

7 Differential Forms

§ 7.2 Exterior Derivatives

Definition 7.14 (Exterior derivative). The **exterior derivative** is a map $d_r : \Omega^r(\mathcal{M}) \rightarrow \Omega^{r+1}(\mathcal{M})$ such that

$$\begin{aligned} \omega &= \frac{1}{r!} \omega_{\mu_1 \dots \mu_r} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_r} \\ \mapsto \quad d_r \omega &\equiv \frac{1}{r!} \left(\frac{\partial}{\partial x^\nu} \right) dx^\nu \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_r} \end{aligned}$$

We usually drop the subscript r if there is no need of specification.

Remark. The exterior derivative maps r -forms to $r+1$ -forms. By definition $d_r \omega$ is properly and automatically antisymmetrized.

Example 7.15. There are the following r -forms in a three-dimensional space.

- (i) $\omega_0 = f(x, y, z) \in \Omega^0(\mathcal{M})$
- (ii) $\omega_1 = w_x(x, y, z) dx + w_y(x, y, z) dy + w_z(x, y, z) dz \in \Omega^1(\mathcal{M})$
- (iii) $\omega_2 = \omega_{xy}(x, y, z) dx \wedge dy + \omega_{yz}(x, y, z) dy \wedge dz + \omega_{zx}(x, y, z) dz \wedge dx \in \Omega^2(\mathcal{M})$
- (iv) $\omega_3 = \omega_{xyz}(x, y, z) dx \wedge dy \wedge dz \in \Omega^3(\mathcal{M})$

Digression. Do you remember the *axial vectors* (or often called as *pseudovectors*)¹?

$$\alpha^\mu = \epsilon^{\mu\nu\lambda} \omega_{\nu\lambda}$$

As you can see, a *two-form* may be regarded as a *vector*. The **Levi-Civita symbol** provides the isomorphism between $\mathcal{X}(\mathcal{M})$ and $\Omega^2(\mathcal{M})$.

The action of exterior derivative gives

- (i') $d\omega_0$ gives **gradient**.

$$d\omega_0 = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

- (ii') $d\omega_1$ gives **curl**.

$$d\omega_1 = \left(\frac{\partial \omega_y}{\partial x} - \frac{\partial \omega_x}{\partial y} \right) dx \wedge dy + \left(\frac{\partial \omega_z}{\partial y} - \frac{\partial \omega_y}{\partial z} \right) dy \wedge dz + \left(\frac{\partial \omega_x}{\partial z} - \frac{\partial \omega_z}{\partial x} \right) dz \wedge dx$$

- (iii') $d\omega_2$ gives **divergence**.

$$d\omega_2 = \left(\frac{\partial \omega_{yz}}{\partial x} + \frac{\partial \omega_{zx}}{\partial y} + \frac{\partial \omega_{xy}}{\partial z} \right) dx \wedge dy \wedge dz$$

- (iv') $d\omega_3$ vanishes: $d\omega_3 = 0$.

¹In contrast to *polar* vectors (or normal vectors).

Exercise 7.16. Let $\xi \in \Omega^q(\mathcal{M})$ and $\omega \in \Omega^r(\mathcal{M})$. Show that

$$d(\xi \wedge \omega) = d\xi \wedge \omega + (-1)^q \xi \wedge d\omega$$

Proof. Since

$$\xi \wedge \omega = \frac{1}{q!r!} \sum_{P \in S_{q+r}} \xi_{\mu_{P(1)} \dots \mu_{P(q)}} \omega_{\mu_{P(q+1)} \dots \mu_{P(q+r)}} dx^{\mu_{P(1)}} \wedge \dots \wedge dx^{\mu_{P(q)}} \wedge dx^{\mu_{P(q+1)}} \wedge \dots \wedge dx^{\mu_{P(q+r)}}$$

Applying the exterior derivative gives

$$d(\xi \wedge \omega) = \frac{1}{q!r!(q+r)!} \sum_{P \in S_{(q+r)}} \underbrace{\frac{\partial(\xi_{\mu_{P(1)} \dots \mu_{P(q)}} \omega_{\mu_{P(q+1)} \dots \mu_{P(q+r)}})}{\partial x^\nu} dx^\nu \wedge dx^{\mu_{P(1)}} \wedge \dots \wedge dx^{\mu_{P(q+r)}}}_{(*)}$$

