V над поле F ($\dim V = n$) $\underline{F^n}$, $e_1, \dots, e_n$ базис

<u>Опр.</u> $f : \underbrace{V \times \dots \times V}_{k} \to F$ е полилинейна

$s = 1, \dots, k$, $a_i \in V$

$f(a_1, \dots, a_s' + a_s'', \dots, a_k) = f(a_1, \dots, a_s', \dots, a_k) + f(a_1, \dots, a_s'', \dots, a_k)$

$f(a_1, \dots, \lambda a_s, \dots, a_k) = \lambda f(a_1, \dots, a_s, \dots, a_k)$

$a_i = a_{i1} e_1 + \dots + a_{in} e_n)$

$$f(a_1, \dots, a_k) = \sum_{j_1=1}^{n} \dots \sum_{j_k=1}^{n} a_{1j_1} a_{2j_2} \dots a_{kj_k} \boxed{f(e_{j_1}, e_{j_2}, \dots, e_{j_k})}$$

<u>Опр.</u> $f : \underbrace{V \times \dots \times V}_{k} \to F$ смяна местата на два аргум. смяна знака

$f : V^3 \to F$ $a_1, a_2, a_3 \in V$

$$f(a_1, a_2, a_3) = \begin{cases} -f(a_2, a_1, a_3) \\ -f(a_3, a_2, a_1) \\ -f(a_1, a_3, a_2) \end{cases} = +f(a_2, a_3, a_1)$$

$[2, 3, 1] = 2$

I/ Нека $f : \underbrace{V \times \dots \times V}_{} \to F$ е полилинейна и антисиметрична ф-я; $e_1, \dots, e_n$ - базис на V

$a_i = (a_{i1}, \dots, a_{in}) = a_{i1}e_1 + \dots + a_{in}e_n$

$$\Rightarrow \boxed{\sum_{j_1} \dots \sum_{j_k} a_{1j_1} \dots a_{kj_k} f(e_{j_1}, \dots, e_{j_k})}$$

$$f(a_1, a_2, \dots, a_k) = \sum_{\substack{j_1 \dots j_k \\ \text{различни}}} a_{1j_1} \dots a_{kj_k} f(e_{j_1} \dots e_{j_k})$$

ако $j_s = j_t$

$f(e_1 \dots e_{j_s}, \dots e_{j_t} \dots e_{j_k}) = 0$

за $k > n$ $\quad f(a_1, \dots, a_k) = 0$

за $k = n$ $\quad \sum_{\substack{j_1 \dots j_n \\ \text{перм.}}} a_{1j_1} \dots a_{nj_n} f(e_{j_1} \dots e_{j_n})$

$k < n$

вариации $\sum_{j_1 \dots j_k} a_{1j_1} \dots a_{kj_k} f(e_{j_1} \dots e_{j_k})$

$$f(a_1, \ldots, a_n) = \sum_{\substack{j_1 \ldots j_n \\ \text{перм.}}} a_{1j_1} \ldots a_{nj_n} (-1)^{[j_1 \ldots j_n]} f(e_1, \ldots, e_n)$$

$k = n$

$$= \left(\sum_{j_1 \ldots j_n} a_{1j_1} \cdots a_{nj_n} (-1)^{[j_1 \ldots j_n]} \right) f(e_1, \ldots, e_n)$$

$f: \underbrace{V\times\dots\times V}_{n} \longrightarrow F$ $\qquad e_1\dots e_n$ базис

$$f(a_1,\dots a_n)=\left(\sum_{\substack{j_1\dots j_n\\ \text{перм}}} a_{1j_1}\dots a_{nj_n}(-1)^{[j_1\dots j_n]}\right) f(e_1\dots e_n)$$

$A \in M_{n\times n}(F)$

$$A=\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn}\end{pmatrix} \qquad \begin{array}{l} a_1=(a_{11},\dots,a_{1n}) \\ \vdots \\ a_n=(a_{n1},\dots,a_{nn}) \end{array} \in F^n$$

Опр./ Детерминантата на кв. матр. $A_{n\times n}$ е полилинейна и антисиметрична ф-я на редовете на матрицата, която за E дава стойност 1.

