BİL3014

Algoritma Analizi

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Recursive Algoritmalar ve Recurrence Bağıntısı

Rekurans Bağıntısı

- Bir dizinin içinde kendisinden bir parça bulunuyorsa o diziye rekurans bağıntısı (özyineli dizi) denir

$a_n = 5n+1 \Longrightarrow a_{20} = 101$ (dizi) sequence

$\boxed{a_n = 2a_{n-1} + 3}$ (a_{n-1} kendisinden parçadır) Rekürans Bağıntısı

$\downarrow$

$a_0 = 3, \; n \geq 1$

$$\left.\begin{aligned} a_n &= a_{n-1} - 2a_{n-2} + 5^n \\ a_n &= 3a_{n-1} + 2a_{n-2} \end{aligned}\right\} \text{rek. bağıntısı} \quad \longrightarrow a_1 = 3, \; a_0 = 5, \; n \geq 2$$

Rekurans Bağıntısı

Örn: $a_n = 2a_{n-1} - a_{n-2} + 2^{n-1}$, $a_0 = 1$, $a_1 = 2$, $n \geq 2$ olarak veriliyor. Buna göre $a_5 = ?$

$n = 2 \Rightarrow a_2 = 2a_1 - a_0 + 2 = 4 - 1 + 2 \Rightarrow \boxed{a_2 = 5}$

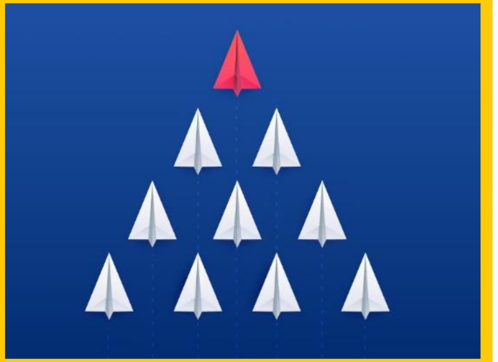
$n = 3 \Rightarrow a_3 = 2a_2 - a_1 + 4 = 10 - 2 + 4 \Rightarrow \boxed{a_3 = 12}$

$n = 4 \Rightarrow a_4 = 2a_3 - a_2 + 8 = 24 - 5 + 8 \Rightarrow \boxed{a_4 = 27}$

$n = 5 \Rightarrow a_5 = 2a_4 - a_3 + 16 = 54 - 12 + 16 \Rightarrow a_5 = 58$

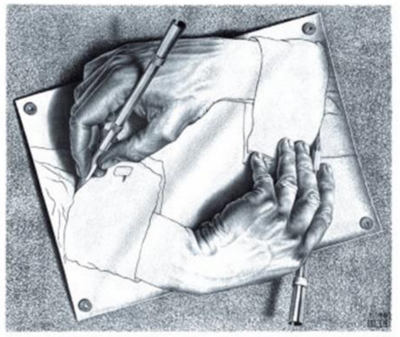
Böl-Yönet Yaklaşımı

- Problem çözümlemede problemin daha küçük alt problemlere parçalanarak çözümlenmesine **böl ve yönet (devide and conquer) yaklaşımı** denir.
- Problem kendisine benzer küçük boyutlu alt problemlere bölünür.
- Alt problemler çözülür ve bulunan çözümler birleştirilir.
    - **Divide:** Problem iki veya daha fazla alt probleme bölünür
    - **Conquer:** Divide-and-conquer özyineleme (recursively) kullanılarak alt problemleri çözer.
    - **Combine:** Alt problemlerin çözümleri alınır ve orijinal problemin çözümü olacak şekilde birleştirilir.

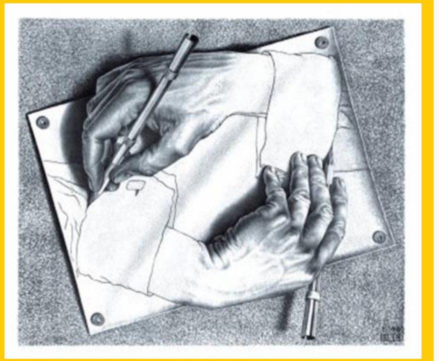


Özyineleme (Recursion)

- Özyineleme bir metodun kendisini çağırdığı bir programlama tekniğidir.
- Birçok algoritmanın uygulanmasında özyinelemeli yaklaşımın uygulanabilirliği söz konusudur.
- Genellikle bu yöntem ***n*** boyutlu problemin daha küçük boyutları ele alınarak benzeri şekilde çözümünde kullanılmaktadır.



Özyineleme (Recursion)



- Özyinelemenin sonsuz olarak devam etmesinin önlenmesi için bitiş koşulunun olması gerekmektedir.
- Dolayısıyla özyinelemeli programlarda bu sürecin ne zaman veya hangi koşullarda durması gerektiği önceden bilinmelidir.
- Tanım gereği fonksiyon kendi kendini çağırdığında genellikle her seferinde daha küçük boyutlu problem ele alınacağından sürecin sonsuz devam edemeyeceği anlaşılmaktadır.
- Özyineleme işlemi durdurma durumu sağlanınca sonlandırılır.

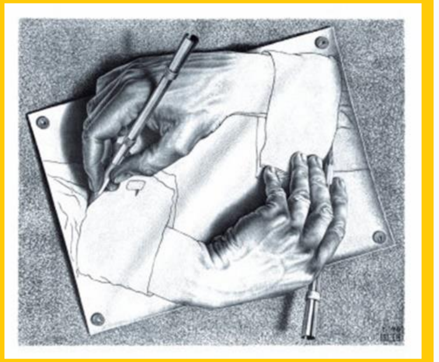
Özyineleme (Recursion)

- Çarpımı toplamla ifade ettiğimizde veya toplamı ardışık olarak 1 ler şeklinde gösterdiğimizde rekursifliğe başvurmaktayız:

6 * 5 = (6 * 4) + 6 = ((6 * 3) + 6) + 6 = (((6 * 2) + 6) + 6) + 6
= ((((6 * 1) + 6) + 6) + 6) + 6 = 6 + 6 + 6 + 6 + 6 = 30

- Veya

6 + 5 = (6 + 4) + 1 = ((6 + 3) + 1) + 1) = (((6 + 2) + 1) + 1) + 1
= ((((6 + 1) + 1) + 1) + 1) + 1 = 6 + 1 + 1 + 1 + 1 + 1 = 11

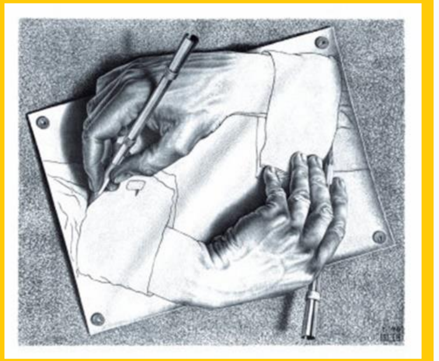


Özyineleme (Recursion)

- Özyinelemeli algoritmalar anlaşıldığı gibi iki temel kısımdan oluşmaktadır. Bunlar;
- Algoritmanın sonlandırılmasını sağlayan ***bitiş kısmı (base case)*** ve,
- Fonksiyonun belirtilen sayıda tekrarlanmasını sağlayan ***gövde kısmı (recursive case)*** dır.

- Genel yazımı:

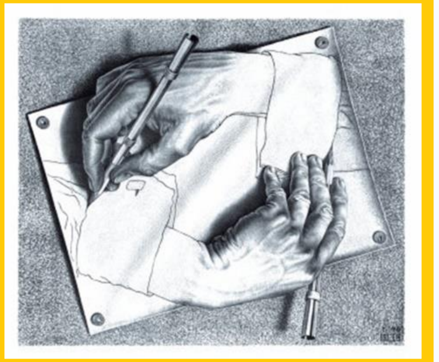
```
if (durdurma durumu sağlandıysa)
        çözümü yap
else
        problemi özyineleme kullanarak indirge
```



Özyineleme (Recursion)

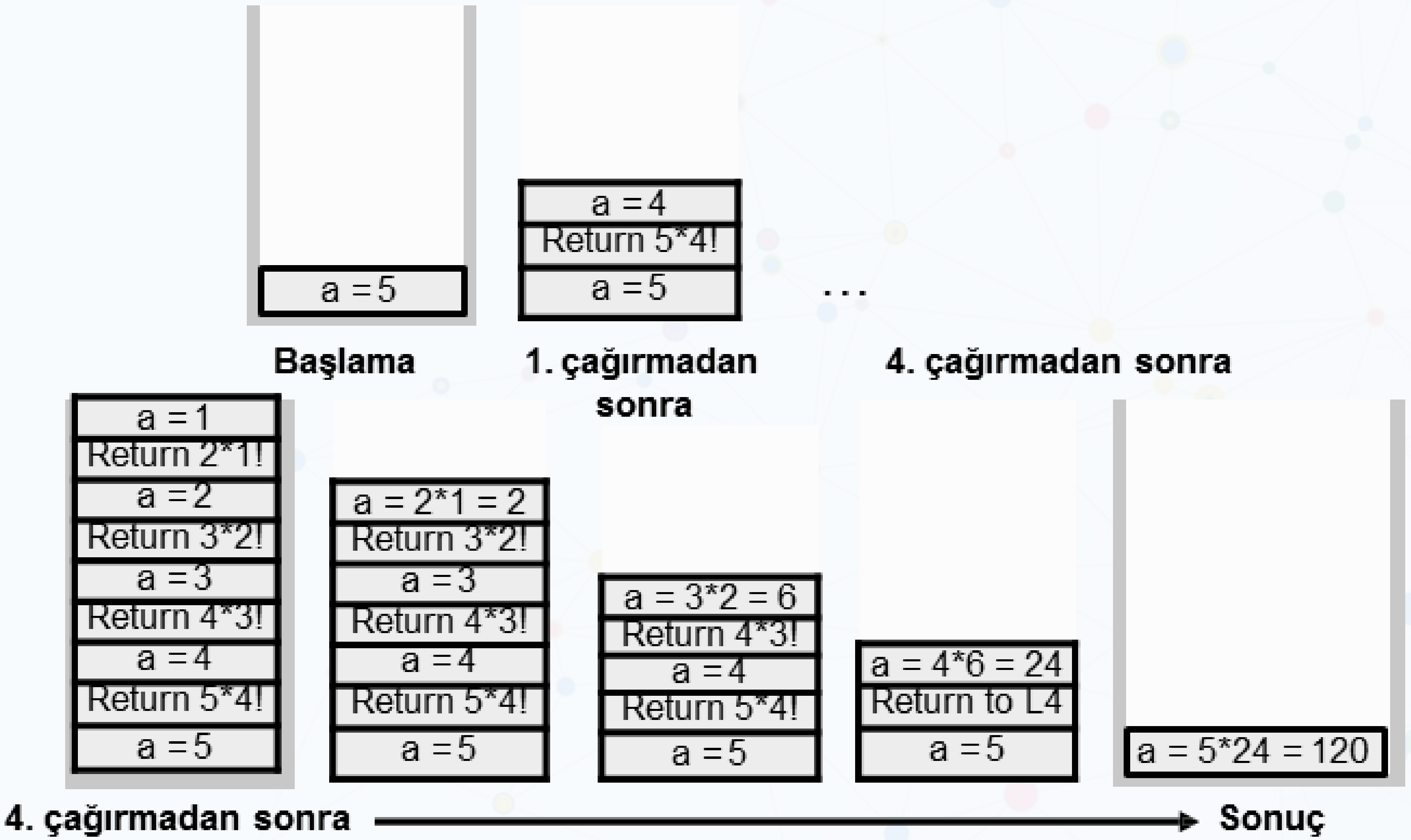
- **Faktöriyelin özyineli olarak hesaplanması**
- 0! = 1 olduğundan n >= 1 ise n! = n * (n - 1)!
- Burada 0! = 1 bitiş koşulu olmaktadır. Fakat bu koşul programcıya bağlı olarak örneğin 2! = 2 şeklinde de değiştirilebilir.

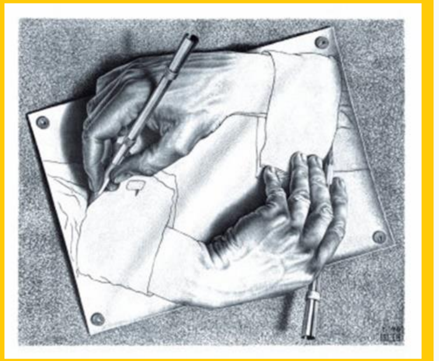
```
int faktoriyel(int n){
        if (n==0)
                return 1;
        return n * faktoriyel(n-1);
}
```



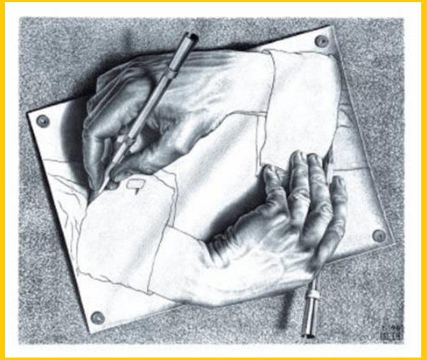
Özyineleme (Recursion)

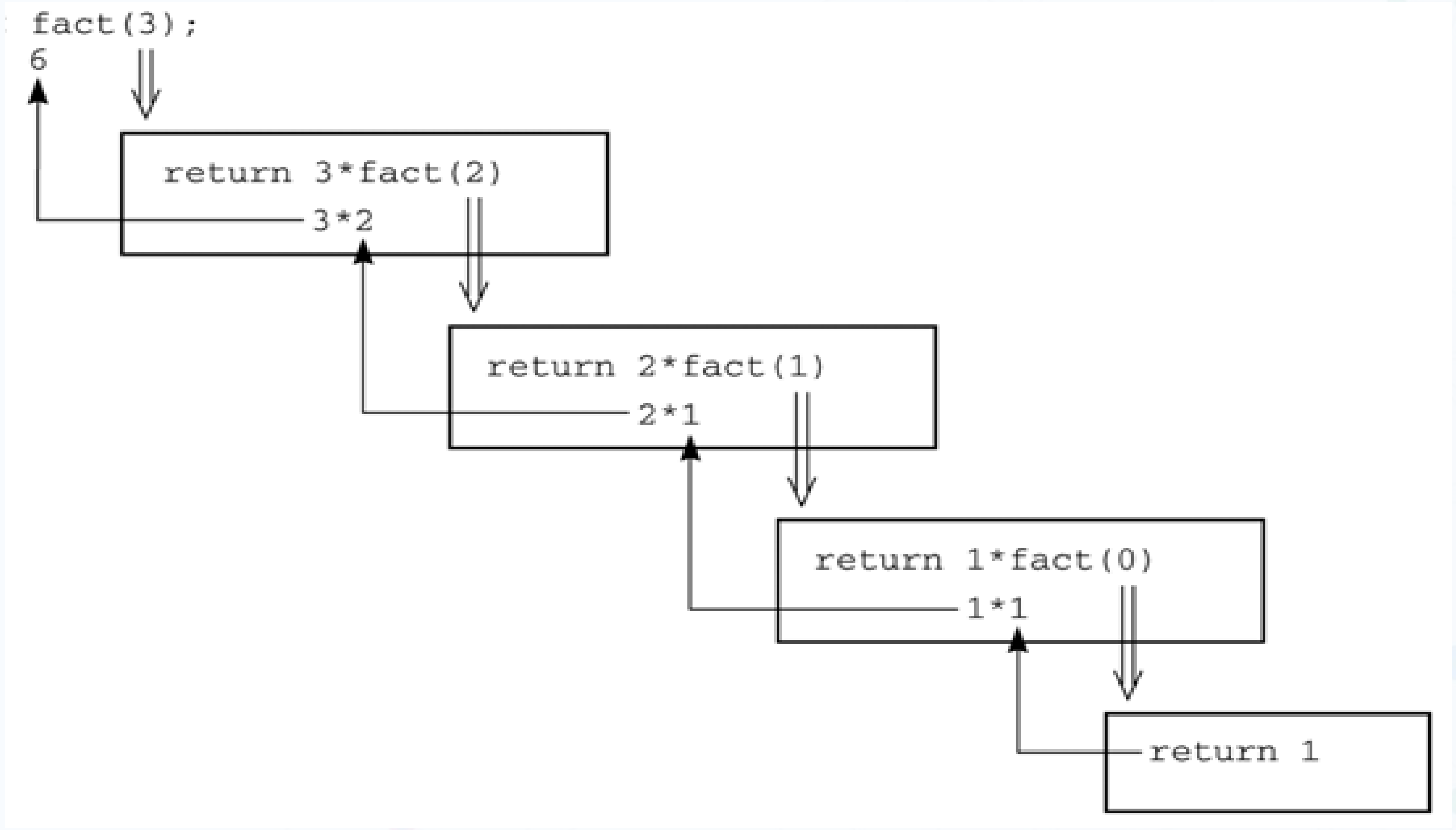
- faktoriyel(5) için çalışmasının stack ile gösterimi.





Özyineleme (Recursion)





Özyineleme (Recursion)

- Özyineleme programlamada bazı algoritmaların anlaşılmasında kolaylıklar sağlasa da birçok durumlarda olumsuz yönlerini de göstermektedir.
- Devamlı olarak fonksiyonun kendi kendini çağırması belirli işlem yüküne de sebep olacaktır.
- Çünkü her fonksiyonun tekrar çağırılması birçok atama işleminin gerçekleştirilmesinin ve yeni değişkenlere uygun bölge ayırımını gerektirmektedir.
- Bu nedenle aynı özyinelemede süreç kendisini çok sayıda çağırdığından bellek ve zaman problemi ortaya çıkacaktır.
- Dolayısıyla eğer programın özyinelemesiz çözümü mümkünse bu çözüm biçimi tercih edilebilir.

Insertion Sort

```
procedure insertionSort(list A):
   for i = 0 to length(A) - 1
      let j = i
      while j > 0 and A[j - 1] > A[j]:
         swap A[j - 1] and A[j]
         j = j - 1
```

O(n) work per iteration

O(n) iterations

Total work done:

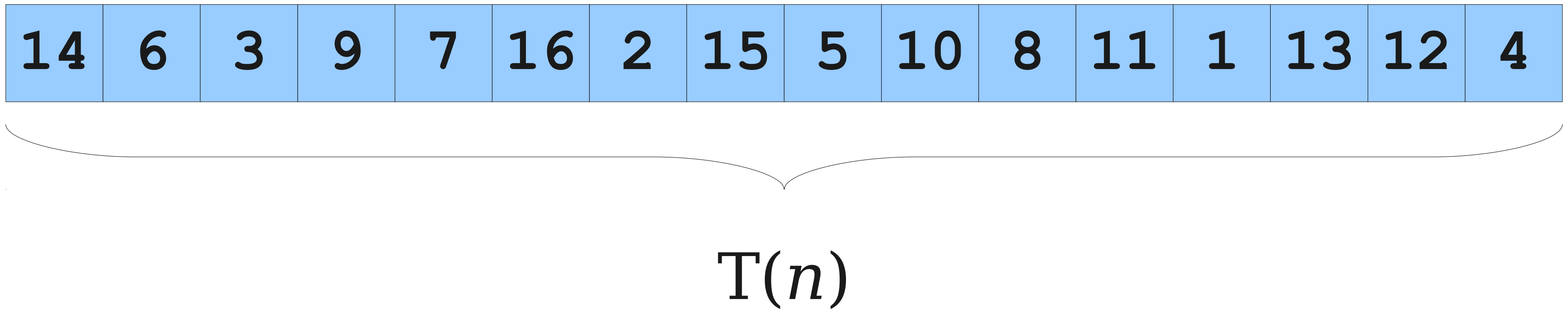
O(n^2)

Daha iyi bir sıralama algoritması olarak; **Mergesort**

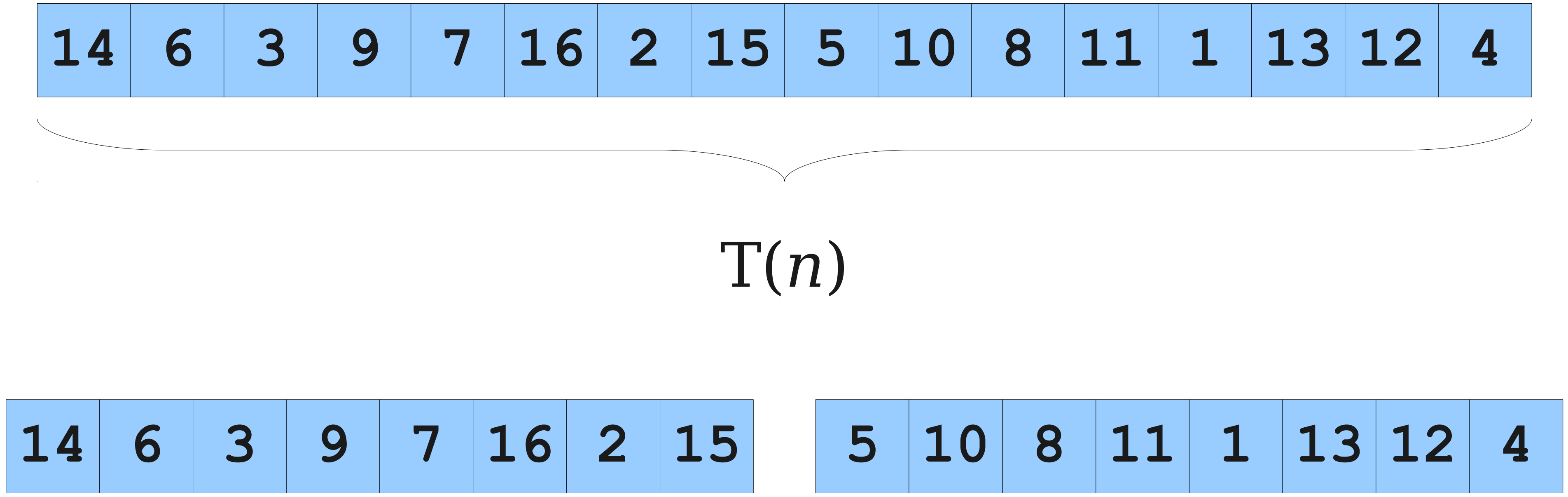
Thinking About O(n^2)

