Math 720 - Measure and Integration

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we measure

Contents

1 Riemann Integration

We motivate the study of measures by the deficiencies of the Riemann integral. We are restricted to integrating over compact intervals and integrands must be bounded without too many discontinuities. Integral and limit cannot be interchanged. Rather than using intervals to approximate our function, we want to break the domain of integration into more general subsets. To do this, we must be able to "measure" these subsets.

1.1 Definition and properties

Definition (Lower and upper Riemann sums): Let $f : [a, b] \to \mathbb{R}$ be a bounded function and suppose P is a partition $x_0 = a, \ldots, x_n = b$ of $[a, b]$. Then, define the *lower and upper Riemann sums* as

$$\begin{aligned} L(f, P, [a, b]) &= \sum_{i=1}^{n} (x_i - x_{i-1}) \inf_{[x_{i-1}, x_i]} f \\ U(f, P, [a, b]) &= \sum_{i=1}^{n} (x_i - x_{i-1}) \sup_{[x_{i-1}, x_i]} f. \end{aligned}$$

Theorem: For any partitions P_1, P_2 of $[a, b]$,

$$L(f, P_1, [a, b]) \leq U(f, P_2, [a, b]).$$

Proof. Use the common refinement $P_1 \cup P_2$. □

Definition (Lower and upper Riemann integrals):

$$\begin{aligned} L(f, [a, b]) &= \sup_P L(f, P, [a, b]) \\ U(f, [a, b]) &= \inf_P U(f, P, [a, b]). \end{aligned}$$

Theorem: $L(f, [a, b]) \leq U(f, [a, b])$.

Proof. By definitions. □

Definition (Riemann integral): A bounded function f on $[a, b]$ is *Riemann integrable* if

$$L(f, [a, b]) = U(f, [a, b]).$$

We define $\int_a^b f(x)\,\mathrm{d}x$ as this quantity.

Theorem (Integrability criterion): f is Riemann integrable if and only if for every $\varepsilon > 0$ there exists a partition P such that

$$U(f, P, [a, b]) - L(f, P, [a, b]) < \varepsilon.$$

Proof. If f is integrable, simply use definitions of sup and inf and common refine to grab a partition. The other direction is true since upper sums bound lower sums and the gap with a partition upper bounds the gap on the integrals. $\square$

Theorem: Continuous functions are integrable.

Proof. Invoke uniform continuity on the compact interval to bound the delta within a fine enough subinterval. $\square$

1.2 Shortcomings of Riemann integral

The Riemann integral doesn't handle well...

Example (Functions with many discontinuities):

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \cap [0,1]; \\ 0, & \text{otherwise.} \end{cases}$$

This is not Riemann integrable, though we'd like this to have integral 0.

Example (Unbounded functions):

$$f(x) = \begin{cases} \frac{1}{\sqrt{x}}, & \text{if } x \in (0,1]; \\ 0, & \text{otherwise.} \end{cases}$$

This is not Riemann integrable, though we'd like this to have integral equal to $\lim_{a\to 0^+} \int_a^1 f = 2$.

Example (Limits of functions):

$$f_n(x) = \begin{cases} n, & \text{if } x \in [0,1/n]; \\ 0, & \text{otherwise.} \end{cases}$$

f_n converges pointwise to 0 but the limit of the integral is 1.

$$f_n(x) = \begin{cases} 1, & \text{if } x \in \{r_1, \dots, r_n\}; \\ 0, & \text{otherwise.} \end{cases}$$

Where $\{r_1, \dots, r_n\}$ denotes the first n rationals within $[0,1]$, f_n converges pointwise to $f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \cap [0,1]; \\ 0, & \text{otherwise.} \end{cases}$. The limit of the integral is well defined as 0 but f is not Riemann integrable.

2 Outer Measures

We motivate outer measures through some bare minimum properties we would like a measure to have: the measure of empty set is 0, monotonicity, and subadditivity. To do this we take elementary sets for which we can define measures (like rectangles) and define μ^* as the infimal sum of a covering by these sets.

A desire for additivity over disjoint sets has us restrict the power set to μ^*-measurable sets called m^*. Caratheodory later tells us that m^* is a σ-algebra and μ^* is indeed an additive complete measure on (X, m^*).

2.1 Definitions and examples

Definition (Outer measure): Let $X \neq \varnothing$. Then $\mu^* : \mathcal{P}(x) \to [0, \infty]$ is an *outer measure* if

(i) $\mu^*(\varnothing) = 0$.

(ii) If $E \subseteq F$, then $\mu^*(E) \leq \mu^*(F)$.

(iii) μ^* is subadditive, i.e. for sets (E_n),

$$\mu^*\left(\bigcup_n E_n\right) \leq \sum_n \mu^*(E_n).$$

Let X be a set. The goal of outer measures is to approximate any subset $E \subseteq X$ through countably many elementary "outer" sets for which a measure is easy to define. Let $\mathcal{E} \subseteq \mathcal{P}(X)$ be a family of elementary sets. We require $\varnothing \in \mathcal{E}$ and $(X_n) \subset \mathcal{E}$, where $X = \bigcup_n X_n$. We also define a function $\rho : \mathcal{E} \to [0, \infty]$ with $\rho(\varnothing) = 0$. This gives us an outer measure μ^*:

$$\mu^*(E) = \inf\left\{\sum_n \rho(E_n) : E_n \in \mathcal{E}, E \subseteq \bigcup_n E_n\right\}.$$

Theorem: μ^* is an outer measure

Proof.

(i) $\mu^*(\varnothing) \leq \rho(\varnothing) = 0$.

(ii) Let $E \subseteq F$. For any (F_n) such that $F \subseteq \bigcup_n F_n$, it follows that $E \subseteq \bigcup_n F_n$. Thus $\mu^*(E) \leq \sum_n \mu^*(F_n) \implies \mu^*(E) \leq \mu^*(F)$.

(iii) Consider family (E_n) and fix $\varepsilon > 0$. We want to show

$$\mu^*\left(\bigcup_n E_n\right) \leq \sum_n \mu^*(E_n) + \varepsilon.$$

Consider a fixed E_k. By infimum there exists some family of elementary sets $F_{k,i}$ such that $\mu^*(E_k) + \frac{\varepsilon}{2^k} > \sum_i \rho(F_{k,i})$. So

$$\begin{aligned}
\sum_n \mu^*(E_n) + \varepsilon &= \sum_n (\mu^*(E_n) + \frac{\varepsilon}{2^n}) \\
&> \sum_n \sum_i \rho(F_{n,i}) \\
&= \sum_{n,i} \rho(F_{n,i}). && \text{(Tonelli)}
\end{aligned}$$

Since $(F_{n,i})$ is a valid covering of $\bigcup_n E_n$, we are done.

□

Example (Lebesgue outer measure):

(a) $X = \mathbb{R}^N$.

(b) $\mathcal{E}$ is family of rectangles $R = I_1 \times \cdots \times I_N$.

(c) $\rho(E) = \ell(I_1) \times \cdots \times \ell(I_N)$.

We define the corresponding outer measure the *Lebesgue outer measure*, denoted $\mathcal{L}_0^N$.

Example (Lebesgue-Stiljes outer measure):

(a) $X = I$, where $I \subset \mathbb{R}$ is an open interval.

(b) $\mathcal{E} = \{(a, b) : a, b \in I, a \leq b\}$.

(c) $\rho((a, b)) = f(b) - f(a)$, where $f : I \to \mathbb{R}$ is increasing.

We define the corresponding outer measure the *Lebesgue-Stiljes outer measure generated by f*.

Example (Hausdorff outer measures):

(a) $X = \mathbb{R}^N$, $0 \leq s < \infty$, $0 < \delta \leq \infty$.

(b) $\mathcal{E}_\delta = \{E \subset \mathbb{R}^N : \operatorname{diam} E < \delta\}$.

(c) $\rho(E) = \alpha_s(\frac{\operatorname{diam} E}{2})^s$, where $\alpha_s = \frac{\pi^{s/2}}{\Gamma(\frac{s}{2}+1)}$.

The *Hausdorff outer measure* H_δ^s allows us to measure sets of lower dimension than $\mathbb{R}^N$. As an example, suppose $s = 1$ in $\mathbb{R}^2$. A curve will be measured as the infimal sum of diameters of "balls" with diameter $< \delta$. As we take δ to 0, the outer measure takes infimum over more fewer sets and thus monotonically increases. So we define

$$H^s(E) = \lim_{\delta \to 0^+} H_\delta^s(E).$$

Theorem (Easy properties on $\mathbb{R}$):

(i) Countable sets have outer measure 0.

(ii) Outer measure is translation invariant.

Proof.

(i) Assume our elementary sets are open intervals. Let (x_n) be a countable sequence. Fix $\varepsilon > 0$. Then, the union of $(x_n - \frac{\varepsilon}{2^n}, x_n + \frac{\varepsilon}{2^n})$ contains $\bigcup_n x_n$ and thus $|\bigcup_n x_n| \leq 2\varepsilon$. Because ε is arbitrary, we are done.

(ii) Translating intervals does not affect length.

□

Theorem (Outer measures on elementary sets): For any $E \in \mathcal{E}$, $\mu^*(E) = \rho(E)$ if and only if ρ is countably subadditive over $\mathcal{E}$.

Proof. Suppose ρ is subadditive over $\mathcal{E}$, i.e. for all $E \subseteq \bigcup_n E_n$,

$$\rho(E) \leq \sum_n \rho(E_n).$$

We know $\mu^*(E) \leq \rho(E)$ by definition of μ^*. It remains to show $\mu^*(E) \geq \rho(E)$, i.e. that for all families of elementary sets E_n such that $E \subseteq \bigcup_n E_n$, $\sum_n \rho(E_n) \geq \rho(E)$. This is exactly from subadditivity.

On the other hand, if ρ is not countably subadditive we have some set $E \subseteq \bigcup_n E_n$ with $\rho(E) > \sum_n \rho(E_n)$, and in this case $\mu^*(E) < \rho(E)$. □

Theorem (Nonadditivity of outer measure): There exist disjoint subsets A, B of $\mathbb{R}$ such that

$$|A \cup B| \neq |A| + |B|.$$

Proof. We construct the *Vitali set*. Let $X = [-1,1]$, $\mu^* = \mathcal{L}_0$. Define an equivalence relation such that $a \sim b$ if $a - b \in \mathbb{Q}$ such that $[a]$ is the equivalence class of a. Use the axiom of choice to construct a set $V \subseteq [-1,1]$ such that V contains one element from each equivalence class.

Order the rationals in $[-2,2]$ as $\{r_1, r_2, \dots\}$. It follows that $[-1,1] \subseteq \bigcup_k (r_k + V)$. By monotonicity and subadditivity,

$$2 = |[-1,1]| \leq \left| \bigcup_k (r_k + V) \right| \leq \sum_k |r_k + V|.$$

Translation invariance tells us that $\sum_k |V| \geq 2$, so $|V|$ has positive measure. But then since

$$\left| \bigcup_{k=1}^{n} |r_k + V| \right| \leq n|V|,$$

for large enough n the outer measure of the union of the n disjoint sets must be strictly less than the sum of their outer measures, since $\bigcup_k (r_k + V) \subseteq [-3,3]$. $\square$

Theorem ("Length" can't be extended to all subsets of $\mathbb{R}$): There does not exist a function $\mu : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ satisfying

(i) The measure of an open interval is its length.

(ii) Measure is translation invariant.

(iii) Measure is countably additive (over disjoint sets).

Proof. We show that the three given properties yield additionally monotonicity, subadditivity, and that $\mu(I) = \sup I - \inf I$. If so, by the previous example, additivity over disjoint sets will fail.

(a) Monotonicity: (i) gives $\mu(\varnothing) = 0$. Suppose $A \subseteq B$. Then $B = A \cup (B \setminus A)$. So by (iii), $\mu(A \cup B) = \mu(A) + \mu(B \setminus A) \geq \mu(A)$.

(b) Subadditivity: Consider a family of sets A_n. We make them disjoint: write $A_1' = A_1, A_2' = A_2 \setminus A_1, A_k' = A_k \setminus (\bigcup_j A_j)$. Then,

$$\bigcup_n A_n = \bigcup_n A_n' = \sum_n A_n' \leq \sum_n A_n.$$

(c) Length of closed interval: for all $\varepsilon > 0$,

$$(a,b) \subset [a,b] \subset (a - \varepsilon, a + \varepsilon).$$

So by monotonicity, since this holds for all $\varepsilon > 0$, the length of a closed interval is $b - a$.

□

Thus this leads us to restrict μ^* to a smaller family of sets–μ^*-measurable sets–to allow these properties to hold.

2.2 μ^*-measurable sets

Suppose $X \neq \varnothing$ is associated with an outer measure $\mu^* : \mathcal{P}(X) \to [0, \infty]$.

Definition (μ^*-measurable): A set $E \subseteq X$ is said to be μ^*-*measurable* if for all $F \subseteq X$,

$$\mu^*(F) = \mu^*(F \cap E) + \mu^*(F \setminus E).$$

Remark: If E is μ^*-measurable, then if $A \cap E = \varnothing$, then

$$\mu^*(A \cup E) = \mu^*(A) + \mu^*(E),$$

since

$$\mu^*(A \cup E) = \mu^*((A \cup E) \cap E) + \mu^*((A \cup E) \setminus E).$$

So μ^*-measurable sets are additive. We will also see they form a σ-algebra satisfying our three desired properties.

Definition (m^*): Define m^* as the set of μ^*-measurable sets of X. We will show later with Caratheodory that m^* is a σ-algebra and μ^* is additive over m^*.

Definition (σ-algebra): A collection of sets m is said to be a σ-algebra if

(a) $\varnothing \in m$.

(b) $E \in m \implies X \setminus E \in M$.

(c) $(E_n) \in m \implies \bigcup_n E_n \in m$.

Definition (Regular outer measure): An outer measure μ^* is said to be *regular* if for all $F \subseteq X$, there exists $E \in m^*$ such that $F \subseteq E$, and $\mu^*(F) = \mu^*(E)$. So we can enlarge any arbitrary set to make it measurable.

Definition (Measurable space): A pair (X, m) is called a *measurable space* if m is a σ-algebra on X. Then sets $E \subseteq m$ are called m-measurable.

Theorem (Basic properties): Let (X, m) be a measurable space.

(a) $X \in m$

(b) $D, E \in m \implies D \cup E \in m, D \cap E \in m, D \setminus E \in m$.

(c) $E_n \in m \implies \bigcap_n E_n \in m$.

Proof.

(a) $\varnothing \in m \implies X \setminus \varnothing = X \in m$.

(b) $D \cup E$ is the union of $D, E, \varnothing, \ldots$. $D \cap E$ is the complement of the union of $X \setminus D, X \setminus E, \varnothing, \ldots$ $D \setminus E = D \cap (X \setminus E)$.

(c) $\bigcap_n E_n$ is the complement of the union $\bigcup_n X \setminus E_n$.

□

Example (Examples of σ-algebras):

(a) $\{\varnothing, X\}$.

(b) $\mathcal{P}(X)$

(c) $m := \{E \subseteq X : E \text{ countable or } X \setminus E \text{ countable}\}$.

(d) In $\mathbb{R}^N$, with outer measure $\mathcal{L}_0^N$, the corresponding m^* (a σ-algebra by Caratheodory) of Lebesgue measurable sets is defined

$$\mathcal{L}^N(E) := \mathcal{L}_0^N(E).$$

(e) Let X be a topological space. Then $\mathcal{B}(X)$ is the smallest σ-algebra containing all the open subsets of X (but we must show this exists, below). Then $\mathcal{B}(X)$ contains all the closed sets as well.

Theorem (Smallest σ-algebra containing $\mathcal{A}$): Let X be a set and $\mathcal{A} \subseteq \mathcal{P}(X)$. The intersection of all σ-algebras containing $\mathcal{A}$ is a σ-algebra.

Proof. It is not an empty intersection, since $A \subseteq \mathcal{P}(X)$. Let m_A be the intersection. Then $A \subseteq m_A$. For the three properties of σ-algebras:

(a) Since all σ-algebras contain $\varnothing$, m_A contains $\varnothing$.

(b) If $E \in m_A$, E is in all σ-algebras and thus $X \setminus E$ is in all σ-algebras, so $X \setminus E \in m_A$.

(c) Similar to (b).

□

3 Measurable functions

Measurable functions are mappings between measurable spaces that respect the σ-algebras. These are the functions we will integrate over, since as we show later measurable functions are the increasing limit of simple functions.

We discuss the special cases of Lebesgue and Borel measurable functions when $f : \mathbb{R}^N \to \mathbb{R}^N$. We show key operations on measurable functions and sequences of measurable functions preserve measurability.

3.1 Definitions and basic properties

Definition (Measurable functions): Let X, Y be nonempty sets, m, n σ-algebras on X, Y respectively. Then, $f : X \to Y$ is said to be *measurable* if for all $F \in n$,

$$f^{-1}(F) \in m.$$

If $Y = \mathbb{R}^N$, n is assumed to be $\mathcal{B}(\mathbb{R}^N)$. If further $X = \mathbb{R}^d$, we classify measurable functions into two categories (where the latter is assumed when not supplied):

(a) *Lebesgue measurable*: $B \in \mathcal{B}(\mathbb{R}^N)$ implies $f^{-1}(B) \in m^*$.

(b) *Borel measurable*: $B \in \mathcal{B}(\mathbb{R}^N)$ implies $f^{-1}(B) \in \mathcal{B}(\mathbb{R}^d)$.

Proposition (Generating sets): Suppose (X, m) and (Y, n). Suppose n is the smallest σ-algebra containing a family $\mathcal{F} \subseteq \mathcal{P}(Y)$. Then $f : X \to Y$ is measurable if and only if for all $F \in \mathcal{F}$, $f^{-1}(F) \in m$.

Proof. Suppose f is measurable. So the inverse images of all measurable sets in Y (including $\mathcal{F}$) is measurable.

Suppose for $F \in \mathcal{F}$, $f^{-1}(F) \in m$. Let $Z = \{F \in n : f^{-1}(F) \in m\}$. First, Z is a σ-algebra:

(a) The inverse image of the empty set is the empty set.

(b) Suppose $F \in Z$. So $f^{-1}(F) \in m$. Then $(f^{-1}(F))^c = f^{-1}(F^c) \in m$, so $F^c \in Z$.

(c) Suppose $F_1, F_2, \cdots \in Z$. So $f^{-1}(F_1), f^{-1}(F_2), \cdots \in m \implies \bigcup_n f^{-1}(F_n) = f^{-1}(\bigcup_n F_n) \in m$, so $\bigcup_n F_n \in m$.

By hypothesis $\mathcal{F} \subseteq Z$. Since Z is a σ-algebra, Z contains the smallest σ-algebra generated by $\mathcal{F}$. Thus Z contains n. □

Theorem (Condition for $\mathbb{R}$-valued measurable functions): A function $f : (X, m) \to \mathbb{R}$ is measurable if for all $a \in \mathbb{R}$,

$$f^{-1}((a, \infty)) \in m.$$

Proof. It suffices to show $\mathcal{B}(\mathbb{R})$ is the smallest σ-algebra that contains $\{(a, \infty) : a \in \mathbb{R}\}$, as then the inverse image of all Borel sets are in m by the previous proposition. We show smallest σ-algebra containing $\{(a, \infty) : a \in \mathbb{R}\}$ must contain all open sets, and thus is equivalent to $\mathcal{B}(\mathbb{R})$.

With set minus we can obtain all half open intervals $(a, b]$, which allows $(a, b) = \bigcup_n (a, b - 1/n]$. With complement we obtain all $(-\infty, a]$, which allows $(-\infty, b) = \bigcup_n (-\infty, a - 1/n]$. Since all open sets in $\mathbb{R}$ can be expressed as a countable disjoint union of open intervals (take maximal open interval around each point and union), we are done.

□

Example (Trivial cases for measurability):

(a) Let $m = \{\varnothing, X\}$. $f : X \to \mathbb{R}$ is measurable if and only if f is constant.

(b) Let $m = \mathcal{P}(X)$. Then any f is measurable.

Proposition (Continuity): Let (X, Z) and (Y, Z') be topological spaces. Then if $f : X \to Y$ continuous, f is measurable.

Proof. Since $\mathcal{B}(Z')$ is generated by the open subsets of Y, it suffices to show that the inverse image of open subsets of Y are open in X. This comes directly from continuity. □

Proposition (Increasing): If $f : \mathbb{R} \to \mathbb{R}$ increasing, then f is measurable.

Proof. WTS $f^{-1}((a, \infty)) \in \mathcal{B}(\mathbb{R})$ for all a. Let $b = \inf f^{-1}((a, \infty))$. Then either

$$f^{-1}((a, \infty)) = (b, \infty) \text{ or } f^{-1}((a, \infty)) = [b, \infty),$$

which in either case is in $\mathcal{B}(\mathbb{R})$. □

Example: Let (X, m) be a measurable space. Let $E \subseteq X$, and let $\chi_E : X \to \{0, 1\}$ be the characteristic function of E.

$$\chi_E^{-1}(B) = \begin{cases} X, & \text{if } 0, 1 \in B; \\ E, & \text{if } 1 \in B, 0 \notin B; \\ X \setminus E, & \text{if } 0 \in B, 1 \notin B; \\ \varnothing, & \text{otherwise.} \end{cases}$$

Thus χ_E is measurable if and only if E is measurable.

