1 Vectors and Vector Spaces

§ 1.1 Vector Space: Definitions

Definition 1.1 (Vector spaces)**.** Let F be a field (such as $\mathbb{R}$, $\mathbb{C}$, etc.). Suppose that there is a set V equipped with addition $+ : V \times V \to V$ and scalar multiplication $\cdot : F \times V \to V$. If V satisfies the following 8 properties (for $a, b \in F$ and $u, v, w \in V$),

(1) $(u + v) + w = u + (v + w)$

(2) $v + w = w + v$

(3) $\exists 0 \in V$ such that $v + 0 = 0 + v = v$

(4) For given $v \in V, \exists -v \in V$ such that $v + (-v) = 0$

(5) $(a + b)v = av + bv$

(6) $a(v + w) = av + aw$

(7) $a(bv) = (ab)v$

(8) $1 \cdot v = v$

then V is a **vector space** over F. Elements of F and V are called **scalars** and **vectors**, respectively.

Sets such as F^n (or something like $\mathbb{R}^n$) are familiar example of vector spaces.

Exercise 1.2. Prove the following.

(1) The **identity element** $0 \in V$ exists uniquely.

(2) For $v \in V$, **inverse element** $-v \in V$ exists uniquely.

Proof. (1) Let $0'$ be another identity element. Then

$$\begin{aligned} 0 &= 0 + 0' \quad (0' \text{ is an identity}) \\ &= 0' \quad (0 \text{ is an identity}) \end{aligned}$$

(2) Let w be another inverse element ($v + w = 0$). Then

$$-v = -v + 0 = -v + (v + w) = (-v + v) + w = 0 + w = w$$

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Exercise 1.3. Prove the following. $(v \in V, a \in F)$

(1) $0v = 0$

(2) $a0 = 0$

(3) $-v = (-1)v$

(4) If $v \neq 0$ and $a \neq 0$, then $av \neq 0$.

Proof. (1) $0v = (0+0)v = 0v + 0v \implies 0v = 0$

(2) $a0 = a(0+0) = a0 + a0 \implies a0 = 0$

(3) $v + (-1)v = 1v + (-1)v = (1-1)v = 0v = 0 \implies -v = (-1)v$

(4) Suppose that $av = 0$. Then

$$v = 1v = \left(\frac{1}{a} \cdot a\right) v = \frac{1}{a}(av) = \frac{1}{a}0 = 0 \quad \Rightarrow\Leftarrow$$

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Definition 1.4 (Subspaces)**.** Let W be a subset of F-vector space V. If W becomes a F-vector space for the addition and scalar multiplication operations inherited from V, W is a **subspace** of V. We denote this by $W \leq V$.

Observation 1.5. For $\varnothing \neq W \subseteq V$, $W \leq V$ if and only if

(1) $w_1, w_2 \in W \implies w_1 + w_2 \in W$

(2) $a \in F,\ w \in W \implies aw \in W$

The proof is almost trivial (you may check it by yourself). In short, W becomes a subspace of V if it is closed for the addition and scalar multiplication.

Lastly, we introduce an extremely important concept.

Definition 1.6 (Isomorphisms)**.** Let V and V' be F-vector spaces. If there exists a bijection $\varphi : V \to V'$ such that

$$\varphi(v + w) = \varphi(v) + \varphi(w), \quad \varphi(av) = a\varphi(v) \quad (a \in F,\ v, w \in V)$$

then V and V' are **isomorphic** ($V \simeq V'$). The bijection φ is called an **isomorphism**.

Remark. Isomorphism φ preserves the addition and scalar multiplication, as you can see in its definition. Moreover, if there is an isomorphism between two vector spaces, those two spaces are **identical** *de facto*.

Definition 1.7. Basis of a vector space A non-empty subset of V, $\mathfrak{B} \subseteq V$ is called a **basis** of V if it satisfies the following properties.

(1) $\langle \mathfrak{B} \rangle = V$, where $\langle \mathfrak{B} \rangle$ denotes the set of all linear combinations of vectors in $\mathfrak{B}$. In other words, $\mathfrak{B}$ generates (or **spans**) V.

(2) $\mathfrak{B}$ is linearly independent.

Remark. When we say a set of vectors $v_1, \cdots, v_n$ are *linearly independent*, this implies

$$\text{If } a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0, \text{ then all } a_i = 0 \ (i = 1, 2, \cdots, n)$$

Example 1.8. Consider $F = \mathbb{R}$ and $V = \mathbb{R}^3$. Then we have **standard (Euclidean) basis**

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The basis is not unique; actually, there exists multiple different basis that spans $\mathbb{R}^3$.

