①

Упражнение 2
Определени интеграли, част 2

Зад. 1 Пресметнете $I_m = \int_0^{\pi/2} \sin^m x\, dx$ и $J_m = \int_0^{\pi/2} \cos^m x\, dx$, където $m \in \mathbb{N}$.

Решение: Имаме, че

$$I_m \underset{x=\frac{\pi}{2}-t}{=} \int_{\pi/2}^{0} \sin^m\left(\frac{\pi}{2}-t\right) d\left(\frac{\pi}{2}-t\right) = \int_0^{\pi/2} \cos^m t\, dt = J_m.$$

Сл. $I_m = J_m$.

Освен това при $m \geq 2$ имаме, че

$$I_m = \int_0^{\pi/2} \sin^{m-1} x . \sin x\, dx = -\int_0^{\pi/2} \sin^{m-1} x\, d\cos x =$$

$$= -\underbrace{\left(\sin^{m-1} x . \cos x\right)\Big|_0^{\pi/2}}_{0} + \int_0^{\pi/2} \cos x\, d\sin^{m-1} x =$$

$$= (m-1)\int_0^{\pi/2} \sin^{m-2} x . \cos^2 x\, dx =$$

$$= (m-1)\int_0^{\pi/2} \sin^{m-2} x . (1-\sin^2 x)\, dx = (m-1) I_{m-2} - (m-1) I_m$$

И така, оказа се, че $I_m = (m-1)I_{m-2} - (m-1)I_m$.

Сл. $I_m = \frac{m-1}{m} I_{m-2}$ за $m \geq 2$.

Ако $m = 2n$, където $n \in \mathbb{N}$, то $I_{2n} = \frac{2n-1}{2n} I_{2n-2} =$

$$= \frac{2n-1}{2n} . \frac{2n-3}{2n-2} I_{2n-4} = \frac{2n-1}{2n} . \frac{2n-3}{2n-2} . \frac{2n-5}{2n-4} I_{2n-6} = \cdots$$

$$\cdots = \frac{2n-1}{2n} . \frac{2n-3}{2n-2} . \frac{2n-5}{2n-4} \cdots \frac{1}{2} I_0 = \frac{(2n-1)!!}{(2n)!!} \frac{\pi}{2},$$ защото

$$I_0 = \int_0^{\pi/2} 1\, dx = x\Big|_0^{\pi/2} = \frac{\pi}{2}.$$

По определение, ако $n \in \mathbb{N}$, то

$(2n)!! = 2n . (2n-2) . (2n-4) \ldots 6 . 4 . 2$,

$(2n+1)!! = (2n+1) . (2n-1) . (2n-3) \ldots 5 . 3 . 1$.

Също по определение $0!! = 1$.

например
$6!! = 6.4.2 = 48$
$5!! = 5.3.1 = 15$

Ако $m = 2n+1$, където $n \in \mathbb{N}$, то $I_{2n+1} = \frac{2n}{2n+1} I_{2n-1} =$

$$= \frac{2n}{2n+1} . \frac{2n-2}{2n-1} I_{2n-3} = \frac{2n}{2n+1} . \frac{2n-2}{2n-1} . \frac{2n-4}{2n-3} I_{2n-5} = \cdots$$

$$\cdots = \frac{2n}{2n+1} . \frac{2n-2}{2n-1} . \frac{2n-4}{2n-3} \cdots \frac{2}{3} I_1 = \frac{(2n)!!}{(2n+1)!!},$$ защото

② $I_1 = \int_0^{\pi/2} \sin x \, dx = (-\cos x)\Big|_0^{\pi/2} = \cos x\Big|_{\pi/2}^{0} = 1 - 0 = 1.$

Отг. на зад. 1: $I_m = J_m = \begin{cases} \frac{(m-1)!!}{m!!}\frac{\pi}{2}, & \text{ако } m \text{ е четно} \\ \frac{(m-1)!!}{m!!}, & \text{ако } m \text{ е нечетно} \end{cases}$

Зад. 2 Пресметнете:

а) $I_n = \int_0^{\pi/2} \cos^n x \sin nx \, dx$, където $n \in \mathbb{N}$;

б) $J_n = \int_0^{\pi/2} \cos^n x \cos nx \, dx$, където $n \in \mathbb{N}$.

