HW 1 Submission

AE-6505 - Kalman Filtering

Kadhir Umasankar

Spring 2022

01-19-22 - HW 1

Wednesday, January 19, 2022 8:10 PM

1.1.6. $\begin{bmatrix} A & A \\ B & A \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix}$

$AA + AC = 0$ — ①

$BA + AC = I$ — ②

From ①:

$AC = -AA$

$C = -A^{-1}AA = \underline{-A}$

From ②:

$BA - AA = I$

$BA = I + AA$

$B = (I + AA)A^{-1} = \underline{A^{-1} + A}$

1.7. Let $A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 2 \\ 2 & -4 \end{bmatrix}$

$A \cdot B = \begin{bmatrix} 14 & -8 \\ 20 & -10 \end{bmatrix}$

$AB \neq (AB)^T$

$\because$ Proved by contradiction

1.8. $\det\left(\begin{bmatrix} a-\lambda & b \\ b & c-\lambda \end{bmatrix}\right) = (a-\lambda)(c-\lambda) - b^2 = ac - a\lambda - c\lambda + \lambda^2 - b^2$

$= \lambda^2 + \lambda(-a-c) + (ac - b^2) = 0$

Using quadratic formula:

$$\lambda = \frac{a+c \pm \sqrt{a^2 + 2ac + c^2 - 4ac + 4b^2}}{2}$$

eig(A) is real if $\underline{a^2 + c^2 + 4b^2 > 2ac}$

b. If A is positive semidefinite, $\mathrm{eig}(A) \geq 0$

$$\therefore \quad a+c \pm \sqrt{a^2+2ac+c^2-4ac+4b^2} \geq 0$$

$$a+c \geq \sqrt{a^2+2ac+c^2-4ac+4b^2}$$

$$\cancel{a^2} + 2ac + \cancel{c^2} \geq \cancel{a^2} - 2ac + \cancel{c^2} + 4b^2$$

$$\cancel{4}ac \geq \cancel{4}b^2$$

$$ac \geq b^2$$

A is positive semi definite $\forall\, b$ s.t. $-\sqrt{ac} \leq b \leq \sqrt{ac}$