$\det: \underbrace{F^n\times\dots\times F^n}_{n} \longrightarrow F \qquad \det(E)=\det(e_1,\dots,e_n)=1$

$$\det A = \det(a_1, \dots, a_n) = |A| = \sum_{\substack{j_1 \dots j_n \\ \text{перм. на } 1,\dots,n}} (-1)^{[j_1 \dots j_n]} a_{1j_1} a_{2j_2} \dots a_{nj_n}$$

Т/ Съществува единствена ф-я детерминанта

Д-во/ Ако допуснем, че $\varphi_1, \varphi_2 : \underbrace{F^n \times \dots \times F^n}_{n} \to F$

φ_1, φ_2 полилинейни антисимет. нормирани $\quad a_1, \dots, a_n \in F^n$

$$\Rightarrow \varphi_1 = \sum_{j_1 \dots j_n} (-1)^{[j_1 \dots j_n]} a_{1j_1} \dots a_{nj_n} = \varphi_2(a_1, \dots, a_n)$$

$\Rightarrow \varphi_1 = \varphi_2 \Rightarrow$ единственост

Нека $f = \sum_{j_1 \dots j_n \in S_n} (-1)^{[j_1 \dots j_n]} a_{1j_1} \dots a_{nj_n}$

$e_1 \dots e_n$

$e_{ii} = 1$

$e_{ij} = 0$ за $i \neq j$

$f(e_1 \dots e_n) = \dots = (-1)^{[1 2 \dots n]} e_{11} e_{22} \dots e_{nn} = 1 \Rightarrow f$ нормирана

$$a_s = a_s' + a_s'' \cancel{\in F^n} = (a_{s1}' + a_{s1}'', \dots, a_{sn}' + a_{sn}'')$$

$$f(a_1, \dots, a_s' + a_s'', \dots, a_n) = \sum_{j_1 \dots j_n \in S_n} (-1)^{[j_1 \dots j_n]} a_{1j_1} \cdots (a_{sj_1}' + a_{sj_2}'') \cdots a_{nj_n}$$

$$= \sum (-1)^{[j_1 \dots j_n]} (a_{1j_1} \cdots a_{sj_s}' \cdots a_{nj_n} + a_{1j_1} \cdots a_{sj_s}'' \cdots a_{nj_n}) =$$

$$= \sum (-1)^{[j_1 \dots j_n]} a_{1j_1} \cdots a_{sj_s}' \cdots a_{nj_n} + \sum (-1)^{[j_1 \dots j_n]} a_{1j_1} \cdots a_{sj_s}'' \cdots a_{nj_n}$$

$$= f(a_1, \dots, a_s', \dots, a_n) + f(a_1, \dots, a_s'', \dots, a_n)$$

$$f(a_1, \dots, \lambda a_s, \dots, a_n) = \sum_{j_1 \dots j_n} a_{1j_1} \cdots \lambda a_{sj_s} \cdots a_{nj_n} = \lambda \sum \cdots$$

~~полилинейност~~

$$= \lambda f(a_1, \dots, a_s, \dots a_n)$$

$$a_s = a_s' + a_s'' \cancel{\in F^n} = (a_{s1}' + a_{s1}'', \dots, a_{sn}' + a_{sn}'')$$

$$f(a_1, \dots, a_s' + a_s'', \dots, a_n) = \sum_{j_1 \dots j_n \in S_n} (-1)^{[j_1 \dots j_n]} a_{1j_1} \cdots (a_{sj_1}' + a_{sj_2}'') \cdots a_{nj_n}$$

$$= \sum (-1)^{[j_1 \dots j_n]} (a_{1j_1} \cdots a_{sj_s}' \cdots a_{nj_n} + a_{1j_1} \cdots a_{sj_s}'' \cdots a_{nj_n}) =$$

$$= \sum (-1)^{[j_1 \dots j_n]} a_{1j_1} \cdots a_{sj_s}' \cdots a_{nj_n} + \sum (-1)^{[j_1 \dots j_n]} a_{1j_1} \cdots a_{sj_s}'' \cdots a_{nj_n}$$

$$= f(a_1, \dots, a_s', \dots, a_n) + f(a_1, \dots, a_s'', \dots, a_n)$$

$$f(a_1, \dots, \lambda a_s, \dots, a_n) = \sum_{j_1 \dots j_n} a_{1j_1} \cdots \lambda a_{sj_s} \cdots a_{nj_n} = \lambda \sum \cdots$$

