03-02-22 - HW 3

Wednesday, March 2, 2022 4:38 PM

4.5. a. Using Eq 4.27

$$Q_{k-1} = \int_{t_{k-1}}^{t_k} e^{A(t_k-\tau)} Q_c(\tau) e^{A^T(t_k-\tau)} d\tau$$

$$A = \begin{bmatrix} -F_0/2V_0 & 0 \\ 0 & -F_0/V_0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 1 \\ \frac{c_1-c_0}{V_0} & \frac{c_2-c_0}{V_0} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$Q_c = B \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} B^T = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$$

We know that matrix exponential e^M of a diagonal matrix $M = \begin{bmatrix} m_{11} & 0 \\ 0 & m_{22} \end{bmatrix}$ is of the form $e^M = \begin{bmatrix} e^{m_{11}} & 0 \\ 0 & e^{m_{22}} \end{bmatrix}$

$$\therefore e^{A(t_k-\tau)} = \begin{bmatrix} e^{\tau-t_k} & 0 \\ 0 & e^{2\tau-2t_k} \end{bmatrix} = e^{A^T(t_k-\tau)}$$

$$e^{A(t_k-\tau)} Q_c(\tau) e^{A^T(t_k-\tau)} = \begin{bmatrix} e^{\tau-t_k} & 0 \\ 0 & e^{2\tau-2t_k} \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} e^{\tau-t_k} & 0 \\ 0 & e^{2\tau-2t_k} \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{\tau-t_k} & 3e^{\tau-t_k} \\ 3e^{2\tau-2t_k} & 5e^{2\tau-2t_k} \end{bmatrix} \begin{bmatrix} e^{\tau-t_k} & 0 \\ 0 & e^{2\tau-2t_k} \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{2\tau-2t_k} & 3e^{3\tau-3t_k} \\ 3e^{3\tau-3t_k} & 5e^{4\tau-4t_k} \end{bmatrix}$$

$$\therefore Q_{k-1} = \int_{t_{k-1}}^{t_k} \begin{bmatrix} 2e^{2\tau-2t_k} & 3e^{3\tau-3t_k} \\ 3e^{3\tau-3t_k} & 5e^{4\tau-4t_k} \end{bmatrix} d\tau = \begin{bmatrix} 1-e^{-2t_k+2t_{k-1}} & 1-e^{-3t_k+3t_{k-1}} \\ 1-e^{-3t_k+3t_{k-1}} & \frac{5}{4}\left(1-e^{4t_k-4t_{k-1}}\right) \end{bmatrix}$$

b. Using Eq 4.28

$$Q_{k-1} = Q_c(t_k)\,\Delta t = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}(t_k - t_{k-1}) = \begin{bmatrix} 2(t_k - t_{k-1}) & 3(t_k - t_{k-1}) \\ 3(t_k - t_{k-1}) & 5(t_k - t_{k-1}) \end{bmatrix}$$

c. We can expand the solution of (a) using a Taylor series as follows:

$$Q_{k-1} = \begin{bmatrix} 1-(1-2\Delta t + (2\Delta t)^2/2! + (2\Delta t)^3/3! + \dots) & 1-(1-3\Delta t + (3\Delta t)^2/2! + (3\Delta t)^3/3! + \dots) \\ 1-(1-3\Delta t + (3\Delta t)^2/2! + (3\Delta t)^3/3! + \dots) & \frac{5}{4} - \frac{5}{4}(1-4\Delta t + (4\Delta t)^2/2! + \dots) \end{bmatrix}$$

where $\Delta t = t_k - t_{k-1}$

From this representation, we can see that for small Δt higher order terms can be ignored, making our result approximately equal to the result from part (b).