14	6	3	9	7	16	2	15	5	10	8	11	1	13	12	4

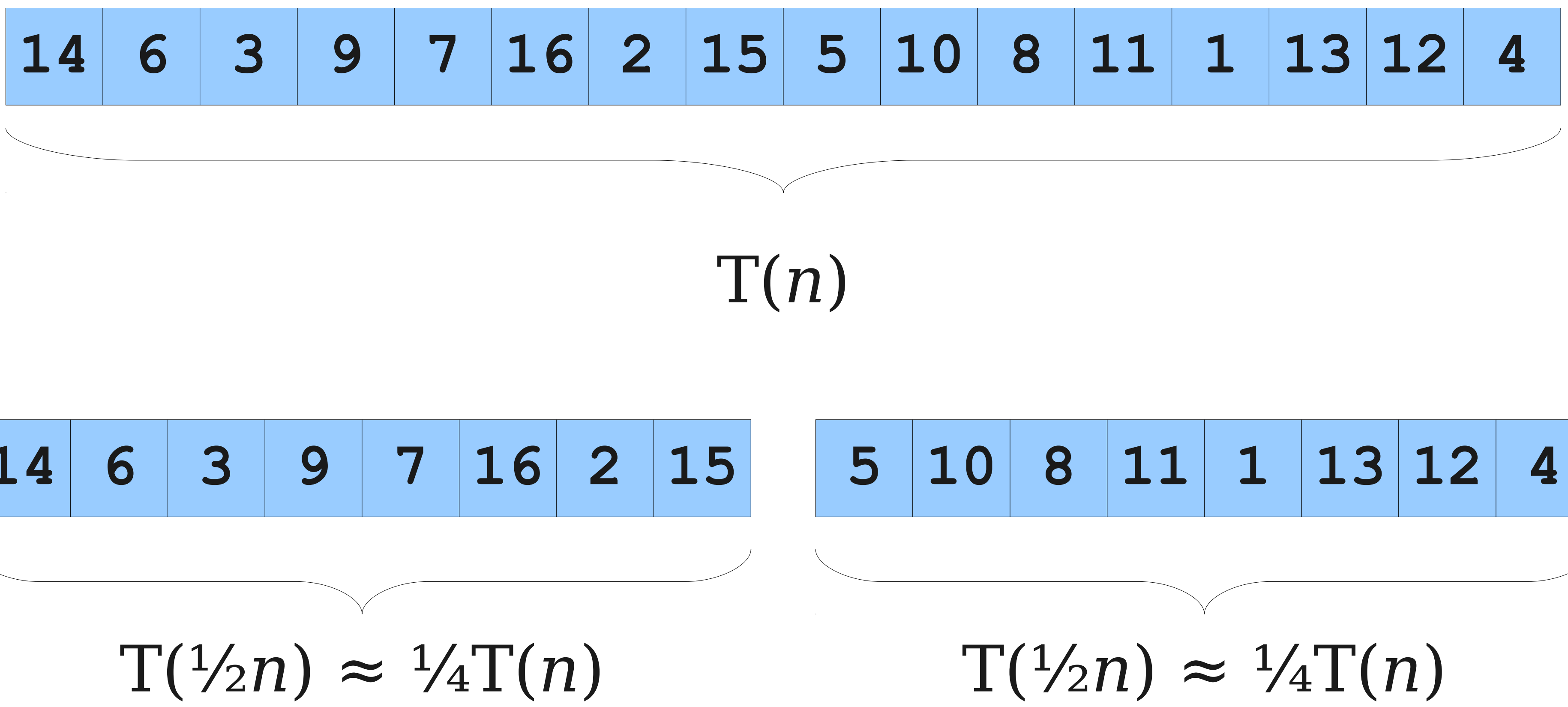
Thinking About O(n^2)



Thinking About O(n^2)



Thinking About $O(n^2)$


14 6 3 9 7 16 2 15 5 10 8 11 1 13 12 4
$T(n)$
14 6 3 9 7 16 2 15
5 10 8 11 1 13 12 4
$T(\frac{1}{2}n) \approx \frac{1}{4}T(n)$
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Thinking About O(n^2)

14	6	3	9	7	16	2	15	5	10	8	11	1	13	12	4

$\underbrace{\hphantom{xxxxxxxxxxxxxxxxxxxxxxxx}}_{}$

$T(n)$

2	3	6	7	9	14	15	16

$\underbrace{\hphantom{xxxxxxxxxxxx}}_{}$

$T(\tfrac{1}{2}n) \approx \tfrac{1}{4}T(n)$

1	4	5	8	10	11	12	13

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$T(\tfrac{1}{2}n) \approx \tfrac{1}{4}T(n)$

The Key Insight: **Merge**

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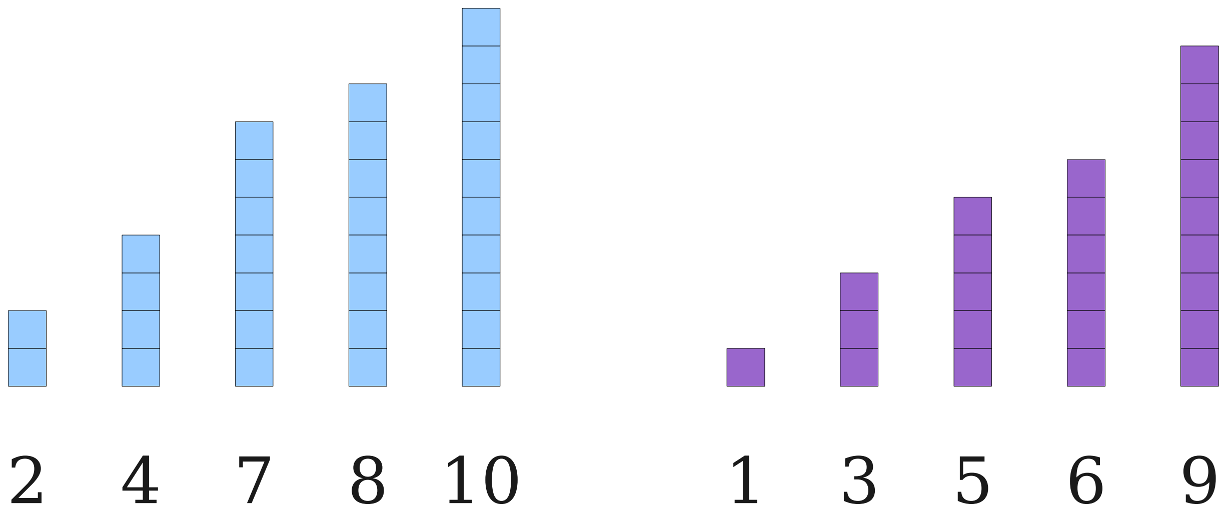

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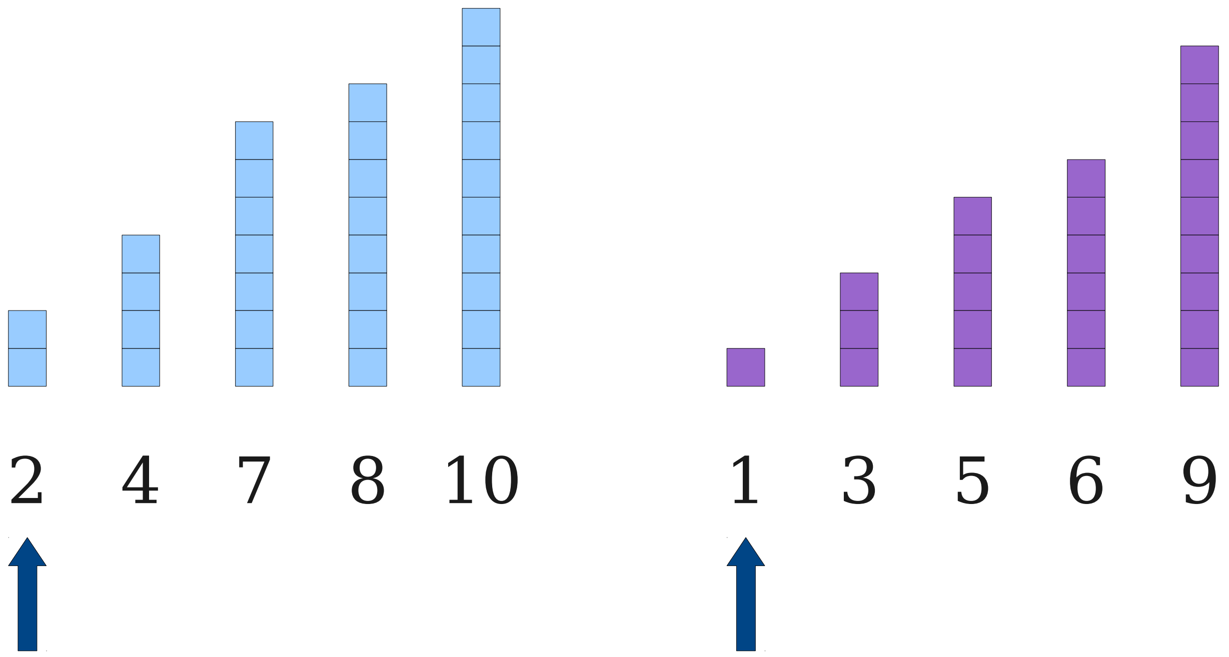

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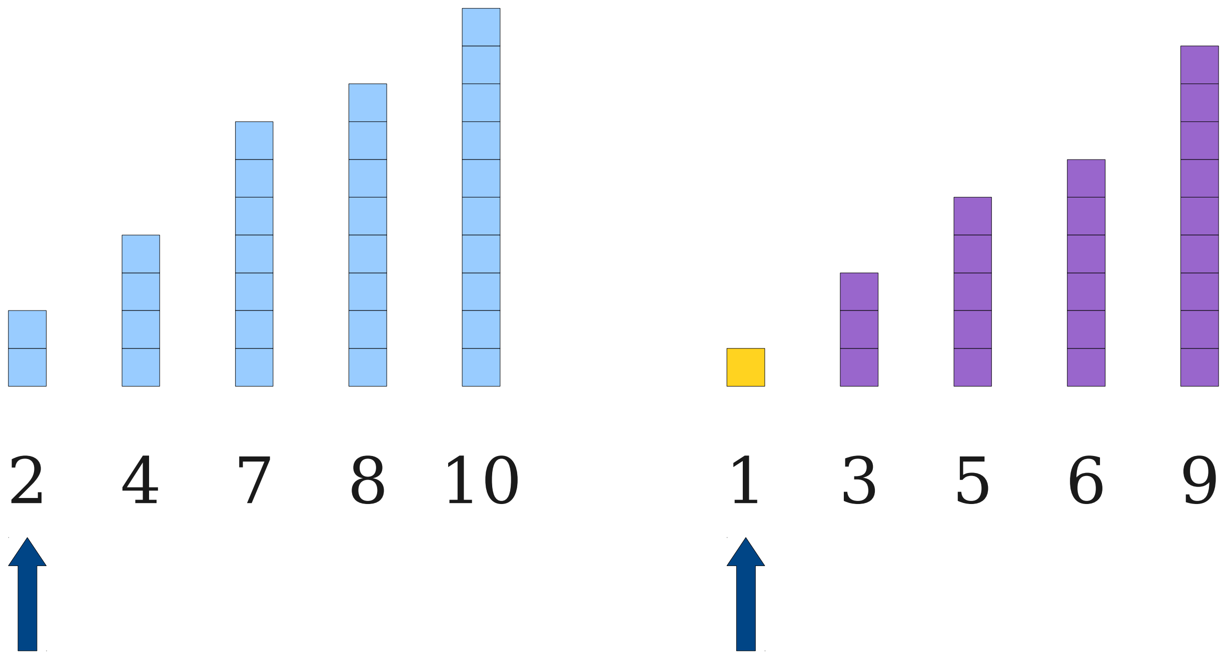

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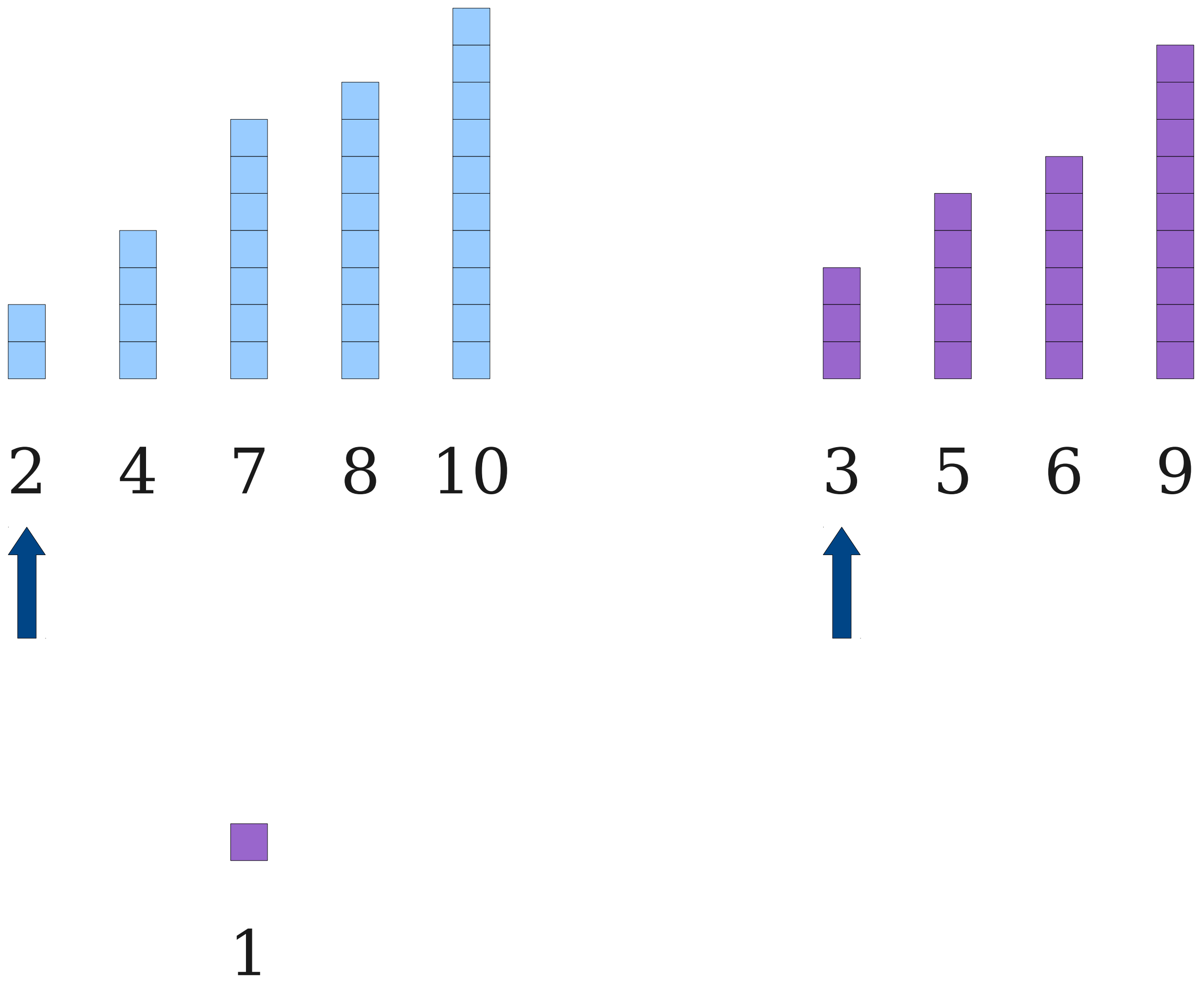

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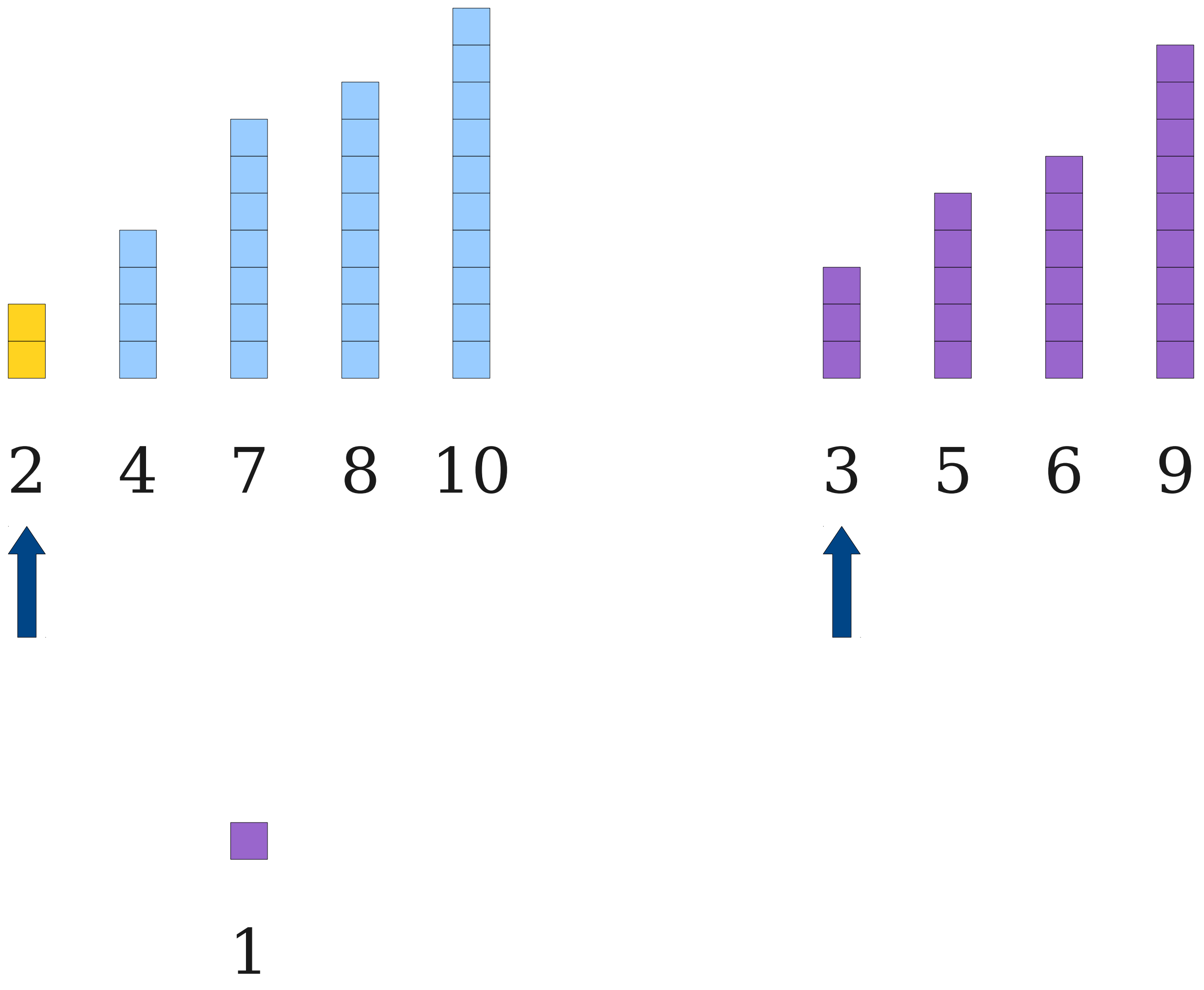

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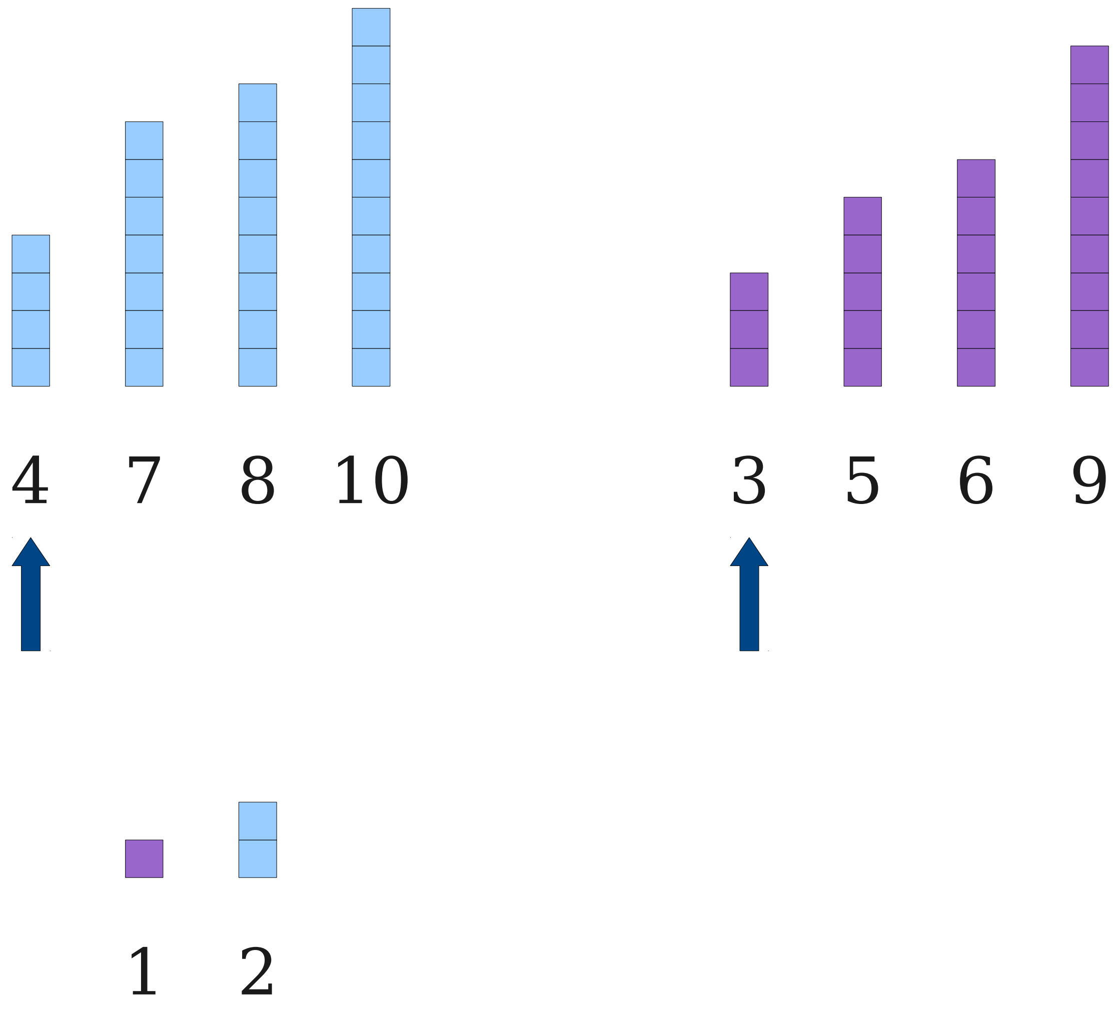

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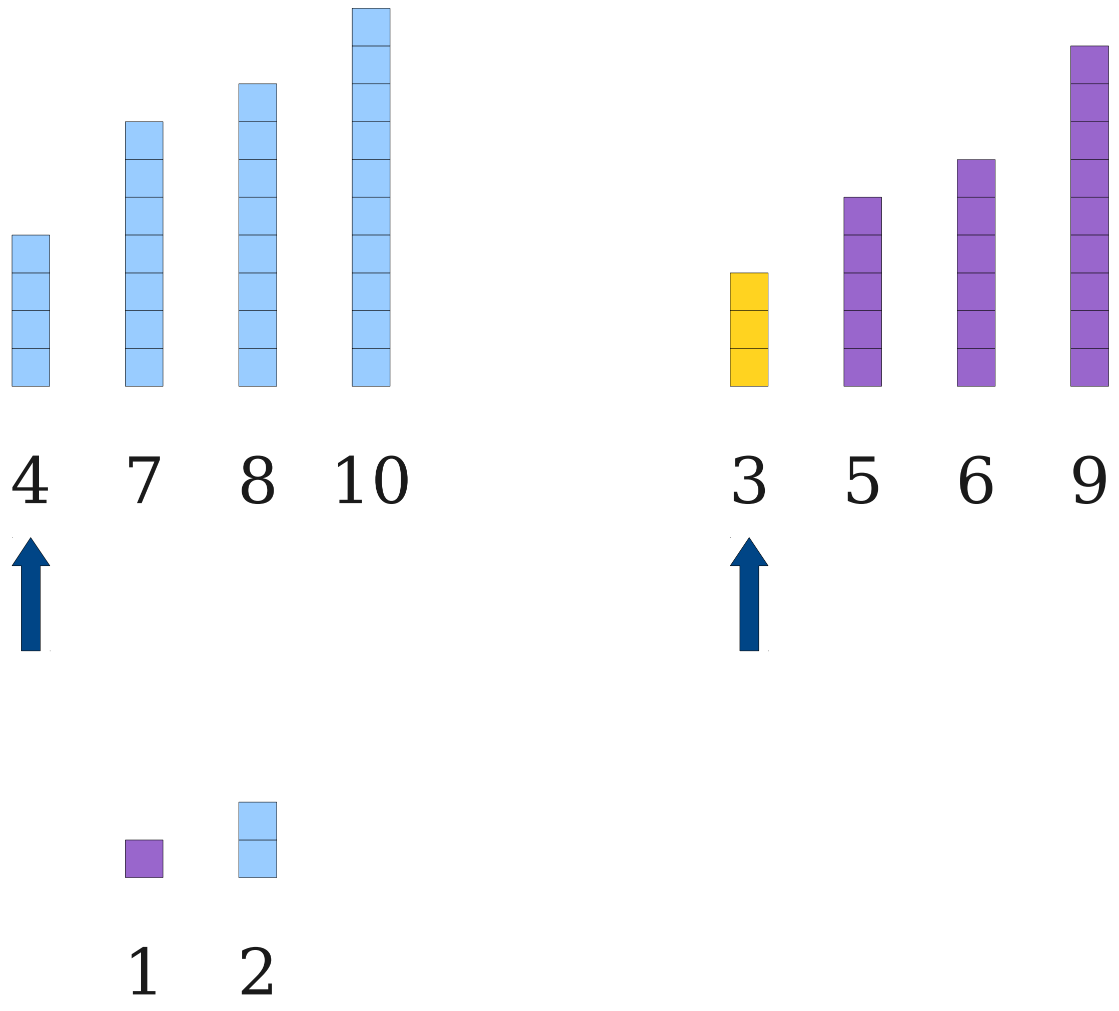

