Convex Sets

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Exercise 2.1

Let $C \subseteq \mathbf{R}^n$ be a convex set, with $x_1, ..., x_k \in C$, and let $\theta_1, ..., \theta_k \in R$ satisfy $\theta_i \geq 0$, $\theta_1 + ... + \theta_k = 1$. Show that $\theta_1 x_1 + ... + \theta_k x_k \in C$. (The definition of convexity is that this holds for $k = 2$; you must show it for arbitrary k.) *Hint.* Use induction on k.

Solution. We know that

$$\theta_1 + \cdots + \theta_{k-2} \; + \; \tilde{\theta}_{k-1} \; + \; \tilde{\theta}_k \; = \; 1 \quad \text{and} \quad \theta_1 x_1 + \cdots + \theta_{k-2} x_{k-2} \; + \; \tilde{\theta}_{k-1} \tilde{x}_{k-1} \; + \; \tilde{\theta}_k \tilde{x}_k \; \in \; C.$$

Next, define

$$\theta_{k-1} \; = \; \tilde{\theta}_{k-1} + \tilde{\theta}_k.$$

From this we see that

$$1 \; = \; \frac{\tilde{\theta}_{k-1} + \tilde{\theta}_k}{\theta_{k-1}},$$

so, using the fact that C is convex, we know that there is a x_{k-1} such that

$$\tilde{\theta}_{k-1} \tilde{x}_{k-1} + \tilde{\theta}_k \tilde{x}_k \; = \; \theta_{k-1} x_{k-1}.$$

Plugging this back into the original equations, we get

$$\theta_1 + \cdots + \theta_{k-1} \; = \; 1 \quad \text{and} \quad \theta_1 x_1 + \cdots + \theta_{k-1} x_{k-1} \; \in \; C,$$

thus reducing two x_i's into one. Repeating this procedure recursively leaves us with a single point x_0, which must lie in C by its convexity. $\square$

Exercise 2.2

Show that a set is convex if and only if its intersection with any line is convex. Show that a set is affine if and only if its intersection with any line is affine.

Solution. First, let us show that if a set C is convex, then its intersection with any line L is also convex. This follows easily because a line itself is convex, and the intersection of convex sets is convex. Indeed, if $x_1, x_2 \in C \cap L$ they must belong to C and L separately, then by the convexity of C, for any $\theta \in [0, 1]$,

$$\theta x_1 + (1 - \theta)x_2 \in C,$$

and by the convexity of L,

$$\theta x_1 + (1 - \theta)x_2 \in L.$$

Hence,

$$\theta x_1 + (1 - \theta)x_2 \in C \cap L.$$

Implying convexity of $C \cap L$.
For the other direction, let us take any two points x_1 and x_2 in C. Consider the line through x_1 and x_2. Since C intersects every line in a convex set, it follows that this line, intersected with C, contains all points between x_1 and x_2. In other words, for all $\theta \in [0, 1]$,

$$\theta x_1 + (1 - \theta)x_2 \in C.$$

Because x_1 and x_2 were chosen arbitrarily, it is true for all $x_1, x_2 \in C$, it follows that C is convex.
Next, let us turn to the affine case. Suppose A is an affine set and let L be a line, which is also affine. The argument is exactly the same as in the convex case, but now we allow $\theta \in \mathbf{R}$ (not just $[0, 1]$). Thus, the intersection $A \cap L$ is affine if A is affine, and the converse holds by an identical reasoning. $\square$

Exercise 2.5

What is the distance between two parallel hyperplanes $\{x \in \mathbf{R}^n \mid a^T x = b_1\}$ and $\{x \in \mathbf{R}^n \mid a^T x = b_2\}$?

Solution. We know that a is parallel to the plane. Subtracting both equations, we get

$$a^T(x_1 - x_2) = b_1 - b_2.$$

Taking the Euclidean norm on both sides, and noting that the distance is the minimum possible value of $\|x_1 - x_2\|_2$, we write

$$\|a^T(x_1 - x_2)\|_2 = |b_1 - b_2|.$$

From the standard norm inequality,

$$\|a^T(x_1 - x_2)\|_2 \;\le\; \|x_1 - x_2\|_2\,\|a^T\|_2.$$

The equality here represents the minimal distance. Consequently, dividing by constant $\|a^T\|_2$

$$\min\Big(\|x_1 - x_2\|_2\Big) \;=\; \frac{|b_1 - b_2|}{\|a^T\|_2}.$$

This value is the distance between the two hyperplanes. □

Exercise 2.7

Voronoi description of halfspace. Let a and b be distinct points in $\mathbf{R}^n$. Show that the set of all points that are closer (in Euclidean norm) to a than b, i.e., $\{x \mid \|x-a\|_2 \le \|x-b\|_2\}$, is a halfspace. Describe it explicitly as an inequality of the form $c^Tx \le d$. Draw a picture.

