Niech $y = \sum_{i=1}^{k} \langle e_i, v\rangle e_i$ wtedy

$\langle v, y\rangle = \langle v, \langle e_1,v\rangle e_1 + \dots + \langle e_k,v\rangle e_k\rangle = \langle v, \langle e_1,v\rangle e_1\rangle + \dots + \langle v, \langle e_k,v\rangle e_k\rangle = \langle e_1,v\rangle\langle v,e_1\rangle + \dots + \langle e_k,v\rangle \langle v,e_k\rangle$

$= \sum_{i=1}^{k} \langle v,e_i\rangle^2 = \boxed{\sum_{i=1}^{k} |\langle v,e_i\rangle|^2}$.

oraz:

$\|y\|^2 = \langle y,y\rangle = \langle \sum_{i=1}^{k} \langle e_i,v\rangle e_i, \sum_{i=1}^{k} \langle e_i,v\rangle e_i\rangle = \sum_{i=1}^{k} \langle \langle e_i,v\rangle e_i, \langle e_i,v\rangle e_i\rangle + \sum_{i\neq j} \langle \langle e_i,v\rangle e_i, \langle e_j,v\rangle e_j\rangle =$

$= \sum_{i=1}^{k} \langle e_i,v\rangle^2 \underbrace{\langle e_i,e_i\rangle}_{1} + \sum_{i\neq j} \langle e_i,v\rangle \langle e_j,v\rangle \underbrace{\langle e_i,e_j\rangle}_{0} = \sum_{i=1}^{k} \langle e_i,v\rangle^2 = \boxed{\sum_{i=1}^{k} |\langle v,e_i\rangle|^2}$.

rozpatrzmy:

$\|v-y\|^2 = \langle v-y, v-y\rangle = \langle v-y, v\rangle - \langle v-y, y\rangle = \langle v,v\rangle - \langle y,v\rangle - (\langle v,y\rangle - \langle y,y\rangle) =$

$\langle v,v\rangle - 2\langle v,y\rangle + \langle y,y\rangle = \|v\|^2 - 2\underline{\langle v,y\rangle} + \underline{\|y\|^2} = \|v\| - \langle v,y\rangle = \|v\| - \sum_{i=1}^{k} |\langle v,e_i\rangle|^2$.

$\|v-y\|^2 \geq 0$ więc $\|v\|^2 - \sum_{i=1}^{k} |\langle v,e_i\rangle|^2 \geq 0 \iff \|v\|^2 \geq \sum_{i=1}^{k} |\langle v,e_i\rangle|^2$.

□

$e_1,\dots,e_k$ bazą $\iff \|v\|^2 = \sum_{i=1}^{k} |\langle v,e_i\rangle|^2$

1) $\Rightarrow$

$e_1,\dots,e_2$ bazą więc z lematu 10.11

$v = \sum_{i=1}^{k} \langle v,e_i\rangle e_i$.

$0 = \|v-v\|^2 = \|v - \sum_{i=1}^{k} \langle v,e_i\rangle e_i\|^2 = \|v\|^2 - \sum_{i=1}^{k} |\langle v,e_i\rangle|^2 \iff \|v\|^2 = \sum_{i=1}^{k} |\langle v,e_i\rangle|^2$. □

2) $\Leftarrow$

$\|v\|^2 = \sum_{i=1}^{k} |\langle v,e_i\rangle|^2 \iff \|v\|^2 - \sum_{i=1}^{k} |\langle v,e_i\rangle|^2 \iff \|v - \sum_{i=1}^{k} \langle v_i,e_i\rangle e_i\|^2 = 0 \iff v = \sum_{i=1}^{k} \langle v_i,e_i\rangle e_i$

$v = \sum_{i=1}^{k} \alpha_i e_i$
$(\alpha_i = \langle v,e_i\rangle)$

więc dowolny wektor v można przedstawić jako kombinację liniową $e_1,\dots,e_k$ więc $e_1,\dots,e_k$ jest bazą.

□