$\boxed{5.2}$ Wielomian w postaci Lagrange:

$$L(x) = \sum_{i=0}^{n} f(x_i)\, \lambda_i(x) \qquad \lambda_i(x) = \prod_{\substack{j=0\\ j\neq i}}^{n} \frac{x-x_j}{x_i-x_j}$$

niech $w_i = \dfrac{1}{\prod_{\substack{j=0\\ j\neq i}}^{n} (x_i-x_j)}$

wtedy $L(x) = \sum_{i=0}^{n} f(x_i)\cdot w_i \cdot \prod_{\substack{j=0\\ j\neq i}}^{n} (x-x_j)$

Wielomian w postaci Newtona:

$$L(x) = \sum_{i=0}^{n} a_i \prod_{j=0}^{i-1} (x-x_j)$$

$p_{k+1}(x) = \prod_{n=0}^{k}(x-x_n)$

gdzie $a_k = \sum_{i=0}^{k} \dfrac{f(x_i)}{\prod_{\substack{j=0\\ j\neq i}}^{k}(x_i-x_j)} = \sum_{i=0}^{k} \dfrac{f(x_i)}{p'_{k+1}(x_i)}$

$\longrightarrow$ iloraz różnicowy

$i = 0, 1, \dots, n$

weźmy takie f, że $f(x_i) = 1$

wtedy

$$a_k = \sum_{i=0}^{k} \frac{1}{\prod_{\substack{j=0\\ j\neq i}}^{k}(x_i-x_j)} = \sum_{i=0}^{k} \overbrace{\frac{1}{\prod_{\substack{j=0\\ j\neq i}}^{n}(x_i-x_j)}}^{w_i} \cdot \prod_{j=k+1}^{n}(x_i-x_j)$$

$$a_k = \sum_{i=0}^{k} w_i \cdot \prod_{j=k+1}^{n}(x_i-x_j)$$

Zauważmy, że $w_i = \frac{1}{p'_{n+1}(x_i)}$

niech $w_i := \sigma_i$

$$a_k = \sum_{i=0}^{k} \sigma_i \cdot \prod_{j=k+1}^{n} (x_i - x_j)$$

Rozpiszmy macierz:

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix} = \begin{bmatrix} \prod_{j=1}^{n}(x_0-x_j) & 0 & 0 & \dots & 0 \\ \prod_{j=2}^{n}(x_0-x_j) & \prod_{j=2}^{n}(x_1-x_j) & 0 & \dots & 0 \\ \prod_{j=3}^{n}(x_0-x_j) & \prod_{j=3}^{n}(x_1-x_j) & \prod_{j=3}^{n}(x_2-x_j) & \dots & 0 \\ \vdots & & & & \\ x_0-x_n & x_1-x_n & x_2-x_n & \dots\ x_{n-1}-x_n & 0 \\ 1 & 1 & 1 & \dots\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \sigma_0 \\ \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_{n-1} \\ \sigma_n \end{bmatrix}$$

Obliczamy pomocnicze wartości za pomocą wzorów:

$$a_0^{(0)} := 1, \quad a_k^{(0)} := 0 \quad (k = 1, 2, \dots, n),$$

$$\left.\begin{aligned} a_k^{(i)} &:= a_k^{(i-1)}/(x_k - x_i), \\ a_i^{(k+1)} &:= a_i^{(k)} - a_k^{(i)} \end{aligned}\right\} \quad (i = 1, 2, \dots, n;\ k = 0, 1, \dots, i-1),$$

Zauważmy, że będzie to eliminacja Gaussa która doprowadzi nas do macierzy jednostkowej.

Wtedy:

$$\begin{bmatrix} a_0^{(n)} \\ a_1^{(n)} \\ \vdots \\ a_n^{(n)} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 \end{bmatrix} \cdot \begin{bmatrix} \sigma_0 \\ \sigma_1 \\ \vdots \\ \sigma_n \end{bmatrix}$$

otrzymamy, że

$a_0^{(n)} = \sigma_0$; $a_1^{(n)} = \sigma_1$; ...; $a_n^{(n)} = \sigma_n$ czyli

$$\boxed{a_k^{(n)} = \sigma_k}$$

$\blacksquare$