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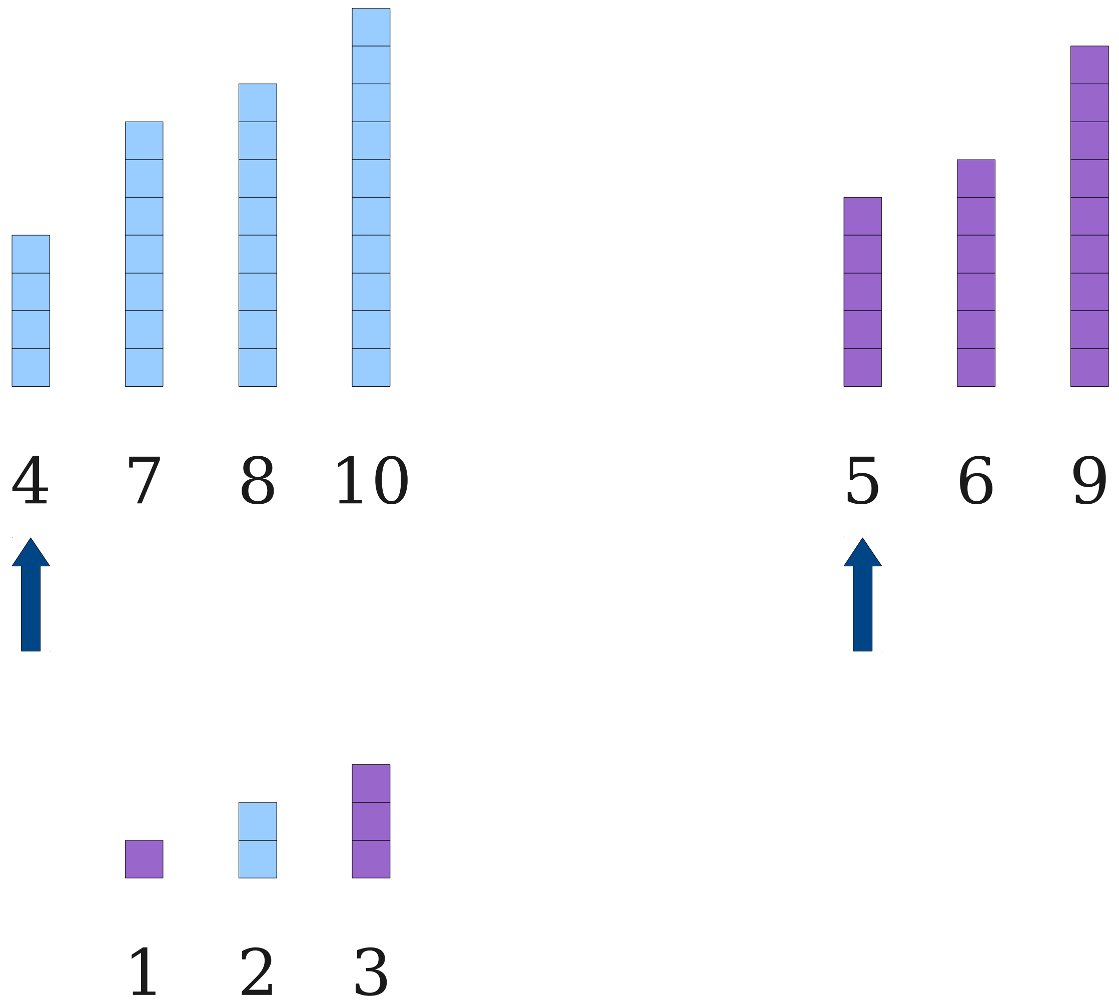

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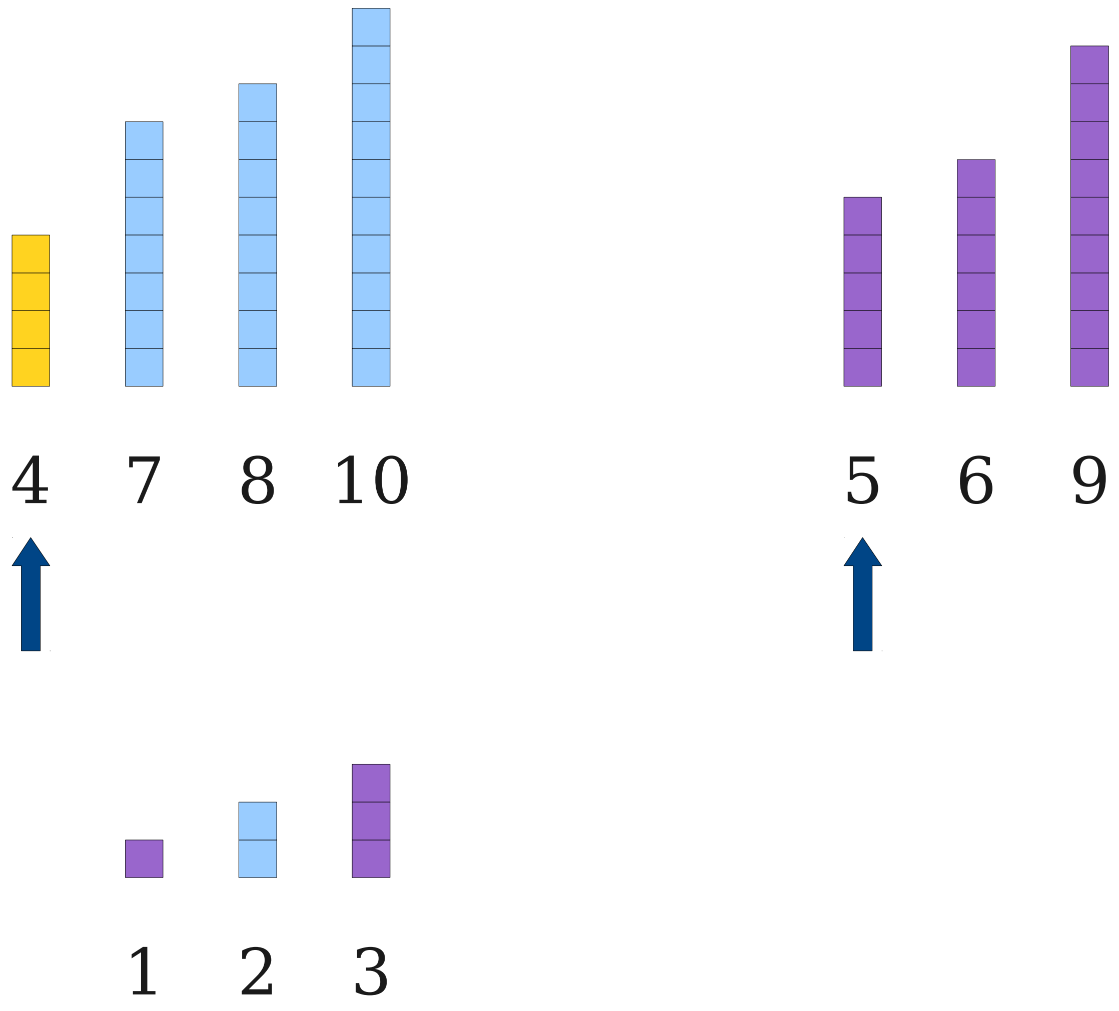

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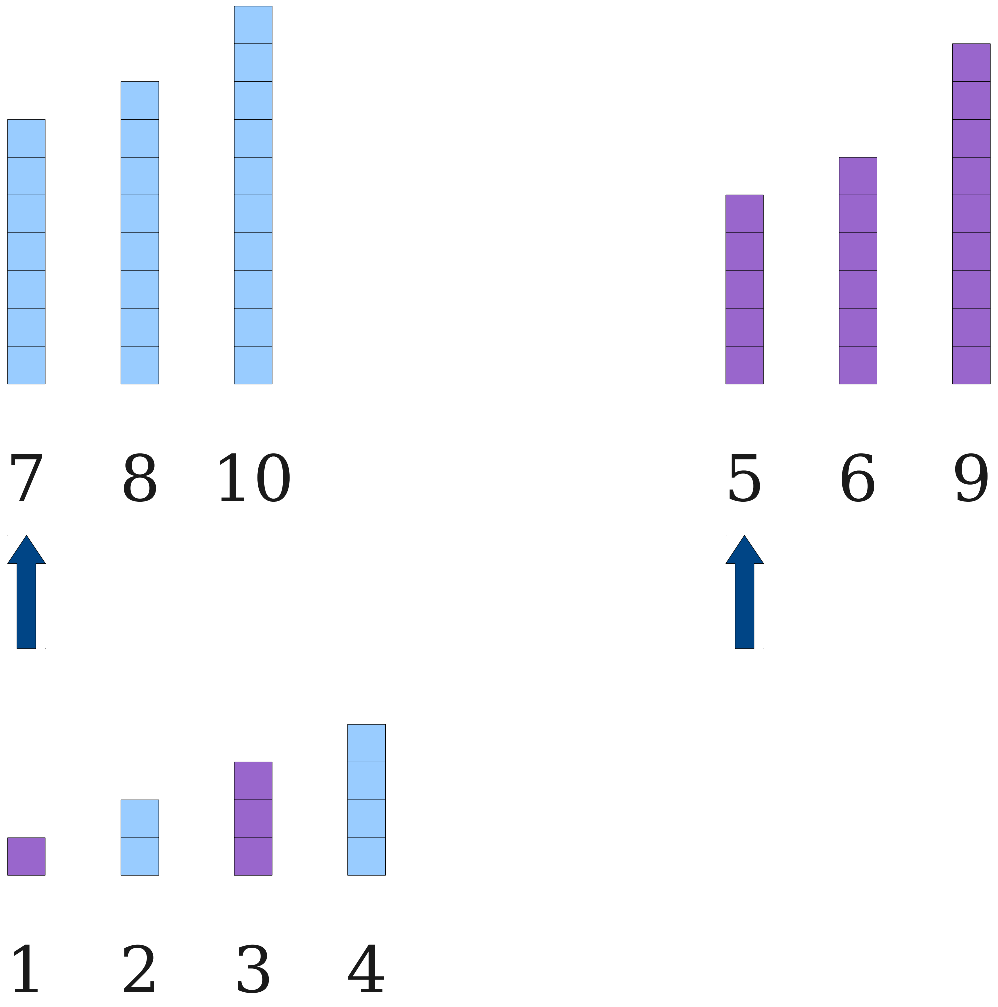

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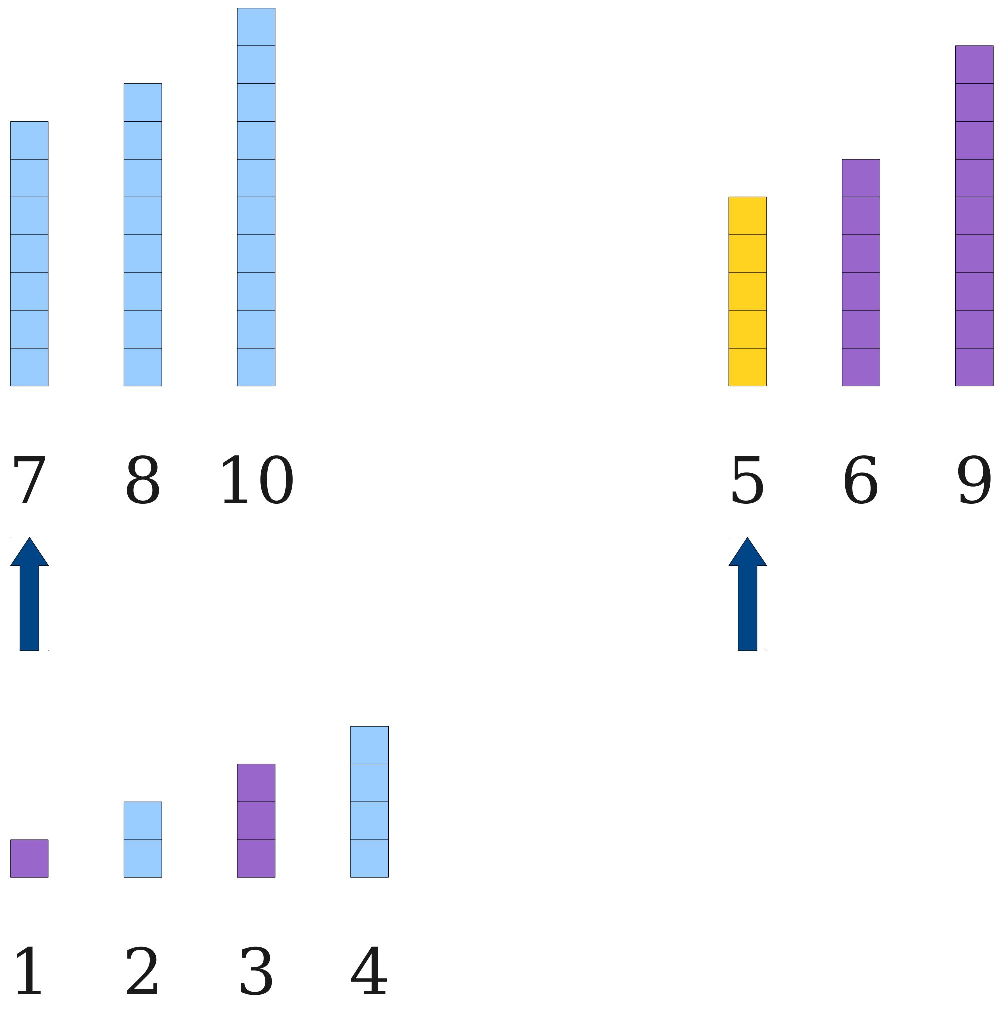

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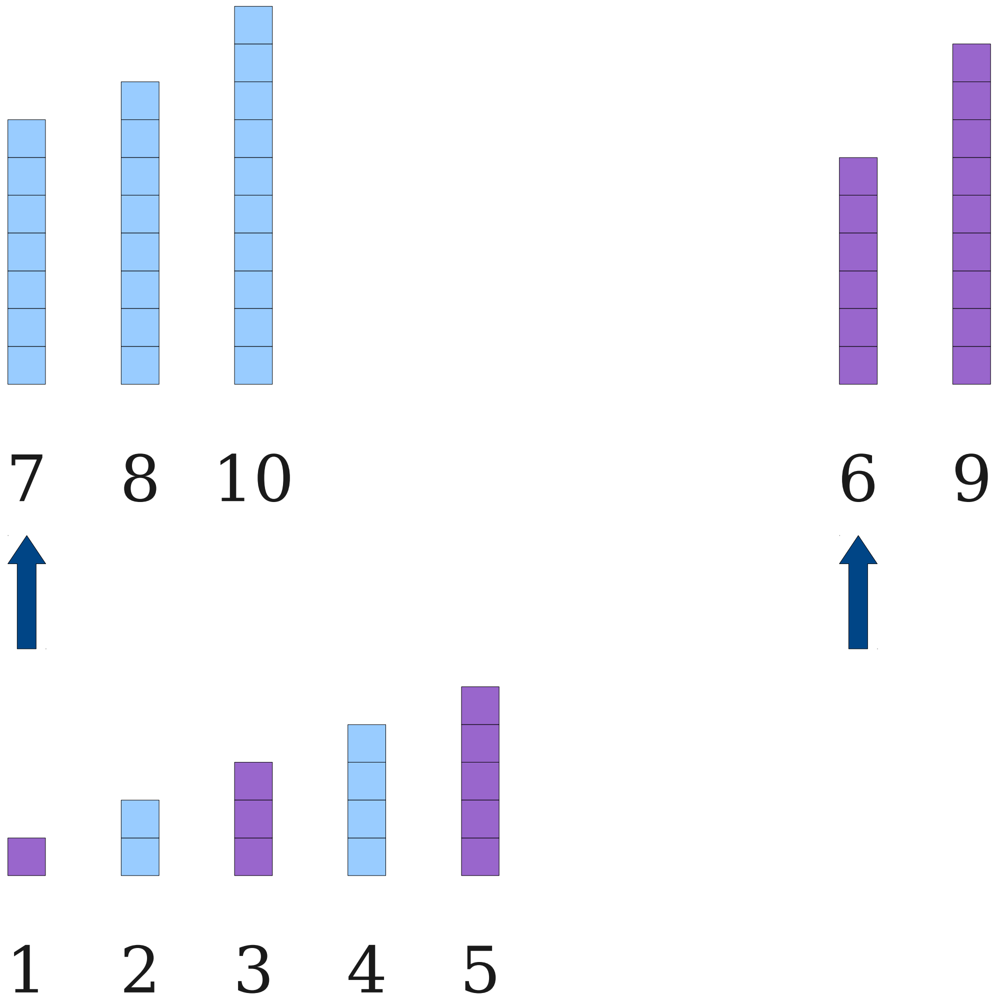

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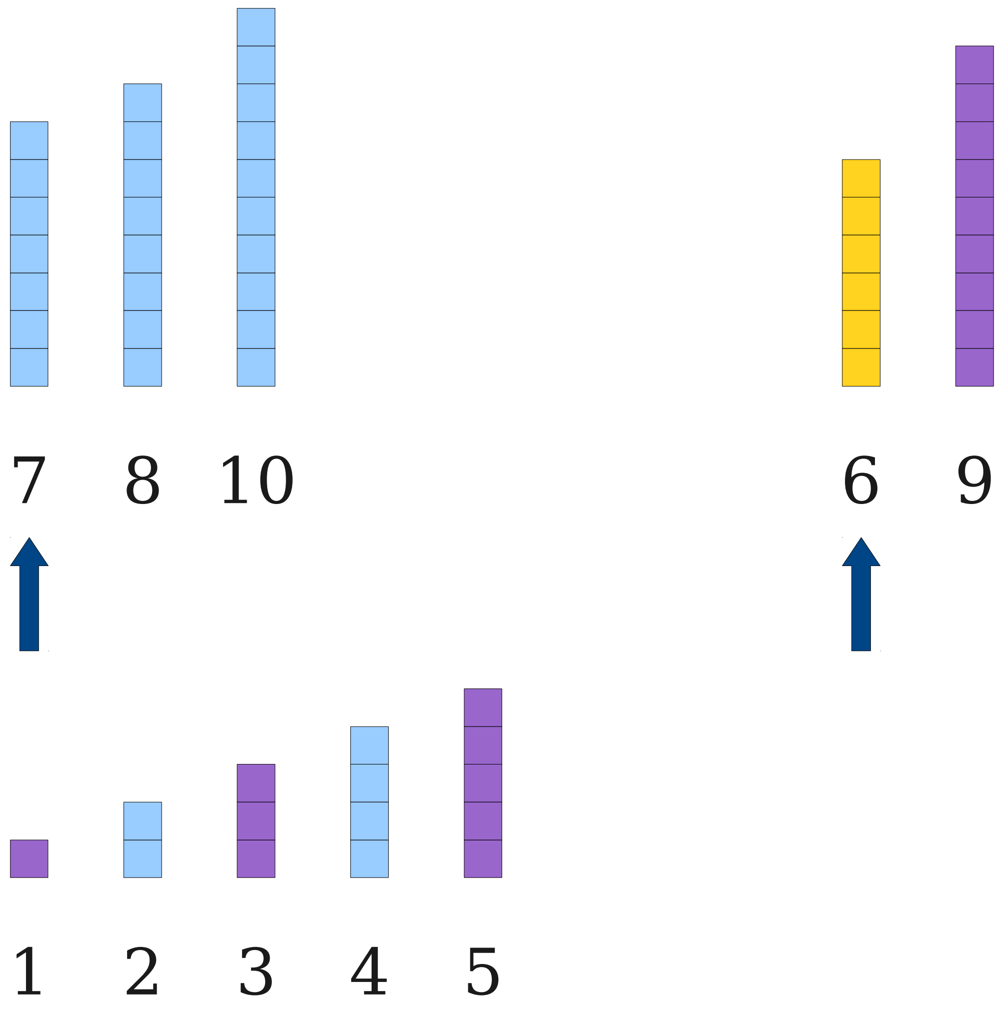

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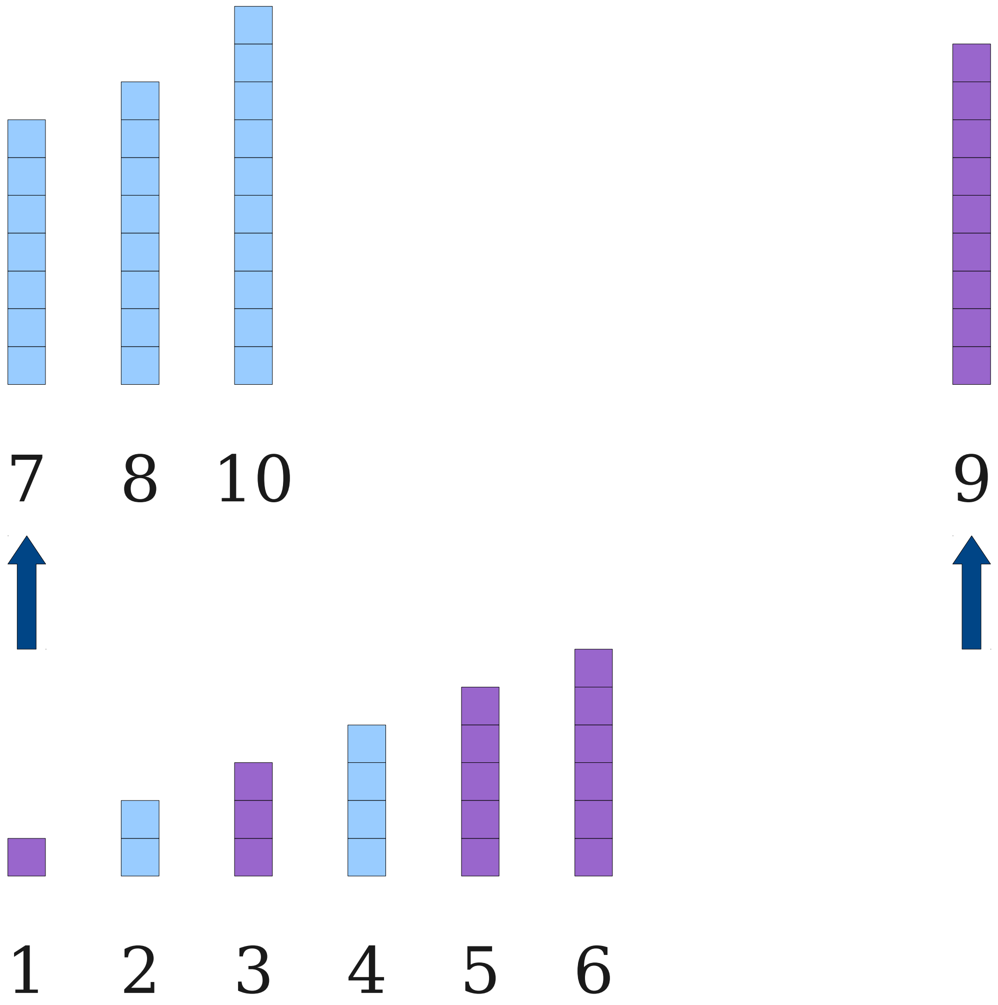

