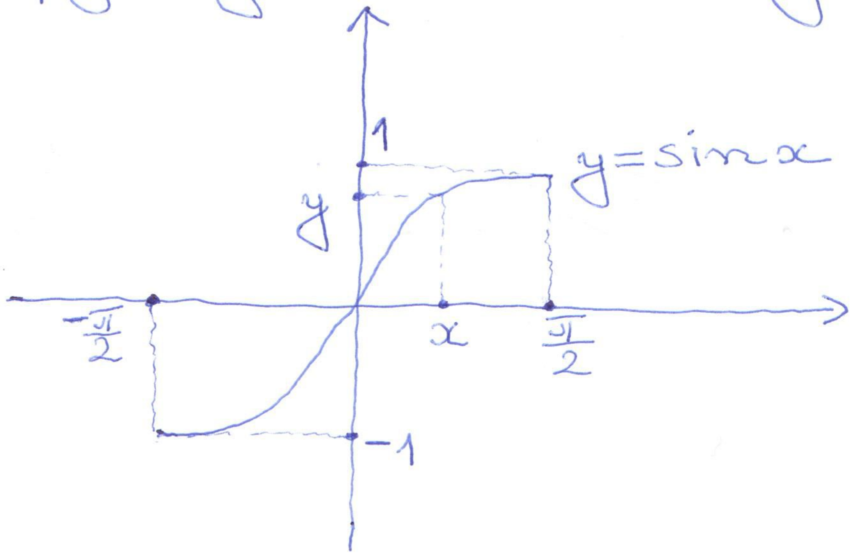
① Упражнение 15 за 1, 2 и 3 група

Обратни тригонометрични функции

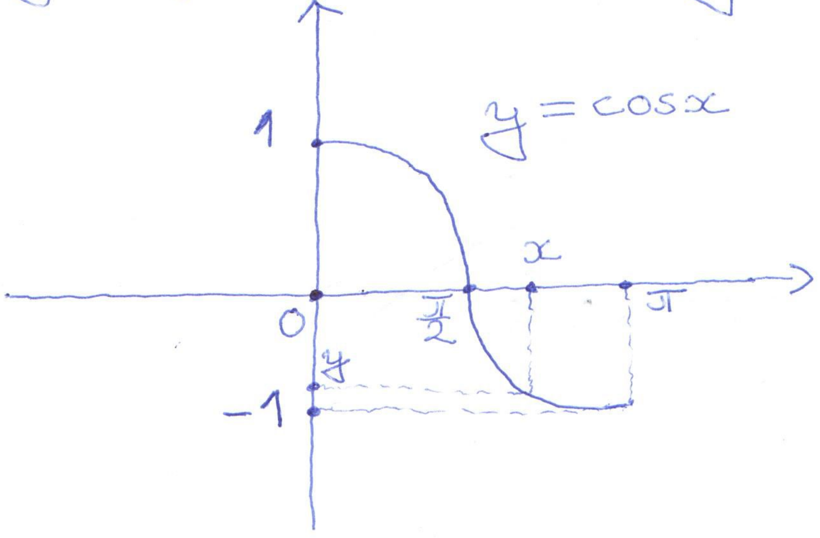
Функцията $y=\sin x:\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\longrightarrow[-1,1]$ е строго растяща и сл. е обратима. Обратната ѝ функция е $x=\arcsin y:[-1,1]\longrightarrow\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.



Така при $y\in[-1,1]$

$$x=\arcsin y \iff \left|\begin{array}{l} x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right] \\ \sin x=y \end{array}\right. .$$

