The Notes Project

Probability Theory

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Part I

Introduction to Probability Theory

Basic Definitions

Sample Space (S/Ω) The set of all possible outcomes.

Event A subset of the sample space.

Independent Events Events A & B defined on Ω are independent if they are not affected by each other.

Mutually Exclusive Events Events A & B defined on Ω are mutually exclusive if they cannot occur simultaneously.

Exhaustive Events Events are exhaustive if their union is equal to the sample space.

Types of Probability

Classical Definition of Probability

If a random experiment has n mutually exclusive, equally likely and exhaustive outcomes, and m of them are favourable to event A, then probability of happening of event A is given by:

$$P(A) = \frac{\text{favourable events}}{\text{total events}} = \frac{m}{n}$$

Properties

1. $0 \leq P(A) \leq 1$
2. $P(A') = 1 - P(A)$

Empirical/Relative Definition of Probability

If an experiment is repeated n times, as n tends to ∞ and it produces m outcomes favourable to event A, then probability of happening of event A is given by:

$$P(A) = \lim_{n \to \infty} \frac{m}{n}$$

It is useful for unequally likely events, or countably infinite sample spaces.

Axiomatic Definition of Probability

Let Ω/S be a sample space, A be any event defined on sample space, then function P is said to be probability function on probability measure if it satisfies the following axioms.

- $P(A) \geq 0$
- $P(\Omega) = 1$
- For A & B, any two mutually exclusive events defined on sample space Ω, $P(A \cup B) = P(A) + P(B)$

Various Theorems and Identities

Addition Theorem:

$$P\left(\bigcup_{i=1}^{n} P(A_i)\right) = \sum_{i=1}^{n} P(A_i) - \sum_{0 \leq i < j = 1}^{n} \sum^{n} P(A_i \cap A_j) + \sum_{0 \leq i < j < k = 1}^{n} \sum^{n} \sum^{n} P(A_i \cap A_j \cap A_k) \ldots$$

Conditional Probability: If A & B are two events, probability of happening of event A if B has already happened is given by:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Similarly,

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Multiplication Theorem:

$$\begin{aligned} P(A \cap B) &= P(A|B) \cdot P(B) \\ &= P(B|A) \cdot P(A) \end{aligned}$$

Bayes' Theorem If $A_1, A_2, \ldots A_n$ are n events defined on sample space such that $\bigcup_{i=1}^{n} P(A_i) = 1$ and $\bigcap_{i=1}^{n} P(A_i) = \phi$, B be an event defined on same sample space such that $B \subseteq \cup A_i, P(B) \neq 0$:

$$P(A_i|B) = \frac{P(B|A_i) \cdot P(A_i)}{\sum P(B|A_i) \cdot P(A_i)}$$

More Definitions

Random Variable A real-valued function defined on sample space.

Probability Distribution The set of pairs of values of a random variable and the probability of those values.

Probability Mass Function Let X be a discrete random variable and $P(X = x)$ be a function defined on X. $P(X = x)$ is said to be its p.m.f. if it satisfies the following conditions:
1) $P(x) \geq 0$
2) $\sum P(x) = 1$

Probability Density Function Let X be a continuous random variable and $f(X = x)$ be a function defined on X. $f(X = x)$ is said to be its p.d.f. if it satisfies the following conditions:
1) $f(x) \geq 0$
2) $\int f(x) = 1$

Cumulative Distribution Function Let X be a random variable and $P(X = x)$ be its p.m.f. Distribution function is given by:

$$f(x) = \begin{cases} \sum_{i=1}^{x} P(X = x), & \text{for discrete r.v.} \\ \\ \int_{-\infty}^{x} f(x)dx, & \text{for continuous r.v.} \end{cases}$$

Expectation of a Random Variable Let X be a r.v. with p.m.f. $P(X = x)$ (for discrete) and $f(x)$ (for continuous). Then, expectation is denoted by $E(X)$ and is given by:

$$E(x) = \begin{cases} \sum x \cdot P(X = x), & \text{for discrete r.v.} \\ \\ \int x \cdot f(x)dx, & \text{for continuous r.v.} \end{cases}$$

Variance of a Random Variable Let X be a r.v. with expectation $E(X)$. Then, variance is denoted by $Var(X)$ and is given by:

$$Var(X) = E(X^2) - [E(X)]^2$$

Moments

Definitions:

Central Moments

$$\mu_r = \frac{\sum(x_i - \mu)^r}{n}$$

Raw Moments

$$\mu'_r = \frac{\sum x_i^r}{n}$$

Arbitrary Moments

$$\mu_{r_A} = \frac{\sum(x_i - A)^r}{n}$$

Functions:

Moment Generating Function

$$M_x(t) = E(e^{tx})$$

Cumulant Generating Function

$$K_x(t) = log(M_x(t)) = log(E(e^{tx}))$$

Characteristic Function

$$\phi_x(t) = E(e^{itx})$$

Part II

Discrete Probability Distributions

Uniform Distribution:

Definition:

Let X be a discrete r.v. It follows uniform distribution if its p.m.f is:

$$P(X = x) = \frac{1}{n};\ x = 1, 2 \ldots n$$

It is denoted by $X \sim D(n)$.

Applications:

1) Tossing a coin, getting heads or tails.
2) Selection of a student from a class.

Expectation and Variance:

$$\begin{aligned}
E(X) &= \sum x \cdot P(x) \\
&= \sum_{x=1}^{n} x \cdot \frac{1}{n} \\
&= \frac{1}{n} \cdot \frac{n(n+1)}{2} \\
E(X) &= \frac{(n+1)}{2}
\end{aligned}$$

$$\begin{aligned}
E(X^2) &= \sum x^2 \cdot P(x) \\
&= \sum_{x=1}^{n} x^2 \cdot \frac{1}{n} \\
&= \frac{1}{n} \cdot \frac{n(n+1)(2n+1)}{6} \\
E(X^2) &= \frac{(n+1)(2n+1)}{6}
\end{aligned}$$

$$\begin{aligned}
Var(x) &= E(X^2) - [E(X)]^2 \\
&= \frac{(n+1)(2n+1)}{6} - \left[\frac{(n+1)}{2}\right]^2 \\
&= \frac{2n^2 + 3n + 2}{6} - \frac{n^2 + 2n + 1}{4} \\
&= \frac{4n^2 + 6n + 4 - 3n^2 - 6n - 3}{12} \\
Var(X) &= \frac{n^2 - 1}{12} = \frac{(n+1)(n-1)}{12}
\end{aligned}$$

Moment Generating Function:

Given $X \sim D(n), f(x) = \dfrac{1}{n}$:

$$\begin{aligned} M_x(t) &= E(e^{tx}) \\ &= \sum e^{tx} \cdot P(x) \\ &= \sum_{x=1}^{n} e^{tx} \cdot \frac{1}{n} \\ &= \frac{1}{n} \sum_{x=1}^{n} e^{tx} \\ M_x(t) &= \frac{e^t(1 - e^{nt})}{n(1 - e^t)} \end{aligned}$$

Bernoulli Distribution:

Definition:

Let X be a discrete r.v. It follows uniform distribution if its p.m.f is:

$$P(x) = \begin{cases} p^x q^{1-x}, & \text{if } x = 0, 1 \\ 0, & \text{otherwise} \end{cases}$$

It is denoted by $X \sim B(p)$.

Applications:

1) Tossing a coin, getting heads or tails.
2) Rolling a die, getting odd or even.
3) Picking a card, getting red or black.
4) Selecting item, defective or not defective.

Expectation and Variance:

$$\begin{aligned} E(X) &= \sum x \cdot P(x) \\ &= \sum_{x=0}^{1} x \cdot p^x q^{1-x} \\ &= 0 \cdot p^0 \cdot q + 1 \cdot p \cdot q^0 \\ E(X) &= p \end{aligned}$$

$$\begin{aligned} E(X^2) &= \sum x^2 \cdot P(x) \\ &= \sum_{x=0}^{1} x^2 \cdot p^x q^{1-x} \\ &= 0 \cdot p^0 \cdot q + 1 \cdot p \cdot q^0 \\ E(X^2) &= p \end{aligned}$$

$$\begin{aligned} Var(x) &= E(X^2) - [E(X)]^2 \\ &= p - p^2 \\ &= p(1-p) \\ Var(X) &= pq \end{aligned}$$

Moment Generating Function:

Given $X \sim B(n), f(x) = p^x q^{1-x}$:

$$\begin{aligned} M_x(t) &= E(e^{tx}) \\ &= \sum e^{tx} \cdot P(x) \\ &= \sum_{x=0}^{1} e^{tx} \cdot p^x q^{1-x} \\ &= e^{0t} \cdot q + e^{1t} \cdot p \\ M_x(t) &= q + pe^t \end{aligned}$$

Binomial Distribution:

Definition:

Let X be a discrete r.v. It follows uniform distribution if its p.m.f is:

$$P(x) = \begin{cases} \binom{n}{x} p^x q^{n-x}, & \text{if } x = 0, 1 \dots n \\ 0, & \text{otherwise} \end{cases}$$

It is denoted by $X \sim B(n, p)$.

