7 Differential Forms

§ 7.2 Exterior Derivatives

Definition 7.14 (Exterior derivative)**.** The **exterior derivative** is a map $\mathrm{d}_r : \Omega^r(\mathcal{M}) \to \Omega^{r+1}(\mathcal{M})$ such that

$$\begin{aligned}\omega =& \frac{1}{r!}\omega_{\mu_1\cdots\mu_r}\, dx^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_r} \\ &\mapsto \quad \mathrm{d}_r\omega \equiv \frac{1}{r!}\left(\frac{\partial}{\partial x^\nu}\omega_{\mu_1\cdots\mu_r}\right) \mathrm{d}x^\nu \wedge \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_r}\end{aligned}$$

We usually drop the subscript r if there is no need of specification.

Remark. The exterior derivative maps r-forms to $r+1$-forms. By definition $\mathrm{d}_r\omega$ is properly and automatically antisymmetrized.

Example 7.15. There are the following r-forms in a three-dimensional space.

(i) $\omega_0 = f(x,y,z) \in \Omega^0(\mathcal{M})$

(ii) $\omega_1 = w_x(x,y,z)\,\mathrm{d}x + \omega_y(x,y,z)\,\mathrm{d}y + \omega_z(x,y,z)\,\mathrm{d}z \in \Omega^1(\mathcal{M})$

(iii) $\omega_2 = \omega_{xy}(x,y,z)\,\mathrm{d}x \wedge \mathrm{d}y + \omega_{yz}(x,y,z)\,\mathrm{d}y \wedge \mathrm{d}z + \omega_{zx}(x,y,z)\,\mathrm{d}z \wedge \mathrm{d}x \in \Omega^2(\mathcal{M})$

(iv) $\omega_3 = \omega_{xyz}(x,y,z)\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \in \Omega^3(\mathcal{M})$

Digression. Do you remember the *axial vectors* (or often called as *pseudovectors*)[1]?

$$\alpha^\mu = \epsilon^{\mu\nu\lambda}\omega_{\nu\lambda}$$

As you can see, a *two-form* may be regarded as a *vector*. The **Levi-Civita symbol** provides the isomorphism between $\mathscr{X}(\mathcal{M})$ and $\Omega^2(\mathcal{M})$.

The action of exterior derivative gives

(i') $\mathrm{d}\omega_0$ gives **gradient**.

$$\mathrm{d}\omega_0 = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y + \frac{\partial f}{\partial z}\,\mathrm{d}z$$

(ii') $\mathrm{d}\omega_1$ gives **curl**.

$$\mathrm{d}\omega_1 = \left(\frac{\partial\omega_y}{\partial x} - \frac{\partial\omega_x}{\partial y}\right)\mathrm{d}x \wedge \mathrm{d}y + \left(\frac{\partial\omega_z}{\partial y} - \frac{\partial\omega_y}{\partial z}\right)\mathrm{d}y \wedge \mathrm{d}z + \left(\frac{\partial\omega_x}{\partial z} - \frac{\partial\omega_z}{\partial x}\right)\mathrm{d}z \wedge \mathrm{d}x$$

(iii') $\mathrm{d}\omega_2$ gives **divergence**.

$$\mathrm{d}\omega_2 = \left(\frac{\partial\omega_{yz}}{\partial x} + \frac{\partial\omega_{zx}}{\partial y} + \frac{\partial\omega_{xy}}{\partial z}\right)\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$$

(iv') $\mathrm{d}\omega_3$ vanishes: $\mathrm{d}\omega_3 = 0$.

[1] In contrast to *polar* vectors (or normal vectors).

Exercise 7.16. Let $\xi \in \Omega^q(\mathcal{M})$ and $\omega \in \Omega^r(\mathcal{M})$. Show that

$$\mathrm{d}(\xi \wedge \omega) = \mathrm{d}\xi \wedge \omega + (-1)^q \xi \wedge \mathrm{d}\omega$$

Proof. Since

$$\xi \wedge \omega = \frac{1}{q!r!} \sum_{P \in S_{q+r}} \xi_{\mu_{P(1)} \cdots \mu_{P(q)}} \omega_{\mu_{P(q+1)} \cdots \mu_{P(q+r)}} \, \mathrm{d}x^{\mu_{P(1)}} \wedge \cdots \wedge \mathrm{d}x^{\mu_{P(q)}} \wedge \mathrm{d}x^{\mu_{P(q+1)}} \wedge \cdots \mathrm{d}x^{\mu_{P(q+r)}}$$

Applying the exterior derivative gives

$$\mathrm{d}(\xi \wedge \omega) = \frac{1}{q!r!(q+r)!} \sum_{P \in S(q+r)} \underbrace{\frac{\partial(\xi_{\mu_{P(1)} \cdots \mu_{P(q)}} \omega_{\mu_{P(q+1)} \cdots \mu_{P(q+r)}})}{\partial x^\nu} \, \mathrm{d}x^\nu \wedge \mathrm{d}x^{\mu_{P(1)}} \wedge \cdots \mathrm{d}x^{\mu_{P(q+r)}}}_{(*)}$$

