7 Differential Forms

§ 7.1 Differential Forms: Basic Definitions

Definition 7.1 (Permutation)**.** Let $\omega \in \mathfrak{T}_p^{(0,r)}(\mathcal{M})$. For a permutation $P \in S_r$[1] and $V_1, V_2, \cdots, V_r \in T_p\mathcal{M}$,

$$P\omega(V_1, V_2, \cdots, V_r) \equiv \omega(V_{P(1)}, V_{P(2)}, \cdots, V_{P(r)})$$

The component of $P\omega$ is given by

$$P\omega(e_{\mu_1}, e_{\mu_2}, \cdots, e_{\mu_r}) = \omega_{\mu_{P(1)}\mu_{P(2)}\cdots\mu_{P(r)}}$$

Definition 7.2 (Symmetrizer and antisymmetrizer)**.**

(1) The **symmetrizer** $\mathcal{S}$ is defined by

$$\mathcal{S}\omega \equiv \frac{1}{r!}\sum_{P\in S_r} P\omega$$

Note that $\mathcal{S}\omega$ is **totally symmetric**: $P\mathcal{S}\omega = \mathcal{S}\omega$ for arbitrary $P \in S_r$.

(2) The **antisymmetrizer** $\mathcal{A}$ is defined by

$$\mathcal{A}\omega \equiv \frac{1}{r!}\sum_{P\in S_r} \mathrm{sgn}(P)P\omega$$

where $\mathrm{sgn}(P) = +1$ for even permutations and $\mathrm{sgn}(P) = -1$ for odd permutations. Note that $\mathcal{A}\omega$ is **totally antisymmetric**: $P\mathcal{A}\omega = \mathrm{sgn}(P)\mathcal{A}\omega$ for arbitrary $P \in S_r$.

Definition 7.3 (Differential forms)**.** A **differential form of order** r (or an r**-form**) is a totally antisymmetric tensor of type $(0, r)$. The vector space of r-forms in $p \in \mathcal{M}$ is denoted by $\Omega_p^r(\mathcal{M})$.

Remark. One-form like $\mathrm{d}x^\mu$ is a differential form of order 1.

Then, how can we construct r-forms from r one-forms?

Definition 7.4 (Wedge product)**.** The **wedge product** $\wedge$ of r one-forms is defined by the totally antisymmetric tensor product.

$$\mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_r} \equiv \sum_{P\in S_r} \mathrm{sgn}(P)\, \mathrm{d}x^{\mu_{P(1)}} \otimes \cdots \otimes \mathrm{d}x^{\mu_{P(r)}}$$

[1] Here, S_r denotes the *symmetric group of order* r.

Example 7.5. Let's construct 2-form and 3-form from the one-form $\mathrm{d}x^\mu$.

(1) Since $|S_2| = 2$, two terms emerge.

$$\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu - \mathrm{d}x^\nu \otimes \mathrm{d}x^\mu$$

(2) Since $|S_3| = 6$,

$$\begin{aligned}\mathrm{d}x^\lambda \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = {} & \mathrm{d}x^\lambda \otimes \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu - \mathrm{d}x^\lambda \otimes \mathrm{d}x^\nu \otimes \mathrm{d}x^\mu \\ & + \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu \otimes \mathrm{d}x^\lambda - \mathrm{d}x^\mu \otimes \mathrm{d}x^\lambda \otimes \mathrm{d}x^\nu \\ & + \mathrm{d}x^\nu \otimes \mathrm{d}x^\lambda \otimes \mathrm{d}x^\mu - \mathrm{d}x^\nu \otimes \mathrm{d}x^\mu \otimes \mathrm{d}x^\lambda\end{aligned}$$

Observation 7.6. The wedge product satisfies:

(1) $\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_r} = 0$ if some index μ_i appears at least twice.

(2) $\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_r} = \mathrm{sgn}(P)\mathrm{d}x^{\mu_{P(1)}} \wedge \cdots \wedge \mathrm{d}x^{\mu_{P(r)}}$ (the constructed r-form is totally antisymmetric).

(3) $\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_r}$ is linear in each $\mathrm{d}x^\mu$ (since r-forms are $(0, r)$-tensors).