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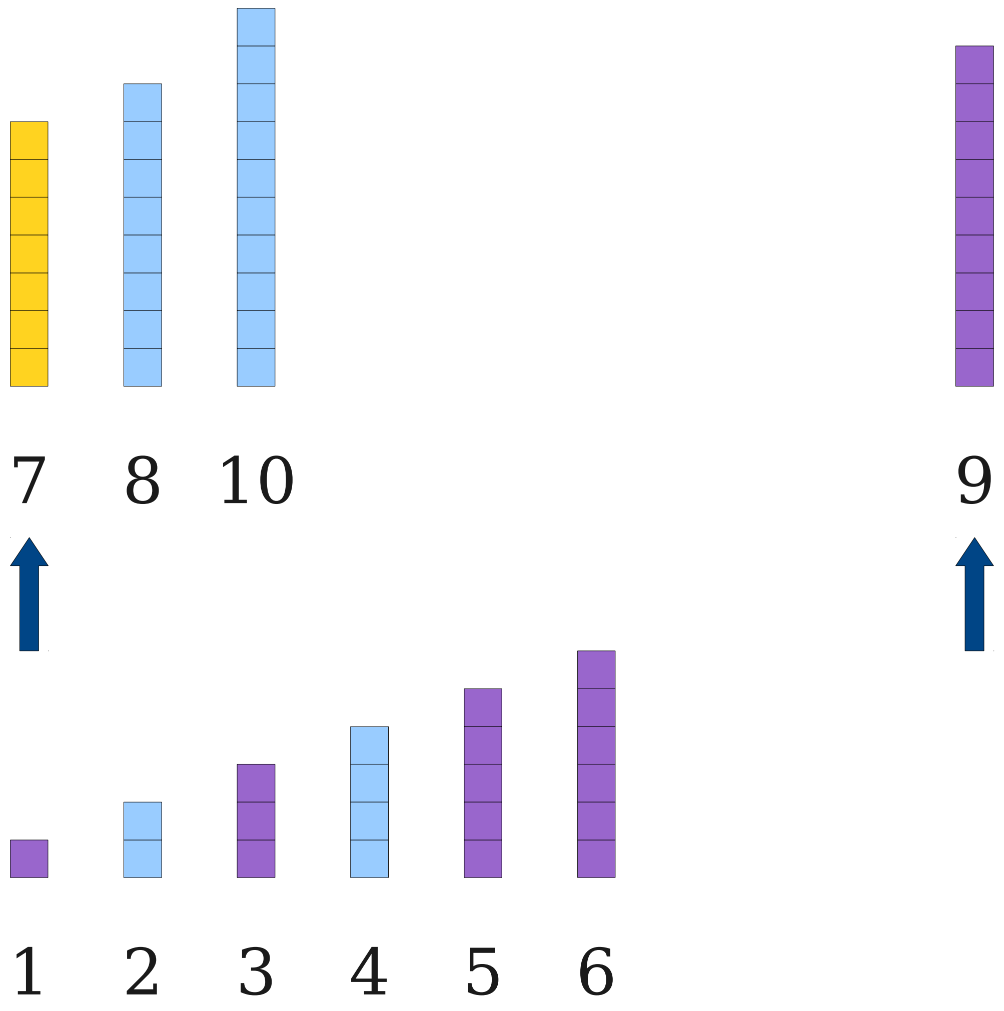

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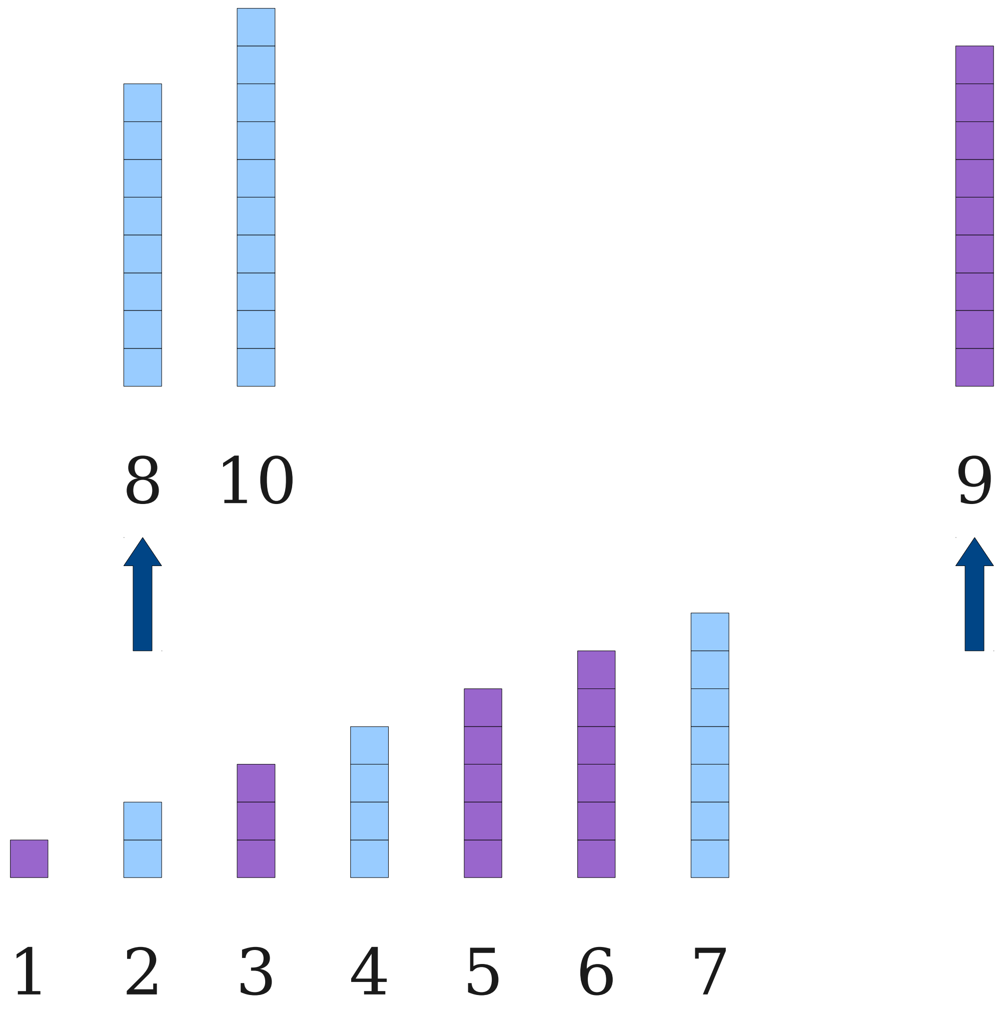

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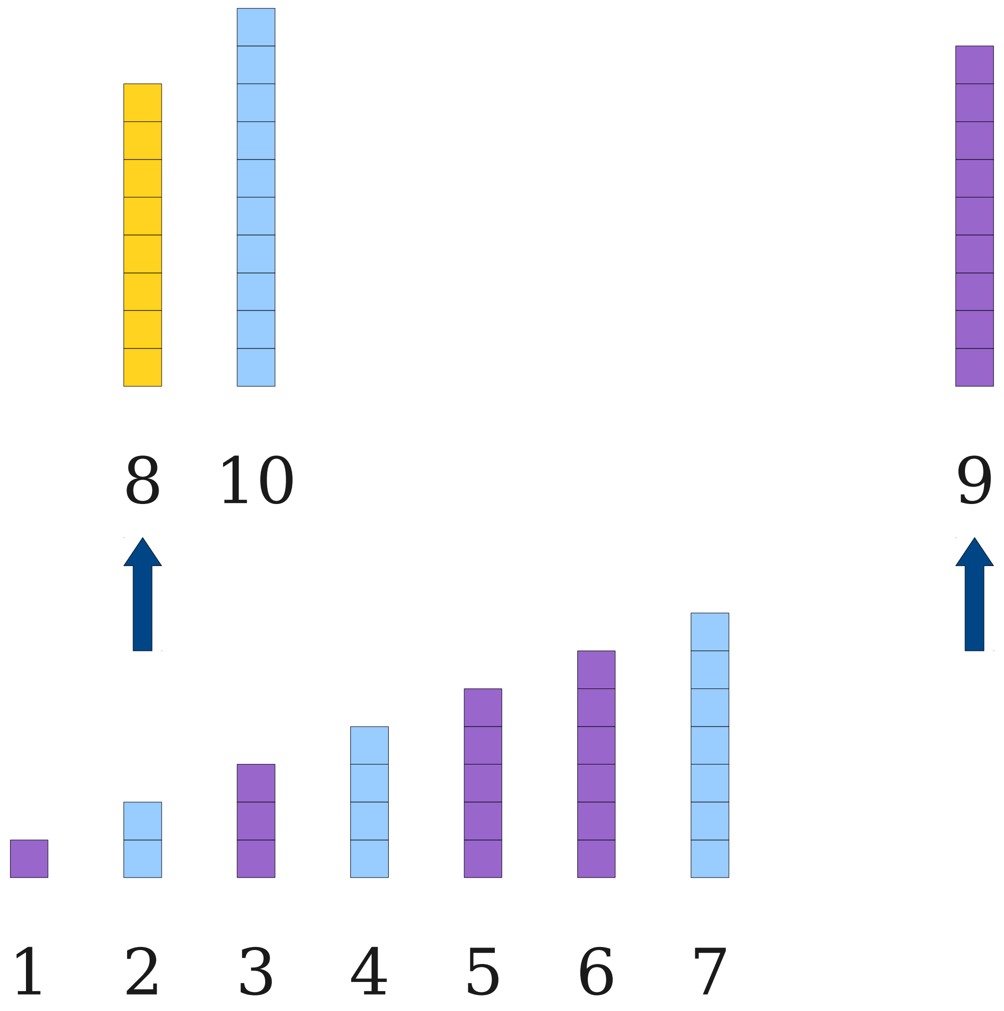

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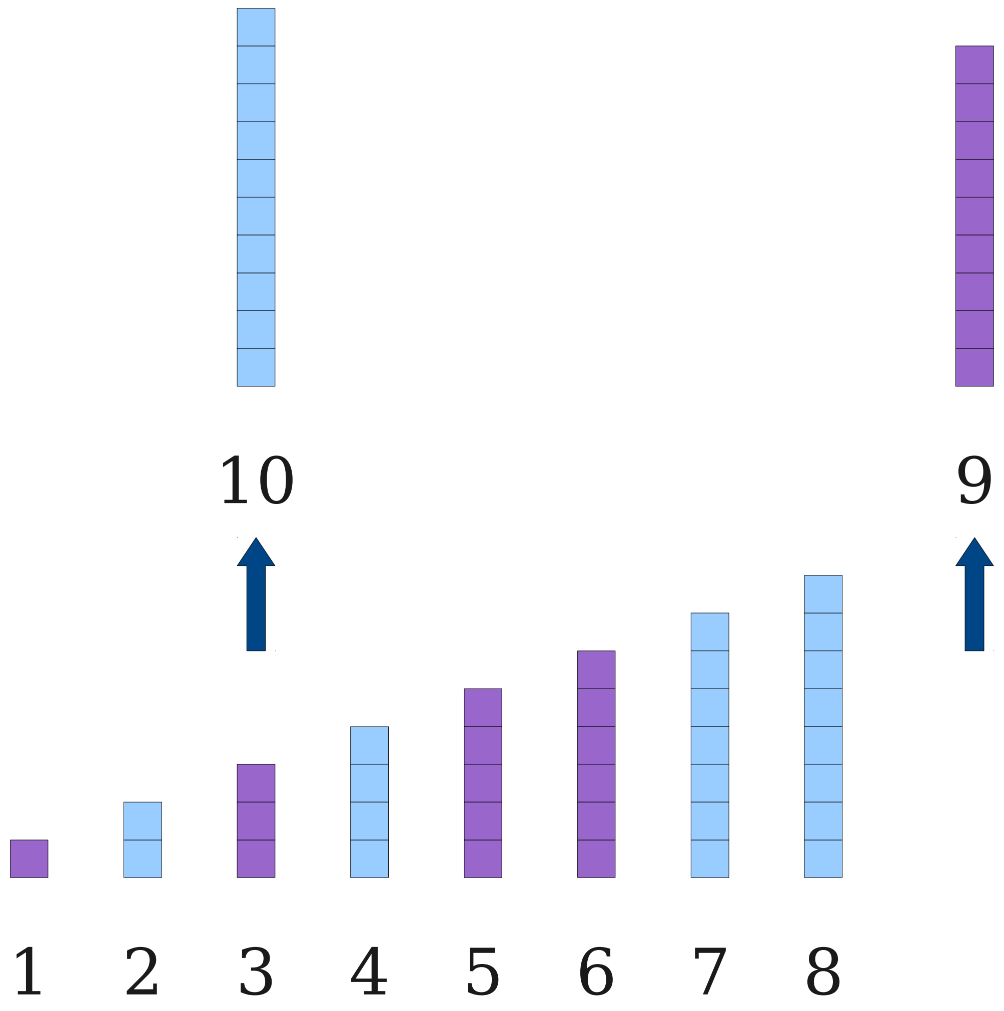

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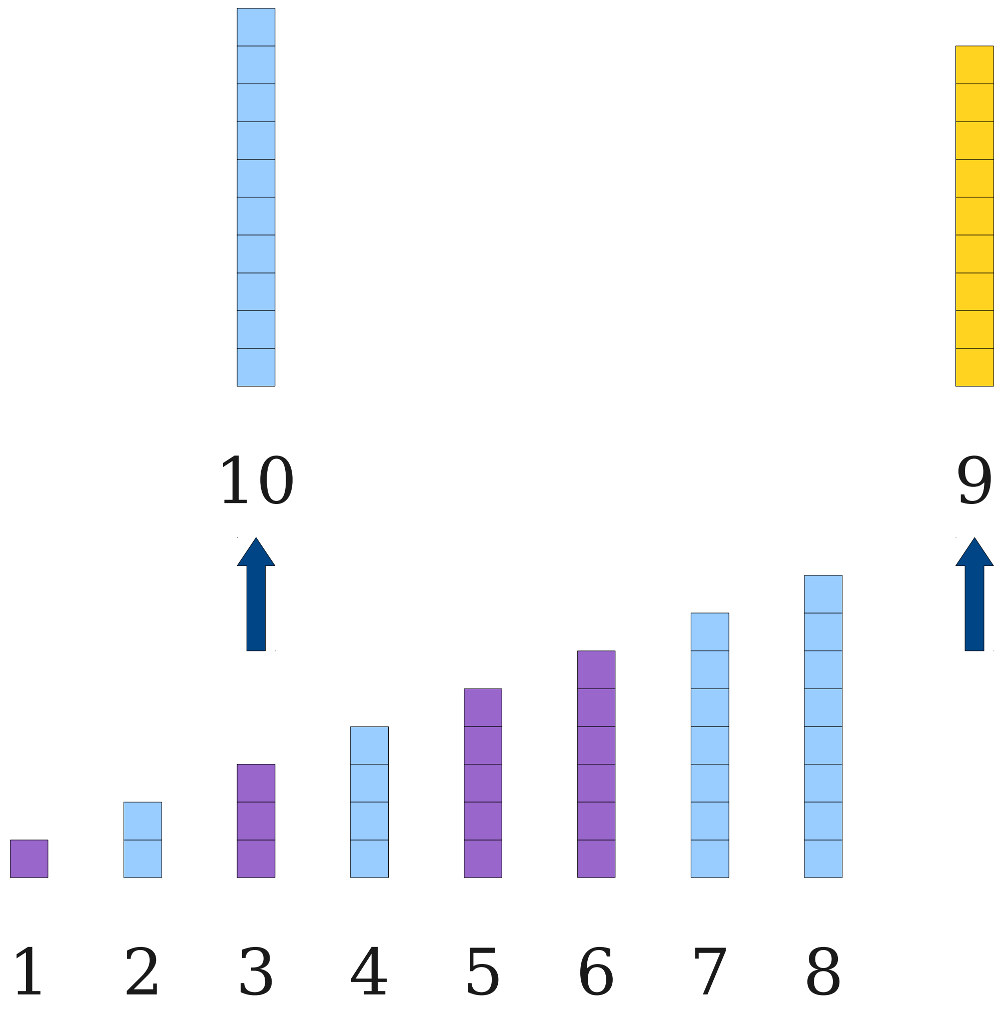

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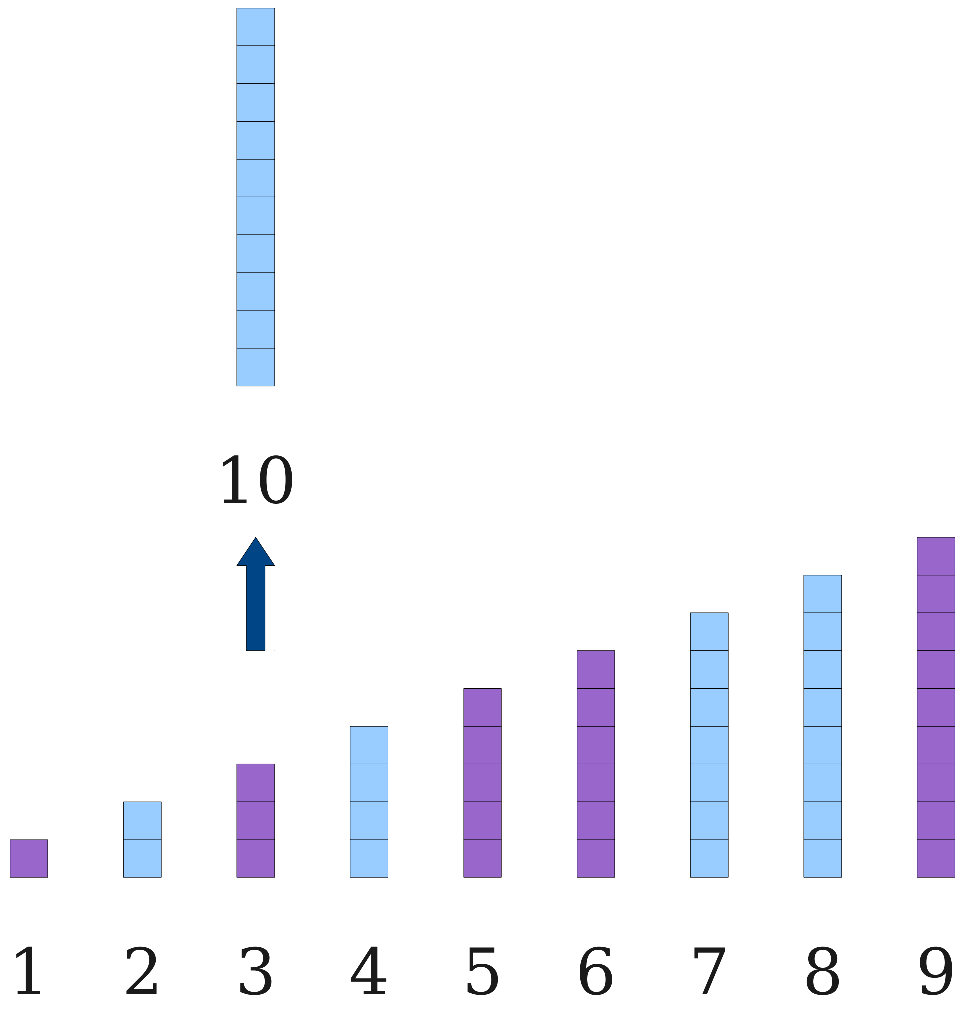

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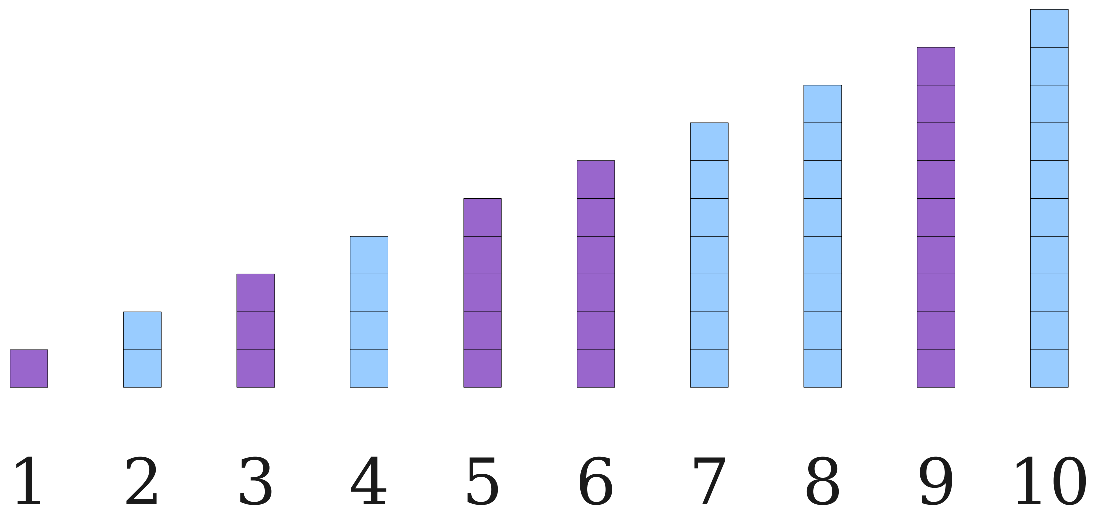


```
procedure merge(list A, list B):
   let result be an empty list.
   while both A and B are nonempty:
      if head(A) < head(B):
         append head(A) to result
         remove head(A) from A
      else:
         append head(B) to result
         remove head(B) from B

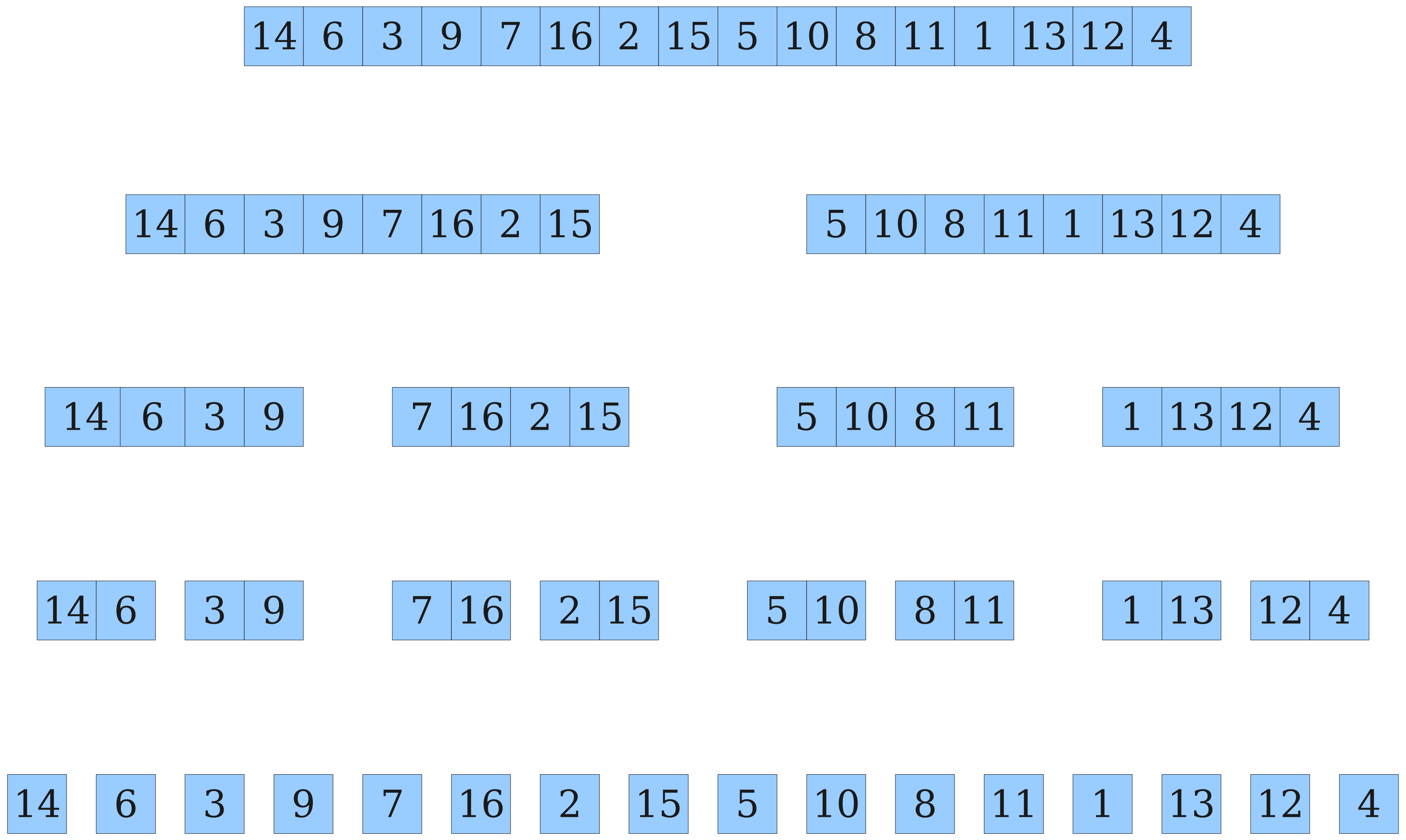
   append all elements remaining in A to result
   append all elements remaining in B to result

   return result
```

Complexity: $\boldsymbol{\Theta(m + n)}$,
where m and n are the lengths of the input lists.