$\therefore$ Answer to part (a) $\approx$ Answer to part (b)

4.8. $F = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$

$Q = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$i=0 \Rightarrow F^0 Q (F^T)^0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$i=1 \Rightarrow \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/4 & 1/4 \\ 0 & 1/4 \end{bmatrix}$

$i=2 \Rightarrow \begin{bmatrix} 1/2^2 & 0 \\ 0 & 1/2^2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/2^2 & 0 \\ 0 & 1/2^2 \end{bmatrix} = \begin{bmatrix} 1/16 & 1/16 \\ 0 & 1/16 \end{bmatrix}$

$i=3 \Rightarrow \begin{bmatrix} 1/2^3 & 0 \\ 0 & 1/2^3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/2^3 & 0 \\ 0 & 1/2^3 \end{bmatrix} = \begin{bmatrix} 1/64 & 1/64 \\ 0 & 1/64 \end{bmatrix}$

$i=4 \Rightarrow \begin{bmatrix} 1/2^4 & 0 \\ 0 & 1/2^4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/2^4 & 0 \\ 0 & 1/2^4 \end{bmatrix} = \begin{bmatrix} 1/256 & 1/256 \\ 0 & 1/256 \end{bmatrix}$

$i=5 \Rightarrow \begin{bmatrix} 1/2^5 & 0 \\ 0 & 1/2^5 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/2^5 & 0 \\ 0 & 1/2^5 \end{bmatrix} = \begin{bmatrix} 1/1024 & 1/1024 \\ 0 & 1/1024 \end{bmatrix}$

$i=6 \Rightarrow \begin{bmatrix} 1/2^6 & 0 \\ \dots \end{bmatrix}\begin{bmatrix} 1 & 1 \\ \dots \end{bmatrix}\begin{bmatrix} 1/2^6 & 0 \\ \dots \end{bmatrix} = \begin{bmatrix} 1/4096 & 1/4096 \\ \dots \end{bmatrix}$

$$i=6 \Rightarrow \begin{bmatrix} 1/2^6 & 0 \\ 0 & 1/2^6 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2^6 & 0 \\ 0 & 1/2^6 \end{bmatrix} = \begin{bmatrix} 1/4096 & 1/4096 \\ 0 & 1/4096 \end{bmatrix}$$

$$i=7 \Rightarrow \begin{bmatrix} 1/2^7 & 0 \\ 0 & 1/2^7 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2^7 & 0 \\ 0 & 1/2^7 \end{bmatrix} = \begin{bmatrix} 1/16384 & 1/16384 \\ 0 & 1/16384 \end{bmatrix}$$

$$\vdots$$

P begins to converge by $i=7$.

Adding the matrices, we find that

$$P = \sum_{i=0}^{\infty} F^i Q (F^T)^i \approx \begin{bmatrix} 1.3333 & 0 \\ 0 & 1.3333 \end{bmatrix}$$

