

LIST A 6

6.1.

$$w(x) = \frac{1}{2} c_0 T_0(x) + c_1 T_1(x) + \dots + c_n T_n(x)$$

$$\begin{cases} B_{n+2} = B_{n+1} = 0 \\ B_k = 2x B_{k+1} - B_{k+2} + c_k, k = n, n-1, \dots, 0 \end{cases}$$

$$w(x) \stackrel{?}{=} \frac{1}{2} (B_0 - B_2)$$

$$w(x) = \sum_{k=0}^3 c_k T_k(x) = \sum_{k=0}^3 (B_k - 2x B_{k+1} + B_{k+2}) \underbrace{T_k(x)}$$

$$= \left( \sum_{k=0}^3 B_k - \sum_{k=0}^3 2x B_{k+1} + \sum_{k=0}^3 B_{k+2} \right) T_k(x)$$

$$\frac{1}{2} B_0 T_0(x) + B_1 T_1(x) - x B_1 T_0(x) + \sum_{k=1}^3 2x B_{k+1} T_k(x) + \sum_{k=0}^3 B_{k+2} T_k(x)$$

$$= \frac{1}{2} B_0 + \cancel{B_1 x} - \cancel{x B_1} + \underbrace{\hspace{10em}}$$

$$\begin{aligned}
& \frac{1}{2}B_0 + \sum_{k=2}^3 B_k T_k(x) - \sum_{k=1}^{3-1} 2x B_{k+1} T_k(x) + \sum_{k=0}^{3-2} B_{k+2} T_k(x) = \\
& = \frac{1}{2}B_0 + \sum_{k=2}^3 B_k T_k(x) - \sum_{k=2}^3 2x B_k T_{k-1}(x) + \sum_{k=2}^3 B_k T_{k-2}(x) = \\
& \frac{1}{2}B_0 + \sum_{k=2}^3 B_k \underbrace{(T_k(x) - 2x T_{k-1}(x) + T_{k-2}(x))}_{=0} - \frac{1}{2}B_2 T_0(x) \\
& = \frac{1}{2}B_0 - \frac{1}{2}B_2 T_0(x) = \frac{1}{2}(B_0 - B_2)
\end{aligned}$$


---

6.3.

$$H_{2n+1}(x) = \sum_{k=0}^n f(x_k) h_k(x) + \sum_{k=0}^n f'(x_k) \bar{h}_k(x)$$

Zauważmy że :

$$p'_{n+1}(x) = \sum_{k=0}^n \prod_{\substack{i=0 \\ i \neq k}}^n (x - x_i)$$

1°  $i \neq k$   $\Rightarrow 0$

$$\mathcal{L}_k(x_i) = \frac{p_{n+1}(x_i)}{(x_i - x_k)(p'_{n+1}(x_i))} = 0$$

2°  $i = k$

$$\mathcal{L}_k(x_i) = \frac{p_{n+1}(x_i)}{(x_i - x_i)(p'_{n+1}(x_i))} = \frac{(x_i - x_0) \dots \cancel{(x_i - x_i)} \dots (x_i - x_n)}{\cancel{(x_i - x_i)} \cdot (x_i - x_0) \dots (x_i - x_{i-1})(x_i - x_{i+1}) \dots (x_i - x_n)} = 1$$

bo:

$$\mathcal{L}_k(x) = \frac{(x-x_0) \cdots (x-x_k) \cdots (x-x_n)}{(x-x_k) \cdot P'_{n+1}(x)} = \frac{\prod_{\substack{g=0 \\ g \neq k}}^n (x-x_g)}{\prod_{\substack{t \neq g \\ t=0}}^n (x-x_t)}$$

wiec:

dla  $i \neq k$

$$\underline{h_k(x_i) = 0}$$

$$\underline{\bar{h}_k(x_i) = 0}$$

dla  $i = k$

$$\underline{h_k(x_i)} = \left[ 1 - 2(x_i - x_k) \mathcal{L}'_k(x_k) \right] \cdot \mathcal{L}_k^2(x_k) =$$

$$= 1 \cdot 1^2 = \underline{1}$$

$$\underline{\bar{h}_k(x_i)} = \underbrace{(x_k - x_k)}_0 \cdot \underbrace{\mathcal{L}_k^2(x_k)}_1 = \underline{0 \cdot 1 = 0}$$