Applications:

1) Tossing multiple coins, getting heads or tails certain number of times.
2) Repeating any Bernoulli event n number of times.

Expectation and Variance:

$$\begin{aligned}
E(X) &= \sum x \cdot P(x) \\
&= \sum_{x=0}^{n} x \cdot \binom{n}{x} p^x q^{1-x} \\
&= 0 + 1 \cdot \binom{n}{1} \cdot p^1 \cdot q^{n-1} + 2 \cdot \binom{n}{2} \cdot p^2 \cdot q^{n-2} + \cdots + n \cdot {}^nC_n \cdot p^n \cdot q^{n-n} \\
&= npq^{n-1} + n(n-1)p^2q^{n-2} + n(n-1)(n-2)p^3q^{n-3} + \cdots + np^n \\
&= np[q^{n-1} + (n-1)pq^{n-2} + \cdots + p^{n-1}] \\
&= np(p+q)^{n-1} && (p + q = 1) \\
E(X) &= np
\end{aligned}$$

$$\begin{aligned}
E(X^2) &= \sum x^2 \cdot P(x) \\
&= \sum_{x=0}^{n} x + x(x-1) \cdot \binom{n}{x} p^x q^{n-x} \\
&= \sum_{x=0}^{n} x \cdot \binom{n}{x} p^x q^{n-x} \sum_{x=0}^{n} x(x-1) \cdot \binom{n}{x} p^x q^{n-x} \\
&= np + \sum_{x=0}^{n} x(x-1) \cdot \binom{n}{x} p^x q^{n-x} \\
&= np + (0 + 0 + n(n-1)p^2q^{n-2} + n(n-1)(n-2)p^3q^{n-3} + \cdots + np^n) \\
&= np + n(n-1)p(p+q)^{n-2} \\
E(X^2) &= np + n^2p^2 - np^2
\end{aligned}$$

$$\begin{aligned}
Var(X) &= E(X^2) - [E(X)]^2 \\
&= np + n^2p^2 - np^2 - n^2p^2 \\
&= np - np^2 \\
&= np(1-p) \\
Var(X) &= npq
\end{aligned}$$

Note: If $X_1, X_2, X_3, \dots, X_k$ are independent B(n, p) then $\sum_{i=1}^{k} X_i \sim B(\sum_{i=1}^{k} n_i, p)$.

Assumptions:

1) The number of trials n is fixed and finite.
2) The probability of success p is the same for every trial.
3) $p + q = 1$, where q is the probability of failure.
4) p is independent for every trial.

Moment Generating Function:

Given $X \sim B(n,p), f(x) = \binom{n}{x} p^x q^{n-x}$:

$$\begin{aligned} M_x(t) &= E(e^{tx}) \\ &= \sum e^{tx} \cdot P(x) \\ &= \sum_{x=0}^{n} e^{tx} \cdot \binom{n}{x} p^x q^{n-x} \\ &= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x \; q^{n-x} \\ M_x(t) &= (q + pe^t)^n \end{aligned}$$

Poisson Distribution:

Let X be a discrete r.v. It follows uniform distribution if its p.m.f is:

$$P(x) = \begin{cases} \frac{e^{-\lambda}\lambda^x}{x!} & \text{if } x = 0, 1 \ldots \\ 0, & \text{otherwise} \end{cases}$$

It is denoted by $X \sim P(\lambda)$.

