7 Differential Forms

§ 7.3 Interior Product

Definition 7.22 (Interior product)**.** The **interior product** is defined by

$$i_X : \Omega^r(\mathcal{M}) \to \Omega^{r-1}(\mathcal{M}), \quad i_X\omega(X_1, \cdots, X_{r-1}) = \omega(X, X_1, \cdots, X_{r-1})$$

for $X \in \mathscr{X}(\mathcal{M})$ and $\omega \in \Omega^r(\mathcal{M})$. In components[1], for $X = X^\mu \partial_\mu$ and $\omega = \frac{1}{r!}\omega_{\mu_1\cdots\mu_r}\,\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_r}$,

$$\begin{aligned} i_X\omega &= \frac{1}{r!}\sum_{s=1}^{r} X^{\mu_s}\omega_{\mu_1\cdots\mu_s\cdots\mu_r}(-1)^{s-1}\,\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \widehat{\mathrm{d}x^{\mu_s}} \wedge \cdots \wedge \mathrm{d}x^{\mu_r} \\ &= \frac{1}{(r-1)!}X^\nu\omega_{\nu\mu_2\cdots\mu_r}\,\mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_r} \end{aligned}$$

where the term with hat denotes the *omitted* term.

Remark. The interior product is the *unique* antiderivation of degree -1 on the exterior algebra.

Observation 7.23. How can we obtain the Nakahara formula? Let me start from the component expression of r-forms.

$$\omega = \omega_{\mu_1\cdots\mu_r}\,\mathrm{d}x^{\mu_1} \otimes \cdots \otimes \mathrm{d}x^{\mu_r} = \frac{1}{r!}\omega_{\mu_1\cdots\mu_r}\,\mathrm{d}x^{\mu_1} \wedge \otimes \wedge \mathrm{d}x^{\mu_r}$$

Interior product is a *contraction* on the first index of ω with X.

$$\begin{aligned} i_X\omega &= X^\nu\omega_{\nu\mu_2\cdots\mu_r}\,\mathrm{d}x^{\mu_2} \otimes \cdots \otimes \mathrm{d}x^{\mu_r} \qquad (\because\ \langle \mathrm{d}x^{\mu_1}, \partial_\nu\rangle = \delta^{\mu_1}{}_\nu) \\ &= \frac{1}{(r-1)!}X^\nu\omega_{\nu\mu_2\cdots\mu_r}\,\mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_r} \end{aligned}$$

But since ω is alternating, we could contract over all of the r indices and get the same thing, yielding a sum with r equal terms.

$$i_X\omega = \frac{1}{r} \cdot \frac{1}{(r-1)!}\sum_{r=1}^{r} X^{\mu_s}\omega_{\mu_1\cdots\mu_s\cdots\mu_r}(-1)^{s-1}\,\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \widehat{\mathrm{d}x^{\mu_s}} \wedge \cdots \wedge \mathrm{d}x^{\mu_r}$$

[1] This formula is from Nakahara's textbook, and it is truly horrible for beginners.

Example 7.24. In $\mathbb{R}^3$, $i_{e_x}(\mathrm{d}x \wedge \mathrm{d}y) = \mathrm{d}y$.

- Note that

$$\begin{aligned}
\mathrm{d}x \wedge \mathrm{d}y &= \frac{1}{2!}\omega_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \\
&= \frac{1}{2}(\omega_{xy}\,\mathrm{d}x \wedge \mathrm{d}y + \cancel{\omega_{yz}\,\mathrm{d}y \wedge \mathrm{d}z} + \cancel{\omega_{zx}\,\mathrm{d}z \wedge \mathrm{d}x} + \omega_{yx}\,\mathrm{d}y \wedge \mathrm{d}x + \cancel{\omega_{zy}\,\mathrm{d}z \wedge \mathrm{d}y} + \cancel{\omega_{xz}\,\mathrm{d}x \wedge \mathrm{d}z}) \\
&= \omega_{xy}\,\mathrm{d}x \wedge \mathrm{d}y \quad (\because\ \omega\text{, antisymmetric})
\end{aligned}$$

Hence, $\mathrm{d}x \wedge \mathrm{d}y$ is a two-form ω with components $\omega_{xy} = -\omega_{yx} = 1$ and zero otherwise.

