

L L S T A 6

6.1.

$$w(x) = \frac{1}{2} c_0 T_0(x) + c_1 T_1(x) + \dots + c_n T_n(x)$$

$$\left\{ \begin{array}{l} B_{n+2} = B_{n+1} = 0 \\ B_k = 2 \times B_{k+1} - B_{k+2} + c_k \end{array} \right. \quad k = n, n-1, \dots, 0$$

$$w(x) = \frac{1}{2} (B_0 - B_2)$$

$$w(x) = \sum_{k=0}^{n-1} c_k T_k(x) = \sum_{k=0}^{n-1} (B_k - 2 \times B_{k+1} + B_{k+2}) T_k(x)$$

$$= \left(\sum_{k=0}^{n-1} B_k - \sum_{k=0}^{n-1} 2 \times B_{k+1} + \sum_{k=0}^{n-1} B_{k+2} \right) T_k(x)$$

$$\frac{1}{2} B_0 T_0(x) + B_1 T_1(x) - \cancel{x B_1 T_0(x)} + \sum_{k=1}^{n-1} 2 \times B_{k+1} T_k(x) + \sum_{k=0}^{n-1} B_{k+2} T_k(x)$$

$$= \frac{1}{2} B_0 + B_1 x - \cancel{B_1} +$$

$$\frac{1}{2}B_0 + \sum_{k=2}^n B_k T_k(x) - \sum_{k=1}^{n-1} 2 \times B_{k+1} \bar{T}_{k+1}(x) + \sum_{k=0}^{n-2} B_{k+2} \bar{T}_k(x) =$$

$$= \frac{1}{2}B_0 + \sum_{k=2}^n B_k T_k(x) - \sum_{k=2}^3 2 \times B_k T_{k-1}(x) + \sum_{k=2}^m B_k \bar{T}_{k-2}(x) =$$

$$\frac{1}{2}B_0 + \sum_{k=2}^3 B_k \left(\underbrace{T_k(x) - 2 \times \bar{T}_{k-1}(x) + \bar{T}_{k-2}(x)}_0 \right) - \frac{1}{2}B_2$$

$$= \frac{1}{2}B_0 - \frac{1}{2}B_2 \bar{T}_0(x) = \frac{1}{2}(B_0 - B_2)$$

6.3.

$$H_{2n+1}(x) = \sum_{k=0}^n f(x_k) h_k(x) + \sum_{k=0}^n f'(x_k) \bar{h}_k(x)$$

Zum weiteren: \dots

$$P_{n+1}'(x) = \sum_{k=0}^n \prod_{\substack{i=0 \\ i \neq k}}^n (x - x_i)$$

$$1^\circ \quad i \neq k \quad \Rightarrow 0$$

$$R_k(x_i) = \frac{P_{n+1}(x_i)}{(x_i - x_k)(P_{n+1}'(x_i))} = \boxed{0}$$

$$2^\circ \quad i = k \quad \boxed{1}$$

$$R_k(x_i) = \frac{P_{n+1}(x_i)}{(x_i - x_i)(P_{n+1}'(x_i))} = \frac{(x_i - x_0) \cdots (\cancel{x_i - x_i}) \cdots (x_i - x_n)}{(x_i - x_i) \cdot (x_i - x_0) \cdots (x_i - x_{i-1})(x_i - x_{i+1}) \cdots (x_i - x_n)} =$$

bo:

$$\pi_k(x) = \frac{(x-x_0) \cdots \cancel{(x-x_k)} \cdots (x-x_n)}{(x-x_k) \cdot P_{n+1}^1(x)} = \frac{\prod_{i=0}^n (x-x_i)}{\prod_{i=0}^{k-1} (x-x_i) \cdot \prod_{i=k+1}^n (x-x_i)}$$

Weg: dla $i \neq k$

$$h_k(x_i) = 0$$

$\bar{h}_k(x_i) = 0$

dla $i = k$

$$h_k(x_k) = [1 - 2(x_k - x_k) \pi_k^1(x_k)] \cdot \pi_k^2(x_k) = 1 \cdot 1^2 = 1$$

$$\bar{h}_k(x_k) = (x_k - x_k) \pi_k^2(x_k) = 0 \cdot 1 = 0$$

Zauważmy, że:

$$H_{2n+1}^1(x) = \sum_{k=0}^n f'(x_k) h_k^1(x) + \underbrace{f(x_k) \cdot \bar{h}_k(x)}_{\text{const.}!}$$

$$h_k^1(x) = [-2 \cdot \pi_k^1(x_k)] \pi_k^2(x) + [1 - 2(x-x_k) \pi_k^1(x_k)] \cdot 2 \pi_k^2(x)$$

$$h_k^1(x_i) = 0 + 0 = 0$$

$$\underbrace{h_k^1(x_i)}_{i=k} = -2 \cdot \pi_k^1(x_k) + 2 \pi_k^1(x_k) = 0$$

$$\bar{h}_k^1(x) = \pi_k^2(x) + (x-x_k) \cdot 2 \pi_k^2(x) \cdot \pi_k^1(x)$$

$$\bar{h}_k^1(x_i) = 0 + 0 = 0$$

$$\bar{h}_k^1(x_i) = 1 + 0 = 1$$

Podsumowanie:

	$i \neq k$	$i = k$
$h_k(x_i)$	0	1
$\bar{h}_k(x_i)$	0	0
$\tilde{h}_k(x_i)$	0	0
$\bar{\tilde{h}}_k(x_i)$	0	1

Uogólnie:

$$H_{2n+1}(x_i) = \sum_{k=0}^m f(x_k) \cdot h_k(x_i) + \sum_{k=0}^n f'(x_k) \cdot \bar{h}_k(x_i) \quad f(x_i)$$

$$\bar{H}_{2n+1}'(x_i) = \sum_{k=0}^m f(x_k) \cdot h_k'(x_i) + \sum_{k=0}^n f'(x_k) \cdot \bar{h}_k'(x_i) = f'(x_i)$$

6.4

$$g(t) = f(t) - H_{2n+1}(t) - k p_{n+1}^2(t)$$

$$g(x) = 0 \Leftrightarrow k p_{n+1}^2(x) = f(x) - H_{2n+1}(x)$$

$$k = (f(x) - H_{2n+1}(x)) \frac{1}{p_{n+1}^2(x)}$$

$$g(t) = f(t) - H_{2n+1}(t) - (f(x) - H_{2n+1}(x)) \frac{p_{n+1}^2(t)}{p_{n+1}^2(x)}$$

$$t := x_i$$

$$g(x_i) = \underbrace{f(x_i) - H_{2n+1}(x_i)}_{\text{residual}} - (f(x) - H_{2n+1}(x)) \frac{\overline{p}_{n+1}^2(x_i)}{\overline{p}_{n+1}^2(x)}$$

$$g(x_i) = 0 \quad ; \quad i = 0, 1, \dots, n$$

oraz

$$g(x) = (f(x) - H_{2n+1}(x)) - (f(x) - H_{2n+1}(x)) \cdot \frac{P_{n+1}^2(x)}{P_{n+1}^2(x)}$$

$$g(x) = 0$$

∴

g ma $n+2$ zer (x, x_0, \dots, x_n) różnych odrębnych.

z twierdzenia Rolle'a g' ma $n+1$ zer różnych od x, x_0, \dots, x_n .

Zauważamy, że $P_{n+1}^2(t) = \prod_{j=0}^n (t - x_j)^2$ ma zero

knotnosci 2 wiec $(P_{n+1}^2(x_i))' = 0$ dla $i = 0, \dots, n$

więc:

$$g'(+) = f'(+) - H_{2n+1}'(+) - (f(x) - H_{2n+1}(x)) \frac{(P_{n+1}^2(+))'}{P_{n+1}^2(x)}$$

$$t := x_i$$

$$g(x_i) = f'(x_i) - H_{2n+1}'(x_i) - (f(x) - H_{2n+1}(x)) \frac{(P_{n+1}^2(x_i))'}{P_{n+1}^2(x)}$$

$$\underline{\underline{g'(x_i) = 0}}$$

więc g' ma $2n+2$ zer (te 2 tw. Rolle'a) oraz x_0, \dots, x_n

2 Uogólnioneżo to. Rolle'a many, zie:

Recall Rolle's Theorem: Let f be continuous on $[a,b]$ and differentiable on (a,b) .

If $f(a) = f(b)$ then there exists $c \in (a,b)$ such that $f'(c) = 0$. The Generalized Rolle's Theorem extends this idea to higher order derivatives:

Generalized Rolle's Theorem: Let f be continuous on $[a,b]$ and n times differentiable on (a,b) . If f is zero at the $n+1$ distinct points $x_0 < x_1 < \dots < x_n$ in $[a,b]$, then there exists a number c in (a,b) such that $f^{(n)}(c) = 0$.