Zauważmy, że:

$$H'_{2n+1}(x) = \sum_{k=0}^n f'(x_k) h'_k(x) + \underbrace{\sum_{k=0}^n f(x_k) \bar{h}'_k(x)}_{\text{const.}}$$

$$h'_k(x) = [-2 \cdot \mathcal{L}'_k(x_k)] \mathcal{L}_k^2(x) + [1 - 2(x - x_k) \mathcal{L}'_k(x_k)] \cdot 2 \mathcal{L}_k(x) \cdot \mathcal{L}'_k(x)$$

$i \neq k$

$$h'_k(x_i) = 0 + 0 = 0$$

$$\underline{h'_k(x_i)}_{i=k} = -2 \cdot \mathcal{L}'_k(x_k) + 2 \mathcal{L}'_k(x_k) = 0$$

$$\bar{h}'_k(x) = \mathcal{L}_k^2(x) + (x - x_k) \cdot 2 \mathcal{L}_k(x) \cdot \mathcal{L}'_k(x)$$

$$\bar{h}'_k(x_i)_{i \neq k} = 0 + 0 = 0$$

$$\bar{h}'_k(x_i)_{i=k} = 1 + 0 = 1$$

Podsumowanie:

	$i \neq k$	$i = k$
$h_k(x_i)$	0	1
$h'_k(x_i)$	0	0
$\bar{h}_k(x_i)$	0	0
$\bar{h}'_k(x_i)$	0	1

Wzór:

$$H_{2n+1}(x_i) = \sum_{k=0}^n f(x_k) \cdot h_k(x_i) + \sum_{k=0}^n f'(x_k) \cdot \bar{h}_k(x_i) = f(x_i)$$

$$H'_{2n+1}(x_i) = \sum_{k=0}^n f(x_k) \cdot h'_k(x_i) + \sum_{k=0}^n f'(x_k) \cdot \bar{h}'_k(x_i) = f'(x_i)$$

6.4

$$g(t) = f(t) - H_{2n+1}(t) - K p_{n+1}^2(t)$$

$$g(x) = 0 \Leftrightarrow K p_{n+1}^2(x) = f(x) - H_{2n+1}(x)$$

$$K = (f(x) - H_{2n+1}(x)) \frac{1}{p_{n+1}^2(x)}$$

$$g(t) = f(t) - H_{2n+1}(t) - (f(x) - H_{2n+1}(x)) \frac{p_{n+1}^2(t)}{p_{n+1}^2(x)}$$

$$t = x_i$$

$$g(x_i) = f(x_i) - H_{2n+1}(x_i) - (f(x) - H_{2n+1}(x)) \frac{p_{n+1}^2(x_i)}{p_{n+1}^2(x)} = 0$$

$$g(x_i) = 0 \quad ; \quad i = 0, 1, \dots, n$$

omaz

$$g(x) = (f(x) - H_{2n+1}(x)) - (f(x) - H_{2n+1}(x)) \cdot \frac{p_{n+1}^2(x)}{p_{n+1}^2(x)}$$

$$g(x) = 0$$

$\Downarrow$

$g$  ma  $n+2$  zer  $(x_0, \dots, x_n)$  różnych od siebie.

z twierdzenia Rolle'a  $g'$  ma  $n+1$  zer różnych od  $x_0, \dots, x_n$ .