Expectation and Variance:

$$\begin{aligned} E(X) &= \sum_{n=0}^{\infty} x \cdot P(x) \\ &= \sum_{n=0}^{\infty} \frac{x\ e^{-\lambda}\ \lambda^x}{x!} \\ &= e^{-\lambda} \sum_{n=0}^{\infty} \frac{x\ \lambda^x}{x!} \\ &= e^{-\lambda} \left(0 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \ldots\right) \\ &= e^{-\lambda} \cdot \lambda \left(1 + \frac{\lambda}{2!} \ldots\right) \\ &= e^{-\lambda} \cdot \lambda \cdot e^{\lambda} \\ E(X) &= \lambda \end{aligned}$$

$$\sum_{n=0}^{\infty} \frac{\lambda^x}{x!} = e^{\lambda}$$

$$\begin{aligned} E(X^2) &= \sum x^2 \cdot P(x) \\ &= \sum_{n=0}^{\infty} \frac{x^2\ e^{-\lambda}\ \lambda^x}{x!} \\ &= e^{-\lambda} \sum_{n=0}^{\infty} \frac{x + x(x-1)\ \lambda^x}{x!} \\ &= e^{-\lambda} \left[\sum_{n=0}^{\infty} \frac{x\ \lambda^x}{x!} + \sum_{n=0}^{\infty} \frac{x(x-1)\ \lambda^x}{x!}\right] \\ &= e^{-\lambda} \left[\lambda e^{\lambda} + \left(0 + 0 + \frac{2\lambda^2}{2!} + \frac{6\lambda^3}{3!} + \ldots\right)\right] \\ &= e^{-\lambda} \left[\lambda e^{\lambda} + \lambda^2 \left(1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} \ldots\right)\right] \\ &= e^{-\lambda} \left[\lambda e^{\lambda} + \lambda^2 e^{\lambda}\right] \\ E(X^2) &= \lambda + \lambda^2 \end{aligned}$$

$$\begin{aligned} Var(X) &= E(X^2) - [E(X)]^2 \\ &= \lambda + \lambda^2 - \lambda^2 \\ Var(X) &= \lambda \end{aligned}$$

Assumptions

1) Mean = Variance = λ.
2) $\sigma = \sqrt{\lambda}$.

3) If $X_1, X_2, X_3, \ldots, X_k$ are independent P(λ_i) then $\sum_{i=1}^{k} X_i \sim P(\sum_{i=1}^{k} \lambda_i)$.

Applications:

1) Probability of rain in many summers.
2) Probability of a misprint in a page across a library.
3) Probability of an accident in a large parking lot.

Moment Generating Function:

Given $X \sim P(\lambda), f(x) = \dfrac{e^{-\lambda}\lambda^x}{x!}$:

$$\begin{aligned}
M_x(t) &= E(e^{tx}) \\
&= \sum e^{tx} \cdot P(x) \\
&= \sum_{x=0}^{\infty} e^{tx} \cdot \frac{e^{-\lambda}\lambda^x}{x!} \\
&= e^{-\lambda} \sum_{x=0}^{\infty} \frac{e^{tx}\lambda^x}{x!} \\
&= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} \\
&= e^{-\lambda} \cdot e^{\lambda e^t} \qquad\qquad \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{\lambda} \\
M_x(t) &= e^{\lambda(e^t - 1)}
\end{aligned}$$

Geometric Distribution:

Definition:

Let X be a discrete r.v. It follows uniform distribution if its p.m.f is:

Type 1:

$$P(x) = \begin{cases} pq^{x-1}, & \text{if } x = 1, 2, \ldots \\ 0, & \text{otherwise} \end{cases}$$

It is denoted by $X \sim G(p)$. In this case, x is the number of trials.

Expectation and Variance:

$$\begin{aligned}
E(X) &= \sum x \cdot P(x) \\
&= \sum x \cdot pq^{x-1} \\
&= p \sum xq^{x-1} \\
&= p\left[0 + 1 + 2q + 3q^2 \ldots\right] \\
&= p \cdot \frac{1}{p^2} \\
E(X) &= \frac{1}{p}
\end{aligned}$$

$$\begin{aligned}
E(X^2) &= \sum x^2 \cdot P(x) \\
&= \sum [x + x(x-1)] \cdot pq^{x-1} \\
&= \sum x \cdot pq^{x-1} + \sum x(x-1) \cdot pq^{x-1} \\
&= \frac{1}{p} + p \sum x(x-1) \cdot q^{x-1} \\
&= \frac{1}{p} + p(0 + 0 + 2q + 6q^2 + 12q^3 \ldots) \\
&= \frac{1}{p} + 2pq(1 + 3q + 6q^2 \ldots) \\
&= \frac{1}{p} + p \cdot q \cdot \frac{1}{p^3} \\
&= \frac{p + 2q}{p^2} \\
&= \frac{p + q + q}{p^2} \\
E(X^2) &= \frac{1 + q}{p^2}
\end{aligned}$$

$$\begin{aligned}
Var(X) &= E(X^2) - [E(X)]^2 \\
&= \frac{q + 1}{p^2} - \frac{1}{p^2} \\
&= \frac{q}{p^2} + \frac{1}{p^2} - \frac{1}{p^2} \\
Var(X) &= \frac{q}{p^2}
\end{aligned}$$

