5 Vectors and Tensors on Manifolds

§ 5.1 Vectors

An *elementary* picture of a vector as an arrow connecting a point and the origin does not work in a manifold.

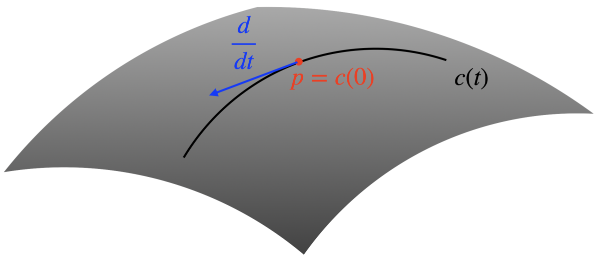
- Where is the origin on the manifold?
- What is a *straight* arrow on the manifold? For example, how can we define a straight arrow that connects London and Los Angeles on the *surface* or the Earth?

Let us look at the **tangent line** to a curve in $\mathbb{R}^2$. If the curve is differentiable, the tangent line at $x = x_0$ becomes

$$y - y(x_0) = a(x - x_0), \quad a = \left.\frac{dy}{dx}\right|_{x_0}$$

The tangent vectors on a manifold $\mathcal{M}$ generalize this tangent line. On a manifold, a vector is defined to be a tangent vector to a **curve** in $\mathcal{M}$.

Definition 5.1. Consider a curve $c : (a,b) \to \mathcal{M}$ and a smooth function $f : \mathcal{M} \to \mathbb{R}$. The **tangent vector** at $c(0)$ is defined as a **directional derivative** of a function $f(c(t))$ along the curve $c(t)$ at $t = 0$.



The rate of change at $t = 0$ along $c(t)$ is $\left.\frac{df(c(t))}{dt}\right|_{t=0}$. In terms of the local coordinates $\{x^\mu\}$,

$$\left.\frac{df(c(t))}{dt}\right|_{t=0} = \underbrace{\frac{\partial f}{\partial x^\mu}}_{(*)} \left.\frac{dx^\mu(c(t))}{dt}\right|_{t=0}$$

We are (again) abusing the notation here. Precisely, $(*)$ should be written as $\frac{\partial(f \circ \varphi^{-1})}{\partial x^\mu}$ where $\varphi : \mathcal{M} \to \mathbb{R}$ is the coordinate function.

In other words, the derivative is obtained by applying the **differential operator** X to f, defined by

$$\left.\frac{df(c(t))}{dt}\right|_{t=0} = \frac{\partial f}{\partial x^\mu} \left.\frac{dx^\mu(c(t))}{dt}\right|_{t=0} = \left.\frac{dx^\mu(c(t))}{dt}\right|_{t=0} \cdot \frac{\partial}{\partial x^\mu} \cdot f = X^\mu \partial_\mu f \equiv X[f]$$

where

$$X = X^\mu \partial_\mu, \quad X^\mu = \left.\frac{dx^\mu(c(t))}{dt}\right|_{t=0}$$

It is $\boxed{X = X^\mu \partial_\mu}$ which we define as the **tangent vector** to $\mathcal{M}$ at $p = c(0)$ along the direction given by the curve $c(t)$!

Remark. Different curves passing through $p = c(0)$ yields different tangent vectors pointing different directions. Even if the two curves are globally different, but if those has same slope at p, they yield same tangent vector.

