2

a) $a_{n+1} = \sqrt{a_n^2 + a_{n-1}^2}$; $a_0 = a_1 = 1$

$$a_{n+1}^2 = b_{n+1}$$

$$b_{n+1} = b_n + b_{n-1}$$

$$E\langle b_n \rangle = \langle b_{n+1} \rangle = \langle b_n \rangle + \langle b_{n-1} \rangle$$

$$E^2\langle b_n \rangle = \langle b_{n+2} \rangle = \langle b_{n+1} \rangle + \langle b_n \rangle$$

$$E^2\langle b_n \rangle = E\langle b_n \rangle + \langle b_n \rangle$$

$$(E^2 - E - 1)\langle b_n \rangle = \langle 0 \rangle$$

$$\Delta = 1 - 4 \cdot 1 \cdot (-1) = 5 \qquad \sqrt{\Delta} = \sqrt{5}$$

$$E_1 = \frac{1-\sqrt{5}}{2} \qquad E_2 = \frac{1+\sqrt{5}}{2}$$

$$\left(E - \left(\frac{1-\sqrt{5}}{2}\right)\right)\left(E - \left(\frac{1+\sqrt{5}}{2}\right)\right)\langle b_n \rangle = \langle 0 \rangle$$

$$b_n = \left(\frac{1-\sqrt{5}}{2}\right)^n \alpha + \left(\frac{1+\sqrt{5}}{2}\right)^n \cdot \beta$$

$$a_0^2 = 1^2 = b_0 = 1 \quad ; \quad a_1^2 = 1^2 = 1 = b_1$$

$$\begin{cases} 1 = \alpha + \beta \\ 1 = \left(\frac{1-\sqrt{5}}{2}\right)\alpha + \left(\frac{1+\sqrt{5}}{2}\right)\beta \end{cases} \Rightarrow \begin{matrix} \alpha = \frac{5-\sqrt{5}}{10} \\ \beta = \frac{5+\sqrt{5}}{10} \end{matrix}$$

$$\frac{\sqrt{5}-1}{2\sqrt{5}} = \frac{5-\sqrt{5}}{10} =$$

$$a_n = \sqrt{b_n} = \sqrt{\frac{5-\sqrt{5}}{10}\cdot\left(\frac{1-\sqrt{5}}{2}\right)^n + \frac{5+\sqrt{5}}{2}\cdot\left(\frac{1+\sqrt{5}}{2}\right)^n}$$

b) $b_{n+1} = \sqrt{b_n^2+3}$ $\qquad b_0 = 8$

$$\underline{b_{n+1}^2 = a_{n+1}}$$

$$a_{n+1} = a_n + 3$$

$$E\langle a_n\rangle = \langle a_{n+1}\rangle = \langle a_n\rangle + \langle 3\rangle$$

$$(E-1)\langle a_n\rangle = \langle 3\rangle \qquad \boxed{(E-1)\langle 3\rangle = 0}$$

$$(E-1)(E-1)\langle a_n\rangle = \langle 0\rangle$$

$(E-1)^2$ anihiluje $a_n = \alpha\cdot n + \beta$

$$a_0 = b_0^2 = 64 \Rightarrow a_0 = 64 = \beta$$

$$\underline{\beta = 64}$$

$$a_1 = b_1^2 = \left(\sqrt{b_0^2+3}\right)^2 = \left(\sqrt{67}\right)^2 = 67$$

$$67 = \alpha + \beta \Rightarrow \alpha = 3$$

$$a_n = 3n+64 \Rightarrow b_n = \sqrt{3n+64}$$

c) $\left.c_{n+1} = (n+1)c_n + (n^2+n)c_{n-1} \quad \right/ : (n+1)! \qquad \begin{cases} c_0 = 0 \\ c_1 = 1 \end{cases}$

$$\frac{c_{n+1}}{(n+1)!} = \frac{c_n}{n!} + \frac{c_{n-1}}{(n-1)!}$$

$$d_n = \frac{c_n}{n!}$$

$$d_{n+1} = d_n + d_{n-1}$$

$$(E^2 - E - 1)\langle d_n \rangle = \langle 0 \rangle$$

$$d_n = \left(\frac{1-\sqrt{5}}{2}\right)^n \alpha + \left(\frac{1+\sqrt{5}}{2}\right)^n \cdot \beta$$