1.17. a. $x_1 = \theta \qquad u = T$

$x_2 = \dot{\theta}$

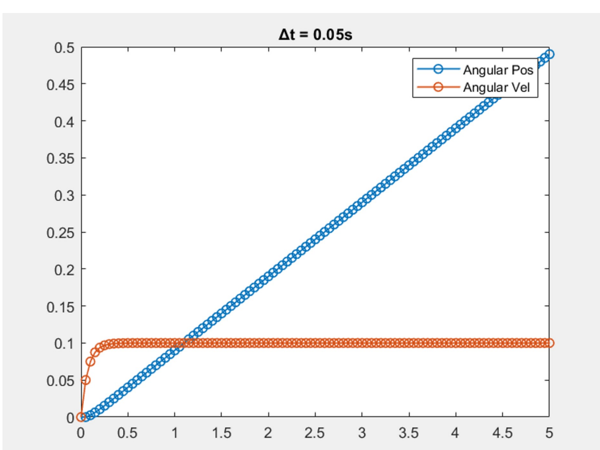
$$\dot{x}_1 = x_2$$

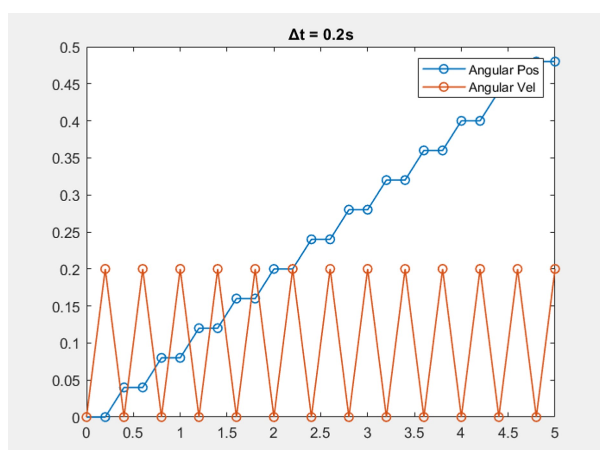
$$\dot{x}_2 = (T - Fx_2)/J$$

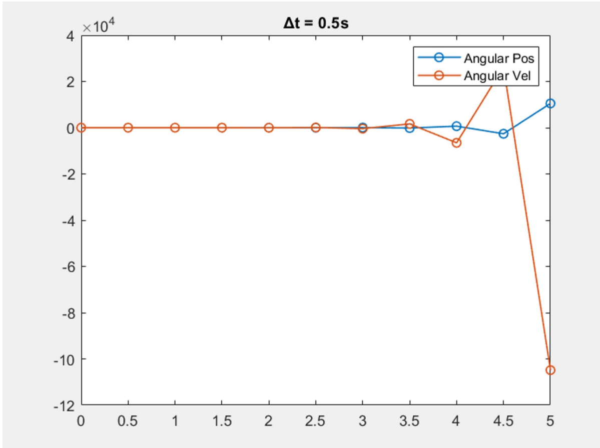
$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -F/J \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/J \end{bmatrix} u$$

b. * Simulating in MATLAB

Using rectangular integration, we get the following plots for $\Delta t = 0.05s$, $\Delta t = 0.2s$ and $\Delta t = 0.5s$ respectively







The output when $\Delta t = 0.05s$ looks correct. As the step size increases, the simulation becomes less accurate and eventually blows up.
The eigenvalues of the A matrix are 0 and -10. From this we can say that the minimum timestep for an accurate simulation $(\text{mean of eigenvalues})^{-1} = 1/5 \, s$

2.8. $Var(W+V) = Var\,W + Var\,V + 2Cov(W,V)$

$$Corr(W,V) = \frac{Cov(W,V)}{\sqrt{Var\,W \cdot Var\,V}} = 0 \quad \text{[Given uncorrelated]}$$

$$Cov(W,V) = 0$$

$\therefore Var(W+V) = Var\,W + Var\,V$

$$\sigma_x = \sqrt{\sigma_w^2 + \sigma_v^2}$$

2.9. a. $$\rho = \frac{\sigma_{XY}^2}{[\sigma_x^2 \sigma_y^2]^{1/2}}$$

$$\sigma_{XY}^2 = r_{XY}^2 - \mu_X \mu_Y$$
$$= \underbrace{E[XY]}_{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\, dx\, dy} - E[X]E[Y]$$

[When X and Y are independent $f(x,y) = f_X(x) f_Y(y)$]

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_X(x)\, f_Y(y)\, dx\, dy$$

$$= \left(\int^{\infty} x f_X(x)\, dx\right)\left(\int^{\infty} y f_Y(y)\, dy\right)$$

$$= \left(\int_{-\infty}^{\infty} x f_X(x)\, dx \right) \left(\int_{-\infty}^{\infty} y f_Y(y)\, dy \right)$$

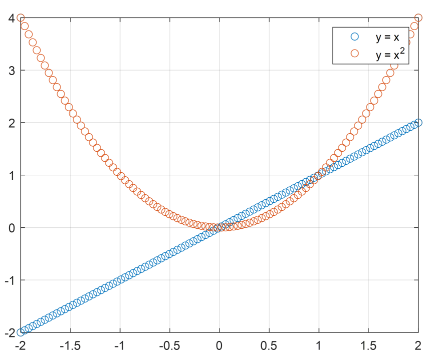
$$= E[X]\, E[Y]$$

$$\therefore \ \sigma_{XY}^2 = E[X]E[Y] - E[X]E[Y] = 0$$

and $\rho = \dfrac{0}{[\sigma_X^2 \sigma_Y^2]^{1/2}} = 0$

$\therefore$ Proven

b. $Y = X^2$. Since Y directly depends on X, they are not independent but their correlation coefficient is 0.