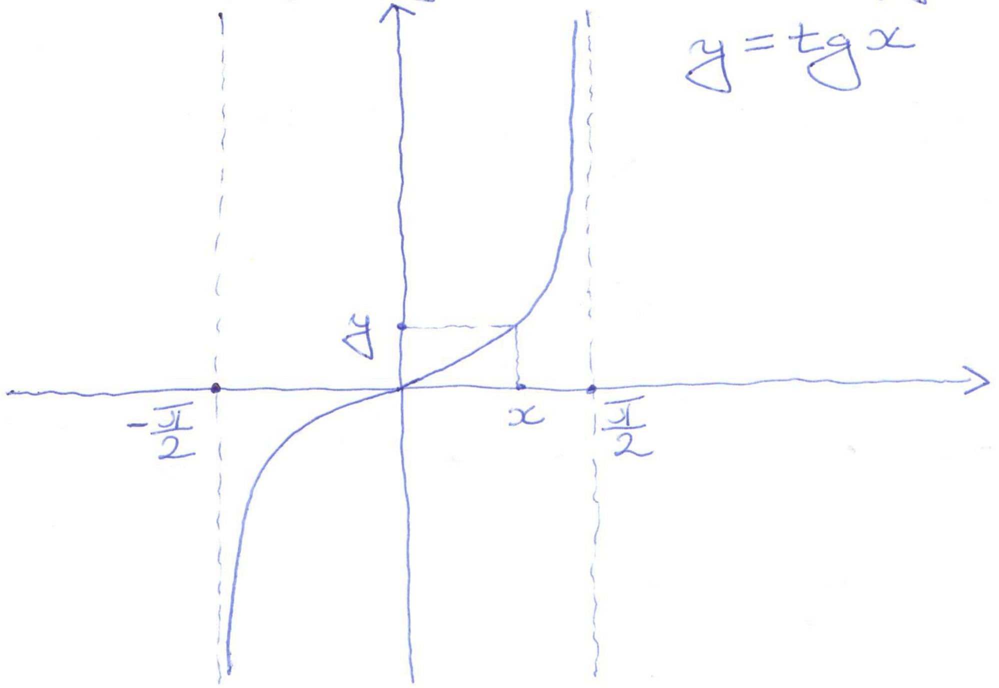
Функцията $y=\cos x:[0,\pi]\longrightarrow[-1,1]$ е строго намаляваща и сл. е обратима. Обратната ѝ функция е $x=\arccos y:[-1,1]\longrightarrow[0,\pi]$.



Така при $y\in[-1,1]$

$$x=\arccos y \iff \left|\begin{array}{l} x\in[0,\pi] \\ \cos x=y \end{array}\right. .$$

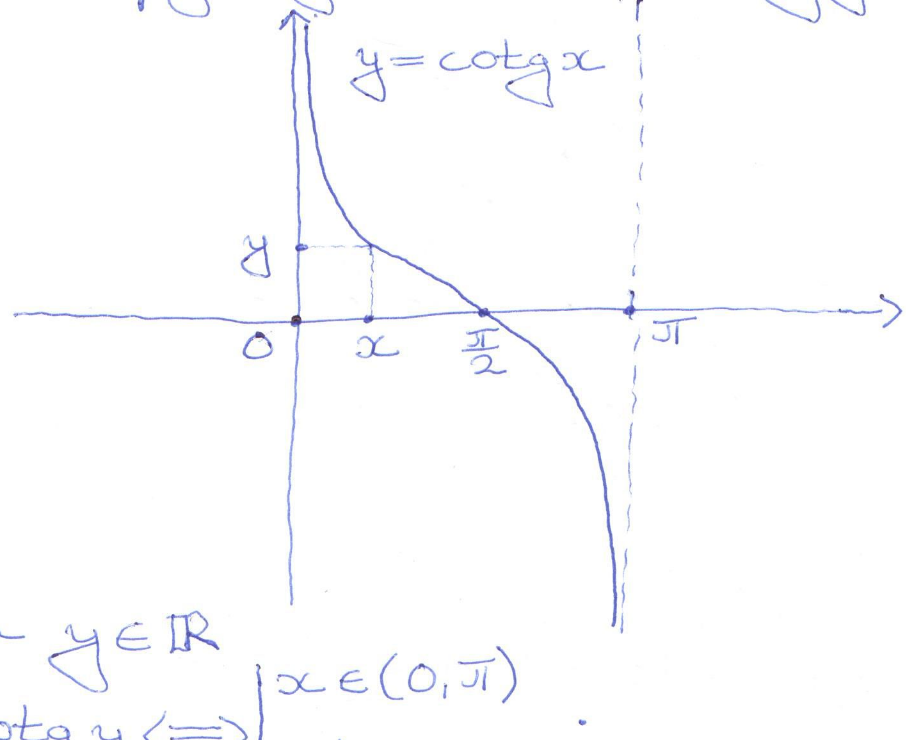
② Функцията $y=\operatorname{tg}x:\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\longrightarrow\mathbb{R}$ е строго растяща и сл. е обратима.
Обратната ѝ функция е $x=\operatorname{arctg}y:\mathbb{R}\longrightarrow\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.



Така при $y\in\mathbb{R}$

$$x=\operatorname{arctg}y \iff \left|\begin{array}{l} x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\\ \operatorname{tg}x=y \end{array}\right. .$$

Функцията $y=\operatorname{cotg}x:(0,\pi)\longrightarrow\mathbb{R}$ е строго намаляваща и сл. е обратима.
Обратната ѝ функция е $x=\operatorname{arccotg}y:\mathbb{R}\longrightarrow(0,\pi)$.