Definition 5.2. Two curves $c_1(t)$ and $c_2(t)$ are equivalent if

$$(i)\ c_1(0) = c_2(0) = p$$
$$(ii)\ \left.\frac{dx^\mu(c_1(t))}{dt}\right|_{t=0} = \left.\frac{dx^\mu(c_2(t))}{dt}\right|_{t=0}$$

In other words, c_1 and c_2 yield the same differential operator. We identify the tangent vector X with the equivalence class of curves.

$$[c(t)] = \left\{ \tilde{c}(t) \mid \tilde{c}(0) = c(0) \text{ and } \left.\frac{dx^\mu(\tilde{c}(t))}{dt}\right|_{t=0} = \left.\frac{dx^\mu(c(t))}{dt}\right|_{t=0} \right\}$$

Definition 5.3 (Tangent spaces)**.** All the equivalence classes of curves at $p \in \mathcal{M}$, namely all the tangent vectors at $p \in \mathcal{M}$, form a vector space called the **tangent space** of $\mathcal{M}$ at p, $T_p\mathcal{M}$.

Evidently, ∂_μ $(\mu = 1, 2, \cdots, m)$ are the *basis vectors* of $T_p\mathcal{M}$.

Remark. 1) If a vector $V \in T_p\mathcal{M}$ is written as $V = V^\mu e_\mu$, V^μ are called the components of V with respect to e_μ.

2) By construction, a vector exists without specifying the coordinates. Coordinate independence property enables us to find the transformation property of the components of the vector. Consider two overlapping charts. For $p \in U_i \cap U_j$, $x = \varphi_i(p)$ and $y = \varphi_j(p)$. Two expressions for $X \in T_p\mathcal{M}$ is

$$X = X^\mu \frac{\partial}{\partial x^\mu} = \tilde{X}^\mu \frac{\partial}{\partial y^\mu}$$

This gives $\tilde{X}^\mu = X^\nu \dfrac{\partial y^\mu}{\partial x^\nu}$

3) The basis of $T_p\mathcal{M}$ need not to be $\{e_\mu\}$. Linear combinations $\hat{e}_i \equiv A_i{}^\mu e_\mu$ where $A = (A_i{}^\mu) \in \mathrm{GL}(m, \mathbb{R})$ is also a basis of $T_p\mathcal{M}$: the **non-coordinate basis**.

§ 5.2 Cotangent Space

Since we defined vectors on the manifold, covectors should follow. The dual vector space to $T_p\mathcal{M}$ should take linear maps from $T_p\mathcal{M}$ to $\mathbb{R}$ as elements.

Definition 5.4 (Cotangent spaces)**.** The **cotangent space** at p, $T_p^*\mathcal{M}$, contains the linear maps $w : T_p\mathcal{M} \to \mathbb{R}$ (**dual vector** or **cotangent vector** or **one-form**).

- The action of a vector V on f is a *directional derivative*.

$$V[f] = V^\mu \frac{\partial f}{\partial x^\mu} \in \mathbb{R}$$

- The action of a one-form $\mathrm{d}f \in T_p^*\mathcal{M}$, which is a **differential** of a smooth function $f \in \mathcal{F}(\mathcal{M})$, on $V \in T_p\mathcal{M}$ is defined by

$$\langle \mathrm{d}f, V\rangle \equiv V[f] = V^\mu \frac{\partial f}{\partial x^\mu} \in \mathbb{R}$$

- $\mathrm{d}f$ is expressed in terms of the coordinate $x = \varphi(p)$ as

$$\mathrm{d}f = \frac{\partial f}{\partial x^\mu}\mathrm{d}x^\mu$$

  Hence, $\{\mathrm{d}x^\mu\}$ forms a basis of $T_p^*\mathcal{M}$ and arbitrary one-form can be expressed as $w = w_\mu \mathrm{d}x^\mu$.

Definition 5.5 (Inner product[1])**.** The **inner product** $\langle\ ,\ \rangle : T_p^*\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R}$ is defined by

$$\langle w, V\rangle = w_\mu V^\mu \left\langle \mathrm{d}x^\mu, \partial_\mu \right\rangle = w_\mu V^\mu \in \mathbb{R}$$

where $w \in T_p^*\mathcal{M}$ and $V \in T_p\mathcal{M}$.

