A Wild Bootstrap for Degenerate Kernel Tests

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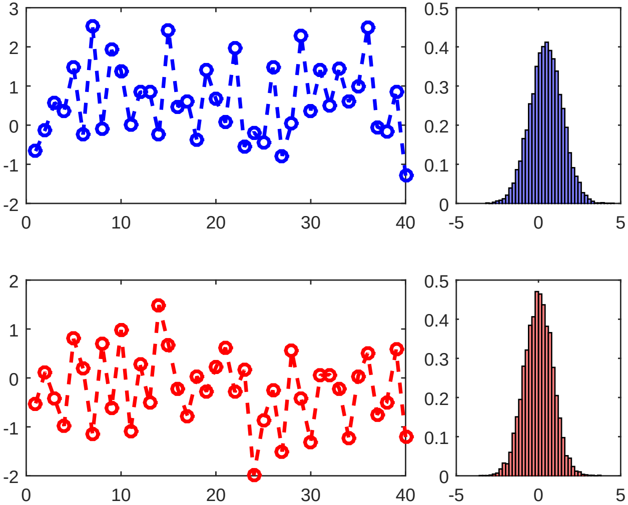
Arthur

Dino

Maximum Mean Discrepancy for Random Processes



Is

P

the same distribution as

Q

?

Statistical Tests for Random Processes

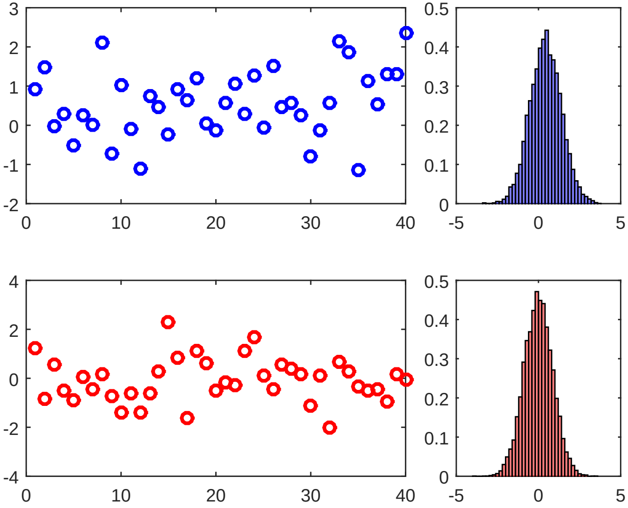
Where one can use Maximum Mean Discrepancy ?

- Markov chains diagnostics
- Change point detection

Other tests

- Hilbert Schmidt Independence Criterion
  - Dependency structure in financial markets
  - Brain region activation
- Three Variables Interaction

Maximum Mean Discrepancy for i.i.d observations



Is

P

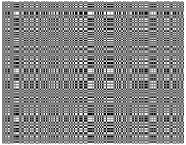
the same distribution as

Q

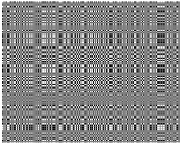
?

Similarity

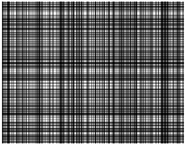
$$K_{P,P} = \begin{bmatrix} k(X_1, X_1) & \dots & k(X_1, X_n) \\ \vdots & \ddots & \vdots \\ k(X_n, X_1) & \dots & k(X_n, X_n) \end{bmatrix} =$$



$$K_{Q,Q} = \begin{bmatrix} k(Y_1, Y_1) & \dots & k(Y_1, Y_n) \\ \vdots & \ddots & \vdots \\ k(Y_n, Y_1) & \dots & k(Y_n, Y_n) \end{bmatrix} =$$

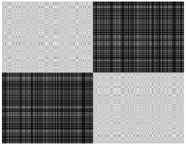


$$K_{P,Q} = \begin{bmatrix} k(X_1, Y_1) & \dots & k(X_1, Y_n) \\ \vdots & \ddots & \vdots \\ k(X_n, Y_1) & \dots & k(X_n, Y_n) \end{bmatrix} =$$



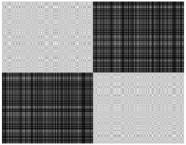
Similarity

$$P \neq Q \Rightarrow \begin{bmatrix} K_{P,P} & K_{P,Q} \\ K_{Q,P} & K_{Q,Q} \end{bmatrix} =$$

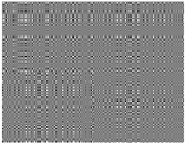


Similarity

$$P \neq Q \Rightarrow \begin{bmatrix} K_{P,P} & K_{P,Q} \\ K_{Q,P} & K_{Q,Q} \end{bmatrix} =$$

