Condensed intuitions for Lebesgue integration

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1 Motivation

Consider the Riemann integral
$$\int_0^1 \frac{1}{\sqrt{x}}\, dx.$$
Recall that Riemann integrals are defined where the upper Riemann integral and lower Riemann integral (that is, the infimum over "overestimates" and supremum over "underestimates" defined by a partitioning of the interval of integration) coincide. But, no matter how we partition $[0,1]$, the function over the first interval will always have no finite supremum. This contradicts the intuition that we should have

$$\int_0^1 \frac{1}{\sqrt{x}}\, dx = \lim_{a\to 0^+} \int_a^1 \frac{1}{\sqrt{x}}\, dx = 2.$$

Consider also integrating the function

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}; \\ 0, & \text{otherwise.} \end{cases}$$

over $[0,1]$. The infimum over any subinterval is 0, while the supremum is 1. The overarching problem is that this partitioning method is too restrictive. We would ideally like to define the integral over arbitary disjoint partition of "nice sets", where we have a notion of "measure" on these sets. Then, for the last example, we could have
$$[0,1] = \{x \in [0,1] : x \in \mathbb{Q}\} \cap \{x \in [0,1] : x \notin \mathbb{Q}\},$$
where the left side of the intersection should have measure 0, making the integral 0. Now, we just have to define these sets and this measure!

2 Constructing a measure

Let X be an abstract space, and m be a collection of subsets of X. We only ask that a measure satisfy two properties, namely that the measure of the empty set is 0, and the measure of a countable disjoint union is the sum of the measures of each set in the union.

Definition (Measure): $\mu : (X, m) \to [0, \infty]$ is a *measure* if

(a) $\mu(\varnothing) = 0$

(b) $\mu\left(\bigcup_n E_n\right) = \sum_n \mu(E_n)$ for disjoint $E_1, E_2, \dots \in m$.

Beginning with some space X, we will construct this measure by first defining an *outer measure* that operates on the powerset of X, then defining the set m which will make countable additivity hold. But first, to even reason about measures, we need to ensure that the collection of subsets that are measured are well behaved:

Definition (σ-algebra): A collection of sets m is said to be a σ-algebra if

(a) $\varnothing \in m$.

(b) $E \in m \implies X \setminus E \in M$.

(c) $(E_n) \in m \implies \bigcup_n E_n \in m$.

Definition (Outer measure): Let $\mathcal{E} \subseteq \mathcal{P}(X)$ be a family of elementary sets. We require $\varnothing \in \mathcal{E}$ and $(X_n) \subset \mathcal{E}$, where $X = \bigcup_n X_n$. We also define a function $\rho : \mathcal{E} \to [0, \infty]$ with $\rho(\varnothing) = 0$. Then define the outer measure as:

$$\mu^*(E) = \inf \left\{ \sum_n \rho(E_n) : E_n \in \mathcal{E}, E \subseteq \bigcup_n E_n \right\}.$$

Essentially we are approximating any subset $E \subseteq X$ through countably many elementary "outer" sets for which a measure is easy to define. In particular, we construct the *Lebesgue outer measure* on $\mathbb{R}^n$ by choosing $\mathcal{E}$ to be the set of rectangles.

Finally, we will take the collection of subsets m to be the set of *μ^*-measurable sets*:

Definition (μ^*-measurable): A set $E \subseteq X$ is said to be *μ^*-measurable* if for all $F \subseteq X$,

$$\mu^*(F) = \mu^*(F \cap E) + \mu^*(F \setminus E).$$

We call this set m^*. Intuitively, E splits sets F "cleanly" into $F \cap E$ and $F \setminus E$.

That μ^* restricted to m^* is a measure and m^* is a *σ-algebra* is a result of Caratheodory's theorem.

3 Integration

Having defined measures and measurable sets, now we can reason about measure spaces (X, m, μ) and functions between these spaces. Measurable functions are mappings between measurable spaces that respect the σ-algebras. These are the functions we will integrate over, since we can show that measurable functions are the increasing limit of simple functions.

Definition (Measurable functions): Let X, Y be nonempty sets, m, n σ-algebras on X, Y respectively. Then, $f : X \to Y$ is said to be *measurable* if for all $F \in n$,

$$f^{-1}(F) \in m.$$

If $Y = \mathbb{R}^N$, n is assumed to be $\mathcal{B}(\mathbb{R}^N)$. If further $X = \mathbb{R}^d$, we classify measurable functions into two categories (where the latter is assumed when not supplied):

(a) *Lebesgue measurable*: $B \in \mathcal{B}(\mathbb{R}^N)$ implies $f^{-1}(B) \in m^*$.

(b) *Borel measurable*: $B \in \mathcal{B}(\mathbb{R}^N)$ implies $f^{-1}(B) \in \mathcal{B}(\mathbb{R}^d)$.

Now we can define the Lebesgue integral. As we wanted, this now considers arbitrary partitions of the domain X into measurable sets.

Definition (Lower Lebesgue sum): Let $P = \{A_1, \ldots, A_m\}$ be a partition of X with $A_i \in m$. The *lower Lebesgue sum* with respect to P is defined by

$$L(f, P) \coloneqq \sum_i^m \mu(A_i) \inf_{A_i} f.$$

Definition (Lebesgue integral):

$$\int_X f d\mu \coloneqq \sup_P L(f, P).$$

Now, we reap the rewards of the Lebesgue integral:

Theorem (Monotone Convergence Theorem): Let (X, m, μ) and consider a sequence of measurable functions

$$0 \leq f_1 \leq \cdots \leq f_n \to f$$

for some $f : X \to [0, \infty]$. Then,

$$\int_X f d\mu = \lim_{k\to\infty} \int_X f_k d\mu.$$

Theorem (Fatou's Lemma): Let (X, m, μ) and $f_n : X \to [0, \infty]$ measurable.

$$\int \liminf_n f_n d\mu \leq \liminf_n \int f_n d\mu.$$

Theorem (Dominated convergence theorem): Let (X, m, μ), $f_n \to \overline{\mathbb{R}}$ measurable functions with limit f μ a.e. Assume there exists an integrable function $g : X \to [0, \infty]$ such that $|f_n(x)| \leq g(x)$ μ a.e. Then f is Lebesgue integrable and

$$\int_X f_n d\mu \to \int_X f d\mu.$$