~~полилинейност~~

$$= \lambda f(a_1, \dots, a_s, \dots a_n)$$

антисиметричност

Нека $s, t \in \{1, \dots, n\}$, $s \neq t$, $a_s = a_t = b$

$$f(a_1, \dots, \underset{b}{\underset{\shortparallel}{a_s}}, \dots \underset{b}{\underset{\shortparallel}{a_t}}, \dots a_n) = \sum_{j_1 \dots j_n} (-1)^{[j_1 \dots j_n]} a_{1j_1} \dots \underset{b_{j_s}}{\underset{\shortparallel}{a_{sj_s}}} \dots \underset{b_{j_t}}{\underset{\shortparallel}{a_{tj_t}}} \dots a_{nj_n}$$

$$\begin{cases} j_1 \dots j_s \dots j_t \dots j_n \to a_{1j_1} \dots \underline{b_{j_s}} \dots b_{j_t} \dots a_{nj_n} \\ j_1 \dots j_t \dots j_s \dots j_n \to a_{1j_1} \dots b_{j_t} \dots b_{j_s} \dots a_{nj_n} \end{cases}$$

стойности съвпадат, но имат разл. знак в $\sum$

елем. се групират по 2-ка

сумата на елем. във всяка двойка е $= 0$

$$\Rightarrow f(a_1, \dots, \underset{b}{\underset{\shortparallel}{a_s}} \dots \underset{b}{\underset{\shortparallel}{a_t}} \dots a_n) = 0$$

$\Rightarrow$ антисиметрична

$$\det A = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ - & - & - \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = \sum_{j_1 \dots j_n \in S_n} (-1)^{[j_1 \dots j_n]} a_{1j_1} a_{2j_2} \dots a_{nj_n}$$

$n!$ събираеми // $\frac{n!}{2}$ знак. + ; $\frac{n!}{2}$ " −"

$a_{1j_1} \dots a_{nj_n}$ по един множ. от всеки ред
по един множ. от всеки стълб.

$n=2$

1 2
2 1

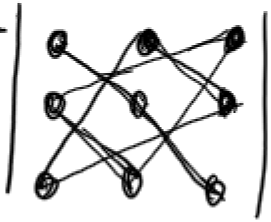
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = (-1)^{[1,2]} ad + (-1)^{[21]} \underset{=b}{a_{12}} \underset{=c}{a_{21}} = ad - bc$$

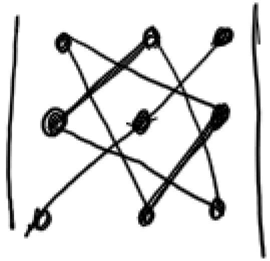
$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \qquad \begin{vmatrix} 1 & i \\ -i & -1 \end{vmatrix}$$

$n = 3$

123
132
213
231
312
321

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (-1)^{[123]} a_{11} a_{22} a_{33} + (-1)^{[132]} a_{11} a_{23} a_{32} + (-1)^{[213]} a_{12} a_{21} a_{33} + (-1)^{[231]} a_{12} a_{23} a_{31} + (-1)^{[312]} a_{13} a_{21} a_{32} + (-1)^{[321]} a_{13} a_{22} a_{31}$$

+ 

 −

$$\begin{vmatrix} 1 & 2 & 3 \\ -2 & 4 & 1 \\ -3 & 5 & 2 \end{vmatrix} = 1 \cdot 4 \cdot 2 + 2 \cdot 1 (-3) + (-2) \cdot 5 \cdot 3 - \left[3 \cdot 4 (-3) + 2 \cdot (-2) \cdot 2 + 5 \cdot 1 \cdot 1 \right]$$

