4 Manifolds: Definitions

§ 4.3 Manifolds

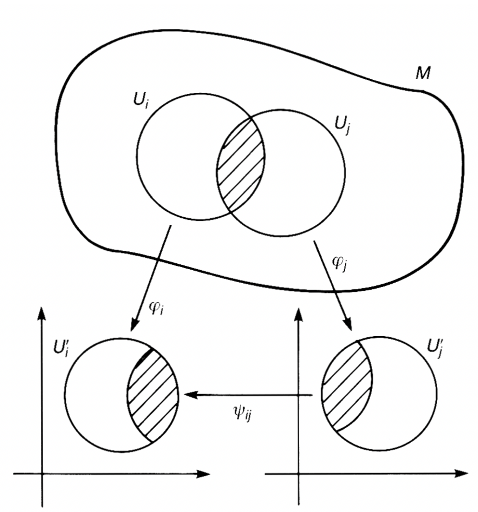
Definition 4.17. M is a m-dimensional **differential manifold** if

(i) M is a *topological space*.

(ii) M is provided with a family of pairs $\{(U_i, \varphi_i)\}$ such that

(iii) $\{U_i\}$ is a family of open sets which covers M ($\cup_i U_i = M$) and φ_i is a *homeomorphism* from U_i onto an open subset $U_i' \subseteq \mathbb{R}^m$.

(iv) For $U_i \cap U_j \neq \varnothing$, the **transition function** $\psi_{ij} = \varphi_i \circ \varphi_j^{-1}$ from $\varphi_j(U_i \cap U_j)$ to $\varphi_i(U_i \cap U_j)$ is C^∞.



Remark.

(1) We call U_i and φ_i **coordinate neighborhood** and **coordinate function**, respectively. The tuple (U_i, φ_i) is called a **chart**. Collection of charts, $\{(U_i, \varphi_i)\}$ is called an **atlas**. The coordinate function is

$$\varphi_i(p) = (x^1(p), x^2(p), \cdots, x^m(p))$$

(2) From (ii) and (iii), M is **LOCALLY** Euclidean(in each coordinate neighborhood U_i, M looks like an open set of $\mathbb{R}^m$ - of course $M \neq \mathbb{R}^m$ globally).

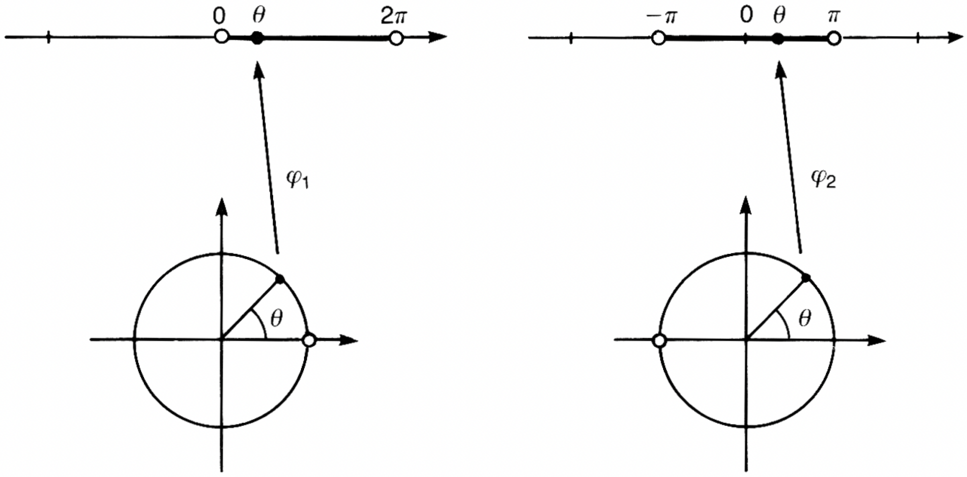
(3) From (iv), transition from one coordinate system to another should be smooth(or C^∞). Consider a point $p \in U_i \cap U_j$. Then corresponding coordinate functions φ_i and φ_j assign x^μ and y^μ respectively. The transition $x^\mu = x^\mu(y^\alpha)$ should be C^∞.

