boldsymbol{b}_2 &=& (1, & 0, & 1)_{\boldsymbol{e}} \\ \varphi(\boldsymbol{a}_3) &=& \varphi(\boldsymbol{e}_1) &+& \varphi(\boldsymbol{e}_2) &+& \varphi(\boldsymbol{e}_3) &=& \boldsymbol{b}_3 &=& (2, & 3, & -1)_{\boldsymbol{e}} \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & (2, & 3, & -1) \\ 0 & 1 & 1 & (1, & 3, & -2) \\ 2 & 0 & 1 & (1, & 0, & 1) \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & (2, & 3, & -1) \\ 0 & 1 & 1 & (1, & 3, & -2) \\ 0 & -2 & -1 & (-3, & -6, & 3) \end{array}\right) \sim$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & (2, & 3, & -1) \\ 0 & 1 & 1 & (1, & 3, & -2) \\ 0 & 0 & 1 & (-1, & 0, & -1) \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & (1, & 0, & 1) \\ 0 & 1 & 0 & (2, & 3, & -1) \\ 0 & 0 & 1 & (-1, & 0, & -1) \end{array}\right)$$

Следователно $\varphi(\boldsymbol{e}_1) = (1,0,1)_{\boldsymbol{e}}$, $\varphi(\boldsymbol{e}_2) = (2,3,-1)_{\boldsymbol{e}}$, $\varphi(\boldsymbol{e}_3) = (-1,0,-1)_{\boldsymbol{e}}$ и φ има матрица

$$\begin{array}{cc} & \begin{array}{ccc} \varphi(\boldsymbol{e}_1) & \varphi(\boldsymbol{e}_2) & \varphi(\boldsymbol{e}_3) \\ \downarrow & \downarrow & \downarrow \end{array} \\ A = & \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & 0 \\ 1 & -1 & -1 \end{pmatrix} \end{array}$$

в базиса $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$.

Нека B е матрицата на φ в базиса $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$. Тогава $B = T^{-1}AT$, където $T = T_{\boldsymbol{e}\to\boldsymbol{a}} = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ Произведението $T^{-1}A$ ще намерим като решение на матричното уравнение $TX = A$.

$$\left(\begin{array}{ccc|ccc} 0 & 2 & 1 & 1 & 2 & -1 \\ 1 & 0 & 1 & 0 & 3 & 0 \\ 1 & 1 & 1 & 1 & -1 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 3 & 0 \\ 0 & 2 & 1 & 1 & 2 & -1 \\ 0 & 1 & 0 & 1 & -4 & -1 \end{array}\right) \sim$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & -1 & 10 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -7 & -1 \\ 0 & 1 & 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & -1 & 10 & 1 \end{array}\right)$$

Следователно

$$B = T^{-1}AT = \begin{pmatrix} 1 & -7 & -1 \\ 1 & -4 & -1 \\ -1 & 10 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -8 & 1 & -7 \\ -5 & 1 & -4 \\ 11 & -1 & 10 \end{pmatrix}.$$

Имаме

$$\varphi(\boldsymbol{v})_{\boldsymbol{e}} = A_{\boldsymbol{e}}\boldsymbol{v}_{\boldsymbol{e}} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & 0 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 7 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 9 \\ 12 \\ -3 \end{pmatrix},$$

т.е. $\varphi(\boldsymbol{v}) = 9\boldsymbol{e}_1 + 12\boldsymbol{e}_2 - 3\boldsymbol{e}_3$.

Имаме $\varphi(\boldsymbol{v})_{\boldsymbol{e}} = T_{\boldsymbol{e}\to\boldsymbol{a}}\varphi(\boldsymbol{v})_{\boldsymbol{a}}$.

$$\left(\begin{array}{ccc|c} 0 & 2 & 1 & 9 \\ 1 & 0 & 1 & 12 \\ 1 & 1 & 1 & -3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & -3 \\ 0 & 1 & 0 & -15 \\ 0 & 2 & 1 & 9 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & -3 \\ 0 & 1 & 0 & -15 \\ 0 & 0 & 1 & 39 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -27 \\ 0 & 1 & 0 & -15 \\ 0 & 0 & 1 & 39 \end{array}\right),$$

т.е. $\varphi(\boldsymbol{v}) = -27\boldsymbol{a}_1 - 15\boldsymbol{a}_2 + 39\boldsymbol{a}_3$.

б)

$$\mathrm{Ker}\varphi : \left|\begin{array}{ccccccc} x_1 & + & 2x_2 & - & x_3 & = & 0 \\ & & 3x_2 & & & = & 0 \\ x_1 & - & x_2 & - & x_3 & = & 0 \end{array}\right.$$

$$\begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & 0 \\ 1 & -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & -3 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Следователно $\mathrm{Ker}\varphi = \{(p,0,p) \mid p \in F\}$. При $p = 1$ получаваме $\boldsymbol{v} = (1,0,1)$ — ФСР, т.е. базис на $\mathrm{Ker}\varphi$. Имаме $\mathrm{Im}\varphi = l(\varphi(\boldsymbol{e}_1), \varphi(\boldsymbol{e}_2), \varphi(\boldsymbol{e}_3))$.

$$\begin{array}{c} \varphi(\boldsymbol{e}_1) \\ \varphi(\boldsymbol{e}_2) \\ \varphi(\boldsymbol{e}_3) \end{array} \begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & -1 \\ -1 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{pmatrix}$$

Следователно $\varphi(\boldsymbol{e}_1)$, $\varphi(\boldsymbol{e}_2)$ е базис на $\mathrm{Im}\varphi$.

Забележка. Нека

$$\begin{array}{cc} & \begin{array}{ccc} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 \\ \downarrow & \downarrow & \downarrow \end{array} \\ A = & \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \end{array} \qquad \begin{array}{cc} & \begin{array}{ccc} \boldsymbol{b}_1 & \boldsymbol{b}_2 & \boldsymbol{b}_3 \\ \downarrow & \downarrow & \downarrow \end{array} \\ B = & \begin{pmatrix} 1 & 1 & 2 \\ 3 & 0 & 3 \\ -2 & 1 & -1 \end{pmatrix} \end{array}$$

Нека $\psi_1, \psi_2 \in \mathrm{Hom}V$ и нека на ψ_1 съответства матрица A, а на ψ_2 — матрица B в базиса $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ (т.е. $\psi_1(\boldsymbol{e}_i) = \boldsymbol{a}_i$, $\psi_2(\boldsymbol{e}_i) = \boldsymbol{b}_i$, $1 \leq i \leq 3$). Нека $\varphi \in \mathrm{Hom}V$ е такъв, че $\varphi(\boldsymbol{a}_i) = \boldsymbol{b}_i$, $1 \leq i \leq 3$, и нека на φ съответства матрица X в базиса $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$. Тогава $\varphi\psi_1 = \psi_2$ (тъй като $\varphi(\psi_1(\boldsymbol{e}_i)) = \varphi(\boldsymbol{a}_i) = \boldsymbol{b}_i = \psi_2(\boldsymbol{e}_i)$, $1 \leq i \leq 3$) и значи $XA = B$.