- Therefore

$$i_{e_x}(\mathrm{d}x \wedge \mathrm{d}y) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} = \mathrm{d}y$$

- Similarly,

$$\begin{aligned}
i_{e_x}\underbrace{(\mathrm{d}y \wedge \mathrm{d}z)}_{\omega_{yz}=1} &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} = 0 \\
i_{e_x}\underbrace{(\mathrm{d}z \wedge \mathrm{d}x)}_{\omega_{zx}=1} &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \end{bmatrix} = -\mathrm{d}z
\end{aligned}$$

Remark. For $\omega \in \Omega^2(\mathcal{M})$, $i_X\omega = X^\mu \omega_{\mu\nu}\,\mathrm{d}x^\nu$.

Exercise 7.25. Let

$$X = y\frac{\partial}{\partial x} + 2x\frac{\partial}{\partial y} + 3xy\frac{\partial}{\partial z} = \begin{bmatrix} y & 2z & 3xy \end{bmatrix}^\mathsf{T} \in T_p\mathcal{M}$$

and

$$\omega = 2z\,\mathrm{d}x + 3x\,\mathrm{d}y - 7zx^2\,\mathrm{d}z = \begin{bmatrix} 2z & 3x & -7zx^2 \end{bmatrix}^\mathsf{T}$$

(1) Compute $\mathrm{d}\omega$, the exterior derivative.

(2) Compute $i_X(\mathrm{d}\omega)$.

(3) Compute $\mathrm{d}(i_X\omega)$ and $\mathrm{d}(i_X\omega) + i_X(\mathrm{d}\omega)$.

(4) Compute $\mathcal{L}_X\omega$ and compare with (3).

Proof.

(1) See **Example 7.15**.

$$\begin{aligned}
\mathrm{d}\omega &= \left[\frac{\partial \omega_y}{\partial x} - \cancel{\frac{\partial \omega_x}{\partial y}}\right] \mathrm{d}x \wedge \mathrm{d}y + \left[\cancel{\frac{\partial \omega_z}{\partial y}} - \cancel{\frac{\partial \omega_y}{\partial z}}\right] \mathrm{d}y \wedge \mathrm{d}z + \left[\frac{\partial \omega_x}{\partial z} - \frac{\partial \omega_z}{\partial x}\right] \mathrm{d}z \wedge \mathrm{d}x \\
&= 3\,\mathrm{d}x \wedge \mathrm{d}y + (14zx + 2)\,\mathrm{d}z \wedge \mathrm{d}x
\end{aligned}$$

(2) Note that $(i_X(\mathrm{d}\omega))_\mu = X^\nu (\mathrm{d}\omega)_{\nu\mu} = (\mathrm{d}\omega)^\mathsf{T}_{\mu\nu} X^\nu$.

$$i_X(\mathrm{d}\omega) = (\mathrm{d}\omega)^\mathsf{T} X = \begin{bmatrix} 0 & -3 & 14zx+2 \\ 3 & 0 & 0 \\ -(14zx+2) & 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ 2z \\ 3xy \end{bmatrix} = \begin{bmatrix} -6z + 6xy + 42x^2yz \end{bmatrix}$$

Hence, $i_X(\mathrm{d}\omega) = (-6z + 6xy + 42x^2yz)\,\mathrm{d}x + 3y\,\mathrm{d}y - (14xyz + 2y)\,\mathrm{d}z$.

(3) $i_X\omega = \omega(X) = 2yz + 6zx - 21x^3yz$. So

$$\mathrm{d}i_X\omega = (6z - 63x^2yz)\,\mathrm{d}x + (2z - 21x^3z)\,\mathrm{d}y + (6x + 2y - 21x^3y)\,\mathrm{d}z$$

and

$$\mathrm{d}i_X\omega + i_X(\mathrm{d}\omega) = (6xy - 21x^2yz)\,\mathrm{d}x + (2z - 21x^3z)\,\mathrm{d}y + (6x - 14xyz - 21x^3y)\,\mathrm{d}z$$

(4) By definition, $\mathcal{L}_X\omega = (X^\nu \partial_\nu \omega_\mu + \partial_\mu X^\nu \omega_\nu)\,\mathrm{d}x^\mu$. Then

$$\partial_\nu \omega_\mu = \begin{bmatrix} 0 & 3 & -14zx \\ 0 & 0 & 0 \\ 2 & 0 & -7x^2 \end{bmatrix} \implies \begin{bmatrix} y & 2z & 3xy \end{bmatrix} \begin{bmatrix} 0 & 3 & -14zx \\ 0 & 0 & 0 \\ 2 & 0 & -7x^2 \end{bmatrix} = \begin{bmatrix} 6xy \\ 3y \\ -14xyz - 21x^3y \end{bmatrix}^\mathsf{T}$$

and

$$\partial_\mu X^\nu = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 3y & 3x & 0 \end{bmatrix} \implies \begin{bmatrix} 2z & 3x & -7zx^2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 3y & 3x & 0 \end{bmatrix} = \begin{bmatrix} -21x^2yz \\ 2z - 21x^3z \\ 6x \end{bmatrix}^\mathsf{T}$$

So, $\mathcal{L}_X\omega = \mathrm{d}i_X\omega + i_X(\mathrm{d}\omega)$.

