6.8

$\boxed{x_{i+1}-x_i=h}$

$[x_i; x_{i+1}]$

$$S_i(x) = \underbrace{\frac{(x-x_i)^3}{6h}M_{i+1}}_{A} + \underbrace{\frac{(x_{i+1}-x)^3}{6h}M_i}_{B}$$

$$+ \underbrace{\left(y_{i+1}-\frac{M_{i+1}}{6}\cdot h^2\right)\frac{x-x_i}{h}}_{C} + \underbrace{\left(y_i-\frac{M_i}{6}h^2\right)\frac{x_{i+1}-x}{h}}_{D}$$

Wtedy
$$s(x_i) = \frac{h^3}{6h}M_i + \left(y_i - \frac{M_i}{6}h^2\right)\frac{h}{h} = y_i$$
$$s(x_{i+1}) = \frac{h^3}{6h}\cdot M_{i+1} + \left(y_{i+1}-\frac{M_{i+1}}{6}h^2\right)\frac{h}{h} = y_{i+1}$$

$$S''(x) = \frac{\cancel{6}(x-x_i)}{\cancel{6}h}\cdot M_{i+1} + \frac{\cancel{6}(x_{i+1}-x)}{\cancel{6}h}\cdot M_i + 0 + 0$$

$$s''(x_i) = 0 + \frac{h}{h}M_i = \underline{M_i}$$

$$s''(x_{i+1}) = \frac{h}{h}M_{i+1} + 0 = \underline{M_{i+1}} \quad \checkmark$$

$$\int_a^b s(x) = \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} s_i(x) =$$

A:
$$\frac{M_{i+1}}{6h}\int_{x_i}^{x_{i+1}}(x-x_i)^3 = \frac{M_{i+1}}{6h}\cdot\frac{(x-x_i)^4}{4}\Bigg|_{x_i}^{x_{i+1}} = \frac{M_{i+1}\cdot h^3}{24}$$

B:
$$\frac{M_i}{6h}\int_{x_i}^{x_{i+1}}(x_{i+1}-x)^3 = \frac{M_i}{6h}\cdot\frac{(x_{i+1}-x)^4}{-4}\Bigg|_{x_i}^{x_{i+1}} = \frac{M_i\cdot h^3}{24}$$

$$\sum_{i=0}^{n-1} A+B = \frac{M_1+\dots+M_{n-1}}{12} h^3 = \sum_{i=0}^{n-1} \frac{M_i}{12} h^3 \quad \boxed{(M_0=0)}$$

$$C: \frac{\left(y_{i+1} - \frac{M_{i+1}}{6} h^2\right)}{h} \int_{x_i}^{x_{i+1}} (x-x_i) =$$

$$= \frac{\left(y_{i+1} - \frac{M_{i+1}}{6} h^2\right)}{h} \cdot \frac{1}{2} (x-x_i)^2 \Big|_{x_i}^{x_{i+1}} =$$

$$C: \frac{\left(y_{i+1} - \frac{M_{i+1}}{6} h^2\right)}{h} \cdot \frac{h^2}{2} = \frac{\left(y_{i+1} - \frac{M_{i+1}}{6} h^2\right) \cdot h}{2}$$

$$C: \frac{f(x_{i+1})h}{2} - \frac{h^3}{12} M_{i+1}$$

$$D: \frac{\left(y_i - \frac{M_i}{6} h^2\right)}{h} \underbrace{\left(\int_{x_i}^{x_{i+1}} (x_{i+1}-x)\right)}_{= \frac{1}{2}h^2} = -\frac{1}{2}(x_{i+1}-x)^2 \Big|_{x_i}^{x_{i+1}} =$$

$$D: \frac{\left(y_i - \frac{M_i}{6} h^2\right) \cdot h}{2} = \frac{h f(x_i)}{2} - \frac{h^3}{12} \cdot M_i$$

$$\Downarrow$$

$$= \sum_{i=0}^{n-1} A+B+C+D = \frac{h^3}{12}\sum_{i=0}^{n-1} M_i + \boxed{h \cdot \sum_{i=0}^{n-1}\frac{f(x_{i+1})+f(x_i)}{2}} +$$

$$-\frac{h^3}{12}\sum_{i=0}^{n-1} M_i + M_{i+1} =$$

$$= h \cdot \sum_{i=0}^{n}{}'' f(x_i)$$

$$= \boxed{h \cdot \sum_{i=0}^{n-1}\frac{f(x_0)+2f(x_1)+\dots+2f(x_{n-1})+f(x_n)}{2}} +$$

$$+ \boxed{\frac{h^3}{12}\sum_{i=0}^{n-1}(M_i - M_i - M_{i+1})}$$

$$\parallel$$

(bo $M_0 = 0$)

$$\frac{h^3}{12}\sum_{i=0}^{n-1} M_{i+1} = -\frac{h^3}{12}\sum_{i=0}^{n} M_i$$

zauważmy, że

$$-\frac{h^3}{12}\sum_{i=0}^{n} M_i = -\frac{h^3}{12}\sum_{i=0}^{n}{}'' M_i \quad (\text{bo } M_0 = M_n = 0)$$