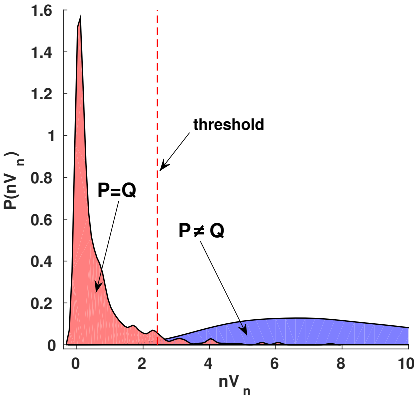


$$P = Q \Rightarrow \begin{bmatrix} K_{P,P} & K_{P,Q} \\ K_{Q,P} & K_{Q,Q} \end{bmatrix} =$$



Quantifying Similarity

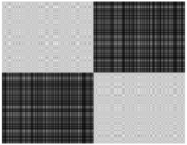
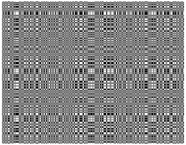
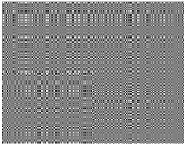
$$V_n = \overline{K_{P,P}} + \overline{K_{Q,Q}} - 2\overline{K_{P,Q}}$$

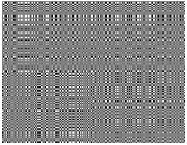
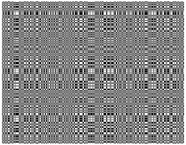
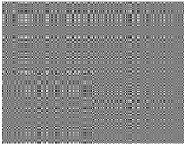


Putting Hands on V_n Distribution

$$W = (-1, 1, 1, \quad \cdots \quad, -1, -1, -1)$$

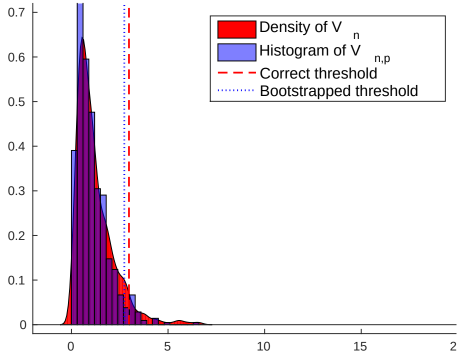
$$\begin{bmatrix} K_{P,P} & K_{P,Q} \\ K_{Q,P} & K_{Q,Q} \end{bmatrix} \odot \left[W^\top W \right] = [?]$$

 $\odot$  $=$ 

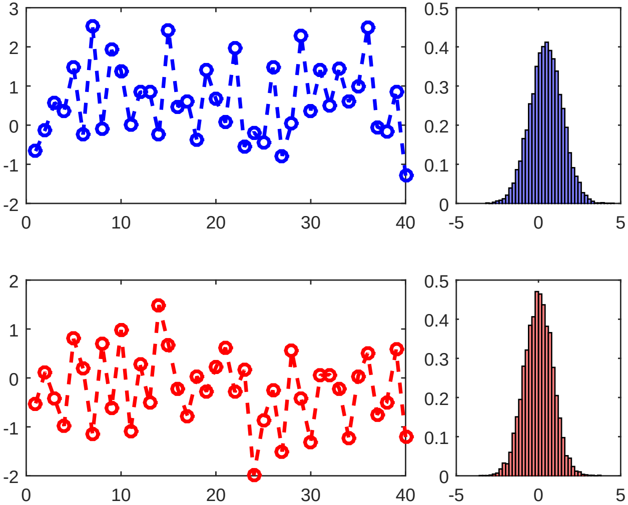
 $\odot$  $=$ 

Estimation of V_n via Permutation

$$V_{n,p} = \overline{\begin{bmatrix} K_{P,P} & K_{P,Q} \\ K_{Q,P} & K_{Q,Q} \end{bmatrix} \odot \left[W^\top W \right]}$$



Back to the Difficult Problem



Is

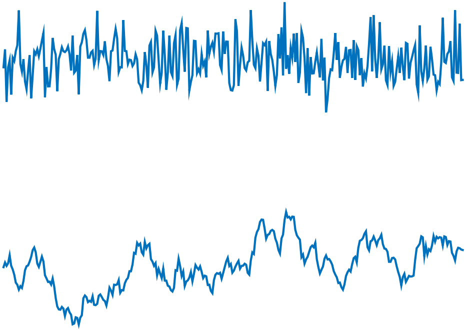
P

the same distribution as

Q

?

Memory of the Processes



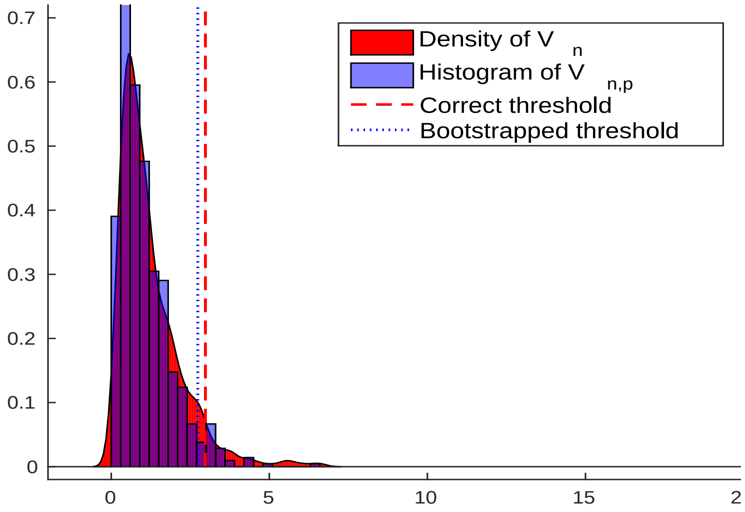
$$Q_t = \mathbf{0.14}\, Q_{t-1} + 0.98\epsilon_t$$