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Is this result a mere coincidence? It is not!

Theorem 7.26 (Cartan's magic formula, Cartan's homotopy formula)**.**

$$\mathcal{L}_X\omega = (\mathrm{d}i_X + i_X\mathrm{d})\omega$$

Proof. **Proof for 1-form.** Note that $i_X(\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu) = X^\mu\,\mathrm{d}x^\nu - X^\nu\,\mathrm{d}x^\mu$.

$$\begin{aligned}
(\mathrm{d}i_X + i_X\mathrm{d})\omega &= \mathrm{d}(X^\mu\omega_\mu) + i_X\left[\frac{\partial\omega_\nu}{\partial x^\mu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\right] \\
&= \frac{\partial(X^\mu\omega_\mu)}{\partial x^\nu}\,\mathrm{d}x^\nu + i_X\left[\frac{1}{2}(\partial_\mu\omega_\nu - \partial_\nu\omega_\mu)\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\right] \\
&= (\omega_\mu\partial_\nu X^\mu + \cancel{X^\mu\partial_\nu\omega_\mu})\,\mathrm{d}x^\mu + \underbrace{\frac{1}{2}(\partial_\mu\omega_\nu - \partial_\nu\omega_\mu)(X^\mu\,\mathrm{d}x^\nu - X^\nu\,\mathrm{d}x^\mu)}_{=X^\mu(\partial_\mu\omega_\nu - \cancel{\partial_\nu\omega_\mu)\,\mathrm{d}x^\nu}} \\
&= (X^\mu\partial_\mu\omega_\nu + \partial_\nu X^\mu\omega_\mu)\,\mathrm{d}x^\nu = \mathcal{L}_X\omega
\end{aligned}$$

Proof for r-form. Lie derivative of r-form is given by

$$\begin{aligned}
\mathcal{L}_X\omega &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}[(\sigma_\epsilon)^*\omega|_{\sigma_\epsilon(x)} - \omega|_x] \\
&= X^\nu\frac{1}{r!}\partial_\nu\omega_{\mu_1\cdots\mu_r}\,\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_r} + \sum_{s=1}^{r}\partial_{\mu_s}X^\nu\frac{1}{r!}\omega_{\mu_1\cdots\underset{\substack{\uparrow\\ \mu_s}}{\nu}\cdots\mu_r}\,\mathrm{d}x^{\mu_1}\wedge\cdots\mathrm{d}x^\nu\wedge\cdots\wedge\mathrm{d}x^{\mu_r}
\end{aligned}$$

$$\begin{aligned}
(\mathrm{d}i_X + i_X\mathrm{d})\omega &= \mathrm{d}\left[\frac{1}{r!}\sum_{s=1}^{r}X^{\mu_s}\omega_{\mu_1\cdots\nu\cdots\mu_r}(-1)^{s-1}\,\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_r}\right] \\
&\qquad + i_X\left[\frac{1}{r!}\partial_\nu\omega_{\mu_1\cdots\mu_r}\,\mathrm{d}x^\nu\wedge\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\cdots\mathrm{d}x^{\mu_r}\right] \\
&= \frac{1}{r!}\sum_{s=1}^{r}(\partial_\nu X^{\mu_s}\omega_{\mu_1\cdots\mu_r} + \cancel{X^{\mu_s}\partial_\nu\omega_{\mu_1\cdots\mu_r}})(-1)^{s-1}\,\mathrm{d}x^{\mu_1}\wedge\cdots\widehat{\mathrm{d}x^{\mu_s}}\wedge\cdots\wedge\mathrm{d}x^{\mu_r} \\
&\quad + \frac{1}{r!}[X^\nu\partial_\nu\omega_{\mu_1\cdots\mu_r}\,\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_r} + \cancel{\sum_{s=1}^{r}X^{\mu_s}\partial_\nu\omega_{\mu_1\cdots\mu_s\cdots\mu_r}(-1)^s\,\mathrm{d}x^\nu\wedge\cdots\wedge\widehat{\mathrm{d}x^{\mu_s}}\wedge\cdots\wedge\mathrm{d}x^{\mu_r}}] \\
&= \mathcal{L}_X\omega
\end{aligned}$$

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Exercise 7.27. Let $X, Y \in \mathscr{X}(\mathcal{M})$ and $\omega \in \Omega^r(\mathcal{M})$. Show that

(1) $i_{[X,Y]}\omega = X(i_Y\omega) - Y(i_X\omega)$

(2) $i_X^2 = 0$ [a]

(3) $\mathcal{L}_X i_X \omega = i_X \mathcal{L}_X \omega$.

