6 Lie Derivatives

§ 6.2 Lie Derivatives (cont'd)

Exercise 6.9. Show that the Lie bracket satisfies ($c_1, c_2 \in \mathbb{R}$, $X_1, X_2, Y_1, Y_2, X, Y, Z \in \mathscr{X}(\mathcal{M})$)

(a) bilinearity:

$$[X, c_1 Y_1 + c_2 Y_2] = c_1[X, Y_1] + c_2[X, Y_2]$$
$$[c_1 X_1 + c_2 X_2, Y] = c_1[X_1, Y] + c_2[X_2, Y]$$

(b) skew-symmetry: $[X, Y] = -[Y, X]$

(c) the **Jacobi identity** $[[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] = 0$.

Proof. Straightforward. ■

Exercise 6.10. Let $X, Y \in \mathscr{X}(\mathcal{M})$ be vector fields on $\mathcal{M}$. Show that

(a) For $f \in \mathcal{F}(\mathcal{M})$,

$$\mathcal{L}_{fX} Y = f[X, Y] - Y[f]X \text{ and } \mathcal{L}_X(fY) = f[X, Y] + X[f]Y$$

(b) For $f : \mathcal{M} \to \mathcal{N}$,

$$f_*[X, Y] = [f_* X, f_* Y]$$

Proof. (a) For $g \in \mathcal{F}(\mathcal{M})$,

$$\mathcal{L}_{fX} Y[g] = fX[Y[g]] - Y[fX[g]] = fX[Y[g]] - Y[f]X[g] - fY[X[g]] = (f[X, Y] - Y[f]X)[g]$$
$$\mathcal{L}_X(fY)[g] = X[fY[g]] - fY[X[g]] = X[f]Y[g] + fX[Y[g]] - fY[X[g]] = (f[X, Y] + X[f]Y)[g]$$

(b) Let x^μ and y^ν denote the local coordinates in $\mathcal{M}$ and $\mathcal{N}$, respectively. In this setup,

$$\begin{aligned} f_*[X, Y] &= f_* \left(X^\mu \frac{\partial}{\partial x^\mu} Y^\nu - Y^\mu \frac{\partial}{\partial x^\mu} X^\nu \right) \frac{\partial}{\partial x^\nu} \\ &= \left(X^\mu \frac{\partial}{\partial x^\mu} Y^\nu - Y^\mu \frac{\partial}{\partial x^\mu} X^\nu \right) \frac{\partial y^\lambda}{\partial x^\nu} \cdot e_\lambda \quad (\leftarrow\ e_\lambda = \frac{\partial}{\partial x^\lambda}) \end{aligned}$$

Since

$$f_* X = \underbrace{\left[X^\nu \frac{\partial y^\alpha}{\partial x^\nu} \right]}_{(f_* X)^\alpha} \frac{\partial}{\partial y^\alpha} \text{ and } f_* Y = \underbrace{\left[Y^\mu \frac{\partial y^\beta}{\partial x^\mu} \right]}_{(f_* X)^\beta} \frac{\partial}{\partial y^\beta}$$