Processes with different memory

$$Z_t = \mathbf{0.97}\, Z_{t-1} + 0.22\epsilon_t.$$

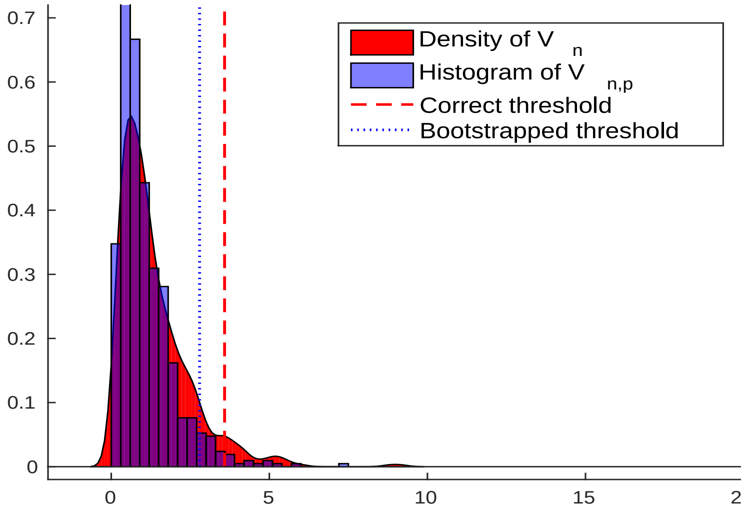
The distribution of the V-statistics is primary driven by memory

Memory 0.1



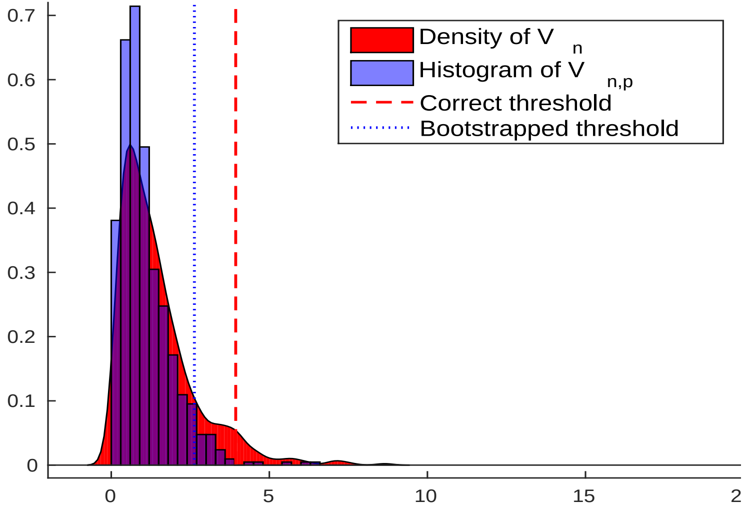
Density of V_n
Histogram of $V_{n,p}$
Correct threshold
Bootstrapped threshold
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
5
10
15
2

Memory 0.2



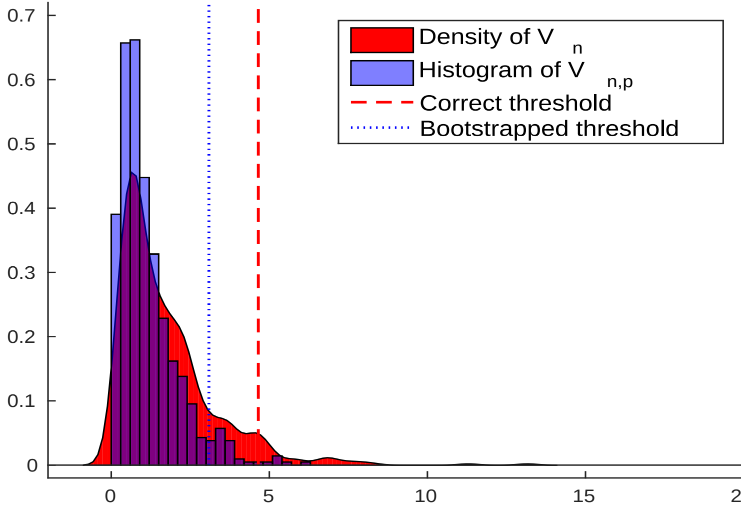
Density of V_n
Histogram of $V_{n,p}$
Correct threshold
Bootstrapped threshold
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
5
10
15
2