Type 2:

$$P(x) = \begin{cases} pq^x, & \text{if } x = 0, 1, 2, \dots \\ 0, & \text{otherwise} \end{cases}$$

It is denoted by $X \sim G(p)$. In this case, x is the number of failures.

Expectation and Variance:

$$\begin{aligned}
E(X) &= \sum x \cdot P(x) \\
&= \sum x \cdot pq^x \\
&= p \sum xq^x \\
&= p[q + 2q^2 + 3q^3 \dots] \\
&= pq\left[1 + 2q + 3q^2 \dots\right] \\
&= pq \cdot \frac{1}{p^2} \\
E(X) &= \frac{q}{p}
\end{aligned}$$

$$\begin{aligned}
E(X^2) &= \sum x^2 \cdot P(x) \\
&= \sum [x + x(x-1)] \cdot pq^x \\
&= \sum x \cdot pq^x + \sum x(x-1) \cdot pq^x \\
&= \frac{q}{p} + p \sum x(x-1) \cdot q^x \\
&= \frac{q}{p} + p(0 + 0 + 2q^2 + 6q^3 + 12q^4 \dots) \\
&= \frac{q}{p} + 2pq^2(1 + 3q + 6q^2 \dots) \\
&= \frac{q}{p} + 2p \cdot q^2 \cdot \frac{1}{p^3} \\
E(X^2) &= \frac{q}{p} + \frac{2p^2}{q^2}
\end{aligned}$$

$$\begin{aligned}
Var(X) &= E(X^2) - [E(X)]^2 \\
&= \frac{q}{p} + \frac{2p^2}{q^2} - \frac{q^2}{p^2} \\
&= \frac{q}{p} + \frac{q^2}{p^2} \\
&= \frac{pq + q^2}{p^2} \\
&= \frac{q(p+q)}{p^2} \\
Var(X) &= \frac{q}{p^2}
\end{aligned}$$

Moment Generating Function: Given $X \sim G(p), f(x) = pq^x$:

$$\begin{aligned} M_x(t) &= E(e^{tx}) \\ &= \sum e^{tx} \cdot P(x) \\ &= \sum_{x=0}^{\infty} e^{tx} \cdot pq^x \\ &= p \sum_{x=0}^{\infty} (qe^t)^x \\ M_x(t) &= \frac{p}{1 - qe^t} \end{aligned}$$

Negative Binomial Distribution:

Definition:

Let X be a discrete r.v. It follows uniform distribution if its p.m.f is:

$$P(x) = \begin{cases} \binom{k+r-1}{r-1} p^r q^x, & \text{if } x = 0, 1, 2, \dots \\ 0, & \text{otherwise} \end{cases}$$

It is denoted by $X \sim G(p)$.

Part III

Continuous Probability Distributions

Continuous Probability Distributions

Rectangular Distribution:

Definition: If c.r.v $X \sim U(a,b)$, then its p.d.f is:

$$f(x) = \begin{cases} \dfrac{1}{b-a}, & \text{if } a \le x \le b \\ \\ 0, & \text{otherwise} \end{cases}$$

