8 Lie Groups and Lie Algebras

§ 8.1 Lie Groups

Definition 8.1 (Lie group)**.** A **Lie group** G is a differentiable manifold which is endowed with group operations

$$\bullet : G \times G \to G, \quad (g_1, g_2) \mapsto g_1 \cdot g_2$$
$$^{-1} : G \to G, \quad g \mapsto g^{-1}$$

are smooth($\mathcal{C}^\infty$).

Remark. The identity element of a Lie group is written as e.

Exercise 8.2. Show that the following groups are Lie groups.

(1) $(\mathbb{R}^+, \times)$

(2) $(\mathbb{R}, +)$

(3) $(\mathbb{R}^2, +)$ with $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

(4) $S^1 = \{e^{i\theta} \,|\, \theta \in \mathbb{R} \pmod{2\pi}\}$ with[a]

$$e^{i\theta} e^{i\varphi} = e^{i(\theta+\varphi)}, \; (e^{i\theta})^{-1} = e^{-i\theta}$$

[a] We call this group $\mathrm{U}(1)$.

Proof. The proof is simple. Hence, it is left for the readers. ∎

Example 8.3. Let's take a look at **matrix groups**.

(1) The group of $n \times n$ invertible matrices is called the **general linear group**, $\mathrm{GL}(n,\mathbb{R})$ and $\mathrm{GL}(n,\mathbb{C})$.

(2) If we add additional $\det M = +1$ constraint, it is called the **special linear group**.

$$\mathrm{SL}(n,\mathbb{R}) = \{M \in \mathrm{GL}(n,\mathbb{R}) \mid \det M = +1\}$$
$$\mathrm{SL}(n,\mathbb{C}) = \{M \in \mathrm{GL}(n,\mathbb{C}) \mid \det M = +1\}$$

(3) The groups of isometries (or *rigid motions*) of $\mathbb{R}^n$ are called **orthogonal groups** and **special orthogonal groups**.

$$\mathrm{O}(n) = \{M \in \mathrm{GL}(n,\mathbb{R}) \mid MM^\mathsf{T} = M^\mathsf{T} M = I_n\}$$
$$\mathrm{SO}(n) = \mathrm{O}(n) \cap \mathrm{SL}(n,\mathbb{R})$$

(4) In $\mathbb{C}^n$, groups with the same analogy are called **unitary groups** and **special unitary groups**.

$$\mathrm{U}(n) = \{M \in \mathrm{GL}(n,\mathbb{C}) \mid MM^\dagger = M^\dagger M = I_n\}$$
$$\mathrm{SU}(n) = \mathrm{U}(n) \cap \mathrm{SL}(n,\mathbb{C})$$

(5) The Lorentz group is also a Lie group,

$$\mathrm{O}(1,3) = \{M \in \mathrm{GL}(4,\mathbb{R}) \mid M\eta M^\mathsf{T} = \eta\}$$

where $\eta = \mathrm{diag}(-1,1,1,1)$ is the Minkowski metric.

Although we do not prove this, but I state an important theorem here.

Theorem 8.4. *Every closed subgroup H of a Lie group G is a Lie subgroup.*

Let G be a group and $H \leq G$. For $g \in G$, a left(right) **coset** is defined by

$$gH = \{gh \mid h \in H\},\ Hg = \{hg \mid h \in H\}$$

We say that H is a **normal subgroup** of G (or $H \trianglelefteq G$) if

$$g \in G, h \in H \implies ghg^{-1} \in H$$

In terms of cosets, if $N \trianglelefteq G$, $gN = Ng$ for $g \in G$. If N is a normal subgroup of G, we call G/N, the set of *cosets* of N, the **quotient group**.

Let's consider $\mathbb{Z}$ and $2\mathbb{Z}$ as examples. Here, $2\mathbb{Z}$ is the set of even numbers.

$$\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$$
$$2\mathbb{Z} = \{\cdots, -4, -2, 0, 2, 4, \cdots\}$$

What are the cosets of $2\mathbb{Z}$? Take several elements from $\mathbb{Z}$ to compute cosets. Then,

$$\begin{aligned}
0 + 2\mathbb{Z} &= \{\cdots, -4, -2, 0, 2, 4, \cdots\} \\
1 + 2\mathbb{Z} &= \{\cdots, -3, -1, 1, 3, 5, \cdots\} \\
2 + 2\mathbb{Z} &= \{\cdots, -2, 0, 2, 4, 6, \cdots\} = 0 + 2\mathbb{Z} \\
3 + 2\mathbb{Z} &= \{\cdots, -1, 1, 3, 5, 7, \cdots\} = 1 + 2\mathbb{Z}
\end{aligned}$$

We can see that $\mathbb{Z}$ is *partitioned* by $0 + 2\mathbb{Z}$ and $1 + 2\mathbb{Z}$! So we say

$$\boxed{\mathbb{Z}/2\mathbb{Z} \simeq \{0,1\}}$$

Definition 8.5 (Coset space)**.** Let G be a Lie group and H be a Lie subgroup of G. Define an

- **equivalence relation** $\sim$ by $g \sim g'$ if $\exists h \in H$ s.t. $g' = gh$
- **equivalence class** $[g] = \{gh \,|\, h \in H\} = gH$.