Memory 0.3

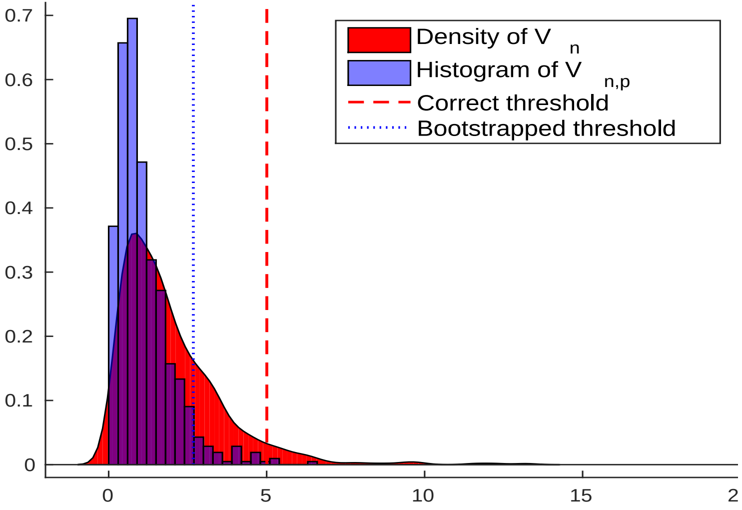


Density of V n
Histogram of V n,p
Correct threshold
Bootstrapped threshold
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
5
10
15
2

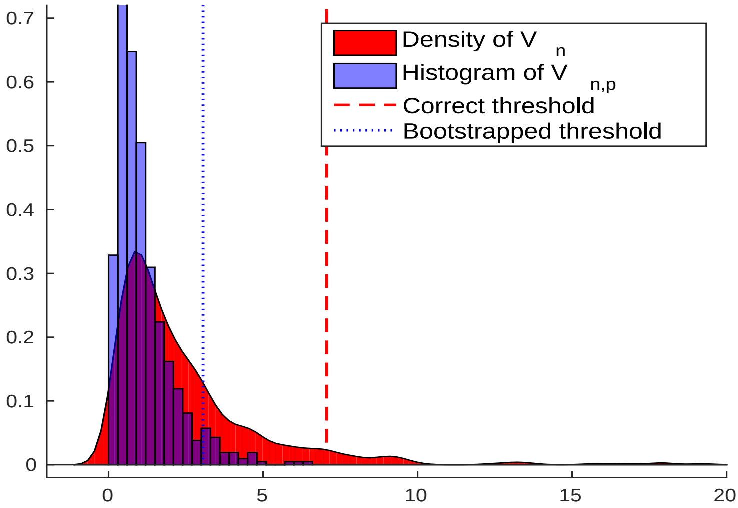
Memory 0.4


Density of V_n
Histogram of $V_{n,p}$
Correct threshold
Bootstrapped threshold
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
5
10
15
2

Memory 0.5

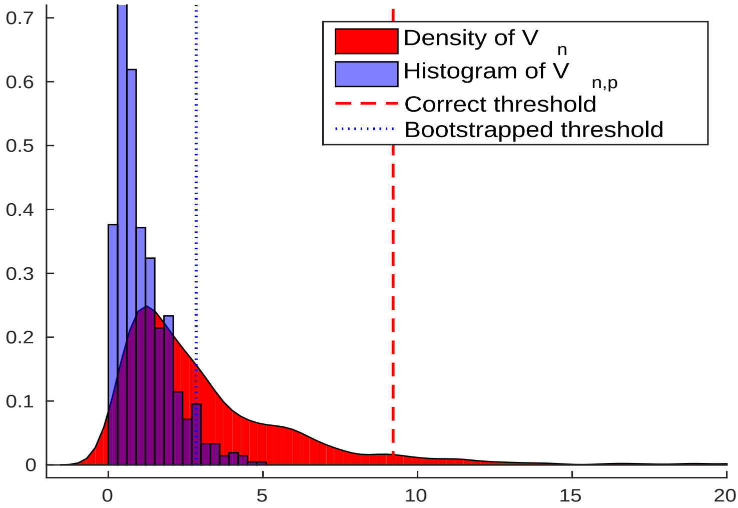

0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
5
10
15
2
Density of V n
Histogram of V n,p
Correct threshold
Bootstrapped threshold

Memory 0.6

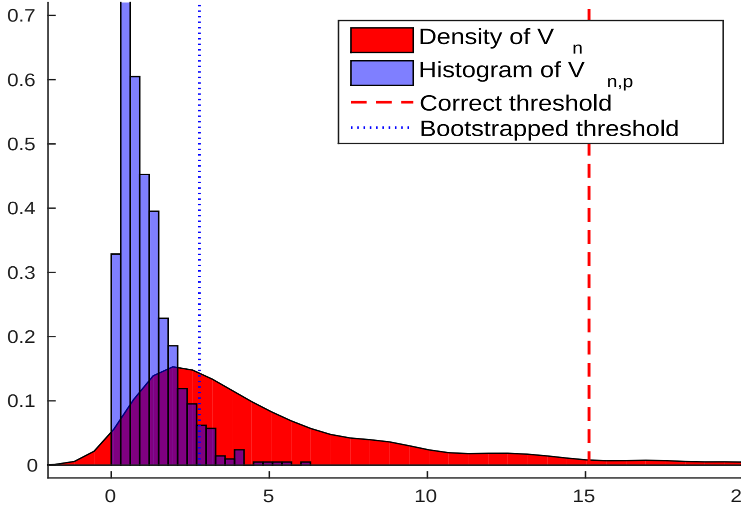


Density of V n
Histogram of V n,p
Correct threshold
Bootstrapped threshold
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
5
10
15
20