We often denote a vector in $\mathbb{R}^3$ as $v = [5, 2, 3]^\mathsf{T}$. In fact, this means

$$v = 5 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 5e_1 + 2e_2 + 3e_3$$

where 5, 2 and 3 indicates the **components** of v.

Definition 1.9 (Components of vectors)**.** Let $\mathfrak{B} = \{e_1, \cdots e_n\}$ be a F-basis of V. Then any vector $v \in V$ can be represented by the linear combination of $\{e_\mu\}_{\mu=1}^n$ (by definition).

$$v = v^1 e_1 + v^2 e_2 + \cdots + v^n e_n = \sum_{\mu=1}^n v^\mu e_\mu$$

We call these v^μ's as **components**.

Remark. Note that basis of a vector space, e_μ, has lower index, while component of a vector v^μ has upper index.

Remark. In differential geometry, we often encounter summations on overlapping indices.

Einstein notation simplies this by (simply) omitting summation signs.

$$\sum_{\mu=1}^n v^\mu e_\mu := v^\mu e_\mu$$

If we encounter overlapping indices - one in superscript and one in subscript, we assume there is an omitted summation.

Definition 1.10 (Dimension)**.** $\dim V = |\mathfrak{B}|$.

§ 1.2 Linear Maps and Dual Vector Spaces

Definition 1.11 (Linear map)**.** Let V and W be F-vector spaces. If a map $L : V \to W$ satisfies

$$L(v+w) = L(v) + L(w), \quad L(av) = a \cdot L(v) \quad (a \in F,\, v, w \in V)$$

we call L a **linear map**.

Exercise 1.12. Let L be a linear map and $u, v \in V$. Prove the followings.

(1) $L(0) = 0$

(2) $L(-v) = -L(v)$

(3) $L(u-v) = L(u) - L(v)$

Proof. (1) $L(0) = L(0+0) = L(0) + L(0) \implies L(0) = 0$

(2) $L(0) = L(v-v) = L(v) + L(-v) \implies -L(v) = L(-v)$

(3) $L(u-v) = L(u) + L(-v) = L(u) - L(v)$

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While talking about maps, we should mention these two.

Definition 1.13 (Kernel and image)**.** Let $L : V \to W$ be a linear map.

(1) The **kernel** of L is defined by $\ker L = L^{-1}(0) = \{v \in V \mid L(v) = 0\}$.

(2) The **image** of L is defined by $\operatorname{im} L = L(V) = \{L(V) \in W \mid v \in V\}$.

Exercise 1.14. Show that (1) $\ker L \leq V$ and (2) $\operatorname{im} L \leq W$.

Proof. By **Observation 1.3**, it suffices to show that $\ker L$ and $\operatorname{im} L$ are closed.

(1) Let $v, w \in \ker L$. Then

$$L(v+w) = L(v) + L(w) = 0 + 0 = 0, \quad L(av) = aL(v) = 0 \ (a \in F)$$

Hence, $v + w, av \in \ker L \implies \ker L \leq V$.

(2) Let $L(v), L(w) \in \operatorname{im} L$. Then

$$L(v) + L(w) = L(\underbrace{v+w}_{\in V}) \in \operatorname{im} L, \quad aL(v) = L(\underbrace{av}_{\in V}) \in \operatorname{im} L \ (a \in F)$$

gives $\operatorname{im} L \leq W$.

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Theorem 1.15 (Dimension theorem)**.** *Let V be a finite dimensional vector space and L be a linear map $L : V \to W$. Then*

$$\dim V = \dim \ker L + \dim \operatorname{im} L$$

Proof. Let basis of $\ker L$ be $\{e_1, \cdots, e_r\}$ (in other words, $\dim \ker L := r$).

Since $\ker L \leq V$, we can *extend* the basis of $\ker L$ to the basis of V[1] as following.

$$\mathfrak{B} = \{e_1, \cdots, e_r, e_{r+1}, \cdots, e_n\}$$

Here, note that $L(e_1) = \cdots = L(e_r)$ (they are elements of $\ker L$).

WTS: $\{L(e_{r+1}), \cdots, L(e_n)\}$ is a basis of $\operatorname{im} L$.

(1) $\{L(e_{r+1}), \cdots, L(e_n)\}$ are linearly independent. In the following equation, we expect all a^k $(k = r+1, \cdots, n)$ should be zero.

$$\sum_{k=r+1}^{n} a^k L(e_k) = 0 = L\left(\sum_{k=r+1}^{n} a^k e_k\right)$$

The last equal sign holds since L is a linear map. Since substituting $a^k e_k$[2] into L yields zero, $a^k e_k \in \ker L$. Hence, $a^k e_k$ can be represented by the linear combination of $\{e_1, \cdots, e_r\}$.

$$\sum_{k=r+1}^{n} a^k e_k = \sum_{k=1}^{r} b^k e_k$$

This equation can be arranged into a single summation.

$$\sum_{k=1}^{n} c^k e_k = 0$$

where $c^k = b^k$ for $k = 1, \cdots, r$ and $c^k = -a^k$ for $k = r+1, \cdots, n$. Since $\{e_k\}_{k=1}^n$ is a basis of V, these vectors are linearly independent - hence, all c^k are zero. Therefore, all a^k $(k = r+1, \cdots, n)$ becomes zero automatically.

