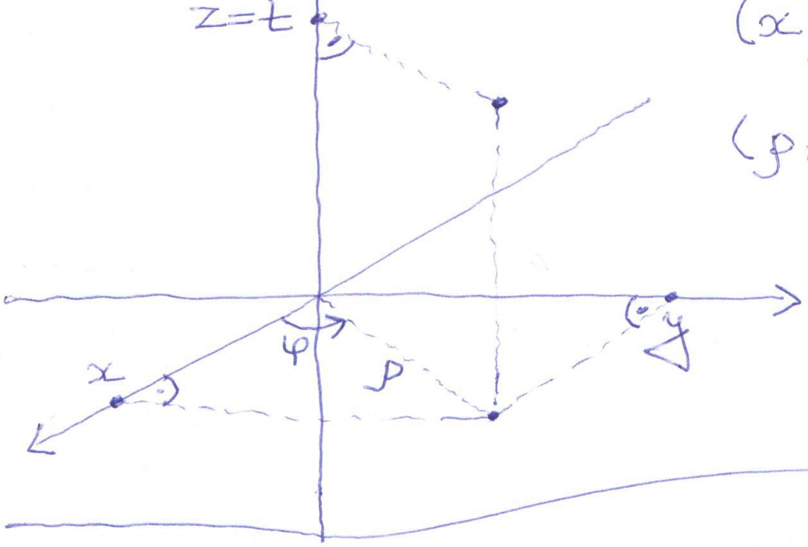
1

Тройни интеграли, част 2

Най-често срещаните смени на променливите при тройните интеграли са цилиндричната смяна и сферичната смяна.



(x, y, z) – правоъгълни координати

(ρ, φ, t) – цилиндрични координати

$x = \rho\cos\varphi$

$y = \rho\sin\varphi$

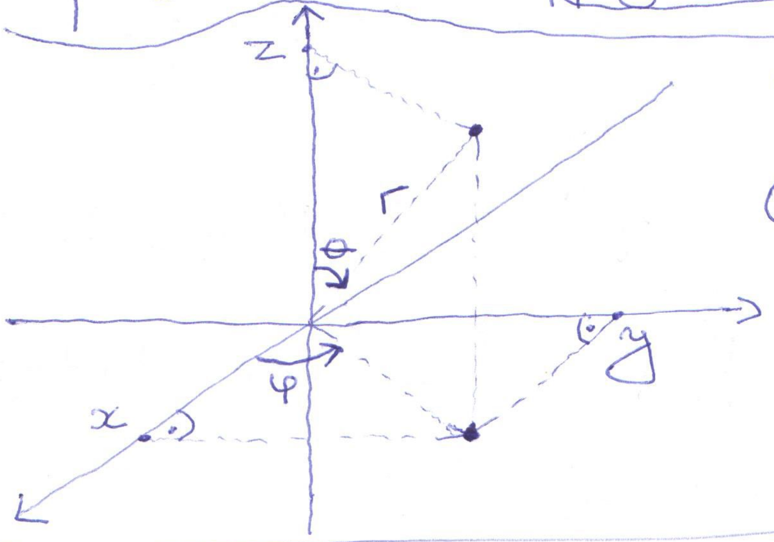
$z = t$

Цилиндрична смяна наричаме смяната

$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \\ z = t \\ \rho \geq 0,\ 0 \leq \varphi \leq 2\pi,\ t \in \mathbb{R} \end{cases}$$

. За нея $\Delta = \begin{vmatrix} x'_\rho & x'_\varphi & x'_t \\ y'_\rho & y'_\varphi & y'_t \\ z'_\rho & z'_\varphi & z'_t \end{vmatrix} =$

$$= \begin{vmatrix} \cos\varphi & -\rho\sin\varphi & 0 \\ \sin\varphi & \rho\cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix} \underset{\text{развиване по 3 ред}}{=} \begin{vmatrix} \cos\varphi & -\rho\sin\varphi \\ \sin\varphi & \rho\cos\varphi \end{vmatrix} = \rho\cos^2\varphi + \rho\sin^2\varphi = \rho.$$



