4 Manifolds: Definitions

§ 4.1 Topological Spaces

Definition 4.1 (Topological space)**.** Let X be any set and $\mathcal{T} = \{U_i : i \in I\}$ be a *collection* of subsets of X. The pair $(X, \mathcal{T})$ is a **topological space** if $\mathcal{T}$ satisfies

(i) $\varnothing \in \mathcal{T}$ and $X \in \mathcal{T}$.

(ii) $\bigcup_{j \in J} U_j \in \mathcal{T}$ where J is a subcollection of I. Here, union can be infinite.

(iii) $\bigcap_{k \in K} U_k \in \mathcal{T}$ where K is a *finite* subcollection of I.

Here U_i are called the **open set** and $\mathcal{T}$ is said to give a **topology** to X.

Remark. Do you remember the definition of *open sets* in analysis?

(i) U is *open* if all point of U is an *interior* point of U.

(ii) $x \in U$ is an *interior* point of U if there exists some *neighborhood* $N(x)$ of x such that $N(x) \in U$.

(iii) The *neighborhood* of x with radius r is $N_r(x) = \{y \in U \,|\, d(x, y) < r\}$.

The definition of open set requires the concept of **metric space**.

Example 4.2. Example of topologies.

(1) Let X be a set and $\mathcal{T}$ be the collection of all subsets of X. This gives **discrete topology** to X.

(2) Let X be a set and $\mathcal{T} = \{\varnothing, X\}$. This gives **trivial topology** to X.

(3) Let $X = \mathbb{R}$ and $\mathcal{T}$ be all open intervals and their unions. This defines a **usual topology** to $\mathbb{R}$. This can be extended to $\mathbb{R}^n$.

Remark. A subset A of X is *closed* if its complement in X is an open set $X - A \in \mathcal{T}$.

Definition 4.3 (Metric)**.** A metric $d : X \times X \to \mathbb{R}$ is a map that satisfies

(i) $d(x, y) = d(y, x)$

(ii) $d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = y$

(iii) $d(x, y) + d(y, z) \geq d(x, z)$ (triangle inequality)

for $x, y, z \in X$.

Definition 4.4 (Metric space and metric topology)**.** If X is endowed with a metric d, X is made into a topological space whose open sets are given by *open discs*, $U_\epsilon(x) = \{y \in X \mid d(x,y) < \epsilon\}$ and their possible unions. Then $\mathcal{T}$ is called a **metric topology** and $(X, \mathcal{T})$ a **metric space**.

Definition 4.5 (Relative topology)**.** Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. Then $\mathcal{T} = \{U_i : i \in I\}$ induces the **relative topology** in A by

$$\mathcal{T}' = \{U_i \cap A \mid U_i \in \mathcal{T}\}$$

Definition 4.6 (Continuous map)**.** Let X and Y be topological spaces. A map $f : X \to Y$ is **continuous** if the inverse image of an open set in Y is an open set in X.

Definition 4.7 (Neighborhood)**.** N is a **neighborhood**[1] of a point $x \in X$ if $N \subseteq X$ and N contains at least one open set U_i which contains x.

Definition 4.8 (Hausdorff space)**.** A topological space $(X, \mathcal{T})$ is a **Hausdorff space** if: for $x, x' \in X$, there exist neighbors U_x of x and $U_{x'}$ of x' such that $U_x \cap U_{x'} = \varnothing$.

Exercise 4.9. Prove the following statements.

(1) $\mathbb{R}$ with the usual topology is a Hausdorff space.

(2) Every metric space is a Hausdorff space.

§ 4.2 Introductory Topology

Definition 4.10 (Compactness)**.** Let X be a set.

(1) A family $\{A_i\}$ of subsets of X is called a **covering** of X if $\bigcup_{i \in I} A_i = X$. If all the A_i happen to be the open sets of the topology $\mathcal{T}$, the covering is called an **open covering**.

(2) The set X is **compact** if: for every open covering $\{U_i : i \in I\}$, there exists finite subcovering $\{U_j : j \in J \subseteq I\}$.

Theorem 4.11 (Heine-Borel theorem)**.** *For a subset $X \subseteq \mathbb{R}^n$, X is compact if and only if X is closed and bounded.*

Remark. A point in $\mathbb{R}^n$ is compact.

[1] The neighborhood needs not be an open set.

