01-19-22 - HW 1

Wednesday, January 19, 2022 8:10 PM

1.1.6. $\begin{bmatrix} A & A \\ B & A \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix}$

$AA + AC = 0$ — ①

$BA + AC = I$ — ②

From ①:

$AC = -AA$

$C = -A^{-1}AA = \underline{-A}$

From ②:

$BA - AA = I$

$BA = I + AA$

$B = (I + AA)A^{-1} = \underline{A^{-1} + A}$

1.7. Let $A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 2 \\ 2 & -4 \end{bmatrix}$

$A \cdot B = \begin{bmatrix} 14 & -8 \\ 20 & -10 \end{bmatrix}$

$AB \neq (AB)^T$

∵ Proved by contradiction

1.8. $\det\left(\begin{bmatrix} a-\lambda & b \\ b & c-\lambda \end{bmatrix}\right) = (a-\lambda)(c-\lambda) - b^2 = ac - a\lambda - c\lambda + \lambda^2 - b^2$

$= \lambda^2 + \lambda(-a-c) + (ac - b^2) = 0$

Using quadratic formula:

$$\lambda = \frac{a + c \pm \sqrt{a^2 + 2ac + c^2 - 4ac + 4b^2}}{2}$$

eig(A) is real if $\underline{a^2 + c^2 + 4b^2} > 2ac$

b. If A is positive semidefinite, $\text{eig}(A) \geq 0$

$$\therefore \quad a+c \pm \sqrt{a^2+2ac+c^2-4ac+4b^2} \geq 0$$

$$a+c \geq \sqrt{a^2+2ac+c^2-4ac+4b^2}$$

$$\cancel{a^2}+2ac+\cancel{c^2} \geq \cancel{a^2}-2ac+\cancel{c^2}+4b^2$$

$$\cancel{4}ac \geq \cancel{4}b^2$$

$$ac \geq b^2$$

A is positive semi definite $\forall\, b$ s.t. $-\sqrt{ac} \leq b \leq \sqrt{ac}$

