Convex Sets

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Exercise 2.1

Let $C \subseteq \mathbf{R}^n$ be a convex set, with $x_1, ..., x_k \in C$, and let $\theta_1, ..., \theta_k \in R$ satisfy $\theta_i \geq 0$, $\theta_1 + ... + \theta_k = 1$. Show that $\theta_1 x_1 + ... + \theta_k x_k \in C$. (The definition of convexity is that this holds for $k = 2$; you must show it for arbitrary k.) *Hint.* Use induction on k.

Solution. We know that

$$\theta_1 + \cdots + \theta_{k-2} \;+\; \tilde{\theta}_{k-1} \;+\; \tilde{\theta}_k \;=\; 1 \quad \text{and} \quad \theta_1 x_1 + \cdots + \theta_{k-2} x_{k-2} \;+\; \tilde{\theta}_{k-1}\tilde{x}_{k-1} \;+\; \tilde{\theta}_k \tilde{x}_k \;\in\; C.$$

Next, define

$$\theta_{k-1} \;=\; \tilde{\theta}_{k-1} + \tilde{\theta}_k.$$

From this we see that

$$1 \;=\; \frac{\tilde{\theta}_{k-1} + \tilde{\theta}_k}{\theta_{k-1}},$$

so, using the fact that C is convex, we know that there is a x_{k-1} such that

$$\tilde{\theta}_{k-1}\tilde{x}_{k-1} + \tilde{\theta}_k \tilde{x}_k \;=\; \theta_{k-1} x_{k-1}.$$

Plugging this back into the original equations, we get

$$\theta_1 + \cdots + \theta_{k-1} \;=\; 1 \quad \text{and} \quad \theta_1 x_1 + \cdots + \theta_{k-1} x_{k-1} \;\in\; C,$$

thus reducing two x_i's into one. Repeating this procedure recursively leaves us with a single point x_0, which must lie in C by its convexity. $\square$

Exercise 2.2

Show that a set is convex if and only if its intersection with any line is convex. Show that a set is affine if and only if its intersection with any line is affine.

Solution. First, let us show that if a set C is convex, then its intersection with any line L is also convex. This follows easily because a line itself is convex, and the intersection of convex sets is convex. Indeed, if $x_1, x_2 \in C \cap L$ they must belong to C and L separately, then by the convexity of C, for any $\theta \in [0, 1]$,

$$\theta x_1 + (1-\theta)x_2 \in C,$$

and by the convexity of L,

$$\theta x_1 + (1-\theta)x_2 \in L.$$

Hence,

$$\theta x_1 + (1-\theta)x_2 \in C \cap L.$$

Implying convexity of $C \cap L$.
For the other direction, let us take any two points x_1 and x_2 in C. Consider the line through x_1 and x_2. Since C intersects every line in a convex set, it follows that this line, intersected with C, contains all points between x_1 and x_2. In other words, for all $\theta \in [0, 1]$,

$$\theta x_1 + (1-\theta)x_2 \in C.$$

Because x_1 and x_2 were chosen arbitrarily, it is true for all $x_1, x_2 \in C$, it follows that C is convex.
Next, let us turn to the affine case. Suppose A is an affine set and let L be a line, which is also affine. The argument is exactly the same as in the convex case, but now we allow $\theta \in \mathbf{R}$ (not just $[0, 1]$). Thus, the intersection $A \cap L$ is affine if A is affine, and the converse holds by an identical reasoning. $\square$