Consider the terms inside the summation. It can be decomposed into

$$\begin{aligned} (*) = &\left[\frac{\partial \xi_{\mu_{P(1)} \cdots \mu_{P(q)}}}{\partial x^\nu} \, \mathrm{d}x^\nu \wedge \mathrm{d}x^{\mu_{P(1)}} \wedge \cdots \wedge \mathrm{d}x^{\mu_{P(q)}}\right] \wedge \left[\omega_{\mu_{P(q+1)} \cdots \mu_{P(q+r)}} \, \mathrm{d}x^{\mu_{P(q+1)}} \wedge \cdots \mathrm{d}x^{\mu_{P(q+r)}}\right] \\ &+ (-1)^q \left[\xi_{\mu_{P(1)} \cdots \mu_{P(q)}} \, \mathrm{d}x^{\mu_{P(1)}} \wedge \cdots \wedge \mathrm{d}x^{\mu_{P(q)}}\right] \wedge \left[\frac{\partial \omega_{\mu_{P(q+1)} \cdots \mu_{P(q+r)}}}{\partial x^\nu} \, \mathrm{d}x^\nu \wedge \mathrm{d}x^{\mu_{P(q+1)}} \wedge \cdots \mathrm{d}x^{\mu_{P(q+r)}}\right] \end{aligned}$$

Let's take a look at the first term of $(*)$. Since

$$\mathrm{d}\xi = \frac{1}{q!} \partial_\nu \xi_{\mu_1 \cdots \mu_q} \, \mathrm{d}x^\nu \wedge \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_q} \text{ and } \omega = \frac{1}{r!} \omega_{\mu_{q+1} \cdots \mu_{q+r}} \, \mathrm{d}x^{\mu_{q+1}} \wedge \cdots \wedge \mathrm{d}x^{\mu_{q+r}}$$

The wedge product between $\mathrm{d}\xi$ and ω gives

$$\begin{aligned} \mathrm{d}\xi \wedge \omega &= \frac{1}{q!r!(q+r+1)!} \sum_{P' \in S_{q+r+1}} \partial_{P'(\nu)} \xi_{\mu_{P'(1)} \cdots \mu_{P'(q)}} \omega_{\mu_{P'(q+1)} \cdots \mu_{P'(q+r)}} \, \mathrm{d}x^{P'(\nu)} \wedge \cdots \wedge \mathrm{d}x^{\mu_{P'(q+r)}} \\ &= (q+r+1) \cdot \frac{1}{q!r!(q+r+1)!} \sum_{P \in S_{q+r}} \left[\frac{\partial \xi_{\mu_{P(1)} \cdots \mu_{P(q)}}}{\partial x^\nu} \, \mathrm{d}x^\nu \wedge \mathrm{d}x^{\mu_{P(1)}} \wedge \cdots \wedge \mathrm{d}x^{\mu_{P(q)}}\right] \\ &\qquad \wedge \left[\omega_{\mu_{P(q+1)} \cdots \mu_{P(q+r)}} \, \mathrm{d}x^{\mu_{P(q+1)}} \wedge \cdots \mathrm{d}x^{\mu_{P(q+r)}}\right] \end{aligned}$$

the first term in $(*)$. For the second term, the same logic can be applied. Note that the term $(q+r+1)$ arises by excluding ν from the permutation in S_{q+r+1}. ■

Observation 7.17. Let $X = X^\mu \dfrac{\partial}{\partial x^\mu}$ and $Y = Y^\mu \dfrac{\partial}{\partial x^\mu}$ be vector fields and $\omega = \omega_\mu \, dx^\mu \in \Omega^1(\mathcal{M})$. Then

$$
\begin{aligned}
X[\omega(Y)] - Y[\omega(X)] - \omega([X,Y]) \\
&= X^\mu \partial_\mu \left\langle \omega_\nu \, dx^\nu, Y^\lambda \partial_\lambda \right\rangle - Y^\mu \partial_\mu \left\langle \omega_\nu \, dx^\nu, X^\lambda \partial_\lambda \right\rangle - \left\langle \omega_\mu \, dx^\mu, X^\nu \partial_\nu Y^\lambda \partial\lambda - Y^\nu \partial_\nu X^\lambda \partial_\lambda \right\rangle \\
&= X^\mu \partial_\mu (\omega_\nu Y^\nu) - Y^\mu \partial_\mu (\omega_\nu X^\nu) - \omega_\mu X^\nu \partial_\nu Y^\mu + \omega_\mu Y^\nu \partial_\nu X^\mu \\
&= X^\mu Y^\nu \partial_\mu \omega_\nu - X^\nu Y^\mu \partial_\mu \omega_\nu \\
&= \frac{\partial \omega_\nu}{\partial x^\mu} (X^\mu Y^\nu - X^\nu Y^\mu) = d\omega(X, Y)
\end{aligned}
$$

Note that $d\omega \in \Omega^2(\mathcal{M})$.