Consider the terms inside the summation. It can be decomposed into

$$\begin{aligned} (*) &= \left[\frac{\partial \xi_{\mu_{P(1)} \dots \mu_{P(q)}}}{\partial x^\nu} dx^\nu \wedge dx^{\mu_{P(1)}} \wedge \dots \wedge dx^{\mu_{P(q)}} \right] \wedge \left[\omega_{\mu_{P(q+1)} \dots \mu_{P(q+r)}} dx^{\mu_{P(q+1)}} \wedge \dots \wedge dx^{\mu_{P(q+r)}} \right] \\ &\quad + (-1)^q \left[\xi_{\mu_{P(1)} \dots \mu_{P(q)}} dx^{\mu_{P(1)}} \wedge \dots \wedge dx^{\mu_{P(q)}} \right] \wedge \left[\frac{\partial \omega_{\mu_{P(q+1)} \dots \mu_{P(q+r)}}}{\partial x^\nu} dx^\nu \wedge dx^{\mu_{P(q+1)}} \wedge \dots \wedge dx^{\mu_{P(q+r)}} \right] \end{aligned}$$

Let's take a look at the first term of (*). Since

$$d\xi = \frac{1}{q!} \partial_\nu \xi_{\mu_1 \dots \mu_q} dx^\nu \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_q} \text{ and } \omega = \frac{1}{r!} \omega_{\mu_{q+1} \dots \mu_{q+r}} dx^{\mu_{q+1}} \wedge \dots \wedge dx^{\mu_{q+r}}$$

The wedge product between $d\xi$ and ω gives

$$\begin{aligned} d\xi \wedge \omega &= \frac{1}{q!r!(q+r+1)!} \sum_{P' \in S_{q+r+1}} \partial_{P'(v)} \xi_{\mu_{P'(1)} \dots \mu_{P'(q)}} \omega_{\mu_{P'(q+1)} \dots \mu_{P'(q+r)}} dx^{P'(v)} \wedge \dots \wedge dx^{\mu_{P'(q+r)}} \\ &= (q+r+1) \cdot \frac{1}{q!r!(q+r+1)!} \sum_{P \in S_{q+r}} \left[\frac{\partial \xi_{\mu_{P(1)} \dots \mu_{P(q)}}}{\partial x^\nu} dx^\nu \wedge dx^{\mu_{P(1)}} \wedge \dots \wedge dx^{\mu_{P(q)}} \right] \\ &\quad \wedge \left[\omega_{\mu_{P(q+1)} \dots \mu_{P(q+r)}} dx^{\mu_{P(q+1)}} \wedge \dots \wedge dx^{\mu_{P(q+r)}} \right] \end{aligned}$$

the first term in (*). For the second term, the same logic can be applied. Note that the term $(q+r+1)$ arises by excluding ν from the permutation in S_{q+r+1} . ■

Observation 7.17. Let $X = X^\mu \frac{\partial}{\partial x^\mu}$ and $Y = Y^\mu \frac{\partial}{\partial x^\mu}$ be vector fields and $\omega = \omega_\mu dx^\mu \in \Omega^1(\mathcal{M})$. Then

$$\begin{aligned} X[\omega(Y)] - Y[\omega(X)] - \omega([X, Y]) &= X^\mu \partial_\mu \langle \omega_\nu dx^\nu, Y^\lambda \partial_\lambda \rangle - Y^\mu \partial_\mu \langle \omega_\nu dx^\nu, X^\lambda \partial_\lambda \rangle - \langle \omega_\mu dx^\mu, X^\nu \partial_\nu Y^\lambda \partial_\lambda - Y^\nu \partial_\nu X^\lambda \partial_\lambda \rangle \\ &= X^\mu \partial_\mu (\omega_\nu Y^\nu) - Y^\mu \partial_\mu (\omega_\nu X^\nu) - \omega_\mu X^\nu \partial_\nu Y^\mu + \omega_\mu Y^\nu \partial_\nu X^\mu \\ &= X^\mu Y^\nu \partial_\mu \omega_\nu - X^\nu Y^\mu \partial_\mu \omega_\nu \\ &= \frac{\partial \omega_\nu}{\partial x^\mu} (X^\mu Y^\nu - X^\nu Y^\mu) = d\omega(X, Y) \end{aligned}$$

Note that $d\omega \in \Omega^2(\mathcal{M})$.