Solution. Let us take a point between a and b:

$$x_0 \;=\; \frac{a+b}{2}.$$

Then let c be the vector pointing from a to b:

$$c \;=\; b - a.$$

If we move in a direction perpendicular to c, we neither move closer to a nor to b. Thus, we can describe a hyperplane defining the boundary between the two halfspaces:

$$\{\, x : c^T(x - x_0) = 0\}.$$

Therefore, the halfspace consisting of points closer to a is

$$\{\, x : c^Tx \;\le\; c^Tx_0\}.$$

(A figure may be provided later.) □

Exercise 2.8

Which of the following sets S are polyhedra? If possible, express S in the form $S = \{x \mid Ax \preceq b,\ Fx = g\}$.

(a)

$$S = \{\, y_1a_1 + y_2a_2 \mid -1 \le y_1 \le 1,\ -1 \le y_2 \le 1\}, \quad \text{where } a_1, a_2 \in \mathbf{R}^n.$$

Solution. Let us look at the boundary, where y_1 and y_2 are equal to their constraints:

$$C = \{ a_1 + a_2,\ -a_1 + a_2,\ a_1 - a_2,\ -a_1 - a_2 \}.$$

We see the swept region is a kind of parallelogram, which is a polyhedron. $\square$

(b)

$$S = \{ x \in \mathbf{R}^n \mid x \succeq 0,\ \mathbf{1}^T x = 1,\ \sum_{i=1}^n x_i a_i = b_1,\ \sum_{i=1}^n x_i a_i^2 = b_2 \},$$

where $a_1, \ldots, a_n \in \mathbf{R}$ and $b_1, b_2 \in \mathbf{R}$.

Solution. We see that it is a polyhedron by writing it as

$$-\mathbf{I}\, x \preceq 0$$

and

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ a_1^2 & a_2^2 & \cdots & a_n^2 \end{bmatrix} x = \begin{bmatrix} 1 \\ b_1 \\ b_2 \end{bmatrix}.$$

$\square$

(c)

$$S = \left\{ x \in \mathbf{R}^n \;\middle|\; x \succeq 0,\ x^T y \le 1 \text{ for all } y \text{ with } \|y\|_2 = 1 \right\}.$$

Solution. The term $x^T y$ is the dot product with a unit vector. Since x is chosen arbitrarily, we can set $y = \dfrac{x}{\|x\|_2}$ to examine the boundary of the set, which gives

$$\|x\|_2 \le 1.$$

Along with $x \succeq 0$, this describes a portion of the n-dimensional sphere lying in the nonnegative orthant. Hence, it is not a polyhedron. $\square$

(d)

$$S = \left\{ x \in \mathbf{R}^n \;\middle|\; x \succeq 0,\ x^T y \le 1 \text{ for all } y \text{ with } \sum_{i=1}^n |y_i| = 1 \right\}.$$

Solution. Let us again look at the boundary. The quantity $x^T y$ is largest when $y_i = 1$ for the coordinate i that maximizes x_i. Thus the shape is bounded by $\max_i x_i \le 1$ and must lie in the nonnegative orthant. Hence, it can be described by

$$\begin{bmatrix} \mathbf{I}_n \\ -\mathbf{I}_n \end{bmatrix} x \preceq \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

so it is a polyhedron. $\square$

Exercise 2.11

Hyperbolic sets. Show that the *hyperbolic* set $\{ x \in \mathbf{R}_+^2 \mid x_1 x_2 \geq 1\}$ is convex. More generally, show that $\{ x \in \mathbf{R}_+^n \mid \prod_{i=1}^n x_i \geq 1\}$ is convex.
Hint. If $a, b \geq 0$ and $0 \leq \theta \leq 1$, then $a^\theta b^{1-\theta} \leq \theta a + (1-\theta) b$.

Solution. Let us go directly to the general case. Suppose x, $y \succeq 0$ lie in the set, which means:

$$\prod_{i=1}^n x_i \;\geq\; 1 \quad \text{and} \quad \prod_{i=1}^n y_i \;\geq\; 1.$$

We want to show that for any convex combination $z = \theta x + (1-\theta)\, y$, with $0 \leq \theta \leq 1$, the point z also lies in the set; in other words, its components' product is at least 1:

$$\prod_{i=1}^n z_i \;=\; \prod_{i=1}^n \Big(\theta x_i + (1-\theta)\, y_i\Big) \;\geq\; \prod_{i=1}^n x_i^\theta \, y_i^{1-\theta} \;\geq\; 1^\theta \, 1^{1-\theta} \;\geq\; 1.$$

The first inequality uses the hint $\Big(\theta a + (1-\theta)\, b\Big) \geq a^\theta b^{1-\theta}$ for $a, b \geq 0$ and $0 \leq \theta \leq 1$. Thus, $\prod_{i=1}^n z_i \geq 1$, so z is in the set. □