Proposition (Composition): Let (X, m), (Y, n), (Z, d) measure spaces, and $f : X \to Y, g : Y \to Z$ both measurable. Then $g \circ f : X \to Z$ is measurable.

Proof. Let $F \in d$. Then $(g \circ f)^{-1}(f) = f^{-1}(g^{-1}(F))$. Apply measurability of g and f. □

Example (Caution with composition): Let $f : (X, \mathcal{B}(X)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be a Borel measurable function. Let $g : \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable. This does not imply that $g \circ f : X \to \mathbb{R}$ is measurable. Suppose $F \subseteq \mathbb{R}$ is Borel, then

$$(g \circ f)^{-1}(F) = f^{-1}(g^{-1}(F)).$$

We know $g^{-1}(F)$ is Lebesgue measurable, however we have no guarantee that it is a Borel set, and Borel measurability of f gives us no guarantees on the inverse image of Lebesgue measurable sets. But if g is Borel, then $g \circ f$ is measurable.

Definition (Borel in extended reals): We define $B \subseteq [-\infty, \infty]$ as Borel if $B \cap \mathbb{R}$ is Borel. It follows easily that $f : (X, m) \to [-\infty, \infty]$ is measurable if $f^{-1}((a, \infty]) \in m$ for all $a \in \mathbb{R}$.

Corollary: Let (X, m) be a measurable space. If $f : X \to \overline{\mathbb{R}}$, then

$$f^2, |f|, f^+, f^-, \lambda f \text{ for } \lambda \in \mathbb{R}.$$

are measurable by showing the preimage of $(a, \infty]$ under the new function is equal to the preimage of some set under f.

Definition (Product σ-algebra): Let $(X, m), (Y, n)$ be measurable spaces. Then $m \otimes n \subseteq \mathcal{P}(X \times Y)$ is the smallest σ-algebra that contains $m \times n$.

Proposition (Measurability of product σ-algebra functions): Let (X, m), (Y_1, n_1), $\dots, (Y_n, n_n)$. Then $f : X \to Y_1 \times \cdots \times Y_n$ is measurable if and only if $\pi_i \circ f : X \to Y_i$ with

$$\pi_i : (y_1, \dots, y_i, \dots, y_n) \mapsto y_i$$

is measurable for all $i = 1, \dots, n$.

Proof. Observe π_i is measurable since for all $F \in n_i$, $\pi_i^{-1}(F) = Y_1 \times \cdots \times F \times \cdots \times Y_n \in n_1 \otimes \cdots \otimes n_n$. So if f is measurable, $\pi_i \circ f$ is a measurable composition. Assume $\pi_i \circ f$ is measurable for all i. Let $F_1 \times \cdots \times F_n \in n_1 \otimes \cdots \otimes n_n$. Then,

$$f^{-1}(F_1 \times \cdots \times F_n) = (\pi_1 \circ f)^{-1}(F_1) \cap \cdots \cap (\pi_n \circ f)^{-1}(F_n) \in m,$$

since each $(\pi_i \circ f)^{-1}(F_i)$ is in m by measurability of $\pi_i \circ f$. $\square$

Proposition (Basic closure properties of measurable functions): Let (X, m) be a measure space where $f, g : X \to \mathbb{R}$ are measurable. Then,

$$f + g, fg, \min\{f, g\}, \max\{f, g\}$$

are measurable.

Proof. By previous proposition, the function $f : (X, m) \to (\mathbb{R}^2, \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}))$ defined by $x \mapsto (f(x), g(x))$ is measurable. Consider the continuous and thus measurable mappings $\mathbb{R}^2 \to \mathbb{R}$: $(s, t) \mapsto s + t$, $(s, t) \mapsto st$, $(s, t) \mapsto \min\{s, t\}$, $(s, t) \mapsto \max\{s, t\}$. These mappings composed with f are measurable. $\square$

Proposition (Agreeing almost everywhere with measurable function): Let $(X, m), (Y, n)$ be measurable spaces with $f, g : X \to Y$ and f measurable. Assume $\mu^* : m \to [0, \infty]$ is a complete outer measure. If $f = g$ *almost everywhere*, then g is also measurable.

Proof. Let $F \in n$. Suppose $f = g$ except for on E with $\mu(E) = 0$. By completeness,

$$\begin{aligned} g^{-1}(F) &= \{x \in X \setminus E : g(x) \in F\} \cup \{x \in E : g(x) \in F\} \\ &= (f^{-1}(F) \setminus E) \cup (g^{-1}(F) \cap E) \in m. \end{aligned}$$

□

3.2 Sequences of measurable functions

Proposition (Sup, inf, limsup, liminf): Let (X, m) with measurable functions $f_n : X \to \overline{\mathbb{R}}$.

$$\sup_n f_n, \inf_n f_n, \limsup_n f_n, \liminf_n f_n$$

are measurable.

Proof.

$$(\sup_n f_n)^{-1}((a, \infty]) = \{x \in X : \sup_n f_n(x) > a\} = \bigcup_n f_n^{-1}((a, \infty]) \in m$$

$$(\inf_n f_n)^{-1}([-\infty, a)) = \{x \in X : \inf_n f_n(x) < a\} = \bigcup_n f_n^{-1}([-\infty, a)) \in m.$$

It follows that $\limsup_n f_n = \inf_n(\sup_{k \geq n} f_k)$ and $\liminf_n f_n = \sup_n(\inf_{k \geq n} f_k)$ are measurable. □

Corollary: The pointwise limit of measurable functions is measurable.

Proposition (Set of limit points): Let (X, m) with measurable functions $f_n : X \to \overline{\mathbb{R}}$.

$$\{x \in X : \lim_{n \to \infty} f_n(x) \text{ exists in } \mathbb{R}\} \in m.$$

Proof. Using Cauchy's criterion for sequence convergence, the set of limit points is equal to

$$\bigcap_k \bigcup_N \bigcap_{n,m \geq N} \{x \in X : |f_n(x) - f_m(x)| < 1/k\},$$

which is measurable since the inner set is preimage of an open set under measurable function $|f_n(x) - f_m(x)|$. Alternatively, using lim sup and lim inf,

$$\{x \in X : \limsup_n f_n(x) = \liminf_n f_n(x)\} = (\limsup_n f_n - \liminf_n f_n)^{-1}(\{0\}).$$

□

4 Measures and σ-algebras

Measures on a measurable space only need satisfy that the measure of an empty set is 0 and that measures of disjoint sets add. We show measures work well with infinite unions and intersections. We prove Caratheodory, which tells us that μ^* is a complete measure on (X, m^*).

We discuss the special cases of the Borel σ-algebra, which is generated by the open sets, and the Lebesgue σ-algebra, which is generated from the Lebesgue outer measure. In particular we show Borel sets are Lebesgue measurable. Moreover there exists a non Lebesgue measurable set and a set which is Lebesgue measurable but not Borel.

4.1 Definitions and basic properties

Definition (Measure): $\mu : (X, m) \to [0, \infty]$ is a *measure* if

(a) $\mu(\varnothing) = 0$

(b) $\mu\left(\bigcup_n E_n\right) = \sum_n \mu(E_n)$ for disjoint $E_1, E_2, \dots \in m$.

We then call the triple (X, m, μ) a *measure space.*

Example:

(a) Counting measure in $\mathbb{R}$:

$$\mu(E) = \begin{cases} \#E, & \text{if } E \text{ finite;} \\ \infty, & \text{otherwise.} \end{cases}$$

(b) Dirac measure for (X, m), $x_o \in X$ defined by

$$\delta_{x_o} : m \to \mathbb{R}, E \mapsto \begin{cases} 1, & \text{if } x_o \in E; \\ 0, & \text{otherwise.} \end{cases}$$

For the below propositions, assume a measure space (X, m, μ).

Proposition: Let $D, E \in m$, $D \subseteq E$.

(a) $\mu(D) \leq \mu(E)$.

(b) $\mu(E \setminus D) = \mu(E) - \mu(D)$ if $\mu(D) < \infty$.

Proof. $E = D \cup (E \setminus D)$, then $\mu(E) = \mu(D) + \mu(E \setminus D) \geq \mu(D)$. □

Proposition (Subadditivity): Let $E_n \in m$.

$$\mu\left(\bigcup_n E_n\right) \leq \sum_n \mu(E_n).$$

Proof. Let $E_1' = E_1, E_2' = E_2 \setminus E_1$, etc. Since these are disjoint,

$$\mu\left(\bigcup_n E_n\right) = \mu\left(\bigcup_n E_n'\right) = \sum_n \mu(E_n') \le \sum_n \mu(E_n).$$

$\square$

Proposition (Measure of increasing set union is limit): Let $E_n \in m$ be an increasing sequence of sets. Then

$$\mu\left(\bigcup_n E_n\right) = \lim_{n\to\infty} \mu(E_n).$$

Proof. WLOG, assume all E_n have finite measure. Write $\bigcup_n E_n = \bigcup_n (E_n \setminus E_{n-1})$. Then

$$\begin{aligned}
\mu\left(\bigcup_n E_n\right) &= \mu\left(\bigcup_n (E_n \setminus E_{n-1})\right) \\
&= \sum_n \mu(E_n \setminus E_{n-1}) \\
&= \sum_n (\mu(E_n) - \mu(E_{n-1})) \\
&= \lim_{n\to\infty} \mu(E_n).
\end{aligned}$$

$\square$

Proposition (Measure of decreasing set intersection is limit): Let $E_n \in m$ be a decreasing sequence of sets with $\mu(E_1) < \infty$.

$$\mu\left(\bigcap_n E_n\right) = \lim_{n\to\infty} \mu(E_n).$$

Proof. Since E_n is decreasing, $E_1 \setminus E_n$ is increasing, and we can use the previous proposition.

$$\mu\left(E_1 \setminus \bigcap_n E_n\right) = \mu\left(\bigcup_n (E_1 \setminus E_n)\right) = \lim_{n\to\infty} \mu(E_1 - E_n).$$

Thus

$$\mu(E_1) - \mu\left(\bigcap_n E_n\right) = \lim_{n\to\infty} \mu(E_1 - E_n) \implies \mu\left(\bigcap_n E_n\right) = \lim_{n\to\infty} \mu(E_n).$$

$\square$

Proposition (Inclusion exclusion): Let $D, E \in m$, $\mu(D \cap E) < \infty$. Then,

$$\mu(D \cup E) = \mu(D) + \mu(E) - \mu(D \cap E).$$

Proof. Write $D \cup E = (D \setminus (D \cap E)) \cup (E \setminus (D \cap E)) \cup (D \cap E)$. These are disjoint, so using additivity and set minus rules we arrive at the desired equation. $\square$

4.2 Caratheodory's Theorem

Definition (Complete measure): Given measure space (X, m, μ), we call μ a *complete measure* if for all sets $E \in m$ with $\mu(E) = 0$, it follows that every $F \subseteq E$ belongs to m.

Theorem (Caratheodory's Theorem): Let $X \neq \varnothing$ and μ^* be an outer measure on X.

(a) m^* is a σ-algebra.

(b) μ^* is a complete measure on (X, m^*).

Proof of (a). We know $\mu^*(\varnothing) = 0$. Let $F \subseteq X$. Then, $\mu^*(F \cap \varnothing) + \mu^*(F \setminus \varnothing) = 0 + \mu^*(F)$, so $\varnothing \in m^*$. Let $E \in m^*$. Fix $F \subseteq X$. Then, $X \setminus E \in m^*$ since

$$\mu^*(F \cap (X \setminus E)) + \mu^*(F \setminus (X \setminus E)) = \mu^*(F \setminus E) + \mu^*(F \cap E) = \mu^*(F),$$

We show m^* is closed under countable union by first showing closure and additivity of finite disjoint unions, using these two properties to extend closure to countable disjoint union, then extending it to non-disjoint sets.

(a) Let $E_1, E_2 \in m^*$. We claim $E_1 \cup E_2 \in m^*$. Fix $F \subseteq X$. We want to show that
$$\mu^*(F) \geq \mu^*(F \cap (E_1 \cup E_2)) + \mu^*(F \setminus (E_1 \cup E_2)).$$
Assume WLOG $\mu^*(F) < \infty$. Applying measurability of E_1 then E_2, then subadditivity,
$$\begin{aligned}
\mu^*(F) &\geq \mu^*(F \cap E_1) + \mu^*(F \setminus E_1) \\
&\geq \mu^*(F \cap E_1) + \mu^*((F \setminus E_1) \cap E_2) + \mu^*((F \setminus E_1) \setminus E_2) \\
&\geq \mu^*((F \cap E_1) \cup ((F \setminus E_1) \cap E_2)) + \mu^*((F \setminus E_1) \setminus E_2) \\
&= \mu^*(F \cap (E_1 \cup E_2)) + \mu^*(F \setminus (E_1 \cup E_2)).
\end{aligned}$$
Now by induction we have closure under finite union.

(b) Let $E_1, E_2 \in m^*$ disjoint. Let $F \subseteq X$. Then,
$$\begin{aligned}
\mu^*(F \cap (E_1 \cup E_2)) &= \mu^*((F \cap (E_1 \cup E_2)) \cap E_1) + \mu^*((F \cap (E_1 \cup E_2)) \setminus E_1) \\
&= \mu^*(F \cap E_1) + \mu^*(F \cap E_2).
\end{aligned}$$
Taking $F = X$, we have additivity of E_1, E_2 and thus finite additivity of disjoint sets by induction.

(c) Let $E_n \in m^*$ mutually disjoint. Let $F \subseteq X$. Then by finite disjoint closure and additivity,
$$\begin{aligned}
\mu^*(F) &= \mu^*(F \cap (\bigcup_{n=1}^{m} E_n)) + \mu^*(F \setminus (\bigcup_{n=1}^{m} E_n)) \\
&= \sum_{n=1}^{m} \mu^*(F \cap E_n) + \mu^*(F \setminus (\bigcup_{n=1}^{m} E_n)) \\
&\geq \sum_{n=1}^{m} \mu^*(F \cap E_n) + \mu^*(F \setminus (\bigcup_{n} E_n)) \qquad \text{(monotonicity)}
\end{aligned}$$

Taking $m \to \infty$, we have that

$$\begin{aligned}\mu^*(F) &\geq \sum_n \mu^*(F \cap E_n) + \mu^*(F \setminus (\bigcup_n E_n)) \\ &\geq \mu^*(F \cap (\bigcup_n E_n)) + \mu^*(F \setminus (\bigcup_n E_n)).\end{aligned}$$

Thus $\bigcup_n E_n \in m^*$.

(d) Let $E_n \in m^*$. We show $\bigcup_n E_n \in m^*$. Write $E'_1 = E_1, E'_{n+1} = E_{n+1} \setminus (\bigcup_k^n E_n)$. Then E'_n are disjoint elements of m^*, whose union are in m^* by (e).

$\square$

Proof of (b). Additivity comes directly from (c) in the previous proof. The outer measure of $\varnothing$ is 0. It remains to show μ^* is a complete measure $m^* \to [0, \infty]$. Let $E \in m^*$ with $\mu^*(E) = 0$. Let $E' \subseteq E$. Let $F \subseteq X$. We want

$$\mu^*(F) \geq \mu^*(F \cap E') + \mu^*(F \setminus E').$$

Consider $\mu^*(F \cap E') + \mu^*(F \setminus E')$. Since $F \cap E' \subseteq E$, by monotonicity $\mu^*(F \cap E') \leq \mu^*(E) = 0$, and since $F \setminus E' \subseteq F$, $\mu^*(F \setminus E') \leq \mu^*(F)$, the result follows. $\square$

4.3 Interlude: Cantor set and function

Definition (Cantor set): Define $G_1 = (1/3, 2/3)$. Remove G_1 from $[0, 1]$. Then define G_2 as the middle-third open interval of each interval remaining in $[0, 1]$. Repeat this process so that at G_n consists of 2^{n-1} open intervals of length $(1/3)^n$.

$G_1 \cup G_2 \cup G_3$ is shown in red.

Then define the *Cantor set* as $C = [0, 1] \setminus (\bigcup_n G_n)$.

Proposition (Basic properties of Cantor set):

(a) C is closed.

(b) $\mathcal{L}(C) = 0$.

Proof.

(a) Each G_n is open, so $\bigcup_n G_n$ is open, its complement is closed, and $[0, 1] \setminus (\bigcup_n G_n)$ is the closed intersection of two closed sets.

(b) Each (disjoint) G_n has measure $2^{n-1}/3^n$, so $\bigcup_n G_n$ has measure

$$\frac{1}{3} \frac{1}{1 - \frac{2}{3}} = 1.$$

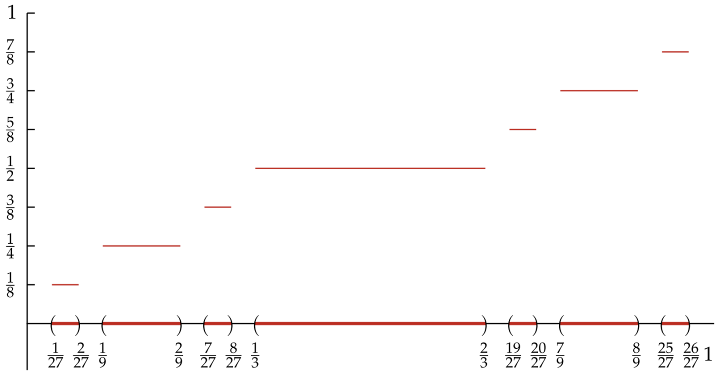
Thus the Cantor set has measure $1 - 1 = 0$.

□

Definition (Cantor function): Define the *Cantor function* $F : [0,1] \to [0,1]$ by

(a) If $x \in C$, then $F(x)$ is the ternary representation of x (consisting of only 0s and 2s) with each 2 replaced by 1 interpreted as as a binary number.

(b) If $x \in [0,1] \setminus C$, then $F(x)$ is the ternary representation of x truncated after the first 1 and with each 2 before the first 1 replaced by 1 interpreted as a binary number.



Graph of the Cantor function on the intervals from first three steps.

Proposition (Cantor function is onto): $F(C) = [0,1]$.

Proof. Let $y \in [0,1]$. Take the binary representation of y, replace 1s with 2s, and interpret it as a ternary number. This new number has only 0s and 2s in ternary so is in C, and maps to y. □

Proposition (Cantor function is increasing and continuous): F is intuitively increasing, and that's enough. Since the image of $[0,1]$ is $[0,1]$, the fact that F is increasing tells us F is continuous.

Proposition (Cantor set is uncountable):

Proof. Suppose C was countable. Then $F(C)$ would be countable. However $F(C) = [0,1]$. □

4.4 Borel and Lebesgue-measurable sets

Let (X, Z) be a topological space, and $\mathcal{B}(X)$ be the Borel σ-algebra (smallest containing all open subsets of X). Let μ^* be an outer measure on X, and define m^* as the σ-algebra of μ^*-measurable sets. The important relationship we will show is that

$$\mathcal{B}(X) \subseteq m^*.$$

Definition (Metric outer measure): Let (X, d) be a metric space and μ^* be an outer measure on X. μ^* is said to be a *metric outer measure* if

$$\mu^*(E \cup F) = \mu^*(E) + \mu^*(F)$$

for all $E, F \subseteq X$ such that $D(x,y) := \inf\{d(x,y) : x \in E, y \in F\} > 0$.

Proposition: Let (X, d) be a metric space and μ^* a metric outer measure. Then every Borel subset of X is μ^*-measurable.

Proof. $\mathcal{B}(X)$ is generated by closed sets, so it suffices to show all closed sets are contained in m^*. Fix $C \subseteq X$ closed. Given $F \subseteq X$, we want to show that

$$\mu^*(F) \geq \mu^*(F \cap C) + \mu^*(F \setminus C).$$

If $\mu^*(F) = \infty$, we are done, so we may assume F has finite measure. Define

$$E_0 \coloneqq \{x \in F \setminus C : D(x, C) \geq 1\}, E_n \coloneqq \{x \in F \setminus C : \frac{1}{n+1} \leq D(x, C) < \frac{1}{n}\}.$$

Since C is closed, $F \setminus C = \bigcup_n E_n$. We can show $\sum_n \mu^*(E_n) < \infty$ (in particular, the tails converge to 0) by summing measures of the even indices and odd indices separately and using the metric outer measure property.

$$\begin{aligned}
\mu^*(F \cap C) + \mu^*(F \setminus C) &= \mu^*(F \cap C) + \mu^*(\bigcup_i E_i) \\
&= \mu^*(F \cap C) + \mu^*(\bigcup_{i=1}^n E_i \cup \bigcup_{j \geq n+1} E_j) \\
&\leq \mu^*(F \cap C) + \mu^*(\bigcup_{i=1}^n E_i) + \mu^*(\bigcup_{j \geq n+1} E_j) \\
&\leq \mu^*(F \cap C) + \mu^*(\bigcup_{i=1}^n E_i) + \sum_{j \geq n+1} \mu^*(E_j).
\end{aligned}$$

It's clear that for any fixed n, $D(F \cap C, \bigcup_{i=1}^n E_i) > 0$. So by metric outer measure,

$$\cdots = \mu^*((F \cap C) \cup (\bigcup_{i=1}^n E_i)) + \sum_{j \geq n+1} \mu^*(E_j) \leq \mu^*(F) + \sum_{j \geq n+1} \mu^*(E_j).$$

Let $n \to \infty$. $\square$

Corollary: Borel sets of $\mathbb{R}^N$ are Lebesgue measurable and Hausdorff measurable.