(x, y, z) – правоъгълни координати

(r, θ, φ) – сферични координати

$x = r\sin\theta\cos\varphi$

$y = r\sin\theta\sin\varphi$

$z = r\cos\theta$

Сферична смяна наричаме смяната

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \\ r \geq 0,\ 0 \leq \theta \leq \pi,\ 0 \leq \varphi \leq 2\pi \end{cases}$$

. За нея $\Delta = \begin{vmatrix} x'_r & x'_\theta & x'_\varphi \\ y'_r & y'_\theta & y'_\varphi \\ z'_r & z'_\theta & z'_\varphi \end{vmatrix} =$

(2)

$$= \begin{vmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} =$$

$$= \underline{r^2\sin^3\theta\sin^2\varphi} + r^2\sin\theta\cos^2\theta\cos^2\varphi +$$

$$+ r^2\sin\theta\cos^2\theta\sin^2\varphi + \underline{r^2\sin^3\theta\cos^2\varphi} =$$

$$= r^2\sin^3\theta + r^2\sin\theta\cos^2\theta = r^2\sin\theta(\sin^2\theta+\cos^2\theta) = r^2\sin\theta.$$

Цилиндричната смяна е удобна, ако в подинтегралната функция или в неравенствата, задаващи множеството, върху което интегрираме, се среща израза x^2+y^2 (в цилиндрични координати този израз се опростява: $x^2+y^2=\rho^2$), а сферичната смяна е удобна, ако се среща израза $x^2+y^2+z^2$ (в сферични координати този израз се опростява: $x^2+y^2+z^2=r^2$).

Зад. 1 Пресметнете $I=\iiint\limits_K (x^2+y^2)\,dx\,dy\,dz$, където

$$K: \left| \begin{array}{l} x^2+y^2 \leqq 2z \\ 0 \leqq z \leqq 2 \end{array} \right. .$$

Решение: Правим цилиндрична смяна

$$\left| \begin{array}{l} x=\rho\cos\varphi \\ y=\rho\sin\varphi \\ z=t \\ \rho\geqq 0,\ 0\leqq\varphi\leqq 2\pi,\ t\in\mathbb{R} \end{array} \right.$$

. Както вече видяхме $\Delta=\rho$.

$$K': \left| \begin{array}{l} \rho^2\cos^2\varphi+\rho^2\sin^2\varphi \leq 2t \\ 0\leqq t\leqq 2 \\ \rho\geqq 0,\ 0\leqq\varphi\leqq 2\pi \end{array} \right. ,\text{ т.е. } K': \left| \begin{array}{l} \left.\begin{array}{l} 0\leqq\varphi\leqq 2\pi \\ 0\leqq t\leqq 2 \end{array}\right\} \mathcal{D}' \\ 0\leq\rho\leq\sqrt{2t} \end{array} \right. .\text{ Тогава}$$

$$I = \iiint\limits_{K'} (\rho^2\cos^2\varphi+\rho^2\sin^2\varphi)\cdot\underbrace{|\rho|}_{\Delta=\rho}\,d\rho\,d\varphi\,dt = \iiint\limits_{K'}\rho^3\,d\rho\,d\varphi\,dt =$$

$$= \iint\limits_{\mathcal{D}'}\left[\int_0^{\sqrt{2t}}\rho^3\,d\rho\right]d\varphi\,dt = \iint\limits_{\mathcal{D}'}\left[\frac{\rho^4}{4}\Big|_0^{\sqrt{2t}}\right]d\varphi\,dt =$$

$$= \frac{1}{\cancel{4}}\iint\limits_{\mathcal{D}'}\cancel{4}t^2\,d\varphi\,dt = \int_0^{2\pi}\left[\int_0^2 t^2\,dt\right]d\varphi = \int_0^{2\pi}\left[\frac{t^3}{3}\Big|_0^2\right]d\varphi =$$

$$= \frac{8}{3}\int_0^{2\pi}1\,d\varphi = \frac{8}{3}\varphi\Big|_0^{2\pi} = \frac{16\pi}{3}.$$

Отг. $I=\frac{16\pi}{3}$.