[a] The interior product is **nilpotent**.

Proof.

(1) Since $[X,Y]^\nu = (\mathcal{L}_X Y)^\nu = X^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu$,

$$\begin{aligned} i_{[X,Y]}\omega &= \frac{1}{(r-1)!}(X^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu)\omega_{\nu\mu_2\cdots\mu_r}\, \mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_r} \\ &= X^\mu \partial_\mu (i_Y\omega) - Y^\mu \partial_\mu (i_X\omega) = X(i_Y\omega) - Y(i_X\omega) \end{aligned}$$

(2) For $X_3, \cdots, X_r \in \mathscr{X}(\mathcal{M})$,

$$i_X i_X \omega(X_3, \cdots, X_r) = i_X\omega(X, X_3, \cdots, X_r) = \omega(X, X, X_3, \cdots, X_r) = 0$$

because ω is *totally antisymmetric*.

(3) By Cartan's magic formula,

$$\begin{aligned} \mathcal{L}_X i_X \omega &= (\cancel{\mathrm{d}i_X} + i_X \mathrm{d})(i_X\omega) = i_X \mathrm{d} i_X \omega \\ i_X \mathcal{L}_X \omega &= i_X(\mathrm{d}i_X + \cancel{i_X \mathrm{d}})\omega = i_X \mathrm{d} i_X \omega \end{aligned}$$

■

Exercise 7.28 (Leibniz rule of interior product). For $X \in \mathscr{X}(\mathcal{M})$, $\omega \in \Omega^r(\mathcal{M})$ and $\eta \in \Omega^s(\mathcal{M})$,

$$i_X(\omega \wedge \eta) = i_X\omega \wedge \eta + (-1)^r \omega \wedge i_X\eta$$

(i_X is an *anti-derivation*.)

Proof. **Elementary proof.** Call $X = V_1$ and let $V_2, \cdots, V_{r+s} \in \mathscr{X}(\mathcal{M})$.

$$\begin{aligned}
&i_{V_1}(\omega \wedge \eta)(V_2, \cdots, V_{r+s}) = (\omega \wedge \eta)(V_1, V_2, \cdots, V_{r+s}) \\
&\quad = \frac{1}{r!s!} \sum_{P \in S_{r+s}} \operatorname{sgn}(P) \omega(V_{P(1)}, \cdots, V_{P(r)}) \eta(V_{P(r+1)} \cdots V_{P(r+s)}) \\
&\quad = \frac{1}{r!s!} [\sum_{P_1 \in S_{r+s}} \operatorname{sgn}(P_1) \omega(V_{P_1(1)}, \cdots, V_{P_1(r)}) \eta(V_{P_1(r+1)} \cdots V_{P_1(r+s)}) \\
&\qquad\qquad + \sum_{P_2 \in S_{r+s}} \operatorname{sgn}(P_2) \omega(V_{P_2(1)}, \cdots, V_{P_2(r)}) \eta(V_{P_2(r+1)} \cdots V_{P_2(r+s)})]
\end{aligned}$$

Here, we split permutations in S_{r+s} into two sets: P_1 such that $1 \in \{P_1(1), \cdots, P_1(r)\}$ and P_2 such that $1 \in \{P_2(r+1), \cdots, P_2(r+s)\}$. In the next step, we view P_1 (and P_2) as a permutation in S_{r+s-1}, by fixing $1 = P_1(1)$ (and $1 = P_2(r+1)$) by exploiting the alternating nature of ω and η. Then

$$\begin{aligned}
i_{V_1}(\omega \wedge \eta)(V_2, \cdots, V_{r+s}) &= \frac{1}{(r-1)!s!} \sum_{P_1' \in S_{r+s-1}} \operatorname{sgn}(P_1') \omega(V_1, V_{P_1'(2)}, \cdots, V_{P_1'(r)}) \eta(V_{P_1'(r+1)}, \cdots, V_{P_1'(r+s)}) \\
&\quad + \frac{1}{r!(s-1)!} \sum_{P_2' \in S_{r+s-1}} \operatorname{sgn}(P_2') (-1)^r \omega(V_{P_2'(1)}, \cdots, V_{P_2'(r)}) \eta(V_1, \cdots, V_{P_2'(r+s)}) \\
&= i_X\omega \wedge \eta + (-1)^r \omega \wedge i_X\eta
\end{aligned}$$