Memory 0.7


Density of V_n
Histogram of $V_{n,p}$
Correct threshold
Bootstrapped threshold
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
5
10
15
20

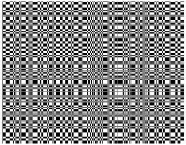
Memory 0.8



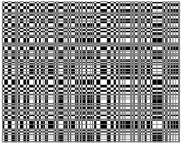
Density of V n
Histogram of V n,p
Correct threshold
Bootstrapped threshold
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
5
10
15
2

Similarity for Random Processes

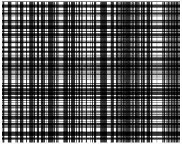
$$K_{P,P} = \begin{bmatrix} k(X_1, X_1) & \dots & k(X_1, X_n) \\ \vdots & \ddots & \vdots \\ k(X_n, X_1) & \dots & k(X_n, X_n) \end{bmatrix} =$$



$$K_{Q,Q} = \begin{bmatrix} k(Y_1, Y_1) & \dots & k(Y_1, Y_n) \\ \vdots & \ddots & \vdots \\ k(Y_n, Y_1) & \dots & k(Y_n, Y_n) \end{bmatrix} =$$

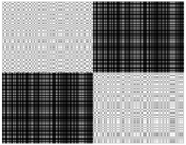


$$K_{P,Q} = \begin{bmatrix} k(X_1, Y_1) & \dots & k(X_1, Y_n) \\ \vdots & \ddots & \vdots \\ k(X_n, Y_1) & \dots & k(X_n, Y_n) \end{bmatrix} =$$



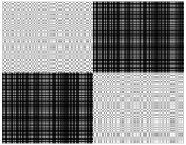
Gram Matrices

$$P \neq Q \quad \Rightarrow \begin{bmatrix} K_{P,P} & K_{P,Q} \\ K_{Q,P} & K_{Q,Q} \end{bmatrix} =$$

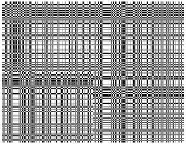


Gram Matrices

$$P \neq Q \quad \Rightarrow \begin{bmatrix} K_{P,P} & K_{P,Q} \\ K_{Q,P} & K_{Q,Q} \end{bmatrix} =$$



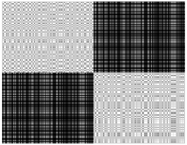
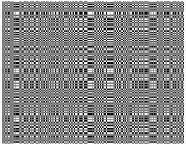
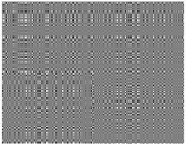
$$P = Q \quad \Rightarrow \begin{bmatrix} K_{P,P} & K_{P,Q} \\ K_{Q,P} & K_{Q,Q} \end{bmatrix} =$$

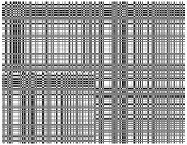
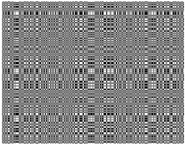
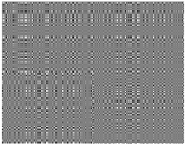


Permutation Test for Random Processes

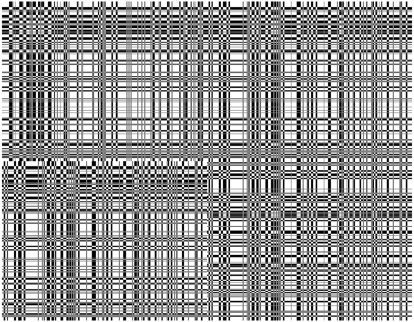
$$W = (-1, 1, 1, \dots, -1, -1, -1)$$

$$\begin{bmatrix} K_{P,P} & K_{P,Q} \\ K_{Q,P} & K_{Q,Q} \end{bmatrix} \odot \left[W^\top W \right] = [?]$$

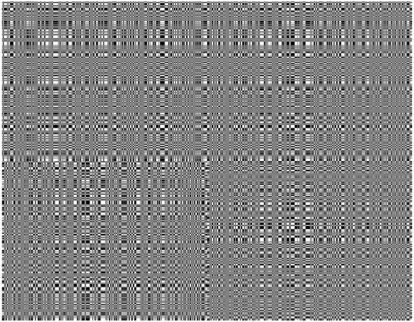
 $\odot$  $=$ 

 $\odot$  $=$ 

If $P = Q$ Permutation Approach Fails

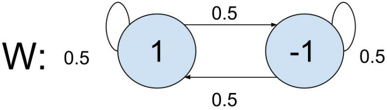


The actual matrix

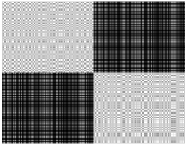
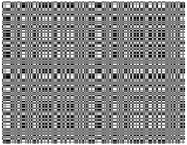
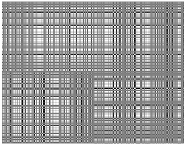


