Умножение на детерминанти

Т// Нека F - поле и $\varphi : F^n \to F^n$ линейно изображение с матрица A спрямо $e_1, \dots, e_n$ - стандартен базис. Ако $a_1, \dots, a_n$ - произволни от F^n, тогава

$$\det(\varphi(a_1), \dots, \varphi(a_n)) = \det A \cdot \det(a_1, \dots, a_n)$$

Д-во// I сл. $\det A = 0 \Rightarrow A$ има ЛЗ стълбове и $r(A) = r(\varphi) < n$

$\Rightarrow \varphi(a_1), \dots, \varphi(a_n)$ са ЛЗ ($\in Im\,\varphi$ и $\dim Im\,\varphi < n$)

$\Rightarrow 0 = \det(\varphi(a_1), \dots, \varphi(a_n)) = \underset{\det A}{0} \cdot \det(a_1, \dots, a_n) \Rightarrow$ изпълнено

II сл. Нека $\det A \neq 0$. Разглеждаме ф-я $f(a_1, \dots, a_n) = \dfrac{\det(\varphi(a_1) \dots \varphi(a_n))}{\det A}$

- $f(a_1, \dots, a_n)$ - полилинейна ($i \in 1, \dots, n$)

$$f(a_1, \dots, \lambda a_i' + \mu a_i'', \dots, a_n) = \frac{1}{\det A} \det(\varphi(a_1), \dots, \varphi(\lambda a_i' + \mu a_i''), \dots, \varphi(a_n)) =$$

$$= \frac{1}{\det A} \det(\varphi(a_1), \dots, \lambda\varphi(a_i') + \mu\varphi(a_i''), \dots, \varphi(a_n)) =$$

$$= \underbrace{\frac{\lambda}{\det A} \det(\varphi(a_1), \dots, \varphi(a_i'), \dots, \varphi(a_n))}_{\lambda f(a_1, \dots, a_i' \dots a_n)} + \underbrace{\frac{\mu}{\det A} \det(\varphi(a_1), \dots, \varphi(a_i''), \dots, \varphi(a_n))}_{\mu f(a_1, \dots, a_i'' \dots a_n)}$$

- антисиметрична $\det(\varphi(a_1) \dots \varphi(a_i) \dots \varphi(a_j) \dots \varphi(a_n))$ е антисим.

- $f(e_1, \dots, e_n) = \dfrac{1}{\det A} \det(\varphi(e_1), \dots, \varphi(e_n)) = \dfrac{\det A^t}{\det A} = 1$

T// Нека $A, B \in M_{n\times n}(F)$. Тогава $\det AB = \det A . \det B$

Д-во// Нека $\varphi, \psi \in Hom(F^n, F^n)$, за които
A - е матрица на φ и B - матрица на ψ
спрямо $e_1, .., e_n \Rightarrow AB$ е матрица на $\varphi \circ \psi$

$$\Rightarrow \det(AB) = \det(\varphi\circ\psi(e_1), .., \varphi\circ\psi(e_n)) = \det(\varphi(\psi(e_1)), \dots, \varphi(\psi(e_n))) =$$

$$= \det A \det(\psi(e_1), .., \psi(e_n)) =$$

$$= \det A . \det B \det(e_1, .., e_n) =$$

$$= \det A . \det B$$

празна стр.

Умножение на детерминанти

Лема 1// Нека $A_{k\times k}$, $B_{s\times s}$, $C_{k\times s}$

$$\det\left(\begin{array}{c|c} A & C \\ \hline 0 & B \end{array}\right)_{(k+s)\times(k+s)} = \det A \cdot \det B = \det D$$

Д-во// индукция по k: (1), за $k=1$ $\det\left(\begin{array}{c|c} a_{11} & C \\ \hline 0 & \\ \vdots & B \\ 0 & \end{array}\right) = a_{11}\det B$ развитие I стълб

Нека е изпълнено за $k-1$:

Нека поддетерминантите на матрицата D ги бележим $\tilde{\Delta}_{ij}$,

развиваме по I стълб

$$\det D = a_{11}(-1)^2\tilde{\Delta}_{11} + a_{21}(-1)^3\tilde{\Delta}_{21} + \dots + a_{k1}(-1)^{k+1}\tilde{\Delta}_{k1} =$$

Δ_{ij} поддетерм. на A

$$= a_{11}(-1)^2\Delta_{11}\det B + a_{21}(-1)^3\Delta_{21}\det B + \dots + a_{k1}(-1)^{k+1}\Delta_{k1}\det B$$

$$= \left(a_{11}(-1)^2\Delta_{11} + a_{21}(-1)^3\Delta_{21} + \dots + a_{k1}(-1)^{k+1}\Delta_{k1}\right)\det B =$$

$$= \det A \cdot \det B$$

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$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 4 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 6 & 7 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = (7-12)\, W(1, 2, 3) =$$

$$= -5\,(2-1)(3-1)(3-2) = -5 \cdot 2 = -10$$

<u>Теорема</u> Нека $A, B \in M_{n\times n}(F)$. Тогава $\det AB = \det A \det B$