Така при $y\in\mathbb{R}$

$$x=\operatorname{arccotg}y \iff \left|\begin{array}{l} x\in(0,\pi)\\ \operatorname{cotg}x=y \end{array}\right. .$$

③ Зад. 1 Пресметнете $\arcsin 0$, $\arccos\frac{\sqrt{2}}{2}$, $\operatorname{arctg}(-1)$, $\operatorname{arccotg}\sqrt{3}$.

Решение: $\arcsin 0 = 0$, защото $0 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ и $\sin 0 = 0$.

$\arccos\frac{\sqrt{2}}{2} = \frac{\pi}{4}$, защото $\frac{\pi}{4} \in [0, \pi]$ и $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$.

$\operatorname{arctg}(-1) = -\frac{\pi}{4}$, защото $-\frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ и $\operatorname{tg}\left(-\frac{\pi}{4}\right) = -1$.

$\operatorname{arccotg}\sqrt{3} = \frac{\pi}{6}$, защото $\frac{\pi}{6} \in (0, \pi)$ и $\operatorname{cotg}\frac{\pi}{6} = \sqrt{3}$.

Зад. 2 Докажете, че:

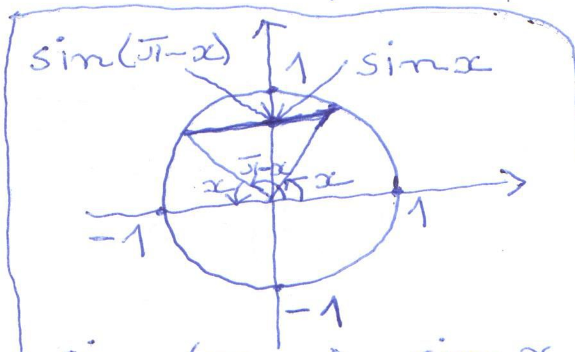
а) $\arcsin(\sin\alpha) = \pi - \alpha$, $\alpha \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$;

б) $\arccos(\cos\alpha) = 2\pi - \alpha$, $\alpha \in [\pi, 2\pi]$.

Решение: а) Трябва да док. че $(\pi - \alpha) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ и че $\sin(\pi - \alpha) = \sin\alpha$.

$$\frac{\pi}{2} \leqq \alpha \leqq \frac{3\pi}{2} \Rightarrow -\frac{3\pi}{2} \leqq -\alpha \leqq -\frac{\pi}{2} \Rightarrow -\frac{\pi}{2} \leqq \pi - \alpha \leqq \frac{\pi}{2}.$$

$\sin(\pi - \alpha) = \sin\alpha$ е известна тригонометрична формула.



$\sin(\pi - x) = \sin x$

б) Трябва да док. че $(2\pi - \alpha) \in [0, \pi]$ и че $\cos(2\pi - \alpha) = \cos\alpha$.

$\pi \leqq \alpha \leqq 2\pi \Rightarrow -2\pi \leqq -\alpha \leqq -\pi \Rightarrow$
$\Rightarrow 0 \leqq 2\pi - \alpha \leqq \pi$.

$$\cos(2\pi - \alpha) = \cos(-\alpha) = \cos\alpha.$$

cos x е 2π-периодична функция

cos x е четна функция

Зад. 3 Док. че: а) $\arcsin(-x) = -\arcsin x$, $x \in [-1, 1]$;

б) $\arccos(-x) = \pi - \arccos x$, $x \in [-1, 1]$;

в) $\operatorname{arctg}(-x) = -\operatorname{arctg} x$ $(x \in \mathbb{R})$; г) $\operatorname{arccotg}(-x) = \pi - \operatorname{arccotg} x$ $(x \in \mathbb{R})$.

Забележка: От а) и в) следва, че $\arcsin x$ и $\operatorname{arctg} x$ са нечетни функции.

(4) Решение: а) Нека $\arcsin x = \alpha$, т.е.
$\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ и $\sin\alpha = x$.
Трябва да док. че $\arcsin(-x) = -\alpha$, т.е. че
$(-\alpha) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ и че $\sin(-\alpha) = -x$.

$-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2} \Rightarrow -\frac{\pi}{2} \leq -\alpha \leq \frac{\pi}{2}$.

$\sin(-\alpha) = -\sin\alpha = -x$.

sin x е нечетна функция

б) Нека $\arccos x = \alpha$, т.е. $\alpha \in [0, \pi]$ и $\cos\alpha = x$.
Трябва да док. че $\arccos(-x) = \pi - \alpha$, т.е. че
$(\pi - \alpha) \in [0, \pi]$ и че $\cos(\pi - \alpha) = -x$.

