

6 Lie Derivatives

§ 6.2 Lie Derivatives (cont'd)

Exercise 6.9. Show that the Lie bracket satisfies ($c_1, c_2 \in \mathbb{R}$, $X_1, X_2, Y_1, Y_2, X, Y, Z \in \mathcal{X}(\mathcal{M})$)

(a) bilinearity:

$$\begin{aligned} [X, c_1 Y_1 + c_2 Y_2] &= c_1 [X, Y_1] + c_2 [X, Y_2] \\ [c_1 X_1 + c_2 X_2, Y] &= c_1 [X_1, Y] + c_2 [X_2, Y] \end{aligned}$$

(b) skew-symmetry: $[X, Y] = -[Y, X]$

(c) the **Jacobi identity** $[[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] = 0$.

Proof. Straightforward. ■

Exercise 6.10. Let $X, Y \in \mathcal{X}(\mathcal{M})$ be vector fields on \mathcal{M} . Show that

(a) For $f \in \mathcal{F}(\mathcal{M})$,

$$\mathcal{L}_X fY = f[X, Y] - Y[f]X \text{ and } \mathcal{L}_X(fY) = f[X, Y] + X[f]Y$$

(b) For $f : \mathcal{M} \rightarrow \mathcal{N}$,

$$f_*[X, Y] = [f_*X, f_*Y]$$

Proof. (a) For $g \in \mathcal{F}(\mathcal{M})$,

$$\begin{aligned} \mathcal{L}_{fX}Y[g] &= fX[Y[g]] - Y[fX[g]] = fX[Y[g]] - Y[f]X[g] - fY[X[g]] = (f[X, Y] - Y[f]X)[g] \\ \mathcal{L}_X(fY)[g] &= X[fY[g]] - fY[X[g]] = X[f]Y[g] + fX[Y[g]] - fY[X[g]] = (f[X, Y] + X[f]Y)[g] \end{aligned}$$

(b) Let x^μ and y^ν denote the local coordinates in \mathcal{M} and \mathcal{N} , respectively. In this setup,

$$\begin{aligned} f_*[X, Y] &= f_* \left(X^\mu \frac{\partial}{\partial x^\mu} Y^\nu - Y^\mu \frac{\partial}{\partial x^\mu} X^\nu \right) \frac{\partial}{\partial x^\nu} \\ &= \left(X^\mu \frac{\partial}{\partial x^\mu} Y^\nu - Y^\mu \frac{\partial}{\partial x^\mu} X^\nu \right) \frac{\partial y^\lambda}{\partial x^\nu} \cdot e_\lambda \quad (\leftarrow e_\lambda = \frac{\partial}{\partial x^\lambda}) \end{aligned}$$

Since

$$f_*X = \underbrace{\left[X^\nu \frac{\partial y^\alpha}{\partial x^\nu} \right]}_{(f_*X)^\alpha} \frac{\partial}{\partial y^\alpha} \text{ and } f_*Y = \underbrace{\left[Y^\mu \frac{\partial y^\beta}{\partial x^\mu} \right]}_{(f_*Y)^\beta} \frac{\partial}{\partial y^\beta}$$