Proof: The argument uses mathematical induction. If $n=1$ then we have the original Rolle's Theorem. To see how the induction argument works, consider the next case, $n=2$. Then $f(x)=0$ at three points $x_0 < x_1 < x_2$. Applying Rolle's Theorem on the interval $[x_0, x_1]$, there exists $c_1 \in (x_0, x_1)$ such that $f'(c_1) = 0$. Similarly, there exists $c_2 \in (x_1, x_2)$ such that $f'(c_2) = 0$. Now, using Rolle's Theorem again, $f'(c_1) = f'(c_2) = 0$ implies that there exists $c \in (a,b)$ such that $f''(c) = 0$. The full mathematical induction argument follows.

istnieje wiele $\xi \in (a, b)$ tzn

$$(g')^{(2n+1)}(\xi) = 0 \Leftrightarrow g^{(2n+2)}(\xi) = 0$$

Zauważamy, że $p_{n+1}^2(+)$ $\in \overline{\mathcal{P}}_{2n+2}$ wiec

$$(p_{n+1}^2(+))^{(2n+2)} \leftarrow \text{pochodna} = (2n+2)!$$

więcmy, że $H_{2n+1}(+)\in \overline{\mathcal{P}}_{2n+1}$ wiec

$$H_{2n+1}(+) = 0 \quad \text{wiec:}$$

$$g^{(2n+2)}(\xi) = f^{(2n+2)}(\xi) - H_{2n+1}^{(2n+2)}(\xi) - (f(x) - H_{2n+1}(x)) \frac{(2n+2)!}{p_{n+1}^2(x)}$$

$$0 = f^{(2n+2)}(\xi) - (f(x) - H_{2n+1}(x)) \frac{(2n+2)!}{p_{n+1}^2(x)}$$

$$f(x) - P_{2n+m}(x) = \frac{f^{(2n+1)}(\xi)}{(2n+2)!} \cdot P_{n+1}^2(x)$$

6. 7.

$$\alpha = x_0 < x_1 < \dots < x_n = b$$

$$s''(x_0) = s''(x_n) = 0$$

$$\int_{x_0}^{x_n} [s''(x)]^2 dx = \sum_{k=0}^{n-1} \left[\int_{x_k}^{x_{k+1}} s''(x) \cdot s''(x) \right] = \\ = \sum_{k=0}^{n-1} \left[s''(x) \cdot s'(x) \Big|_{x_k}^{x_{k+1}} - \int_{x_k}^{x_{k+1}} s'''(x) \cdot s'(x) \right] (*)$$

$$s(x) \in \Pi_3; s'(x) \in \Pi_2; s''(x) \in \Pi_1; s'''(x) \in \Pi_0$$

$$\underbrace{s''(x_k) = M_k; s''(x_{k+1}) = M_{k+1}}$$

$$s''(x) = M_k \cdot \frac{x - x_{k+1}}{x_k - x_{k+1}} + M_{k+1} \cdot \frac{x - x_k}{x_{k+1} - x_k}$$

$$\text{Intervall } s''(x_k) = M_k; s''(x_{k+1}) = M_{k+1} \quad \checkmark$$

$$s'''(x) = \frac{M_k}{x_k - x_{k+1}} + \frac{M_{k+1}}{x_{k+1} - x_k} = \frac{M_{k+1} - M_k}{x_{k+1} - x_k}$$

$$(*) = \sum_{k=0}^{n-1} \left[\underbrace{\left(s''(x_{k+1}) \cdot s'(x_{k+1}) - s''(x_k) \cdot s'(x_k) \right)}_0 - \underbrace{s'''(x) s(x)}_{\text{const}} \right]$$

$$= \sum_{k=0}^{n-1} - \frac{M_{k+1} - M_k}{x_{k+1} - x_k} \cdot [s(x_{k+1}) - s(x_k)] \quad \begin{array}{l} s(x_k) = f(x_k) \\ k=0, \dots, n \end{array}$$

$$= - \sum_{k=0}^{n-1} \frac{M_{k+1}}{x_{k+1} - x_k} \left[f(x_{k+1}) - f(x_k) \right] + \sum_{k=0}^{n-1} \frac{M_k}{x_{k+1} - x_k} \left[f(x_{k+1}) - f(x_k) \right]$$

$$= \sum_{k=1}^{n-1} \left(\frac{M_k}{x_k - x_{k-1}} \left[f(x_k) - f(x_{k-1}) \right] \right) - \frac{M_n}{x_n - x_{n-1}} \cdot \left[f(x_n) - f(x_{n-1}) \right]$$

$$+ \sum_{k=0}^{n-1} \frac{M_k}{x_{k+1} - x_k} \left[f(x_{k+1}) - f(x_k) \right] = 0$$

$$M_0 = M_n = s''(x_0) = s''(x_n) = 0$$

$$\sum_{k=1}^{n-1} M_k \left[\frac{f(x_{k+1}) - f(x_k)}{x_{k+1} - x_k} - \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} \right] =$$