Proof. It suffices to show $\mathcal{L}_0^N, H_0^s$ are metric outer measures in $\mathbb{R}^N$. Suppose $E, F \subseteq \mathbb{R}^N$ with $D(E, F) > 0$. Then,

(a) $\mathcal{L}_0^N(E \cup F) = \mathcal{L}_0^N(E) + \mathcal{L}_0^N(F)$:
Fix $\varepsilon > 0$ and let R_n rectangles such that $E \cup F \subseteq \bigcup_n R_n$ and $\sum_n \operatorname{vol} R_n < \mathcal{L}_0(E \cup F) + \varepsilon$. Partition each R_n into smaller rectangles with diameter less than $D(E, F)$. Then if $R_m \cap E \neq \varnothing$, then $R_m \cap F = \varnothing$, and vice versa. So we may subdivide rectangles into two sets:

$$\sum_n \operatorname{vol} R_n = \sum_{R_m \cap E \neq \varnothing} \operatorname{vol} R_m + \sum_{R_m \cap F \neq \varnothing} \operatorname{vol} R_m \geq \mathcal{L}_0^N(E) + \mathcal{L}_0^N(F).$$

Thus we have shown that for arbitrary $\varepsilon > 0$,

$$\mathcal{L}_0^N(E \cup F) + \varepsilon \geq \mathcal{L}_0^N(E) + \mathcal{L}_0^N(F).$$

(b) $\mathcal{H}_0^N(E \cup F) \geq \mathcal{H}_0^N(E) + \mathcal{H}_0^N(F)$:
It suffices to show that for δ close enough to 0 and arbitrary $\varepsilon > 0$,

$$\mathcal{H}_\delta^H(E \cup F) + \varepsilon \geq \mathcal{H}_\delta^N(E) + \mathcal{H}_\delta^N(F).$$

Let $\delta < D(E, F)$. Then, we may find a covering $E \cup F \subseteq \bigcup_n A_n$ such that $\operatorname{diam} A_n < \delta$ and $\sum_n \rho(A_n) < \mathcal{H}_\delta^N(E \cup F) + \varepsilon$. By similar reasoning as in (a), we can partition these sets into those intersecting with E and those intersecting with F, and the result holds when we take $\delta \to 0$.

□

Example (Set that is not measurable): Let $V \subseteq [-1, 1]$ be the Vitali set, (r_k) the rationals in $[-2, 2]$. If V was Lebesgue measurable, then since finitely many disjoint μ^*-measurable sets are additive,

$$\left| \bigcup_{k=1}^n (r_k + V) \right| = \sum_{i=1}^n |r_k + V| = n|V|.$$

But since we showed $|V| > 0$, we can choose n large enough so that $n|V| > 6$, a contradiction to monotonicity.

Example (Set that is Lebesgue but not Borel): [HW2] The continuous preimage of a Borel set is Borel, so it suffices to find a Lebesgue measurable set whose preimage under a continuous function isn't Borel. Let Λ be the Cantor function and C be the cantor set. If E is non measurable (Vitali of some sort),

$$\Lambda(\Lambda^{-1}(E) \cap C) = E.$$

So if Λ were invertible, we would be done. Instead, we define $F(x) = \Lambda(x) + x$ and construct Vitali within $F(C)$ with measure 1.

Example (Remarks):

(a) If $E \subseteq \mathbb{R}$ countable, then $\mathcal{L}_0^N(E) = 0$. But there also exist $D \subseteq \mathbb{R}$ uncountable with $\mathcal{L}_0^N(D) = 0$ (Cantor set).

(b) There exists $E \subseteq \mathbb{R}$ Lebesgue measurable such that $E + E$ is not Lebesgue measurable. (not true with Borel).

Theorem (Continuous image of Borel is Leb): If X complete, separable metric space, $f : X \to \mathbb{R}^d$ is continuous, then if $E \subseteq X$ Borel, then $f(E)$ is Lebesgue measurable.

Theorem (Continuous image of Leb is not always Leb): Inverse image of Vitali set under Cantor function intersect Cantor set maps to Vitali set.

Theorem (Continuous inverse image of Borel is Borel): hw2

5 $\mathcal{L}(\mathbb{R}^N)$ completes $\mathcal{B}(\mathbb{R}^N)$

We continue our discussion of Borel and Lebesgue σ-algebras in $\mathbb{R}^N$. The Lebesgue measure μ on the Borel σ-algebra, $(X, \mathcal{B}(\mathbb{R}^N), \mu)$ is not complete, meaning there Borel sets of measure 0 with subsets that aren't Borel. We show that the Lebesgue measurable sets $\mathcal{L}(\mathbb{R}^N)$ complete $\mathcal{B}(\mathbb{R}^N)$ with respect to the Lebesgue measure, i.e. it's what we get after throwing in all sets of measure 0.

5.1 Borel Regularity

Theorem (Inner and outer regularity): Let $A \subseteq \mathbb{R}^N$.

(a) Outer regularity:

$$\mathcal{L}_0(A) = \inf\{\mathcal{L}(G) : A \subseteq G, G \text{ open}\}.$$

(b) Inner regularity: If A is Lebesgue measurable then

$$\mathcal{L}(A) = \sup\{\mathcal{L}(K) : K \subseteq A, K \text{ compact}\}.$$

Proof.

(a) WLOG $\mathcal{L}_0(A) < \infty$. The $\leq$ is always true by monotonicity. It remains to show
$$\mathcal{L}_0(A) \geq \inf\{\mathcal{L}(G) : A \subseteq G, G \text{ open}\}.$$
We show that the covering with total volume less than $\mathcal{L}(A)$ padded by a little is close to being an open covering and so greater than the infimal open covering measure. Fix $\varepsilon > 0$ and choose rectangles R_n by definition of outer measure such that $A \subseteq \bigcup_n R_n$ and
$$\sum_n \operatorname{vol}(R_n) \leq \mathcal{L}_0(A) + \varepsilon/2.$$
Let S_n be rectangles such that $R_n \subseteq S_n^o$ and $\operatorname{vol}(S_n) \leq \operatorname{vol}(R_n) + \frac{\varepsilon}{2^{n+1}}$.
$$\begin{aligned}
\mathcal{L}_0(A) + \varepsilon &\geq \sum_n \operatorname{vol}(R_n) + \varepsilon/2 \\
&\geq \sum_n \operatorname{vol}(S_n) \\
&\geq \operatorname{vol}\left(\bigcup_n S_n\right) && \text{(subadditivity of volume)} \\
&\geq \mathcal{L}\left(\bigcup_n S_n\right) \\
&\geq \mathcal{L}\left(\bigcup_n S_n^o\right) && \text{(monotonicity)} \\
&\geq \inf\{\mathcal{L}(G) : A \subseteq G, G \text{ open}\}.
\end{aligned}$$
As ε is arbitrary we are done.

(b) First assume A is a bounded measurable set (thus with finite measure). It suffices to show

$$\mathcal{L}(A) \leq \sup\{\mathcal{L}(K) : K \subseteq A, K \text{ compact}\},$$

since $\geq$ is given by monotonicity. Let F be a compact set containing A. Fix $\varepsilon > 0$. By outer regularity we can find G open such that $F \setminus A \subseteq G$ and

$$\mathcal{L}(G) \leq \mathcal{L}(F \setminus A) + \varepsilon.$$

Since A is measurable, $\mathcal{L}(F) = \mathcal{L}(F \cap A) + \mathcal{L}(F \setminus A)$, so

$$\mathcal{L}(A) = \mathcal{L}(F) - \mathcal{L}(F \setminus A) \leq \mathcal{L}(F) - \mathcal{L}(G) + \varepsilon.$$

Let $K \coloneqq F \setminus G$ compact such that $K \subseteq A$. Then,

$$\mathcal{L}(A) \leq \mathcal{L}(F) - \mathcal{L}(G) + \varepsilon = \mathcal{L}(K) + \varepsilon \leq \sup\{\dots\} + \varepsilon.$$

As ε is arbitrary we have proved inner regularity for bounded sets. Suppose on the other hand A isn't bounded. Define $A_k \coloneqq A \cap B(0, k)$ so that $A_k \to A$. Then, by earlier proposition

$$\mathcal{L}(A_k) \to \mathcal{L}(A).$$

If $\mathcal{L}(A) = \infty$, then $\mathcal{L}(A_k) \to \infty$ and by what we've just proved we can find $K_k \subseteq A_k$ compact such that $\mathcal{L}(K_k) + 1 \geq \mathcal{L}(A_k)$. So $\mathcal{L}(K_k) \to \infty$, therefore

$$\sup\{\mathcal{L}(K) : K \subseteq A, K \text{ compact}\} = \infty = \mathcal{L}(A).$$

If $\mathcal{L}(A) < \infty$, then for fixed $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that $\mathcal{L}(A_k) \geq \mathcal{L}(A) - \varepsilon/2$. Choose $K \subseteq A_k$ compact by above result such that $\mathcal{L}(K) \geq \mathcal{L}(A_k) - \varepsilon/2$, then

$$\mathcal{L}(A) \leq \mathcal{L}(A_k) + \varepsilon/2 \leq \mathcal{L}(K) + \varepsilon.$$

Since ε was arbitrary we are done.

□

5.2 Conditions for Lebesgue measurability

Proposition (Outer condition for Lebesgue measurability): $A \subseteq \mathbb{R}^N$ is Lebesgue measurable if and only if for every $\varepsilon > 0$ there exists open G, $A \subseteq G$,

$$\mathcal{L}_0(G \setminus A) < \varepsilon.$$

Proof. Suppose A is Lesbesgue measurable with finite measure. Fix $\varepsilon > 0$. By outer regularity there exists G open, $A \subseteq G$ such that

$$\mathcal{L}(G) \leq \mathcal{L}(A) + \varepsilon.$$

So by measurability A,

$$\mathcal{L}(G \setminus A) = \mathcal{L}(G) - \mathcal{L}(G \cap A) \leq \varepsilon,$$

since $\mathcal{L}(G \cap A) \leq \mathcal{L}(A)$.

Assume $\mathcal{L}(A) = \infty$. We define $A_k := A \cap \{x \in \mathbb{R}^N : k \leq |x| < k+1\}$. Using the previous argument, for all $k \in N$ there exists G_k open, $A_k \subseteq G_k$,

$$\mathcal{L}(G_k \setminus A_k) < \varepsilon/2^k.$$

Then $G = \bigcup_k G_k$ is open, $A = \bigcup_k A_k$, so $A \subseteq G$.

$$\mathcal{L}(G \setminus A) = \mathcal{L}(\bigcup_k G_k \setminus A) \leq \sum_k \mathcal{L}(G_k \setminus A) \leq \sum_k \mathcal{L}(G_k \setminus A_k) = \varepsilon.$$

Now suppose the latter. Fix $E \subseteq \mathbb{R}^N$. We want to show

$$\mathcal{L}_0(E) \geq \mathcal{L}_0(E \cap A) + \mathcal{L}_0(E \setminus A).$$

Fix $\varepsilon > 0$ and select open G by hypothesis such that

$$\mathcal{L}_0(G \setminus A) < \varepsilon.$$

Write $E \setminus A = (E \setminus G) \cup (E \cap (G \setminus A))$ by measurability G.

$$\begin{aligned}
\mathcal{L}(E \cap A) + \mathcal{L}(E \setminus A) &\leq \mathcal{L}(E \cap A) + \mathcal{L}(E \setminus G) + \mathcal{L}(E \cap (G \setminus A)) \\
&\leq \mathcal{L}(E \cap G) + \mathcal{L}(E \setminus G) + \varepsilon \\
&= \mathcal{L}(E) + \varepsilon.
\end{aligned}$$

□

Remark: We cannot make a similar argument for inner regularity since the countable union of closed sets is not necessarily closed.

Theorem (Approximating Leb set by closed and open sets): $A \subseteq \mathbb{R}^N$ is Lebesgue measurable if and only if for all $\varepsilon > 0$ there exists G open and F closed,

$$F \subseteq A \subseteq G,$$

such that $\mathcal{L}(G \setminus F) < \varepsilon$. If $\mathcal{L}(A) < \infty$ then F can be chosen to be compact.

Proof. HW3. □

The next theorem allows us to express Lebesgue sets as the union of a Borel set with a set of measure 0 from both above and below.

Definition (G_δ, F_σ): A G_δ in $\mathbb{R}^N$ is a countable intersection of open sets. A F_σ in $\mathbb{R}^N$ is a countable union of closed sets.

Theorem (Leb sets are measure 0 set away from Borel sets): $A \subseteq \mathbb{R}^N$ is Lebesgue measurable if and only if there exists a G_δ G and a F_σ F such that

$$F \subseteq A \subseteq G,$$

and $\mathcal{L}(G \setminus F) = 0$.

Proof. Suppose A is Lebesgue measurable. For every k we can find G_k open, F_k closed, $F_k \subseteq A \subseteq G_k$, $\mathcal{L}(G_k \setminus F_k) < 1/k$. Take $G = \bigcap_k G_k$, $F = \bigcup_k F_k$ and we still have $F \subseteq A \subseteq G$ and $\mathcal{L}(G \setminus F) = 0$.

Suppose the latter. Then $\mathcal{L}(A \setminus F) = 0$ so by completeness $A \setminus F$ is measurable. Thus $A = F \cup (A \setminus F)$ is measurable since F is Borel. □

5.3 Lebesgue completes Borel

Recall that a complete measure μ on a sigma algebra m of subsets of X has that all subsets of sets $\mu(E) = 0$ are in m. The Lebesgue measure on the Borel σ-algebra is not complete, since there exist subsets of measure 0 sets (like Cantor set) which are not Borel. The following theorem tells us that to complete the Borel σ-algebra, we quite literally toss in all sets of measure 0.

Theorem (Leb is completion of Borel): The Lebesgue σ-algebra $\mathcal{L}(\mathbb{R}^N)$ is the completion of the Borel σ-algebra $\mathcal{B}(\mathbb{R}^N)$, i.e.

$$\mathcal{L}(\mathbb{R}^N) = \{B \cup M, B \in \mathcal{B}(\mathbb{R}^N), \mathcal{L}(M) = 0\}.$$

Proof. If $A \subseteq \mathcal{L}(\mathbb{R}^N)$, it contains a F_σ a set of measure 0 away by previous proposition.

Let $E = B \cup M$ for B borel and M measure 0. Let $F \subseteq \mathbb{R}^N$, then by measurability B,

$$\begin{aligned}
\mathcal{L}(F) &= \mathcal{L}(F \cap B) + \mathcal{L}(F \setminus B) \\
&\geq \mathcal{L}(F \cap B) + \mathcal{L}(F \setminus E) \\
&\geq \mathcal{L}(F \cap E) - \mathcal{L}(F \cap M) + \mathcal{L}(F \setminus E) && \text{(subadditivity)} \\
&= \mathcal{L}(F \cap E) + \mathcal{L}(F \setminus E). && \text{(monotonicity)}
\end{aligned}$$

□

6 Geometric measure theory

We show the Lebesgue measure is equal to the Hausdorff measure ($s = N$) in $\mathbb{R}^N$. The Isodiametric inequality states that the Lebesgue measure of a set with given diameter is maximized when it's a ball, immediately showing $\mathcal{H} \geq \mathcal{L}$.

Vitali-Besicovich covering lemma says that given a Vitali's covering of a set (each point is contained in balls of arbitrarily small size), there exists a disjoint subfamily only omitting a set of zero measure. This means we can break any covering into balls with radius less than δ over which $\mathcal{H}_\delta \leq \mathcal{L}$, and take $\delta \to 0$ to get $\mathcal{H} \leq \mathcal{L}$.

6.1 Isoparimetric Problems

These are relations between volume and surface area.

What is the shape that minimizes surface area for a fixed volume?

Theorem (Brunn-Minkowski Inequality): Suppose $E, F \subseteq \mathbb{R}^N$ are measurable sets such that $E + F = \{x + y : x \in E, y \in F\}$ is measurable (not always true). Then,

$$\mathcal{L}(E)^{1/n} + \mathcal{L}(F)^{1/n} \leq \mathcal{L}(E + F)^{1/n}.$$

Proof. Omitted. □

Remark: Fix $\theta \in (0, 1)$. suppose E, F are measurable sets such that $\theta E + (1 - \theta)F$ are measurable. Then,

$$\begin{aligned} \theta\mathcal{L}(E)^{1/n} + (1-\theta)\mathcal{L}(F)^{1/n} &= \mathcal{L}(\theta E)^{1/n} + \mathcal{L}((1-\theta)F)^{1/n} && \text{(N-homogeneity)} \\ &\leq (\mathcal{L}(\theta E + (1-\theta)F))^{1/n}. && \text{(B-M)} \end{aligned}$$

In particular, $f : t \mapsto (\mathcal{L}(tE + (1-t)F))^{1/n}$ is concave.

Theorem (Isodiametric inequality): Let $E \subseteq \mathbb{R}^N$ lebesgue measurable. Then,

$$\mathcal{L}(E) \leq \alpha_n \left(\frac{\operatorname{diam} E}{2}\right)^N,$$

where $\alpha_n = \mathcal{L}(B(0, 1))$.

Proof. If E is unbounded, then the right hand side is infinite. Supposing E is finite, WLOG suppose $\operatorname{diam} E = 1$ (this is finite, since both Lebesgue measure and diameter are N-homogenous).

Let $F = \{-x : x \in E\}$. We claim that $\frac{1}{2}E + \frac{1}{2}F \subseteq B(0, \frac{1}{2})$, where $\mathcal{L}(E) \leq \mathcal{L}(\frac{1}{2}E + \frac{1}{2}F)$ by the above remark. Then,

$$\mathcal{L}(E) \leq \mathcal{L}(1/2E + 1/2F) \leq \mathcal{L}(B(0, 1/2)) = \alpha_N \frac{1}{2^N}.$$

□

6.2 Hausdorff measure

Theorem: $\mathcal{L}_0 \leq \mathcal{H}_0^N$ in $\mathbb{R}^N$

Proof. Fix $\delta > 0$. Let $E \subseteq \bigcup_n E_n$ be a covering with closed (WLOG) rectangles, splitting if necessary to obtain $\operatorname{diam} E_n < \delta$.

$$\mathcal{L}_0(E) \leq \mathcal{L}(\bigcup_n E_n) \leq \sum_n \mathcal{L}(E_n) \leq \sum_n \alpha_N \left(\frac{\operatorname{diam} E_n}{2}\right)^N.$$

Take the inf over all admissible E_n and sup over δ, then $\mathcal{L}_0(E) \leq \mathcal{H}_0(E)$. □

Theorem (Vitali's covering lemma): Suppose $\mathcal{F}$ is a family of balls contained in a bounded set. Then there exists a countable subfamily of (closed) balls $\{B(x_n, r_n)\}$ pairwise disjoint and such that

$$\bigcup_{B \in \mathcal{F}} B \subseteq \bigcup_n B(x_n, 5r_n).$$

Proof. Omitted. □

Definition: Let $E \subseteq \mathbb{R}^N$. A family of closed balls $\mathcal{F}$ is said to be a *Vitali's cover* of E if for all $x \in E$ and all $\varepsilon > 0$, there exists a ball such that $x \in B$ and $\operatorname{diam} B < \varepsilon$.

Corollary: There exists a family $\mathcal{F}$ of closed balls contained in a bounded set $E \subseteq \mathbb{R}^N$ where $\mathcal{F}$ is a Vitali's cover of E. Let $\{B(x_n, r_n)\}$ as in above thm. Then for all $\ell \in \mathbb{N}$,

$$E \setminus \bigcup_{m=1}^{\ell} B(x_n, y_n) \subseteq \bigcup_{n \geq \ell+1} B(x_n, 5y_n).$$

Theorem (Vitali-Besicovich covering lemma): Let $\mathcal{F}$ be a Vitali's cover of E. Then there exists a countable subfamily of balls $\{B(x_n, r_n)\}$ pairwise disjoint such that

$$\mathcal{L}_0(E \setminus \bigcup_n B(x_n, r_n)) = 0.$$

Proof (E bounded). Suppose $E \subseteq B(0, R)$ with $R > 0$. Take $\mathcal{F}_1 := \{B \in \mathcal{F} : B \subseteq B(0, R)\}$ still a Vitali's cover of E. By Vitali's covering lemma, there exists countable family $\{B(x_n, r_n)\}$ pairwise disjoint such that

$$\bigcup_{B \in \mathcal{F}_1} B \subseteq \bigcup_n B(x_n, 5r_n).$$

We claim that in fact $\bigcup_n B(x_n, r_n)$ covers essentially everything. Since

$$\sum_n \mathcal{L}(B(x_n, 5r_n)) = 5^N \sum_n \mathcal{L}(B(x_n, r_n))$$

and the disjoint balls are bounded in $B(0, R)$, the series on the LHS converges. So for all $\varepsilon > 0$ there exists $\ell \in \mathbb{N}$ such that

$$\sum_{n \geq \ell+1} \mathcal{L}^N(B(x_n, 5r_n)) \leq \varepsilon.$$

So,

$$\begin{aligned}
\mathcal{L}_0(E \setminus \bigcup_n B(x_n, r_n)) &\leq \mathcal{L}_0(E \setminus \bigcup_{n=1}^{\ell} B(x_n, r_n)) \\
&\leq \mathcal{L}_0(\bigcup_{n \geq \ell+1} B(x_n, 5r_n)) \\
&\leq \sum_{n \geq \ell+1} \mathcal{L}(B(x_n, 5r_n)) < \varepsilon,
\end{aligned}$$

and we can take $\varepsilon \to 0$. □

Proof (E unbounded). Let $E_1 := E \cap B(0, 1)$ and $E_k := (E \cap B(0, k)) \setminus B(0, k-1)$. Then $E = \bigcup_n E_n$ (ignoring the boundaries) and E_n are mutually disjoint and bounded. Take

$$\mathcal{F}_1 = \{B \in \mathcal{F}, B \subseteq B(0, 1)\},$$

so $\mathcal{F}_1$ is a Vitali's cover for E_1. By previous part, we can find countably many balls $\{B_{n,1}\} \subseteq \mathcal{F}_1$ such that

$$\mathcal{L}_0(E_1 \setminus \bigcup_n B_{n,1}) = 0.$$

Similarly define Vitali cover for E_k by $\mathcal{F}_k := \{B \in \mathcal{B} : B \subseteq B(0, k) \setminus B(0, k-1)\}$, and we can find countably many disjoint balls $\{B_{n,k}\}_{n \in \mathbb{N}} \subseteq \mathcal{F}_k$ such that $\mathcal{L}_0(E_k \setminus (\bigcup_n B_{n,k})) = 0$. Finally,

$$\mathcal{L}_0(E \setminus \bigcup_k \bigcup_n B_{n,k}) \leq \mathcal{L}_0(\bigcup_k (E_k \setminus \bigcup_n B_{n,k})) \leq \sum_k \mathcal{L}_0(E_k \setminus \bigcup_n B_{n,k}) = 0.$$

□

Remark: We use closed balls in our coverings and open balls in "rings". It does not matter in $\mathbb{R}^N$ as boundaries have measure 0 but sometimes boundaries are charged.