Зад. 2 Пресметнете $I=\iiint\limits_K xyz\,dx\,dy\,dz$, където

$$K:\begin{cases} x^2+y^2\leq 4z^2\\ x\geq 0,\ y\geq 0\\ 0\leq z\leq 1\end{cases}.$$

Решение: Правим цилиндрична смяна

$$\begin{cases} x=\rho\cos\varphi\\ y=\rho\sin\varphi\\ z=t\\ \rho\geq 0,\ 0\leq\varphi\leq 2\pi,\ t\in\mathbb{R}\end{cases}.$$ Както вече знаем $\Delta=\rho$.

$$K':\begin{cases} \rho^2\cos^2\varphi+\rho^2\sin^2\varphi\leq 4t^2\\ \rho\cos\varphi\geq 0,\ \rho\sin\varphi\geq 0\\ 0\leq t\leq 1\\ \rho\geq 0,\ 0\leq\varphi\leq 2\pi\end{cases},\ \text{т.е. } K':\begin{cases} \left.\begin{matrix}0\leq\varphi\leq\frac{\pi}{2}\\ 0\leq t\leq 1\end{matrix}\right\}\mathcal{D}'\\ 0\leq\rho\leq 2t\end{cases}.$$

Тогава $I=\iiint\limits_{K'}\rho^2\cos\varphi\sin\varphi\cdot t\,\underbrace{|\rho|}_{\Delta=\rho}\,d\rho\,d\varphi\,dt=$

$$=\iint\limits_{\mathcal{D}'}\Big[\int_0^{2t}\rho^3\cos\varphi\sin\varphi\cdot t\,d\rho\Big]d\varphi\,dt=$$

интегрирането е по ρ при фиксирана точка $(\varphi,t)\in\mathcal{D}'$

$$=\iint\limits_{\mathcal{D}'}\Big[t\cos\varphi\sin\varphi\int_0^{2t}\rho^3\,d\rho\Big]d\varphi\,dt=$$

$$=\iint\limits_{\mathcal{D}'}\Big[t\cos\varphi\sin\varphi\,\frac{\rho^4}{4}\Big|_0^{2t}\Big]d\varphi\,dt=4\iint\limits_{\mathcal{D}'}t^5\cos\varphi\sin\varphi\,d\varphi\,dt=$$

$$=4\int_0^{\pi/2}\Big[\int_0^1 t^5\cos\varphi\sin\varphi\,dt\Big]d\varphi=4\int_0^{\pi/2}\Big[\cos\varphi\sin\varphi\int_0^1 t^5\,dt\Big]d\varphi$$

