3 Manifolds: Heuristic Introduction

Remark. Both Euclidean space($\mathbb{R}^3$ with metric $\delta_{\mu\nu}$) and Minkowski space($\mathbb{R}^4$ with metric $\eta_{\mu\nu}$) are *flat*. In other words, those spaces have **constant metric**, regardless of the point you are interested in.

Definition 3.1 (Polar coordinates in 2D)**.** The **polar coordinates** in $\mathbb{R}^2$ is represented by (r, ϕ) with $0 \leq r < \infty$ and $0 \leq \phi \leq 2\pi$. Then

$$x = r\cos\phi, \quad y = r\sin\phi$$

and

$$r = \sqrt{x^2 + y^2}, \quad \phi = \arctan\frac{y}{x}$$

Note that the coordinate system is well-defined except at $r = 0$. Let's compute the metric.

$$\begin{aligned} \mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 &= (\mathrm{d}r\cos\phi - r\sin\phi\mathrm{d}\phi)^2 + (\mathrm{d}r\sin\phi + r\cos\phi\mathrm{d}\phi)^2 \\ &= \mathrm{d}r^2 + r^2\mathrm{d}\phi^2 = \begin{bmatrix}\mathrm{d}r & \mathrm{d}\phi\end{bmatrix}\begin{bmatrix}1 & 0 \\ 0 & r^2\end{bmatrix}\begin{bmatrix}\mathrm{d}r \\ \mathrm{d}\phi\end{bmatrix} \end{aligned}$$

Hence,

$$g_{\mu\nu} = \begin{bmatrix}1 & 0 \\ 0 & r^2\end{bmatrix} \quad \text{or} \quad g_{rr} = 1,\ g_{r\phi} = g_{\phi r} = 0,\ g_{\phi\phi} = r^2$$

The metric changes upon the location of point $p \in \mathbb{R}^3$.

Definition 3.2 (Spherical polar coordinates)**.** The **spherical polar coordinates** in $\mathbb{R}^3$ is represented by (r, θ, ϕ) with $0 \leq r < \infty$, $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. Then

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$$

and

$$r = \sqrt{x^2 + y^2 + z^2},\ \theta = \arctan\frac{\sqrt{x^2+y^2}}{z},\ \phi = \arctan\frac{y}{x}$$

The interval becomes

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2 \\ &= \begin{bmatrix}\mathrm{d}r & \mathrm{d}\theta & \mathrm{d}\phi\end{bmatrix}\begin{bmatrix}1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta\end{bmatrix}\begin{bmatrix}\mathrm{d}r \\ \mathrm{d}\theta \\ \mathrm{d}\phi\end{bmatrix} \end{aligned}$$

Hence,

$$g_{\mu\nu} = \begin{bmatrix}1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta\end{bmatrix} \quad \text{or} \quad g_{rr} = 1,\ g_{\theta\theta} = r^2,\ g_{\phi\phi} = r^2\sin^2\theta$$

where all off-diagonal terms are zero.

Observation 3.3. Consider $\mathbb{R}^n$ with coordinate system x^μ and metric g ($\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$). Suppose that we define new *local* coordinate system $\tilde{x}^\mu$ on a subspace of $\mathbb{R}^n$.

$$\tilde{x}^\mu = h^\mu(x^1, \cdots, x^n)$$

We call this new coodinate system *nice* if $\{h^\mu\}$ are invertible (in other words, $\{\tilde{h}^\mu\}$ exists such that $x^\mu = \tilde{h}^\mu(\tilde{x}^1, \cdots, \tilde{x}^n)$). We assume that $\{h^\mu\}$ and $\{\tilde{h}^\mu\}$ are sufficiently differentiable. The **transformation matrix** (or **Jacobian**) between those coordinates is

$$\frac{\partial \tilde{x}^\mu}{\partial x^\nu} = \frac{\partial h^\mu}{\partial x^\nu} := \Lambda^\mu{}_\nu$$