1.17. a. $x_1 = \theta \qquad u = T$

$x_2 = \dot{\theta}$

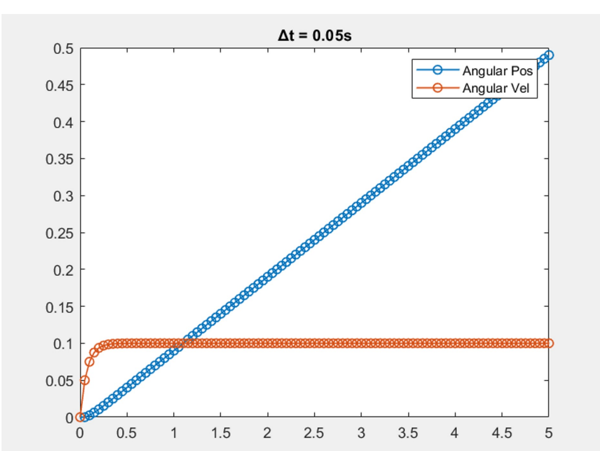
$$\dot{x}_1 = x_2$$

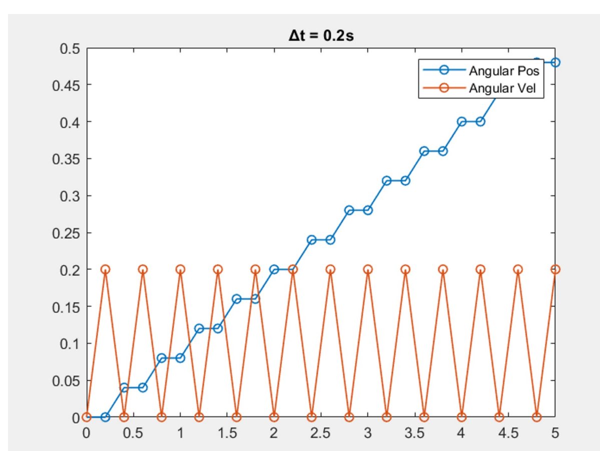
$$\dot{x}_2 = (T - Fx_2)/J$$

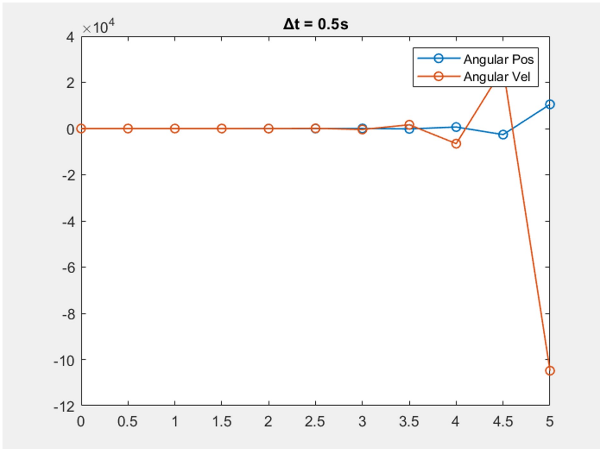
$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -F/J \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/J \end{bmatrix} u$$

b. * Simulating in MATLAB

Using rectangular integration, we get the following plots for $\Delta t = 0.05s$, $\Delta t = 0.2s$ and $\Delta t = 0.5s$ respectively







The output when $\Delta t = 0.05s$ looks correct. As the step size increases, the simulation becomes less accurate and eventually blows up.
The eigenvalues of the A matrix are 0 and -10. From this we can say that the minimum timestep for an accurate simulation (mean of eigenvalues)$^{-1}$ $= 1/5$ s

2.8. $\mathrm{Var}(W+V) = \mathrm{Var}\,W + \mathrm{Var}\,V + 2\,\mathrm{Cov}(W,V)$

$$\mathrm{Corr}(W,V) = \frac{\mathrm{Cov}(W,V)}{\sqrt{\mathrm{Var}\,W \cdot \mathrm{Var}\,V}} = 0 \quad \text{[Given uncorrelated]}$$

$$\mathrm{Cov}(W,V) = 0$$

$$\therefore \mathrm{Var}(W+V) = \mathrm{Var}\,W + \mathrm{Var}\,V$$

$$\underline{\sigma_x = \sqrt{\sigma_w^2 + \sigma_v^2}}$$

2.9. a. $$\rho = \frac{\sigma_{XY}^2}{[\sigma_X^2\,\sigma_Y^2]^{1/2}}$$

$$\sigma_{XY}^2 = r_{XY}^2 - \mu_X \mu_Y$$
$$= \underbrace{E[XY]}_{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\,dx\,dy} - E[X]E[Y]$$

[When X and Y are independent $f(x,y) = f_x(x) f_y(y)$]

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_x(x)\, f_y(y)\,dx\,dy$$

$$= \left(\int^{\infty} x f_x(x)\,dx\right)\left(\int^{\infty} y f_Y(y)\,dy\right)$$

$$= \left(\int_{-\infty}^{\infty} x f_X(x)\,dx\right)\left(\int_{-\infty}^{\infty} y f_Y(y)\,dy\right)$$

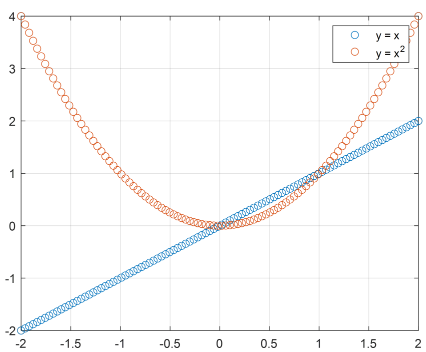
$$= E[X]\,E[Y]$$

$$\therefore \quad \sigma_{XY}^2 = E[X]E[Y] - E[X]E[Y] = 0$$

and $\rho = \dfrac{0}{[\sigma_X^2 \sigma_Y^2]^{1/2}} = 0$

$\therefore$ Proven

b. $Y = X^2$. Since Y directly depends on X, they are not independent but their correlation coefficient is 0.



