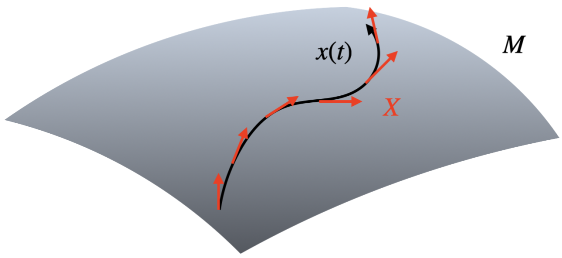
6 Lie Derivatives

§ 6.1 Flows

Definition 6.1 (Integral curves)**.** Let X be a vector field in $\mathcal{M}$. An **integral curve** $x(t)$ of X is a curve in $\mathcal{M}$, whose tangent vector at $x(t)$ is $X|_x$.



Given a chart (U, φ), this means

$$\boxed{\frac{\mathrm{d}x^\mu(t)}{\mathrm{d}t} = X^\mu(x(t))}$$

Here, $x^\mu(t)$ denotes the μ-th component of $\varphi \circ x(t)$[1] and X^μ denotes the μ-th component of $X|_x$.

Remark. Finding an integral curve is equivalent to solving the system of ODEs with the initial condition $x_0^\mu = x^\mu(0)$. Hence, unique solution is guaranteed.

Definition 6.2 (Flows)**.** Let $\sigma(t, x_0)$ be an integral curve of X, which passes a point x_0 at $t = 0$. Then σ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma^\mu(t, x_0) = X^\mu(\sigma(t, x_0)) \text{ and } \sigma^\mu(0, x_0) = x_0^\mu$$

The map $\sigma : \mathbb{R} \times \mathcal{M} \to \mathcal{M}$ is called a **flow** generated by $X \in \mathscr{X}(\mathcal{M})$.

Remark. Flows satisfy $\sigma(t, \sigma(s, x)) = \sigma(t + s, x)$ for all t and s.

Definition 6.3. For fixed $t \in \mathbb{R}$, a flow $\sigma(t, x)$ is a *diffeomorphism* from $\mathcal{M}$ to $\mathcal{M}$, $\sigma_t : \mathcal{M} \to \mathcal{M}$. σ_t is made into a *commutative group* by the following rules.

(i) $\sigma_t \circ \sigma_s = \sigma_{t+s} = \sigma_s \circ \sigma_t$

(ii) σ_0 is the identity map.

(iii) $\sigma_{-t} = (\sigma_t)^{-1}$.

This group is the **one-parameter group of transformations**.

Remark. One-parameter group of transformations is *locally* isomorphic to $(\mathbb{R}, +)$, but not globally.

[1] abuse of notation.

Observation 6.4. With an infinitesimal ϵ,

$$\sigma_\epsilon^\mu(x) = \sigma^\mu(\epsilon, x) = x^\mu + \epsilon X^\mu(x)$$

In this context, the vector field X is called the **infinitesimal generator** of σ_t. The flow σ is often referred to as the **exponentiation** of X.

$$\begin{aligned}\sigma^\mu(t,x) &= x^\mu + t\frac{\mathrm{d}}{\mathrm{d}s}\,\sigma^\mu(s,x)|_{s=0} + \frac{t^2}{2!}\frac{\mathrm{d}^2}{\mathrm{d}s^2}\,|\sigma^\mu(s,x)|_{s=0} + \cdots \\ &= \exp\left(t\frac{\mathrm{d}}{\mathrm{d}s}\right)\sigma^\mu(s,x)|_{s=0} = e^{tX}x_0^\mu\end{aligned}$$

The flow satisfies the following *exponential properties*.

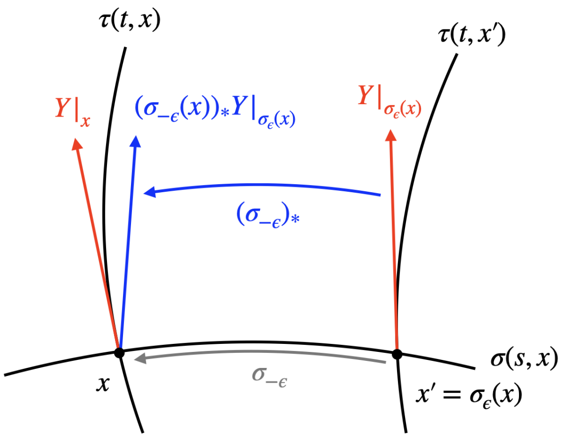
(i) $\sigma(0,x) = x = \exp(0x)x$

(ii) $\dfrac{\mathrm{d}\sigma(t,x)}{\mathrm{d}t} = X\exp(tX)x$

(iii) $\sigma(t,\sigma(s,x)) = \sigma(t,\exp(sX)x) = e^{tX}e^{sX}x = e^{(t+s)X}x = \sigma(t+s,x)$

§ 6.2 Lie Derivatives

Observation 6.5. Let $\sigma(t,x)$ and $\tau(t,x)$ be two flows generated by the vector fields X and Y.



$$\frac{\mathrm{d}\sigma^\mu(s,x)}{\mathrm{d}s} = X^\mu(\sigma(s,x)) \text{ and } \frac{\mathrm{d}\tau^\mu(t,x)}{\mathrm{d}t} = Y^\mu(\sigma(t,x))$$

Then what is the change of the vector field Y along $\sigma(s,x)$?