c. Let $Y = aX + b$

$$\rho = \frac{E[XY] - E[X]E[Y]}{\sqrt{\operatorname{Var} X \ \operatorname{Var} Y}}$$

$$E[XY] = E[X(aX+b)] = E[aX^2 + bX] = aE[X^2] + bE[X]$$

$$E[Y] = E[aX+b] = aE[X] + b$$

$$\operatorname{Var} X = E[X^2] - E[X]^2$$

$$\operatorname{Var} Y = E[Y^2] - E[Y]^2$$

$$= E[a^2 X^2 + 2ab X + b^2] - (aE[X]+b)^2$$

$$= a^2 E[X^2] + \cancel{2ab E[X]} + \cancel{b^2} - (a^2 E[X]^2 + \cancel{2ab E[X]} + \cancel{b^2})$$

$$= a^2 (E[X^2] - E[X]^2)$$

$$= a^2 E[X^2] + 2ab\cancel{E[X]} + \cancel{b^2} - (a^2 E[X]^2 + \cancel{2abE[X]} + \cancel{b^2})$$

$$= a^2 (E[X^2] - E[X]^2)$$

$$\rho = \frac{aE[X^2] + \cancel{bE[X]} - (aE[X] + \cancel{b})E[X]}{\sqrt{a^2 (E[X^2] - E[X]^2)^2}}$$

$$= \frac{a(E[X^2] - E[X])}{\pm a (E[X^2] - E[X])}$$

$$\rho = \pm 1$$

$\therefore$ Proven

2.10. $$\underbrace{\int_0^\infty \int_0^\infty a e^{-2x} e^{-3y} \, dx \, dy}_{\left[\frac{-ae^{-2x-3y}}{2}\right]_0^\infty} = 1$$

$$\int_0^\infty \frac{ae^{-3y}}{2} dy = \left[\frac{-ae^{-3y}}{6}\right]_0^\infty$$

$$= \frac{a}{6} = 1$$

$$a = 6$$

b. $$\bar{x} = \int_0^\infty x f_x(x) \, dx$$

$$f_x(x) = \int_0^\infty 6 e^{-2x} e^{-3y} dy$$

$$= 2e^{-2x}$$

$$\bar{x} = \int_0^\infty 2xe^{-2x} dx = 1/2$$

$$\bar{y} = \int_0^\infty y f_y(y) dy$$

$$f_y(y) = \int_0^\infty 6 e^{-2x} e^{-3y} dx$$

$$= 3e^{-3y}$$

$$\bar{y} = \int_0^\infty 3ye^{-3y} dy = 1/3$$

c. $$E[X^2] = \int_0^\infty x^2 f_x(x) dx = \int_0^\infty x^2 (2e^{-2x}) dx = 1/2$$

c. $E[X^2] = \int_0^{\infty} x^2 f_X(x)\,dx = \int_0^{\infty} x^2 \left(2e^{-2x}\right) dx = \frac{1}{2}$

$E[Y^2] = \int_0^{\infty} y^2 f_Y(y)\,dy = \int_0^{\infty} y^2 \left(3e^{-3y}\right) dy = \frac{2}{9}$

$$E[XY] = \int_0^{\infty}\int_0^{\infty} xy\left(6e^{-2x}e^{-3y}\right) dx\,dy$$

$$= \int_0^{\infty} \frac{3}{2} y e^{-3y}\,dy$$

$$= \frac{1}{6}$$

d. $r_{XY}^2 = \begin{bmatrix} E[X^2] & E[XY] \\ E[XY] & E[Y^2] \end{bmatrix}$

$r_{XY}^2 = \begin{bmatrix} 1/2 & 1/6 \\ 1/6 & 2/9 \end{bmatrix}$

e. $\sigma_X^2 = E[X^2] - E[X]^2$

$= 1/2 - 1/4 = 1/4$

$\sigma_Y^2 = E[Y^2] - E[Y]^2$

$= 2/9 - 1/9 = 1/9$

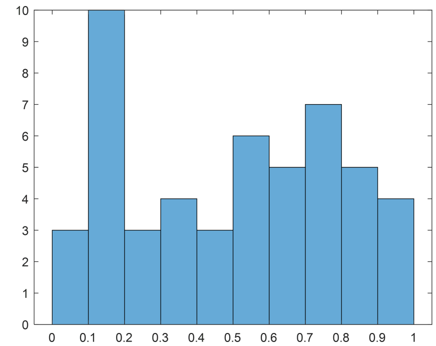
$C_{XY} = E[XY] - E[X]E[Y]$

$= \frac{1}{6} - \frac{1}{2}\cdot\frac{1}{3} = 0$

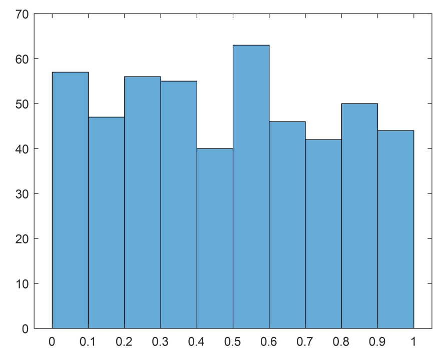
f. $C = \begin{bmatrix} 1/4 & 0 \\ 0 & 1/9 \end{bmatrix}$