Observation 7.7. The set of r-forms $\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_r}$ forms a basis of $\Omega^r_p(\mathcal{M})$ and for $\omega \in \Omega^r_p(\mathcal{M})$,

$$\begin{aligned}\omega &= \omega_{\mu_1\cdots\mu_r}\, \mathrm{d}x^{\mu_1} \otimes \mathrm{d}x^{\mu_2} \otimes \cdots \otimes \mathrm{d}x^{\mu_r} \\ &= \boxed{\frac{1}{r!}\omega_{\mu_1\cdots\mu_r}\, \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_r}}\end{aligned}$$

Observation 7.8. Consider a (0,2)-tensor $\omega_{\mu\nu}$. This tensor can be decomposed into symmetric and anti-symmetric parts: the symmetric part

$$\sigma_{\mu\nu} = \frac{\omega_{\mu\nu} + \omega_{\nu\mu}}{2} \quad \to \quad \sigma_{\mu\nu}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = 0$$

and antisymmetric part

$$\alpha_{\mu\nu} = \frac{\omega_{\mu\nu} - \omega_{\nu\mu}}{2} \quad \to \quad \alpha_{\mu\nu}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \frac{\omega_{\mu\nu} - \omega_{\nu\mu}}{2}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \omega_{\mu\nu}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$$

Observe that only antisymmetric part of $\omega_{\mu\nu}$ can contribute to two-form.

Observation 7.9. Since there are $\binom{m}{r}$ choices in the set $(\mu_1, \cdots, \mu_r)$ from $(1, \cdots, m)$,

$$\dim \Omega_p^r(\mathcal{M}) = \binom{m}{r}$$

- Define $\Omega_p^0(\mathcal{M}) \equiv \mathbb{R}$.
- $\Omega_p^1(\mathcal{M}) = T_p^*\mathcal{M}$ ($\dim T_p^*\mathcal{M} = m$).
- If $r \ngeq m$, some index appears at least twice in the antisymmetrical sum, so such differential form vanishes.
- Since $\binom{m}{r} = \binom{m}{m-r}$, $\dim \Omega_p^r(\mathcal{M}) = \dim \Omega_p^{m-r}(\mathcal{M})$.
- Moreover, since $\Omega_p^r(\mathcal{M})$ is a vector space, $\Omega_p^r(\mathcal{M}) \simeq \Omega_p^{m-r}(\mathcal{M})$.

Until now, we used *wedge product* to construct r-forms. Can we make higher-order forms by combining q-forms and r-forms? Yes.

Definition 7.10. The **exterior product** of a q-form and a r-form

$$\wedge : \Omega_p^q(\mathcal{M}) \times \Omega_p^r(\mathcal{M}) \to \Omega_p^{q+r}(\mathcal{M})$$

is defined by a trivial extension. For $\omega \in \Omega_p^q(\mathcal{M})$ and $\xi \in \Omega_p^r(\mathcal{M})$, the action of the $(q+r)$-form $\omega \wedge \xi$ on $(q+r)$ vectors is defined by

$$(\omega \wedge \xi)(V_1, \cdots, V_{q+r}) = \frac{1}{q!r!} \sum_{P \in S_{q+r}} \mathrm{sgn}(P)\omega(V_{P(1)}, \cdots, V_{P(q)})\xi(V_{P(q+1)}, \cdots, V_{P(q+r)})$$

Remark. If $q + r > m$, $\omega \wedge \xi$ vanishes naturally.

Remark. With this exterior product, we define an algebra

$$\Omega_p^*(\mathcal{M}) \equiv \Omega_p^0(\mathcal{M}) \oplus \Omega_p^1(\mathcal{M}) \oplus \cdots \oplus \Omega_p^m(\mathcal{M})$$

$\Omega_p^*(\mathcal{M})$ denotes the space of all differential forms at p. This space is closed under the exterior product. Moreover, we may assign an r-form *smoothly* at each point on a manifold $\mathcal{M}$. We denote the space of smooth r-forms on $\mathcal{M}$ by $\Omega^r(\mathcal{M})$.

r-form	Basis	Dimension
$\Omega^0(\mathcal{M}) = \mathcal{F}(\mathcal{M})$	$\{1\}$	1
$\Omega^1(\mathcal{M}) = T^*\mathcal{M}$	$\{\mathrm{d}x^\mu\}$	m
$\Omega^2(\mathcal{M})$	$\{\mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2}\}$	$m(m-1)/2!$
$\Omega^3(\mathcal{M})$	$\{\mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \mathrm{d}x^{\mu_3}\}$	$m(m-1)(m-2)/3!$
$\vdots$	$\vdots$	$\vdots$
$\Omega^m(\mathcal{M})$	$\{\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m\}$	1

Table 1: The space of smooth r-forms.