c. Let $Y = aX + b$

$$\rho = \frac{E[XY] - E[X]E[Y]}{\sqrt{\text{Var}\,X \ \text{Var}\,Y}}$$

$$E[XY] = E[X(aX+b)] = E[aX^2 + bX] = aE[X^2] + bE[X]$$

$$E[Y] = E[aX+b] = aE[X] + b$$

$$\text{Var}\,X = E[X^2] - E[X]^2$$

$$\text{Var}\,Y = E[Y^2] - E[Y]^2$$

$$= E[a^2X^2 + 2abX + b^2] - (aE[X]+b)^2$$

$$= a^2E[X^2] + \cancel{2abE[X]} + \cancel{b^2} - (a^2E[X]^2 + \cancel{2abE[X]} + \cancel{b^2})$$

$$= a^2(E[X^2] - E[X]^2)$$

$$= a^2 E[X^2] + 2ab\cancel{E[X]} + \cancel{b^2} - (a^2 E[X]^2 + \cancel{2abE[X]} + \cancel{b^2})$$

$$= a^2 (E[X^2] - E[X]^2)$$

$$\rho = \frac{aE[X^2] + \cancel{bE[X]} - (aE[X] + \cancel{b})E[X]}{\sqrt{a^2 (E[X^2] - E[X]^2)^2}}$$

$$= \frac{a(E[X^2] - E[X])}{\pm a(E[X^2] - E[X])}$$

$$\rho = \pm 1$$

$\therefore$ Proven

2.10. $$\int_0^\infty \underbrace{\int_0^\infty ae^{-2x} e^{-3y} \, dx\, dy}_{\left[\frac{-ae^{-2x-3y}}{2}\right]_0^\infty} = 1$$

$$\int_0^\infty \frac{ae^{-3y}}{2} dy = \left[\frac{-ae^{-3y}}{6}\right]_0^\infty$$

$$= \frac{a}{6} = 1$$

$$a = 6$$

b $$\bar{x} = \int_0^\infty x f_X(x)\, dx$$

$$f_X(x) = \int_0^\infty 6e^{-2x} e^{-3y} \, dy = 2e^{-2x}$$

$$\bar{x} = \int_0^\infty 2xe^{-2x} dx = 1/2$$

$$\bar{y} = \int_0^\infty y f_Y(y) dy$$

$$f_Y(y) = \int_0^\infty 6e^{-2x} e^{-3y} \, dx = 3e^{-3y}$$

$$\bar{y} = \int_0^\infty 3ye^{-3y} dy = 1/3$$

c. $$E[X^2] = \int_0^\infty x^2 f_X(x)\, dx = \int_0^\infty x^2 (2e^{-2x})\, dx = 1/2$$

c. $E[X^2] = \int_0^\infty x^2 f_X(x)\,dx = \int_0^\infty x^2 \left(2e^{-2x}\right) dx = \frac{1}{2}$

$E[Y^2] = \int_0^\infty y^2 f_Y(y)\,dy = \int_0^\infty y^2 \left(3e^{-3y}\right) dy = \frac{2}{9}$

$$E[XY] = \int_0^\infty \int_0^\infty xy\left(6e^{-2x}e^{-3y}\right) dx\,dy$$

$$= \int_0^\infty \frac{3}{2} y e^{-3y}\,dy$$

$$= \frac{1}{6}$$

d. $r_{XY}^2 = \begin{bmatrix} E[X^2] & E[XY] \\ E[XY] & E[Y^2] \end{bmatrix}$

$r_{XY}^2 = \begin{bmatrix} 1/2 & 1/6 \\ 1/6 & 2/9 \end{bmatrix}$

e. $\sigma_X^2 = E[X^2] - E[X]^2$

$= 1/2 - 1/4 = 1/4$

$\sigma_Y^2 = E[Y^2] - E[Y]^2$

$= 2/9 - 1/9 = 1/9$

$C_{XY} = E[XY] - E[X]E[Y]$

$= \frac{1}{6} - \frac{1}{2}\cdot\frac{1}{3} = 0$

f. $C = \begin{bmatrix} 1/4 & 0 \\ 0 & 1/9 \end{bmatrix}$

g. $\rho = \frac{C_{XY}}{\sqrt{\mathrm{Var}X\,\mathrm{Var}Y}} = \frac{0}{\sqrt{\ldots}} = 0$