Here, the metric g and coordinate transformation Λ are position-dependent. Those two are related via

$$\begin{aligned} \mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu &= g_{\mu\nu}\frac{\partial x^\mu}{\partial \tilde{x}^\lambda}\frac{\partial x^\nu}{\partial \tilde{x}^\sigma}\mathrm{d}\tilde{x}^\lambda \mathrm{d}\tilde{x}^\sigma \\ &= \underbrace{(\Lambda^{-1})^\mu{}_\lambda g_{\mu\nu}(\Lambda^{-1})^\nu{}_\sigma}_{:=\tilde{g}_{\lambda\sigma}}\mathrm{d}\tilde{x}^\lambda \mathrm{d}\tilde{s}^\sigma \end{aligned}$$

In matrix notation, $(\Lambda^{-1})^\mathsf{T} g \Lambda^{-1} = \tilde{g}$.

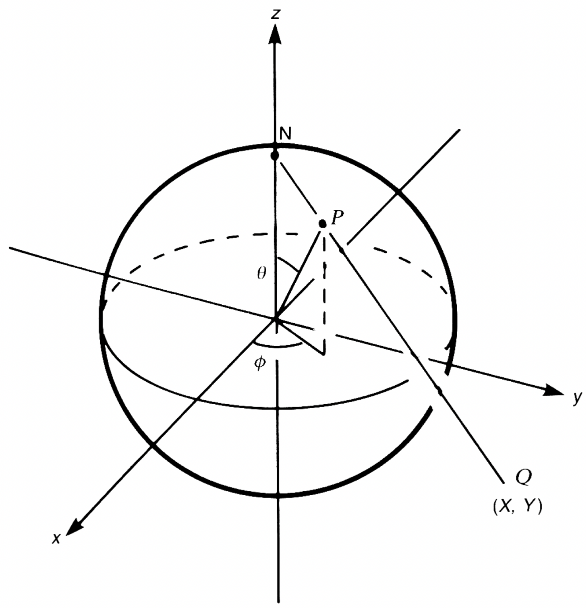
Remark. Multiple local coordinate system can exist in one space!

Example 3.4 (Unit sphere)**.** Consider a unit sphere in $\mathbb{R}^3$, S^2.

1. S^2 can be parametrized with spherical polar coordinates. However, we cannot represent entire S^2 with this coordinate system.
   - At $\phi = 0$ (or $\phi = 2\pi$), *discontinuity* arises.
   - If we choose ϕ to increase more as we cross the $\phi = 2\pi$ line, *uniqueness* problem arises (how do we know a point has ϕ or $\phi + 2\pi$, $\phi + 4\pi, \cdots$?).
   - At poles, ϕ values are ill-defined.
2. **Stereographic projection** maps a point on S^2, $P(x, y, z)$ to the point on xy-plane, $Q(X, Y, 0)$ by drawing a straight line connecting a pole and P. The point where that line meets with xy-plane becomes Q. Such Q is related to Cartesian (and spherical polar) coordinates by

$$X = \frac{x}{1-z} = \cot\frac{\theta}{2}\cos\phi,\ Y = \frac{y}{1-z} = \cot\frac{\theta}{2}\sin\phi$$

   - If we use North Pole as a reference, North Pole itself cannot be projected.
   - If we choose two points lying close to the pole, their stereographic projections will be located on widely different points.



Observation 3.5. We cannot label the points on the sphere with a single coordinate system such that

(i) Nearby points are mapped to nearby coordinates.

(ii) Every point has unique coordinates.

Instead, define coordinates that satisfy requirements on a part of S^2 (by introducing two or more overlapping coordinate systems).

(i) Nearby points are mapped to nearby coordinates(*in at least one coordinate system*).

(ii) Every point has unique coordinates(*in each system that contains that point*).

(iii) If two coordinate systems overlap, they are related to each other in a sufficiently *smooth* way.