Решение: а) $I_n = \frac{1}{n}\int_0^{\pi/2} \cos^n x \sin nx \, d(nx) =$

$= -\frac{1}{n}\int_0^{\pi/2} \cos^n x \, d\cos nx =$ $\quad$ $\sin u \, du = d(-\cos u)$, $u = nx$

$= -\frac{1}{n}\left(\cos^n x \cos nx\right)\Big|_0^{\pi/2} + \frac{1}{n}\int_0^{\pi/2} \cos nx \, d\cos^n x =$

$= -\frac{1}{n}(0-1) + \frac{1}{\cancel{n}}\int_0^{\pi/2} \cos nx . \cancel{n} \cos^{n-1} x . (-\sin x) dx =$

$= \frac{1}{n} - \int_0^{\pi/2} \cos^{n-1} x \cos nx \sin x \, dx.$

Събираме равенствата

$$I_n = \int_0^{\pi/2} \cos^n x \sin nx \, dx \quad \text{и}$$

$$I_n = \frac{1}{n} - \int_0^{\pi/2} \cos^{n-1} x \cos nx \sin x \, dx$$

и получаваме, че

$2I_n = \frac{1}{n} + \int_0^{\pi/2} \cos^{n-1} x \left(\sin nx \cos x - \sin x \cos nx\right) dx =$

$= \frac{1}{n} + \int_0^{\pi/2} \cos^{n-1} x \sin(n-1)x \, dx = \frac{1}{n} + I_{n-1}.$

Оказа се, че $I_n = \frac{1}{2}\left(\frac{1}{n} + I_{n-1}\right).$

Тогава $I_n = \frac{1}{2}\left(\frac{1}{n} + I_{n-1}\right) = \frac{1}{2}\left[\frac{1}{n} + \frac{1}{2}\left(\frac{1}{n-1} + I_{n-2}\right)\right] =$

$= \frac{1}{2^2}\left(\frac{2}{n} + \frac{1}{n-1} + I_{n-2}\right) = \frac{1}{2^2}\left[\frac{2}{n} + \frac{1}{n-1} + \frac{1}{2}\left(\frac{1}{n-2} + I_{n-3}\right)\right] =$

$= \frac{1}{2^3}\left(\frac{2^2}{n} + \frac{2}{n-1} + \frac{1}{n-2} + I_{n-3}\right) = \dots$

$\dots = \frac{1}{2^n}\left(\frac{2^{n-1}}{n} + \frac{2^{n-2}}{n-1} + \frac{2^{n-3}}{n-2} + \dots + \frac{1}{1} + \overset{0}{\overset{\parallel}{I_0}}\right).$

Отг. на а): $I_n = \frac{1}{2^{n+1}}\left(\frac{2}{1} + \frac{2^2}{2} + \frac{2^3}{3} + \dots + \frac{2^{n-1}}{n-1} + \frac{2^n}{n}\right).$

③ б) $J_n = \frac{1}{n}\int_0^{\pi/2} \cos^n x \cos nx \, d(nx) =$ $\left(\begin{matrix}\cos u\, du = d\sin u\\ u = nx\end{matrix}\right)$

$= \frac{1}{n}\int_0^{\pi/2} \cos^n x \, d\sin nx =$

$= \frac{1}{n}\underbrace{\left(\cos^n x \sin nx\right)\Big|_0^{\pi/2}}_{=0} - \frac{1}{n}\int_0^{\pi/2} \sin nx \, d\cos^n x =$

$= -\frac{1}{\cancel{n}}\int_0^{\pi/2} \sin nx \, \cancel{n} \cos^{n-1} x (-\sin x) dx =$

$= \int_0^{\pi/2} \cos^{n-1} x \sin nx \sin x \, dx.$

Събираме равенствата

$J_n = \int_0^{\pi/2} \cos^n x \cos nx \, dx$ и

$J_n = \int_0^{\pi/2} \cos^{n-1} x \sin nx \sin x \, dx$

и получаваме, че

$2J_n = \int_0^{\pi/2} \cos^{n-1} x (\cos nx \cos x + \sin nx \sin x) dx =$

$= \int_0^{\pi/2} \cos^{n-1} x \cos (n-1)x \, dx = J_{n-1}.$

Оказа се, че $J_n = \frac{1}{2} J_{n-1}.$

Тогава $J_n = \frac{1}{2} J_{n-1} = \frac{1}{2^2} J_{n-2} = \frac{1}{2^3} J_{n-3} = \dots$

$\dots = \frac{1}{2^n} J_0 = \frac{1}{2^n}\int_0^{\pi/2} 1 \, dx = \frac{1}{2^n} x\Big|_0^{\pi/2} = \frac{\pi}{2^{n+1}}.$