Then the **coset space** G/H is a manifold (not necessarily a Lie group) with $\dim G/H = \dim G - \dim H$.

Observation 8.6. G/H is a Lie group if $H \trianglelefteq G$[1].

- Take $gH, g'H \in G/H$. If the group structure is well-defined in G/H, the product $(gH)(g'H)$ must be independent of the choice of representatives.
- Let gh and $g'h'$ be the representatives of gH and $g'H$, respectively. Then

  $$ghg'h' = gg'h''h' \in (gg')H$$

  Since there exists $h'' \in H$ such that $h'' = (g')^{-1}h((g')^{-1})^{-1}$. Therefore, group multiplication is well-defined.
- Let gh and $g'h'$ be the representation of gH and $(gH)^{-1}$. Then $(gH)(gH)^{-1} = eH$ should hold.

  $$ghg'h' = gg'h''h' = \underbrace{gg'}_{e}\underbrace{h''h'}_{\in H} \implies (gH)^{-1} = g^{-1}H$$

  Hence, inverse element is well-defined.

[1]In other words, for arbitrary $g \in G$ and $h \in H$, $ghg^{-1} \in H$.

§ 8.2 Lie Algebras

Definition 8.7 (Translations)**.** Let a and g be elements of a Lie group G. The **right-(left-)translation** of g by a

$$R_a : G \to G, \quad R_a g = ga$$

and

$$L_a : G \to G, \quad L_a g = ag$$

are diffeomorphisms[2]. Hence, these maps induce

$$R_{a*} : T_g G \to T_{ga} G, \quad L_{a*} : T_g G \to T_{ag} G$$

Remark. Both right- and left-translation are suitable for the further discussion, but we will mainly consider the left-translations.

Observation 8.8. The diffeomorphism L_a takes the identity e to the element a, and induces $L_{a*,e} : T_e G \to T_a G$. Hence, if we can describe $T_e G$ at identity, then $L_{a*} T_e G$ will give a description of the tangent space $T_g G$ at any point $g \in G$.

Definition 8.9 (Left-invariant vector fields)**.** Let X be a vector field on a Lie group G. For any $g \in G$, because left-translation $L_a : G \to G$ is a diffeomorphism, the *pushforward* $L_{a*} X$ is a well-defined vector field on G. We say that the vector field X is **left-invariant** if

$$\boxed{L_{a*} X = X} \text{ or } \boxed{L_{a*} X|_g = X|_{ag}}$$

Observation 8.10. A left-invariant vector field X is completely determined by its value $X|_e$ at the identity, since

$$\boxed{X|_g = L_{g*} X|_e} \cdots (*)$$

Conversely, given a tangent vector $X|_e \in T_e G$, we can define a vector field X on G by $(*)$. So defined, the vector field X is left-invariant.

$$\begin{aligned} L_{a*} X|_g &= L_{a*} L_{g*} X|_e = (L_a \circ L_g)_* X|_e \quad (\because \text{ Exercise 5.9}) \\ &= (L_{ag})_* X|_e = X|_{ag} \end{aligned}$$

Thus, there is a **one-to-one-correspondence** $(X|_e \leftrightarrow X)$

$$T_e G \leftrightarrow \mathfrak{g} := \{\text{left-invariant vector fields on } G\}$$

Remark. If $X|_g = L_{g*} X|_e$ for all $g \in G$, we call X the *left-invariant vector field on G generated by $X|_e$*.

Remark. The map $T_e G : \mathfrak{g}$ defined by $V \mapsto X_V$ is an *isomorphism*: $\dim G = \dim \mathfrak{g}$. Hence, $\mathfrak{g}$ is a vector space isomorphic to $T_e G$.

[2] why?

Exercise 8.11. Verify that a left-invariant vector field X satisfies

$$\begin{aligned} L_{a*}X|_g &= X^\mu(g)\frac{\partial x^\nu(ag)}{\partial x^\mu(g)}\partial_\nu|_{ag} \quad (\because \text{ definition of pushforward}) \\ &= X^\nu(ag)\partial_\nu|_{ag} \end{aligned}$$

Proof. The exercise itself is the proof. ■

Observation 8.12. Since $\mathfrak{g}$ is a set of vector fields, $\mathfrak{g} \subseteq \mathscr{X}(G)$. Let's show that $\mathfrak{g}$ is closed under the *Lie bracket*. Take two points g and $ag = L_a g \in G$. Then,

$$L_{a*}[X,Y]|_g = [L_{a*}X|_g, L_{a*}Y|_g] = [X,Y]|_{ag}$$

by **Exercise 6.10(b)**. Hence, $[X,Y]$ is an another left-invariant vector field: $[X,Y] \in \mathfrak{g}$.

Definition 8.13 (Lie algebra)**.** The set of left-invariant vector fields $\mathfrak{g}$ with the Lie bracket

$$[\,,\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

is called the **Lie algebra** of a Lie group G.