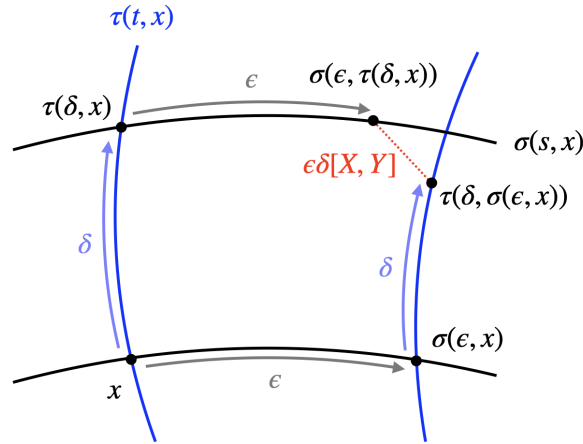
we have

$$\begin{aligned}
 f_*[X, Y] &= \left[X^\mu \frac{\partial y^\alpha}{\partial x^\mu} \cdot \frac{\partial}{\partial y^\alpha} \left((f_*Y)^\beta \frac{\partial x^\nu}{\partial y^\beta} \right) - Y^\mu \frac{\partial y^\alpha}{\partial x^\mu} \cdot \frac{\partial}{\partial y^\alpha} \left((f_*X)^\beta \frac{\partial x^\nu}{\partial y^\beta} \right) \right] \frac{\partial y^\lambda}{\partial x^\nu} \cdot e_\lambda \\
 &= \left[(f_*X)^\alpha \frac{\partial}{\partial y^\alpha} (f_*Y)^\beta - (f_*Y)^\alpha \frac{\partial}{\partial y^\alpha} (f_*X)^\beta \right] \underbrace{\frac{\partial x^\nu}{\partial y^\beta} \frac{\partial y^\lambda}{\partial x^\nu}}_{=\delta_\beta^\lambda} \cdot e_\lambda \\
 &\quad + \left[(f_*X)^\alpha (f_*Y)^\beta \frac{\partial^2 x^\nu}{\partial y^\alpha \partial y^\beta} - (f_*Y)^\alpha (f_*X)^\beta \frac{\partial^2 x^\nu}{\partial y^\alpha \partial y^\beta} \right] \frac{\partial y^\lambda}{\partial x^\nu} \cdot e_\lambda \\
 &= [f_*X, f_*Y]
 \end{aligned}$$

■

Observation 6.11 (Geometrical meaning of Lie brackets).

Lie bracket = non-commutativity of two flows
= failure of the closure of the parallelogram.



Consider two flows $\sigma(s, x)$ and $\tau(t, x)$ generated from $X, Y \in \mathcal{X}(\mathcal{M})$, respectively. Suppose that we move for ϵ along σ and then move for δ along τ .

$$\begin{aligned}
 \tau^\mu(\delta, \sigma(\epsilon, x)) &\simeq \tau^\mu(\delta, x^\nu + \epsilon X^\nu(x)) \\
 &\simeq x^\mu + \epsilon X^\mu(x) \delta Y^\mu(x^\nu + \epsilon X^\nu(x)) \\
 &\simeq x^\mu + \epsilon X^\mu(x) + \delta Y^\mu(x) + \epsilon \delta X^\nu(x) \partial_\nu Y^\mu(x)
 \end{aligned}$$

If we go for δ along τ first (and ϵ along σ later),

$$\sigma^\mu(\epsilon, \tau(\delta, x)) \simeq x^\mu + \epsilon X^\mu(x) + \delta Y^\mu(x) + \epsilon \delta Y^\nu(x) \partial_\nu X^\mu(x)$$

The *failure* of closure is

$$\tau^\mu(\delta, \sigma(\epsilon, x)) - \sigma^\mu(\epsilon, \tau(\delta, x)) = \epsilon \delta (X^\nu \partial_\nu Y^\mu - Y^\nu \partial_\nu X^\mu) = \boxed{\epsilon \delta [X, Y]^\mu}$$

Remark.

$$\mathcal{L}_X Y = [X, Y] = 0 \iff \sigma(s, \tau(t, x)) = \tau(t, \sigma(s, x))$$

In other words, if two flows commute then Lie derivative vanishes.

Definition 6.12 (Lie derivative of one-forms). The **Lie derivative of an one-form** $\omega \in \Omega^1(\mathcal{M})$ ¹ along $X \in \mathcal{X}(\mathcal{M})$ is defined by

$$\mathcal{L}_X \omega = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} [(\sigma_\epsilon)^* \omega|_{\sigma_\epsilon(x)} - \omega|_x]$$

where $(\sigma_\epsilon)^* : T_{\sigma_\epsilon(x)}^* \mathcal{M} \rightarrow T_x^* \mathcal{M}$ is a *pullback* map of σ_ϵ .²

Observation 6.13. Put $\omega = \omega_\mu dx^\mu$. Then

$$\omega|_{\sigma_\epsilon(x)} \simeq \omega_\mu(x^\nu + \epsilon X^\nu(x)) dx^\mu|_{x+\epsilon X} \simeq [\omega_\mu(x) + \epsilon X^\nu(x) \partial_\nu \omega_\mu(x)] dx^\mu|_{x+\epsilon X}$$