Theorem (Leb is equal to Haus in $\mathbb{R}^N$): $\mathcal{L}_0 = \mathcal{H}_0^N$ in $\mathbb{R}^N$.

Proof. We claim $\mathcal{H} \le \mathcal{L}$. Fix $\delta > 0$. First we show there exists $c_N > 0$ such that $\mathcal{H}_\delta \le c_N \mathcal{L}_0$. This way when a set has Lebesgue measure 0 it also has Hausdorff measure 0. Let E be covered by cubes $Q(x_n, r_n)$ subdivided to have diameter less than δ.

$$\mathcal{H}_\delta(E) \le \sum_n \alpha_N \left(\frac{\operatorname{diam} Q(x_n, r_n)}{2} \right)^N = \sum_n \alpha_N \left(\frac{\sqrt{N} r_n}{2} \right)^N = c_N \sum_n r_n^N.$$

Thus $\mathcal{H}_\delta(E) \le c_N \mathcal{L}_0(E)$ when we take infimum over cube coverings Q_n.

On the other hand for all n define $\mathcal{F}_n$ as the family of all closed balls in $Q(x_n, r_n)$, clearly a Vitali's cover of $Q(x_n, r_n)$. Apply Vitali-Besicovich to find countable pairwise disjoint family $B(x_{i,n} r_{i,n}) \subseteq \mathcal{F}_n$ such that

$$\mathcal{L}(Q(x_n, r_n) \setminus \bigcup_i B(x_{i,n}, r_{i,n})) = 0,$$

and since $\mathcal{H} \le c_N \mathcal{L}$, $\mathcal{H}(Q(x_n, r_n) \setminus \bigcup_i B(x_{i,n}, r_{i,n})) = 0$. Thus

$$\begin{aligned}
\mathcal{H}_\delta(E) &\le \sum_n \mathcal{H}_\delta(Q(x_n, r_n)) \\
&= \sum_n \mathcal{H}_\delta(\bigcup_i B(x_{i,n}, r_{i,n})) \\
&\le \sum_n \sum_i \mathcal{H}_\delta(B(x_{i,n}, r_{i,n})).
\end{aligned}$$

Since the diameters were defined $< \delta$,

$$\mathcal{H}_\delta(B(x_{i,n}, r_{i,n})) \le \alpha_N \left(\frac{\operatorname{diam} B(x_{i,n}, r_{i,n})}{2} \right)^N = \mathcal{L}(B(x_{i,n}, r_{i,n})).$$

Now, since the balls are disjoint and contained in $Q(x_n, r_n)$, the sum is bounded by

$$\sum_n \sum_i \mathcal{L}(B(x_{i,n}, r_{i,n})) \le \sum_n \mathcal{L}(Q(x_n, r_n)).$$

By infimizing over the cubes and then δ we are done. □

Example: Within $\mathcal{R}^3$, H^3 is volume, H^2 is surface measure, H^1 is length, and H^0 is counting measure.

**Use H_0.

Proposition: Let $E \subseteq \mathbb{R}^N, \delta > 0$, with $0 \le s < t < \infty$.

(a) H^0 is counting measure.

(b) $H^s = 0$ if $s > N$.

(c) $H^s(x + E) = H^s(E)$.

(d) $H^s(\lambda E) = \lambda^s H^s(E)$.

(e) If $H^s_\delta(E) = 0$ for some $0 < \delta < \infty$, then $H^s(E) = 0$.

(f) $H^s(E) < \infty \implies H^t(E) = 0$.

(g) $H^t(E) > 0 \implies H^s(E) = \infty$.

Definition (Hausdorff dimension): The *Hausdorff dimension* of a set $E \subseteq \mathbb{R}^N$ is given by

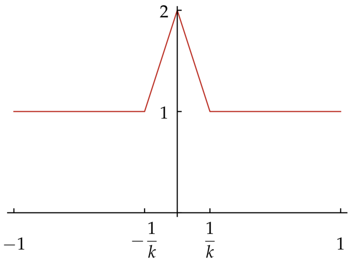
$$\dim_{\mathcal{H}}(E) = \inf\{0 \leq s < \infty : H^s(E) = 0\}.$$

7 Convergence of measurable functions

Egorov's theorem says that on a set X with finite measure, measurable functions f_n converging pointwise will converge almost uniformly. Measurable functions are the increasing limit of simple functions. Lusin's theorem says that a Lebesgue measurable function is continuous restricted to a closed set omitting less than ε measure from $\mathbb{R}$. Lebesgue measurable functions are equal to some Borel measurable function almost everywhere.

7.1 Uniform convergence: Egorov

Pointwise convergence generally has undesirable properties. We cannot interchange limit and integral (consider traveling spike of height n on $[0, 1]$), and the limit of continuous functions need not be continuous:



Egorov's theorem states that for any $\varepsilon > 0$ we can find a set omitting less than ε measure from $[-1, 1]$ on which these functions converge uniformly. Indeed, uniform convergence holds for all $[-1, -\varepsilon/4] \cup [\varepsilon/4, 1]$.

Theorem (Egorov's Theorem): Let (X, m, μ) and $\mu(X) < \infty$. Suppose $f_n : X \to \mathbb{R}$ are measurable functions converging to f pointwise. Then for all $\varepsilon > 0$ there exists $E \in m$ such that $\mu(X \setminus E) < \varepsilon$ and $f_n \to f$ uniformly on E.

Proof. Fix $\varepsilon > 0$ and $n \in N$. By pointwise convergence we can express

$$X = \bigcup_m \bigcap_{k \geq m} \{x \in X : |f_k(x) - f(x)| < 1/n\}.$$

Define the inner intersection $A_{m,n}$, which is an increasing sequence with $X = \bigcup_m A_{m,n}$. By monotone convergence,

$$\mu(X) = \lim_{m \to \infty} \mu(A_{m,n}).$$

For fixed n choose $m_n \in \mathbb{N}$ such that $\mu(X) - \mu(A_{m_n,n}) < \frac{\varepsilon}{2^n}$. So m_n is the "uniform threshold" for basically all points being within $1/n$ of f. Then, the intersection of all sets $A_{m_n,n}$ (which we define as E) gives us a set of points that converge uniformly within $1/n$ at m_n for all n that omits very few points by our selection via monotone convergence:

$$\mu(X \setminus E) = \mu(\bigcup_n (X \setminus A_{m_n,n})) \leq \sum_n \mu(X \setminus A_{m_n,n}) < \varepsilon.$$

By our construction, f_n converges uniformly E: for fixed $\varepsilon' > 0$, find some n large enough that $1/n < \varepsilon'$, then

$$E \subseteq A_{m_n,n},$$

meaning for all points x in E, $f_k(x)$ is within $1/n < \varepsilon'$ of $f(x)$ for all $k \geq m_n$. □

7.2 Simple function approximation

Definition (Simple function): A function $f : X \to \mathbb{R}$ is a *simple function* if f takes only finitely many values. In *standard representation*, where $c_1, \ldots, c_n$ are the distinct nonzero values of f,

$$f(x) = c_1\chi_{E_1}(x) + \cdots + c_n\chi_{E_n}(x),$$

where $E_i = f^{-1}(\{c_i\})$. We see that if $f : (X, m) \to \mathbb{R}$, f is measurable if and only if $E_i \in m$ for all $i = 1, \ldots, n$.

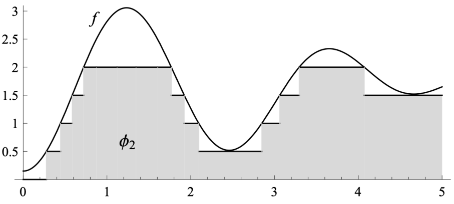
Theorem (Approximation by simple functions): Let $f : X \to \mathbb{R}$ a measurable function on (X, m). There exist measurable simple functions $f_n : X \to \mathbb{R}$ such that

(a) $|f_k(x)| \leq |f_{k+1}(x)| \leq |f(x)|$.

(b) $\lim_{k\to\infty} f_k(x) = f(x)$ for all $x \in X$.

(c) If f is bounded then $f_k \to f$ uniformly.

Proof. At the kth function we round down $f(x)$ to the nearest multiple of 2^{-k} (round up if negative), capping things at k so we ensure functions only take finitely many values. Here is a visual of $k = 2$ mistakenly rounded to $1/2$ rather than $1/4$.



Formally,

$$f_k(x) = \begin{cases} m/2^k, & \text{if } 0 \leq f(x) \leq k, f(x) \in [m/2^k, (m+1)/2^k); \\ (m+1)/2^k, & \text{if } -k \leq f(x) < 0, f(x) \in [m/2^k, (m+1)/2^k); \\ k, & \text{if } f(x) > k; \\ -k, & \text{if } f(x) < -k. \end{cases}$$

Since f is measurable and the inverse images of each value of these step functions are inverse images of Borel sets, f_k is measurable. (a) is clear from definition. (b) follows since for large enough k, $f(x)$ will be within $[-k, k]$ and then within $1/2^k$ of $f_k(x)$. For (c), if f is bounded then for large enough k, all $f(x)$ will be within $[-k, k]$ and thus within $1/2^k$ of $f_k(x)$. □

7.3 Continuity: Lusin

Theorem (Lusin's Theorem): Suppose that $g : \mathbb{R}^N \to \mathbb{R}$ is Lebesgue measurable. Then for every $\varepsilon > 0$ there exists a closed set $F \subseteq \mathbb{R}$, $\mathcal{L}(\mathbb{R}^N \setminus F) < \varepsilon$, and $g|_F$ is continuous on F.

Proof. **Step 1: g is step function.** First assume

$$g = d_1\chi_{D_1} + \cdots + d_n\chi_{D_n}.$$

for some distinct d_i, $D_i \in m$. Fix $\varepsilon > 0$. For $k \in \{1, \dots, n\}$ applying measurability of D_k with regularity there exists an open set G_k and closed set F_k,

$$F_k \subseteq D_k \subseteq G_k$$

with $\mathcal{L}(G_k \setminus F_k) < \varepsilon/n$. Set $F := (\bigcup_{k=1}^n F_k) \cup (\bigcap_{k=1}^n (\mathbb{R}^N \setminus G_k))$. Note that the F_ks are pairwise disjoint and disjoint with the intersection, and F is closed. Then,

$$\mathbb{R}^N \setminus F \subseteq \bigcup_{k=1}^n (G_k \setminus F_k) \implies \mathcal{L}(\mathbb{R}^N \setminus F) \leq \sum_{k=1}^n \mathcal{L}(G_k \setminus F_k) < \varepsilon.$$

Since $F_k \subseteq D_k$, $g|_F$ is constant on F_k. On the other hand, $\bigcap_{k=1}^n (\mathbb{R}^N \setminus G_k) \subseteq \bigcap_{k=1}^n (\mathbb{R}^N \setminus D_k)$, on which $g = 0$. Thus $g|_F$ is continuous on F.

Step 2: g is arbitrary. Let g_n be measurable simple functions converging to g. Fix $\varepsilon > 0$. By above, by any $k \in \mathbb{N}$ there exists a closed set C_k such that

$$\mathcal{L}(\mathbb{R}^N \setminus C_k) < \frac{\varepsilon}{2^{k+1}}$$

and $g_k|_{C_k}$ is continuous on C_k. Then $C = \bigcap_k C_k$ is closed and $g_k|_{C_k}$ is continuous on C for all k with $\mathcal{L}(\mathbb{R}^N \setminus C) < \varepsilon/2$.

Define the disjoint sets $D_1 = B(0,1)$ and $D_m = B(0,m) \setminus B(0, m-1)$. For all m, restricted to D_m, $g_n \to g$. By Egorov's, there exists $E_m \subseteq D_m$ measurable with

$$\mathcal{L}(D_m \setminus E_m) < \frac{\varepsilon}{2^{m+2}},$$

such that $g_n \to g$ on E_m uniformly. So restricting to $C \cap E_m$, the convergence is uniform and each $g_k|_{C\cap E_m}$ is continuous. Since the uniform limit of continuous functions is continuous, for all m, $g|_{C\cap E_m}$ is continuous, thus g is continuous restricted to $D := \bigcup_m (C \cap E_m)$ with

$$\mathbb{R}^N \setminus D \subseteq (\mathbb{R}^N \setminus C) \cup (\bigcup_m (D_m \setminus E_m)).$$

So $\mathcal{L}(\mathbb{R}^N \setminus D) < \varepsilon$. By inner regularity we can find $F \subseteq D$ closed with

$$\mathcal{L}(D \setminus F) < \varepsilon - \mathcal{L}(\mathbb{R}^N \setminus D).$$

g is continuous on F and $\mathcal{L}(\mathbb{R}^N \setminus F) = \mathcal{L}((\mathbb{R}^N \setminus D) \cup (D \setminus F)) \leq \mathcal{L}(\mathbb{R}^N \setminus D) + \varepsilon - \mathcal{L}(\mathbb{R}^N \setminus D) = \varepsilon.$ ☐

7.4 Almost-Borel measurablility

Theorem: If $f : \mathbb{R}^N \to \mathbb{R}$ is Lebesgue measurable then there exists a Borel measurable function $g : \mathbb{R}^N \to \mathbb{R}$ such that $f = g$ almost everywhere.

Proof. Consider increasing simple functions $f_k \to f$. For a given $k \in \mathbb{N}$, suppose $f_k = \sum_{i=1}^n c_i \chi_{A_i}$. For each Lebesgue measurable A_i, there exists a Borel set $B_i \subseteq A_i$ with $\mathcal{L}(B_i \setminus A_i) = 0$. Define the Borel measurable function

$$g_k = \sum_{i=1}^n c_i \chi_{B_i}.$$

So f_k and g_k differ on at most a set of measure 0, call it D_k. Consider the set $\bigcup_k D_k$ with measure 0. If $x \notin \bigcup_k D_k$, $\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} f_n(x) = f(x)$. Consider the Borel measurable set (by earlier proposition)

$$E = \{x \in X : \lim_{n\to\infty} g_n(x) \text{ exists in } \mathbb{R}\}.$$

Note that $\mathbb{R} \setminus E \subseteq \bigcup_k D_k$ so $\mathcal{L}(\mathbb{R} \setminus E) = 0$. Define

$$g(x) = \lim_{n\to\infty} (\chi_E g_n).$$

So that $f = g$ outside the set of measure 0 $(\mathbb{R} \setminus E) \cup (\bigcup_k D_k)$, and g is Borel measurable.

□

8 Integration

The Lebesgue integral of nonnegative measurable functions is the supremum over measurable partitions of the domain of lower Lebesgue sums or the supremum over integrals of upper bounded simple functions.

The monotone convergence theorem says that the integral of increasing limit of nonnegative measurable functions is the limit of the integrals. Fatou's lemma tells us that for arbitrary nonnegative measurable sequences of functions, the integral of the lim inf is bounded above by the lim inf of the integrals (we lose mass in the limit).

An arbitrary sign measurable function is Lebesgue integrable if it has finite L^1 norm. Any integrable function can be approximated by continuous functions up to ε in L^1 norm.

The dominated convergence theorem says that if a sequence of measurable functions has a μ a.e. limit and is μ a.e. bounded by an integrable function, then the limit of the integrals is the integral of the limit.

8.1 Definition and basic properties

Let $(X, m\mu)$ be a measure space and $f : X \to [0, \infty]$ measurable function.

Definition (Lower Lebesgue sum): Let $P = \{A_1, \dots, A_m\}$ be a partition of X with $A_i \in m$. The *lower Lebesgue sum* with respect to P is defined by

$$L(f, P) \coloneqq \sum_i^m \mu(A_i) \inf_{A_i} f.$$

Definition (Integral):

$$\int_X f d\mu \coloneqq \sup_P L(f, P).$$

Proposition (Integral of characteristic function): Let $E \in m$.

$$\int_X \chi_E d\mu = \mu(E).$$

Proof. Let $P = \{E, X \setminus E\}$. Then $L(\chi_E, P) = \mu(E) \le \int_X f d\mu$. Conversely, let

$$P = \{A_1, \dots, A_n\}.$$

Then $L(\chi_E, P) = \sum_{j : A_j \subseteq E} \mu(A_j) = \mu(\bigcup_{j : A_j \subseteq E}) \le \mu(E)$. Take sup over P. □

Example (Integral of $\chi_{\mathbb{Q}}$): If $X = \mathbb{R}$, $\mu = \mathcal{L}$, then

$$\int_{\mathbb{R}} \chi_{\mathbb{Q}} d\mathcal{L} = \mathcal{L}(\mathbb{Q}) = 0.$$

So

$$\int_{\mathbb{R}} \chi_{[0,1] \setminus \mathbb{Q}} d\mathcal{L} = \mathcal{L}([0, 1] \setminus \mathbb{Q}) = 1.$$

Proposition (Integral of simple function): Let (X, m, μ). Take $E_1, \ldots, E_n$ pairwise disjoint measurable sets, and $c_1, \ldots, c_n \in [0, \infty]$. Then

$$\int_X (\sum_{k=1}^n c_k \chi_{E_k}) d\mu = \sum_{k=1}^n c_k \mu(E_k).$$

Proof. WLOG assume $\{E_1, \ldots, E_n\}$ is a partition (since we may add leftovers). Then $\sum_{k=1}^n \mu(E_k) \inf_{E_k} (\sum_{j=1}^k c_j \chi_{E_j}) = \sum_{k=1}^n c_k \mu(E_k)$ lower bounds the integral.

On the other hand, suppose $P = \{A_1, \ldots, A_m\}$ is an arbitrary partition. Then,

$$\begin{aligned} L(\sum_{k=1}^n c_k \chi_{E_k}, P) &= \sum_{j=1}^m \mu(A_j) \min_{i: A_i \cap E_i \neq \varnothing} c_i \\ &= \sum_{j=1}^m \sum_{k=1}^n \mu(A_j \cap E_k) \min_{i: A_i \cap E_i \neq \varnothing} c_i \\ &\leq \sum_{j=1}^m \sum_{k=1}^n \mu(A_j \cap E_k) c_k \\ &= \sum_{k=1}^n c_k \sum_{j=1}^m \mu(A_j \cap E_k) \\ &= \sum_{k=1}^n \mu(E_k) c_k. \end{aligned}$$

□

Proposition (Monotonicity): Let (X, m, μ) be a measure space, $f, g : X \to [0, \infty]$ measurable and $f \leq g$ μ a.e. Then

$$\int_X f d\mu \leq \int_X g d\mu.$$

Proof. Suppose $f(x) \leq g(x)$ for all $x \notin E$ with $\mu(E) = 0$. Let $P = \{A_1, \ldots, A_n\}$

be a partition of X. Then, $q = \{A_1 \setminus E, \dots, A_n \setminus E, E\}$ is also a partition.

$$\begin{aligned} L(f,P) &= \sum_{i=1}^{n} \mu(A_i) \inf_{A_i} f \\ &= \sum_{i=1}^{n} (\mu(A_i \cap E) + \mu(A_i \setminus E)) \inf_{A_i} f \\ &= \sum_{i=1}^{n} \mu(A_i \setminus E) \inf_{A_i} f \\ &\leq \sum_{i=1}^{n} \mu(A_i \setminus E) \inf_{A_i \setminus E} f \\ &\leq \sum_{i=1}^{n} \mu(A_i \setminus E) \inf_{A_i \setminus E} g \\ &= \sum_{i=1}^{n} \mu(A_i \setminus E) \inf_{A_i \setminus E} g + \mu(E) \inf_{E} g \\ &= L(g,q). \end{aligned}$$

Now take sup over P. $\square$

8.2 Monotone convergence theorem

Theorem (Alternate formulation of integral): Let (X, m, μ), $f : X \to [0, \infty]$ a measurable function. Then,

$$\int_X f d\mu = \sup \left\{ \sum_{k=1}^{m} \mu(A_k) c_k \right\},$$

where the sup is taken over $A_1, \dots, A_m$ disjoint measurable, $c_1, \dots, c_m \in [0, \infty)$, and f upper bounds the corresponding simple function.