$$= \sum_{k=1}^{n-1} M_k \cdot (f[x_k, x_{k+1}] - f[x_{k-1}, x_k])$$

6.8

$$S_i(x) = \frac{(x - x_i)^3}{6h} M_{i+1} + \frac{(x_{i+1} - x)^3}{6h} M_i + \left(y_{i+1} - \frac{M_{i+1}}{6} h^2 \right) \frac{x - x_i}{h} + \left(y_i - \frac{M_i}{6} h^2 \right) \frac{x_{i+1} - x}{h}$$

$$\text{Integy } S(x_i) = \frac{h^3}{6h} M_i + \left(y_i - \frac{M_i}{6} h^2 \right) \frac{h}{h} = y_i$$

$$S(x_{i+1}) = \frac{h^3}{6h} \cdot M_{i+1} + \left(y_{i+1} - \frac{M_{i+1}}{6} h^2 \right) \frac{h}{h} = g_{i+1}$$

$$S''(x) = \frac{16}{6h} \cdot M_{i+1} + \frac{6}{8h} \cdot M_i + 0 + 0$$

$$S''(x_i) = 0 + \frac{h}{h} \cdot M_i = \underline{\underline{M_i}}$$

$$S''(x_{i+1}) = \frac{h}{h} M_{i+1} + 0 = \underline{\underline{M_{i+1}}} \quad \checkmark$$

$$\int_a^b S(x) = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} s_i(x) =$$

A:

$$\frac{M_{i+1}}{6h} \int_{x_i}^{x_{i+1}} (x-x_i)^3 = \frac{M_{i+1}}{6h} \cdot \left. \frac{(x-x_i)^4}{4} \right|_{x_i}^{x_{i+1}} = \frac{M_{i+1} \cdot h^3}{24}$$

B:

$$\frac{M_i}{6h} \int_{x_i}^{x_{i+1}} (x_{i+1}-x)^3 = \frac{M_i}{6h} \cdot \left. \frac{(x_{i+1}-x)^4}{-4} \right|_{x_i}^{x_{i+1}} = \frac{M_i \cdot h^3}{24}$$

$$\sum_{i=0}^{n-1} A + B = \frac{M_1 + \dots + M_{n-1}}{12} h^3 = \sum_{i=0}^{n-1} \frac{M_i}{12} h^3$$

C:

$$\left(\frac{y_{i+1} - \frac{M_{i+1}}{6} \cdot h^2}{h} \right) \int_{x_i}^{x_{i+1}} (x-x_i) = \boxed{\underline{\underline{(M_0 = 0)}}}$$

$$= \left(\frac{y_{i+1} - \frac{M_{i+1}}{6} \cdot h^2}{h} \right) \cdot \left. \frac{1}{2} (x-x_i)^2 \right|_{x_i}^{x_{i+1}} =$$

$$C: \left(\frac{y_{i+1} - \frac{M_{i+1}}{6} \cdot h^2}{h} \right) \cdot \frac{h^2}{2} = \frac{(y_{i+1} - \frac{M_{i+1}}{6} \cdot h^2) \cdot h}{2}$$

$$C: \frac{f(x_{i+1})h}{2} - \frac{h^3}{12} M_{i+1}$$

$$D: \frac{(y_i - \frac{M_i}{6}h^2)}{h} \left(\int_{x_i}^{x_{i+1}} (x_{i+1} - x) \right) = -\frac{1}{2} (x_{i+1} - x)^2 \Big|_{x_i}^{x_{i+1}} = \frac{1}{2} h^2$$

$$D: \frac{(y_i - \frac{M_i}{6}h^2) \cdot h}{2} = h \cdot f(x_i) - \frac{h^3}{12} \cdot M_i$$

$$\begin{aligned}
 &= \sum_{i=0}^{n-1} A + B + C + D = \frac{h^3}{12} \sum_{i=0}^{n-1} M_i + h \cdot \sum_{i=0}^{n-1} \frac{f(x_{i+1}) + f(x_i)}{2} \\
 &\quad - \frac{h^3}{12} \sum_{i=0}^{n-1} M_i + M_{i+1} = h \cdot \sum_{i=0}^{n-1} f(x_i) \\
 &= h \cdot \frac{f(x_0) + 2f(x_1) + \dots + 2f(x_{n-1}) + f(x_n)}{2} + \\
 &\quad + \frac{h^3}{12} \sum_{i=0}^{n-1} (M_i - M_{i-1} - M_{i+1}) \\
 &\quad // \\
 &- \frac{h^3}{12} \sum_{i=0}^{n-1} M_{i+1} = -\frac{h^3}{12} \sum_{i=0}^n M_i \quad (b.o. M_0 = 0)
 \end{aligned}$$

Załamajmy, iż

$$-\frac{h^3}{12} \sum_{i=0}^m M_i = -\frac{h^3}{12} \sum_{i=0}^n M_i \quad (bo \quad M_0 = M_n = 0)$$