(1)

a) $t_n = t_{n-1} + 3^n \; ; n > 1 \; ; t_1 = 3$

$t_2 = 3^1 + 3^2$

$t_3 = 3^1 + 3^2 + 3^3$ (geo)

$\vdots$

$t_n = 3^1 + \dots + 3^n = \underbrace{3 \cdot \frac{1-3^n}{1-3}}$

b) $h_n = h_{n-1} + (-1)^{n+1} n \; ; n > 1 ; h_1 = 1$

$h_2 = 1 - n$

$h_3 = 1 - n + n = 1$

$h_4 = 1 - n$

$\vdots$

$$h_n = \begin{cases} 1 & 2 \nmid n \\ 1 - n & 2 \mid n \end{cases}$$

(3)

a) $c_0 = 1$

$c_n - c_{n-1} = c_{n-2}$

$c_n = c_{n-1} + c_{n-2}$ (Fibonacci)

b) $d_0 = 1 \quad d_1 = 2 \quad d_n = \frac{d_{n-1}^2}{d_{n-2}} \Rightarrow$

$\swarrow$ (geo)

$$d_n \cdot d_{n-2} = d_{n-1}^2$$

$$d_n = d_0 \cdot q^n = q^n$$

$$d_1 = 2 = q \Rightarrow \underline{d_n = 2^n}$$

(2)

a) $a_{n+1} = \sqrt{a_n^2 + a_{n-1}^2}$; $a_0 = a_1 = 1$

$$a_{n+1}^2 = b_{n+1}$$

$$b_{n+1} = b_n + b_{n-1}$$

$$E\langle b_n \rangle = \langle b_{n+1} \rangle = \langle b_n \rangle + \langle b_{n-1} \rangle$$

$$E^2\langle b_n \rangle = \langle b_{n+2} \rangle = \langle b_{n+1} \rangle + \langle b_n \rangle$$

$$E^2\langle b_n \rangle = E\langle b_n \rangle + \langle b_n \rangle$$

$$(E^2 - E - 1)\langle b_n \rangle = \langle 0 \rangle$$

fib. $b_n = \dots$

$$a_n = \sqrt{b_n} = \sqrt{\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)}$$

$\boxed{\text{Wzór:}}$ $(E-c)^k$ anihiluje $\langle(\alpha_{k-1}n^{k-1}+\alpha_{k-2}n^{k-2}+\dots+\alpha_1 n+\alpha_0)c^n\rangle$

b) $b_{n+1} = \sqrt{b_n^2+3}$ $\qquad b_0 = 8$

$\underline{b_{n+1}^2 = a_{n+1}}$

$$a_{n+1} = a_n + 3$$

$$E\langle a_n\rangle = \langle a_{n+1}\rangle = \langle a_n\rangle + \langle 3\rangle$$

$$(E-1)\langle a_n\rangle = \langle 3\rangle$$

$\boxed{(E-1)\langle 3\rangle = 0}$

$$(E-1)(E-1)\langle a_n\rangle = \langle 0\rangle$$

$(E-1)^2$ anihiluje $a_n = \alpha\cdot n + \beta$

$a_0 = b_0^2 = 64 \Rightarrow a_0 = 64 = \beta$

$\underline{\beta = 64}$

$$a_1 = b_1^2 = \left(\sqrt{b_0^2+3}\right)^2 = \left(\sqrt{67}\right)^2 = 67$$

$67 = \alpha + \beta \Rightarrow \underline{\alpha = 3}$

$$a_n = 3n + 64$$

$$b_n = \sqrt{a_n} = \sqrt{3n+64}$$

c) $c_{n+1} = (n+1)c_n + (n^2+n)c_{n-1}$ $\left\{ \begin{array}{l} c_0 = 0 \\ c_1 = 1 \end{array} \right.$

$$E^2\langle c_n\rangle = \langle c_{n+2}\rangle = \langle (n+1)c_{n+1} + (n^2+n)c_n\rangle$$

$$E^2\langle c_n\rangle = E\langle c_n \cdot n\rangle + \langle (n^2+m)\cdot c_n\rangle$$

$$E^2\langle c_n\rangle = E$$

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(8) a) $a_{n+2} = 2a_{n+1} - a_n + 3^n$ $a_0 = a_1 = 0$

$$E^2\langle a_n\rangle = \langle a_{n+1}\rangle = 2E\langle a_n\rangle - \langle a_n\rangle + \langle 3^n\rangle$$

$$(E^2 - 2E + 1)\langle a_n\rangle = \langle 3^n\rangle$$

$$(E-3)\langle 3^n\rangle = \langle 0; 0\rangle$$

$$(E-1)^2(E-3)\langle a_n\rangle = \langle 0\rangle$$

$$\underbrace{a_n = \alpha n + B + \gamma \cdot 3^n}$$

b) $a_{n+2} = 4a_{n+1} - 4a_n + n \cdot 2^{n+1}$; $a_0 = a_1 = 1$

$$E^2 \langle a_n \rangle = \langle 4a_{n+1} - 4a_n + n \cdot 2^{n+1} \rangle$$

$$E^2 \langle a_n \rangle = 4E \langle a_n \rangle - 4 \langle a_n \rangle + 2 \langle n \cdot 2^n \rangle$$

$$\underbrace{(E-2)^2 \langle n \cdot 2^{n+1} \rangle} =$$

$$(E-2) \langle (n+1) \cdot 2^{n+2} - n \cdot 2^{n+2} \rangle =$$

$$(E-2) \langle 2^{n+2} \rangle = \langle 2^{n+3} - 2^{n+3} \rangle$$

$$\underbrace{= \langle 0 \rangle}$$

$$(E^2 - 4E + 4)(E-2)^2 \langle a_n \rangle = \langle 0 \rangle$$

$$(E-2)^4 \langle a_n \rangle = \langle 0 \rangle$$

$$\underbrace{2^n (\alpha n^3 + \beta n^2 + \gamma n + \psi) = a_n}$$

c) $a_{n+2} = \frac{1}{2^{n+1}} - 2a_{n+1} - a_n$ $\qquad a_0 = a_1 = 1$

$$E^2 \langle a_n \rangle = \langle \frac{1}{2^{n+1}} \rangle - 2E \langle a_n \rangle - 1$$

$$(E - \tfrac{1}{2}) \langle \frac{1}{2^{n+1}} \rangle = \langle 0 \rangle$$

$$\underbrace{(E^2+2E+1)\left(E-\frac{1}{2}\right)}_{(E+1)^2\left(E-\frac{1}{2}\right)}\langle a_n\rangle = \langle 0\rangle$$

$$\underline{(-1)^n(\alpha n+B)+\left(\frac{1}{2}\right)^n = a_n}$$