4.11. Using $\bar{x}_0 = 1$

$P_0 = 2$

$f = -0.5$

$q_c = 1$

$\Delta t = 0.01s$

$t_f = 5s$

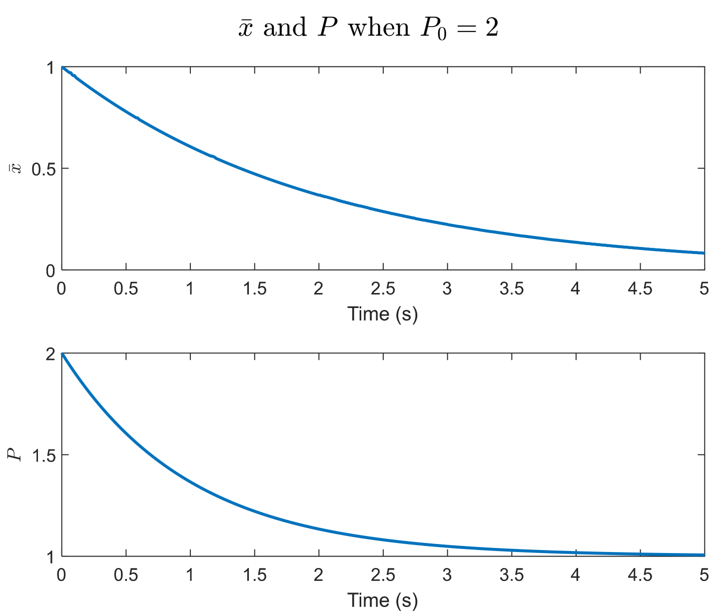
and using Eqn 4.32

$\bar{x}_k = \exp(kf\Delta t)\,\bar{x}_0$

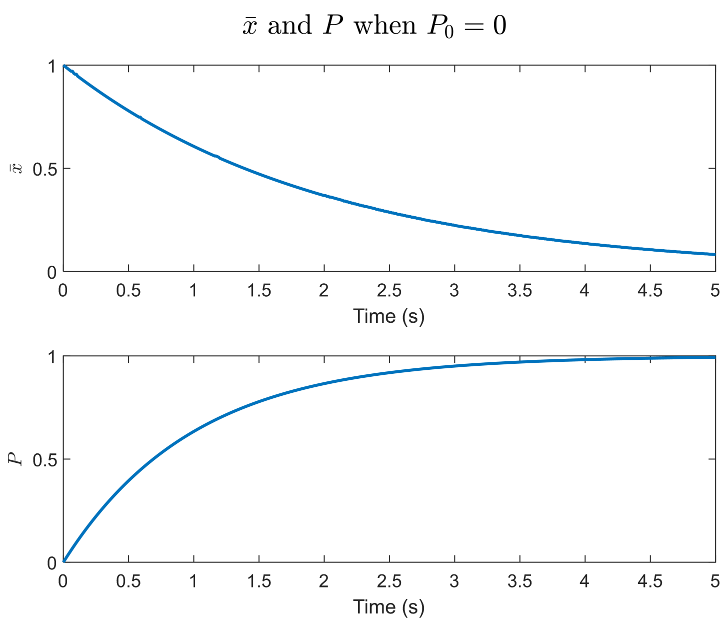
and Eqn 4.33

$P_k = (2fP_{k-1} + q_c)\Delta t + P_{k-1}$

the following plots were found for $\bar{x}$ and P:



With $P_0 = 0$ the following plots were found:



In both plots, the steady-state value of the variance appears to be 1, which is consistent with the result of the analytically determined steady-state variance using the equation from the text:

$$P_{k-1} = -\frac{q_c}{2f} = -\frac{1}{2 * -0.5} = 1$$

4.13.a. Using Eq 4.49

$$\dot{P} = AP + PA^T + Q_c$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} u$$

Using $R=3$, $L=1$, $C=0.5$

$$A = \begin{bmatrix} 0 & 2 \\ -1 & -3 \end{bmatrix}$$

$$Q_c = B q_c B^T = \begin{bmatrix} 0 \\ 1 \end{bmatrix} (1) \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$AP + PA^T + Q_c = 0$$

$$\begin{bmatrix} 0 & 2 \\ -1 & -3 \end{bmatrix} P + P \begin{bmatrix} 0 & -1 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

*Solved using MATLAB

$$P = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/6 \end{bmatrix}$$

b. $A = \begin{bmatrix} 0 & 2 \\ -1 & -3 \end{bmatrix}$

$$\det(A - \lambda I) = \det\begin{bmatrix} -\lambda & 2 \\ -1 & -3-\lambda \end{bmatrix}$$

$$= (-\lambda)(-3-\lambda) + 2$$

$$= \lambda^2 + 3\lambda + 2$$

$$= (\lambda+1)(\lambda+2)$$

$$\ker(A+I) = \ker\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -0.8944 \\ 0.4472 \end{bmatrix}$$

$$\ker(A+2I) = \ker\begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -0.7071 \\ 0.7071 \end{bmatrix}$$

$$\therefore S = \begin{bmatrix} -0.8944 & -0.7071 \\ 0.4472 & 0.7071 \end{bmatrix}$$

$$\hat{A} = S^{-1}AS = \begin{bmatrix} -0.8944 & -0.7071 \\ 0.4472 & 0.7071 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 2 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} -0.8944 & -0.7071 \\ 0.4472 & 0.7071 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

Using Eq 1.71

$$e^{AT} = Q e^{\hat{A}T} Q^{-1}$$

$$= \begin{bmatrix} -0.8944 & -0.7071 \\ 0.4472 & -0.7071 \end{bmatrix} \begin{bmatrix} e^{-T} & 0 \\ 0 & e^{-2T} \end{bmatrix} \begin{bmatrix} -0.8944 & -0.7071 \\ 0.4472 & -0.7071 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 2e^{-T} - e^{-2T} & 2e^{-T} - 2e^{-2T} \\ e^{-2T} - e^{-T} & 2e^{-2T} - e^{-T} \end{bmatrix}$$