$$d_0 = \frac{c_0}{0!} = 0 \qquad d_1 = \frac{1}{1!} = 1$$

$$\begin{cases} 0 = \alpha + \beta \\ 1 = \left(\frac{1-\sqrt{5}}{2}\right)\alpha + \left(\frac{1+\sqrt{5}}{2}\right)\beta \end{cases} \Rightarrow \begin{array}{l} \alpha = -\frac{1}{\sqrt{5}} \\ \beta = \frac{1}{\sqrt{5}} \end{array}$$

$$d_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

$$c_n = \frac{\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)}{n!}$$

4

$$\underbrace{n+1, n+2, \dots, n+k}_{k \text{ kolejnych liczb}}$$

$k! \mid^{?} (n+1)\cdot\ldots\cdot(n+k) = \frac{(n+k)!}{n!}$

pokażmy, że $\frac{(n+k)!}{n!\cdot k!} \in \mathbb{N}$

$$\frac{(n+k)!}{n!\cdot k!} = \binom{n+k}{k} \in \mathbb{N}$$

6

$a_n^2 = 2a_{n-1}^2 + 1 \qquad a_0 = 2 \quad a_n > 0$

$a_n^2 = b_n$

$b_n = 2b_{n-1} + 1$

$(E-2)(E-1)\langle b_n\rangle = \langle 0\rangle$

$b_n = 2^n\cdot\alpha + 1^n\cdot\beta$

$a_0 = 2 \Rightarrow b_0 = 4$

$a_1^2 = 2a_0^2 + 1$

$a_1^2 = 2\cdot 4 + 1 = 9 = b_1$

$$-\begin{cases} 4 = \alpha + \beta \\ 9 = 2\alpha + \beta \end{cases}$$

$-5 = -\alpha \Rightarrow \alpha = 5 \Rightarrow \beta = -1$

$b_n = 5\cdot 2^n - 1 \Rightarrow a_n = \sqrt{5\cdot 2^n - 1}$

8 a) $a_{n+2} = 2a_{n+1} - a_n + 3^n$ $\qquad a_0 = a_1 = 0$

$$E^2\langle a_n\rangle = \langle a_{n+1}\rangle = 2E\langle a_n\rangle - \langle a_n\rangle + \langle 3^n\rangle$$

$$(E^2 - 2E + 1)\langle a_n\rangle = \langle 3^n\rangle$$

$$(E-3)\langle 3^n\rangle = \langle 0;0\rangle$$

$$(E-1)^2(E-3)\langle a_n\rangle = \langle 0\rangle$$

$$a_n = \alpha n + \beta + 3^n\cdot\gamma \qquad a_0 = a_1 = 0$$

$$a_2 = 2a_1 - a_0 + 3^0$$

$$a_2 = 0 - 0 + 1 = 1$$

$$\begin{cases} 0 = \beta + \gamma \\ 0 = \alpha + \beta + 3\gamma \\ 1 = 2\alpha + \beta + 9\gamma \end{cases}$$

$\Downarrow$

$$\alpha = -\frac{1}{2}$$

$$\beta = -\frac{1}{4} \quad\Rightarrow\quad \underline{a_n = -\frac{1}{2}n - \frac{1}{4} + 3^n\cdot\frac{1}{4}}$$

$$\gamma = \frac{1}{4}$$

b) $a_{n+2} = 4a_{n+1} - 4a_n + n\cdot 2^{n+1}$; $a_0 = a_1 = 1$

$$E^2\langle a_n\rangle = \langle 4a_{n+1} - 4a_n + n\cdot 2^{n+1}\rangle$$

$$E^2\langle a_n\rangle = 4E\langle a_n\rangle - 4\langle a_n\rangle + 2\langle n\cdot 2^n\rangle$$

$$(E-2)^2 \langle n \cdot 2^{n+1} \rangle =$$

$$(E-2) \langle (n+1) \cdot 2^{n+2} - n \cdot 2^{n+2} \rangle =$$

$$(E-2) \langle 2^{n+2} \rangle = \langle 2^{n+3} - 2^{n+3} \rangle$$

$$= \langle 0 \rangle$$

$$(E^2 - 4E + 4)(E-2)^2 \langle a_n \rangle = \langle 0 \rangle$$

$$(E-2)^4 \langle a_n \rangle = \langle 0 \rangle$$

$$\underline{2^n(\alpha n^3 + \beta n^2 + \gamma n + \psi) = a_n}$$

c) $a_{n+2} = \frac{1}{2^{n+1}} - 2a_{n+1} - a_n$ $\qquad a_0 = a_1 = 1$

$$E^2 \langle a_n \rangle = \langle \frac{1}{2^{n+1}} \rangle - 2E \langle a_n \rangle - 1$$

$$(E - \tfrac{1}{2}) \langle \tfrac{1}{2^{n+1}} \rangle = \langle 0 \rangle$$

$$\underbrace{(E^2 + 2E + 1)(E - \tfrac{1}{2})}_{(E+1)^2 (E - \frac{1}{2})} \langle a_n \rangle = \langle 0 \rangle$$

$$\underline{(-1)^n (\alpha n + \beta) + \left(\tfrac{1}{2}\right)^n \gamma = a_n}$$