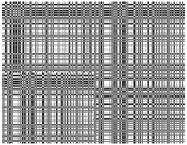
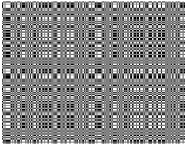
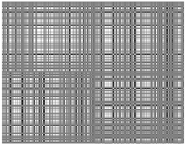
Matrix generated via permutation

Wild Bootstrap

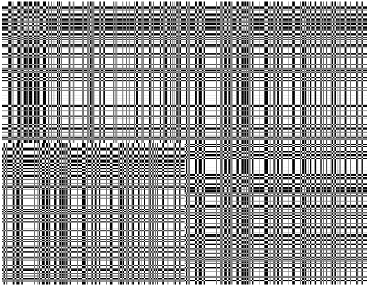


$$\begin{bmatrix} K_{P,P} & K_{P,Q} \\ K_{Q,P} & K_{Q,Q} \end{bmatrix} \odot \begin{bmatrix} W^\top W & -W^\top W \\ -W^\top W & W^\top W \end{bmatrix} = \begin{bmatrix} ? \end{bmatrix}$$

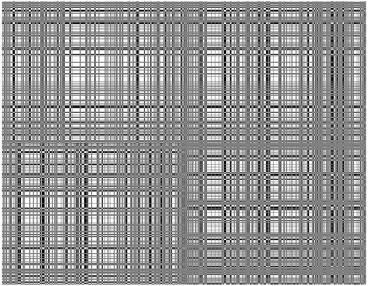
 $\odot$  $=$ 

 $\odot$  $=$ 

If $P = Q$ the Wild Bootstrap Approach Works!

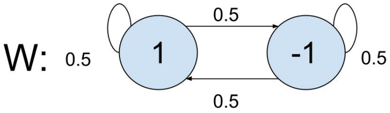


The actual matrix



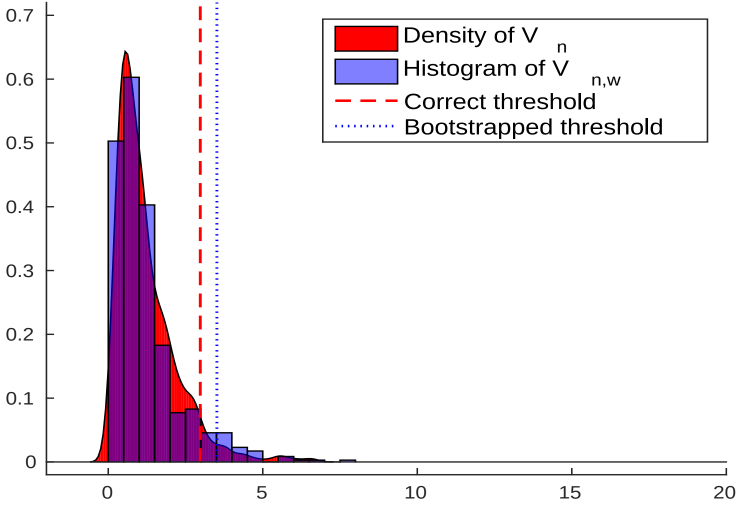
Matrix generated via **wild bootstrap**

Estimation of V_n via the Wild Bootstrap

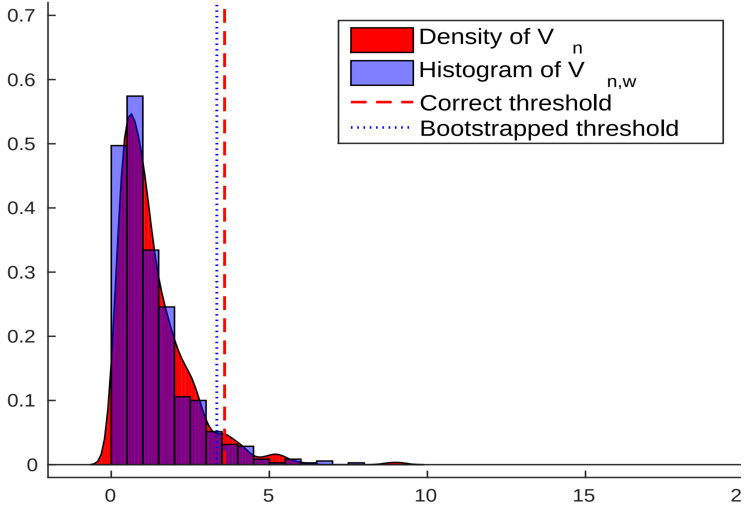


$$V_{n,w} = \overline{\begin{bmatrix} K_{P,P} & K_{P,Q} \\ K_{Q,P} & K_{Q,Q} \end{bmatrix} \odot \begin{bmatrix} W^\top W & -W^\top W \\ -W^\top W & W^\top W \end{bmatrix}}$$