Zauważmy, że  $p_{n+1}^2(t) = \prod_{j=0}^n (t - x_j)^2$  ma zera krotności 2 więc  $(p_{n+1}^2(x_i))' = 0$  dla  $i = 0, \dots, n$

więc:

$$g'(t) = f'(t) - H_{2n+1}'(t) - (f(x) - H_{2n+1}(x)) \frac{(p_{n+1}^2(t))'}{p_{n+1}^2(x)}$$

$$t := x_i$$

$$g(x_i) = \underbrace{f'(x_i) - H_{2n+1}'(x_i)}_{=0} - (f(x) - H_{2n+1}(x)) \frac{(p_{n+1}^2(x_i))'}{p_{n+1}^2(x)} = 0$$

$$\underline{g'(x_i) = 0}$$

więc  $g'$  ma  $2n+2$  zer (te z tw. Rolle'a oraz  $x_0, \dots, x_n$ )

2. Uogólnienie tw. Rolle'a mamy, że:

Recall Rolle's Theorem: Let  $f$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$ .

If  $f(a) = f(b)$  then there exists  $c \in (a, b)$  such that  $f'(c) = 0$ . The Generalized

Rolle's Theorem extends this idea to higher order derivatives:

**Generalized Rolle's Theorem:** Let  $f$  be continuous on  $[a, b]$  and  $n$  times differentiable on  $(a, b)$ . If  $f$  is zero at the  $n+1$  distinct points  $x_0 < x_1 < \dots < x_n$  in  $[a, b]$ , then there exists a number  $c$  in  $(a, b)$  such that  $f^{(n)}(c) = 0$ .

**Proof:** The argument uses mathematical induction. If  $n = 1$  then we have the original Rolle's Theorem. To see how the induction argument works, consider the next case,  $n = 2$ . Then  $f(x) = 0$  at three points  $x_0 < x_1 < x_2$ . Applying Rolle's Theorem on the interval  $[x_0, x_1]$ , there exists  $c_1 \in (x_0, x_1)$  such that  $f'(c_1) = 0$ . Similarly, there exists  $c_2 \in (x_1, x_2)$  such that  $f'(c_2) = 0$ . Now, using Rolle's Theorem again,  $f'(c_1) = f'(c_2) = 0$  implies that there exists  $c \in (a, b)$  such that  $f''(c) = 0$ . The full mathematical induction argument follows.

istnieje więc  $\xi \in (a, b)$  + że

$$(g^{(2n+1)})'(\xi) = 0 \Leftrightarrow g^{(2n+2)}(\xi) = 0$$

Zauważmy, że  $p_{n+1}^2(t) \in \Pi_{2n+2}$  więc

$$(p_{n+1}^2(t))^{(2n+2)} \leftarrow \text{pochodna} = (2n+2)!$$

Wiemy, że  $h_{2n+1}(\xi) \in \Pi_{2n+1}$  więc

$$h_{2n+1}^{(2n+2)}(t) = 0 \text{ więc:}$$

$$\underbrace{g^{(2n+2)}(\xi)}_0 = \underbrace{f^{(2n+2)}(\xi)}_0 - \underbrace{h_{2n+1}^{(2n+2)}(\xi)}_0 - (f(x) - h_{2n+1}(x)) \frac{\overbrace{(p_{n+1}^2(\xi))^{(2n+2)}}^{(2n+2)!}}{\underbrace{p_{n+1}^2(x)}} \\ 0 = f^{(2n+2)}(\xi) - (f(x) - h_{2n+1}(x)) \frac{(2n+2)!}{p_{n+1}^2(x)}$$

$$f(x) - H_{2n+1}(x) = \frac{f^{(2n+2)}(\xi)}{(2n+2)!} \cdot p_{n+1}^2(x)$$

6.7.