$\because$ Proven

2. Estimate X_0 and V_0

$x^* = [3.9]$

2. Estimate X_0 and V_0

$$X_0^* = \begin{bmatrix} 3.9 \\ 0.5 \end{bmatrix}$$

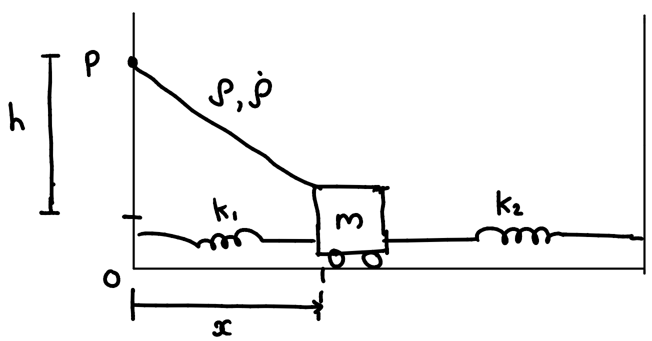
$$\bar{P}_0 = \begin{bmatrix} 1000 & 0 \\ 0 & 100 \end{bmatrix}$$

$$R = \begin{bmatrix} 0.0625 & 0 \\ 0 & 0.01 \end{bmatrix}$$

$k_1 = 2.6$, $k_2 = 3.5$, $m = 1.8$, $h = 6.4$

$t_{span} = 0:1:11$

Let the setup be as follows



$x(t)$ = position $\qquad \bar{x}$ = static equilibrium position

$v(t)$ = velocity

$\dot{x}(t) = v(t)$

$\rho(t) = (x(t)^2 + h^2)^{1/2}$

$$\dot{\rho}(t) = \frac{d}{dt}\left(x(t)^2 + h^2\right)^{1/2} = \frac{1}{2}(x^2+h^2)^{-1/2}\, 2x\dot{x} = \frac{xv}{\rho}$$

We know that for a mass spring system

$$\ddot{x} = -(k_1+k_2)(x-\bar{x})/m = \dot{v}$$

Let $\omega^2 = (k_1+k_2)/m$

Assembling state vector $\bar{X}$:

$$\bar{X} = \begin{bmatrix} x \\ v \end{bmatrix}$$

$$\dot{\bar{X}} = \begin{bmatrix} \dot{x} \\ \end{bmatrix} = \begin{bmatrix} v \\ \end{bmatrix} = F$$

$$\dot{X} = \begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} v \\ -\omega^2(x-\bar{x}) \end{bmatrix} = F$$

Assembling observations Y:

$$Y = \begin{bmatrix} \rho \\ \dot{\rho} \end{bmatrix} + \begin{bmatrix} \varepsilon_\rho \\ \varepsilon_{\dot{\rho}} \end{bmatrix} = G(X(t), t) + \varepsilon(t)$$

where $G = \begin{bmatrix} \rho \\ \dot{\rho} \end{bmatrix} = \begin{bmatrix} [x^2 + h^2]^{1/2} \\ xv/\rho \end{bmatrix}$

Linearizing dynamics:

$$A = \frac{\partial}{\partial X} F = \begin{bmatrix} \partial f_1/\partial X_1 & \partial f_1/\partial X_2 \\ \partial f_2/\partial X_1 & \partial f_2/\partial X_2 \end{bmatrix} = \begin{bmatrix} \partial v/\partial x & \partial v/\partial v \\ \frac{\partial}{\partial x}(-\omega^2(x-\bar{x})) & \frac{\partial}{\partial v}(-\omega^2(x-\bar{x})) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}$$

Linearizing measurement model:

$$\tilde{H} = \frac{\partial G}{\partial X} = \begin{bmatrix} \partial\rho/\partial X_1 & \partial\rho/\partial X_2 \\ \partial\dot{\rho}/\partial X_1 & \partial\dot{\rho}/\partial X_2 \end{bmatrix} = \begin{bmatrix} \partial\rho/\partial x & \partial\rho/\partial v \\ \partial\dot{\rho}/\partial x & \partial\dot{\rho}/\partial v \end{bmatrix}$$

* Computing partials in MATLAB

$$\tilde{H} = \begin{bmatrix} x/\rho & 0 \\ v/\rho - x^2 v/\rho^3 & x/\rho \end{bmatrix}$$

State transition matrix Φ:

$$\dot{\Phi} = A\Phi(t,t_0) \qquad \Phi(t_0,t_0) = I$$

$$\begin{bmatrix} \dot{\Phi}_{11} & \dot{\Phi}_{12} \\ \dot{\Phi}_{21} & \dot{\Phi}_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} = \begin{bmatrix} \Phi_{21} & \Phi_{22} \\ -\omega^2\Phi_{11} & -\omega^2\Phi_{12} \end{bmatrix}$$