2.15.
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N = 50
μ = 0.496384
σ = 0.284838
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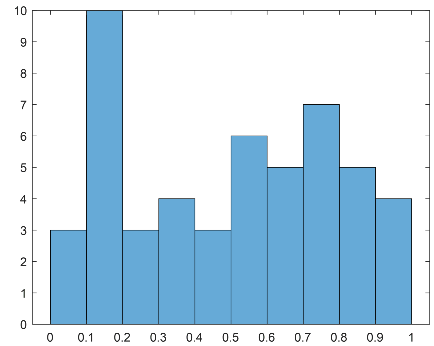
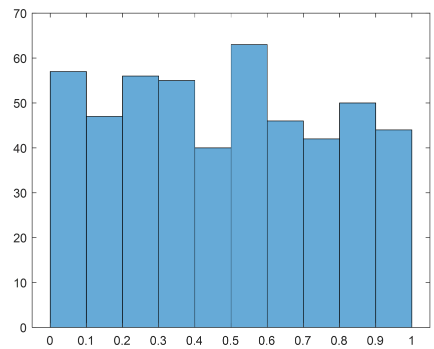
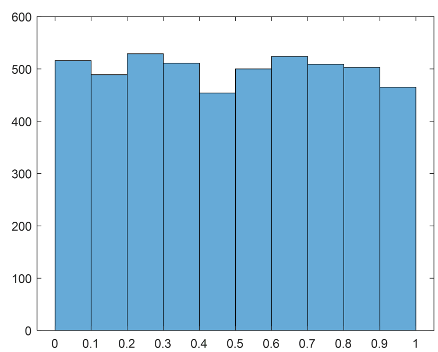


N = 500
μ = 0.482591
σ = 0.286209



N = 5000
μ = 0.496992
σ = 0.288375



Consistent with the formula for the mean and std dev of the Unif(0,1) distribution, I would expect

$$\mu = \frac{0+1}{2} = 0.5 \qquad \sigma = \sqrt{\frac{(0-1)^2}{12}} = 0.2887$$

As N increases, the mean and standard deviation of the sample get closer to the aforementioned values. The histogram starts to look more like a rectangle, i.e. 10 bins of equal height.

2. $P(vvv) = 0.2$

$P(ddd) = 0.2$

$P(vvd, vdv, dvv, vdd, dvd, ddv) = 0.1$

a. $P(N_v = 2) = 0.1 + 0.1 + 0.1 = 0.3$

b. $P(N_v \geq 1) = 0.1 \times 6 + 0.2 = 0.8$

c. $P(vvd \mid N_v = 2) = \dfrac{P(N_v = 2 \mid vvd)\, P(vvd)}{P(N_v = 2)} = \dfrac{1 \cdot 0.1}{0.3} = 0.333$

d. $P(ddv \mid N_v = 2) = \dfrac{P(N_v = 2 \mid ddv) \cdots}{\cdots} = 0$

e. $P(N_v = 2 \mid N_v \geq 1) = \dfrac{P(N_v \geq 1 \mid N_v = 2)\, P(N_v = 2)}{P(N_v \geq 1)} = \dfrac{1 \cdot 0.3}{0.8} = 0.375$

f. $P(N_v \geq 1 \mid N_v = 2) = \dfrac{P(N_v = 2 \mid N_v \geq 1)\, P(N_v \geq 1)}{P(N_v = 2)} = \dfrac{0.375 \cdot 0.8}{0.3} = 1$

3. From the covariance equation given in lecture

$$\sigma_{XY}^2 = r_{XY}^2 - \mu_X \mu_Y$$

$$= \underbrace{E[XY]}_{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\, dx\, dy} - E[X]E[Y]$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\, dx\, dy$$ [When X and Y are independent $f(x,y) = f_X(x) f_Y(y)$]

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_X(x)\, f_Y(y)\, dx\, dy$$

$$= \left(\int_{-\infty}^{\infty} x f_X(x)\, dx\right)\left(\int_{-\infty}^{\infty} y f_Y(y)\, dy\right)$$

$$= E[X]\, E[Y]$$

$\therefore \sigma_{XY}^2 = E[X]E[Y] - E[X]E[Y] = 0$

$\because$ Proven