Remark. Similar with $X \in T_p\mathcal{M}$, consider $p \in U_i \cap U_j$ with coordinate functions $x = \varphi_i(p)$ and $y = \varphi_j(p)$.

$$w = w_\mu \mathrm{d}x^\mu = \tilde{w}_\nu \mathrm{d}y^\nu \implies \tilde{w}_\nu = w_\mu \frac{\partial x^\mu}{\partial y^\nu} \text{ since } \mathrm{d}y^\nu = \frac{\partial y^\nu}{\partial x^\mu}\mathrm{d}x^\mu$$

§ 5.3 Tensors and Tensor Fields

Definition 5.6. A **tensor** of type (q, r) is a multilinear object which maps q elements of $T_p^*\mathcal{M}$ and r elements of $T_p\mathcal{M}$ to a real number.

$$T^{(q,r)} \in \mathfrak{T}_p^{(q,r)}(\mathcal{M}) \text{ and } T^{(q,r)} : (T_p^*\mathcal{M})^{\otimes q} \otimes (T_p\mathcal{M})^{\otimes r} \to \mathbb{R}$$

In component-basis representation,

$$T^{(q,r)} = T^{\mu_1\cdots\mu_q}_{\nu_1\cdots\nu_r} \frac{\partial}{\partial x^{\mu_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{\mu_q}} \otimes \mathrm{d}x^{\nu_1} \otimes \cdots \otimes \mathrm{d}x^{\nu_r}$$

[1] Note that we emphasized that this is not an inner product before. But Nakahara wrote this as an inner product so we follow it.

Remark. For $V_i = V_i^\mu \partial_\mu$ $(i = 1, \cdots, r)$ and $w_i = w_{i\mu} \mathrm{d}x^\mu$ $(i = 1, \cdots, q)$,

$$T(w_1, \cdots, w_q, V_1, \cdots, V_r) = T^{\mu_1 \cdots \mu_q}_{\nu_1 \cdots \nu_r} w_{1\mu_1} \cdots w_{q\mu_q} V_1^{\nu_1} V_r^{\nu_r}$$

We use the notation $\langle w, X \rangle = w(X)$.

Definition 5.7.

(1) If a vector is assigned *smoothly* to each point of $\mathcal{M}$, it is called a **vector field** over $\mathcal{M}$. In other words, V is a vector field if $V[f] \in \mathcal{F}(\mathcal{M})$ for any $f \in \mathcal{F}(\mathcal{M})$. The set of vector fields on $\mathcal{M}$ is $\mathscr{X}(\mathcal{M})$.

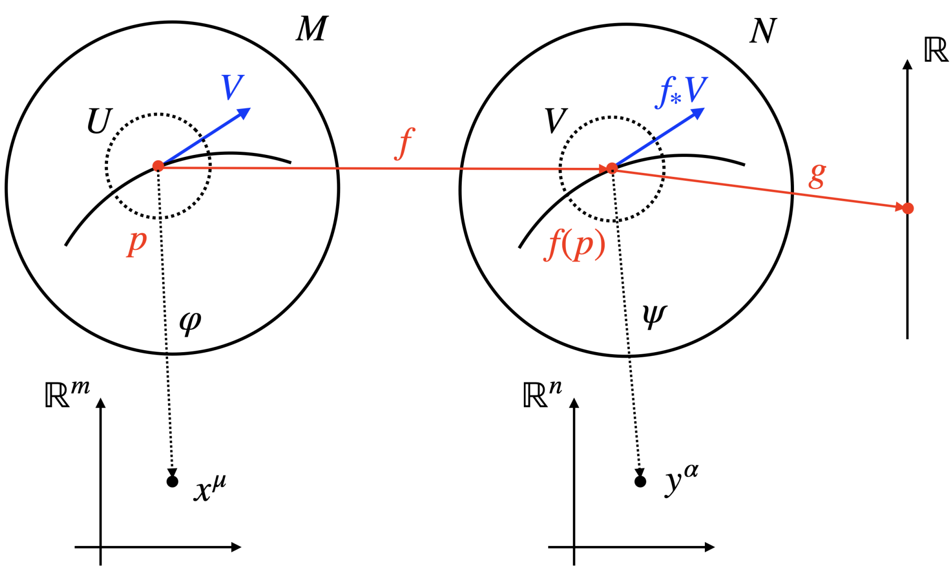
(2) A vector field X restricted at $p \in \mathcal{M}$ is $X|_p \in T_p\mathcal{M}$.

