# Convex Sets

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## Exercise 2.1

Let $C \subseteq \mathbf{R}^n$ be a convex set, with $x_1, ..., x_k \in C$, and let $\theta_1, ..., \theta_k \in R$ satisfy $\theta_i \geq 0$, $\theta_1 + ... + \theta_k = 1$. Show that $\theta_1 x_1 + ... + \theta_k x_k \in C$. (The definition of convexity is that this holds for $k = 2$; you must show it for arbitrary $k$.) *Hint.* Use induction on $k$.

***Solution.*** We know that

$$\theta_1 + \cdots + \theta_{k-2} \ + \ \tilde{\theta}_{k-1} \ + \ \tilde{\theta}_k \ = \ 1 \quad \text{and} \quad \theta_1 x_1 + \cdots + \theta_{k-2} x_{k-2} \ + \ \tilde{\theta}_{k-1} \tilde{x}_{k-1} \ + \ \tilde{\theta}_k \tilde{x}_k \ \in \ C.$$

Next, define

$$\theta_{k-1} \ = \ \tilde{\theta}_{k-1} + \tilde{\theta}_k.$$

From this we see that

$$1 \ = \ \frac{\tilde{\theta}_{k-1} + \tilde{\theta}_k}{\theta_{k-1}},$$

so, using the fact that $C$ is convex, we know that there is a $x_{k-1}$ such that

$$\tilde{\theta}_{k-1} \tilde{x}_{k-1} + \tilde{\theta}_k \tilde{x}_k \ = \ \theta_{k-1} x_{k-1}.$$

Plugging this back into the original equations, we get

$$\theta_1 + \cdots + \theta_{k-1} \ = \ 1 \quad \text{and} \quad \theta_1 x_1 + \cdots + \theta_{k-1} x_{k-1} \ \in \ C,$$

thus reducing two $x_i$'s into one. Repeating this procedure recursively leaves us with a single point $x_0$, which must lie in $C$ by its convexity. $\square$