Отг. на б): $J_n = \frac{\pi}{2^{n+1}}.$

Зад. 3 Пресметнете $\lim_{n\to\infty} S_n$, ако:

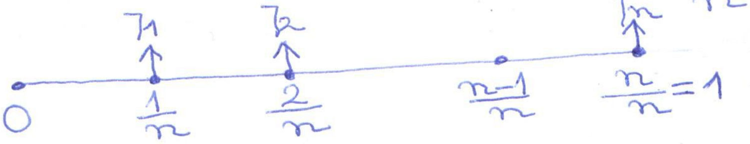
а) $S_n = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n}, \; n \in \mathbb{N};$

б) $S_n = \frac{1}{\sqrt{4n^2-1^2}} + \frac{1}{\sqrt{4n^2-2^2}} + \dots + \frac{1}{\sqrt{4n^2-n^2}}, \; n \in \mathbb{N};$

в) $S_n = \frac{1^p + 2^p + \dots + n^p}{n^{p+1}}, \; n \in \mathbb{N}$ (тук $p > 0$ е константа);

г) $S_n = \frac{\sqrt[n]{n!}}{n}, \; n \in \mathbb{N}.$

4 Решение: а) $S_n = \frac{1}{n}\cdot\frac{1}{1+\frac{1}{n}} + \frac{1}{n}\cdot\frac{1}{1+\frac{2}{n}} + \dots + \frac{1}{n}\cdot\frac{1}{1+1}$



S_n е Риманова интегрална сума на $f(x)=\frac{1}{1+x}$ в $[0,1]$ при разбиването $0<\frac{1}{n}<\frac{2}{n}<\frac{3}{n}<\dots<\frac{n-1}{n}<\frac{n}{n}=1$.
Диаметърът на разбиването е $\frac{1}{n}$ и $\frac{1}{n}\longrightarrow 0$ при $n\longrightarrow+\infty$. Сл. $\lim\limits_{n\to\infty} S_n = \int_0^1 \frac{1}{1+x}\,dx =$
$= \int_0^1 \frac{1}{1+x}\,d(1+x) = \ln(1+x)\Big|_0^1 = \ln 2$.

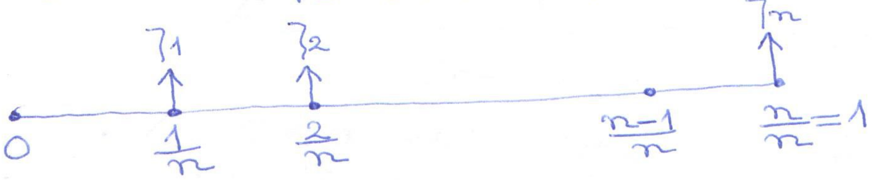
Отг. на а): $\lim\limits_{n\to\infty} S_n = \ln 2$.

б) $S_n = \frac{1}{2n}\cdot\frac{1}{\sqrt{1-\left(\frac{1}{2n}\right)^2}} + \frac{1}{2n}\cdot\frac{1}{\sqrt{1-\left(\frac{2}{2n}\right)^2}} + \dots + \frac{1}{2n}\cdot\frac{1}{\sqrt{1-\left(\frac{n}{2n}\right)^2}}$.

S_n е Риманова интегрална сума на $f(x)=\frac{1}{\sqrt{1-x^2}}$ в $[0,\frac{1}{2}]$ при разбиването $0<\frac{1}{2n}<\frac{2}{2n}<\frac{3}{2n}<\dots<\frac{n-1}{2n}<\frac{n}{2n}=\frac{1}{2}$
Диаметърът на разбиването е $\frac{1}{2n}$ и $\frac{1}{2n}\longrightarrow 0$ при $n\longrightarrow+\infty$. Сл. $\lim\limits_{n\to\infty} S_n = \int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\,dx =$
$= \arcsin x\Big|_0^{1/2} = \frac{\pi}{6} - 0 = \frac{\pi}{6}$.

Отг. на б): $\lim\limits_{n\to\infty} S_n = \frac{\pi}{6}$.

в) $S_n = \frac{1}{n}\cdot\left(\frac{1}{n}\right)^p + \frac{1}{n}\cdot\left(\frac{2}{n}\right)^p + \dots + \frac{1}{n}\cdot\left(\frac{n}{n}\right)^p$



S_n е Риманова интегрална сума на $f(x)=x^p$ в $[0,1]$ при разбиването $0<\frac{1}{n}<\frac{2}{n}<\frac{3}{n}<\dots<\frac{n-1}{n}<\frac{n}{n}=1$.
Диаметърът на разбиването е $\frac{1}{n}$ и $\frac{1}{n}\longrightarrow 0$ при $n\longrightarrow+\infty$.
Сл. $\lim\limits_{n\to\infty} S_n = \int_0^1 x^p\,dx = \frac{x^{p+1}}{p+1}\Big|_0^1 = \frac{1}{p+1}$.

