Lemma 1. *Let $\{Z_t\}$ be a τ-dependent stationary process with $\tau(r) = O(r^{-6-\epsilon})$. Let h be a bounded Lipschitz continuous function of $m \geq 3$ arguments (not necessarily symmetric) such that $\forall j \in \{1, \ldots, m\}$,*

$$\mathbb{E}h(z_1, \ldots, z_{j-1}, Z_j, z_{j+1}, \ldots, z_m) = 0. \tag{1}$$

Then,

$$\sum_{i \in [n]^m} |\mathbb{E}h(Z_{i_1}, \ldots, Z_{i_m})| \quad = \quad O\left(n^{\lfloor \frac{m}{2} \rfloor} + n^{2\lfloor \frac{m}{2} \rfloor - 4 - \epsilon}\right).$$

Proof. The proof uses the same technique as (!reference to Arcones, Lemma 3). We will focus on ordered m-tuples $1 \leq i_1 \leq \ldots \leq i_m \leq n$, and by considering all possible permutations of their indices, we obtain an upper bound

$$\sum_{i \in [n]^m} |\mathbb{E}h(Z_{i_1}, \ldots, Z_{i_m})| \quad \leq \quad \sum_{1 \leq i_1 \leq \ldots \leq i_m \leq n} \sum_{\pi \in S_m} \left|\mathbb{E}h\left(Z_{i_{\pi(1)}}, \ldots, Z_{i_{\pi(m)}}\right)\right|, \tag{2}$$

where (strict) inequality stems from the fact that the m-tuples i with some coinciding entries appear multiple times on the right. Now denote $s = \left\lfloor \frac{m}{2} \right\rfloor + 1$ and

$$j_1 = i_2 - i_1;\; j_l = \min\{i_{2l} - i_{2l-1}, i_{2l-1} - i_{2l-2}\},\; l = 2, \ldots, s-1;\;\; j_s = i_m - i_{m-1}.$$

Let $w(i) = \max\{j_1, \ldots, j_s\}$, i.e., $w(i)$ corresponds to the largest minimum gap between an individual entry in the ordered m-tuple i and its neighbours. For example, $w([1, 2, 5, 9, 9]) = 3$. Note that $w(i) = 0$ means that no entry in i appears exactly once. Let us assume that the maximum $w(i) = w > 0$ is obtained at j_r for some $r \in \{1, \ldots, s\}$. Let us partition the vector $(Z_{i_1}, \ldots, Z_{i_m})$ into three parts:

$$A \quad = \quad \left(Z_{i_1}, \ldots, Z_{i_{2r-2}}\right),\; B = Z_{i_{2r-1}},\; C = \left(Z_{i_{2r}}, \ldots, Z_{i_m}\right).$$

Note that if $r = 1$, A is empty and if $r = s$ and m is odd, C is empty but this does not change our arguments below. Using Lemma 6, we can construct B^* and C^* that are independent of A and independent of each other and

$$\mathbb{E}\left\|(A, B, C) - (A, B^*, C^*)\right\|_1 \leq m\tau(w). \tag{3}$$

Because $B*$ consists of a singleton and is independent of both A and C^*, (1) implies

$$\mathbb{E}h(A, B^*, C^*) = 0. \tag{4}$$

Thus, for $w(i) = w > 0$, we have that

$$\begin{aligned} |\mathbb{E}h(Z_{i_1}, \ldots, Z_{i_m})| &\leq \mathbb{E}\left|h(A, B, C) - h(A, B^*, C^*)\right| + |\mathbb{E}h(A, B^*, C^*)| \\ &\leq \mathrm{Lip}(h)\mathbb{E}\left\|(A, B, C) - (A, B^*, C^*)\right\|_1 + 0 \\ &\leq m\mathrm{Lip}(h)\tau(w). \end{aligned}$$

Finally, if the entries within the ordered m-tuple i are permuted, L_1-norm in (3) remains the same and (4) still holds, so also $\left|\mathbb{E}h\left(Z_{i_{\pi(1)}},\ldots,Z_{i_{\pi(m)}}\right)\right| \leq m\mathrm{Lip}(h)\tau(w)$ $\forall \pi \in S_m$ and

$$\sum_{\pi\in S_m}\left|\mathbb{E}h\left(Z_{i_{\pi(1)}},\ldots,Z_{i_{\pi(m)}}\right)\right| \leq m!m\mathrm{Lip}(h)\tau(w).$$

Let us upper bound the number of ordered m-tuples i with $w(i) = w$. i_1 can take n different values, but since $i_2 \leq i_1 + w$, i_2 can take at most $w+1$ different values. For $2 \leq l \leq s-1$, since $\min\{i_{2l} - i_{2l-1}, i_{2l-1} - i_{2l-2}\} \leq w$, we can either let i_{2l-1} take up to n different values and let i_{2l} take up to $w+1$ different values (if $i_{2l} - i_{2l-1} \leq i_{2l-1} - i_{2l-2}$) or let i_{2l-1} take up to $w+1$ different values and let i_{2l} take up to n different values (if $i_{2l} - i_{2l-1} > i_{2l-1} - i_{2l-2}$), upper bounding the total number of choices for $[i_{2l-1}, i_{2l}]$ by $2n(w+1)$. Finally, the last term i_m can always have at most $w+1$ different values. This brings the total number of m-tuples with $w(i) = w$ to at most $2^{s-2}n^{s-1}(w+1)^s$. Thus, the number of m-tuples with $w(i) = 0$ is $O(n^{s-1})$ and since h is bounded, we have

$$\begin{aligned}
&\sum_{1\leq i_1\leq\ldots\leq i_m\leq n}\sum_{\pi\in S_m}\left|\mathbb{E}h\left(Z_{i_{\pi(1)}},\ldots,Z_{i_{\pi(m)}}\right)\right| \\
&\leq O(n^{s-1}) + \sum_{w=1}^{n-1}\sum_{\substack{i\in[n]^m:\\ w(i)=w}}\sum_{\pi\in S_m}\left|\mathbb{E}h\left(Z_{i_{\pi(1)}},\ldots,Z_{i_{\pi(m)}}\right)\right| \\
&\leq O(n^{s-1}) + 2^{s-2}m!m\mathrm{Lip}(h)n^{s-1}\sum_{w=1}^{n-1}(w+1)^s\tau(w) \\
&\leq O(n^{s-1}) + Cn^{s-1}\sum_{w=1}^{n-1}w^{s-6-\epsilon} \\
&\leq O(n^{s-1}) + O(n^{2s-6-\epsilon}),
\end{aligned}$$

which proves the claim. We have used $\tau(w) = O(w^{-6-\epsilon})$ and collated the constants into C. $\square$