(2) $\langle L(e_{r+1}), \cdots, L(e_n)\rangle = \operatorname{im} L$.

This is straightforward; for arbitrary $L(v) \in \operatorname{im} L$,

$$\begin{aligned}
L(v) &= L\left(\sum_{k=1}^{n} v^k e_k\right) \\
&= L\left(\sum_{k=1}^{r} v^k e_k + \sum_{k=r+1}^{n} v^k e_k\right) \\
&= \cancel{L\left(\sum_{k=1}^{r} v^k e_k\right)} + L\left(\sum_{k=r+1}^{n} v^k e_k\right) \quad (\because \sum_{k=1}^{r} v^k e_k \in \ker L) \\
&= \sum_{k=r+1}^{n} v^k L(e_k)
\end{aligned}$$

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[1] $\because$ Basis extension theorem, which we neither mentioned nor proved.

[2] This is written in Einstein notation.

Definition 1.16 (Vector space of linear maps)**.** Let $\mathfrak{L}(V, W)$ be the set of all linear maps from V to W.

$$\mathfrak{L}(V, W) = \{L : V \to W \mid L \text{ is linear}\}$$

By defining addition and scalar multiplication of linear maps as following,

$$\begin{aligned} \text{addition} : \quad & (L + M)(v) := L(v) + M(v) \\ \text{scalar multiplication} : \quad & (aL)(v) := aL(v) \end{aligned}$$

$(L, M \in \mathfrak{L}(V, W),\ v \in V,\ a \in F)$ $\mathfrak{L}(V, W)$ becomes a **vector space** of linear maps.

Definition 1.17 (Dual vector space)**.** Especially,

$$\mathfrak{L}(V, F) := V^*$$

is called a **dual vector space**. The element of V^*, $w \in V^*$ is called a **dual vector** (or a **covector**).

Remark. An element of a dual vector space is actually a *linear map*. In other words, for $w \in V^*$ and $v \in V$,

$$w : V \to F \quad \text{and} \quad w(v) \in F$$

Now we prove an important theorem that helps us grasp concepts of dual basis and components of dual vectors.

Theorem 1.18. $\dim V^* = \dim V$.

Proof. Let $\mathfrak{B} = \{e_1, \cdots, e_n\} = \{e_\mu\}$ be a basis of V ($\dim V := n$).
Define some linear maps (or dual vectors) $e^\mu : V \to F$ as

$$e^\mu(e_\nu) = \delta^\mu{}_\nu$$

where $\delta^\mu{}_\nu$ is the usual Kroenecker delta[3].
WTS: $\mathfrak{B}^* = \{e^\mu\}$ is a basis of V^*[4].

(1) $\mathfrak{B}^*$ is linearly independent: let's start from $a_\mu e^\mu = 0$. Then

$$(v \in V) \quad a_\mu e^\mu(v) = 0$$

However, since e^μ is a linear map, it suffices to check their behavior at the basis of V, e_{nu}[5]. Hence

$$a_\mu e^\mu(e_\nu) = a_\mu \delta^\mu{}_\nu = a_\nu = 0$$

(2) $\langle \mathfrak{B}^* \rangle = V^*$? For arbitrary $w \in V^*$, it can be expanded into

$$w = w(e_1)e^1 + w(e_2)e^2 + \cdots + w(e_n)e^n := w_\mu e^\mu$$

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[3] You may wonder why one index is superscripted while the other is subscripted. This will be clarified in the next week's lecture.
[4] Proving this statement gives our original claim immediately.
[5] This has deeper context in standard linear algebra course, and can be shown. However, we skip the detail.

Remark. In the last line of proof, we defined the **component** of a dual vector, $w(e_\mu) = w_\mu$. Both the basis and component of dual vectors are closely related to the basis e_μ of original vector space V.

In summary,

- Let V be a F-vector space, and $v \in V$ be a vector. Then v can be expanded into the linear combination of basis vectors.

  $$\boxed{v = v^\mu e_\mu}$$

- The concept of dual space naturally follows from the original vector space V. The dual vector space $V^* = \mathfrak{L}(V, W)$ has dual vectors (or covectors) as elements: $w \in V^*$ and $w : V \to F$. The dual vector can be expanded into dual basis.

  $$\boxed{w = w_\mu e^\mu}$$

  where $w_\mu := w(e_\mu)$.