(3) A **tensor field** of type (q, r) is a smooth assignment of an element of $\mathfrak{T}_p^{(q,r)}(\mathcal{M})$ at each $p \in \mathcal{M}$.

§ 5.4 Induced Maps

Definition 5.8 (Pushforward)**.** A smooth map $f : \mathcal{M} \to \mathcal{N}$ naturally induces a **pushforward** map $f_* : T_p\mathcal{M} \to T_{f(p)}\mathcal{N}$ as

$$(f_*V)[g] \equiv V[g \circ f] \qquad V \in T_p\mathcal{M},\ g \in \mathcal{F}(\mathcal{N})$$



In terms of charts (U, φ) on $\mathcal{M}$ and (V, ψ) on $\mathcal{N}$,

$$f_*V[g \circ \psi^{-1}(y)] = V[g \circ f \circ \varphi^{-1}(x)]$$

where $x = \varphi(p)$ and $y = \psi(p)$.

Remark. Let's compute the components of the pushforward map. Let

$$V = V^\mu \frac{\partial}{\partial x^\mu} \text{ and } f_* V = W^\alpha \frac{\partial}{\partial y^\alpha}$$

Then

$$W^\alpha \frac{\partial}{\partial y^\alpha}[g \circ \psi^{-1}(y)] = V^\mu \frac{\partial}{\partial x^\mu}[g \circ f \circ \varphi^{-1}(x)]$$

Take $g = y^\alpha$. Then

$$W^\alpha = V^\mu \frac{\partial}{\partial x^\mu} y^\alpha$$

where $\partial y^\alpha / \partial x^\mu$ is the Jacobian of $f : \mathcal{M} \to \mathcal{N}$.

Remark. f_* is naturally extended to tensors of type $(q, 0)$.

$$f_* : \mathfrak{T}_p^{(q,0)}(\mathcal{M}) \to \mathfrak{T}_{f(p)}^{(q,0)}(\mathcal{N})$$

Exercise 5.9. Let $f : \mathcal{M} \to \mathcal{N}$ and $g : \mathcal{N} \to \mathcal{P}$. Show that the pushforward of the composite map $g \circ f : \mathcal{M} \to \mathcal{P}$ is $(g \circ f)_* = g_* \circ f_*$.

Proof. Consider a map $h : \mathcal{P} \to \mathbb{R}$.

$$\begin{aligned}(g \circ f)_* V[h] &= V[h \circ g \circ f] \\ &= f_* V[h \circ g] = g_*(f_* V[h]) = (g_* \circ f_*) V[h]\end{aligned}$$

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Definition 5.10 (Pullback). A map $f : \mathcal{M} \to \mathcal{N}$ also induces a map $f^* : T^*_{f(p)}\mathcal{N} \to T^*_p\mathcal{M}$ (**pullback map**) as

$$\langle f^* w,\, V \rangle = \langle w,\, f_* V \rangle \quad \text{where } V \in T^*_p M,\ w \in T^*_{f(p)}\mathcal{N}$$

Remark. Let $w = w_\alpha \, \mathrm{d}y^\alpha \in T^*_{f(p)}\mathcal{N}$ and $f^* w = \xi_\mu \, \mathrm{d}x^\mu \in T^*_p\mathcal{M}$. Then

$$\langle f^* w,\, V \rangle = \left\langle \xi_\mu \, \mathrm{d}x^\mu,\, V^\nu \frac{\partial}{\partial x^\nu} \right\rangle = \xi_\mu V^\mu$$

and

$$\langle w, f_* V \rangle = \left\langle w_\alpha \, \mathrm{d}y^\alpha,\, V^\mu \frac{\partial y^\nu}{\partial x^\mu} \frac{\partial}{\partial y^\nu} \right\rangle = w_\alpha V^\mu \frac{\partial y^\alpha}{\partial x^\mu}$$

gives $\xi_\mu = w_\alpha \dfrac{\partial y^\alpha}{\partial x^\mu}$.