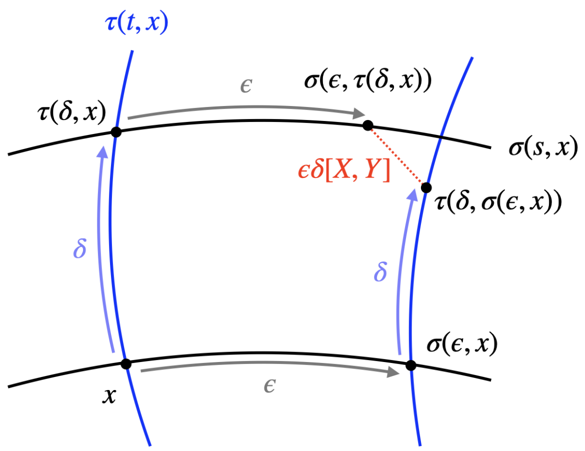
we have

$$\begin{aligned}
f_*[X,Y] &= \left[X^\mu \frac{\partial y^\alpha}{\partial x^\mu}\cdot\frac{\partial}{\partial y^\alpha}\left((f_*Y)^\beta\frac{\partial x^\nu}{\partial y^\beta}\right) - Y^\mu\frac{\partial y^\alpha}{\partial x^\mu}\cdot\frac{\partial}{\partial y^\alpha}\left((f_*X)^\beta\frac{\partial x^\nu}{\partial y^\beta}\right)\right]\frac{\partial y^\lambda}{\partial x^\nu}\cdot e_\lambda \\
&= \left[(f_*X)^\alpha\frac{\partial}{\partial y^\alpha}(f_*Y)^\beta - (f_*Y)^\alpha\frac{\partial}{\partial y^\alpha}(f_*X)^\beta\right]\underbrace{\frac{\partial x^\nu}{\partial y^\beta}\frac{\partial y^\lambda}{\partial x^\nu}}_{=\delta_\beta{}^\lambda}\cdot e_\lambda \\
&\qquad + \left[\cancel{(f_*X)^\alpha(f_*Y)^\beta\frac{\partial^2 x^\nu}{\partial y^\alpha\partial y^\beta}} - \cancel{(f_*Y)^\alpha(f_*X)^\beta\frac{\partial^2 x^\nu}{\partial y^\alpha\partial y^\beta}}\right]\frac{\partial y^\lambda}{\partial x^\nu}\cdot e_\lambda \\
&= [f_*X, f_*Y]
\end{aligned}$$

■

Observation 6.11 (Geometrical meaning of Lie brackets)**.**

$$\begin{aligned}
\text{Lie bracket} &= \text{non-commutativity of two flows} \\
&= \text{failure of the closure of the parallelogram.}
\end{aligned}$$



Consider two flows $\sigma(s,x)$ and $\tau(t,x)$ generated from $X, Y \in \mathscr{X}(\mathcal{M})$, respectively. Suppose that we move for ϵ along σ and then move for δ along τ.

$$\begin{aligned}
\tau^\mu(\delta,\sigma(\epsilon,x)) &\simeq \tau^\mu(\delta, x^\nu + \epsilon X^\nu(x)) \\
&\simeq x^\mu + \epsilon X^\mu(x)\delta Y^\mu(x^\nu + \epsilon X^\nu(x)) \\
&\simeq x^\mu + \epsilon X^\mu(x) + \delta Y^\mu(x) + \epsilon\delta X^\nu(x)\partial_\nu Y^\mu(x)
\end{aligned}$$

If we go for δ along τ first (and ϵ along σ later),

$$\sigma^\mu(\epsilon,\tau(\delta,x)) \simeq x^\mu + \epsilon X^\mu(x) + \delta Y^\mu(x) + \epsilon\delta Y^\nu(x)\partial_\nu X^\mu(x)$$

The *failure* of closure is

$$\tau^\mu(\delta,\sigma(\epsilon,x)) - \sigma^\mu(\epsilon,\tau(\delta,x)) = \epsilon\delta(X^\nu\partial_\nu Y^\mu - Y^\nu\partial_\nu X^\mu) = \boxed{\epsilon\delta[X,Y]^\mu}$$

Remark.

$$\mathcal{L}_X Y = [X,Y] = 0 \quad\Longleftrightarrow\quad \sigma(s,\tau(t,x)) = \tau(t,\sigma(s,x))$$

In other words, if two flows commute then Lie derivative vanishes.

Definition 6.12 (Lie derivative of one-forms)**.** The **Lie derivative of an one-form** $\omega \in \Omega^1(\mathcal{M})$[1] along $X \in \mathscr{X}(\mathcal{M})$ is defined by

$$\mathcal{L}_X\omega = \lim_{\epsilon\to 0}\frac{1}{\epsilon}[(\sigma_\epsilon)^*\omega|_{\sigma_\epsilon(x)} - \omega|_x]$$

where $(\sigma_\epsilon)^* : T^*_{\sigma_\epsilon(x)}\mathcal{M} \to T^*_x\mathcal{M}$ is a *pullback* map of σ_ϵ.[2]

Observation 6.13. Put $\omega = \omega_\mu \mathrm{d}x^\mu$. Then

$$\omega|_{\sigma_\epsilon(x)} \simeq \omega_\mu(x^\nu + \epsilon X^\nu(x))\,\mathrm{d}x^\mu|_{x+\epsilon X} \simeq [\omega_\mu(x) + \epsilon X^\nu(x)\partial_\nu\omega_\mu(x)]\,\mathrm{d}x^\mu|_{x+\epsilon X}$$

Applying the pullback map gives

$$\begin{aligned}(\sigma_\epsilon)^*\omega|_{\sigma_\epsilon(x)} &= [\omega_\mu(x) + \epsilon X^\nu(x)\partial_\nu\omega_\mu(x)]\underbrace{\frac{\partial(\sigma_\epsilon(x))^\alpha}{\partial x^\mu}}_{=\partial_\mu(x^\alpha+\epsilon X^\alpha(x))}\mathrm{d}x^\mu|_x \\ &= \omega_\mu \mathrm{d}x^\mu + \epsilon[X^\nu(x)\partial_\nu\omega_\mu(x) + \omega_\nu(x)\partial_\mu X^\nu(x)]\,\mathrm{d}x^\mu + \mathcal{O}(\epsilon^2)\end{aligned}$$

which leads to

$$\boxed{\mathcal{L}_X\omega = (X^\nu\partial_\nu\omega_\mu + \partial_\mu X^\nu\omega_\nu)\,\mathrm{d}x^\mu} \in T^*_x\mathcal{M}$$