Отг. на в): $\lim\limits_{n\to\infty} S_n = \frac{1}{p+1}$.

(5) г) $S_n = \frac{\sqrt[n]{n!}}{n} = e^{\ln\frac{\sqrt[n]{n!}}{n}} = e^{\ln\sqrt[n]{\frac{n!}{n^n}}} =$

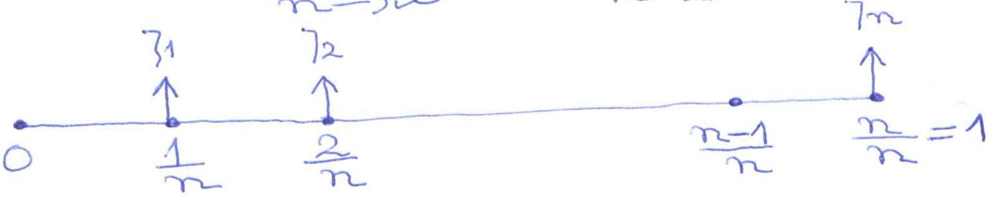
$= e^{\frac{1}{n}\ln\frac{n!}{n^n}} = e^{\frac{1}{n}\ln\frac{1.2.3\ldots(n-1)n}{n.n.n\ldots n.n}} =$ ако $a>0, b>0$, то $\ln(ab)=\ln a+\ln b$

$= e^{\frac{1}{n}\ln\left(\frac{1}{n}\cdot\frac{2}{n}\cdot\frac{3}{n}\cdots\frac{n-1}{n}\cdot\frac{n}{n}\right)} =$

$= e^{\frac{1}{n}\left[\ln\frac{1}{n}+\ln\frac{2}{n}+\ln\frac{3}{n}+\ldots+\ln\frac{n-1}{n}+\ln\frac{n}{n}\right]} =$

$= e^{A_n}$, където $A_n = \frac{1}{n}\ln\frac{1}{n}+\frac{1}{n}\ln\frac{2}{n}+\ldots+\frac{1}{n}\ln\frac{n}{n}$.

Тогава $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} e^{A_n} = e^{\lim\limits_{n\to\infty} A_n}$



A_n е Риманова интегрална сума на $f(x)=\ln x$ в $(0,1]$ при разбиването

$0<\frac{1}{n}<\frac{2}{n}<\frac{3}{n}<\ldots<\frac{n-1}{n}<\frac{n}{n}=1$.

Диаметърът на разбиването е $\frac{1}{n}$ и $\frac{1}{n}\to 0$ при $n\to+\infty$

сл. $\lim\limits_{n\to\infty} A_n = \int_0^1 \ln x\,dx = (x\ln x)\Big|_0^1 - \int_0^1 x\,d\ln x =$

$= 0 - \lim\limits_{x\to 0^+}(x\ln x) - \int_0^1 x\cdot\frac{1}{x}\,dx =$

$= -\lim\limits_{x\to 0^+}\frac{\ln x}{\frac{1}{x}} - \int_0^1 1\,dx \overset{\text{Лоп.}}{=} -\lim\limits_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}} - x\Big|_0^1 =$

$= \lim\limits_{x\to 0^+} x - (1-0) = 0-1 = -1$.

Така $\lim\limits_{n\to\infty} S_n = e^{\lim\limits_{n\to\infty} A_n} = e^{-1} = \frac{1}{e}$. Отг. на г): $\lim\limits_{n\to\infty} S_n = \frac{1}{e}$.

Геометрични приложения на опр. интеграл

(1) Пресмятане на лице на криволинеен трапец

Лицето $S(T)$ на криволинейния трапец

$T: \begin{cases} a\leqq x\leqq b \\ f(x)\leqq y\leqq g(x)\end{cases}$, където $f(x)$ и $g(x)$ са непрекъснати в $[a,b]$, е