$0 \leq \alpha \leq \pi \Rightarrow -\pi \leq -\alpha \leq 0 \Rightarrow 0 \leq \pi - \alpha \leq \pi$

$\cos(\pi - \alpha) = \underset{-1}{\cos\pi}\cos\alpha + \underset{0}{\sin\pi}\sin\alpha = -\cos\alpha = -x$.

в) Аналогично на а); г) Аналогично на б).

Зад. 4 Док. че: а) $\arcsin x + \arccos x = \frac{\pi}{2}$, $x \in [-1, 1]$;
б) $\mathrm{arctg}\, x + \mathrm{arccotg}\, x = \frac{\pi}{2}$, $x \in \mathbb{R}$.

Решение: а) Нека $\arcsin x = \alpha$, т.е. $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
и $\sin\alpha = x$. Трябва да док. че $\arccos x = \frac{\pi}{2} - \alpha$,
т.е. че $\left(\frac{\pi}{2} - \alpha\right) \in [0, \pi]$ и че $\cos\left(\frac{\pi}{2} - \alpha\right) = x$.

$-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2} \Rightarrow -\frac{\pi}{2} \leq -\alpha \leq \frac{\pi}{2} \Rightarrow 0 \leq \frac{\pi}{2} - \alpha \leq \pi$.

$\cos\left(\frac{\pi}{2} - \alpha\right) = \sin\alpha = x$.

б) Нека $\mathrm{arctg}\, x = \alpha$, т.е. $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ и $\mathrm{tg}\,\alpha = x$.
Трябва да док. че $\mathrm{arccotg}\, x = \frac{\pi}{2} - \alpha$, т.е. че
$\left(\frac{\pi}{2} - \alpha\right) \in (0, \pi)$ и че $\mathrm{cotg}\left(\frac{\pi}{2} - \alpha\right) = x$.

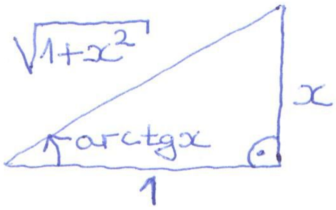
$-\frac{\pi}{2} < \alpha < \frac{\pi}{2} \Rightarrow -\frac{\pi}{2} < -\alpha < \frac{\pi}{2} \Rightarrow 0 < \frac{\pi}{2} - \alpha < \pi$.

$\mathrm{cotg}\left(\frac{\pi}{2} - \alpha\right) = \mathrm{tg}\,\alpha = x$.

(5) Зад. 5 Док. че: а) $\sin(\operatorname{arctg} x) = \dfrac{x}{\sqrt{1+x^2}}, \ x \in \mathbb{R}$;

б) $\operatorname{tg}(\arcsin x) = \dfrac{x}{\sqrt{1-x^2}}, \ x \in (-1, 1)$.

Решение: а) 1 сл. $x > 0$.



$$\sin(\operatorname{arctg} x) = \frac{x}{\sqrt{1+x^2}}.$$

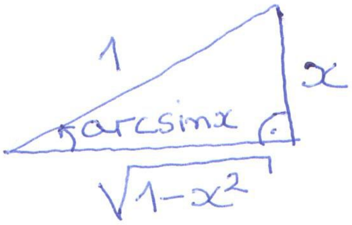
2 сл. $x = 0$. Трябва да док. че $\sin(\underbrace{\operatorname{arctg} 0}_{=0}) = \dfrac{0}{\sqrt{1+0^2}}$ върно е

3 сл. $x < 0$.

Сега $\sin(\operatorname{arctg} x) \overset{\text{от зад. 3в)}}{=} \sin(-\operatorname{arctg}(-x)) =$

$$= -\sin(\operatorname{arctg}(\underbrace{-x}_{>0})) \overset{\text{от 1 сл.}}{=} -\frac{(-x)}{\sqrt{1+(-x)^2}} = \frac{x}{\sqrt{1+x^2}}.$$

б) 1 сл. $x \in (0, 1)$.



$$\operatorname{tg}(\arcsin x) = \frac{x}{\sqrt{1-x^2}}.$$

2 сл. $x = 0$. Трябва да док. че $\operatorname{tg}(\underbrace{\arcsin 0}_{=0}) = \dfrac{0}{\sqrt{1-0^2}}$ върно е

3 сл. $x \in (-1, 0)$.

Сега $\operatorname{tg}(\arcsin x) \overset{\text{от зад. 3а)}}{=} \operatorname{tg}(-\arcsin(-x)) =$

$$= -\operatorname{tg}(\arcsin(\underbrace{-x}_{\in(0,1)})) \overset{\text{от 1 сл.}}{=} -\frac{(-x)}{\sqrt{1-(-x)^2}} = \frac{x}{\sqrt{1-x^2}}.$$