8 Lie Groups and Lie Algebras

§ 8.3 Examples

Example 8.14. Let $G = (\mathbb{R}, +)$. Then, identity is 0 and $L_a x = a + x$. What is the tangent vector at $x = 0$? Let's compute the following equation's both sides

$$L_{a*} \left.\frac{d}{dx}\right|_{x=0} = k \cdot \left.\frac{d}{dx}\right|_{x=a}$$

and find value of the constant k. To evaluate k, apply both sides to x. Then,

$$k \cdot \left.\frac{d}{dx}\right|_{x=a} x = k$$

and

$$L_{a*} \left(\left.\frac{d}{dx}\right|_{x=0} \right) x = \left.\frac{d}{dx}\right|_{x=0} (x \circ L_a) = \left.\frac{d}{dx}\right|_{x=0} (a + x) = 1$$

So, $k = 1$ and

$$\boxed{L_{a*} \left.\frac{d}{dx}\right|_{x=0} = \left.\frac{d}{dx}\right|_{x=a}}$$

Therefore, $\frac{d}{dx}$ is a left-invariant vector field on $\mathbb{R}$.

Moreover, left-invariant vector fields on $\mathbb{R}$ are constant multiples of $\frac{d}{dx}$.

Remark. $X = \frac{\partial}{\partial \theta}$ is the unique left-invariant vector field on $G = \mathrm{SO}(2) = \{e^{i\theta} \mid 0 \le \theta \le 2\pi\}$. $\mathbb{R}$ and $\mathrm{SO}(2)$ share the common Lie algebra.

Example 8.15 (Left-invariant fields on $\mathrm{GL}(n,\mathbb{R})$)**.** An element of $\mathrm{GL}(n,\mathbb{R})$ has n^2 entries, x^{ij}, of the matrix.

- Unit element: $e = \mathbf{I}_n = \delta^{ij}$
- Let $g = \{x^{ij}(g)\}$ and $a = \{x^{ij}(a)\} \in \mathrm{GL}(n,\mathbb{R})$. The left-translation is

$$L_a g = ag = \sum_k x^{ik}(a)x^{kj}(g)$$

- Take a vector $V = V^{ij} \left.\frac{\partial}{\partial x^{ij}}\right|_e \in T_eG$. The left-invariant vector field generated by V is

$$\begin{aligned} X_V|_g = L_{g*}V &= V^{ij} \underbrace{\left(\left.\frac{\partial}{\partial x^{ij}}\right|_e x^{kl}(g)x^{lm}(e) \right)}_{\text{pushforward formula}} \underbrace{\left.\frac{\partial}{\partial x^{km}}\right|_g}_{\text{basis at } g} \\ &= V^{ij}x^{kl}(g)\delta^l{}_i\delta^m{}_j \left.\frac{\partial}{\partial x^{km}}\right|_g = x^{ki}(g)V^{ij} \left.\frac{\partial}{\partial x^{kj}}\right|_g \\ &= \boxed{(gV)^{kj} \left.\frac{\partial}{\partial x^{kj}}\right|_g} \end{aligned}$$

  As we expected, left-translation of *vectors* in $T_e\mathrm{GL}(n,\mathbb{R})$ is represented by usual matrix multiplication.

- Now we compute Lie brackets. Take two vectors

$$V = V^{ij} \left.\frac{\partial}{\partial x^{ij}}\right|_e \quad \text{and} \quad W = W^{ij} \left.\frac{\partial}{\partial x^{ij}}\right|_e$$

  Then, the Lie bracket of two vector fields X_V and X_W generated by V and W, respectively, becomes

$$\begin{aligned} [X_V, X_W]|_g &= x^{ki}(g)V^{ij} \left.\frac{\partial}{\partial x^{kj}}\right|_g x^{ca}W^{ab} \left.\frac{\partial}{\partial x^{cb}}\right|_g - (V \leftrightarrow W) \\ &= x^{ki}(g)[V^{ij}W^{jb} - W^{ij}V^{jb}] \left.\frac{\partial}{\partial x^{kb}}\right|_g \\ &= x^{ij}(g)[V^{jk}W^{kl} - W^{jk}V^{kl}] \left.\frac{\partial}{\partial x^{il}}\right|_g \qquad (\because \text{ dummy index change}) \\ &= (g[V,W])^{ij} \left.\frac{\partial}{\partial x^{ij}}\right|_g \end{aligned}$$