$$a = x_0 < x_1 < \dots < x_n = b$$

$$s''(x_0) = s''(x_n) = 0$$

$$\begin{aligned} \int_{x_0}^{x_n} [s''(x)]^2 dx &= \sum_{k=0}^{n-1} \left[ \int_{x_k}^{x_{k+1}} s''(x) \cdot s''(x) dx \right] = \\ &= \sum_{k=0}^{n-1} \left[ s''(x) \cdot s'(x) \Big|_{x_k}^{x_{k+1}} - \int_{x_k}^{x_{k+1}} s'''(x) \cdot s'(x) dx \right] \quad (*) \end{aligned}$$

$$s(x) \in \Pi_3; \quad s'(x) \in \Pi_2; \quad s''(x) \in \Pi_1; \quad s'''(x) \in \Pi_0$$

$$s''(x_k) = M_k; \quad s''(x_{k+1}) = M_{k+1}$$

$$s''(x) = M_k \cdot \frac{x - x_{k+1}}{x_k - x_{k+1}} + M_{k+1} \cdot \frac{x - x_k}{x_{k+1} - x_k}$$

$$\text{indeed } s''(x_k) = M_k; \quad s''(x_{k+1}) = M_{k+1} \quad \checkmark$$

$$\begin{aligned} s'''(x) &= \frac{M_k}{x_k - x_{k+1}} + \frac{M_{k+1}}{x_{k+1} - x_k} = \frac{M_{k+1} - M_k}{x_{k+1} - x_k} \quad \text{const} \\ (*) &= \sum_{k=0}^{n-1} \left[ \underbrace{s''(x_{k+1}) \cdot s'(x_{k+1}) - s''(x_k) \cdot s'(x_k)}_0 - s'''(x) s(x) \right]_{x_k}^{x_{k+1}} \end{aligned}$$

$$= \sum_{k=0}^{n-1} \frac{M_{k+1} - M_k}{x_{k+1} - x_k} \cdot [s(x_{k+1}) - s(x_k)] \quad \left( s(x_k) = f(x_k) \right. \\ \left. k=0, \dots, n \right)$$

$$= \sum_{k=0}^{n-1} \frac{M_{k+1}}{x_{k+1} - x_k} [f(x_{k+1}) - f(x_k)] + \sum_{k=0}^{n-1} \frac{M_k}{x_{k+1} - x_k} [f(x_{k+1}) - f(x_k)]$$

$$= \sum_{k=1}^n \left( \frac{M_k}{x_k - x_{k-1}} [f(x_k) - f(x_{k-1})] \right) - \frac{M_n}{x_n - x_{n-1}} [f(x_n) - f(x_{n-1})] \\ + \sum_{k=0}^{n-1} \frac{M_k}{x_{k+1} - x_k} [f(x_{k+1}) - f(x_k)] = 0$$

$$M_0 = M_n = s''(x_0) = s''(x_n) = 0$$

$$\sum_{k=1}^{n-1} M_k \left[ \frac{f(x_{k+1}) - f(x_k)}{x_{k+1} - x_k} - \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} \right] =$$

$$= \sum_{k=1}^{n-1} M_k \cdot (f[x_k, x_{k+1}] - f[x_{k-1}, x_k])$$

6.8  $[x_i, x_{i+1}]$   $x_{i+1} - x_i = h$  A B

$$S_i(x) = \frac{(x - x_i)^3}{6h} M_{i+1} + \frac{(x_{i+1} - x)^3}{6h} M_i$$

$$+ \left( y_{i+1} - \frac{M_{i+1}}{6} h^2 \right) \frac{x - x_i}{h} + \left( y_i - \frac{M_i}{6} h^2 \right) \frac{x_{i+1} - x}{h}$$

intedy  $s(x_i) = \frac{h^3}{6h} M_i + \left( y_i - \frac{M_i}{6} h^2 \right) \frac{h}{h} = y_i$

$$s(x_{i+1}) = \frac{h^3}{6h} M_{i+1} + \left( y_{i+1} - \frac{M_{i+1}}{6} h^2 \right) \frac{h}{h} = y_{i+1}$$

$$S''(x) = \frac{6(x-x_i)}{6h} \cdot M_{i+1} + \frac{6(x_{i+1}-x)}{6h} \cdot M_i + 0 + 0$$

$$S''(x_i) = 0 + \frac{h}{h} \cdot M_i = \underline{M_i}$$

$$S''(x_{i+1}) = \frac{h}{h} M_{i+1} + 0 = \underline{M_{i+1}} \quad \checkmark$$

$$\int_a^b S(x) = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} S_i(x) =$$