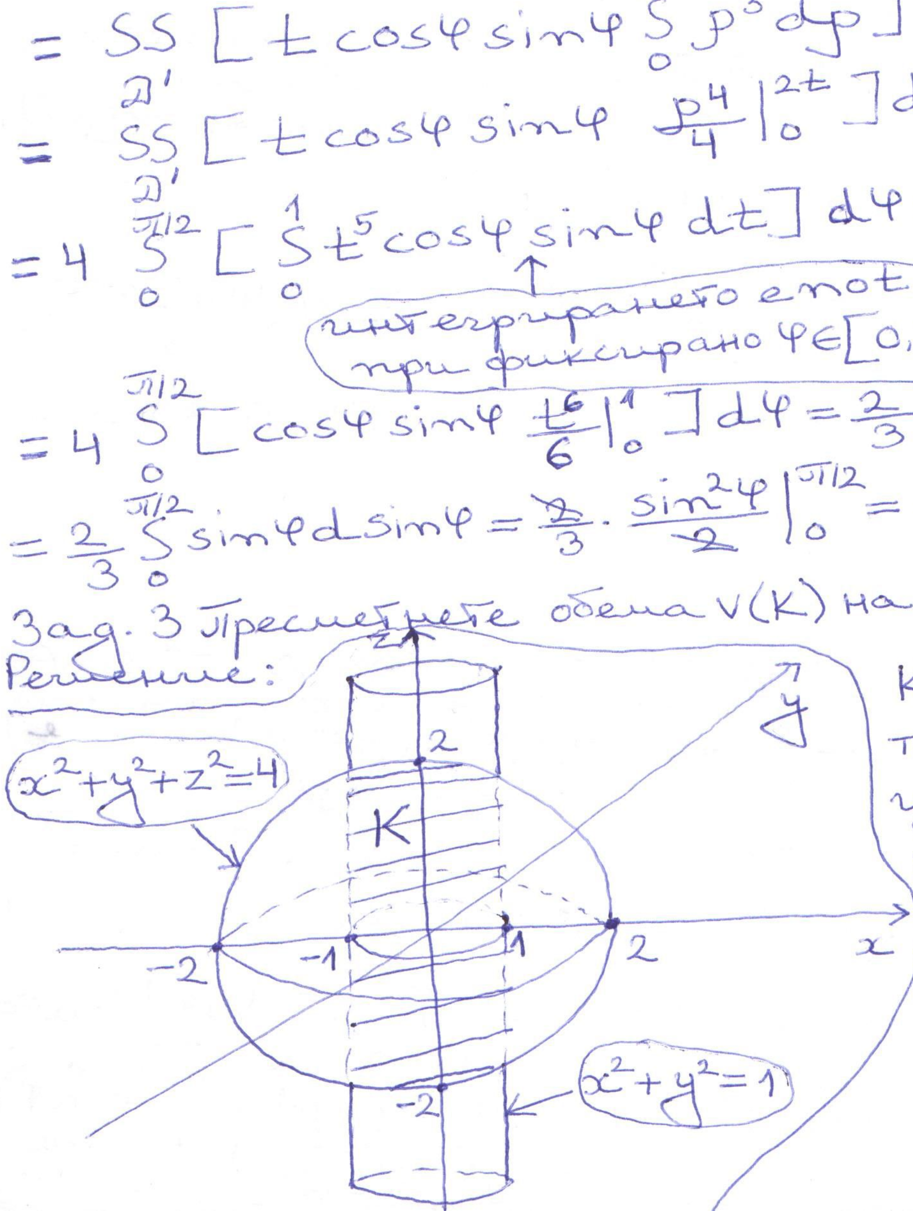
интегрирането е по t при фиксирано $\varphi\in[0,\frac{\pi}{2}]$

$$=4\int_0^{\pi/2}\Big[\cos\varphi\sin\varphi\,\frac{t^6}{6}\Big|_0^1\Big]d\varphi=\frac{2}{3}\int_0^{\pi/2}\cos\varphi\sin\varphi\,d\varphi=$$

$$=\frac{2}{3}\int_0^{\pi/2}\sin\varphi\,d\sin\varphi=\frac{\cancel{2}}{3}\cdot\frac{\sin^2\varphi}{\cancel{2}}\Big|_0^{\pi/2}=\frac{1}{3}.$$ Отг. $I=\frac{1}{3}$.

Зад. 3 Пресметнете обема $V(K)$ на тялото $K:\begin{cases}x^2+y^2\leq 1\\ x^2+y^2+z^2\leq 4\end{cases}$.

Решение:



K е всъщност тази част от цилиндъра $x^2+y^2\leq 1$, която лежи вътре в сферата $x^2+y^2+z^2=4$.

Имаме, че $V(K) = \iiint\limits_K 1\, dx\,dy\,dz$.

В този интеграл правим цилиндрична смяна $\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \\ z = t \\ \rho \geq 0,\ 0 \leq \varphi \leq 2\pi,\ t \in \mathbb{R} \end{cases}$. Както вече видяхме $\Delta = \rho$.

$K': \begin{cases} \rho^2\cos^2\varphi + \rho^2\sin^2\varphi \leq 1 \\ \rho^2\cos^2\varphi + \rho^2\sin^2\varphi + t^2 \leq 4 \\ \rho \geq 0,\ 0 \leq \varphi \leq 2\pi,\ t \in \mathbb{R} \end{cases}$, $K': \begin{cases} \rho^2 \leq 1 \\ \rho^2 + t^2 \leq 4 \\ \rho \geq 0,\ 0 \leq \varphi \leq 2\pi \end{cases}$.