Remark. f^* is naturally extended to tensors of type $(0, r)$.

$$f^* : \mathfrak{T}_{f(p)}^{(0,r)}(\mathcal{N}) \to \mathfrak{T}_p^{(0,r)}(\mathcal{M})$$

Exercise 5.11. Let $f : \mathcal{M} \to \mathcal{N}$ and $g : \mathcal{N} \to \mathcal{P}$. Show that the pullback of the composite map $g \circ f : \mathcal{M} \to \mathcal{P}$ is $(g \circ f)^* = f^* \circ g^*$.

Proof.

$$\begin{aligned} \langle (g \circ f)^* w,\, V \rangle &= \langle w,\, (g \circ f)_* V \rangle \\ &= \langle w,\, (g_* \circ f_*) V \rangle = \langle g^* w,\, f_* V \rangle = \langle (f^* \circ g^*) w,\, V \rangle \end{aligned}$$

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Remark. There is no natural extension of the induced map for a tensor of mixed type. The extension is only possible if $f : \mathcal{M} \to \mathcal{N}$ is a *diffeomorphism*, where the Jacobian of f^{-1} is also defined.

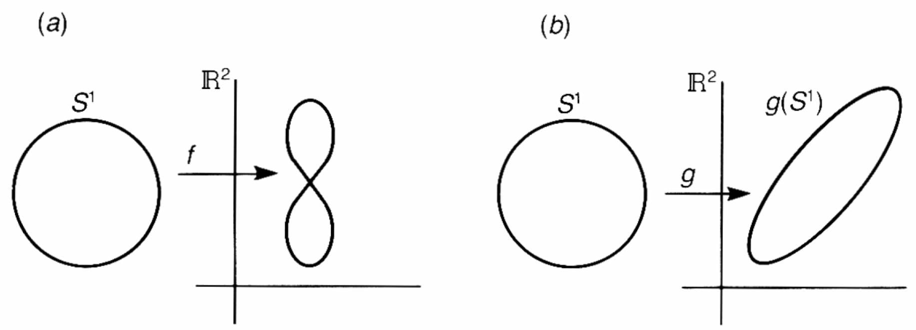
Definition 5.12. Let $f : \mathcal{M} \to \mathcal{N}$ be a smooth map and let $\dim \mathcal{M} \leq \dim \mathcal{N}$.

(a) The map f is called an **immersion** of $\mathcal{M}$ into $\mathcal{N}$ if $f_* : T_p\mathcal{M} \to T_{f(p)}\mathcal{N}$ is an injection[2], that is $\operatorname{rk} f_* = \dim \mathcal{M}$.

(b) The map f is called an **embedding** if f is an injection and an immersion. The image $f(\mathcal{M})$ is called a **submanifold** of $\mathcal{N}$.

Remark. If f is an immersion, f_* maps $T_p\mathcal{M}$ isomorphically to an m-dimensional subspace of $T_{f(p)}\mathcal{N}$.

Example 5.13. Consider two maps $f, g : S^1 \to \mathbb{R}^2$.



- f is an immersion since a 1D tangent space of S^1 is mapped by f_* to a subspace of $T_{f(p)}\mathbb{R}^2$.
- $f(S^1)$ is not a submanifold of $\mathbb{R}^2$ since f is not an injection.
- $g(S^1)$ is a submanifold of $\mathbb{R}^2$, in similar manner.

[2] one-to-one function.