$\boxed{\textit{Problem.}}$ $Y|_x$ (lives in $T_x\mathcal{M}$) and $Y|_{\sigma_\epsilon(x)}$ (lives in $T_{\sigma_\epsilon(x)}\mathcal{M}$) live in different spaces.

$\boxed{\textit{Answer.}}$ To define a sensible derivative, we first map $Y|_{\sigma_\epsilon(x)}$ to $T_x\mathcal{M}$ by **pushforward map** of $\sigma_{-\epsilon}$,

$$(\sigma_{-\epsilon})_* : T_{\sigma_\epsilon(x)}\mathcal{M} \to T_x\mathcal{M}$$

after which we take a difference between two vectors.

Definition 6.6 (Lie derivatives)**.** The **Lie derivative** of a vector field Y along the flow σ of X is defined by

$$\boxed{\mathcal{L}_X Y \equiv \lim_{\epsilon \to 0} \frac{1}{\epsilon}[(\sigma_{-\epsilon})_* Y|_{\sigma_\epsilon(x)} - Y|_x]}$$

Observation 6.7. Let (U, φ) be a chart with the coordinates x^μ and

$$X = X^\mu \frac{\partial}{\partial x^\mu},\ Y = Y^\mu \frac{\partial}{\partial x^\mu}$$

be vector fields defined on U. Here, we use *coordinate basis*

$$\frac{\partial}{\partial x^\mu} := e_\mu|_x$$

where RHS denotes the basis vector at x. Then from **Observation 6.4**,

$$Y|_{\sigma_\epsilon(x)} = Y^\mu(x^\nu + \epsilon X^\nu(x)) \cdot e_\mu|_{x+\epsilon X} \simeq [Y^\mu(x) + \epsilon X^\nu(x)\partial_\nu Y^\mu(x)]e_\mu|_{x+\epsilon X}$$

Now map this vector defined at $\sigma_\epsilon(x)$ to x by $(\sigma_{-\epsilon})_* : T_{\sigma_\epsilon(x)}\mathcal{M} \to T_x\mathcal{M}$.

$$\begin{aligned}
(\sigma_{-\epsilon})_* Y|_{\sigma_\epsilon(x)} &= [Y^\mu(x) + \epsilon X^\lambda(x)\partial_\lambda Y^\mu(x)] \frac{\partial x^\nu}{\partial(\sigma_\epsilon(x))^\mu} e_\nu|_x \\
&= [Y^\mu(x) + \epsilon X^\lambda(x)\partial_\lambda Y^\mu(x)][\delta_\mu{}^\nu - \epsilon\partial_\mu X^\nu]e_\nu|_x \\
&= \underbrace{Y^\mu(x)e_\mu|_x}_{Y|_x} + \epsilon[X^\mu(x)\partial_\mu Y^\nu(x) - Y^\nu(x)\partial_\mu X^\nu(x)]e_\nu|_x + \mathcal{O}(\epsilon^2)
\end{aligned}$$

Since

$$\frac{\partial x^\nu}{\partial(\sigma_\epsilon(x))^\mu} = \frac{\partial(\sigma_{-\epsilon}(x))^\nu}{\partial x^\mu} = \partial_\mu[x^\nu - \epsilon X^\nu] = \delta_\mu{}^\nu - \epsilon\partial_\mu X^\nu$$

In conclusion,

$$\boxed{\mathcal{L}_X Y = [X^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu]e_\nu}$$

This is how we differentiate the vector field on the manifolds.

Definition 6.8 (Lie brackets)**.** Let $X = X^\mu\partial_\mu$ and $Y = Y^\mu\partial_\mu$ be vector fields in $\mathcal{M}$. The **Lie bracket** is defined by

$$[X, Y]f = X[Y[f]] - Y[X[f]] \quad (f \in \mathcal{F}(\mathcal{M}))$$

Then

$$\begin{aligned}
[X, Y]f &= X^\mu\partial_\mu(Y^\nu\partial_\nu f) - Y^\mu\partial_\mu(X^\nu\partial_\nu f) \\
&= X^\mu(\partial_\mu Y^\nu)(\partial_\nu f) + \cancel{X^\mu Y^\nu \partial_\mu\partial_\nu f} - Y^\mu(\partial_\mu X^\nu)(\partial_\nu f) - \cancel{X^\nu Y^\mu \partial_\mu\partial_\nu f} \\
&= (X^\mu\partial_\mu Y^\nu - Y^\mu\partial_\mu X^\nu)\partial_\nu \cdot f = \mathcal{L}_X Y \cdot f
\end{aligned}$$

Hence, Lie derivative is equivalent to Lie bracket.

$$\boxed{\mathcal{L}_X Y = [X, Y]}$$