Окончателно $K': \begin{cases} \left.\begin{matrix} 0 \leq \rho \leq 1 \\ 0 \leq \varphi \leq 2\pi \end{matrix}\right\} \mathcal{D}' \\ -\sqrt{4-\rho^2} \leq t \leq \sqrt{4-\rho^2} \end{cases}$. Тогава

$$V(K) = \iiint\limits_{K'} 1 \cdot |\rho|\, d\rho\, d\varphi\, dt = \iint\limits_{\mathcal{D}'} \Big[\underbrace{\int\limits_{-\sqrt{4-\rho^2}}^{\sqrt{4-\rho^2}} \rho\, dt}_{\text{интегрирането е по } t \text{ при фиксирано } \rho \in [0,1]} \Big] d\rho\, d\varphi =$$

$$= \iint\limits_{\mathcal{D}'} \Big[\rho \cdot t \Big|_{-\sqrt{4-\rho^2}}^{\sqrt{4-\rho^2}} \Big] d\rho\, d\varphi =$$

$$= 2\iint\limits_{\mathcal{D}'} \Big[\rho\sqrt{4-\rho^2}\Big] d\rho\, d\varphi = 2\int\limits_0^1 \Big[\underbrace{\int\limits_0^{2\pi} \rho\sqrt{4-\rho^2}\, d\varphi}_{\text{интегрирането е по } \varphi \text{ при фиксирано } \rho \in [0,1]} \Big] d\rho$$

$$= 2\int\limits_0^1 \rho\sqrt{4-\rho^2}\, \varphi\Big|_0^{2\pi} d\rho = 4\pi\int\limits_0^1 \rho\sqrt{4-\rho^2}\, d\rho =$$

$$= -2\pi\int\limits_0^1 \sqrt{4-\rho^2}\, d(4-\rho^2) = -2\pi\, \frac{(4-\rho^2)^{\frac{3}{2}}}{\frac{3}{2}} \Bigg|_0^1 =$$

$$= -\frac{4\pi}{3}\left(3^{\frac{3}{2}} - 4^{\frac{3}{2}}\right) = \frac{4\pi}{3}\left(\sqrt{64} - \sqrt{27}\right).$$ Отг. $V(K) = \dfrac{4\pi\left(4\sqrt{4} - 3\sqrt{3}\right)}{3}$.

Зад. 4 Пресметнете $I = \iiint\limits_K \dfrac{1}{x^2+y^2+z^2+1}\, dx\,dy\,dz$, където $K: \begin{cases} x^2+y^2+z^2 \leq 1 \\ z \geq 0 \end{cases}$.

Решение: Правим сферична смяна

$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \\ r \geq 0,\ 0 \leq \theta \leq \pi,\ 0 \leq \varphi \leq 2\pi \end{cases}$. Както вече знаем $\Delta = r^2\sin\theta$.

(Да припомним, че в сферични координати $x^2+y^2+z^2 = r^2$ – това се проверява лесно.)

$K': \begin{cases} r^2 \leq 1 \\ r\cos\theta \geq 0 \\ r \geq 0,\ 0 \leq \theta \leq \pi,\ 0 \leq \varphi \leq 2\pi \end{cases}$, т.е. $K': \begin{cases} \left.\begin{matrix} 0 \leq r \leq 1 \\ 0 \leq \theta \leq \frac{\pi}{2} \end{matrix}\right\} \mathcal{D}' \\ 0 \leq \varphi \leq 2\pi \end{cases}$. Тогава

$$I = \iiint\limits_{K'} \frac{1}{r^2+1} \cdot \underbrace{|r^2\sin\theta|}_{\Delta = r^2\sin\theta}\, dr\, d\theta\, d\varphi = \iint\limits_{\mathcal{D}'} \Big[\underbrace{\int\limits_0^{2\pi} \frac{r^2\sin\theta}{r^2+1}\, d\varphi}_{\text{интегрирането е по } \varphi \text{ при фиксирана точка } (r,\theta) \in \mathcal{D}'} \Big] dr\, d\theta =$$