Observation 6.14 (Lie derivative of smooth functions)**.** The **Lie derivative of smooth function** $f \in \mathcal{F}(\mathcal{M})$ along a flow σ generated by a vector field X is

$$\begin{aligned}\mathcal{L}_X f &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}[f(\sigma_\epsilon(x)) - f(x)] = \lim_{\epsilon\to 0}\frac{1}{\epsilon}[f(x^\mu + \epsilon X^\mu) - f(x)] \\ &= X^\mu\partial_\mu f = X[f]\end{aligned}$$

the usual directional derivative.

How can we compute the Lie derivative of *general* (q, r)-tensors?

Proposition 6.15. The Lie derivative satisfies $\mathcal{L}_X(t_1 + t_2) = \mathcal{L}_X t_1 + \mathcal{L}_X t_2$ where t_1 and t_2 are tensor fields of the same type. For any type of tensors t_1 and t_2, the following holds.

$$\mathcal{L}_X(t_1 \otimes t_2) = \mathcal{L}_X t_1 \otimes t_2 + t_1 \otimes \mathcal{L}_X t_2$$

Proof. We do not prove this proposition here - instead, we *embrace* it. ■

[1] $\Omega^1(\mathcal{M}) \equiv T^*_p(\mathcal{M})$.

[2] As the pushforward maps a vector from one tangent space to the other, pullback maps an one-form from one cotangent space to the other.

Example 6.16. Take $Y \in \mathscr{X}(\mathcal{M}), \omega \in \Omega^1(\mathcal{M})$ and construct $Y \otimes \omega$. The Lie derivative of this (1,1)-tensor is

$$\begin{aligned}
\mathcal{L}_X(Y \otimes \omega) &= \lim_{\epsilon \to 0} \frac{1}{\epsilon}[\{(\sigma_{-\epsilon}(x))_* Y \otimes (\sigma_\epsilon)^* \omega\}_{\sigma_\epsilon(x)} - (Y \otimes \omega)|_x] \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon}[(\sigma_{-\epsilon})_* Y \otimes \{(\sigma_\epsilon)^* \omega - \omega\} + \{(\sigma_{-\epsilon})_* Y - Y\} \otimes \omega] \\
&= Y \otimes \mathcal{L}_X \omega + \mathcal{L}_X Y \otimes \omega
\end{aligned}$$

For the general (1,1)-tensor $T = T_\mu{}^\nu \, \mathrm{d}x^\mu \otimes \dfrac{\partial}{\partial x^\nu} \in \mathfrak{T}^{(1,1)}(\mathcal{M})$,

$$\mathcal{L}_X T = X[T_\mu{}^\nu] \, \mathrm{d}x^\nu \otimes \frac{\partial}{\partial x^\nu} + T_\mu{}^\nu (\mathcal{L}_X \mathrm{d}x^\mu) \otimes \frac{\partial}{\partial x^\nu} + T_\mu{}^\nu \, \mathrm{d}x^\mu \otimes \left(\mathcal{L}_X \frac{\partial}{\partial x^\nu}\right)$$

Exercise 6.17. Let T be a tensor field. Show that

$$\mathcal{L}_{[X,Y]} T = \mathcal{L}_X \mathcal{L}_Y T - \mathcal{L}_Y \mathcal{L}_X T$$

Proof. First, note that

$$[X,Y]T = XYT - YXT = XYT - YTX + YTX - YXT = [X, YT] - Y[X,T]$$

Then

$$\begin{aligned}
\mathcal{L}_{[X,Y]} T &= [[X,Y],T] = [X,Y]T - T[X,Y] \\
&= [X,YT] - Y[X,T] - [X,TY] - [X,T]Y \\
&= XYT - YTX - Y[X,T] - XTY + TYX + [X,T]Y \\
&= X[Y,T] - [Y,T]X + [X,T]Y - Y[X,T] \\
&= [X,[Y,T]] - [Y,[X,T]] = \boxed{\mathcal{L}_X \mathcal{L}_Y T - \mathcal{L}_Y \mathcal{L}_X T}
\end{aligned}$$

■