Proof. Monotonicity tells us

$$\int_X f d\mu \geq \int_X \sum_{j=1}^{m} c_j \chi_{A_j} = \sum_{j=1}^{m} c_j \mu(A_j).$$

Take sup over the simple function on the RHS.

On the other hand, assume first that $\mu(A) > 0 \implies \inf_A f < \infty$. Let $P = \{A_1, \dots, A_m\}$ be an arbitrary partition of X. Note we use that $0 \cdot \infty = 0$.

$$L(f,P) = \sum_{k=1}^{m} \mu(A_k) \inf_{A_k} f.$$

Since we can let $c_k := \inf_{A_k} f$, this is a "valid simple function" (by statement), and thus upper bounded by the sup of all "valid simple functions".

If $\mu(A) > 0$ and $\inf_A f = \infty$ for some $A \in m$, then we show the RHS is ∞. Indeed, it must upper bound $\mu(A)c$ for all $c > 0$ since $\chi_{A_k} c$ lower bounds f. $\square$

Theorem (Monotone Convergence Theorem): Let (X, m, μ) and consider a sequence of measurable functions

$$0 \leq f_1 \leq \cdots \leq f_n \to f$$

for some $f : X \to [0, \infty]$. Then,

$$\int_X f d\mu = \lim_{k \to \infty} \int_X f_k d\mu.$$

Proof. We can show the limit is upper bounded by the integral of f directly by monotonicity. For the other side, we prove the statement for a simple function $s \leq f$ and take supremum over all simple functions, which works since f_k is increasing so $\lim_{k \to \infty}$ is equivalent to $\sup_k$ and we can swap sups.

Let f bound some simple function, $f \geq \sum_{j=1}^m c_j \chi_{A_j}$. Let $t \in (0, 1)$ (we will be taking $t \to 1$) and define

$$E_k = \{x \in X : f_k(x) \geq t \sum_{i=1}^m c_i \chi_{A_i}(x)\}$$

such that $\bigcup_k E_k = X$. Then, for fixed $k \in \mathbb{N}$

$$f_k \geq t \sum_{i=1}^m c_i \chi_{A_i \cap E_k},$$

and by monotonicity and integral of simple function,

$$\int_X f_k d\mu \geq t \sum_{i=1}^m c_i \mu(A_i \cap E_k).$$

Taking $k \to \infty, t \to 1$, and using limit of measure of increasing sets,

$$\lim_{k \to \infty} \int_X f_k d\mu \geq \sum_{i=1}^m c_i \mu(A_i),$$

which proves the desired inequality taking supremum over all valid step functions. □

Example (Converse of MCT isn't true): Consider the marching intervals,

$$\begin{aligned} f_1 &= \chi_{[0,1/2]} \\ f_2 &= \chi_{[1/2,1]} \\ f_3 &= \chi_{[0,1/4]} \\ f_4 &= \chi_{[1/4,1/2]} \\ f_5 &= \chi_{[1/2,3/4]} \\ &\vdots \end{aligned}$$

Clearly $\int_{[0,1]} f_n d\mu \to 0$, however these functions don't converge to 0.

Example (MCT may fail for decreasing functions): Let $f_n = \frac{1}{n}\chi_{[n,\infty)}$. Then

$$\lim_{n\to\infty} \int f d\mu = \infty \neq 0 = \int \lim_{n\to\infty} f_n d\mu.$$

8.3 Properties of integral

We need to prove two things along the way to showing linearity of the integral.

Proposition (Simple function representation doesn't matter): Suppose $a_1, \ldots, a_m$, $b_1, \ldots, b_n$ in $[0, \infty]$, with not necessarily disjoint $A_1, \ldots, A_m$ and $B_1, \ldots, B_n$. Suppose the corresponding simple functions are equal, i.e.

$$\sum_{j=1}^{m} a_j \chi_{A_j} = \sum_{j=1}^{n} b_k \chi_{B_k}.$$

Then,

$$\sum_{j=1}^{m} a_j \mu(A_j) = \sum_{k=1}^{n} b_k \mu(B_k).$$

Proof. If $A_i \cap A_j \neq \varnothing$, then

$$a_i \chi_{A_i} + a_j \chi_{A_j} = a_i \chi_{A_i \setminus A_j} + a_j \chi_{A_j \setminus A_i} + (a_i + a_j)\chi_{A_i \cap A_j},$$

and the integral-type sums remain the same, so we can continue making these substitutions until all sets A_i are disjoint. Then, use additivity of measure to union all sets with the same value. This leaves us with the standard representation. The equality of the simple functions implies the equality of their standard representations, and thus the equality of their corresponding integral-type sums. $\square$

Proposition (Integral of arbitrary simple function representation): Let (X, m, μ), $E_1, \ldots, E_n \in m$, and $c_1, \ldots, c_n \in [0, \infty]$. Then,

$$\int \sum_{k=1}^{n} c_k \chi_{E_k} d\mu = \sum_{k=1}^{n} c_k \mu(E_k).$$

Proof. Write the simple function on the LHS in standard form and apply integral of simple function in standard form. By previous proposition this sum is equal to the integral-type sum of the original simple function. □

Proposition (Additivity): Let (X, m, μ) and $f, g : X \to [0, \infty]$ measurable. Then,

$$\int (f + g) d\mu = \int f d\mu + \int g d\mu.$$

Proof. Note that by the previous proposition, simple functions are additive over the integral.

Take f_k increasing simple measurable functions converging to f pointwise, and g_k likewise converging to g. So $(f_k + g_k) \to (f + g)$. By the monotone convergence theorem,

$$\int (f + g) d\mu = \lim_{k\to\infty} \int (f_k + g_k) d\mu.$$

Then by additivity of simple functions over integral and MCT once again,

$$\lim_{k\to\infty} \int (f_k + g_k) d\mu = \lim_{k\to\infty} \int f_k d\mu + \lim_{k\to\infty} \int g_k d\mu = \int f d\mu + \int g d\mu.$$

□

Remark: Recall that the lower Riemann integral isn't additive, as if $f = \chi_{\mathbb{Q}\cap[0,1]}$ and $g = \chi_{[0,1]\setminus\mathbb{Q}}$, $L(f + g, [0, 1]) = 1$ but $L(f, [0, 1]) = L(g, [0, 1]) = 0$. However we have just shown that the lower Lebesgue integral is additive (over measurable functions). Indeed, since $\mathbb{Q} \cap [0, 1]$ has measure 0, $f + g$ has integral 1, f has integral 0, and g has integral 1.

Proposition (Properties of nonnegative integrals): Let (X, m, μ), $f, g : X \to [0, \infty]$ measurable functions. Then,

(a) If $c \in [0, \infty)$, then

$$\int_X c f d\mu = c \int_X f d\mu.$$

(b) $\int_X f d\mu = 0$ if and only if $f = 0$ μ-a.e.

(c) Let $E \in m$, $\mu(E) = 0$. Then,

$$\int_E f d\mu = 0.$$

(d) If $\int_X f d\mu < \infty$, then $f(x) < \infty$ μ-a.e.

(e) If $E \in m$,

$$\int_E f d\mu = \int_X (\chi_E f) d\mu.$$

Proof.

(a) Fix a partition P. It's easy to see that $\mathcal{L}(cf, P) = c\mathcal{L}(f, P)$. It follows that $\int_X cf d\mu = \sup_P \mathcal{L}(cf, P) = \sup_P c\mathcal{L}(f, P) = c\int_X f d\mu$.

(b) Suppose $\int_X f d\mu = 0$ but $f > 0$ on some set E with positive measure. Then $\int_X f d\mu \geq \mu(E) \inf_E f > 0$. On the other hand suppose $f = 0$ μ-a.e. Then for every set E where $\inf_E f > 0$, E is a subset of a set of measure zero so has measure zero. So $\int_X f d\mu = 0$.

(c) For any partition P of E, my monotonicity all sets have measure 0. Thus $\int_E f d\mu = 0$.

(d) Suppose $f(x) = \infty$ on a set E with positive measure. Then $\int_E f d\mu \geq \mu(E) \inf_E f = \infty$.

(e) Consider an arbitrary partition P of E. Then, $q = P \cup \{X \setminus E\}$ is a partition of X with $\mathcal{L}(f, P, E) = \mathcal{L}(f, q, X)$, since over E, $f = \chi_E f$. Thus

$$\int_E f d\mu \leq \int_X (\chi_E f) d\mu.$$

On the other hand, consider an arbitrary partition P of X. $\mathcal{L}(\chi_E f, P, X)$ is determined by the sets in P contained in E. Take q as these sets along with the remainder of E, then $\mathcal{L}(\chi_E f, P, X) \leq \mathcal{L}(f, q, E)$, so

$$\int_X \chi_E f d\mu \leq \int_E f d\mu.$$

□

Corollary (Integral over disjoint sets): Suppose E, F partition X. Then,

$$\int_X f d\mu = \int_E f d\mu + \int_F f d\mu.$$

In particular, this means integrals are equivalent up to μ a.e., since we may decompose into the sum of an integral over the good set and the integral over a set of measure 0, which is 0.

Proof.

$$\int_X f d\mu = \int_X f(\chi_E + \chi_F) d\mu = \int_X f\chi_E d\mu + \int_X f\chi_F d\mu = \int_E f d\mu + \int_F f d\mu.$$

□

Proposition (Integral of nonnegative series): Let (X, m, μ), f_n measurable functions $X \to [0, \infty]$. Then,

$$\sum_n \int_X f_n d\mu = \int_X (\sum_n f_n) d\mu.$$

Proof. Define the increasing partial sums $s_n = \sum_{i=1}^{n} f_i$. By MCT,

$$\lim_{n\to\infty} \int_X s_n d\mu = \int_X (\sum_n f_n) d\mu.$$

Then, working from the LHS and using linearity of integral,

$$\lim_{n\to\infty} \int_X s_n d\mu = \lim_{n\to\infty} \sum_{i=1}^{n} \int_X f_n d\mu = \sum_n \int_X f_n d\mu.$$

$\square$

Corollary (Tonelli's theorem for series): Let $a_{n,k} \geq 0$. Then,

$$\sum_n \sum_k a_{n,k} = \sum_k \sum_n a_{n,k}.$$

Proof. Let $X = \mathbb{N}$, μ the counting measure. Define $f_n : \mathbb{N} \to [0, \infty)$ by $k \mapsto a_{n,k}$.

$$\sum_n \sum_k a_{n,k} = \sum_n \int_X f_n d\mu = \int_X (\sum_n f_n) d\mu = \sum_k \sum_n a_{n,k}.$$

$\square$

Theorem (Fatou's Lemma): Let (X, m, μ) and $f_n : X \to [0, \infty]$ measurable.

$$\int \liminf_n f_n d\mu \leq \liminf_n \int f_n d\mu.$$

Proof. By MCT,

$$\int \liminf_n f_n d\mu = \lim_{n\to\infty} \int \inf_{k\geq n} f_n d\mu.$$

By monotonicity of integral $\int \inf_{k\geq n} f_n d\mu \leq \int f_i d\mu$ for all $i \geq n$, so

$$\lim_{n\to\infty} \int \inf_{k\geq n} f_n d\mu \leq \lim_{n\to\infty} \inf_{k\geq n} \int f_n d\mu = \liminf_n \int f_n d\mu.$$

$\square$

Note that when $f_n \to f$ pointwise, Fatou's lemma tells us

$$\int f d\mu \leq \liminf_n \int f_n d\mu.$$

In general, Fatou's lemma tells us that we only "lose mass" from the integral in the limit. For example, if

$$f_n(x) = n\chi_{[0,1/n]},$$

$f_n \to 0$ so $\int \lim_{n\to\infty} f_n d\mu = 0$ but the integrals of each f_n is always 1.

8.4 Functions of arbitrary sign

Definition (f^+, f^-): Let $f : X \to \overline{\mathbb{R}}$. Define

$$f^+ = \max\{f, 0\}, f^- = \max\{-f, 0\}.$$

It follows that $f = f^+ + f^-$ and $|f| = f^+ + f^-$.

Definition (Lebesgue integrable): Let (X, m, μ), $f : X \to \overline{\mathbb{R}}$ measurable. If at least one of the two integrals

$$\int_X f^+ d\mu, \int_X f^- d\mu$$

is finite then we define the Lebesgue integral of f to be

$$\int_X f d\mu = \int_X f^+ d\mu - \int_X f^- d\mu.$$

If both are finite then f is said to be *Lebesgue integrable.*

If f is a topological space, then f is said to be *locally integrable* if $f|_K$ is integrable for all $K \subseteq X$ finite (wrt $(K, m|_K, \mu|_{m|_K})$).

Remark: $f : X \to \overline{\mathbb{R}}$ is Lebesgue integrable if and only if $\int_X |f| d\mu < \infty$.

Definition ($L_1(X)$): Let (X, m, μ). We define a family of integrable (measurable) functions

$$L^1(x) \coloneqq \{f : X \to \overline{\mathbb{R}} : f \text{ is integrable}\}.$$

We further define the L^1 norm as

$$\|f\|_{L^1} = \int_X |f| d\mu,$$

abbreviated $\|f\|_1$.

The following theorem provides a more robust condition of Riemann integrability than what we see in Analysis 1.

Theorem (Lebesgue-Vitali): A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if

$$\mathcal{L}(\{x \in [a, b] : f \text{ not continuous at } x\}) = 0.$$

Moreover, if f is Riemann integrable, the Riemann integral coincides with the Lebesgue integral.

Proof. Omitted. □

Proposition (Approximation of integral by continuous functions): Let $f : \mathbb{R}^N \to \mathbb{R}$, $f \in L^1(\mathbb{R}^N)$. Let $\varepsilon > 0$. There exists $g : \mathbb{R}^N \to \mathbb{R}$ continuous such that $\|f - g\|_{L^1} < \varepsilon$.

Proof. Let $f = f^+ - f^-$. Fix $\varepsilon > 0$. There exists s_1, s_2 simple such that

$$\int f^+ d\mu < \int s_1 d\mu + \varepsilon/4, \int f^- d\mu \leq \int s_2 d\mu + \varepsilon/4.$$

Take s_1, s_2 in standard form:

$$\int s_1 d\mu = \sum_{k=1}^{m} a_k \mathcal{L}(A_k) < \infty, \int s_2 d\mu = \sum_{k=1}^{n} b_k \mathcal{L}(B_k) < \infty.$$

So in particular each A_k has finite measure, so we can find G_k^1 open and F_k^1 compact, $F_k^1 \subseteq A_k \subseteq G_k^1$ and

$$\mathcal{L}(G_k^1 \setminus F_k^1) < \frac{\varepsilon}{4m} \frac{1}{\max\{a_k\}}.$$

Similarly for each B_k find G_k^2, F_k^2 so that $\mathcal{L}(G_k^2 \setminus F_k^2) < \frac{\varepsilon}{4n} \frac{1}{\max\{b_k\}}$.

For $i = 1, 2$ since $F_k^i, X \setminus G_k^i$ are disjoint closed sets, by Urysohn's lemma there exists a continuous $\phi_k^i : \mathbb{R}^N \to [0, 1]$ (basically a continuous approximation of the characteristic function that uses normality of topological space) such that

$$\phi_k^i|_{F_k^i} = 1, \phi_k^i = 0 \text{ outside } G_k^i.$$

Define the continuous

$$g = \sum_{k=1}^{m} a_k \phi_k^1 - \sum_{j=1}^{n} b_j \phi_j^2.$$

Now,

$$\begin{aligned}
\int |f - g| dx &= \int \left| (f^+ - f^-) - \sum_{k=1}^{m} a_k \phi_k^1 + \sum_{j=1}^{n} b_j \phi_j^2 \right| dx \\
&\leq \int \left| f^+ - \sum_{k=1}^{m} a_k \phi_k^1 \right| + \left| f^- - \sum_{j=1}^{n} b_j \phi_j^2 \right| dx
\end{aligned}$$

Let's consider just the f^+ side for simplicity.

$$\begin{aligned}
\cdots &\leq \int |f^+ - s_1| dx + \int \left| \sum_{k=1}^{m} a_k \chi_{A_k} - \sum_{k=1}^{m} a_k \phi_k^1 \right| dx \\
&\leq \varepsilon/4 + \sum_{k=1}^{m} a_k \int |\chi_{A_k} - \phi_k^1| dx && (\Delta, \text{linearity}) \\
&\leq \varepsilon/4 + \sum_{k=1}^{m} a_k \int_{G_k^1 \setminus F_k^1} |\chi_{A_k} - \phi_k^1| dx && (\text{def } \phi) \\
&\leq \varepsilon/4 + \sum_{k=1}^{m} a_k \mathcal{L}(G_k^1 \setminus F_k^1) \\
&< \varepsilon/4 + \varepsilon/4.
\end{aligned}$$

Thus overall

$$\int |f - g| dx < \varepsilon.$$

□

Example: Let $f : [\pi, \infty) \to \mathbb{R}$ by $x \mapsto \frac{\sin x}{x}$ which is bounded and continuous. We claim this is not Lebesgue integrable since

$$\int_{[\pi,\infty)} |\frac{\sin x}{x}| d\mu = \sum_k \int_{(2k-1)\pi}^{2k\pi} \frac{\sin x}{x}\,\mathrm{d} \geq \sum_k \frac{1}{(2k-1)\pi} \int_{(2k-1)\pi}^{2k\pi} |\sin x|\,\mathrm{d}x$$

By periodicity,

$$= \sum_k \frac{1}{(2k-1)\pi} \int_{\pi}^{2\pi} |\sin x|\,\mathrm{d}x = \infty.$$

We want to show $\lim_{l\to\infty} \int_\pi^l \frac{\sin x}{x}\,\mathrm{d}x$ is finite. Use integration by parts. Then, claim $l \mapsto \int_\pi^l \frac{\cos x}{x^2}\,\mathrm{d}x$ is a Cauchy sequence.

Proposition (Properties of integral for real functions): Let (X, m, μ) with $f, g : X \to \overline{\mathbb{R}}$ measurable functions.

(a) If f, g integrable and $\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g$ is integrable,

$$\int_X (\alpha f + \beta g) d\mu = \alpha \int_X f d\mu + \beta \int_X g d\mu,$$

(b) If f, g integrable and $f \leq g$ μ-a.e.,

$$\int_X f d\mu \leq \int_X g d\mu.$$

(c) $|\int_X f d\mu| \leq \int_X |f| d\mu$.

(d) If f integrable, $\mu(\{x \in X : |f(x)| = \infty\}) = 0$.

(e) If $f = g$ μ-a.e., then f is integrable if and only if g is integrable, and

$$\int_X f d\mu = \int_X g d\mu.$$

Proof.

(a) Write $f = f^+ - f^-$, $g = g^+ - g^-$, etc.

(b) Let $f < g$ on a set E of measure 0.

$$\int_X g d\mu - \int_X f d\mu = \int_X (g - f) d\mu = \int_{X\setminus E} (g - f) d\mu + \int_E (g - f) d\mu \geq 0.$$

(c) Use $f = f^+ - f^-$, then

$$\left|\int_X f^+ - f^- d\mu\right| \leq \int_X f d\mu + \int_X f^- d\mu = \int_X |f| d\mu.$$

(d) Since $\int_X |f| d\mu < \infty$, $|f|$ can only be ∞ on a set of measure 0 by property of nonnegative integrals.

(e) Let E be the set on which $f = g$. If f is integrable, then

$$\int_X |f| d\mu = \int_E |f| d\mu = \int_E |g| d\mu = \int_X |g| d\mu < \infty,$$

and vice versa. And similar argument of restricting the integral shows that $\int_X f d\mu = \int_X g d\mu$.

$\square$

Corollary: Let (X, m, μ). Then $(L^1(X), \|.\|_1)$ is a normed space.

Proof.

(a) $\int_X |f| d\mu = 0$ if and only if $f = 0$ almost everywhere.

(b) $\int_X |\lambda f| d\mu = \lambda \int_X |f| d\mu$ by linearity.

(c) $\int_X |f + g| d\mu \leq \int_X |f| + |g| d\mu = \int_X |f| d\mu + \int_X |g| d\mu$.