g. $\rho = \frac{C_{XY}}{\sqrt{\mathrm{Var}X\,\mathrm{Var}Y}} = \frac{0}{\sqrt{\ldots}} = 0$

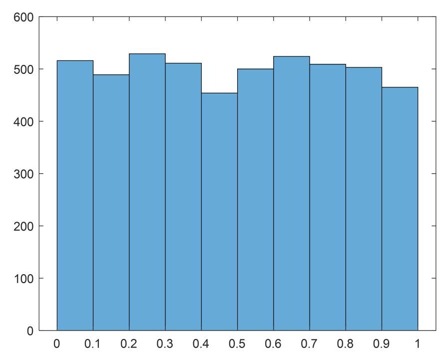
2.15.
N = 50
μ = 0.496384
σ = 0.284838



N = 500
μ = 0.482591
σ = 0.286209



N = 5000
μ = 0.496992
σ = 0.288375



Consistent with the formula for the mean and std dev of the Unif(0,1) distribution, I would expect

$$\mu = \frac{0+1}{2} = 0.5 \qquad \sigma = \sqrt{\frac{(0-1)^2}{12}} = 0.2887$$

As N increases, the mean and standard deviation of the sample get closer to the aforementioned values. The histogram starts to look more like a rectangle, i.e. 10 bins of equal height.

2. $P(vvv) = 0.2$
$P(ddd) = 0.2$

$P(vvd, vdv, dvv, vdd, dvd, ddv) = 0.1$

a. $P(N_v = 2) = 0.1 + 0.1 + 0.1 = 0.3$

b. $P(N_v \geq 1) = 0.1 \times 6 + 0.2 = 0.8$

c. $P(vvd \mid N_v = 2) = \dfrac{P(N_v = 2 \mid vvd)\, P(vvd)}{P(N_v = 2)} = \dfrac{1 \cdot 0.1}{0.3} = 0.333$

d. $P(ddv \mid N_v = 2) = \dfrac{P(N_v = 2 \mid ddv) \cdots}{\cdots} = 0$

e. $P(N_v = 2 \mid N_v \geq 1) = \dfrac{P(N_v \geq 1 \mid N_v = 2)\, P(N_v = 2)}{P(N_v \geq 1)} = \dfrac{1 \cdot 0.3}{0.8} = 0.375$

f. $P(N_v \geq 1 \mid N_v = 2) = \dfrac{P(N_v = 2 \mid N_v \geq 1)\, P(N_v \geq 1)}{P(N_v = 2)} = \dfrac{0.375 \cdot 0.8}{0.3} = 1$

3. From the covariance equation given in lecture

$$\sigma_{XY}^2 = r_{XY}^2 - \mu_X \mu_Y$$
$$= \underbrace{E[XY]}_{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy\, f(x,y)\, dx\, dy} - E[X]E[Y]$$

[When X and Y are independent $f(x,y) = f_X(x) f_Y(y)$]

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy\, f_X(x)\, f_Y(y)\, dx\, dy$$
$$= \left(\int_{-\infty}^{\infty} x f_X(x)\, dx \right) \left(\int_{-\infty}^{\infty} y f_Y(y)\, dy \right)$$
$$= E[X]\, E[Y]$$