■

Proof. **Fancy proof.** It is sufficient to prove where ω and η are *decomposable*. Suppose $\alpha^1, \cdots, \alpha^{r+s} \in \Omega^1(\mathcal{M})$ and

$$\omega = \alpha^1 \wedge \cdots \wedge \alpha^r \quad \text{and} \quad \eta = \alpha^{r+1} \wedge \cdots \wedge \alpha^{r+s}$$

Then $i_X(\omega \wedge \eta) = i_X(\alpha^1 \wedge \cdots \wedge \alpha^{r+s})$. Let $X_2, \cdots, X_{r+s} \in T_p\mathcal{M}$.

$$\begin{aligned}
i_X(\alpha^1 \wedge \cdots \wedge \alpha^{r+s})(X_2, \cdots, X_{r+s}) &= (\alpha^1 \wedge \cdots \wedge \alpha^{r+s})(X, X_2, \cdots, X_{r+s}) \\
&= (r+s)!\mathcal{A}(\alpha^1 \otimes \cdots \otimes \alpha^{r+s})(X_1, \cdots, X_{r+s}) \quad (X \equiv X_1) \\
&= (r+s)! \sum_{P \in S_{r+s}} \operatorname{sgn}(P) \prod_{k=1}^{r+s} \alpha^k(X_{P(k)}) \\
&= (r+s)! \det[\alpha^i(X_j)] \\
&= (r+s)! \det \begin{bmatrix} \alpha^1(X_1) & \cdots & \alpha^1(X_{r+s}) \\ \alpha^2(X_1) & \cdots & \alpha^2(X_{r+s}) \\ \vdots & \ddots & \vdots \\ \alpha^{r+s}(X_1) & \cdots & \alpha^{r+s}(X_{r+s}) \end{bmatrix} \quad \text{(expand along the first column)} \\
&= (r+s)! \sum_{i=1}^{r+s} (-1)^{i+1} \alpha^i(X) \det[\alpha^l(v_j)]_{1 \le l \le r+s, l \ne i, 2 \le j \le k} \\
&= \sum_{i=1}^{r+s} (-1)^{i+1} \alpha^i(X) \alpha^1 \wedge \cdots \wedge \widehat{\alpha^i} \wedge \cdots \wedge \alpha^{r+s}(X_2, \cdots, X_{r+s})
\end{aligned}$$

Hence,

$$\begin{aligned}
i_X(\alpha^1 \wedge \cdots \wedge \alpha^{r+s}) &= \left(\sum_{i=1}^{r} (-1)^{i+1} \alpha^i(X) \alpha^1 \wedge \cdots \widehat{\alpha^i} \wedge \cdots \wedge \alpha^r \right) \wedge \alpha^{r+1} \wedge \cdots \wedge \cdots \alpha^{r+s} \\
&\quad + (-1)^r \alpha^1 \wedge \cdots \wedge \alpha^r \wedge \left(\sum_{i=1}^{s} \alpha^{r+i}(X) \alpha^{r+1} \wedge \cdots \wedge \widehat{\alpha^{r+i}} \wedge \cdots \wedge \alpha^{r+s} \right) \\
&= i_X \omega \wedge \eta + (-1)^r \omega \wedge i_X \eta
\end{aligned}$$

■

Exercise 7.29. Let $T = \mathfrak{T}^{(n,m)}(\mathcal{M})$. Show that

$$(\mathcal{L}_X T)^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\nu_m} = X^\lambda \partial_\lambda T^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\nu_m} + \sum_{s=1}^{m} \partial_{\nu_s} X^\lambda T^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots \underset{\underset{\nu_s}{\uparrow}}{\lambda} \cdots\nu_m} - \sum_{s=1}^{n} \partial_\lambda X^{\mu_s} T^{\mu_1\cdots \overset{\lambda}{\underset{\mu_s}{\uparrow}} \cdots\mu_n}{}_{\nu_1\cdots\nu_m}$$