Remark. For an r-form $\omega \in \Omega^r(\mathcal{M})$,

$$
\begin{aligned}
d\omega(X_1, \cdots, X_{r+1}) = \sum_{i=1}^{r+1} (-1)^{i+1} X_i \omega(X_1, \cdots, \hat{X}_i, \cdots, X_{r+1}) \\
+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_{r+1})
\end{aligned}
$$

The proof is left for the reader.

Observation 7.18. $d^2 = 0$. More explicitly, $d_{r+1} d_r = 0$. Take $\omega = \dfrac{1}{r!} \omega_{\mu_1 \cdots \mu_r} \, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r} \in \Omega^r(\mathcal{M})$. The action of d^2 gives

$$
d^2 \omega = \frac{1}{r!} \underbrace{\frac{\partial^2 \omega_{\mu_1 \cdots \mu_r}}{\partial x^\nu \partial x^\lambda}}_{\text{sym. under} \nu \leftrightarrow \lambda} \underbrace{dx^\nu \wedge dx^\lambda}_{\text{antisym. under} \nu \leftrightarrow \lambda} \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r} = 0
$$

Definition 7.19. The **pullback** of an r-form is defined by

$$
f^* : \Omega^r_{f(p)}(\mathcal{N}) \to \Omega^r_p(\mathcal{M}), \ (f^* \omega)(X_1, \cdots, X_r) = \omega(f_* X_1, \cdots, f_* X_r)
$$

where $f : \mathcal{M} \to \mathcal{N}$, $\omega \in \Omega^r(\mathcal{N})$ and $X_i \in T_p \mathcal{M}$.

Exercise 7.20. Let $\xi, \omega \in \Omega^r(\mathcal{N})$ and $f : \mathcal{M} \to \mathcal{N}$. Show that

(1) $\mathrm{d}(f^*\omega) = f^*(\mathrm{d}\omega)$

(2) $f^*(\xi \wedge \omega) = (f^*\xi) \wedge (f^*\omega)$

Proof.

(i) $\mathrm{d}(f^*\omega)(X_1, \cdots, X_r) = \mathrm{d}\omega(f_*X_1, \cdots, f_*X_r) = f^*\mathrm{d}\omega(X_1, \cdots, X_r)$

(ii) Use the definition of wedge product:

$$\begin{aligned} f^*(\xi \wedge \omega)(X_1, \cdots, X_{2r}) &= (\xi \wedge \omega)(f_*X_1, \cdots, f_*X_{2r}) \\ &= \frac{1}{(r!)^2} \sum_{P \in S_{2r}} \mathrm{sgn}(P)\xi(f_*X_{P(1)}, \cdots, f_*X_{P(r)})\omega(f_*X_{P(r+1)}, \cdots, f_*X_{P(2r)}) \\ &= \frac{1}{(r!)^2} \sum_{P \in S_{2r}} \mathrm{sgn}(P)f^*\xi(X_{P(1)}, \cdots, X_{P(r)})f^*\omega(X_{P(r+1)}, \cdots, X_{P(2r)}) \\ &= (f^*\xi) \wedge (f^*\omega)(X_1, \cdots, X_{2r}) \end{aligned}$$

■

Definition 7.21 (De Rham Complex and de Rham cohomology)**.** The exterior derivative d_r induces the sequence (**de Rham complex**)

$$0 \xrightarrow{i} \Omega^0(\mathcal{M}) \xrightarrow{\mathrm{d}_0} \Omega^1(\mathcal{M}) \xrightarrow{\mathrm{d}_1} \cdots \xrightarrow{\mathrm{d}_{m-2}} \Omega^{m-1}(\mathcal{M}) \xrightarrow{\mathrm{d}_{m-1}} \Omega^m(\mathcal{M}) \xrightarrow{\mathrm{d}_m} 0$$

- $\mathrm{d}^2 = 0$ implies $\mathrm{im}\,\mathrm{d}_r \subset \ker \mathrm{d}_{r+1}$.

$$\omega \in \Omega^r(\mathcal{M}) \implies \mathrm{d}_r\omega \in \mathrm{im}\,\mathrm{d}_r, \quad \mathrm{d}_{r+1}(\mathrm{d}_r\omega) = 0 \implies \mathrm{d}_r\omega \in \ker \mathrm{d}_{r+1}$$

- An element of $\ker \mathrm{d}_r$ and $\mathrm{im}\,\mathrm{d}_{r-1}$ are called **closed r-form** and **exact r-form**, respectively.

- Namely, $\omega \in \Omega^r(\mathcal{M})$ is *closed* if $\mathrm{d}_r\omega = 0$ and *exact* if there exists an $(r-1)$-form ψ such that $\omega = \mathrm{d}\psi$.

- The quotient space $\ker \mathrm{d}_r / \mathrm{im}\,\mathrm{d}_{r-1}$ is called the r-th **de Rham cohomology group**.

We won't go further on this.