Remark. For an r -form $\omega \in \Omega^r(\mathcal{M})$,

$$\begin{aligned} d\omega(X_1, \dots, X_{r+1}) &= \sum_{i=1}^{r+1} (-1)^{i+1} X_i \omega(X_1, \dots, \hat{X}_i, \dots, X_{r+1}) \\ &\quad + \sum_{i < j} (-1)^{i+j} \omega([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{r+1}) \end{aligned}$$

The proof is left for the reader.

Observation 7.18. $d^2 = 0$. More explicitly, $d_{r+1}d_r = 0$. Take $\omega = \frac{1}{r!} \omega_{\mu_1 \dots \mu_r} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_r} \in \Omega^r(\mathcal{M})$. The action of d^2 gives

$$d^2\omega = \frac{1}{r!} \underbrace{\frac{\partial^2 \omega_{\mu_1 \dots \mu_r}}{\partial x^\nu \partial x^\lambda}}_{\text{sym. under } \nu \leftrightarrow \lambda} \underbrace{dx^\nu \wedge dx^\lambda}_{\text{antisym. under } \nu \leftrightarrow \lambda} \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_r} = 0$$

Definition 7.19. The **pullback** of an r -form is defined by

$$f^* : \Omega_{f(p)}^r(\mathcal{N}) \rightarrow \Omega_p^r(\mathcal{M}), \quad (f^*\omega)(X_1, \dots, X_r) = \omega(f_*X_1, \dots, f_*X_r)$$

where $f : \mathcal{M} \rightarrow \mathcal{N}$, $\omega \in \Omega^r(\mathcal{N})$ and $X_i \in T_p\mathcal{M}$.

Exercise 7.20. Let $\xi, \omega \in \Omega^r(\mathcal{N})$ and $f : \mathcal{M} \rightarrow \mathcal{N}$. Show that

$$(1) \quad d(f^*\omega) = f^*(d\omega)$$

$$(2) \quad f^*(\xi \wedge \omega) = (f^*\xi) \wedge (f^*\omega)$$

Proof.

$$(i) \quad d(f^*\omega)(X_1, \dots, X_r) = d\omega(f_*X_1, \dots, f_*X_r) = f^*d\omega(X_1, \dots, X_r)$$

(ii) Use the definition of wedge product:

$$\begin{aligned} f^*(\xi \wedge \omega)(X_1, \dots, X_{2r}) &= (\xi \wedge \omega)(f_*X_1, \dots, f_*X_{2r}) \\ &= \frac{1}{(r!)^2} \sum_{P \in S_{2r}} \text{sgn}(P) \xi(f_*X_{P(1)}, \dots, f_*X_{P(r)}) \omega(f_*X_{P(r+1)}, \dots, f_*X_{P(2r)}) \\ &= \frac{1}{(r!)^2} \sum_{P \in S_{2r}} \text{sgn}(P) f^*\xi(X_{P(1)}, \dots, X_{P(r)}) f^*\omega(X_{P(r+1)}, \dots, X_{P(2r)}) \\ &= (f^*\xi) \wedge (f^*\omega)(X_1, \dots, X_{2r}) \end{aligned}$$

■

Definition 7.21 (De Rham Complex and de Rham cohomology). The exterior derivative d_r induces the sequence (**de Rham complex**)

$$0 \xrightarrow{i} \Omega^0(\mathcal{M}) \xrightarrow{d_0} \Omega^1(\mathcal{M}) \xrightarrow{d_1} \dots \xrightarrow{d_{m-2}} \Omega^{m-1}(\mathcal{M}) \xrightarrow{d_{m-1}} \Omega^m(\mathcal{M}) \xrightarrow{d_m} 0$$

- $d^2 = 0$ implies $\text{im } d_r \subset \ker d_{r+1}$.

$$\omega \in \Omega^r(\mathcal{M}) \implies d_r\omega \in \text{im } d_r, \quad d_{r+1}(d_r\omega) = 0 \implies d_r\omega \in \ker d_{r+1}$$

- An element of $\ker d_r$ and $\text{im } d_{r-1}$ are called **closed r -form** and **exact r -form**, respectively.

- Namely, $\omega \in \Omega^r(\mathcal{M})$ is *closed* if $d_r\omega = 0$ and *exact* if there exists an $(r-1)$ -form ψ such that $\omega = d\psi$.

- The quotient space $\ker d_r / \text{im } d_{r-1}$ is called the r -th **de Rham cohomology group**.

We won't go further on this.