$\therefore \sigma_{XY}^2 = E[X]E[Y] - E[X]E[Y] = 0$

$\because$ Proven

1 Appendix: MATLAB Code

```
%%
clc; clear all; close all;
A = [2 3; 3 4];
B = [4 2; 2 -4];
A*B

%% 1.8
clc; clear all; close all;
syms a b c L
A = [a b; b c];
det([a-L b; b c-L])
det([L-a b; b L-c])
aa = 1;
bb = -a-c;
cc = a*c - b^2;
(-bb + sqrt(bb^2 - 4*aa*cc))/(2*aa)

%% 1.17
clc; clear all; close all;

[T,X]=ode45(@fcneom,0:0.05:5,[0 0]);
figure()
plot(T, X(:, 1), '-o', 'LineWidth', 1)
hold on
plot(T, X(:, 2), '-o', 'LineWidth', 1)
legend("Angular Pos", "Angular Vel")
title("t = 0.05s")
hold off

[T,X]=ode45(@fcneom,0:0.2:5,[0 0]);
figure()
plot(T, X(:, 1), '-o', 'LineWidth', 1)
hold on
plot(T, X(:, 2), '-o', 'LineWidth', 1)
legend("Angular Pos", "Angular Vel")
title("t = 0.2s")
hold off

[T,X]=ode45(@fcneom,0:0.5:5,[0 0]);
figure()
plot(T, X(:, 1), '-o', 'LineWidth', 1)
hold on
plot(T, X(:, 2), '-o', 'LineWidth', 1)
legend("Angular Pos", "Angular Vel")
title("t = 0.5s")
hold off

%% 1.17 v2
clc; clear all; close all;

F = 100;
J = 10;
T = 10;
x1(1) = 0;
x2(1) = 0;
tspan = 0:0.05:5;
for i = 2:length(tspan)
    xdot = [0 1; 0 -F/J]*[x1(i-1); x2(i-1)] + [0 1/J]' * T;
    x1(i) = x1(i-1) + xdot(1)*0.05;
    x2(i) = x2(i-1) + xdot(2)*0.05;
end
figure()
```

```
plot(tspan, x1, '-o', 'LineWidth', 1)
hold on
plot(tspan, x2, '-o', 'LineWidth', 1)
legend("Angular Pos", "Angular Vel")
title("t = 0.05s")
hold off

x1 = [];
x2 = [];
x1(1) = 0;
x2(1) = 0;
tspan = 0:0.2:5;
for i = 2:length(tspan)
    xdot = [0 1; 0 -F/J]*[x1(i-1); x2(i-1)] + [0 1/J]' * T;
    x1(i) = x1(i-1) + xdot(1)*0.2;
    x2(i) = x2(i-1) + xdot(2)*0.2;
end
figure()
plot(tspan, x1, '-o', 'LineWidth', 1)
hold on
plot(tspan, x2, '-o', 'LineWidth', 1)
legend("Angular Pos", "Angular Vel")
title("t = 0.2s")
hold off

deltaT = 0.5;
x1 = [];
x2 = [];
x1(1) = 0;
x2(1) = 0;
tspan = 0:deltaT:5;
for i = 2:length(tspan)
    xdot = [0 1; 0 -F/J]*[x1(i-1); x2(i-1)] + [0 1/J]' * T;
    x1(i) = x1(i-1) + xdot(1)*deltaT;
    x2(i) = x2(i-1) + xdot(2)*deltaT;
end
figure()
plot(tspan, x1, '-o', 'LineWidth', 1)
hold on
plot(tspan, x2, '-o', 'LineWidth', 1)
legend("Angular Pos", "Angular Vel")
title("t = 0.5s")
hold off

%% 2.9
clc; clear all; close all;
x = linspace(-2, 2);
plot(x, x, 'o')
hold on
plot(x, x.^2, 'o')
grid on
axis on
legend('y = x', 'y = x^2')
hold off

%% q2.15
clc; clear all; close all;

x = rand(50, 1);
figure()
histogram(x, 10)
fprintf("N = 50\n = %f\n = %f\n\n", mean(x), std(x))

x = rand(500, 1);
figure()
```

```
histogram(x, 10)
fprintf("N = 500\n = %f\n = %f\n\n", mean(x), std(x))

x = rand(5000, 1);
figure()
histogram(x, 10)
fprintf("N = 5000\n = %f\n = %f\n\n", mean(x), std(x))

%% functions
function eom = fcneom(t, state)
    F = 100;
    J = 10;
    T = 10;
    xdot = [0 1; 0 -F/J]*state + [0 1/J]' * T;
    eom = xdot;
end
```