In summary, the following holds for any matrix groups.

$$\boxed{L_{g*}V = gV} \quad \text{and} \quad \boxed{[X_V, X_W]|_g = L_{g*}[V,W] = g[V,W]}$$

Example 8.16 (Lie groups and algebras of matrix groups)**.**

(a) $\mathfrak{gl}(n,\mathbb{R})$: Lie algebra of $\mathrm{GL}(n,\mathbb{R})$.

- Consider the parametrized curve $c : (-\epsilon,\epsilon) \to \mathrm{GL}(n,\mathbb{R})$ with $c(0) = \mathbf{I}_n$.
- If ϵ is small enough, this curve can be approximated by $c(s) = \mathbf{I}_n + sA + \mathcal{O}(s^2)$ near $s = 0$, where A is an $n \times n$ matrix of real entries (without further constraints).
- For small s, $\det c(s)$ cannot vanish, so $c(s) \in \mathrm{GL}(n,\mathbb{R})$.
- The tangent vector to $c(s)$ at $\mathbf{I}_n$ is $c'(s)|_{s=0} = A$. Therefore, $\mathfrak{gl}(n,\mathbb{R})$ is the set of $n \times n$ matrices with dimension $\dim \mathfrak{gl}(n,\mathbb{R}) = n^2 = \dim \mathrm{GL}(n,\mathbb{R})$

(b) $\mathfrak{sl}(n,\mathbb{R})$: Lie algebra of $\mathrm{SL}(n,\mathbb{R})$. Let's use the same setup as in (a). Now we have additional constraint: for the curve $c(s)$ to be in $\mathrm{SL}(n,\mathbb{R})$, $\det c(s) = +1$ should be satisfied.

$$\det c(s) = 1 + s\operatorname{tr} A = 1 \implies \boxed{\operatorname{tr} A = 0}$$

Hence, $\mathfrak{sl}(n,\mathbb{R})$ is the set of $n \times n$ *traceless* matrices with $\dim \mathfrak{sl}(n,\mathbb{R}) = n^2 - 1$.

(c) $\mathfrak{o}(n)$: Lie algebra of $\mathrm{O}(n)$. Now, condition for $c(s)$ becomes $c(s)^\top c(s) = \mathbf{I}_n$. Differentiating both sides with respect to s yields

$$c'(s)^\top c(s) + c(s)^\top c'(s) = 0$$

At $s = 0$, this reduces into $A^\top + A = 0$, or equivalently, $A^\top = -A$. Therefore, $\mathfrak{o}(n)$ is the set of skew-symmetric matrices with $\dim \mathfrak{o}(n) = \binom{n}{2}$.

(d) $\mathfrak{so}(n) = \mathfrak{o}(n)$ since all skew-symmetric matrices are traceless[1].

(e) We can think of same analogy for complex matrices, except they have $2n^2$ entries(n^2 for real part and n^2 for imaginary part).

- $\mathfrak{gl}(n,\mathbb{C})$: the set of $n \times n$ matrices with complex entries ($\dim \mathfrak{gl}(n,\mathbb{C}) = 2n^2$).
- $\mathfrak{sl}(n,\mathbb{C})$: the set of $n \times n$ traceless matrices with complex entries ($\dim \mathfrak{sl}(n,\mathbb{C}) = 2(n^2 - 1)$).
- $\mathfrak{u}(n)$: the set of $n \times n$ skew-Hermitian[2] matrices with complex entries ($\dim \mathfrak{u}(n) = n + 2\binom{n}{2} = n^2$, where n comes from the imaginary part of diagonal elements).
- $\mathfrak{su}(n)$: the set of $n \times n$ traceless skew-Hermitian matrices with complex entries ($\dim \mathfrak{su}(n) = n^2 - 1$).

[1] Note that we are interested only in the vicinity of the unit element, so $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ shows no difference here.
[2] $A^\dagger = -A$