Motivating Mergesort

- Splitting the input array in half, sorting each half, and merging them back together will take roughly half as long as soring the original array.
- So why not split the array into fourths? Or eighths?
- **Question**: What happens if we *never stop splitting?*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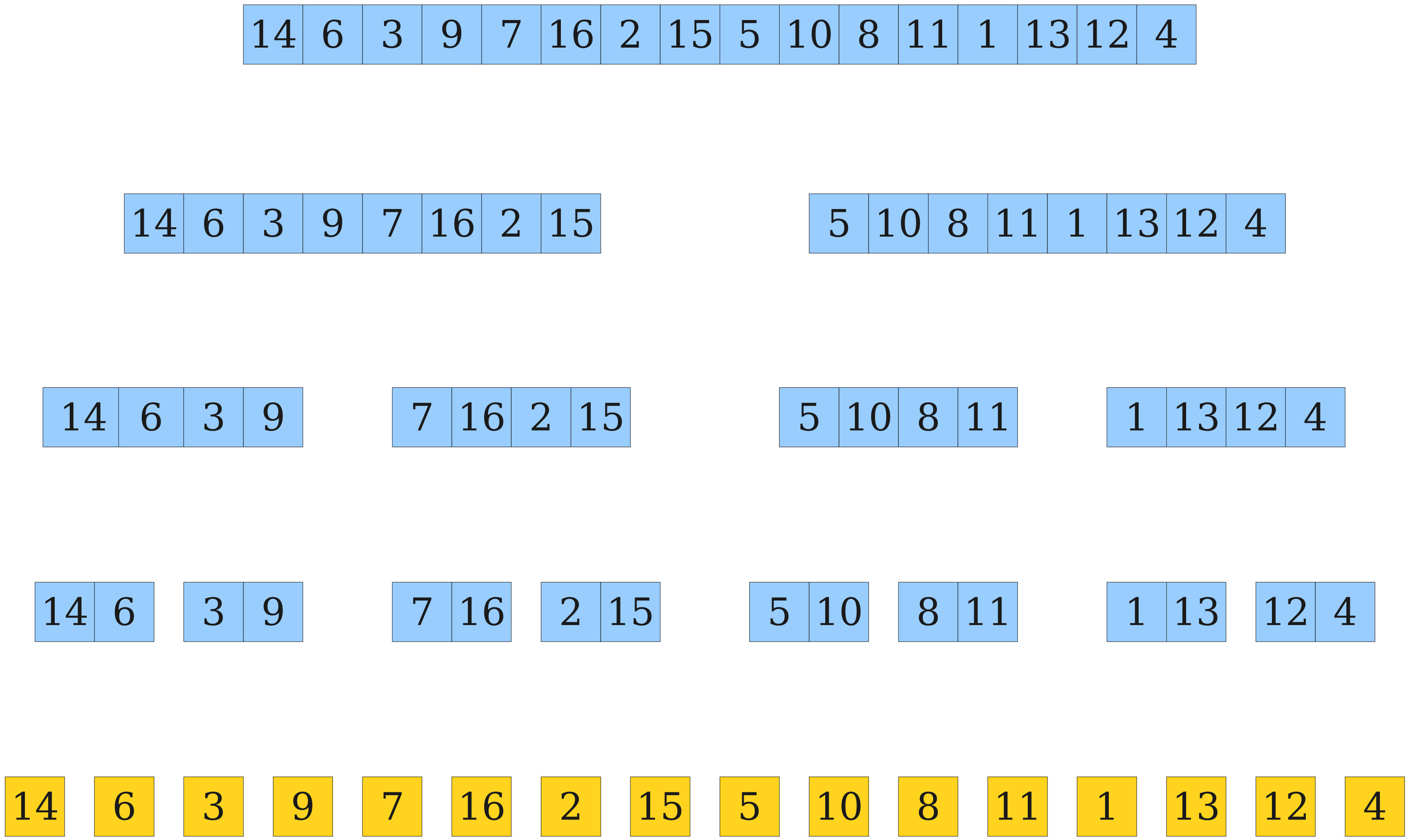
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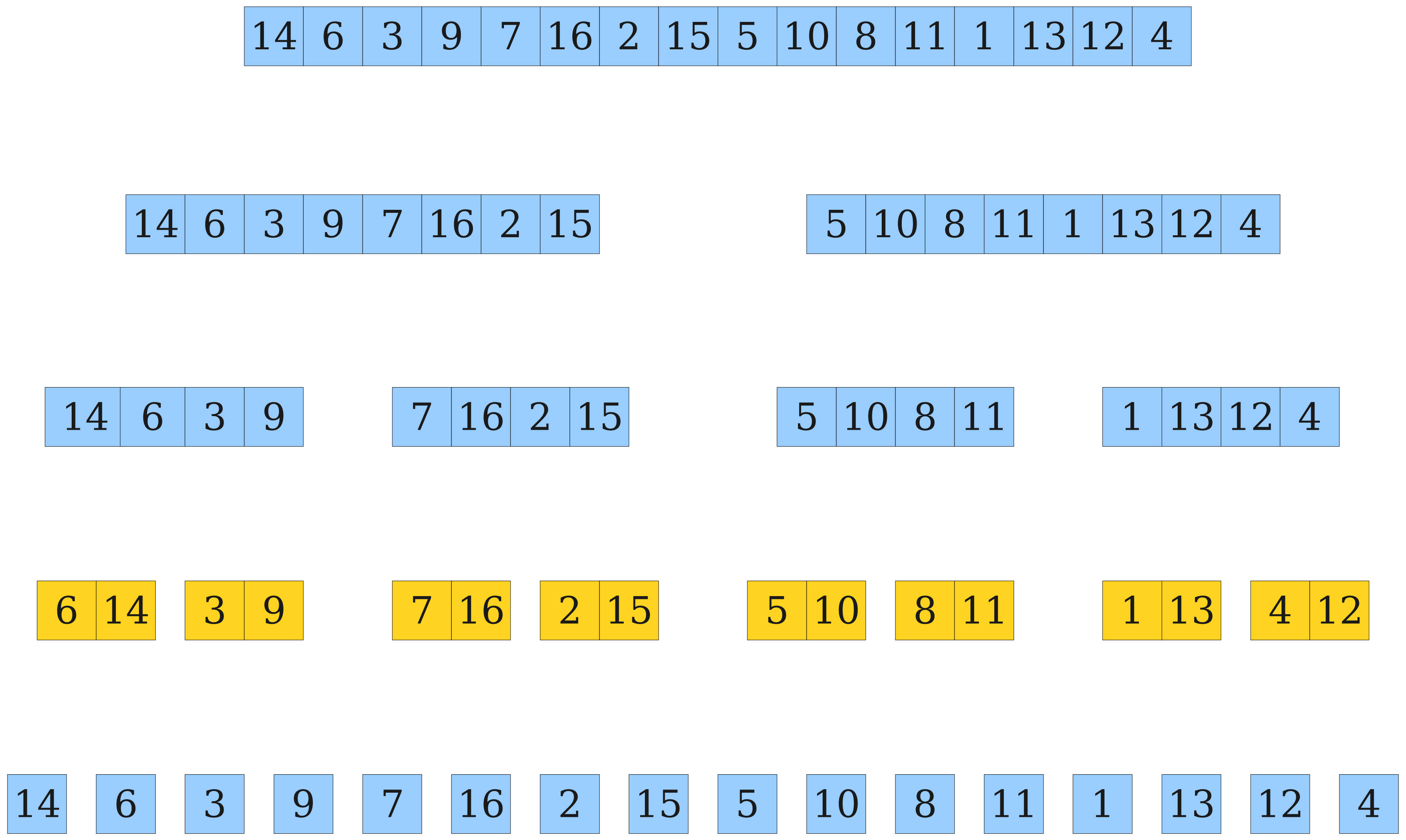
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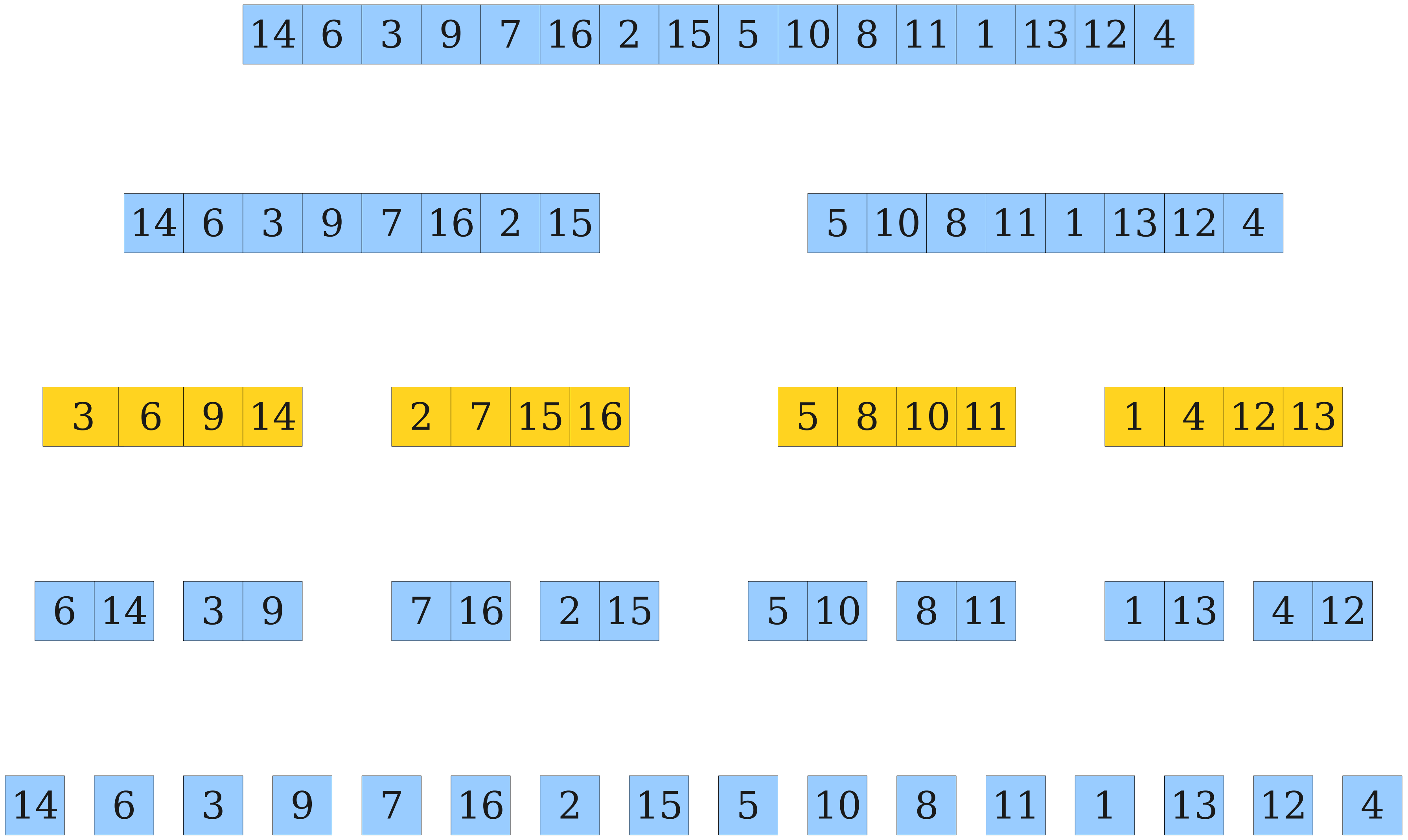

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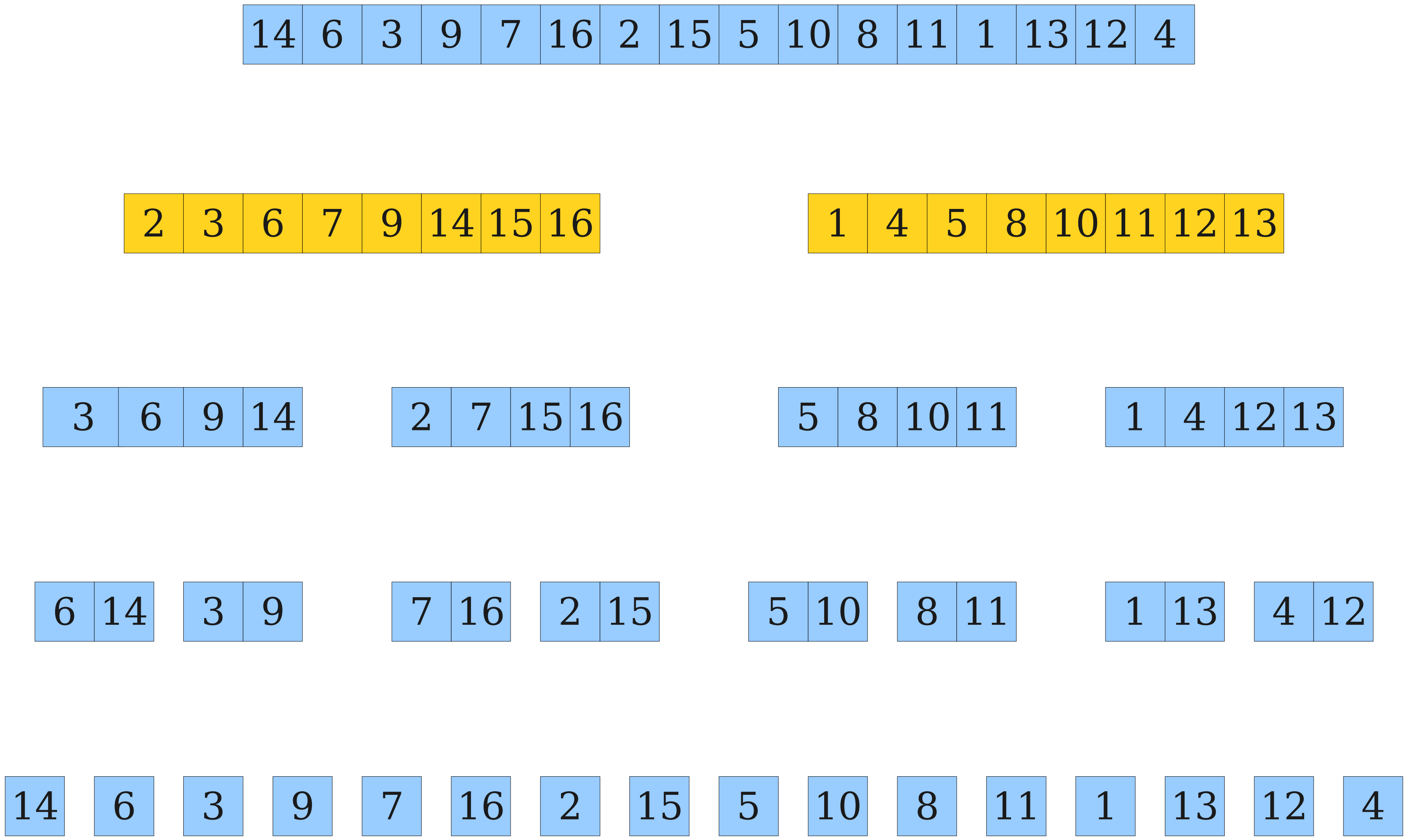
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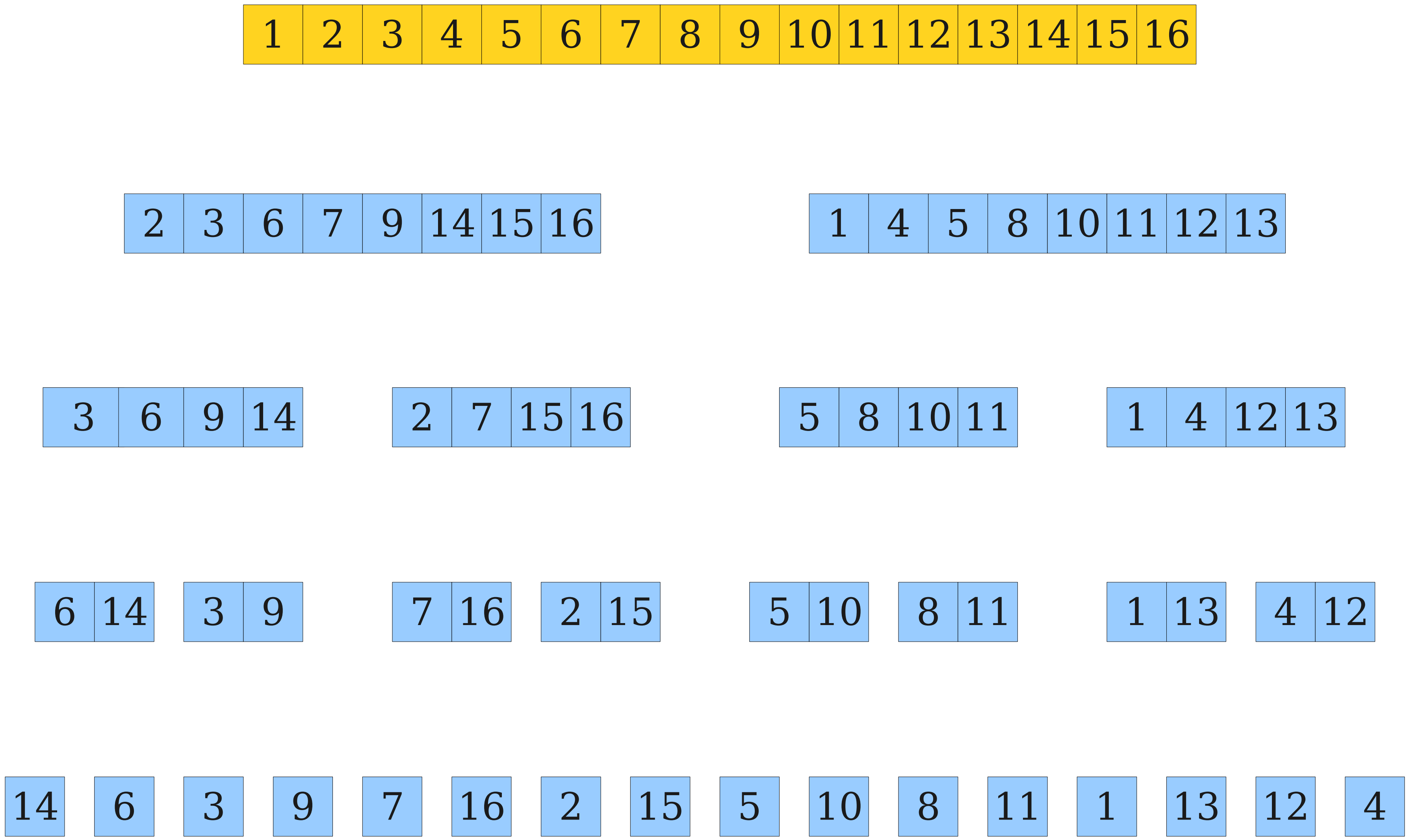
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14 6 3 9 7 16 2 15 5 10 8 11 1 13 12 4

High-Level Idea

- A recursive sorting algorithm!
- **Base Case**:
  - An empty or single-element list is already sorted.
- **Recursive step**:
  - Break the list in half and recursively sort each part.
  - Merge the sorted halves back together.
- This algorithm is called *mergesort*.

```
procedure mergesort(list A):
   if length(A) ≤ 1:
      return A

   let left  be the first  half of the elements of A
   let right be the second half of the elements of A

   return merge(mergesort(left), mergesort(right))
```

What is the complexity of mergesort?