$$= \iint_{D'} \left[\frac{r^2\sin\theta}{r^2+1}\cdot \varphi\Big|_0^{2\pi}\right] dr\,d\theta = 2\pi \iint_{D'} \frac{r^2 \sin\theta}{r^2+1}\, dr\,d\theta =$$

$$= 2\pi \int_0^1 \left[\int_0^{\pi/2} \frac{r^2}{r^2+1}\sin\theta\, d\theta\right] dr = 2\pi\int_0^1 \left[\frac{r^2}{r^2+1}(-\cos\theta)\Big|_0^{\frac{\pi}{2}}\right]dr =$$

$$= 2\pi\int_0^1 \frac{r^2}{r^2+1}\,dr = 2\pi \int_0^1 \frac{(r^2+1)-1}{r^2+1}\,dr = 2\pi\int_0^1\left(1-\frac{1}{r^2+1}\right)dr =$$

$$= 2\pi(r-\operatorname{arctg} r)\Big|_0^1 = 2\pi\left(1-\frac{\pi}{4}\right).$$ Отг. $I=\frac{\pi(4-\pi)}{2}$.

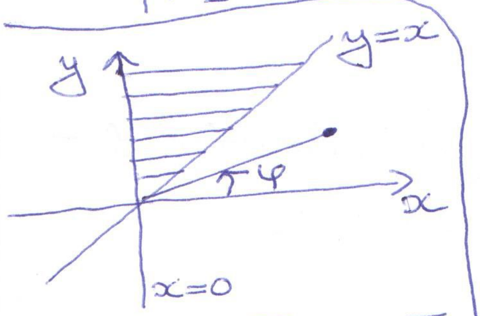
Зад. 5 Пресметнете $I=\iiint_K \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz$, където

$K: \begin{cases} x^2+y^2+z^2 \leq 9 \\ x \geq 0,\ z \geq 0 \\ y \geq x \end{cases}$.

Решение: Правим сферична смяна $\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \\ r\geq 0,\ 0\leq\theta\leq\pi,\ 0\leq\varphi\leq 2\pi \end{cases}$.

Както вече знаем $\Delta = r^2\sin\theta$.

$K': \begin{cases} r^2 \leq 9 \\ \frac{\pi}{4} \leq \varphi \leq \frac{\pi}{2},\ r\cos\theta \geq 0 \\ r \geq 0,\ 0 \leq \theta \leq \pi \end{cases}$, т.е. $K': \begin{cases} \left.\begin{matrix} 0 \leq r \leq 3 \\ 0 \leq \theta \leq \frac{\pi}{2} \end{matrix}\right\} D' \\ \frac{\pi}{4} \leq \varphi \leq \frac{\pi}{2} \end{cases}$. Тогава



$\begin{cases} x \geq 0 \\ y \geq x \end{cases} \Longleftrightarrow \frac{\pi}{4} \leq \varphi \leq \frac{\pi}{2}$

$$I = \iiint_{K'} r\cdot|r^2\sin\theta|\,dr\,d\theta\,d\varphi =$$

$$= \iint_{D'}\left[\int_{\pi/4}^{\pi/2} r^3\sin\theta\,d\varphi\right]dr\,d\theta =$$

интегрирането е по φ при фиксирана точка $(r,\theta)\in D'$

$$= \iint_{D'}\left[r^3\sin\theta\cdot\varphi\Big|_{\pi/4}^{\pi/2}\right]dr\,d\theta =$$

$$= \frac{\pi}{4}\iint_{D'} r^3\sin\theta\,dr\,d\theta = \frac{\pi}{4}\int_0^3\left[\int_0^{\pi/2} r^3\sin\theta\,d\theta\right]dr =$$