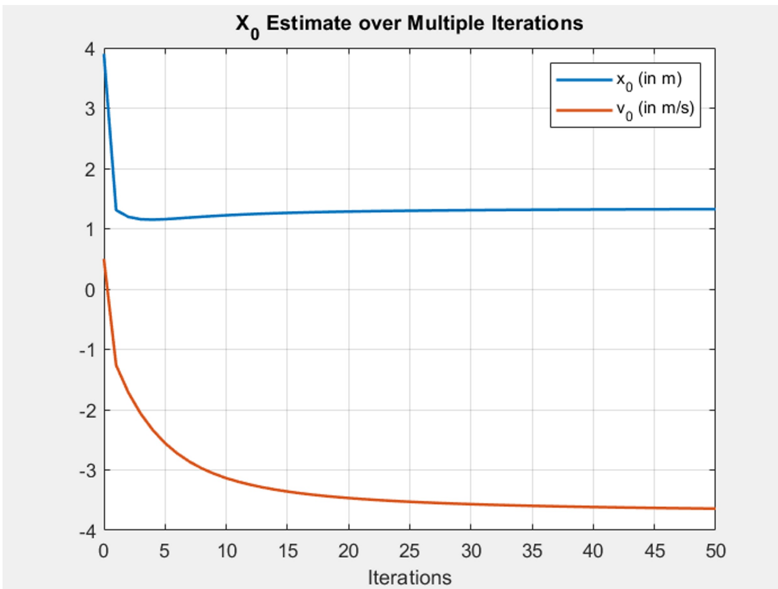
* Solving using dsolve in MATLAB

$$\Phi(t,t_0) = \begin{bmatrix} \cos(\omega t) & \sin(\omega t)/\omega \\ -\omega\sin(\omega t) & \cos(\omega t) \end{bmatrix}$$

After implementing this setup in MATLAB, X_0 after 3 iterations is

$$\begin{bmatrix} 1.157 \\ \underline{\underline{-2.065}} \end{bmatrix}$$

When running more iterations, the values for X_0 changed as follows:



3. Assembling state vector $\underline{X}$:

$$\underline{X} = \begin{bmatrix} x & y & z & V_x & V_y & V_z \end{bmatrix}'$$

$$\dot{\underline{X}} = \begin{bmatrix} \dot{x} & \dot{y} & \dot{z} & \dot{V_x} & \dot{V_y} & \dot{V_z} \end{bmatrix}'$$

$$= \begin{bmatrix} V_x & V_y & V_z - gt & 0 & 0 & -g \end{bmatrix} = F$$

Assembling observations $\underline{Y}$:

$$\underline{Y} = \begin{bmatrix} \rho \\ \alpha \\ \beta \end{bmatrix} + \begin{bmatrix} \varepsilon_\rho \\ \varepsilon_\alpha \\ \varepsilon_\beta \end{bmatrix} = G(\underline{X}(t), t) + \varepsilon(t)$$

where

$$G = \begin{bmatrix} \rho \\ \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} (x^2 + y^2 + z^2)^{1/2} \\ \tan^{-1}(y/x) \\ \tan^{-1}(z/\sqrt{x^2+y^2}) \end{bmatrix}$$

Answer for part (b)

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Linearizing dynamics:

$$A = \frac{\partial}{\partial \bar{X}} F = \begin{bmatrix} \partial f_1/\partial X_1 & \cdots & \partial f_1/\partial X_6 \\ \partial f_2/\partial X_1 & \ddots & \vdots \\ \partial f_3/\partial X_1 & & \vdots \\ \partial f_4/\partial X_1 & & \vdots \\ \partial f_5/\partial X_1 & & \vdots \\ \partial f_6\,\partial X_1 & \cdots & \partial f_6/\partial X_6 \end{bmatrix}$$

* Computing in MATLAB

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Linearizing measurement model:

$$\tilde{H} = \frac{\partial G}{\partial \bar{X}} = \begin{bmatrix} \partial \rho/\partial X_1 & \cdots & \partial \rho/\partial X_6 \\ \partial \alpha/\partial X_1 & \cdots & \partial \alpha/\partial X_6 \\ \partial \beta/\partial X_1 & \cdots & \partial \beta/\partial X_6 \end{bmatrix}$$