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   let left  be the first  half of the elements of A
   let right be the second half of the elements of A

   return merge(mergesort(left), mergesort(right))
```

$$T(0) = \Theta(1)$$

```
procedure mergesort(list A):
   if length(A) ≤ 1:
      return A

   let left  be the first  half of the elements of A
   let right be the second half of the elements of A

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$$T(1) = \Theta(1)$$

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   return merge(mergesort(left), mergesort(right))
```

$$T(0) = \Theta(1)$$
$$T(1) = \Theta(1)$$
$$T(n) = T(\lceil n / 2 \rceil) + T(\lfloor n / 2 \rfloor) + \Theta(n)$$

Recurrence Relations

- A **recurrence relation** is a function or sequence whose values are defined in terms of earlier values.
- In our case, we get this recurrence for the runtime of mergesort:

$$\boxed{\begin{array}{l} T(0) = \Theta(1) \\ T(1) = \Theta(1) \\ T(n) = T(\lceil n / 2 \rceil) + T(\lfloor n / 2 \rfloor) + \Theta(n) \end{array}}$$

- We can **solve** a recurrence by finding an explicit expression for its terms, or by finding an asymptotic bound on its growth rate.
- How do we solve this recurrence?

Simplifying our Recurrence

- It is often difficult to solve recurrences involving floors and ceilings, as ours does.

$$\boxed{\begin{aligned}&T(0) = \Theta(1)\\&T(1) = \Theta(1)\\&T(n) = T(\lceil n / 2\rceil) + T(\lfloor n / 2\rfloor) + \Theta(n)\end{aligned}}$$

- Note that if we only consider $n = 1, 2, 4, 8, 16, \ldots$, then the floors and ceilings are always equivalent to standard division.
- **Simplifying Assumption 1:** We will only consider the recurrence as applied to powers of two.
- We need to justify why this is safe, which we'll do later.

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Simplifying our Recurrence

- It is often difficult to solve recurrences involving floors and ceilings, as ours does.

$$\boxed{\begin{array}{l} T(1) = \Theta(1) \\ T(n) = T(n / 2) + T(n / 2) + \Theta(n) \end{array}}$$

- Note that if we only consider $n = 1, 2, 4, 8, 16, \ldots$, then the floors and ceilings are always equivalent to standard division.
- **Simplifying Assumption 1:** We will only consider the recurrence as applied to powers of two.
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Simplifying our Recurrence

- It is often difficult to solve recurrences involving floors and ceilings, as ours does.

$$\boxed{\begin{aligned} &T(1) = \Theta(1) \\ &T(n) = 2T(n / 2) + \Theta(n) \end{aligned}}$$

- Note that if we only consider $n = 1, 2, 4, 8, 16, \ldots$, then the floors and ceilings are always equivalent to standard division.
- **Simplifying Assumption 1:** We will only consider the recurrence as applied to powers of two.
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Simplifying our Recurrence

- Without knowing the actual functions hidden by the Θ notation, we cannot get an exact value for the terms in this recurrence.

$$\boxed{\begin{aligned} &T(1) = \Theta(1) \\ &T(n) = 2T(n / 2) + \Theta(n) \end{aligned}}$$

- If the Θ(1) just hides a constant and Θ(n) just hides a multiple of n, this would be a lot easier to manipulate!

- **Simplifying Assumption 2:** We will pretend that Θ(1) hides some constant and Θ(n) hides a multiple of n.
- We need to justify why this is safe, which we'll do later.

Simplifying our Recurrence

- Without knowing the actual functions hidden by the Θ notation, we cannot get an exact value for the terms in this recurrence.

$$\boxed{\begin{aligned} &T(1) = c_1 \\ &T(n) = 2T(n / 2) + c_2 n \end{aligned}}$$

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Simplifying our Recurrence

- Working with two constants c_1 and c_2 is most accurate, but it makes the math a *lot* harder.

$$\boxed{\begin{aligned} &T(1) = c_1 \\ &T(n) = 2T(n / 2) + c_2 n \end{aligned}}$$

- If all we care about is getting an asymptotic bound, these constants are unlikely to make a noticeable difference.
- **Simplifying Assumption 3:** Set $c = \max\{c_1, c_2\}$ and replace the equality with an upper bound.
- We need to justify why this is safe, which we'll do later.

Simplifying our Recurrence

- Working with two constants c_1 and c_2 is most accurate, but it makes the math a *lot* harder.

$$\boxed{\begin{aligned} &T(1) \leq c \\ &T(n) \leq 2T(n / 2) + cn \end{aligned}}$$

- If all we care about is getting an asymptotic bound, these constants are unlikely to make a noticeable difference.
- **Simplifying Assumption 3:** Set $c = \max\{c_1, c_2\}$ and replace the equality with an upper bound.
- This is less exact, but is easier to manipulate.

The Final Recurrence

- Here is the final version of the recurrence we'll be working with:

$$\boxed{\begin{aligned}&\mathbf{T(1)} \le c\\&\mathbf{T(}n\mathbf{)} \le \mathbf{2T(}n\mathbf{ / 2) +} cn\end{aligned}}$$

- As before, we will justify why all of these simplifications are safe later on.
- The analysis we're about to do (without justifying the simplifications) is at the level we will expect for most of our discussion of divide-and-conquer algorithms.

Getting an Intuition

- Simple recurrence relations often give rise to surprising results.
- It is often useful to build up an intuition for what the recursion solves to before trying to formally prove it.
- We will explore two methods for doing this:
  - The *iteration method*.
  - The *recursion-tree method*.

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We will explore two methods for doing this:

- The *iteration method*.
- The *recursion-tree method*.

$$\boxed{\begin{aligned} &T(1) \le c \\ &T(n) \le 2T(n / 2) + cn \end{aligned}}$$

$$T(n) \leq 2T\left(\frac{n}{2}\right) + cn$$

$T(1) \leq c$
$T(n) \leq 2T(n / 2) + cn$

$$\begin{aligned} T(n) &\le 2T\left(\frac{n}{2}\right) + c\,n \\ &\le 2\left(2T\left(\frac{n}{4}\right) + \frac{c\,n}{2}\right) + c\,n \end{aligned}$$

$$\boxed{\begin{aligned} &T(1) \le c \\ &T(n) \le 2T(n/2) + cn \end{aligned}}$$

$$\begin{aligned}
T(n) &\le 2T\left(\frac{n}{2}\right) + c\,n \\
&\le 2\left(2T\left(\frac{n}{4}\right) + \frac{c\,n}{2}\right) + c\,n \\
&= 4T\left(\frac{n}{4}\right) + c\,n + c\,n
\end{aligned}$$

$$\boxed{\begin{array}{l} T(1) \le c \\ T(n) \le 2T(n / 2) + cn \end{array}}$$

$$\boxed{\begin{array}{l}T(1) \le c \\ T(n) \le 2T(n / 2) + cn\end{array}}$$

$$\begin{aligned}
T(n) &\le 2T\left(\frac{n}{2}\right) + cn \\
&\le 2\left(2T\left(\frac{n}{4}\right) + \frac{cn}{2}\right) + cn \\
&= 4T\left(\frac{n}{4}\right) + cn + cn \\
&= 4T\left(\frac{n}{4}\right) + 2cn
\end{aligned}$$

$$\boxed{\begin{aligned}&T(1) \le c\\ &T(n) \le 2T(n/2) + cn\end{aligned}}$$

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T(n) &\le 2T\left(\frac{n}{2}\right) + cn\\
&\le 2\left(2T\left(\frac{n}{4}\right) + \frac{cn}{2}\right) + cn\\
&= 4T\left(\frac{n}{4}\right) + cn + cn\\
&= 4T\left(\frac{n}{4}\right) + 2cn\\
&\le 4\left(2T\left(\frac{n}{8}\right) + \frac{cn}{4}\right) + 2cn
\end{aligned}$$