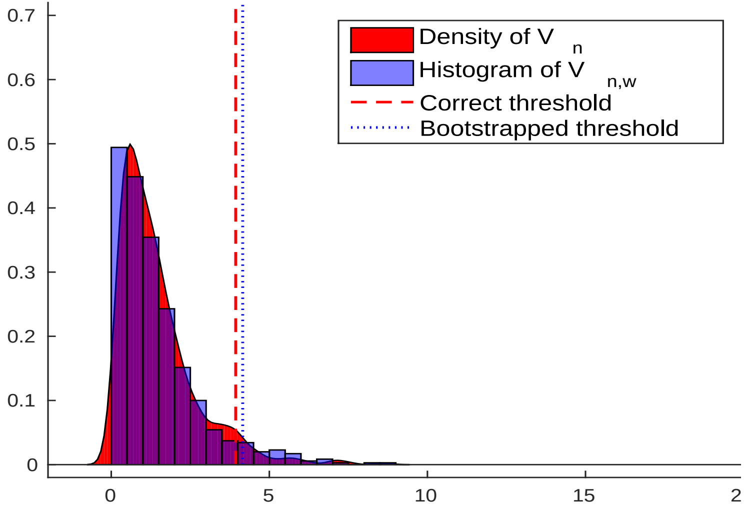
Memory 0.1


Density of V_n
Histogram of $V_{n,w}$
Correct threshold
Bootstrapped threshold
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
5
10
15
20

Memory 0.2

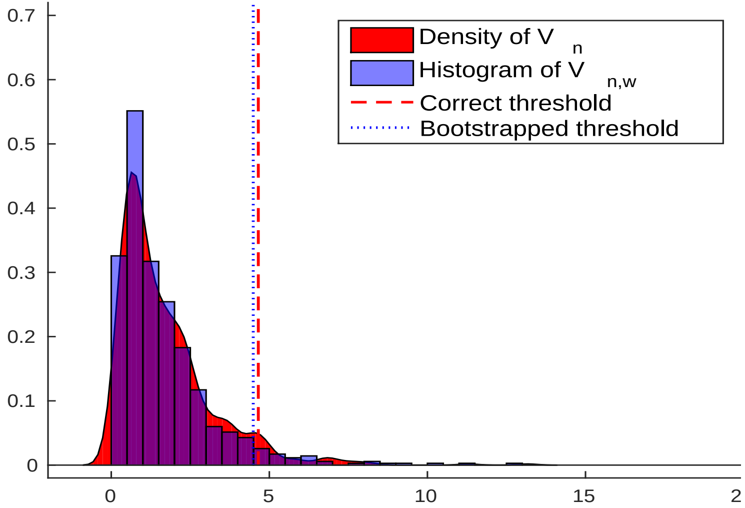

0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
5
10
15
2
Density of V n
Histogram of V n,w
Correct threshold
Bootstrapped threshold

Memory 0.3



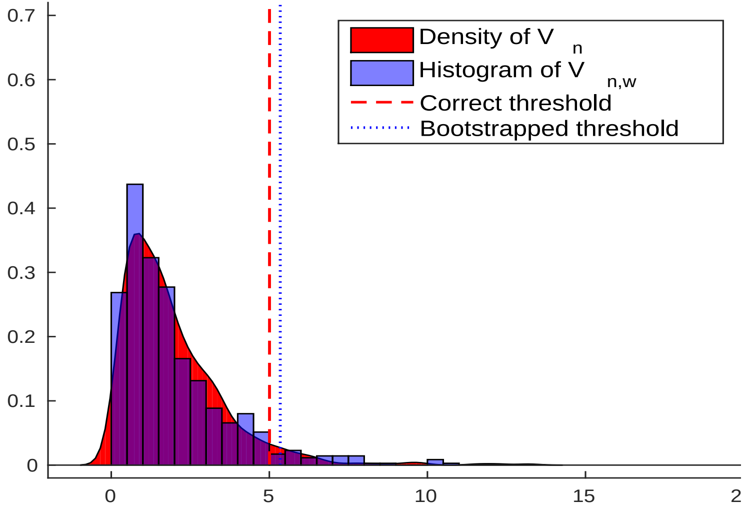
Density of V n
Histogram of V n,w
Correct threshold
Bootstrapped threshold
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
5
10
15
2