$$S(T) = \int_a^b [g(x)-f(x)]\,dx.$$

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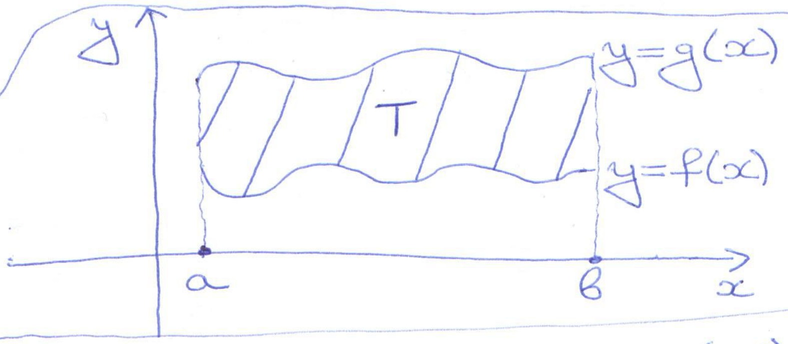
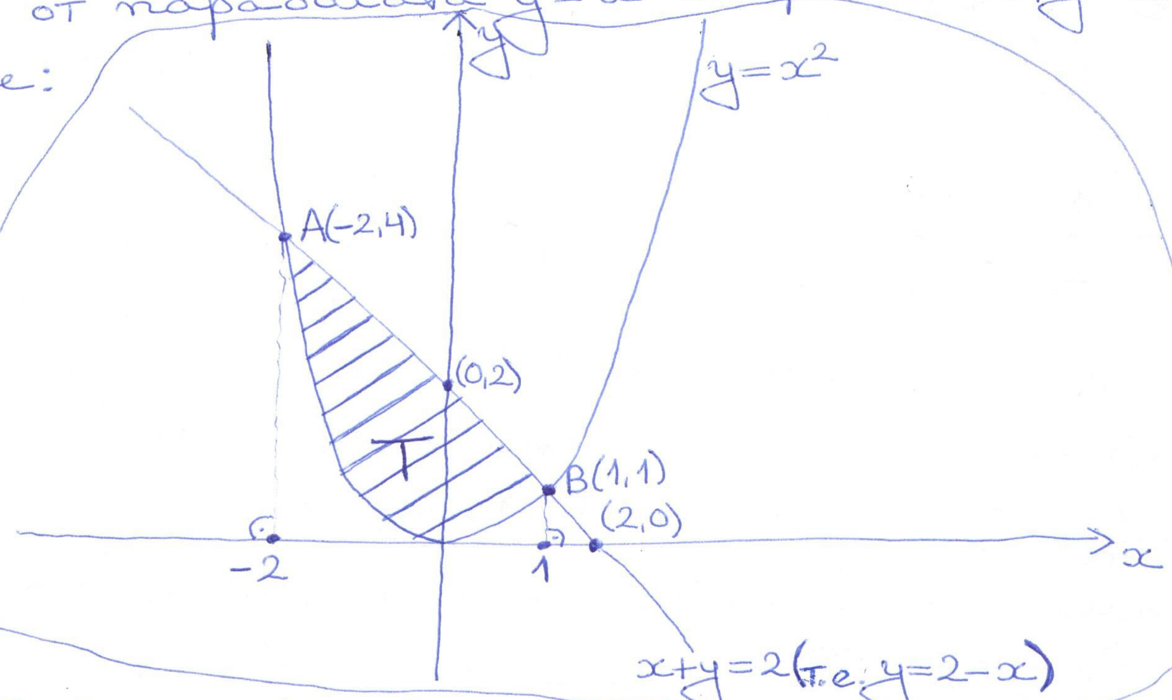


$$s(T)=\int_a^в [g(x)-f(x)]\,dx$$

Зад. 1 Намерете лицето $s(T)$ на фигурата T, заградена от параболата $y=x^2$ и правата $x+y=2$.

Решение:



Координатите на A и B са решенията на системата

$$\begin{cases} y=x^2 \\ x+y=2 \end{cases} \iff \begin{cases} 2-x=x^2 \\ y=2-x \end{cases} \iff \begin{cases} x^2+x-2=0 \\ y=2-x \end{cases},\quad \begin{matrix} B(1,1) \\ A(-2,4) \end{matrix}$$

$$s(T)=\int_{-2}^{1}\left[(2-x)-x^2\right]dx=$$

$$=\left(2x-\frac{x^2}{2}-\frac{x^3}{3}\right)\Big|_{-2}^{1}=$$

$$=\left(2-\frac{1}{2}-\frac{1}{3}\right)-\left(-4-2+\frac{8}{3}\right)=$$

$$=8-\frac{1}{2}-3=5-\frac{1}{2}=\frac{9}{2}$$

Отг. на зад. 1: $s(T)=\frac{9}{2}$.

В следващото упражнение ще разгледаме и още примери за пресмятане на лице на криволинеен трапец.