Definition 4.12 (Connectedness)**.**

(a) A topological space X is **connected** if it cannot be written as $X = X_1 \cup X_2$ where X_1 and X_2 are both open and $X_1 \cap X_2 = \varnothing$.

(b) A topological space is **arcwise connected** if for any points $x, y \in X$ there exists a continuous map $f : [0,1] \to X$ such that $f(0) = x$ and $f(1) = y$.

(c) A **loop** in a topological space X is a continuous map $f : [0,1] \to X$ such that $f(0) = f(1)$. If any loop in X can be continuously shrunk to a point, then X is called **simply connected**.

Example 4.13.

(a) $\mathbb{R}$ is arcwise connected while $\mathbb{R} - \{0\}$ is not. Meanwhile, both $\mathbb{R}^n$ and $\mathbb{R}^n - \{0\}$ are arcwise connected for $n \geq 2$.

(b) S^n is arcwise connected. S^1 is not simply connected while S^n $(n \geq 2)$ is simply connected. The n-dimensional torus

$$T^n = \underbrace{S^1 \times \cdots \times S^1}_{n}$$

is arcwise connected but not simply connected.

(c) $\mathbb{R}^2 - \mathbb{R}$ is not arcwise connected. $\mathbb{R}^2 - \{0\}$ is arcwise connected but not simply connected. However, $\mathbb{R}^3 - \{0\}$ is both arcwise connected and simply connected.

Topology is about classifying the spaces (are they equal or different?). In topology, two figures are *equivalent* if it can be deformed continuously from one to the other.

Definition 4.14 (Homeomorphisms)**.** Let X_1 and X_2 be topological spaces. A map $f : X_1 \to X_2$ is a **homeomorphism** if it is continuous and has an inverse $f^{-1} : X_2 \to X_1$ which is also continuous. Then, X_1 and X_2 are **homeomorphic**. In other words,

$$\exists\, f : X_1 \to X_2,\ g : X_2 \to X_1 \quad \text{s.t.} \quad f \circ g = \mathrm{id}_{X_2} \text{ and } g \circ f = \mathrm{id}_{X_1}$$

Remark. Homeomorphisms give equivalence relations, which divide all topological spaces into equivalence classes(Is it possible to deform one space into other by a homeomorphism).

Exercise 4.15.

(1) $X = \{1, 1/2, \cdots, 1/n\} \subseteq \mathbb{R}$ is not closed in $\mathbb{R}$.

(2) $Y = X \cup \{0\}$ is closed in $\mathbb{R}$.

(3) Y is compact.

Proof.

(1) Firstly, X is not open because it is a set of isolated points. Considering the complement of X,

$$(-\infty, 0] \cup (1, \infty) \cup (1/2, 1) \cup (1/3, 1/2) \cup \cdots$$

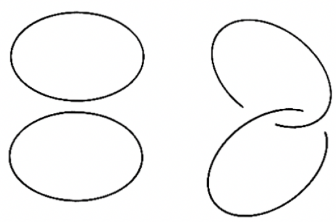
Since $\mathbb{R} - X$ contains *accumulation point* 0, $(-\infty, 0]$ makes it not open. Therefore, X is not closed.

(2) Trivial from (1).

(3) Heine-Borel theorem (theorem 4.11).

■

Exercise 4.16. Show that these two figures are homeomorphic. Show that those linked rings are separable (or deformable into separated rings) in $\mathbb{R}^4$.



Proof. Let two rings in separated and linked forms be (A, B) and (A′, B′), respectively. If we take homeomorphisms from A to A′ and B to B′ and combine them, separated rings and linked rings are homeomorphic.

Existence of homotopy. Consider a homotopy $\{0,1\} \times (S^1 \sqcup S^1) \to \mathbb{R}^4$, where the image of $(0, \bullet, \bullet) : S^1 \sqcup S^1 \to \mathbb{R}^4$ denotes separated rings and $(1, \bullet, \bullet) : S^1 \sqcup S^1 \to \mathbb{R}^4$ denotes linked rings. Assuming separated and linked rings are embedded in $\mathbb{R}^3 \times \{0\}$, we can use the fourth coordinate to lift one of the two rings in separated rings to the $(\bullet, \bullet, \bullet, 1)$ plane. Move it over to the corresponding ring on the linked ring and move it back to $(\bullet, \bullet, \bullet, 0)$ plane. ■