$$\boxed{\begin{array}{l} T(1) \le c \\ T(n) \le 2T(n/2) + cn \end{array}}$$

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&= 4T\left(\frac{n}{4}\right) + 2cn \\
&\le 4\left(2T\left(\frac{n}{8}\right) + \frac{cn}{4}\right) + 2cn \\
&= 8T\left(\frac{n}{8}\right) + cn + 2cn
\end{aligned}$$

$$\boxed{\begin{array}{l}\mathrm{T}(1) \le c \\ \mathrm{T}(n) \le 2\mathrm{T}(n / 2) + cn\end{array}}$$

$$\begin{aligned}
\mathrm{T}(n) &\le 2\mathrm{T}\left(\frac{n}{2}\right)+cn \\
&\le 2\left(2\mathrm{T}\left(\frac{n}{4}\right)+\frac{cn}{2}\right)+cn \\
&= 4\mathrm{T}\left(\frac{n}{4}\right)+cn+cn \\
&= 4\mathrm{T}\left(\frac{n}{4}\right)+2cn \\
&\le 4\left(2\mathrm{T}\left(\frac{n}{8}\right)+\frac{cn}{4}\right)+2cn \\
&= 8\mathrm{T}\left(\frac{n}{8}\right)+cn+2cn \\
&= 8\mathrm{T}\left(\frac{n}{8}\right)+3cn
\end{aligned}$$

$$\boxed{\begin{aligned}&\mathrm{T}(1) \le c\\ &\mathrm{T}(n) \le 2\mathrm{T}(n/2) + cn\end{aligned}}$$

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&= 8\mathrm{T}\left(\frac{n}{8}\right) + cn + 2cn \\
&= 8\mathrm{T}\left(\frac{n}{8}\right) + 3cn \\
&\ldots \\
&\le 2^k\mathrm{T}\left(\frac{n}{2^k}\right) + kcn
\end{aligned}$$

$$\boxed{\begin{array}{l} T(1) \le c \\ T(n) \le 2T(n/2) + cn \end{array}}$$

$$\boxed{n / 2^k = 1}$$

$$\begin{aligned}
T(n) &\le 2T\left(\frac{n}{2}\right) + cn \\
&\le 2\left(2T\left(\frac{n}{4}\right) + \frac{cn}{2}\right) + cn \\
&= 4T\left(\frac{n}{4}\right) + cn + cn \\
&= 4T\left(\frac{n}{4}\right) + 2cn \\
&\le 4\left(2T\left(\frac{n}{8}\right) + \frac{cn}{4}\right) + 2cn \\
&= 8T\left(\frac{n}{8}\right) + cn + 2cn \\
&= 8T\left(\frac{n}{8}\right) + 3cn \\
&\dots \\
&\le 2^k T\left(\frac{n}{2^k}\right) + kcn
\end{aligned}$$

$$\boxed{\begin{array}{l}T(1) \le c \\ T(n) \le 2T(n / 2) + cn\end{array}}$$

$$\boxed{\begin{array}{c}n / 2^k = 1 \\ n = 2^k\end{array}}$$

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&\ldots \\
&\le 2^k T\left(\frac{n}{2^k}\right) + kcn
\end{aligned}$$

$$\boxed{\begin{array}{l} T(1) \le c \\ T(n) \le 2T(n/2) + cn \end{array}}$$

$$\boxed{\begin{array}{c} n / 2^k = 1 \\ n = 2^k \\ \log_2 n = k \end{array}}$$

$$\begin{aligned}
T(n) &\le 2T\left(\frac{n}{2}\right) + cn \\
&\le 2\left(2T\left(\frac{n}{4}\right) + \frac{cn}{2}\right) + cn \\
&= 4T\left(\frac{n}{4}\right) + cn + cn \\
&= 4T\left(\frac{n}{4}\right) + 2cn \\
&\le 4\left(2T\left(\frac{n}{8}\right) + \frac{cn}{4}\right) + 2cn \\
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&\dots \\
&\le 2^k T\left(\frac{n}{2^k}\right) + kcn
\end{aligned}$$

$$\boxed{\begin{array}{l} \mathrm{T}(1) \le c \\ \mathrm{T}(n) \le 2\mathrm{T}(n / 2) + cn \end{array}}$$

$$\mathrm{T}(n) \quad \le \quad 2^k\,\mathrm{T}\left(\frac{n}{2^k}\right) + k\,c\,n$$

$$\boxed{\begin{aligned}&T(1) \le c\\&T(n) \le 2T(n/2) + cn\end{aligned}}$$

$$\begin{aligned}T(n) &\le 2^k\, T\!\left(\frac{n}{2^k}\right) + k\,c\,n\\&= 2^{\log_2 n}\, T(1) + c\,n\log_2 n\end{aligned}$$

$$\boxed{\begin{array}{l} T(1) \le c \\ T(n) \le 2T(n / 2) + cn \end{array}}$$

$$\begin{aligned} T(n) &\le 2^k T\left(\frac{n}{2^k}\right) + k\,c\,n \\ &= 2^{\log_2 n} T(1) + c\,n \log_2 n \\ &= n\,T(1) + c\,n \log_2 n \end{aligned}$$

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$$\begin{aligned} T(n) &\le 2^k T\left(\frac{n}{2^k}\right) + k\,c\,n \\ &= 2^{\log_2 n} T(1) + c\,n\log_2 n \\ &= n\,T(1) + c\,n\log_2 n \\ &\le c\,n + c\,n\log_2 n \end{aligned}$$

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$$\begin{aligned}
T(n) &\le 2^k T\left(\frac{n}{2^k}\right) + kcn\\
&= 2^{\log_2 n} T(1) + cn\log_2 n\\
&= nT(1) + cn\log_2 n\\
&\le cn + cn\log_2 n\\
&= O(n\log n)
\end{aligned}$$

The Iteration Method

- What we just saw is an example of the *iteration method*.
- Keep plugging the recurrence into itself until you spot a pattern, then try to simplify.
- Doesn't always give an exact answer, but useful for building up an intuition.

Getting an Intuition

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$$\boxed{\begin{array}{l} T(1) \le c \\ T(n) \le 2T(n / 2) + cn \end{array}}$$

$$\boxed{\begin{array}{l} T(1) \leq c \\ T(n) \leq 2T(n / 2) + cn \end{array}}$$

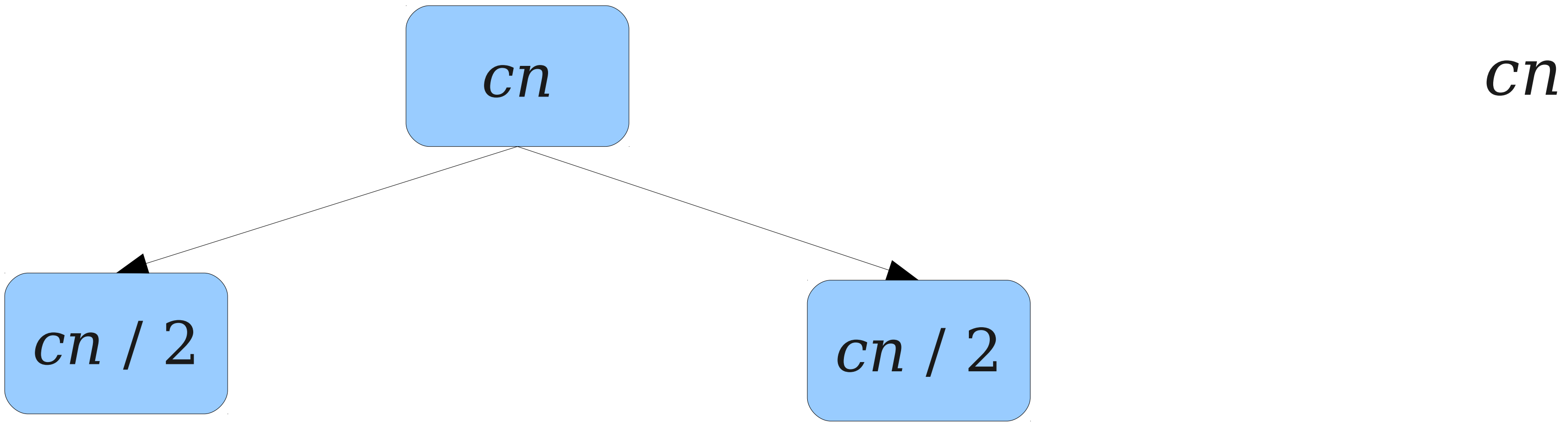
cn

$$T(1) \leq c$$
$$T(n) \leq 2T(n / 2) + cn$$

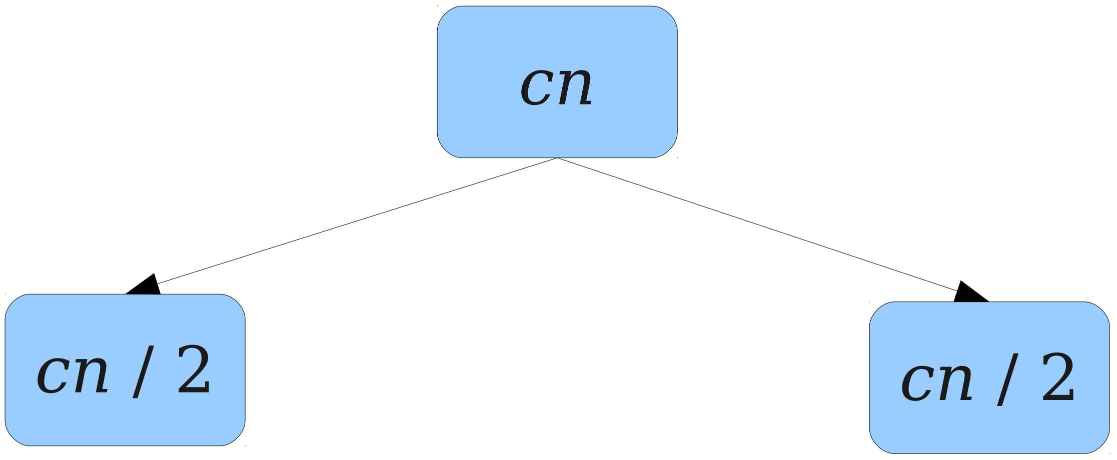
cn

cn

$$T(1) \leq c$$
$$T(n) \leq 2T(n / 2) + cn$$



$$T(1) \le c$$
$$T(n) \le 2T(n / 2) + cn$$



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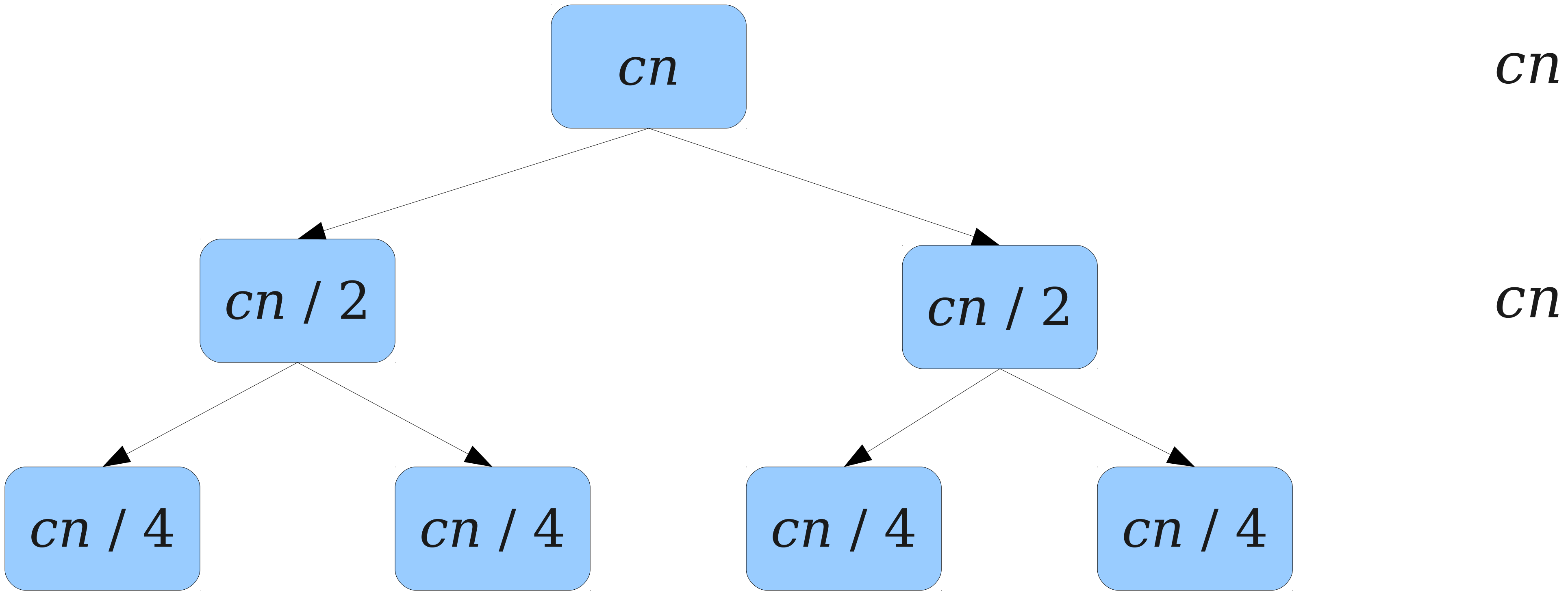

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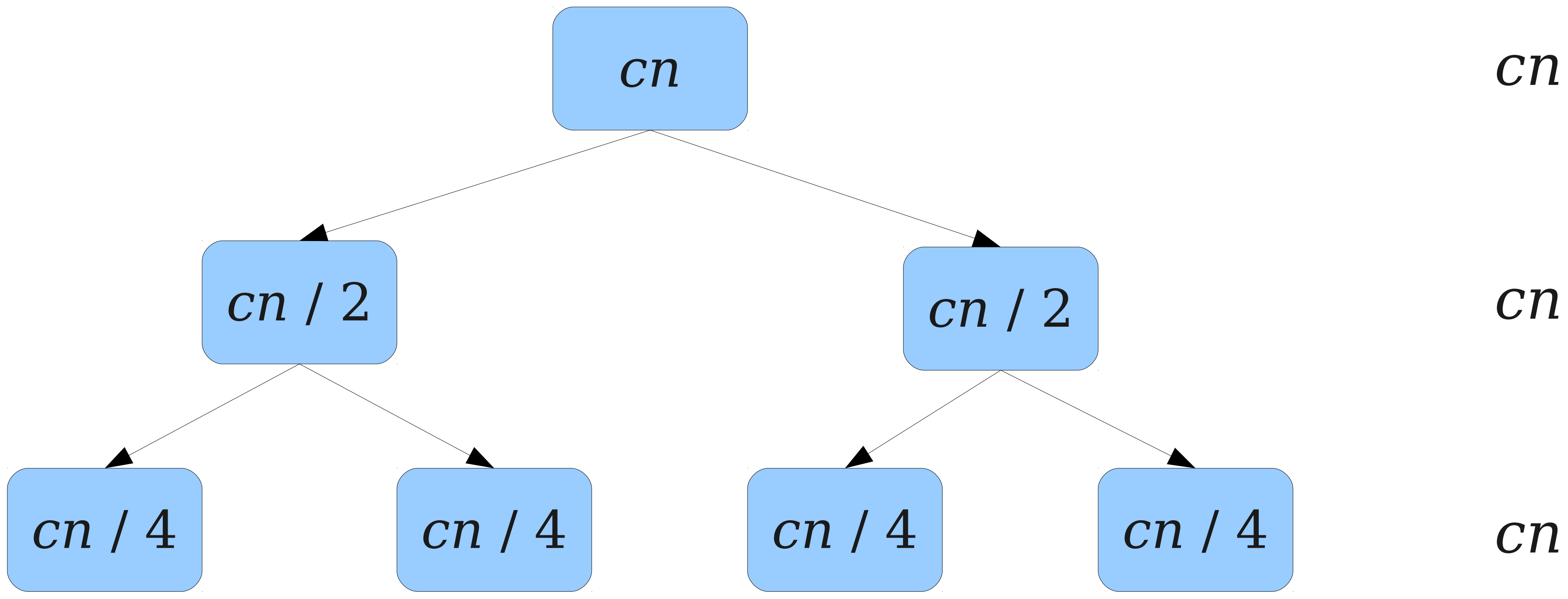
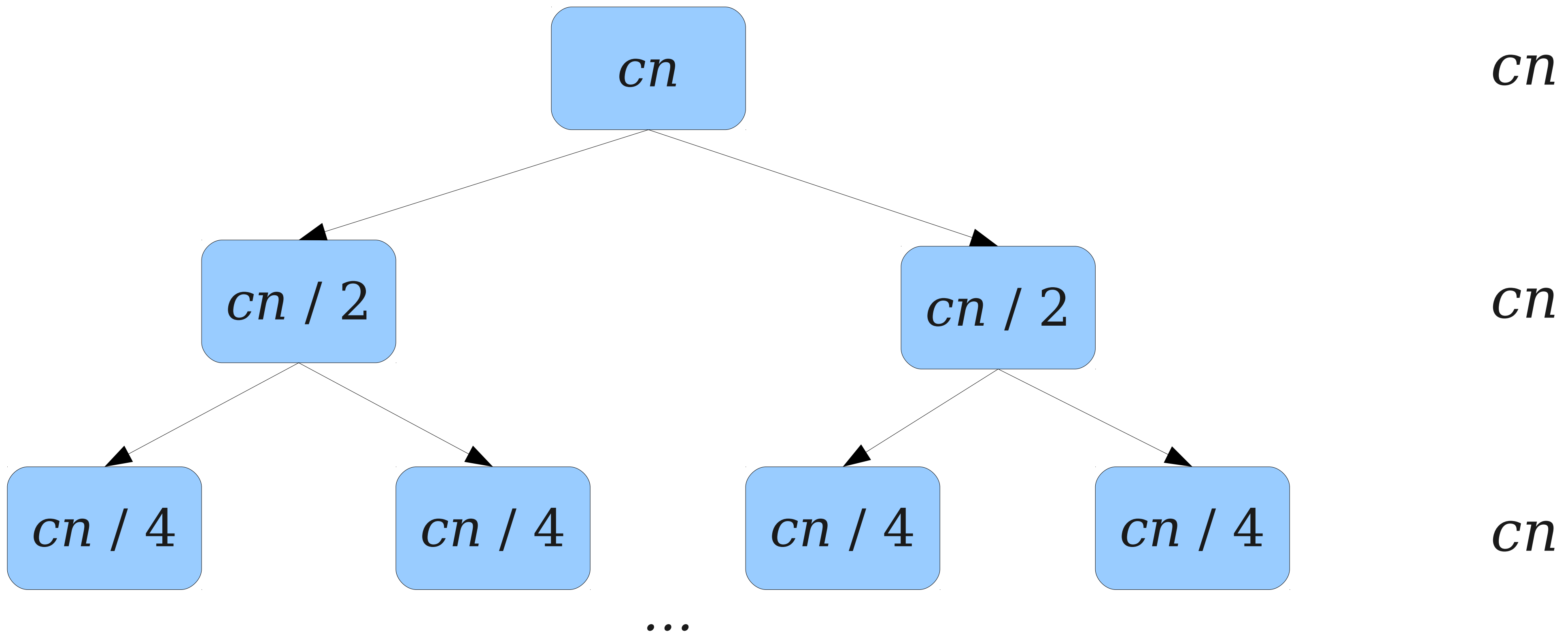


$T(1) \le c$
$T(n) \le 2T(n / 2) + cn$



$T(1) \le c$

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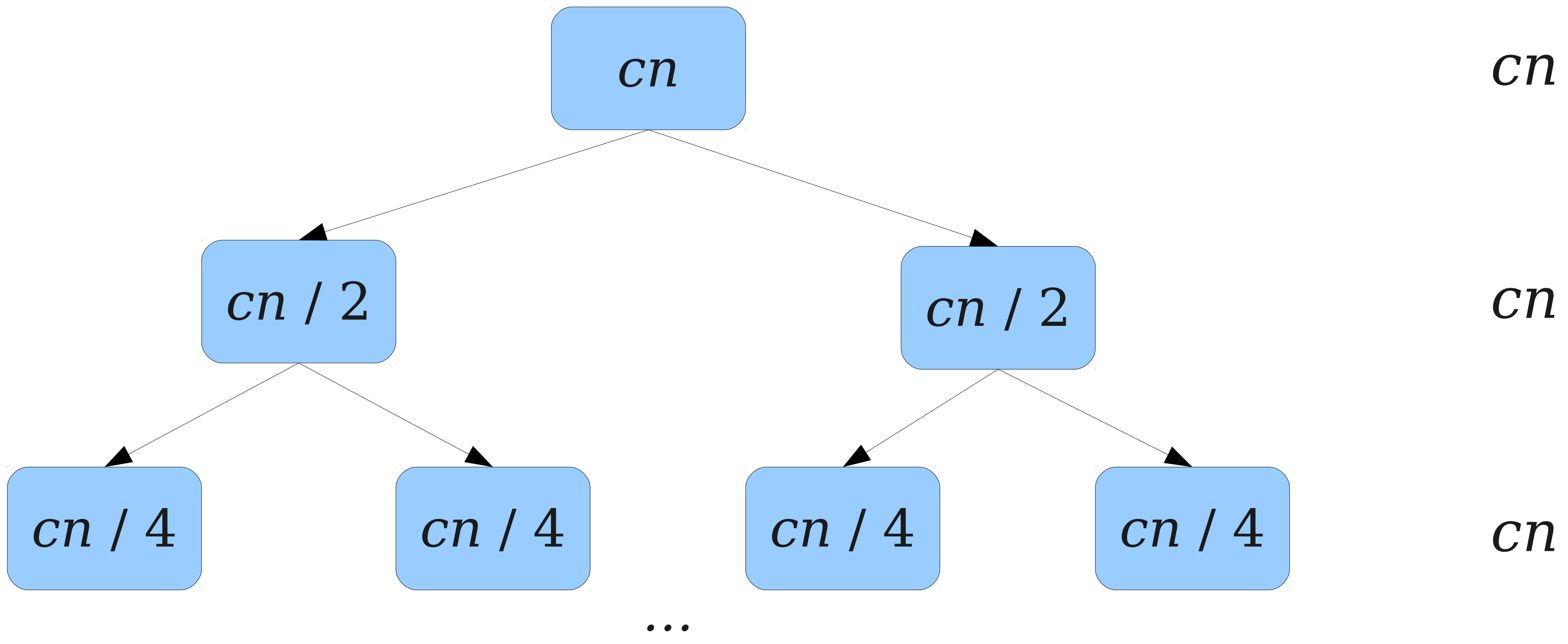
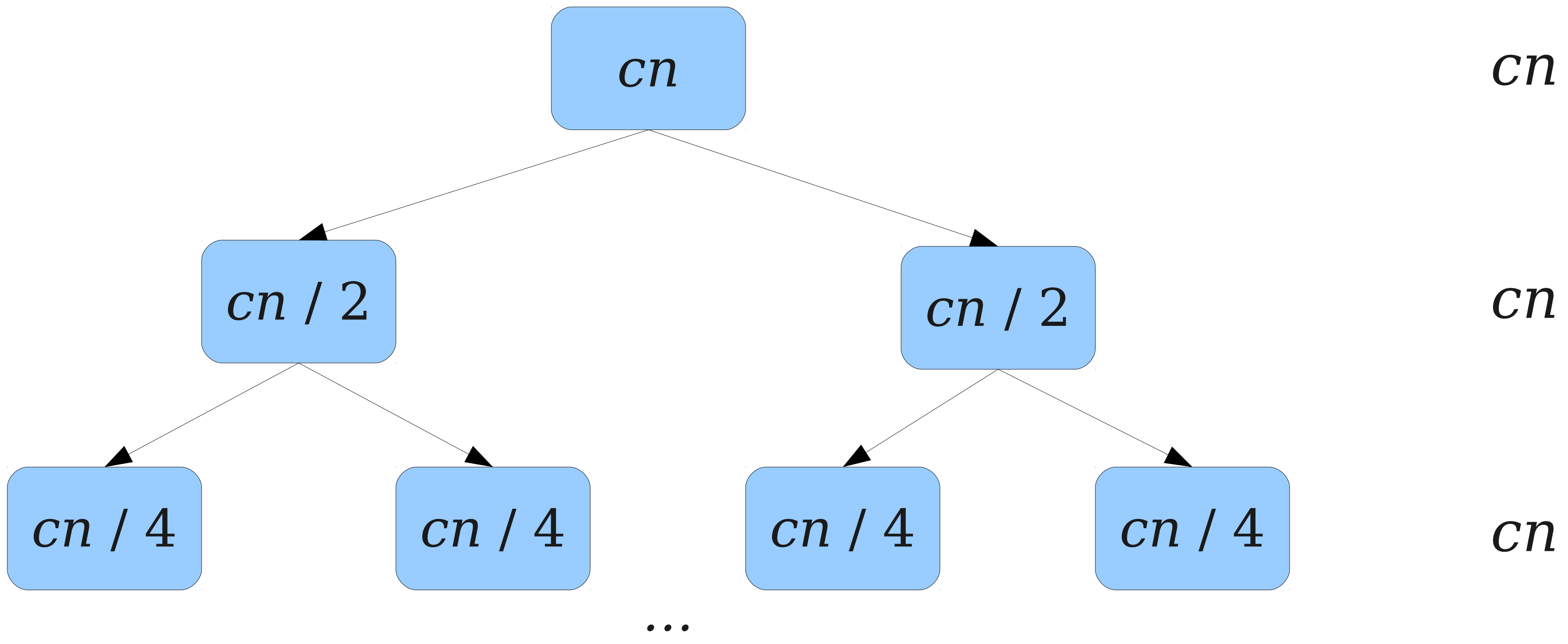


There are **$\log_2 n + 1$** layers in the tree (numbered 0, 1, 2, …, $\log_2 n$).

The first $\log_2 n$ of them are the recursive case. The last one consists purely of base cases.

$$T(1) \le c$$
$$T(n) \le 2T(n / 2) + cn$$



$T(1) \le c$
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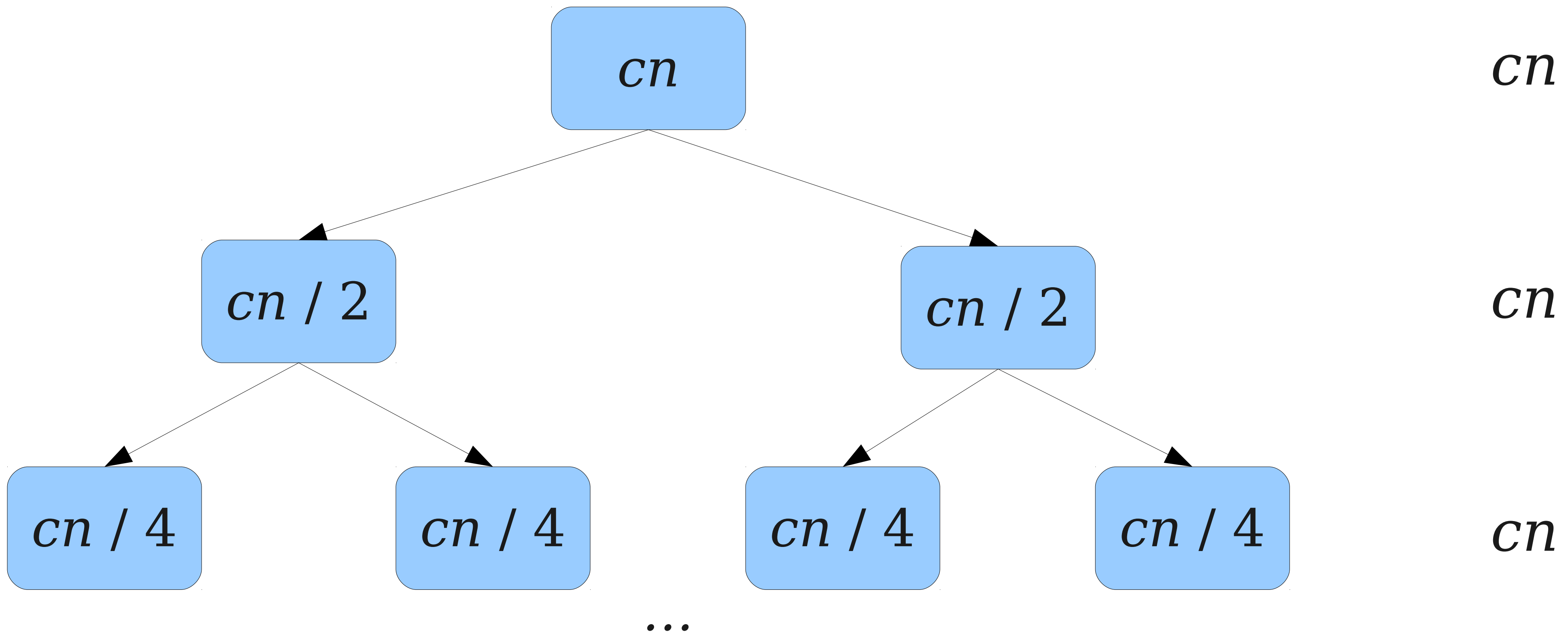
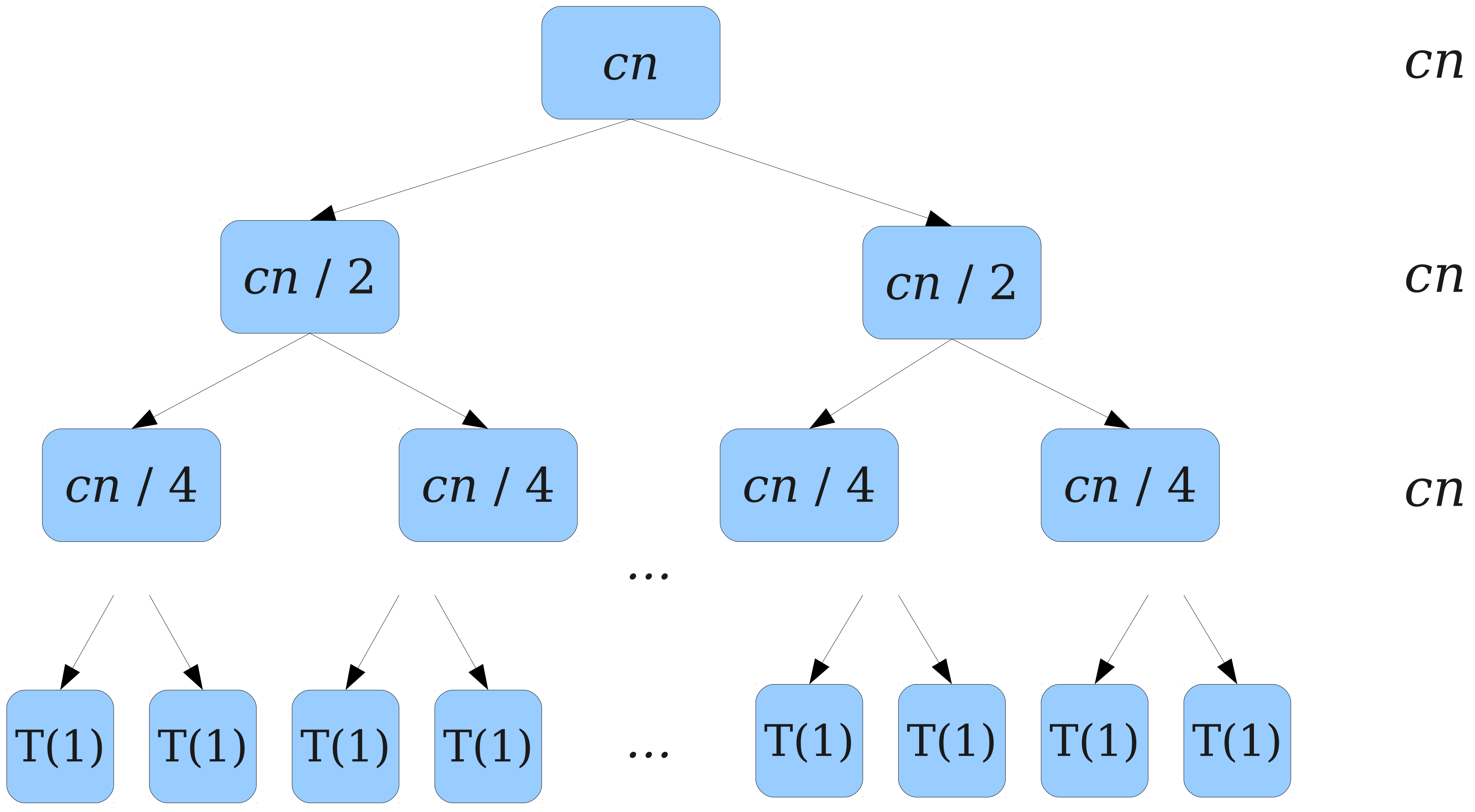


$cn \log_2 n$

$T(1) \le c$
$T(n) \le 2T(n / 2) + cn$



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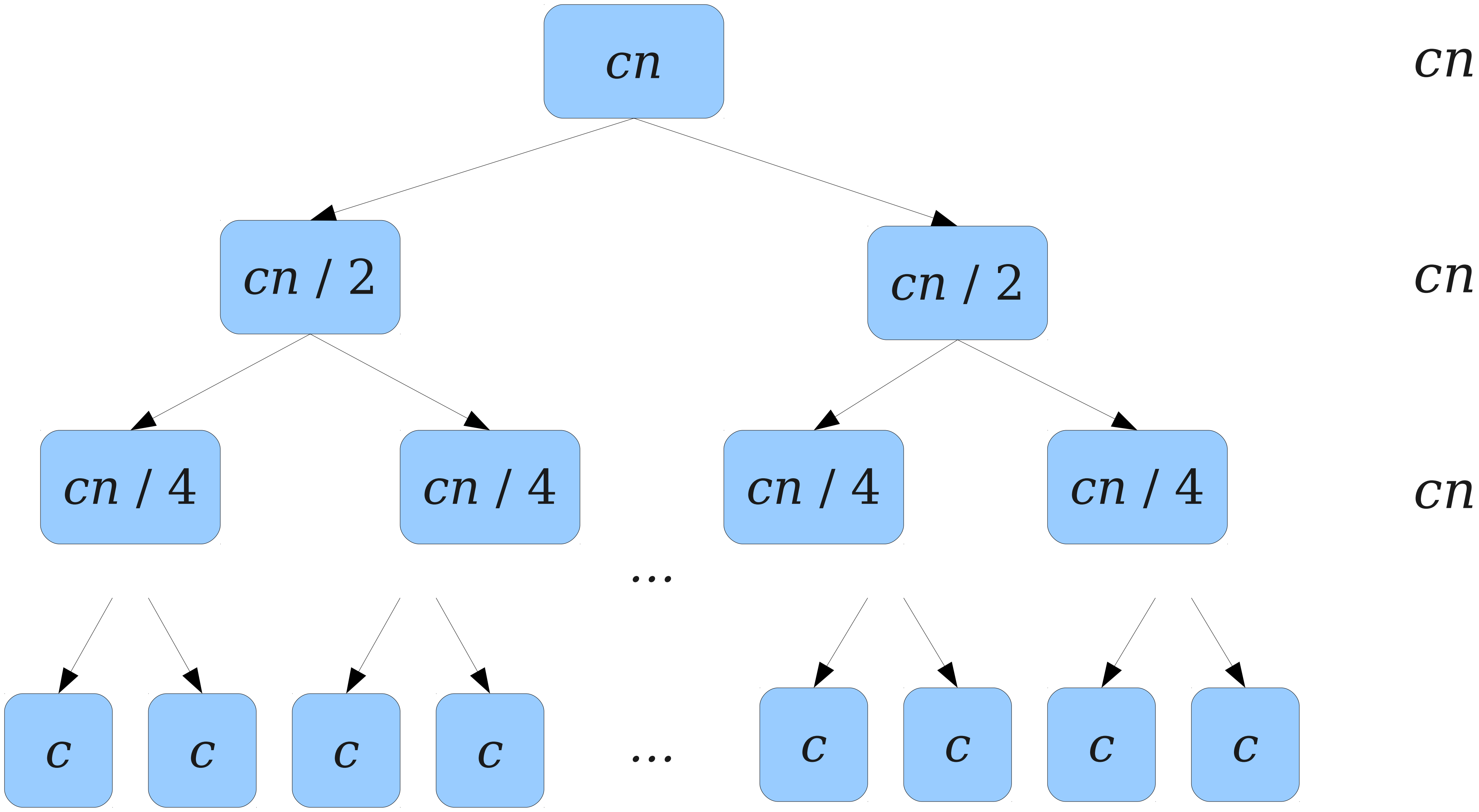
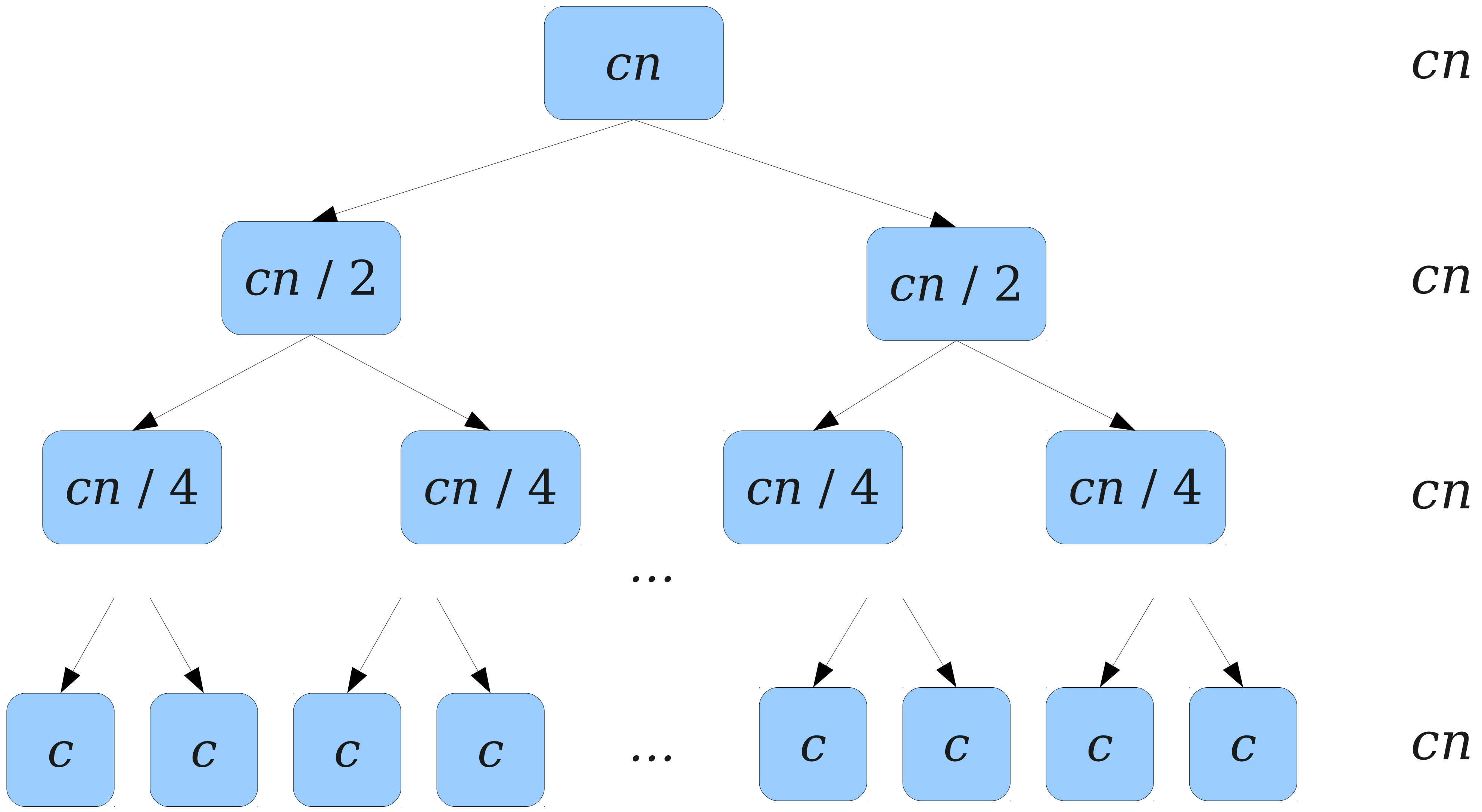


$$cn \log_2 n$$

$$\boxed{\begin{array}{l} T(1) \le c \\ T(n) \le 2T(n/2) + cn \end{array}}$$



$cn \log_2 n$

$T(1) \le c$
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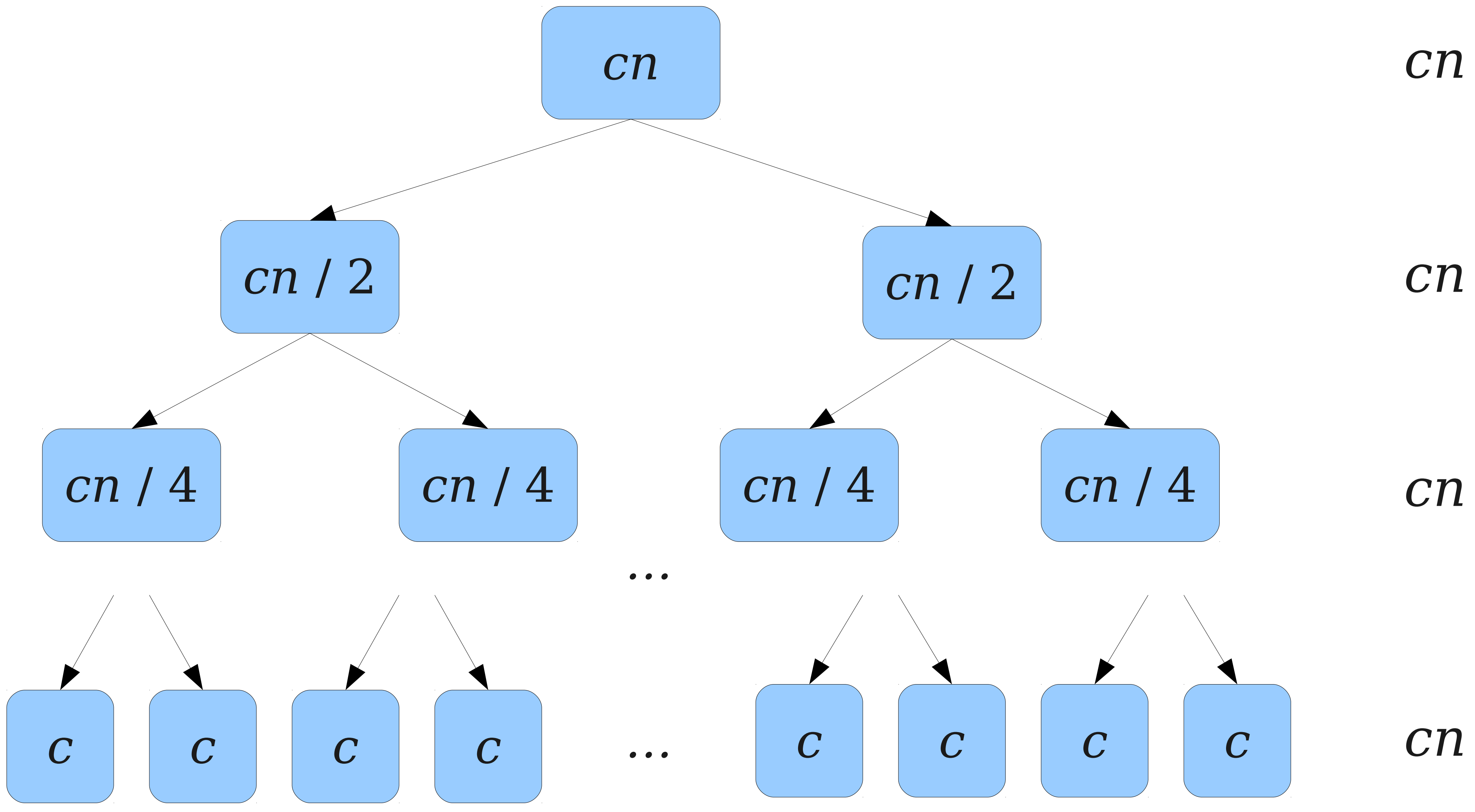


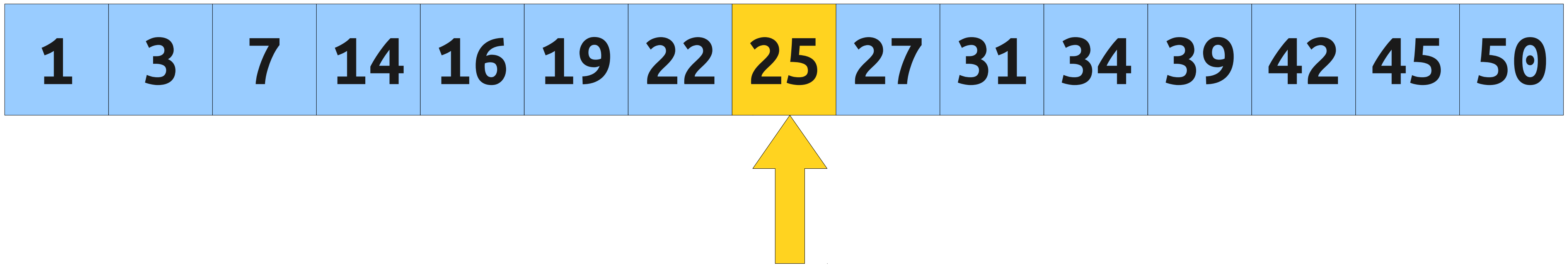
$cn \log_2 n + cn$

The Recursion Tree Method

- This diagram is called a **recursion tree** and accounts for how much total work each recursive call makes.
- Often useful to sum up the work across the layers of the tree.

Another Algorithm: **Binary Search**

1	3	7	14	16	19	22	25	27	31	34	39	42	45	50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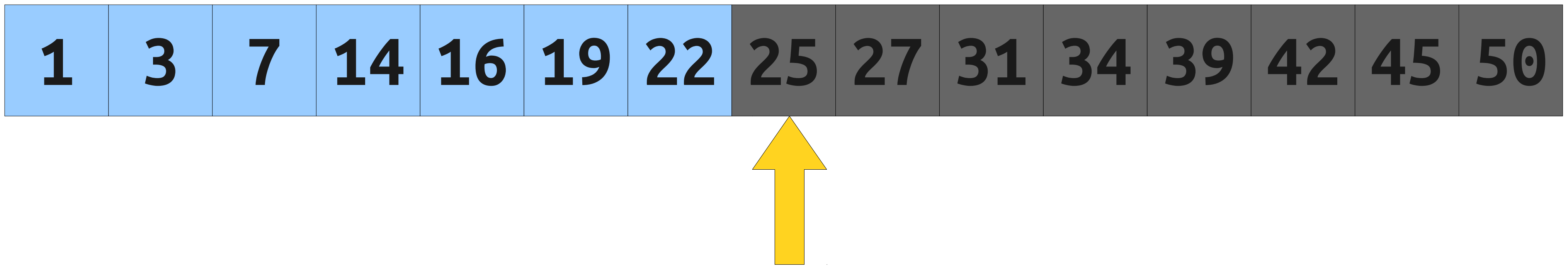
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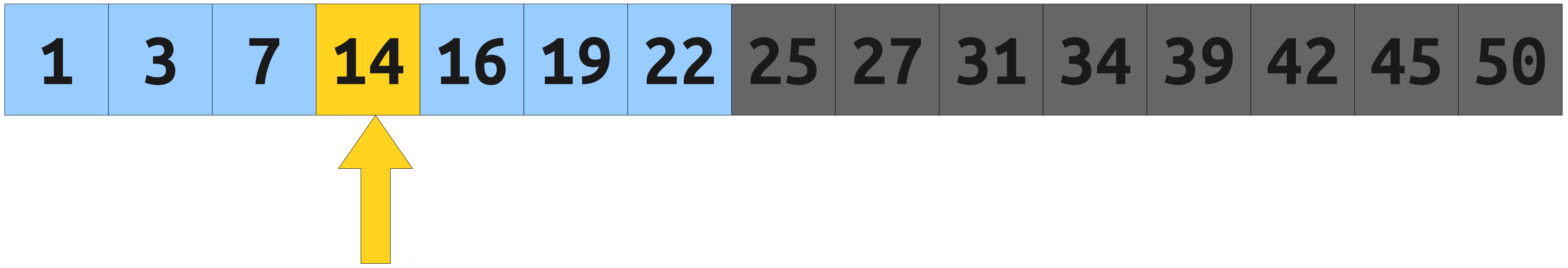
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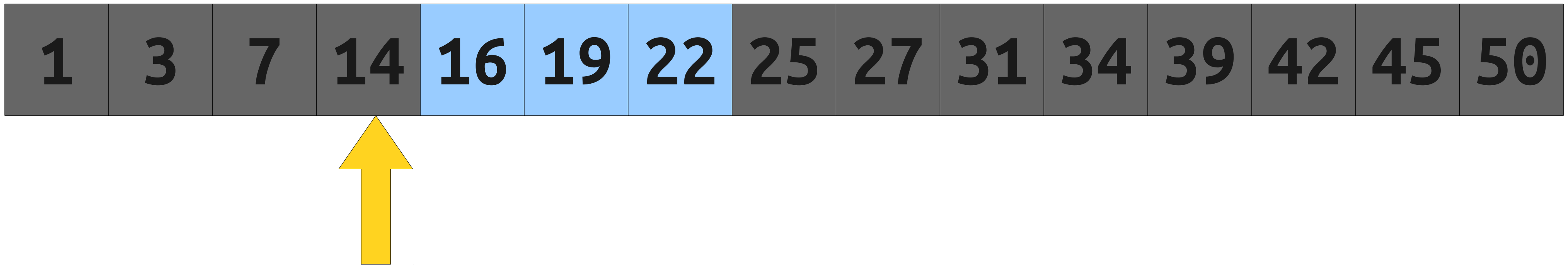
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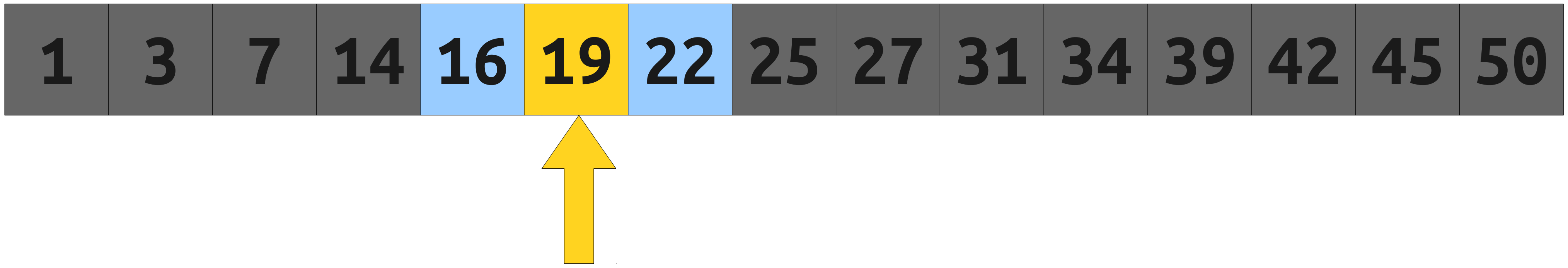
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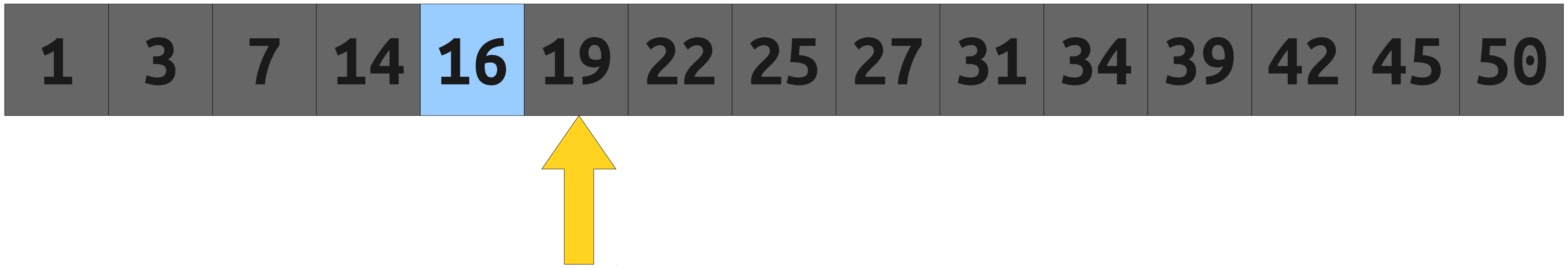
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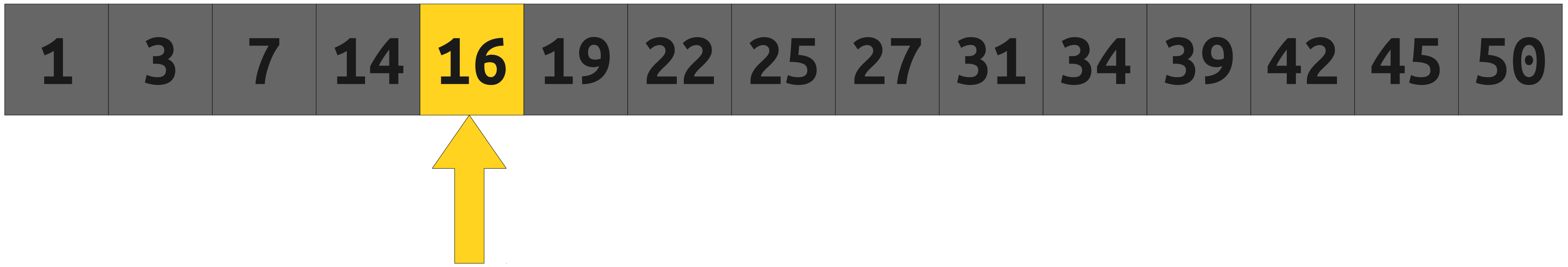
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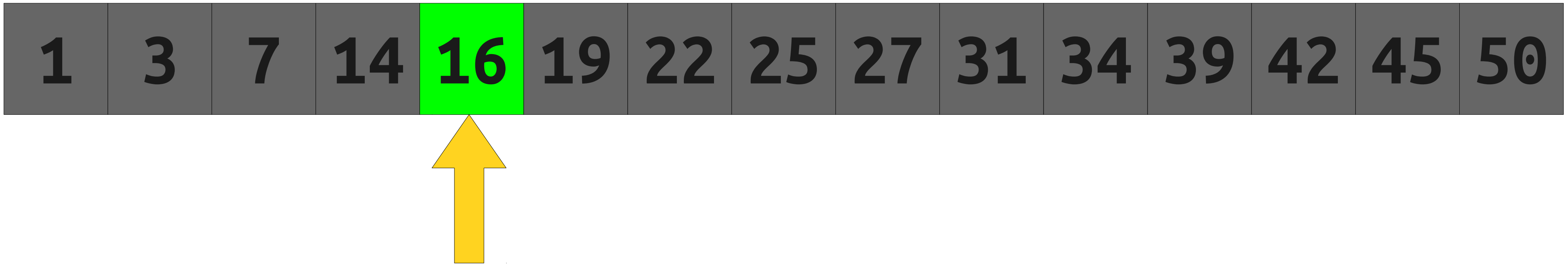
1
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39
42
45
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```
procedure binarySearch(list A, int low, int high,
                                 value key):
  if low ≥ high:
     return false

  let mid = ⌊(high + low) / 2⌋
  if A[mid] = key:
     return true
  else if A[mid] > key:
     return binarySearch(a, low, mid)
  else (A[mid] < key):
     return binarySearch(a, mid + 1, high)
```