Exercise 7.11. Consider a Cartesian coordinates (x, y) in $\mathbb{R}^2$. Show that

$$\mathrm{d}x \wedge \mathrm{d}y = r\,\mathrm{d}r \wedge \mathrm{d}\theta$$

Proof.

$$\begin{aligned}
\mathrm{d}x \wedge \mathrm{d}y &= \mathrm{d}x \otimes \mathrm{d}y - \mathrm{d}y \otimes \mathrm{d}x \\
&= (\cos\theta\,\mathrm{d}r - r\sin\theta\,\mathrm{d}\theta) \otimes (\sin\theta\,\mathrm{d}r + r\cos\theta\,\mathrm{d}\theta) \\
&\qquad - (\sin\theta\,\mathrm{d}r + r\cos\theta\,\mathrm{d}\theta) \otimes (\cos\theta\,\mathrm{d}r - r\sin\theta\,\mathrm{d}\theta) \\
&= r\,dr \otimes \mathrm{d}\theta - r\,\mathrm{d}\theta \otimes \mathrm{d}r = r\,\mathrm{d}r \wedge \mathrm{d}\theta
\end{aligned}$$

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We conclude this lecture by proving several important properties of exterior products. Before we do that, we prove this useful lemma first.

Lemma 7.12. *Let $\xi \in \Omega^q_p(\mathcal{M})$ and $\eta \in \Omega^r_p(\mathcal{M})$. Then*

$$\mathcal{A}(\mathcal{A}(\xi) \otimes \eta) = \mathcal{A}(\xi \otimes \eta)$$

Proof. By definition,

$$\mathcal{A}(\mathcal{A}(\xi) \otimes \eta) = \frac{1}{(q+r)!} \sum_{\sigma \in S_{q+r}} \mathrm{sgn}(\sigma)\sigma \left(\frac{1}{q!} \sum_{\tau \in S_q} \mathrm{sgn}(\tau)\tau\xi \otimes \eta \right)$$

If we interpret $\tau \in S_q$ as a permutation in S_{q+r} such that $\tau(i) = i$ $(i = q+1, \cdots, q+r)$, $\tau\xi \otimes \eta = \tau(\xi \otimes \eta)$.

$$= \frac{1}{q!(q+r)!} \sum_{\sigma \in S_{q+r}} \sum_{\tau \in S_q} \mathrm{sgn}(\sigma)\mathrm{sgn}(\tau)(\sigma\tau)(\xi \otimes \eta)$$

Let $\mu = \sigma\tau \in S_{q+r}$. For each μ, there are $q!$ ways to write $\mu = \sigma\tau$ with $\sigma \in S_{q+r}$ and $\tau \in S_q$, because each $\tau \in S_q$ determines a unique σ by $\sigma = \mu\tau^{-1}$. Hence,

$$= q! \cdot \frac{1}{q!(q+r)!} \sum_{\mu \in S_{q+r}} \mathrm{sgn}(\mu)\mu(\xi \otimes \eta) = \mathcal{A}(\xi \otimes \eta)$$

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Exercise 7.13. Let $\xi \in \Omega_p^q(\mathcal{M})$, $\eta \in \Omega_p^r(\mathcal{M})$ and $\omega \in \Omega_p^s(\mathcal{M})$. Show that

(1) $\xi \wedge \xi = 0$ if q is odd.

(2) $\xi \wedge \eta = (-1)^{qr} \eta \wedge \xi$.

(3) $(\xi \wedge \eta) \wedge \omega = \xi \wedge (\eta \wedge \omega)$.

Proof.

(1) Let $V_1, \cdots, V_{2q} \in T_p\mathcal{M}$.

$$\begin{aligned}(\xi \wedge \xi)(V_1, \cdots, V_{2q}) &= \frac{1}{(q!)^2} \sum_{P \in S_{2q}} \operatorname{sgn}(P) \xi(V_{P(1)}, \cdots, V_{P(q)}) \xi(V_{P(q+1)}, \cdots, V_{P(2q)}) \\ &= -\frac{1}{(q!)^2} \sum_{P \in S_{2q}} \operatorname{sgn}(P) \xi(V_{P(q+1)}, \cdots, V_{P(2q)}) \xi(V_{P(1)}, \cdots, V_{P(q)}) = 0\end{aligned}$$

Here, changing permutation

$$(P(1), \cdots, P(q), P(q+1), \cdots, P(2q))$$

to

$$(P(q+1), \cdots, P(2q), P(1), \cdots, P(q))$$

requires q^2 swaps. Hence, if q is odd, additional (-1) factor arises.

(2) In the exactly same manner, you can show that $(-1)^{qr}$ factor arises from changing the order of two forms.

(3) By definition,

$$\begin{aligned}(\xi \wedge \eta) \wedge \omega) &= \frac{(q+r+s)!}{(q+r)!s!} \mathcal{A}((\xi \wedge \eta) \otimes \omega) \\ &= \frac{(q+r+s)!}{(q+r)!s!} \frac{(q+r)!}{q!r!} \mathcal{A}(\mathcal{A}(\xi \otimes \eta) \otimes \omega) \\ &= \frac{(q+r+s)!}{q!r!s!} \mathcal{A}(\xi \otimes \eta \otimes \omega) \quad (\because \textbf{Lemma 7.12})\end{aligned}$$

You can yield same expression for $\xi \wedge (\eta \wedge \omega)$.

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