Св-ва I

1) Ако ред на det е $\theta \Rightarrow \det = 0$

2) Ако разменим местата на два реда $\Rightarrow$ det сменя знак

3) Ако има два равни реда $\Rightarrow \det = 0$

4) Ако умножим ред по число $\lambda \Rightarrow \det = \lambda \cdot$ преди

$A_{3\times3}$ $\quad \det 3A = 27 \det A$

5) Ако има два пропорционални реда $\det = 0$

6) $a_s = a_s' + a_s''$

$$\det(a_1, \dots, a_s' + a_s'', \dots, a_n) = \det(a_1, \dots, a_s', \dots, a_n) + \det(a_1, \dots, a_s'', \dots, a_n)$$

$$\begin{array}{ccc} 110+1 & 110+2 & 110+3 \end{array}$$

$$\begin{vmatrix} 111 & 112 & 113 \\ 51 & 49 & 61 \\ 2021 & 2022 & 2023 \end{vmatrix} = \begin{vmatrix} 110 & 110 & 110 \\ 51 & 49 & 61 \\ 2021 & 2022 & 2023 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 3 \\ 51 & 49 & 61 \\ 2021 & 2022 & 2023 \end{vmatrix}$$

$$= 110 \begin{vmatrix} 1 & 1 & 1 \\ 51 & 49 & 61 \\ 2021 & 2022 & 2023 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 3 \\ 51 & 49 & 61 \\ 2020 & 2020 & 2020 \end{vmatrix} + \underbrace{\begin{vmatrix} 1 & 2 & 3 \\ 51 & 49 & 61 \\ 1 & 2 & 3 \end{vmatrix}}_{=0}$$

$$\underbrace{110 \begin{vmatrix} 1 & 1 & 1 \\ 51 & 49 & 61 \\ 2020 & 2020 & 2020 \end{vmatrix}}_{=0} + 110 \begin{vmatrix} 1 & 1 & 1 \\ 51 & 49 & 61 \\ 1 & 2 & 3 \end{vmatrix} + 2020 \begin{vmatrix} 1 & 2 & 3 \\ 51 & 49 & 61 \\ 1 & 1 & 1 \end{vmatrix}$$

$$(110-2020) \begin{vmatrix} 1 & 1 & 1 \\ 51 & 49 & 61 \\ 1 & 2 & 3 \end{vmatrix}$$

(7) Ако към един ред на det прибавим друг ред умножен по число $\Rightarrow$ det не се променя

$$a_s' = a_s + \lambda a_t$$

$$\det(a_1, \dots, \underset{\substack{\parallel \\ a_s + \lambda a_t}}{a_s'}, \dots, a_t, \dots, a_n) = \det(a_1, \dots, a_s, \dots, a_t \dots a_n) +$$

$$+ \underbrace{\det(a_1, \dots, \underset{s}{\lambda a_t}, \dots, a_t, \dots, a_n)}_{=0}$$

$$(110-2020)\begin{vmatrix} 1 & 1 & 1 \\ 51 & 49 & 61 \\ 1 & 2 & 3 \end{vmatrix} = (\dots) \begin{vmatrix} 1 & 1 & 1 \\ 0 & -2 & 10 \\ 0 & 1 & 2 \end{vmatrix}$$

$$\underbrace{-4 - 10}_{-14}$$

Тв

$$\begin{vmatrix} a_{11} & & * \\ & a_{22} & \\ & & \ddots \\ 0 & & & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & & 0 \\ & \ddots & \\ * & & a_{nn} \end{vmatrix} = a_{11}a_{22}\cdots a_{nn}$$

a_{st} <
- $s=t$ в/у гл. диаг
- $s<t$ над гл. диаг
- $s>t$ отдолу под гл. диаг

$a_{1j_1}\cdots a_{nj_n}$ имаме $\exists\, k, j_k \neq k, k$

$j_1\cdots j_n \neq 1,\cdots,n$

$\Rightarrow$ имаме елем. под гл. диагонала

има и елем. ~~над~~ над гл. диаг.