<u>Д-во</u> $D \in M_{2n\times 2n}(F)$; $D = \left(\begin{array}{c|c} A & 0 \\ \hline -E & B \end{array}\right)$; $\det D = \det A . \det B$

работа по стълбове!

$$\det D = \left|\begin{array}{cccc|cccc} a_{11} & \dots & & a_{1n} & 0 & \dots & & 0 \\ & & & & & & & \\ a_{n1} & \dots & & a_{nn} & 0 & \dots & & 0 \\ \hline -1 & 0 & \dots & 0 & b_{11} & \dots & & b_{1n} \\ 0 & -1 & \dots & 0 & b_{21} & \dots & & b_{2n} \\ & & & & & & & \\ 0 & \dots & & -1 & b_{n1} & \dots & & b_{nn} \end{array}\right| = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \hline b_{11} \\ \vdots \\ b_{n1} \end{pmatrix} + b_{11}\begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \\ \hline -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + b_{21}\begin{pmatrix} a_{12} \\ \vdots \\ a_{n2} \\ \hline 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \dots + b_{n1}\begin{pmatrix} a_{1n} \\ \vdots \\ a_{nn} \\ \hline 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix}$$

анулираме

$$= \left|\begin{array}{cccc|cccc} a_{11} & \dots & & a_{1n} & c_{11} & \dots & & c_{1n} \\ & & & & & & & \\ a_{n1} & \dots & & a_{nn} & c_{n1} & \dots & & c_{nn} \\ \hline -1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & -1 & \dots & 0 & 0 & 0 & \dots & 0 \\ & & & & & & & \\ 0 & 0 & \dots & 0\ -1 & 0 & 0 & \dots & 0 \end{array}\right| =$$

$c_{11} = a_{11}b_{11} + a_{12}b_{21} + \dots + a_{1n}b_{n1}$

$\vdots$

$c_{n1} = a_{n1}b_{11} + a_{n2}b_{21} + \dots + a_{nn}b_{nn}$

$c_{ij} = a_{i1}b_{1j} + \dots + a_{in}b_{nj}$

$$\det D = \left|\begin{array}{c|c} A & AB \\ \hline -E & 0 \end{array}\right|$$

$$\det A \cdot \det B = \left|\begin{array}{c|c} A & 0 \\ \hline -E & B \end{array}\right| = \left|\begin{array}{c|c} A & AB \\ \hline -E & 0 \end{array}\right| = \left\{ \begin{array}{l} \text{разместваме} \\ \text{стълбовете} \\ i \Longleftrightarrow n+i \ // \ i=1..n \end{array}\right.$$

$$= (-1)^n \left|\begin{array}{c|c} AB & A \\ \hline 0 & -E \end{array}\right| = (-1)^n \cdot \det(AB) \cdot \det(-E) = (-1)^n \cdot (-1)^n \det(AB)$$

$$\Rightarrow \det A \cdot \det B = \det AB$$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Пр/

$$\begin{vmatrix} (1+1)^4 & (1+2)^4 & (1+3)^4 & (1+4)^4 & (1+5)^4 \\ (2+1)^4 & (2+2)^4 & (2+3)^4 & (2+4)^4 & (2+5)^4 \\ (3+1)^4 & (3+2)^4 & (3+3)^4 & (3+4)^4 & (3+5)^4 \\ (4+1)^4 & (4+2)^4 & (4+3)^4 & (4+4)^4 & (4+5)^4 \\ (5+1)^4 & (5+2)^4 & (5+3)^4 & (5+4)^4 & (5+5)^4 \end{vmatrix} = \begin{vmatrix} 1^4 & 4.1^3 & 6.1^2 & 4.1 & 1 \\ 2^4 & 4.2^3 & 6.2^2 & 4.2 & 1 \\ 3^4 & 4.3^3 & 6.3^2 & 4.3 & 1 \\ 4^4 & 4.4^3 & 6.4^2 & 4.4 & 1 \\ 5^4 & 4.5^3 & 6.5^2 & 4.5 & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1^2 & 2^2 & 3^2 & 4^2 & 5^2 \\ 1^3 & 2^3 & 3^3 & 4^3 & 5^3 \\ 1^4 & 2^4 & 3^4 & 4^4 & 5^4 \end{vmatrix}$$

$$= 4.6.4\, W(1,2,3,4,5) \cdot W(1,2,3,4,5) =$$

$$4.6.4\,(2.3.4.2.3.2)^2 = 4^4.6^5.4 = (24)^5$$

празна ст.