Remark. If the union of two atlases $\{(U_i, \varphi_i)\}$ and $\{(V_j, \psi_j)\}$ is again an atlas, these two atlases are said to be **compatible**. Such compatibility gives rise to *equivalence relations*. Equivalence classes are called **differentiable structures** on M.

Example 4.18. Examples of manifolds.

(a) $\mathbb{R}^n$ is a trivial example: single chart covers whole $\mathbb{R}^n$, with φ as an identity map.

(b) S^1, a circle $x^2 + y^2 = 1$ in the xy-plane is another example.



For two open sets $U_1 = (0, 2\pi)$ and $U_2 = (-\pi, \pi)$,

$$\varphi_1^{-1} : U_1 \to S^1 - \{(1,0)\} \quad \text{s.t.} \quad \theta \mapsto (\cos\theta, \sin\theta)$$
$$\varphi_2^{-1} : U_2 \to S^1 - \{(-1,0)\} \quad \text{s.t.} \quad \theta \mapsto (\cos\theta, \sin\theta)$$

One can show that:

- φ_1^{-1} and φ_2^{-1} are invertible.
- φ_1, φ_1^{-1}, φ_2, φ_2^{-1} are continuous[1].
- $\psi_{12} = \varphi_1 \circ \varphi_2^{-1}$ and $\psi_{21} = \varphi_2 \circ \varphi_1^{-1}$ are smooth.

(c) S^n realized in $\mathbb{R}^{n+1}$: $\sum_{i=0}^{n} (x^i)^2 = 1$. Define coordinate neighborhoods

$$U_{i+} = \{(x^0, x^1, \cdots, x^n) \mid x^i > 0\}$$
$$U_{i-} = \{(x^0, x^1, \cdots, x^n) \mid x^i < 0\}$$

and coordinate maps by

$$\varphi_{i+} : U_{i+} \to \mathbb{R}^n, \quad \varphi_{i+}(x^0, \cdots, x^n) = (x^0, \cdots, x^{i-1}, x^{i+1}, \cdots, x^n)$$
$$\varphi_{i-} : U_{i-} \to \mathbb{R}^n, \quad \varphi_{i-}(x^0, \cdots, x^n) = (x^0, \cdots, x^{i-1}, x^{i+1}, \cdots, x^n)$$

These are projections of the hemispheres $U_{i\pm}$ to the plane $x^i = 0$. For example, we can consider $U_{x\pm}$, $U_{y\pm}$ and $U_{z\pm}$ for S^2. Then, one transition function of this atlas becomes

$$\psi_{y-x+} := \varphi_{y-} \circ \varphi_{x+}^{-1}$$
$$\psi_{y-x+}(y, z) = (\sqrt{1 - y^2 - z^2}, z)$$

which is $\mathcal{C}^\infty$ on $U_{x+} \cap U_{y-}$.

[1] From these two conditions, φ_1 and φ_2 become *homeomorphisms*

(d) **Stereographic projections**. Construct the atlas by

- $S^2 - \{\text{South Pole}\}$ with stereographic projection from South Pole
- $S^2 - \{\text{North Pole}\}$ with stereographic projection from North Pole

Then, transition function is smooth (show this as an exercise).

Example 4.19. The **real projective space** $\mathbb{R}P^n$ is the set of lines through the origin in $\mathbb{R}^{n+1}$.