A:

$$\frac{M_{i+1}}{6h} \int_{x_i}^{x_{i+1}} (x-x_i)^3 = \frac{M_{i+1}}{6h} \cdot \left. \frac{(x-x_i)^4}{4} \right|_{x_i}^{x_{i+1}} = \frac{M_{i+1} \cdot h^3}{24}$$

B:

$$\frac{M_i}{6h} \int_{x_i}^{x_{i+1}} (x_{i+1}-x)^3 = \frac{M_i}{6h} \cdot \left. \frac{(x_{i+1}-x)^4}{-4} \right|_{x_i}^{x_{i+1}} = \frac{M_i \cdot h^3}{24}$$

$$\sum_{i=0}^{n-1} A+B = \frac{M_1 + \dots + M_{n-1}}{12} h^3 = \sum_{i=0}^{n-1} \frac{M_i}{12} h^3$$

C:

$$\frac{\left( y_{i+1} - \frac{M_{i+1}}{6} h^2 \right)}{h} \int_{x_i}^{x_{i+1}} (x-x_i) = \boxed{(M_0 = 0)}$$

$$= \frac{\left( y_{i+1} - \frac{M_{i+1}}{6} h^2 \right)}{h} \cdot \left. \frac{1}{2} (x-x_i)^2 \right|_{x_i}^{x_{i+1}} =$$

C:

$$\frac{\left( y_{i+1} - \frac{M_{i+1}}{6} h^2 \right)}{h} \cdot \frac{h^2}{2} = \frac{\left( y_{i+1} - \frac{M_{i+1}}{6} h^2 \right) \cdot h}{2}$$

$$C: \frac{f(x_{i+1})h}{2} - \frac{h^3}{12} M_{i+1}$$

$$D: \frac{\left(y_i - \frac{M_i}{6}h^2\right)}{h} \left( \int_{x_i}^{x_{i+1}} (x_{i+1}-x) \right) = -\frac{1}{2}(x_{i+1}-x)^2 \Big|_{x_i}^{x_{i+1}} = \frac{1}{2}h^2$$

$$D: \frac{\left(y_i - \frac{M_i}{6}h^2\right) \cdot h}{2} = \frac{h \cdot f(x_i)}{2} - \frac{h^3}{12} M_i$$

$$= \sum_{i=0}^{n-1} A+B+C+D = \frac{h^3}{12} \sum_{i=0}^{n-1} M_i + h \sum_{i=0}^{n-1} \frac{f(x_{i+1})+f(x_i)}{2} - \frac{h^3}{12} \sum_{i=0}^{n-1} M_i + M_{i+1} = -h \sum_{i=0}^{n-1} f(x_i)$$

$$= h \cdot \sum_{i=0}^{n-1} \frac{f(x_0) + 2f(x_1) + \dots + 2f(x_{n-1}) + f(x_n)}{2} +$$

$$+ \frac{h^3}{12} \sum_{i=0}^{n-1} (M_i - M_i - M_{i+1})$$

(b.c.  $M_0 = 0$ )

$$- \frac{h^3}{12} \sum_{i=0}^{n-1} M_{i+1} = - \frac{h^3}{12} \sum_{i=0}^n M_i$$

zauważmy, że

$$-\frac{\hbar^2}{12} \sum_{i=0}^n M_i = -\frac{\hbar^2}{12} \sum_{i=0}^n M_i \quad (60 \quad M_0 = M_n = 0)$$