$\square$

8.5 Dominated convergence theorem

Theorem (Dominated convergence theorem): Let (X, m, μ), $f_n \to \overline{\mathbb{R}}$ measurable functions with limit f μ a.e. Assume there exists an integrable function $g : X \to [0, \infty]$ such that $|f_n(x)| \leq g(x)$ μ a.e. Then f is Lebesgue integrable and

$$\int_X f_n d\mu \to \int_X f d\mu.$$

Proof. Using $\liminf_n f_n = f$ μ-a.e.,

$$\int_X (f + g) d\mu = \int_X \liminf_n (f_n + g) d\mu \leq \liminf_n \int_X (f_n + g) d\mu.$$

Subtracting $\int_X g d\mu$ from both sides gives $\int_X f d\mu \leq \liminf_n \int_X f_n d\mu$. It remains to show $\int_X f d\mu \geq \limsup_n \int_X f_n d\mu$. Now

$$\int_X (g - f) d\mu = \int_X \liminf_n (g - f_n) d\mu \leq \liminf_n \int_X (g - f_n) d\mu.$$

Subtracting $\int_X g d\mu$ from both sides, we have

$$-\int_X f d\mu \leq \liminf_n \int_X -f_n d\mu = -\limsup_n \int_X f_n d\mu.$$

Negating both sides completes the proof. $\square$

Remark (DCT implies convergence in L^1): Note that $|f_n - f|$ is dominated by $2g$, thus

$$\lim_{n \to \infty} \int_X |f_n - f| d\mu = 0.$$

Example (*g* is necessary): If $X = [0,1]$, $f_n = n\chi_{[0,1/n]}$, then $f_n \to 0$ pointwise a.e. on $[0,1]$, but

$$\lim_{n\to\infty}\int_{[0,1]} f_n dx = 1 \neq 0 = \int_{[0,1]} \lim_{n\to\infty} f_n dx.$$

Example (Application to Riemann integration): Recall that if f_n is Riemann integrable and $|f_n(x)| \leq M$ for $x \in [a,b], n \in \mathbb{N}$, and if $f_n \to f$, then

$$\int_a^b f\,\mathrm{d}x = \lim_{n\to\infty}\int_a^b f_n\,\mathrm{d}x.$$

We can (almost) prove this by Lebesgue-Vitali. Since f_n bounded and Riemann integrable, then f_n are Lebesgue integrable. It follows that

$$\lim_{n\to\infty}\int_a^b f_n\,\mathrm{d}x = \lim_{n\to\infty}\int_{[a,b]} f_n dx = \int_{[a,b]} f dx.$$

We cannot conclude that the Lebesgue and Riemann integrals are equal since the pointwise limit f is not necessarily Riemann integrable.

We now give an analog of the theorem we proved using MCT that the series of integrals of nonnegative functions is equal to the integral of the series.

Proposition (Integral of arbitrary series): Let (X, m, μ), $f_n : X \to \overline{\mathbb{R}}$ measurable functions with

$$\sum_n \int_X |f_n| d\mu < \infty.$$

Then the series $\sum_n f_n$ converges μ a.e., is integrable, and

$$\sum_n \int_X f_n d\mu = \int_X (\sum_n f_n) d\mu.$$

Proof. Using the result for nonnegative functions,

$$\sum_n \int |f_n| d\mu = \int (\sum_n |f_n|) d\mu.$$

Since this is finite by assumption, the series is integrable. In particular, $\sum_n |f_n| < \infty$ μ a.e., so $\sum_n f_n$ converges absolutely μ a.e.

Define $g = \sum_n |f_n|$ and $s_n = \sum_{k=1}^n f_k$ with clearly $s_n \to \sum_n f_n$. By DCT, since g bounds $|s_n|$ μ a.e.,

$$\lim_{n\to\infty}\int_X s_n d\mu = \int_X \sum_n f_n d\mu.$$

Rewriting the LHS completes the proof:

$$\lim_{n\to\infty}\int_X s_n d\mu = \lim_{n\to\infty}\sum_{k=1}^{n}\int_X f_k d\mu = \sum_n \int_X f_n d\mu.$$

□

8.6 More examples

Example (3A.4): Consider the positive, Borel measurable $f : [0,1] \to (0,\infty)$,

$$f(x) = \begin{cases} \dfrac{1}{n}, & \text{if } x \in \mathbb{Q}, x = \dfrac{m}{n}; \\ 1, & \text{otherwise.} \end{cases}$$

Then $L(f,[0,1]) = 0$ since the inf over any interval will approach 0. But $\int_{[0,1]} f d\mu > 0$ since $\mu\{x \in X : f(x) > 0\} > 0$.

9 Modes of convergence

9.1 Definitions and examples

Definition: Let (X, m, μ), $f_n, f : X \to \mathbb{R}$ measurable functions.

(a) $f_n \to f$ pointwise μ-a.e. if it converges for all but on a set of measure 0.

(b) $f_n \to f$ *almost uniformly* if for all $\varepsilon > 0$ there exists $E \in m$, $\mu(E) < \varepsilon$ such that $f_n \to f$ uniformly on $X \setminus E$, i.e.

$$\lim_{n\to\infty} \sup_{x \in X\setminus E} |f_n(x) - f(x)| = 0.$$

(c) $f_n \to f$ *in measure* if for all $\varepsilon > 0$,

$$\lim_{n\to\infty} \mu(\{x \in X : |f_n(x) - f(x)| > \varepsilon\}) = 0.$$

(d) $f_n \to f$ in $L_1(X)$ if

$$\|f_n - f\|_1 = \int_X |f_n - f| d\mu \to 0.$$

(e) $f_n \to f$ *weakly in $L^1(X)$* if for all $g : X \to \mathbb{R}$ measurable and bounded,

$$\int_X f_n g d\mu \to \int_X f g d\mu.$$

Example (Almost uniform convergence): Let $X = [0, 1]$,

$$f_n(x) = x^n \to f(x) = \begin{cases} 1, & \text{if } x = 1; \\ 0, & \text{otherwise.} \end{cases}$$

The limit is discontinuous so continuous functions cannot converge uniformly (in fact even if $X = [0, 1)$ with $f = 0$, there is no uniform convergence).

But for fixed $\varepsilon > 0$, if we choose $E = [1 - \varepsilon/2, 1]$, then f_n converges uniformly on $[0, 1] \setminus E$.

Example (Pointwise μ a.e. does not imply others): Let $X = \mathbb{R}$,

$$f_n(x) = \chi_{[n,\infty)} \to 0.$$

But the set

$$\{x \in X : |f_n(x) - f(x)| > 1/2\} = [n, \infty)$$

will always have infinite measure, so f_n cannot converge to 0 in measure and hence in $L_1(\mathbb{R})$. Fix $\varepsilon > 0$ and n. For any set E with $\mu(E) < \varepsilon$, $\mathbb{R} \setminus E$ must contain some point in $[n, \infty)$, so the convergence isn't uniform, and f_n cannot converge to 0 almost uniformly. Note that this also shows we can't have convergence in $L^1(\mathbb{R})$.

Proposition (Convergence in $L_1(X)$ is strongest): Suppose $f_n \to f$ in $L_1(X)$. Then,

(a) $f_n \to f$ in measure, which means there is a subsequence $f_{n_k} \to f$ almost uniformly, which means $f_{n_k} \to f$ pointwise μ-a.e.

(b) $f_n \to f$ weakly.

Theorem (Chebyshev inequality): Let $f \in L_1(X)$, $c > 0$. Then

$$\mu(\{x \in X : |f(x)| > c\}) \leq \frac{1}{c}\int_X |f| d\mu.$$

Proof. Let $E = \{x \in X : |f| \geq c\}$. Then

$$\int_X |f| d\mu \geq \int_E |f| d\mu \geq \int_E c d\mu = c\mu(E).$$

Divide both sides by c. □

Theorem:

(a) $f_n \to f$ almost uniformly implies $f_n \to f$ in measure and pointwise μ-a.e.

(b) $f_n \to f$ in measure implies there is subsequence f_{n_k} such that $f_{n_k} \to f$ almost uniformly.

(c) $f_n \to f$ in $L_1(X)$ implies $f_n \to f$ weakly in $L_1(X)$.

(d) $f_n \to f$ in $L_1(X)$ implies $f_n \to f$ in measure.

Proof.

(a) Suppose $f_n \to f$ almost uniformly. We claim $f_n \to f$ in measure, i.e. for fixed $\varepsilon > 0$

$$\lim_{n\to\infty} \mu(\{x \in X : |f_n(x) - f(x)| \geq \varepsilon\}) = 0.$$

So fix $\delta > 0$. We find an N after which the measure is less than δ. By almost uniform convergence, there exists $E_\delta \in m$, $\mu(E_\delta) < \delta$, $f_n \to f$ uniformly on $X \setminus E_\delta$, i.e. there exists an N such that for $n \geq N$,

$$\sup_{x \in X\setminus E_\delta} |f_n(x) - f(x)| < \varepsilon.$$

In turn,

$$\mu(\{x \in X : |f_n(x) - f(x)| \geq \varepsilon\}) \leq \mu(E_\delta) < \delta.$$

To show pointwise almost everywhere convergence, for each $n \in \mathbb{N}$ we can choose some $\mu(E_n) < 1/n$ such that $f_n \to f$ uniformly on $X \setminus E_n$. Let $E = \bigcap_n E_n$ clearly with $\mu(E) = 0$ and $f_n \to f$ on $X \setminus E$.

(b) Suppose $f_n \to f$ in measure. Fix $k \in \mathbb{N}$ and use convergence in measure to choose n_k such that for $n \geq n_k$,

$$\mu(\{x \in X : |f_n(x) - f(x)| > 1/2^k\}) \leq \frac{1}{2^{k+1}}.$$

We claim $\{f_{n_k}\}$ converges to f almost uniformly. Define

$$E_k = \{x \in X : |f_{n_k}(x) - f(x)| > 1/2^k\}, F_k = \bigcup_{j \geq k} E_k$$

so that $\mu(F_k) \leq \frac{1}{2^k}$. Fix $\varepsilon > 0$ and choose k such that $\frac{1}{2^k} < \varepsilon$. We claim $f_{n_k} \to f$ uniformly on $X \setminus F_k$. Indeed, fix $\delta > 0$ and choose $\frac{1}{2^m} < \delta$ so that for $j \geq m$, $|f_{n_j} - f| < \delta$ on $X \setminus F_k$.

(c) Suppose $f_n \to f$ in L^1, and suppose $g : X \to \mathbb{R}$ is measurable and bounded by M. So

$$\int_X f_n g - f g d\mu \leq \int_X M|f_n - f| d\mu,$$

which tends to 0 as $n \to \infty$ by convergence of f_n in L^1.

(d) Immediately by Chebyshev:

$$\mu(\{x \in X : |f_n(x) - f(x)| > \varepsilon\}) \leq \frac{1}{\varepsilon} \int_X |f_n(x) - f(x)| d\mu \to 0.$$

□

Proposition: Suppose $f_n \to f$ in L^1. Then there exists $g : X \to [0, \infty]$ integrable and subsequence f_{n_k} such that $|f_{n_k}(x)| \leq g(x)$ μ a.e.

Proof. By L^1 convergence,

$$\lim_{n \to \infty} \int_X |f_n(x) - f(x)| d\mu = 0.$$

Let n_k be threshold for

$$\int_X |f_{n_k}(x) - f(x)| d\mu < 1/2^k.$$

Define $w(x) = \sum_k |f_{n_k}(x) - f(x)|$ such that by series interchange $\int_X w d\mu \leq 1$, i.e. $w < \infty$ μ a.e. (redefine to finite as necessary). Define $g(x) = w(x) + |f(x)|$. Then,

$$|f_{n_k}(x)| \leq |f_{n_k}(x) - f(x)| + |f(x)| \leq w(x) + |f(x)| = g(x).$$

□

Example (In measure does not imply pointwise): Let $X = [0, 1)$. Consider the cascading intervals

$$[0, \frac{1}{2}), [\frac{1}{2}, 1), [0, \frac{1}{3}), [\frac{1}{3}, \frac{2}{3}), \ldots$$

Let f_n be the characteristic function of the nth interval. Then

$$\int_{[0,1)} |f_n - f| dx = \int_{[0,1)} f_n dx \to 0.$$

But f_n clearly cannot converge to 0 pointwise.

9.2 Equi-integrability

Definition: Let (X, m, μ) and $\mathcal{F}$ a family of measurable functions $f : X \to \overline{\mathbb{R}}$. $\mathcal{F}$ is said to be *equi-integrable* if for all $\varepsilon > 0$ there is a $\delta > 0$ such that when $E \in m, \mu(E) < \delta$,

$$\sup_{f \in \mathcal{F}} \int_E |f| d\mu < \varepsilon.$$

Proposition (Equi-integrability):

(a) If f is integrable, then $\{f\}$ is equi-integrable.

(b) A finite family of integrable functions are equi-integrable.

(c) Any family $\mathcal{F}$ where there exists an integrable g with

$$|f(x)| \leq g(x),$$

for μ a.e. $x \in X$ and all $f \in \mathcal{F}$ is equi-integrable.

Proof.

(a) Suppose $f \in L^1$. By Chebyshev, $\mu(\{x \in X : |f(x)| > n\}) \to 0$. Denote the inner sets E_n. Then, by DCT, since the functions $\chi_{E_n}|f|$ are bounded by integrable f and converge μ a.e. to 0,

$$\int_{E_n} |f| d\mu \to 0.$$

So choose n_ε such that for $n \geq n_\varepsilon$,

$$\int_{E_n} |f| d\mu < \varepsilon/2.$$

Let $\delta = \frac{\varepsilon}{2n_\varepsilon}$. Let $E \in m$ with $\mu(E) < \delta$. Then

$$\int_E |f| d\mu \leq \int_{E_{n_\varepsilon}} |f| d\mu + \int_{E \setminus E_{n_\varepsilon}} |f| d\mu \leq \varepsilon/2 + n_\varepsilon \mu(E \setminus E_{n_\varepsilon}) < \varepsilon.$$

(b) Let $f_1, \ldots f_n$ integrable. Then, $f_1 + \cdots + f_n$ is integrable and we can use (a).

(c) Directly by (a).

□

Proposition: If $f \in L^1$, then $\{f\}$ is reverse equi-integrable.

Proof. Let $G_n = \{x \in X : 1/n \leq f(x) \leq n\}$ so that $\chi_{G_n \cup \{f=0\}}|f| \to |f|$. By MCT,

$$\int_{G_n} |f| d\mu \to \int_X |f| d\mu.$$

So

$$\int_{X \setminus G_n} |f| d\mu \to 0.$$

By Chebyshev,

$$\mu(G_n) \leq n \int_X |f| d\mu < \infty.$$

□

Example (Equi-integrability but not integrable): Suppose that there is M such that $|f(x)| \leq M$ μ a.e. Then for fixed ε we can choose $\delta = \varepsilon/M$. For example, let $\mathcal{F} = \{f(x) = 1\}$, then $\mathcal{F}$ is equi-integrable but f is not integrable.

Theorem (Vitali's convergence theorem): Let (X, m, μ). Let $f_n, f : X \to \mathbb{R}$ integrable functions (this can also be $\overline{\mathbb{R}}$ by μ a.e. property). Then $f_n \to f$ in L^1 if and only if

(a) $f_n \to f$ in measure.

(b) $\{f_n\}$ is equi-integrable.

(c) $\{f_n\}$ is reverse equi-integrable, i.e. for every $\varepsilon > 0$ we can find $G \in m$ such that $\mu(G) < \infty$ and

$$\sup_n \int_{X \setminus G} |f_n| d\mu \leq \varepsilon.$$

Note that this is trivially satisfied if $\mu(X) < \infty$.

Proof. **Step 1:** Suppose we have (a), (b), and (c). Using (a) and (b), fixing $\varepsilon > 0$ we may assume that for $\mu(E) < \delta$,

$$\int_E |f_n| d\mu < \varepsilon, \int_E |f| d\mu < \varepsilon.$$

Using (b) and (c), there is E_ε with finite measure so that

$$\int_{X \setminus E_\varepsilon} |f_n| d\mu < \varepsilon, \int_{X \setminus E_\varepsilon} |f| d\mu < \varepsilon.$$

If $\mu(E_\varepsilon) = 0$:

$$\int_X |f_n - f| d\mu = \int_{X \setminus E_\varepsilon} |f_n - f| d\mu + 0 \leq \int_{X \setminus E_\varepsilon} |f_n| d\mu + \int_{X \setminus E_\varepsilon} |f| d\mu \leq 2\varepsilon$$

Otherwise invoke (a) to find N such that for $n \geq N$,

$$\mu(x \in X : |f_n - f| > \frac{\varepsilon}{\mu(E_\varepsilon)}) < \delta.$$

Then for $n \geq N$, defining the inner set above as A,

$$\begin{aligned}\int_X |f_n - f| d\mu &= \int_{X \setminus E_\varepsilon} |f_n - f| d\mu + \int_{E_\varepsilon} |f_n - f| d\mu \\ &= 2\varepsilon + \int_{E \cap A} |f_n - f| d\mu + \int_{E \cap A^c} |f_n - f| d\mu \\ &= 5\varepsilon.\end{aligned}$$

Step 2: Suppose $f_n \to f$ in L^1. (a) is satisfied by previous theorem. We show properties (b) and (c). We showed that a finite family of integrable functions is equi-integrable and reverse equi-integrable (c). Let N be such that for $n \geq N$,

$$\int_X |f_n - f| d\mu < \varepsilon.$$

Find $\varepsilon, \delta, E_\varepsilon$ on finite family $\{f_1, \ldots, f_N, f\}$ so that for $n = 1, \ldots, N$, $\mu(E) < \delta$,

$$\int_E |f_n| d\mu < \varepsilon, \int_E |f| d\mu < \varepsilon,$$

and

$$\int_{X \setminus E_\varepsilon} |f_n| d\mu < \varepsilon, \int_{X \setminus E_\varepsilon} |f| d\mu < \varepsilon.$$

For $n = 1, \ldots, N$ we are done. For $n > N$,

$$\int_E |f_n| d\mu \leq \int_E |f_n - f| d\mu + \int_E |f| d\mu = 2\varepsilon,$$

and

$$\int_{X \setminus E_\varepsilon} |f_n| d\mu = \int_{X \setminus E_\varepsilon} |f_n - f| d\mu + \int_{X \setminus E_\varepsilon} |f| d\mu < 2\varepsilon.$$

□

Example: $f_n = n\chi_{[0,1/n]}$ converges in measure to 0, satisfies VCT (c), but isn't equi-integrable. On the other hand, $f_n = \frac{1}{n}\chi_{[n,2n]}$ converges in measure to 0, is equi-integrable, and doesn't satisfy VCT (c).

Example: If $X = [0, 1]$, the sequence $f_n = \sin(nx)$ satisfies VCT (b) and (c) but doesn't converge in measure.

Theorem (Conditions for equi-integrability): Let (X, m, μ) and $\mathcal{F}$ a family of functions $f : X \to \overline{\mathbb{R}}$. Consider the conditions:

(a) $\mathcal{F}$ is equi-integrable.

(b) $\lim_{t\to\infty} \sup_{f\in\mathcal{F}} \int_{\{x\in X:|f|>t\}} |f| d\mu = 0$.

(c) (*De la Valle Poussin*) There exists an increasing function $\gamma : [0, \infty) \to [0, \infty]$ with

$$\lim_{t\to\infty} \frac{\gamma(t)}{t} = \infty$$

such that

$$\sup_{f\in\mathcal{F}} \int_X \gamma(|f|) d\mu < \infty.$$

(b) and (c) are equivalent and imply (a). If $\sup_{f\in\mathcal{F}} \int_X |f| d\mu < \infty$, then all three are equivalent.

Proof. **Step 1: (b) $\implies$ (a).** Fix $\varepsilon > 0$. Choose t_ε such that

$$\sup_f \int_{\{|f|>t\}} |f| d\mu < \varepsilon.$$

Let $E \in m$, $f \in \mathcal{F}$.

$$\int_E |f| d\mu = \int_{\{x\in E:|f|>t_\varepsilon\}} |f| d\mu + \int_{\{x\in E:|f|\le t_\varepsilon\}} |f| d\mu \le \varepsilon + \mu(E) t_\varepsilon.$$

So let $\delta = \frac{\varepsilon}{t_\varepsilon}$.

Step 2: (b) $\iff$ (c). Define k_i as increasing threshold so that for all f,

$$\int_{|f|>k_i} |f| d\mu < \frac{1}{2^i}.$$

Define b_l as the number of ks at most l. Then, define

$$\gamma(t) = b_{\lfloor t\rfloor} t, \frac{\gamma(t)}{t} \to \infty.$$

Now for any f

$$\int_X \gamma(|f|) d\mu = \sum_i \int_{\gamma(x)=i} |f| d\mu \le \sum_i \int_{\gamma(x)\ge i} |f| d\mu \le \sum_i \frac{1}{2^i} < \infty.$$

Conversely, define $M = \int_X \gamma(|f|) d\mu < \infty$. Fix $\varepsilon > 0$. Choose t_0 such that for $t \ge t_0$,

$$\frac{\gamma(t)}{t} > \frac{M}{\varepsilon}.$$

Then

$$\int_{|f|>t} |f| d\mu \le \frac{\varepsilon}{M} \int_{|f|>t} \gamma(t) d\mu \le \varepsilon.$$

Step 3: Suppose $\int_X |f| d\mu < C$. We show (a) $\implies$ (b), so all three statements are equivalent. Fix $\varepsilon > 0$ and find δ such that $\int_E |f| d\mu < \varepsilon$ for all $\mu(F) < \delta$. By Chebyshev, for fixed t,

$$\mu(x \in X : |f| > t) \leq \frac{1}{t} \int_X |f| d\mu.$$

So choose $t > \frac{C}{\delta}$ and we are done by equi-integrability. $\square$

10 Product Measures

10.1 Definitions

Definition (Product of two σ-algebras): Let $(X, m), (Y, n)$. Let $m \otimes n$ be defined as the smallest σ-algebra that contains $\{F \times G : F \in m, G \in n\}$.

Proposition: If X, Y are topological spaces,

$$\mathcal{B}(X) \otimes \mathcal{B}(Y) \subseteq \mathcal{B}(X \times Y).$$

If X, Y are separable metric spaces,

$$\mathcal{B}(X) \otimes \mathcal{B}(Y) = \mathcal{B}(X \times Y).$$

In particular,

$$\mathcal{B}(\mathbb{R}^N) = \mathcal{B}(\mathbb{R}) \otimes \cdots \otimes \mathcal{B}(\mathcal{R}).$$

Definition (Product outer measure): Let $(X, m, \mu), (Y, n, \nu)$. We define a *product outer measure* $(\mu \times \nu)^*$ on $X \times Y$ as follows:

(a) Elementary sets are $\{F \times G : F \in m, G \in n\}$, "rectangles" whose sides are measurable sets in F and G.