- If $x = (x^0, \cdots, x^n) \neq 0$, x defines a line through the origin.
- $y \in \mathbb{R}^{n+1}$ defines the same line as x if $\exists\, a \neq 0$ s.t. $y = ax$.
- Hence, we can define an *equivalence relation* by

$$x \sim y \iff \exists\, a \in \mathbb{R} - \{0\} \text{ s.t. } y = ax$$

- From the equivalence relation, a quotient space with equivalence classes can be defined.

$$\mathbb{R}P^n := (\mathbb{R}^{n+1} - \{0\}) / \sim$$

- Since $\mathbb{R}P^n \subset \mathbb{R}^{n+1}$, it has $(n+1)$-coordinates $x^0, \cdots, x^n$, which is not good(since $\mathbb{R}P^n$ is an n-dimensional manifold). These are **homogeneous** coordinates.

Let U_i be set of lines with $x^i \neq 0$. Then, **inhomogeneous** coordinates on U_i is defined by

$$\xi^j_{(i)} = \frac{x^j}{x^i},\ \xi_{(i)} = (\xi^0_{(i)}, \xi^1_{(i)}, \cdots, \xi^{i-1}_{(i)}, \xi^{(i+1)}_{(i)}, \cdots, \xi^n_{(i)})$$

This coordinate system is independent of the choice of the representative.

$$\frac{x^j}{x^i} = \frac{y^j}{y^i} \text{ where } y = ax$$

The coordinate map corresponding to the chart becomes

$$\varphi_i : (x^0, \cdots, x^n) \mapsto (\frac{x^0}{x^i}, \cdots, \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \cdots, \frac{x^n}{x^i})$$

For $x \in U_i \cap U_j$ (with inhomogeneous coordinates $\xi^k_{(i)} = \frac{x^k}{x^i}$ and $\xi^k_{(j)} = \frac{x^k}{x^j}$), transition function

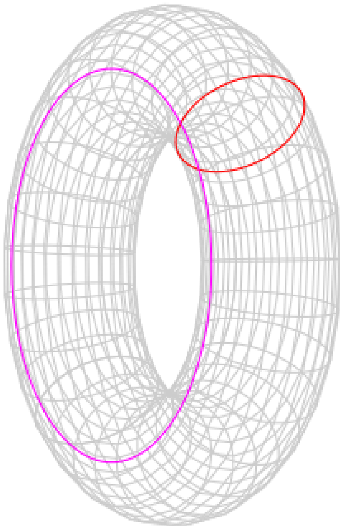
$$\psi_{ij} : \xi^k_{(j)} \to\mapsto \xi^k_{(i)} = \frac{x^j}{x^i}\xi^k_{(j)}$$

is smooth.

Remark. The real projective space can be generalized into k-dimensional planes in $\mathbb{R}^n$. Such planes also form **Grassmann manifold** $G_{k,n}(\mathbb{R})$.

Definition 4.20 (Product manifold). Let M and N be m- and n-dimensional manifold with atlases $\{(U_i, \varphi_i)\}$ and $\{(V_j, \psi_j)\}$, respectively. The **product manifold** $M \times N$ is a $(m+n)$-dimensional manifold whose atlas is $\{(U_i \times V_j), (\varphi_i, \psi_j)\}$.

Example 4.21. Consider the **torus** $T^2 = S^1 \times S^1$.



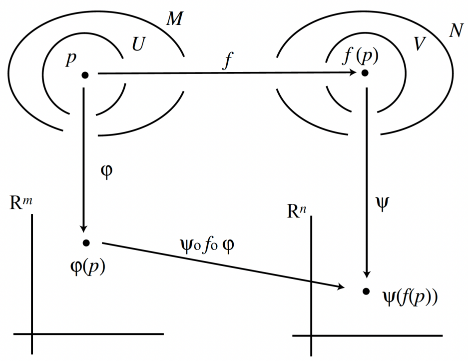
- The torus can be described by two coordinates (θ_1, θ_2).
- Since S^1 is embedded in $\mathbb{R}^2$, we can imagine T^2 embedded in $\mathbb{R}^4$. This is the torus as a *flat* manifold.
- However, we often imagine T^2 as the surface of a doughnut in $\mathbb{R}^3$, in which case, however, we inevitably have to introduce bending of the surface.
- This is the **extrinsic** feature brought about by the embedding.
- In the **intrinsic** point of view, we forget about the embedding. More on this later.