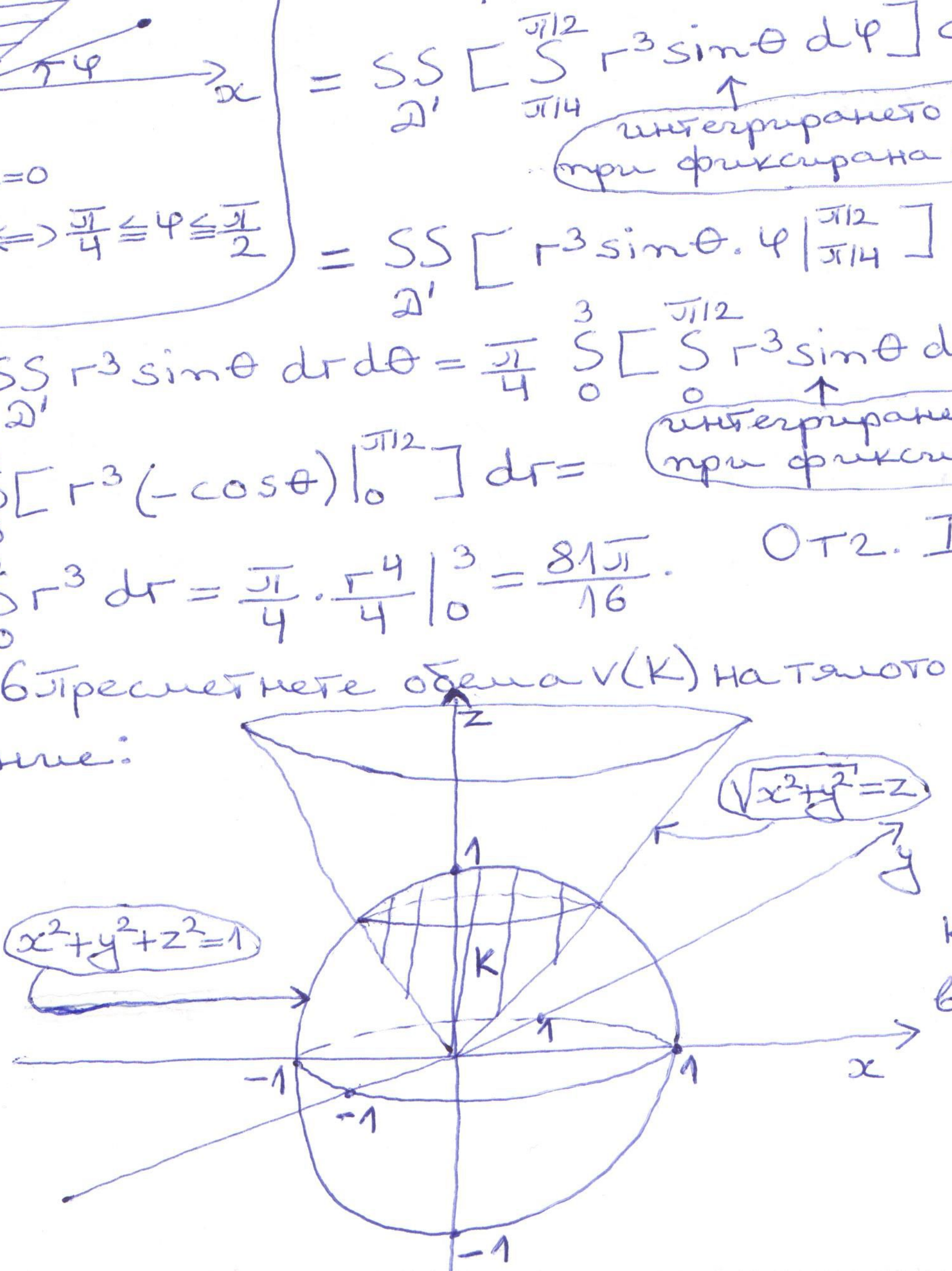
интегрирането е по θ при фиксирано $r\in[0,3]$

$$= \frac{\pi}{4}\int_0^3\left[r^3(-\cos\theta)\Big|_0^{\pi/2}\right]dr =$$

$$= \frac{\pi}{4}\int_0^3 r^3\,dr = \frac{\pi}{4}\cdot\frac{r^4}{4}\Big|_0^3 = \frac{81\pi}{16}.$$ Отг. $I = \frac{81\pi}{16}$.

Зад. 6 Пресметнете обема $V(K)$ на тялото $K: \begin{cases} \sqrt{x^2+y^2} \leq z \\ x^2+y^2+z^2 \leq 1 \end{cases}$

Решение:



Всъщност K е частта от конуса $\sqrt{x^2+y^2}\leq z$, която лежи вътре в сферата $x^2+y^2+z^2=1$.