(b) $\rho(F \times G) = \mu(F)\nu(G)$.

By Caratheodory, define

$$m \times n := \{(u \times v)^* - \text{measurable sets}\}$$

such that $\mu \times \nu$ is a measure restricted to the $(\mu \times \nu)^*$-measurable sets.

Theorem: Let $(X, m, \mu), (Y, n, \nu)$. Let $F \in m, G \in n$. Then $F \times G$ is $(\mu \times \nu)^*$-measurable and $(\mu \times \nu)(F \times G) = \mu(F)\nu(G)$. In particular,

$$m \otimes n \subseteq m \times n.$$

Proof. Let $F_1, F_2 \in m, G_1, G_2 \in n$. Note that

$$(F_1 \times G_1) \cap (F_2 \times G_2) = (F_1 \cap F_2) \times (G_1 \cap G_2)$$
$$(F_1 \times G_1) \setminus (F_2 \times G_2) = ((F_1 \setminus F_2) \times G_1) \cup ((F_1 \cap F_2) \times (G_1 \setminus G_2)).$$

Fix $E \subseteq X \times Y$. Assume $E \subseteq (\bigcup_n F_n \times G_n)$ (for $F_n \in m, G_n \in n$). We then show

$$(\mu \times \nu)^*(E) \leq (\mu \times \nu)^*(E \cap (F \times G)) + (\mu \times \nu)^*(E \setminus (F \times G)),$$

using the above identities and taking infimum over F_n, G_n. Note that we already by definition $\mu \times \nu$ that

$$(\mu \times \nu)(F \times G) \leq \mu(F)\mu(G).$$

The other direction is subadditivity of ρ. $\square$

Proposition (ρ is subadditive): If $F, F_n \in m, G, G_n \in n, F \times G \subseteq \bigcup_n F_n \times G_n$, then

$$\mu(F)v(G) \leq \sum_n \mu(F_n)v(G_n).$$

Proof. Write

$$F \times G = \bigcup_n ((F_n \cap F) \times (G_n \cap G)).$$

We turn these into functions:

$$\chi_{F\times G}(x,y) \leq \sum_n \chi_{(F_n\cap F)\times(G_n\cap G)}(x,y) = \sum_n \chi_{F_n\cap F}(x)\chi_{G_n\cap G}(y).$$

Fix y and integrate with respect to x:

$$\mu(F)\chi_G(y) \leq \sum_n \mu(F_n \cap F)\chi_{G_n\cap G}(y).$$

Now integrate with respect to y:

$$\mu(F)\nu(G) \leq \sum_n \mu(F_n \cap F)\nu(G_n \cap G) \leq \sum_n \mu(F_n)\nu(G_n).$$

□

Corollary (Outer regularity): If $(X, m, \mu), (Y, n, v)$, $E \subseteq X \times Y$, then there exists $C \in m \otimes n, E \subseteq C$ such that

$$(u \times v)^*(E) = (u \times v)(C).$$

Proof. For each $k \in \mathbb{N}$ consider a covering of E such that $\sum_n \mu(F_n^{(k)})v(G_n^{(k)}) \leq (u \times v)^*(E) + 1/k$. Take C as the intersection of all coverings. □

Definition (finite and σ-finite measures): Let (X, m, μ). Then X is finite if $\mu(X) < \infty$. X is σ-finite if $X = \bigcup_n X_n$, $X_n \in m$, $\mu(X_n) < \infty$.

Proposition: Let $f : X \times Y \to \mathbb{R}$ be $m \otimes n$ measurable. Define

$$[f]_x : y \mapsto f(x,y), [f]_y : x \mapsto f(x,y).$$

Then $[f]_x, [f]_y$ are measurable.

Proof. Let $B \subseteq \mathbb{R}$ Borel. Then

$$([f]_x)^{-1}(B) = \{y \in Y : f(x,y) \in B\} = (f^{-1}(B))_x,$$

where $(f^{-1}(B))_x$ is a section as defined above. (? TODO)

Note that if $E \in m \otimes n$, then $E_x \in n, E_y \in m$ (Exercise). □

10.2 Tonelli and Fubini

Definition (Dynkin class): Let $X \neq \varnothing$. A collection $D \subseteq \mathcal{P}(X)$ is called a *Dynkin class* on X if

(a) $X \in D$

(b) $E, F \in D, E \subseteq F$, then $F \setminus E \in D$.

(c) $E_n \in D$ increasing, then $\bigcup_n E_n \in D$.

Theorem: Let $\mathcal{F} \in \mathcal{P}(X)$ be a family closed under finite intersections. Then, the σ-algebra generated by $\mathcal{F}$ coincides with the Dynkin class generated by $\mathcal{F}$.

Proof. Omitted. □

Theorem (Order of integration on characteristic function): Let $(X, m, \mu), (Y, n, \nu)$ with μ, ν complete. Suppose $E \subseteq m \times n$ has σ-finite $\mu \times v$ measure.

(a) For μ a.e. $x \in X$, the section

$$E_x := \{y \in Y : (x, y) \in E\} \in n,$$

and for ν a.e. $y \in Y$,

$$E_y := \{x \in X : (x, y) \in E\} \in m.$$

(b) The maps

$$x \mapsto \nu(E_x), y \mapsto \mu(E_y)$$

are measurable.

(c)

$$(\mu \times \nu)(E) = \int_Y \mu(E_y) d\nu(y) = \int_X \nu(E_x) d\mu(x).$$

Proof. Omitted. □

Corollary: We have Tonelli's theorem for characteristic functions:

$$\begin{aligned}\int_{X \times Y} \chi_E(x, y) d(\mu \times \nu)(x, y) &= \int_Y \left(\int_X \chi_{E_y}(x) d\mu(x) \right) d\nu(y) \\ &= \int_X \left(\int_Y \chi_{E_x}(y) d\nu(y) \right) d\mu(x).\end{aligned}$$

and by linearity it follows that we have Tonelli's theorem for nonnegative simple functions.

Example: Let $\mu = \nu = \mathcal{L}^N$. Let $G \subseteq \mathbb{R}^N$ not Lebesgue measurable. Then, $(\mu \times \nu)^*(\{0\} \times G) = 0$ but if G was $\mu \times \nu$ was measurable then G would be measurable by section.

Proposition: Let μ, ν σ-finite, μ, ν complete. Then, μ, ν are complete.

Proof. Let $X = \bigcup_n X_n, Y = \bigcup_n Y_n$ with $\mu(X_n) < \infty, \mu(Y_n) < \infty$. Let $F \in m, \mu(F) = 0, F' \subseteq F$. We want to show $F' \in m$.

Let

$$m_n = \{E \cap X_n : E \in m\}, n_m = \{G \cap Y_m : G \in n\}.$$

Let

$$\mu_n = \mu|_{m_n}, \nu_m = \nu|_{n_m},$$

since $\mu \times \nu$ complete, $(F' \cap X_n) \times Y_m \in m \times n$, and it is in particular in $m_n \otimes n_m$. By part 1 of previous proof, for ν-a.e.,

$$((F' \cap X_n) \times Y_m)_y \in m_n$$

$\square$

Theorem (Tonelli): Let $(X, m, \mu), (Y, n, \nu)$ with μ, ν complete and σ-finite. Let $f : X \times Y \to [0, \infty]$ is $\mu \times \nu$-measurable function. Then,

(a) $f(x, .)$ is measurable for μ a.e. $x \in X$ (and same for $f(., y)$).

(b) $\int_Y f(., y) d\nu(y)$ is measurable (and same for $\int_X f(x, .) d\mu(x)$).

(c) We can integrate f in either order:

$$\begin{aligned}\int_{X \times Y} f(x, y) d(\mu \times \nu)(x, y) &= \int_X \left(\int_Y f(x, y) d\nu(y) \right) d\mu(x) \\ &= \int_Y \left(\int_X f(x, y) d\mu(x) \right) d\nu(y).\end{aligned}$$

Proof. To show (d), there exists simple functions $s_n : X \times Y \to [0, \infty]$ nonnegative increasing to f. Repeatedly applying MCT,

$$\begin{aligned}\int_{X \times Y} f(x, y) d(\mu \times \nu)(x, y) &= \lim_{n \to \infty} \int_{X \times Y} s_n(x, y) d(\mu \times \nu)(x, y) \\ &= \lim_{n \to \infty} \int_X \left(\int_Y s_n(x, y) d\nu(y) \right) d\mu(x) \\ &= \int_X \left(\lim_{n \to \infty} \int_Y s_n(x, y) d\nu(y) \right) d\mu(x) \\ &= \int_X \left(\int_Y \lim_{n \to \infty} s_n(x, y) d\nu(y) \right) d\mu(x) \\ &= \int_X \left(\int_Y f(x, y) d\nu(y) \right) d\mu(x).\end{aligned}$$

And we can similarly integrate out X first. For (a), the map $y \mapsto f(x,y)$ is the limit of simple functions $s_n(x,y)$ which are sums of sections of measurable sets $(E_k)_x$ and thus measurable (μ a.e. $x \in X$) by last theorem. Next for (b) consider $g : x \mapsto \int_Y f(x,y)d\nu(y)$.

$$\begin{aligned} g(x) &= \lim_{n\to\infty} \int_Y s_n(x,y)d\nu(y) && \text{(MCT)} \\ &= \lim_{n\to\infty} g_n, \end{aligned}$$

where g_n is measurable so g is measurable. □

Theorem (Fubini): Let $(X, m, \mu), (Y, n, \nu)$ measure spaces, μ, ν complete. Let $f : X \times Y \to \overline{\mathbb{R}}$ be $\mu \times \nu$-integrable. Then,

(a) For μ a.e. $x \in X$, $f(x, .)$ is ν-integrable (and same for $f(., y)$).

(b) $\int_Y f(., y)d\nu(y)$ is μ-integrable (and same for $\int_X f(x, .)d\mu(x)$).

(c) We can integrate f in either order:

$$\begin{aligned} \int_{X\times Y} f(x,y)d(\mu\times\nu)(x,y) &= \int_X \left(\int_Y f(x,y)d\nu(y)\right)d\mu(x) \\ &= \int_Y \left(\int_X f(x,y)d\mu(x)\right)d\nu(y). \end{aligned}$$

Proof. Let $f = f^+ - f^-$. Since

$$\int_{X\times Y} |f(x,y)|d(\mu\times\nu)(x,y) < \infty,$$

$E = \{(x,y) : |f(x,y)| > 0\}$ is σ-finite with respect to $\mu \times \nu$, since

$$E = \bigcup_n E_n, E_n = \{\frac{1}{n} < |f(x,y)| < n\}.$$

Now

$$\begin{aligned} \int_{X\times Y} fd(\mu\times\nu) &= \int_E f^+d(\mu\times\nu) - \int_E f^-d(\mu\times\nu) \\ &= \int_X\int_{E_x} f^+d\nu(y)d\mu(x) - \int_X\int_{E_x} f^-d\nu(y)d\mu(x) \\ &= \int_X\int_{E_x} (f^+ - f^-)d\nu(y)d\mu(x) \\ &= \int_X\int_{E_x} fd\nu(y)d\mu(x) \\ &= \int_X\int_Y fd\nu(y)d\mu(x). \end{aligned}$$

□

10.3 Applications

Theorem (Converting to Lebesgue measure): Let (X, m, μ) measure space, μ complete, $1 \leq p < \infty$. Let $f : X \to \mathbb{R}$ measurable. Then,

$$\int_X |f(x)|^p d\mu = p \int_0^\infty s^{p-1} \mu(\{x \in X : |f(x)| > s\}) \, \mathrm{d}s.$$

Proof. If there is $s_0 > 0$ such that

$$\mu(\{x \in X : |f(x)| > s_0\}) = \infty,$$

then $\mu(\{x \in X : |f(x)| > s\}) = \infty$ for $0 \leq s < s_0$ and both sides are ∞.

Assume now the contrary. Restrict f to $X_0 = \{|f| > 0\}$, which is μ σ-finite by earlier argument. □

Theorem: Consider the gamma function

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} \, \mathrm{d}x, t > 0.$$

For $N \geq 1$, $\mathcal{L}^N(B(0, R)) = \alpha_N R^N$, where

$$\alpha_N = \frac{\pi^{N/2}}{\Gamma(1 + N/2)}.$$

Proof.

$$\mathcal{L}^N(B(0, R)) = R^N \mathcal{L}^N(B(0, 1)).$$

Let

$$D = \{(x, y) \in \mathbb{R}^{N+1} : \|x\|^2 < y\}.$$

$$\begin{aligned}
\int_D e^{-y} dx dy &= \int_0^\infty \left(\int_{B(0,\sqrt{y})} e^{-y} dx \right) \mathrm{d}x \\
&= \int_0^\infty e^{-y} \mathcal{L}^N(B(0, \sqrt{y})) \, \mathrm{d}y \\
&= \mathcal{L}^N(B(0, 1)) \int_0^\infty y^{N/2} e^{-y} \, \mathrm{d}y \\
&= \mathcal{L}^N(B(0, 1)) \Gamma(1 + N/2).
\end{aligned}$$

□

Remark: By Caratheodory, $\mu \times \nu$ is always complete, regardless of whether μ or ν are. But this

11 L^p spaces

11.1 Definitions and properties

Definition (Normed space): A normed space $(X, \|.\|)$ is a vector space X endowed with a norm $\|\| : X \to [0, \infty)$, i.e.

(a) $\|x\| = 0 \iff x = 0$.

(b) $\|tx\| = t\|x\|, t \in \mathbb{R}, x \in X$.

(c) $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in X$.

Definition (Distance): Distance on a normed space is defined as $d(x, y) = \|x - y\|$.

Definition: Let (X, m, μ), $1 \leq p < \infty$. Define

$$M^p(E) := \{f : E \to \overline{\mathbb{R}} : E \in m, f \text{ measurable and } \int_E |f|^p d\mu < \infty\}.$$

Define

$$\|f\|_{M^p(E)} := \left(\int_E |f|^p d\mu\right)^{1/p}.$$

Define

$$M^\infty(E) := \{f : E \to \overline{\mathbb{R}} : \text{measurable and bounded}\}.$$

Define

$$\|f\|_{M^\infty(E)} := \sup_{x \in E} |f(x)|.$$

Definition (Holder conjugate exponent): Define the *Holder conjugate exponent* of $1 \leq p \leq \infty$ as

$$q := \begin{cases} \dfrac{p}{p-1}, & \text{if } 1 < p < \infty; \\ \infty, & \text{if } p = 1; \\ 1, & \text{otherwise.} \end{cases}$$

Note that this means

$$\frac{1}{p} + \frac{1}{q} = 1.$$

11.2 Holder and Minkowski inequalities

Theorem (Young's inequality):

$$\frac{1}{p}a^p + \frac{1}{q}b^q \geq ab.$$

Proof. Since ln is concave,

$$\ln(\theta a^p + (1-\theta)b^q) \geq \theta \ln(a^p) + (1-\theta)\ln b^q.$$

Letting $\theta = \frac{1}{p}, 1-\theta = \frac{1}{q}$,

$$= \ln a + \ln b = \ln(ab).$$

So,

$$\frac{1}{p}a^p + \frac{1}{q}b^q \geq ab.$$

$\square$

Theorem (Holder inequality): Let (X, m, μ), $1 \leq p \leq \infty$. Let $f, g : X \to \overline{\mathbb{R}}$ measurable.

(a) $1 < p < \infty$:

$$\int_X |fg| d\mu \leq \left(\int_X |f|^p d\mu\right)^{1/p} \left(\int_X |g|^q d\mu\right)^{1/q}.$$

(b) $p = 1$:

$$\int_X |fg| d\mu \leq \sup_X |g| \int_X |f| d\mu.$$

(c) $p = \infty$:

$$\int_X |fg| d\mu \leq \sup_X |f| \int_X |g| d\mu.$$

In particular, if $f \in M^p(X), g \in M^q(X)$, $fg \in M^1(X)$.

Proof. If $\|f\|_{M^p(X)} = 0$ or $\|g\|_{M^q(X)} = 0$ then $fg = 0$ μ a.e. and there is nothing to prove. The inequality similarly holds when either norm is ∞. So suppose both are finite. By Young's inequality,

$$|fg| \leq \frac{1}{p}|f|^p + \frac{1}{q}|g|^q.$$

Integrating,

$$\begin{aligned} \int_X |fg| d\mu &\leq \frac{1}{p}\int_X |f|^p d\mu + \frac{1}{q}\int_X |g|^q d\mu \\ &= \frac{1}{p}\|f\|_p^p + \frac{1}{q}\|g\|_q^q. \end{aligned}$$

If $\|f\|_p = \|g\|_q = 1$, the RHS is $1 = \|f\|_p \|g\|_q$ as required. In the general case, substitute $f \leftarrow \frac{f}{\|f\|_p}$ and $g \leftarrow \frac{g}{\|g\|_q}$. $\square$

Theorem (Minkowski's inequality): Let (X, m, μ), $1 \le p \le \infty$. Let $f, g \in M^p(X)$. Then

$$\|f+g\|_{M^p} \le \|f\|_{M^p} + \|g\|_{M^p}.$$

Proof. If $p = 1$ or ∞, we are done. So assume $1 < p < \infty$. Note that t^p is convex. Let $a, b > 0$,

$$(a+b)^p = 2^p(\frac{1}{2}a + \frac{1}{2}b)^p \le 2^p\left(\frac{1}{2}a^p + \frac{1}{2}b^p\right) = 2^{p-1}a^p + 2^{p-1}b^p.$$

So

$$\int_X |f+g|^p d\mu \le \int_X (|f|+|g|)^p d\mu \le 2^{p-1}\left(\int_X |f|^p d\mu + \int_X |g|^p d\mu\right).$$

So $f + g \in M^p(X)$. To show the inequality,

$$\begin{aligned}\int_X |f+g|^p d\mu \le& \int_X |f||f+g|^{p-1} d\mu + \int_X |g||f+g|^{p-1} d\mu \\ \le& \left(\int_X |f|^p d\mu\right)^{1/p}\left(\int_X |f+g|^{(p-1)q} d\mu\right)^{1/q} \\ &+ \left(\int_X |g|^p d\mu\right)^{1/p}\left(\int_X |f+g|^{(p-1)q} d\mu\right)^{1/q} \\ =& (\|f\|_p + \|g\|_p)\, \|f+g\|_p^{p/q}.\end{aligned}$$

Since

$$p - p\frac{p-1}{p} = 1,$$

divide both sides by $\|f+g\|_p^{p/q}$ (if it's $0, \infty$, we have nothing to prove). $\square$

11.3 L^p spaces and properties

Definition ($L^p(X)$): Define the equivalence relation in $M^p(X)$ $f \sim g$ iff $f = g$ μ-a.e. Then, for $1 \le p < \infty$ define

$$L^p(X) := M^p(X)/\sim.$$

In particular, the norm is well defined since integrals don't see sets of measure 0. Now $L^p(X)$ is a normed space since if the norm $\|f\|_p$ is 0, f is in the equivalence class of $[0]$.

On the other hand, if $p = \infty$, M^∞ does see sets of measure 0. So we introduce the *essential supremum*

$$espf := \inf\{t \in \mathbb{R} : f(x) \le t, \mu \text{ a.e.}\}.$$

And define

$$L^\infty(X) := \{[f] : f : X \to \overline{\mathbb{R}}, f \text{ measurable}, esp|f| < \infty\}.$$

Proposition: Suppose (X, m, μ) is a measure space and $\mu(X) < \infty$. Let $1 \leq p < q \leq \infty$. Then $L^q(X) \subseteq L^p(X)$.

Proof. Let $f \in L^q(X)$. Then,

$$\begin{aligned} \int_X |f|^p d\mu &= \int_X |f|^p \cdot 1 d\mu \\ &\leq \left\| |f|^p \right\|_{q/p} \|1\|_{(q/p)'} && \text{(Holder)} \\ &= \|f\|_q^p \mu(X)^{\frac{p-q}{p}} < \infty. \end{aligned}$$

□

Proposition: Let (X, m, μ) and $f \in L^\infty \cap L^p$ for large p. Then

$$\|f\|_{L^p} \to \|f\|_{L^\infty}.$$

Proof 1. Assume $\mu(X) < \infty$. Since $|f| \leq \|f\|_\infty$ μ a.e.,

$$\|f\|_p = \left(\int_X |f|^p d\mu \right)^{1/p} \leq \|f\|_\infty \mu(X)^{1/p} \to \|f\|_\infty.$$

So $\limsup_p \|f\|_p \leq \|f\|_\infty$. On the other hand, let $r < \|f\|_\infty$, $E := \{x \in X : |f(x)| > r\}$.

$$\|f\|_p^p = \int_X |f|^p d\mu \geq \int_E |f|^p d\mu \geq r^p \mu(E).$$

Raise the inequality to $1/p$, let $r \to \|f\|_\infty$. Since $\mu(E) < \infty$ it follows that $\liminf_p \|f\|_p \geq \|f\|_\infty$, completing the proof. □

Proof 2. Now suppose $\mu(X)$ is arbitrary. Fix $\varepsilon > 0$. Let $D = \{x : |f| > \|f\|_\infty - \varepsilon\}$. Then, □

Theorem: Let (X, m, μ), $1 \leq p < q < \infty$. Then,

(a) $L^p(X)$ is not contained in $L^q(X)$ if and only if X contains measurable sets of arbitrarily small measure.

