Problem Statement:

Find the function *u* : [0, 2] -> *R* satisfying the equation:

$$-\frac{d}{dx}\left(k(x)\,\frac{du(x)}{dx}\right) = 100x$$

Where function *k* is defined as follows:

$$k(x) = \begin{cases} 1\ for\ x\ \in [0,1] \\ 2x\ for\ x\ \in (1,2] \end{cases}$$

With boundary conditions:

$$u(2) = 0$$

$$\frac{du(0)}{dx} + u(0) = 20$$

Weak Formulation:

Multiply the equation on both sides by a certain test function $v(x)$:

$$-\frac{d}{dx}\left(k(x)\,\frac{du(x)}{dx}\right) = 100x\ /\cdot\ v(x)$$

$$where\ v \in V,\ \ V = \{f\ \in H^1, f(2) = 0\}$$

$$-\ \frac{d}{dx}\left(k(x)\frac{du(x)}{dx}\right)v(x) = 100x\ v(x)$$

Integrate on both sides within the interval [0, 2]:

$$-\int_0^2 \frac{d}{dx}\left(k(x)\frac{du(x)}{dx}\right)v(x)dx = \int_0^2 100x\ v(x)dx$$

Now let's write the left-hand side of the equation in a simpler form:

$$-\int_0^2 \frac{d}{dx}\left(k(x)\frac{du(x)}{dx}\right)v(x)dx = -|k(x)\frac{du(x)}{dx}v(x)|_0^2 + \int_0^2 k(x)\frac{du(x)}{dx}\ \frac{dv(x)}{dx}\ dx =$$

$$= -k(2)\frac{du(2)}{dx}v(2) + k(0)\frac{du(0)}{dx}v(0) + \int_0^2 k(x)\frac{du(x)}{dx}\frac{dv(x)}{dx}dx = *$$

We use the relation $v(2) = 0$

$$* = k(0)\frac{du(0)}{dx}v(0) + \int_0^2 k\frac{du(x)}{dx}\frac{dv(x)}{dx}dx = **$$

We know that $\frac{du(0)}{dx} + u(0) = 20 \Leftrightarrow \frac{du(0)}{dx} = 20 - u(0)$

$$** = k(0)\big(20 - u(0)\big)v(0) + \int_0^2 k(x)\frac{du(x)}{dx}\frac{dv(x)}{dx}dx\ =$$

$$= 20k(0)v(0) - k(0)u(0)v(0) + \int_0^2 k(x)\frac{du(x)}{dx}\frac{dv(x)}{dx}dx =$$

$$= 20v(0) - u(0)v(0) + \int_0^1 \frac{du(x)}{dx}\frac{dv(x)}{dx}dx + \int_1^2 2x\frac{du(x)}{dx}\frac{dv(x)}{dx}dx$$

Therefore:

$$20v(0) - u(0)v(0) + \int_0^1 \frac{du(x)}{dx}\frac{dv(x)}{dx}dx + \int_1^2 2x\frac{du(x)}{dx}\frac{dv(x)}{dx}dx = \int_0^2 100x\ v(x)dx$$

$$-u(0)v(0) + \int_0^1 \frac{du(x)}{dx}\frac{dv(x)}{dx}dx + \int_1^2 2x\frac{du(x)}{dx}\frac{dv(x)}{dx}dx = \int_0^2 100x\ v(x)dx -\ 20v(0)$$

Where:

$$B(u,v) = -u(0)v(0) + \int_0^1 \frac{du(x)}{dx}\frac{dv(x)}{dx}dx + \int_1^2 2x\frac{du(x)}{dx}\frac{dv(x)}{dx}dx$$

$$L(v) = \int_0^2 100x\, v(x)dx - 20v(0)$$