9

x_n – ciągi z {0,1,2} kończące się na 0
y_n – ciągi z {0,1,2} kończące się na 1
z_n – ciągi z {0,1,2} kończące się na 2
} takie, że nie ma w nich 00 i 11, czyli spełniają warunki zadania).

$$c_n = x_n + y_n + z_n$$

$$x_{n+1} = y_n + z_n$$

$$\underbrace{\qquad\qquad\qquad}_{y_n}\,\boxed{0} \quad \text{lub} \quad \underbrace{\qquad\qquad\qquad}_{z_n}\,\boxed{0}$$

analogicznie $y_{n+1} = x_n + z_n$

$$z_{n+1} = x_n + y_n + z_n$$

$$\underbrace{\qquad\qquad\qquad}_{x_n}\,\boxed{2} + \underbrace{\qquad\qquad\qquad}_{y_n}\,\boxed{2} + \underbrace{\qquad\qquad\qquad}_{z_n}\,\boxed{2}$$

$$\begin{cases} c_n = x_n + y_n + z_n & (1) \\ x_{n+1} = y_n + z_n & (2) \\ y_{n+1} = x_n + z_n & (3) \\ z_{n+1} = x_n + y_n + z_n & (4) \end{cases}$$

$(1)-(4)$

$\Downarrow$

$c_n - z_{n+1} = 0$

$\Downarrow$

$$c_n = z_{n+1}$$

$$(2)-(4) \Rightarrow x_{n+1} - z_{n+1} = -x_n$$

$$z_{n+1} = x_{n+1} + x_n$$

$$z(3)\ y_{n+1} = x_n + z_n = 2x_n + x_{n-1}$$

$$z(4)\ z_{n+1} = x_n + y_n + z_n$$

$$x_{n+1} + \cancel{x_n} = \cancel{x_n} + 2x_{n-1} + x_{n-2} + x_n + x_{n-1}$$

$$x_{n+1} = x_n + 3x_{n-1} + x_{n-2}$$

$$E^3 \langle x_n \rangle = \langle x_{n+3} \rangle = \langle x_{n+2} + 3x_{n+1} + x_n \rangle$$

$$E^3 \langle x_n \rangle = E^2 \langle x_n \rangle + 3E \langle x_n \rangle + \langle x_n \rangle$$

$$(E^3 - E^2 - 3E - 1)\langle x_n \rangle = \langle 0 \rangle$$

$$(E-(-1))(E-(1-\sqrt{2}))(E-(1+\sqrt{2}))\langle x_n \rangle = \langle 0 \rangle$$

$$x_n = (-1)^n \cdot \alpha + (1-\sqrt{2})^n \cdot \beta + (1+\sqrt{2})^n \gamma$$

$$z_{n+1} = x_{n+1} + x_n = c_n$$

$$c_n = \underbrace{(-1)^n \alpha (1-1)}_{0} + (1-\sqrt{2})^n \beta \left(1 + (1-\sqrt{2})\right) + (1+\sqrt{2})^n \gamma \left(1 + (1+\sqrt{2})\right)$$

$$c_n = (1-\sqrt{2})^n (2-\sqrt{2}) \beta + (1+\sqrt{2})^n (2+\sqrt{2}) \gamma$$

$$c_0 = 1 \quad ; \quad c_1 = 3 \qquad (60 \ \{1\}, \{2\}, \{0\} \ .)$$

$$\begin{cases} 1 = (2-\sqrt{2})\beta + (2+\sqrt{2})\gamma \\ 3 = (1-\sqrt{2})(2-\sqrt{2})\beta + (1+\sqrt{2})(2+\sqrt{2})\gamma \end{cases}$$

$$\Downarrow$$

$$\beta = \frac{-\sqrt{2}}{4}$$

$$\gamma = \frac{\sqrt{2}}{4}$$

$$\Downarrow$$

$$c_n = (1-\sqrt{2})^n (2-\sqrt{2}) \left(\frac{-\sqrt{2}}{4}\right) + (1+\sqrt{2})^n (2+\sqrt{2}) \left(\frac{\sqrt{2}}{4}\right)$$