Applying the pullback map gives

$$\begin{aligned} (\sigma_\epsilon)^* \omega|_{\sigma_\epsilon(x)} &= [\omega_\mu(x) + \epsilon X^\nu(x) \partial_\nu \omega_\mu(x)] \underbrace{\frac{\partial(\sigma_\epsilon(x))^\alpha}{\partial x^\mu}}_{=\partial_\mu(x^\alpha + \epsilon X^\alpha(x))} dx^\mu|_x \\ &= \omega_\mu dx^\mu + \epsilon [X^\nu(x) \partial_\nu \omega_\mu(x) + \omega_\nu(x) \partial_\mu X^\nu(x)] dx^\mu + \mathcal{O}(\epsilon^2) \end{aligned}$$

which leads to

$$\boxed{\mathcal{L}_X \omega = (X^\nu \partial_\nu \omega_\mu + \partial_\mu X^\nu \omega_\nu) dx^\mu} \in T_x^* \mathcal{M}$$

Observation 6.14 (Lie derivative of smooth functions). The **Lie derivative of smooth function** $f \in \mathcal{F}(\mathcal{M})$ along a flow σ generated by a vector field X is

$$\begin{aligned} \mathcal{L}_X f &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} [f(\sigma_\epsilon(x)) - f(x)] = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} [f(x^\mu + \epsilon X^\mu) - f(x)] \\ &= X^\mu \partial_\mu f = X[f] \end{aligned}$$

the usual directional derivative.



How can we compute the Lie derivative of *general* (q, r) -tensors?

Proposition 6.15. The Lie derivative satisfies $\mathcal{L}_X(t_1 + t_2) = \mathcal{L}_X t_1 + \mathcal{L}_X t_2$ where t_1 and t_2 are tensor fields of the same type. For any type of tensors t_1 and t_2 , the following holds.

$$\mathcal{L}_X(t_1 \otimes t_2) = \mathcal{L}_X t_1 \otimes t_2 + t_1 \otimes \mathcal{L}_X t_2$$

Proof. We do not prove this proposition here - instead, we *embrace* it. ■

¹ $\Omega^1(\mathcal{M}) \equiv T_p^*(\mathcal{M})$.

²As the pushforward maps a vector from one tangent space to the other, pullback maps an one-form from one cotangent space to the other.

Example 6.16. Take $Y \in \mathcal{X}(\mathcal{M})$, $\omega \in \Omega^1(\mathcal{M})$ and construct $Y \otimes \omega$. The Lie derivative of this (1,1)-tensor is

$$\begin{aligned}\mathcal{L}_X(Y \otimes \omega) &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} [\{(\sigma_{-\epsilon}(x))_* Y \otimes (\sigma_\epsilon)^* \omega\}_{\sigma_\epsilon(x)} - (Y \otimes \omega)|_x] \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} [(\sigma_{-\epsilon})_* Y \otimes \{(\sigma_\epsilon)^* \omega - \omega\} + \{(\sigma_{-\epsilon})_* Y - Y\} \otimes \omega] \\ &= Y \otimes \mathcal{L}_X \omega + \mathcal{L}_X Y \otimes \omega\end{aligned}$$

For the general (1,1)-tensor $T = T_\mu{}^\nu dx^\mu \otimes \frac{\partial}{\partial x^\nu} \in \mathfrak{T}^{(1,1)}(\mathcal{M})$,

$$\mathcal{L}_X T = X[T_\mu{}^\nu] dx^\mu \otimes \frac{\partial}{\partial x^\nu} + T_\mu{}^\nu (\mathcal{L}_X dx^\mu) \otimes \frac{\partial}{\partial x^\nu} + T_\mu{}^\nu dx^\mu \otimes \left(\mathcal{L}_X \frac{\partial}{\partial x^\nu} \right)$$

Exercise 6.17. Let T be a tensor field. Show that

$$\mathcal{L}_{[X,Y]} T = \mathcal{L}_X \mathcal{L}_Y T - \mathcal{L}_Y \mathcal{L}_X T$$

Proof. First, note that

$$[X, Y]T = XYT - YXT = XYT - YTX + YTX - YXT = [X, YT] - Y[X, T]$$

Then

$$\begin{aligned}\mathcal{L}_{[X,Y]} T &= [[X, Y], T] = [X, Y]T - T[X, Y] \\ &= [X, YT] - Y[X, T] - [X, TY] - [X, T]Y \\ &= XYT - YTX - Y[X, T] - XTY + TYX + [X, T]Y \\ &= X[Y, T] - [Y, T]X + [X, T]Y - Y[X, T] \\ &= [X, [Y, T]] - [Y, [X, T]] = \boxed{\mathcal{L}_X \mathcal{L}_Y T - \mathcal{L}_Y \mathcal{L}_X T}\end{aligned}$$

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