Convex Sets

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Exercise 2.1

Let $C \subseteq \mathbf{R}^n$ be a convex set, with $x_1, ..., x_k \in C$, and let $\theta_1, ..., \theta_k \in R$ satisfy $\theta_i \geq 0$, $\theta_1 + ... + \theta_k = 1$. Show that $\theta_1 x_1 + ... + \theta_k x_k \in C$. (The definition of convexity is that this holds for $k = 2$; you must show it for arbitrary k.) *Hint.* Use induction on k.

Solution. We know that

$$\theta_1 + \cdots + \theta_{k-2} \ + \ \tilde{\theta}_{k-1} \ + \ \tilde{\theta}_k \ = \ 1 \quad \text{and} \quad \theta_1 x_1 + \cdots + \theta_{k-2} x_{k-2} \ + \ \tilde{\theta}_{k-1} \tilde{x}_{k-1} \ + \ \tilde{\theta}_k \tilde{x}_k \ \in \ C.$$

Next, define

$$\theta_{k-1} \ = \ \tilde{\theta}_{k-1} + \tilde{\theta}_k.$$

From this we see that

$$1 \ = \ \frac{\tilde{\theta}_{k-1} + \tilde{\theta}_k}{\theta_{k-1}},$$

so, using the fact that C is convex, we know that there is a x_{k-1} such that

$$\tilde{\theta}_{k-1} \tilde{x}_{k-1} + \tilde{\theta}_k \tilde{x}_k \ = \ \theta_{k-1} x_{k-1}.$$

Plugging this back into the original equations, we get

$$\theta_1 + \cdots + \theta_{k-1} \ = \ 1 \quad \text{and} \quad \theta_1 x_1 + \cdots + \theta_{k-1} x_{k-1} \ \in \ C,$$

thus reducing two x_i's into one. Repeating this procedure recursively leaves us with a single point x_0, which must lie in C by its convexity. $\square$

Exercise 2.2

Show that a set is convex if and only if its intersection with any line is convex. Show that a set is affine if and only if its intersection with any line is affine.

Solution. First, let us show that if a set C is convex, then its intersection with any line L is also convex. This follows easily because a line itself is convex, and the intersection of convex sets is convex. Indeed, if $x_1, x_2 \in C \cap L$ they must belong to C and L separately, then by the convexity of C, for any $\theta \in [0, 1]$,

$$\theta x_1 + (1 - \theta)x_2 \in C,$$

and by the convexity of L,

$$\theta x_1 + (1 - \theta)x_2 \in L.$$

Hence,

$$\theta x_1 + (1 - \theta)x_2 \in C \cap L.$$

Implying convexity of $C \cap L$.
For the other direction, let us take any two points x_1 and x_2 in C. Consider the line through x_1 and x_2. Since C intersects every line in a convex set, it follows that this line, intersected with C, contains all points between x_1 and x_2. In other words, for all $\theta \in [0, 1]$,

$$\theta x_1 + (1 - \theta)x_2 \in C.$$

Because x_1 and x_2 were chosen arbitrarily, it is true for all $x_1, x_2 \in C$, it follows that C is convex.
Next, let us turn to the affine case. Suppose A is an affine set and let L be a line, which is also affine. The argument is exactly the same as in the convex case, but now we allow $\theta \in \mathbf{R}$ (not just $[0, 1]$). Thus, the intersection $A \cap L$ is affine if A is affine, and the converse holds by an identical reasoning. □

Exercise 2.5

What is the distance between two parallel hyperplanes $\{x \in \mathbf{R}^n \mid a^T x = b_1\}$ and $\{x \in \mathbf{R}^n \mid a^T x = b_2\}$?

Solution. We know that a is parallel to the plane. Subtracting both equations, we get

$$a^T(x_1 - x_2) = b_1 - b_2.$$

Taking the Euclidean norm on both sides, and noting that the distance is the minimum possible value of $\|x_1 - x_2\|_2$, we write

$$\|a^T(x_1 - x_2)\|_2 = |b_1 - b_2|.$$

From the standard norm inequality,

$$\|a^T(x_1 - x_2)\|_2 \;\le\; \|x_1 - x_2\|_2\,\|a^T\|_2.$$

The equality here represents the minimal distance. Consequently, dividing by constant $\|a^T\|_2$

$$\min\Big(\|x_1 - x_2\|_2\Big) \;=\; \frac{|b_1 - b_2|}{\|a^T\|_2}.$$

This value is the distance between the two hyperplanes. □

Exercise 2.7

Voronoi description of halfspace. Let a and b be distinct points in $\mathbf{R}^n$. Show that the set of all points that are closer (in Euclidean norm) to a than b, i.e., $\{x \mid \|x-a\|_2 \le \|x-b\|_2\}$, is a halfspace. Describe it explicitly as an inequality of the form $c^T x \le d$. Draw a picture.

Solution. Let us take a point between a and b:

$$x_0 \;=\; \frac{a+b}{2}.$$

Then let c be the vector pointing from a to b:

$$c \;=\; b - a.$$

If we move in a direction perpendicular to c, we neither move closer to a nor to b. Thus, we can describe a hyperplane defining the boundary between the two halfspaces:

$$\{\, x : c^T(x - x_0) = 0\}.$$

Therefore, the halfspace consisting of points closer to a is

$$\{\, x : c^T x \;\le\; c^T x_0\}.$$

(A figure may be provided later.) □