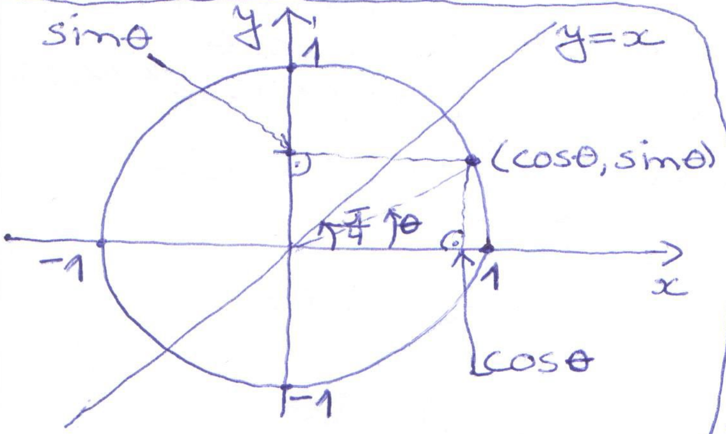
Имаме, че $V(K) = \iiint\limits_K 1\,dx\,dy\,dz$.

В този интеграл правим сферична смяна

$$\left|\begin{array}{l} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \\ r \geqq 0,\ 0 \leqq \theta \leqq \pi,\ 0 \leqq \varphi \leqq 2\pi \end{array}\right.$$

. Както вече знаем $\Delta = r^2\sin\theta$.

$$K': \left|\begin{array}{l} \sqrt{r^2\sin^2\theta} \leqq r\cos\theta \\ r^2 \leqq 1 \\ r \geqq 0,\ 0 \leqq \theta \leqq \pi,\ 0 \leqq \varphi \leqq 2\pi \end{array}\right. ,\quad K': \left|\begin{array}{l} 0 \leqq r \leqq 1 \\ \sin\theta \leqq \cos\theta \\ 0 \leqq \theta \leqq \pi,\ 0 \leqq \varphi \leqq 2\pi \end{array}\right. .$$



$$\left|\begin{array}{l} \sin\theta \leqq \cos\theta \\ 0 \leqq \theta \leqq \pi \end{array}\right. \Longleftrightarrow 0 \leqq \theta \leqq \frac{\pi}{4}$$

Окончателно

$$K': \left|\begin{array}{l} \left.\begin{array}{l} 0 \leqq r \leqq 1 \\ 0 \leqq \theta \leqq \frac{\pi}{4} \end{array}\right\} D' \\ 0 \leqq \varphi \leqq 2\pi \end{array}\right. .$$

Тогава

$$I = \iiint\limits_{K'} |r^2\sin\theta|\,dr\,d\theta\,d\varphi =$$

$\boxed{\Delta = r^2\sin\theta}$

$$= \iint\limits_{D'} \left[\int_0^{2\pi} r^2\sin\theta\,d\varphi\right] dr\,d\theta = \iint\limits_{D'} \left[r^2\sin\theta\cdot\varphi\Big|_0^{2\pi}\right] dr\,d\theta =$$

(интегрирането е по φ при фиксирана точка $(r,\theta)\in D'$)

$$= 2\pi \iint\limits_{D'} r^2\sin\theta\,dr\,d\theta =$$

$$= 2\pi \int_0^1 \left[\int_0^{\pi/4} r^2\sin\theta\,d\theta\right] dr = 2\pi\int_0^1 \left[r^2(-\cos\theta)\Big|_0^{\pi/4}\right] dr =$$

(интегрирането е по θ при фиксирано $r\in[0,1]$)

$$= 2\pi\left(1-\frac{\sqrt{2}}{2}\right)\int_0^1 r^2\,dr =$$

$$= (2-\sqrt{2})\pi\,\frac{r^3}{3}\Big|_0^1 = \frac{(2-\sqrt{2})\pi}{3}.$$

Отг. $V(K) = \dfrac{(2-\sqrt{2})\pi}{3}$.