Memory 0.4



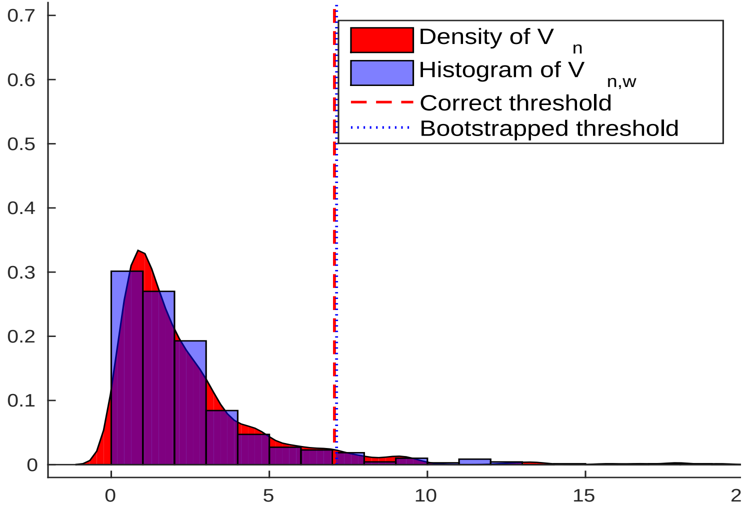
Density of V_n
Histogram of $V_{n,w}$
Correct threshold
Bootstrapped threshold
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
5
10
15
2

Memory 0.5

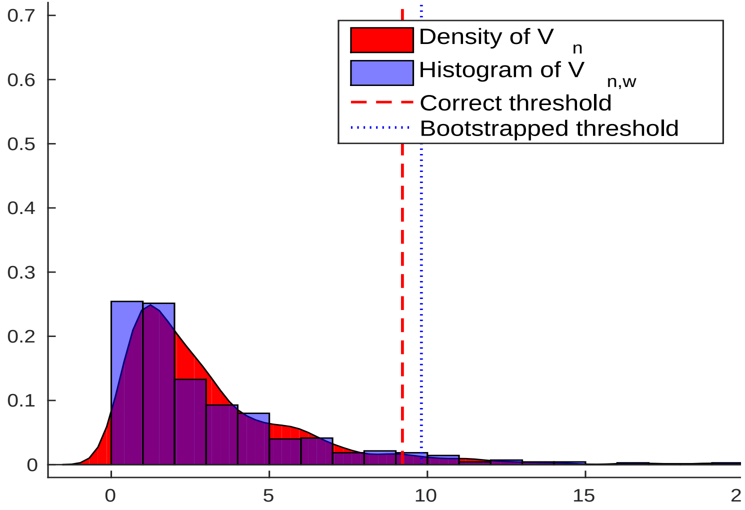


Density of V_n
Histogram of $V_{n,w}$
Correct threshold
Bootstrapped threshold
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
5
10
15
2

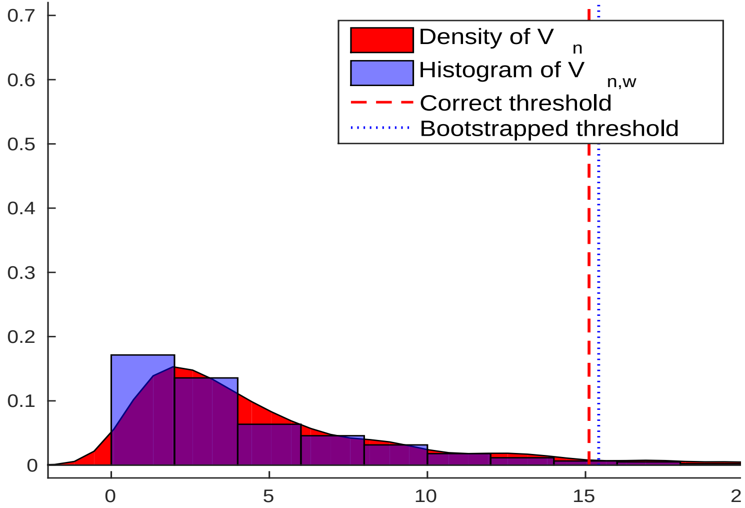
Memory 0.6


0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
5
10
15
2
Density of V_n
Histogram of $V_{n,w}$
Correct threshold
Bootstrapped threshold

Memory 0.7

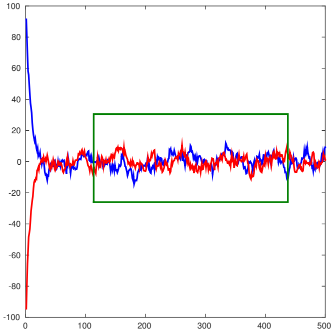

Density of V n
Histogram of V n,w
Correct threshold
Bootstrapped threshold
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
5
10
15
2

Memory 0.8



Density of V_n
Histogram of $V_{n,w}$
Correct threshold
Bootstrapped threshold
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
5
10
15
2

MCMC M.D. Experiment



Is P the same distribution as Q ?

Test - MMD	Type one error
Permutation	68 %
Wild Bootstrap	6 %

Indie Pop Group Predicts the Volume of the Dow Jones!

Indie Pop Group Predicts the Volume of the Dow Jones!

Test	p-value	Dependent
Permutation HSIC	0.003	Yes !

Indie Pop Group Predicts the Volume of the Dow Jones!

Test	p-value	Dependent
Permutation HSIC	0.003	Yes !
Wild Bootstrap HSIC	0.231	No