* Computing partials in MATLAB:

$$= \begin{bmatrix} x/\rho & y/\rho & z/\rho & 0 & 0 & 0 \\ -y/(y^2+x^2) & x/(x^2+y^2) & 0 & 0 & 0 & 0 \\ -xz/\rho^2\sqrt{x^2+y^2} & -yz/\rho^2\sqrt{x^2+y^2} & \sqrt{x^2+y^2}/\rho^2 & 0 & 0 & 0 \end{bmatrix}$$

Answer for part (c)

State transition matrix Φ:

$$\dot{\Phi}(t,t_0) = A(t,t_0) \qquad \Phi(t_0,t_0) = I$$

$$\begin{bmatrix} \dot{\Phi}_{11} & \cdots & \dot{\Phi}_{16} \\ \vdots & \ddots & \vdots \\ \dot{\Phi}_{61} & \cdots & \dot{\Phi}_{66} \end{bmatrix} = \begin{bmatrix} A_{11} & \cdots & A_{16} \\ \vdots & \ddots & \vdots \\ A_{61} & \cdots & A_{66} \end{bmatrix} \begin{bmatrix} \Phi_{11} & \cdots & \Phi_{16} \\ \vdots & \ddots & \vdots \\ \Phi_{61} & \cdots & \Phi_{66} \end{bmatrix}$$

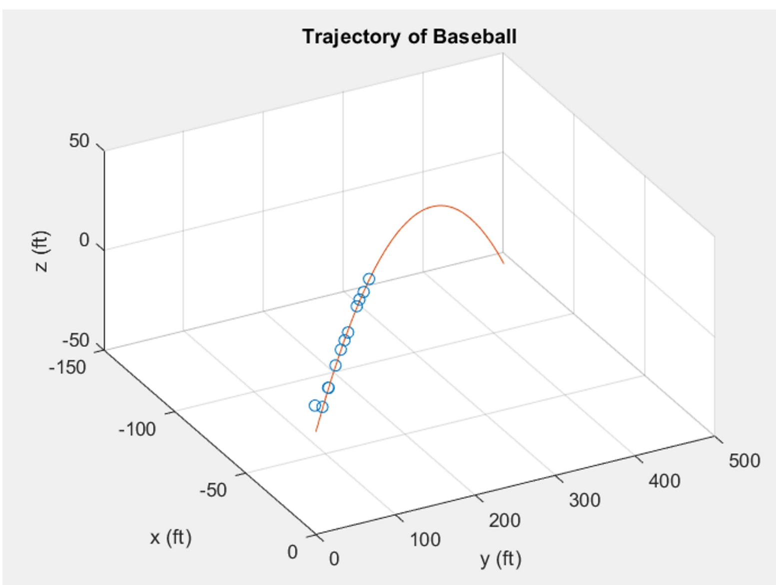
$$\begin{bmatrix} \vdots & \ddots & \vdots \\ \dot{\Phi}_{61} & \cdots & \dot{\Phi}_{66} \end{bmatrix} = \begin{bmatrix} \vdots & \ddots & \vdots \\ A_{61} & \cdots & A_{66} \end{bmatrix} \begin{bmatrix} \vdots & \ddots & \vdots \\ \Phi_{61} & \cdots & \Phi_{66} \end{bmatrix}$$

* Solving using dsolve in MATLAB

$$\Phi(t, t_0) = \begin{bmatrix} 1 & 0 & 0 & t & 0 & 0 \\ 0 & 1 & 0 & 0 & t & 0 \\ 0 & 0 & 1 & 0 & 0 & t \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Answer for part (a)

d. The process was repeated 4 times (as gone over in class), and the following trajectory was obtained:



Using the distance formula, it was found that the baseball landed 405.71 ft from the origin.

e. To find the uncertainty of the estimate, the last P_0 term was propagated to the final time using $\Phi(t, 0)$.

$t_f = 3.41s$

$$P = \Phi P_0 \Phi^T = * \text{Evaluated in MATLAB}$$

The Frobenius norm of this matrix was found

$$\sqrt{\sum_{j=1}^{6} \sum_{i=1}^{6} a_{ij}^2} = 1.3289$$