Proof. Contract T with arbitrary vectors $Y^1, \cdots, Y^m$ and covectors $Z^1, \cdots, Z^n$ to yield

$$(*) = T^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\nu_m}(Y^1)^{\nu_1}\cdots(Y^m)^{\nu_m}(Z^1)_{\mu_1}\cdots(Z^n)_{\mu_n}$$

and apply the Lie derivative.

$$\begin{aligned}
\mathcal{L}_X(*) = &\underbrace{\boxed{\mathcal{L}_X(T^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\nu_m})}}_{X^\lambda\partial_\lambda T^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\nu_m})}(Y^1)^{\nu_1}\cdots(Y^m)^{\nu_m}(Z^1)_{\mu_1}\cdots(Z^n)_{\mu_n} \\
&+ T^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\nu_m})\left[\sum_{s=1}^{m}\cdots \underbrace{(\mathcal{L}_X(Y^s))^{\nu_s}}_{X^\lambda\partial_\lambda(Y^s)^{\nu_s} - (Y^s)^\lambda\partial_\lambda X^{\nu_s}} \cdots\right](Z^1)_{\mu_1}\cdots(Z^n)_{\mu_n} \\
&+ T^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\nu_m})(Y^1)^{\nu_1}\cdots(Y^m)^{\nu_m}\left[\sum_{s=1}^{n}\cdots \underbrace{(\mathcal{L}_X(Z^s))_{\mu_s}}_{X^\lambda\partial_\lambda(Z^s)_{\mu_s} + \partial_{\nu_s}X^\lambda(Z^s)_\lambda} \cdots\right]
\end{aligned}$$

Now choose $(Y^s)^{\nu_s} = \delta^{\nu_s}{}_a$ and $(Z^s)_{\mu_s} = \delta^b{}_{\mu_s}$ for fixed a and b. Then blue terms disappear and the proof is complete. ■

§ 7.4 Hamiltonian Mechanics and Symplectic Geometry

Consider a single particle in $\mathbb{R}^3$. To describe the motion of that particle, we require $\mathbb{R}^6$ phase space. In case of N particles, those particles *live* in $3N$-dimensional space while they enjoy $6N$-dimensional phase space. *Even-dimensional spaces have something special!*

$$\text{Position } (q_1, q_2, q_3) \qquad + \qquad \text{Momentum } (p_1, p_2, p_3) \qquad \Longrightarrow \qquad \text{Hamiltonian } H(\mathbf{q}, \mathbf{p})$$

Definition 7.30. The **symplectic two-form** $\omega = \mathrm{d}p_\mu \wedge \mathrm{d}q^\mu$[2] is

- antisymmetric: $\omega(X, Y) = -\omega(Y, X)$.
- nondegenerate: $i_X\omega = 0 \implies X = 0$.
- closed: $\mathrm{d}\omega = 0$.

Remark. The one-form $\theta = p_\mu \,\mathrm{d}q^\mu$ gives

$$\mathrm{d}\theta = \mathrm{d}p_\mu \wedge \mathrm{d}q^\mu + p_\mu \cancel{\mathrm{d}^2 q^\mu} = \omega$$

[2]Equation in Nakahara is wrong.

Definition 7.31 (Hamiltonian vector fields). Given a function $f(\mathbf{q},\mathbf{p})$ in the phase space, define

$$X_f = \frac{\partial f}{\partial p_\mu}\frac{\partial}{\partial q^\mu} - \frac{\partial f}{\partial q^\mu}\frac{\partial}{\partial p_\mu}$$

Then

$$i_{X_f}\omega = \frac{\partial f}{\partial p_\mu}(-\mathrm{d}p_\mu) - \frac{\partial f}{\partial q^\mu}(\mathrm{d}q^\mu) = -\mathrm{d}f$$

Remark. The symplectic two-form ω is **left-invariant** along the flow generated by X_H.

$$\mathcal{L}_{X_H}\omega = \mathrm{d}i_{X_H}\omega + i_{X_H}\cancel{(\mathrm{d}\omega)} = -\mathrm{d}^2 H = 0$$

Remark. A vector field X that satisfies $\mathcal{L}_X\omega = 0$ is called a **Hamiltonian vector field**. The space of such vector fields on $\mathbb{R}^{2n}$ is denoted as $\mathrm{Vect}(\mathbb{R}^{2n},\omega)$ (By Poincaré's Lemma[3], $\exists\, H(\mathbf{q},\mathbf{p})$ such that $i_X\omega = -\mathrm{d}H$).

[3] I did not mentioned this before.