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```

$T(0) = \Theta(1)$
$T(1) = \Theta(1)$

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$$T(0) = \Theta(1)$$
$$T(1) = \Theta(1)$$
$$T(n) \leq T(\lfloor n / 2 \rfloor) + \Theta(1)$$

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                       value key):
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```

$$T(1) \leq c$$
$$T(n) \leq T(n / 2) + c$$

The Iteration Method

$$\boxed{\begin{array}{l} T(1) \leq c \\ T(n) \leq T(n / 2) + c \end{array}}$$

The Iteration Method

T(1) ≤ c
T(n) ≤ T(n / 2) + c

$$\mathrm{T}(n) \le T\left(\frac{n}{2}\right) + c$$

The Iteration Method

T(1) ≤ c
T(n) ≤ T(n / 2) + c

$$
\begin{aligned}
\mathrm{T}(n) &\le T\left(\frac{n}{2}\right)+c \\
&\le \left(T\left(\frac{n}{4}\right)+c\right)+c
\end{aligned}
$$

The Iteration Method

$T(1) \le c$
$T(n) \le T(n / 2) + c$

$$\begin{aligned} \mathrm{T}(n) &\le T\left(\frac{n}{2}\right)+c \\ &\le \left(T\left(\frac{n}{4}\right)+c\right)+c \\ &= T\left(\frac{n}{4}\right)+2c \end{aligned}$$

The Iteration Method

T(1) ≤ c
T(n) ≤ T(n / 2) + c

$$\begin{aligned} \mathrm{T}(n) &\le T\left(\frac{n}{2}\right)+c \\ &\le \left(T\left(\frac{n}{4}\right)+c\right)+c \\ &= T\left(\frac{n}{4}\right)+2c \\ &\le \left(T\left(\frac{n}{8}\right)+c\right)+2c \end{aligned}$$

The Iteration Method

T(1) ≤ c
T(n) ≤ T(n / 2) + c

$$\begin{aligned}
\mathrm{T}(n) &\le T\left(\frac{n}{2}\right)+c \\
&\le \left(T\left(\frac{n}{4}\right)+c\right)+c \\
&= T\left(\frac{n}{4}\right)+2c \\
&\le \left(T\left(\frac{n}{8}\right)+c\right)+2c \\
&= T\left(\frac{n}{8}\right)+3c
\end{aligned}$$

The Iteration Method

$$\boxed{\begin{array}{l} T(1) \le c \\ T(n) \le T(n/2) + c \end{array}}$$

$$\begin{aligned}
T(n) &\le T\left(\frac{n}{2}\right) + c \\
&\le \left(T\left(\frac{n}{4}\right) + c\right) + c \\
&= T\left(\frac{n}{4}\right) + 2c \\
&\le \left(T\left(\frac{n}{8}\right) + c\right) + 2c \\
&= T\left(\frac{n}{8}\right) + 3c \\
&\dots \\
&\le T\left(\frac{n}{2^k}\right) + kc
\end{aligned}$$

The Iteration Method

$$\boxed{\begin{array}{l} T(1) \le c \\ T(n) \le T(n / 2) + c \end{array}}$$

$$T(n) \le T\left(\frac{n}{2^k}\right) + k\,c$$

The Iteration Method

$\mathrm{T}(1) \le c$
$\mathrm{T}(n) \le \mathrm{T}(n / 2) + c$

$$\begin{aligned} \mathrm{T}(n) &\le T\left(\frac{n}{2^k}\right) + k\,c \\ &= T(1) + c\log_2 n \end{aligned}$$

The Iteration Method

T(1) ≤ c
T(n) ≤ T(n / 2) + c

$$
\begin{aligned}
\mathrm{T}(n) &\le T\left(\frac{n}{2^k}\right) + k\,c \\
&= T(1) + c\log_2 n \\
&\le c + c\log_2 n
\end{aligned}
$$

The Iteration Method

$$\boxed{\begin{array}{l}T(1) \le c \\ T(n) \le T(n/2) + c\end{array}}$$

$$\begin{aligned} T(n) &\le T\left(\frac{n}{2^k}\right) + kc \\ &= T(1) + c\log_2 n \\ &\le c + c\log_2 n \\ &= O(\log n) \end{aligned}$$

The Recursion Tree Method

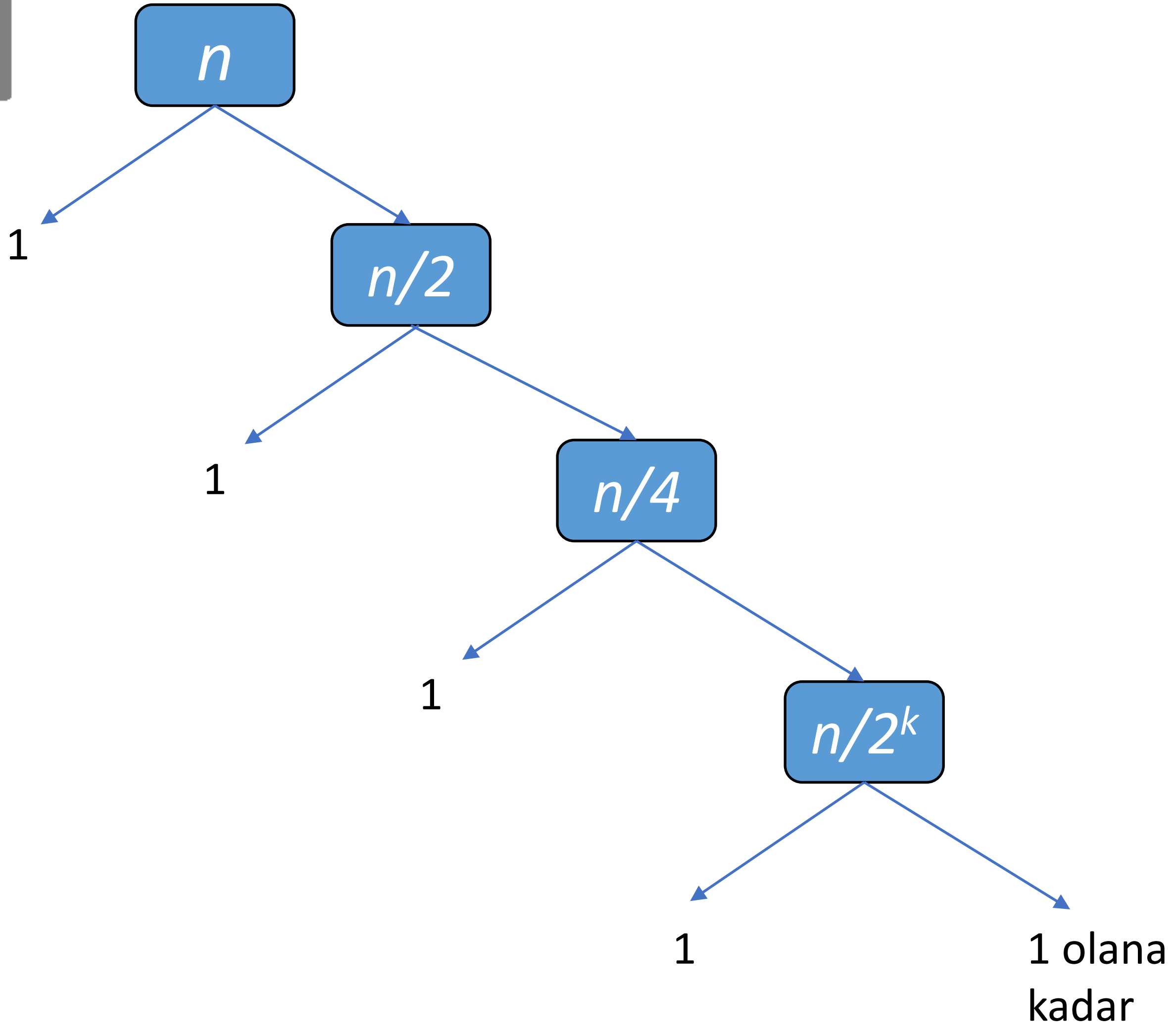
$$\boxed{\begin{array}{l} T(1) \le c \\ T(n) \le T(n / 2) + c \end{array}}$$

$$T(n) = ck$$

$$T\left(\frac{n}{2^k}\right) = 1$$

$$k = log_2 n$$

$$T(n) = clog_2 n$$


n
1
n/2
1
n/4
1
n/2^k
1
1 olana kadar