$\Rightarrow$ остава $a_{11}a_{22}\cdots a_{nn}$

T// $\det A = 0 \Leftrightarrow$ редовете на матрицата са линейно зависими

1) Нека редов. са ЛЗ $\Rightarrow \exists a_k \in l(a_1, \dots, a_{k-1}, a_{k+1}, \dots a_n)$

Нека $a_n \in l(a_1, \dots, a_{n-1}) \Rightarrow a_n = \lambda_1 a_1 + \dots + \lambda_{n-1} a_{n-1}$

$$\left.\begin{aligned} &\det(a_1, \dots, a_{n-1}, \lambda_1 a_1 + \dots + \lambda_{n-1} a_{n-1}) = \\ &= \lambda_1 \det(a_1, \dots, a_{n-1}, a_1) + \\ &+ \dots \\ &\lambda_{n-1} \det(a_1, \dots, a_{n-1}, a_{n-1}) \end{aligned}\right\} = 0$$

2) $a_1, \dots, a_n$ са ЛНЗ образуват базис

$e_1, \dots, e_n$ $\quad e_k = \alpha_1 a_1 + \dots + \alpha_n a_n$

$$\underbrace{\det(e_1, \dots, e_n)}_{=1} = \underbrace{\left(\sum_{i_1 \dots i_n} (-1)^{[i_1 \dots i_n]} \alpha_{1 i_1} \dots \alpha_{n i_n}\right)}_{t} \det(a_1, \dots, a_n)$$

$1 = t \det(a_1 \dots a_n) \Rightarrow \det(a_1 \dots a_n) \neq 0$

I) При транспониране детермин. не се променя

транспонира $a_{ij} \to ji$

$$\sum_{j_1 \dots j_n \in S_n} (-1)^{[j_1 \dots j_n]} a_{1j_1} \dots a_{nj_n}$$

Лема $k_1 \dots k_n$ пермутация $\{1, \dots n\}$
$s_1 \dots s_n$ пермутация

$\boxed{a_{k_1 s_1} a_{k_2 s_2} \dots a_{k_n s_n}}$ участва в det със знак $(-1)^{[s_1 \dots s_n] + [k_1 \dots k_n]}$ в произв.

$a_{k_i s_i} \longleftrightarrow a_{k_j s_j}$

$(-1)^{[s_1 \dots s_n] + [k_1 \dots k_n]}$ смяна $s_i \leftrightarrow s_j$, $k_i \leftrightarrow k_j$

и двете пермутации си сменят четността

$\Rightarrow [s_1, \dots, s_n] + [k_1 \dots k_n]$ не сменя четността си

$\Rightarrow a_{k_1 s_1} \dots a_{k_n s_n} \longleftrightarrow a_{1t_1} \dots a_{nt_n}$ $(-1)^{[t_1 \dots t_n] + [1, \dots n]}$

Пр $a_{31} a_{45} a_{12} a_{67} a_{54} a_{76} a_{23}$

3 4 1 6 5 7 2
1 5 2 7 4 6 3

$\det A \qquad A = (a_{ij}) \qquad B = A^t \qquad b_{ij} = a_{ji}$

$$\det B = \sum_{j_1 \dots j_n} (-1)^{[j_1 \dots j_n]} b_{1j_1} \dots b_{nj_n} = \sum_{j_1 \dots j_n} (-1)^{[j_1 \dots j_n]} a_{j_1,1} \dots a_{j_n,n}$$

$a_{j_1,1} \dots a_{j_n n}$ в $\det A \longrightarrow (-1)^{[j_1 \dots j_n] + [1, 2 \dots n]}$

участва с еднакъв знак в $\det A$, в $\det A^t$

$\Rightarrow \det A = \det A^t$

Сл. Всички св-ва на det важат и по стълбове