(b) $L^q(X)$ is not contained in $L^p(X)$ if and only if X contains measurable sets of arbitrarily large measure.

Proof.

(a) Assume there is $f \in L^p$ with $f \notin L^q$. Define for $n \in \mathbb{N}$

$$E_n := \{x \in X : |f(x)| \geq n\}.$$

Then,

$$\infty > \int_X |f|^p d\mu \geq \int_{E_n} |f|^p d\mu \geq n^p \mu(E_n).$$

Dividing by n^p, it follows that $\mu(E_n) \to 0$. We show that $\mu(E_n) > 0$ for all n, in which case we have sets of arbitrarily small measure as required. Suppose for some n_0, $\mu(E_{n_0}) = 0$. Then

$$\int_X |f|^q d\mu = \int_{X\setminus E_{n_0}} |f|^q d\mu \leq \int_{X\setminus E_{n_0}} n_0^{q-p}|f|^p d\mu < \infty,$$

a contradiction. Conversely, take a sequence by hypothesis $F_n \in m$ such that $0 < \mu(F_0) \leq 1$, and

$$0 < \mu(F_n) < \frac{1}{3}\mu(F_{n-1}).$$

Define pairwise disjoint

$$E_n := F_n \setminus \left(\bigcup_{k\geq n+1} F_k\right),$$

with $\mu(E_n) \leq \mu(F_n) \leq \frac{1}{3^n}\mu(F_0)$. Note that we must have $\mu(E_n) > 0$. So define

$$c_n^q := \frac{1}{n\mu(E_n)},$$

and we have

$$\sum_n c_n^q \mu(E_n) = \sum_n \frac{1}{n} = \infty,$$

and

$$\sum_n c_n^p \mu(E_n) = \sum_n \frac{1}{n^{p/q}} \frac{1}{\mu(E_n)^{p/q}} \mu(E_n) = \sum_n \cdots.$$

(b) Suppose $f \in L^q, f \notin L^p$. Define

$$F_n := \{x \in X : \frac{1}{n+1} < |f(x)| \leq \frac{1}{n}\}, F_\infty := \bigcup_n F_n.$$

If $\mu(F_\infty) < \infty$ then

$$\int_X |f|^p d\mu = \int_{|f|\leq 1} |f|^p d\mu + \int_{|f|>1} |f|^p d\mu \leq \mu(F_\infty) + \int_{|f|>1} |f|^q d\mu < \infty.$$

Otherwise,

$\square$

Definition (Banach space): A normed space $(X, \|.\|)$ is a *Banach space* if Cauchy sequences converge.

Lemma: A normed space $(V, \|.\|)$ is Banach if and only if every absolutely convergent series in V converges in V, i.e. for any sequence $\{v_n\}$ such that $\sum_n \|v_n\| < \infty$, there exists a limit v for the partial sums:

$$\lim_{\ell\to\infty} \left\| \sum_{i=1}^{\ell} v_i - v \right\| = 0.$$

Proof. Suppose V is Banach and $\sum_n \|v_n\| < \infty$. We show $s_n := \sum_{i=1}^n v_n$ is convergent. Fix $\varepsilon > 0$. Choose N such that $\sum_{i=N}^\infty \|v_n\| < \varepsilon$. Then for $n, m \geq N$, $\|s_n - s_m\| < \varepsilon$. So s_n is Cauchy and s_n converges.

Suppose on the other hand every absolutely convergent series converges. Let v_n be a Cauchy sequence, and choose subsequence v_{n_k} such that

$$\left\|v_{n_k} - v_{n_{k+1}}\right\| < \frac{1}{2^k}$$

for all k (we can always do this by Cauchy property). So a subsequence converges, meaning v_n converges. $\square$

Theorem: $L^p(X)$ be a Banach space for $1 \leq p \leq \infty$.

Proof. Using the lemma, let $f_n \in L^p(X)$ such that $\sum_n \|f_n\|_p < \infty$. Let

$$f(x) = \sum_n f_n.$$

We show $f \in L^p$ by dominating f by a L^p function g, then use dominated convergence theorem to prove the statement. Define

$$g = \left(\sum_n |f_n|\right)^p.$$

By MCT and Minkowski, for $l \in \mathbb{N}$,

$$\int_X g d\mu = \lim_{n\to\infty} \int_X \left(\sum_{n=1}^l |f_n|\right)^p d\mu \leq \lim_{n\to\infty} \left(\sum_{n=1}^l \|f_n\|_p\right)^p < \infty.$$

Since $|f|^p \leq g$, $f \in L^p$ and we can use DCT with g to show $|\sum_{n=1}^l f_n - f|$ converges in L^p to 0. $\square$

Theorem: Let (X, m, μ). Then the family of simple functions in L^p, $1 \leq p \leq \infty$, is dense in L^p.

In particular, for all $f \in L^p$, there exists s_n simple functions in L^p such that $s_n \to f$ in L^p.

Proof. HW8. $\square$

Proposition: Let $g \in L^{p'}(X)$. Show

$$\|g\|_{p'} = \sup_{f \in L^p(X), \|f\|_p \leq 1} \left\{\left|\int_X fg d\mu\right|\right\}.$$

Proof. Let $f = |g|^{p'-1} \|g\|_{p'}^{1-p'}$. We can verify $\|f\|_p = 1$, and

$$\int_X fg d\mu = \left(\int_X |g|^{p'} d\mu\right)^{\frac{1-p'}{p'}} \int_X |g|^{p'} d\mu = \|g\|_{p'}.$$

The sup cannot be greater than $\|g\|_{p'}$ immediately by Holder's inequality. $\square$

11.4 Continuous functions and differentiation

Definition: Let (X, z) be a topological space. X is *normal* if disjoint closed sets are "separate", i.e. for closed sets $C_1, C_2 \subseteq X$ closed, $C_1 \cap C_2 = \varnothing$, there exists open sets $U_1, U_2 \subseteq X$ with $C_1 \subseteq U_1, C_2 \subseteq U_2$, and $U_1 \cap U_2 = \varnothing$.

Remark: Metric spaces are normal, as we can define $U_1 \coloneqq \{x \in X : D(x, C_1) < d(x, C_2)\}$, and similarly for U_2.

Theorem (Uryshon): (X, z) normal iff for C_1, C_2 closed, $C_1 \cap C_2 = \varnothing$, there exists continuous function $\phi : X \to [0, 1]$ such that $\phi = 1$ on C_1, $\phi = 0$ on C_2.

Theorem: Let (X, m, μ) with X normal and $\mathcal{B}(X) \subseteq m$. Assume for all $E \in M$ with finite measure we have inner and outer regularity:

$$\mu(E) = \sup\{\mu(C) : C \text{ closed}, C \subseteq E\} = \inf\{\mu(A) : A \text{ open}, E \subseteq A\}.$$

Then $L^p \cap C_b(X)$ (continuous and bounded) is dense in $L^p(X)$. Furthermore if X is a metric space then $L^p \cap C_c(X)$ (uniformly continuous) is dense in $L^p(X)$.

Proof. Since simple functions are dense in L^p it suffices to approximate simple functions χ_E, $E \in m$, with functions in $C_b(X)$. By inner and outer regularity find

$$C \subseteq E \subseteq A, \mu(A \setminus C) < \varepsilon^p.$$

Then by Uryshon there is continuous function $g : X \to [0, 1]$ such that $g = 1$ on C and $g = 0$ on $X \setminus A$, and

$$\int_X |\chi_E - g|^p d\mu = \int_{A \setminus C} |\chi_E - g|^p d\mu \leq \mu(A \setminus C) = \varepsilon^p.$$

□

Definition (Hardy-Littlewood maximal function): Let $f : \mathbb{R}^N \to \mathbb{R}$ be locally integrable. the *Hardy-Littlewood maximal function* of f is defined as

$$M(f)(x) = \sup_{r \geq 0} ⨍_{B(x,r)} f d\mu.$$

Definition (M_1): Let $\mu : \mathcal{B}(\mathbb{R}^N) \to [0, \infty]$ finite on compact sets. Define

$$M_1 \coloneqq \{x \in \mathbb{R}^N : \mu(B(x, r)) = 0 \text{ for some } r > 0\}.$$

Lemma: Let $f \in L^p$ for $1 \leq p < \infty$.

(a) If $p = 1$,

$$\mu(\{x \in \mathbb{R}^N \setminus M_1 : M(f)(x) > t\}) \leq \frac{C(N)}{t} \int_X |f| d\mu.$$

(b) If $p > 1$, $M(f) \in L^p$ and

$$\|M(f)\|_p \leq C(N,p)\,\|f\|_p\,.$$

Theorem (Lebesgue differentiation theorem): Let $f : \mathbb{R}^N \to \mathbb{R}$ locally integrable and $\mu : \mathcal{B}(\mathbb{R}^N) \to [0,\infty]$ finite on compact sets. Then,

$$\lim_{r\to 0^+} \fint_{B(x,r)} |f(y) - f(x)|d\mu(y) = 0.$$

Proof. Assume f is integrable. Since $C_c(\mathbb{R}^N)$ is dense in L^p we can find g_ε uniformly continuous such that $\int_X |f - g_\varepsilon| d\mu < \varepsilon$. By uniform continuity

$$\lim_{r\to 0^+} \fint_{B(x,r)} |g_\varepsilon(y) - g_\varepsilon(x)| = 0.$$

Now,

$$\begin{aligned}
&\limsup \fint_{B(x,r)} |f(y) - f(x)|d\mu(y) \\
&= \limsup \fint_{B(x,r)} |f(y) - g(y) + g(y) - g(x) + g(x) - f(x)|d\mu(y) \\
&\leq M(f-g)(x) + 0 + |g(x) - f(x)|.
\end{aligned}$$

Define

$$\begin{aligned}
G_t &= \{x \in \mathbb{R}^N \setminus M_1 : \limsup \fint_{B(x,r)} |f(y) - f(x)|d\mu(y) > t\} \\
E_{t,\varepsilon} &= \{x \in \mathbb{R}^N \setminus M_1 : M(f - g_\varepsilon)(x) > t\} \\
F_{t,\varepsilon} &= \{x \in \mathbb{R}^N : |g_\varepsilon(x) - f(x)| > t\}.
\end{aligned}$$

By the last inequality,

$$G_{2t} \subseteq E_{t,\varepsilon} \cup F_{t,\varepsilon}.$$

By the last lemma,

$$\mu(E_{t,\varepsilon}) \leq \frac{C(N)}{t} \int_X |f - g_\varepsilon| d\mu \leq \frac{C(N)\varepsilon}{t}.$$

By Chebyshev,

$$\mu(F_{t,\varepsilon}) \leq \frac{1}{t} \int_X |g_\varepsilon - f| d\mu \leq \frac{\varepsilon}{t}.$$

So letting $\varepsilon \to 0$, using the containment we have that $\mu(G_{2t}) = 0$ for all t. Let $E = \bigcup_n G_{1/n}$, and it follows that if $x \in \mathbb{R}^N \setminus E$, the desired limit is 0. □

12 Decomposition theorems

12.1 Radon-Nikodym

Definition (Absolutely continuous): Let μ, ν be measures defined on (X, m). ν is *absolutely continuous* with respect to μ, and we write $\nu \ll \mu$, if for every $E \in m$ with $\mu(E) = 0$, $\nu(E) = 0$.

Note that given a measure μ and a measurable function f we can define

$$\nu(E) = \int_E f d\mu$$

and it follows that $\nu \ll \mu$. Radon-Nikodym says that when μ is σ-finite, and $\nu \ll \mu$, we can always find a function f (called the Radon-Nikodym derivative) that converts between the measures in the same fashion as above. For example, when μ is the Lebesgue measure and ν is a probability measure, the R-N derivative $\frac{d\nu}{d\mu}$ is the pdf of the probability measure.

To prove Radon-Nikodym we need two lemmas.

Definition: Let μ, ν measures on (X, m). Define

$$\begin{aligned}\nu_a(E) \coloneqq \sup\{ \int_E f d\mu, f : X \to [0, \infty] \text{ measurable}, \\ \int_{E'} f d\mu \le \nu(E'), \text{ for all } E' \subseteq E, E' \in m\}.\end{aligned}$$

Lemma: ν_a is a measure with $\nu_a \ll \mu$, and for each $E \in m$ the supremum over f is admissible. Moreover, if ν_a is σ-finite, we may choose f independently of E.

Proof. Omitted. x □

Definition (Signed measure): Let (X, m) a measurable space. A *signed measure* is a function $\lambda : m \to \overline{\mathbb{R}}$ such that

(a) $\lambda(\varnothing) = 0$.

(b) λ cannot take both $-\infty$ and ∞.

(c) For every countable collection of pairwise disjoint sets E_n,

$$\lambda\left(\bigcup_n E_n\right) = \sum_n \lambda(E_n).$$

Definition (Positive/negative sets wrt signed measure): Let $E \in m$ such that for all $E' \subseteq E$ measurable, $\mu(E') \ge 0$. Then we call E *positive*.

Lemma: Let (X, m) and $\lambda : m \to \overline{\mathbb{R}}$ a signed measure. For $E \in m$ define

$$\begin{aligned}\lambda^+(E) &= \sup\{\lambda(F) : F \subseteq E, F \in m\} \\ \lambda^-(E) &= -\inf\{\lambda(F) : F \subseteq E, F \in m\} = (-\lambda)^+(E).\end{aligned}$$

Then λ^+, λ^- are measures. Moreover if $\lambda : [-\infty, \infty)$, then

$$\lambda^+(E) = \sup\{\lambda(F) : F \subseteq E, F \in m, \lambda^-(F) = 0\},$$

λ^+ is finite, and $\lambda = \lambda^+ - \lambda^-$.

Proof. **Step 1:** λ^+ is a measure:

(a) $\lambda^+(\varnothing) = \lambda(\varnothing) = 0$.

(b) Note that λ^+ is monotone. We claim for pairwise disjoint E_n,

$$\lambda^+(\bigcup_n E_n) \geq \sum_n \lambda^+(E_n).$$

If one $\lambda^+(E_n)$ is infinite then we are done. Otherwise, for each n select $F_n \subseteq E_n$ such that $\lambda(F_n) \geq \lambda^+(E_n) - \frac{\varepsilon}{2^n}$. Then

$$\lambda^+(E) \geq \lambda(\bigcup_n F_n) = \sum_n \lambda(F_n) \geq \sum_n \lambda^+(E_n) - \varepsilon.$$

Let $\varepsilon \to 0$. Conversely, for any $F \subseteq E$,

$$\lambda(F) = \lambda(\bigcup_n F \cap E_n) = \sum_n \lambda(F \cap E_n) \leq \sum_n \lambda^+(E_n).$$

Take supremum over sets $F \subseteq E$.

Step 2: fuck no $\square$

Theorem (Radon-Nikodym): Let $\nu \ll \mu$ measures on (X, m) with μ σ-finite. There exists a unique (up to sets of measure 0) measurable function $f : X \to [0, \infty]$ such that

$$\nu(E) = \int_E f d\mu$$

for all measurable E.

Proof. **Step 1**: Suppose μ, ν are finite. ν finite implies ν_a finite implies ν_a σ-finite. By earlier lemma there exists $f : X \to [0, \infty]$ measurable such that $\nu_a(E) = \int_E f d\mu$ for all $E \in m$. We show f also works for ν: $\nu(E) = \int_E f d\mu$.

Define $\nu'(E) = \nu(E) - \int_E f d\mu$. By definition ν_a, $\nu'(E) \geq 0$. ν' is a measure and $\nu' \ll \mu$. We WTS $\nu' = 0$. Suppose not, so let $E_0 \in m$ with $\nu'(E_0) > 0$. So $\mu(E_0) > 0$. We can find $\varepsilon > 0$ such that

$$\nu'(E_0) > \varepsilon\mu(E_0) \implies (\nu' - \varepsilon\mu)^+(E_0) > 0.$$

By above lemma there is $E_0' \subseteq E_0$ such that $(\nu' - \varepsilon\mu)^-(E_0') = 0$ and $\nu'(E_0') > \varepsilon\mu(E_0') > 0$. On the other hand,

$$(\varepsilon\mu - \nu')^+(E_0') = 0$$

implies that for all $E'' \subseteq E_0'$, $\varepsilon\mu(E'') \leq \nu'(E'') = \nu(E'') - \int_{E''} f d\mu$, so

$$\int_{E''} f + \varepsilon\chi_{E_0'} d\mu \leq \nu(E'').$$

Then for $E \in m$, write

$$\int_E f + \varepsilon\chi_{E_0'} d\mu = \int_{E\setminus E_0'} f d\mu + \int_{E\cap E_0'} f + \varepsilon\chi_{E_0'} d\mu$$
$$\leq \nu(E \setminus E_0') + \nu(E \cap E_0') = \nu(E).$$

Thus $f + \varepsilon\chi_{E_0'}$ is admissible for $\nu_a(X)$ and

$$\nu_a(X) \geq \int_X f + \varepsilon\chi_{E_0'} d\mu = \nu_a(X) + \varepsilon\mu(E_0') > \nu_a(X),$$

a contradiction.

Suppose now that there are two functions f, g such that for all $E \in m$,

$$\nu(E) = \int_E f d\mu = \int_E g d\mu.$$

Then

$$\int_X (f - g) d\mu = 0,$$

so $f = g$ μ a.e. □

12.2 Lebesgue decomposition theorem

Definition (Mutually singular): We call $\nu \perp \mu$ (mutually singular) if $X = X_\mu \cup X_\nu$ disjoint such that for all $E \in m$, $\mu(E) = \mu(E \cap X_\mu)$ and $\nu(E) = \nu(E \cap X_\nu)$.

Definition (ν_s): Define

$$\nu_s(E) = \sup\{\nu(F) : F \subseteq E, F \in m, \mu(F) = 0\},$$

where $\nu_s \leq \nu$.

Lemma:

(a) ν_s is a measure.

(b) For all $E \in m$, the supremum in ν_s is attained by a measurable set, i.e. there exists $F \in m$ such that $\mu(F) = 0$ and $\nu_s(F) = \nu(F)$.

(c) If ν_s is σ-finite then $\nu_s \perp \mu$.

Proof.

(a) $\nu_s(\varnothing) = 0$. Suppose E_n are disjoint. Then, WTS

$$\nu_s(\bigcup_n E_n) = \sum_n \nu_s(E_n).$$

(b) If $\nu_s(E) = 0$, choose $F = \varnothing$. Otherwise, find an increasing sequence $t_n \to \nu_s(E)$ and use supremum to get measurable sets F_n with $\nu_s(E) > \nu(F_n) > t_n$ and $\mu(F_n) = 0$. The set $\bigcup_n F_n$ attains the supremum.

(c) First assume ν_s is finite. Using (b) find set X_s such that $\nu_s(X) = \nu(X)$ and $\mu(X_s) = 0$.

□

Theorem (Lebesgue decomposition theorem): Let μ, ν measures on (X, m) with μ σ-finite. Then,

$$\nu = \nu_a + \nu_s,$$

with $\nu_a \ll \mu$. Further, if ν is σ-finite, then $\nu_s \perp \mu$ and the decomposition is unique.

Proof. **Step 1:** We show $\nu = \nu_a + \nu_s$. Fix $E \in m$. Let $E_s \subseteq E$ be a measurable set such that $\mu(E_s) = 0$ and $\nu(E_s) = \nu_s(E_s)$. If $\nu(E_s) = \infty$ we are done, so assume $\nu(E_s) < \infty$.

We claim $\nu|_{E \setminus E_s}$ is absolutely continuous with respect to $\mu|_{E \setminus E_s}$. Let $F \subseteq E \setminus E_s$ with $\mu(F) = 0$. Suppose for contradiction that $\nu(F) > 0$. Then,

$$\infty > \nu_s(E) \geq \nu(E_s \cup F) = \nu(E_s) + \nu(F) > \nu(E_s) = \nu_s(E),$$

a contradiction. Then by Radon-Nikodym ν_a attains its bound on $E \setminus E_s$ and $\nu(F) = \nu_a(F)$. Thus

$$\nu(E \setminus E_s) = \nu_a(E \setminus E_s).$$

Since $\nu_a \ll \mu$ and $\mu(E_s) = 0$,

$$\nu_a(E \setminus E_s) = \nu_a(E).$$

Thus

$$\nu(E) = \nu(E \setminus E_s) + \nu(E_s) = \nu_a(E) + \nu_s(E).$$

Step 2: Suppose ν is σ-finite. We show the decomposition is unique. By above lemma $\nu_s \perp \mu$. Suppose

$$\nu = \nu_a + \nu_s = \overline{\nu_a} + \overline{\nu_s}.$$

Let $X_s \in m$ such that $\mu(X_s) = 0$ and $\overline{\nu_s}(E) = \overline{\nu_s}(E \cap X_s)$. Then for every $E \subseteq X \setminus X_s$,

$$\nu(E) = \overline{\nu_a}(E) \implies \nu|_{X \setminus X_s} \ll \mu|_{X \setminus X_s}.$$

Then for fixed $E \in m$,

$$\overline{\nu_a}(E) = \overline{\nu_a}(E \setminus X_s) = \nu(E \setminus X_s) \geq \nu_a(E \setminus X_s).$$

Repeat this argument starting from ν_a, and we have $\nu_a = \overline{\nu_a}$. If ν is finite, the singular measures are equal as well and we are done. If ν is σ-finite, we have equality of singular measures over each block thus overall. $\square$