In summary,

Manifolds are LOCALLY homeomorphic to Euclidean spaces $\mathbb{R}^n$,
enabling differentiation and integration on them.

§ 4.4 Differentiable Maps

Definition 4.22 (Differentiable maps)**.** Let $f : M \to N$ be a map from an m-dimensional manifold M to an n-dimensional manifold N.



Take a chart (U, φ) on M and (V, ψ) on N. Then f has the coordinate representation

$$\psi \circ f \circ \varphi^{-1} : \mathbb{R}^m \to \mathbb{R}^n$$

If we write $\varphi(p) = (x^\mu)$ and $\psi(f(p)) = (y^\alpha)$, $\psi \circ f \circ \varphi^{-1}(x) = y$. We often *abuse* the notation as

$$y = f(x), \; y^\alpha = f^\alpha(x^\mu)$$

If $\psi \circ f \circ \varphi^{-1}$ is smooth with respect to each x^μ, f is said to be **differentiable**(or **smooth**) at p.

Remark. The differentiability of f is independent of coordinate system. Consider a overlapping charts (U_1, φ_1) and (U_2, φ_2) with $p \in U_1 \cap U_2$ and

$$\varphi_1(p) = (x_1^\mu), \; \varphi_2(p) = (x_2^\nu)$$

where f is differentiable with first chart. Then

$$\psi \circ f \circ \varphi_2^{-1} = \underbrace{(\psi \circ f \circ \varphi_1^{-1})}_{C^\infty} \circ \underbrace{(\varphi_1 \circ \varphi_2^{-1})}_{=\psi_{12},\, C^\infty}$$

f is also differentiable in second chart.

Definition 4.23 (Diffeomorphism)**.** Let $f : M \to N$ be a homeomorphism and ψ and φ be coordinate functions as previously defined. If $\psi \circ f \circ \varphi^{-1}$ is invertible and both $y = \psi \circ f \circ \varphi^{-1}(x)$ and $x = \varphi \circ f \circ \psi^{-1}(y)$ are C^∞, f is called a **diffeomorphism** and M is said to be **diffeomorphic** to N and vice versa, denoted by $M \equiv N$.

Remark. $M \equiv N$ implies $\dim M = \dim N$.

Remark. *Homeomorphisms* classify spaces according to whether it is possible to deform one space into another *continuously*. *Diffeomorphisms* classify spaces into equivalence classes according to whether it is possible to deform one space into another *smoothly*.

Definition 4.24. The set of diffeomorphisms $f : M \to M$ is a group denoted by $\mathrm{Diff}(M)$.

Remark. Active and passive transformation point of view

- Take a point p in a chart (U, φ) such that $\varphi(p) = x^\mu(p)$. Under $f \in \mathrm{Diff}(M)$, p is mapped into $f(p)$ and $\varphi(f(p)) = y^\mu(f(p))$. Clearly y is a differentiable function of x(**active** transformation).
- If (U, φ) and (V, ψ) are overlapping charts with $x^\mu = \varphi(p)$ and $y^\mu = \psi(p)$ ($p \in U \cap V$), the map $x \mapsto y$ is differentiable by the assumed smoothness of the manifold(**passive** transformation).

Before we finish, let me define two important entities for later.

Definition 4.25 (Curves)**.** An **open curve** in an m-dimensional manifold M is a map $c : (a, b) \to M$ such that

- c does not intersect with itself.
- For simplicity, we assume 0 is included in (a, b).
- a and b can be $-\infty$ and ∞, respectively.

Definition 4.26 (Functions on the manifold)**.** A **function** f on M is a smooth map $f : M \to \mathbb{R}$. The set of smooth functions on M is denoted by $\mathcal{F}(M)$.