Expectation, Median, Mode and Variance:

$$\begin{aligned} E(X) &= \int_a^b x \cdot f(x)dx \\ &= \int_a^b \frac{x}{b-a}dx \\ &= \frac{x^2}{2(b-a)}\Bigg|_a^b \\ &= \frac{b^2-a^2}{2(b-a)} \\ &= \frac{(b+a)(b-a)}{2(b-a)} \\ E(X) &= \frac{b+a}{2} \end{aligned}$$

$$\begin{aligned} E(X^2) &= \int_a^b x^2 \cdot f(x)dx \\ &= \int_a^b \frac{x^2}{b-a}dx \\ &= \frac{x^3}{3(b-a)}\Bigg|_a^b \\ &= \frac{b^3-a^3}{2(b-a)} \\ &= \frac{(b^2+ba+a^2)(b-a)}{3(b-a)} \\ E(X^2) &= \frac{b^2+ba+a^2}{3} \end{aligned}$$

$$\begin{aligned} Var(x) &= E(X^2) - [E(X)]^2 \\ &= \frac{b^2+ba+a^2}{3} - \left[\frac{b+a}{2}\right]^2 \\ &= \frac{4a^2+4ab+4b^2-3a^2-6ab-3b^2}{12} \\ &= \frac{b^2-2ab+a^2}{12} \\ Var(X) &= \frac{(b-a)^2}{12} \end{aligned}$$

Standard Deviation σ:

$$\sigma = \sqrt{Var(x)}$$
$$\sigma = \frac{(b-a)}{\sqrt{12}}$$

To find median M:

$$\int_a^M f(x)dx = \frac{1}{2}$$
$$\frac{M-a}{b-a} = \frac{1}{2}$$
$$2M - 2a = b - a$$
$$M = \frac{b+a}{2}$$

Mode of Rectangular Distribution is every x such that $a \leq x \leq b$ as:

$$\frac{d}{dx}\frac{1}{b-a} = 0$$

Therefore, all points are its maxima and minima.

Moment Generating Function:

$$M_x(t) = E(e^{tx})$$
$$= \int_a^b e^{tx} \cdot f(x)dx$$
$$= \frac{1}{b-a}\left.\frac{e^{tx}}{t}\right|_a^b$$
$$= \frac{1}{t(b-a)} \cdot (e^{bt} - e^{at})$$
$$M_x(t) = \frac{e^{bt} - e^{at}}{t(b-a)}$$

First Raw Moment:

$$\mu_r' = \int_a^b x^r f(x)dx$$
$$\mu_r' = \frac{1}{b-a}\left[\frac{b^{r+1} - a^{r+1}}{r+1}\right]$$

Cumulant Generating Function:

$$K_x(t) = log\left[\frac{e^{bt} - e^{at}}{t(b-a)}\right]$$

C.G.F for $t = 1$:

$$K_x(1) = log\left[\frac{e^b - e^a}{b-a}\right]$$

Characteristic Function:

$$\phi_x(t) = E(e^{itx})$$
$$\phi_x(t) = \frac{e^{ibt} - e^{iat}}{it(b-a)}$$

Triangular Distribution:

Definition: If c.r.v $X \sim T(a,b)$ with mode c, then its p.d.f is:

$$f(x) = \begin{cases} \dfrac{2(x-a)}{(b-a)(c-a)}, & \text{if } a \le x \le c \\[2ex] \dfrac{2(b-x)}{(b-a)(b-c)}, & \text{if } c \le x \le b \\[2ex] 0, & \text{otherwise} \end{cases}$$

$$f(c) = \frac{2}{b-a}$$

Moment Generating Function: Moment Generating Function of $X \sim T(a,b)$ with mode c is:

$$M_x(t) = \frac{2}{t^2}\left[\frac{e^{at}}{(a-b)(a-c)}\frac{e^{ct}}{(c-a)(c-b)}\frac{e^{bt}}{(b-a)(b-c)}\right]$$

If X & Y are i.i.d. $U(-a,a)$, then addition of X & Y, i.e. $X+Y \sim T(-2a,2a)$, with mode 0. Additionally, $X - Y \sim T(-2a,2a)$, with mode 0.

Properties: 1) $-\infty < a < b < \infty, c \in [a,b]$
2) if $C < E(X)$, distribution is positively skewed.
if $C > E(X)$, distribution is negatively skewed.
if $C = E(X)$, distribution is symmetric.

Gamma Distribution:

One Parameter: If c.r.v $X \sim \gamma(\lambda)$, then its p.d.f is:

$$f(x) = \begin{cases} \dfrac{e^{-x}x^{\lambda-1}}{\Gamma\lambda}, & \text{if } 0 < x < \infty,\ \lambda > 0 \\ \\ 0, & \text{otherwise} \end{cases}$$

Properties:

$$\Gamma n = \int_0^\infty e^{-x}x^{n-1}dx$$
$$\Gamma(n+1) = n\Gamma n$$
$$\Gamma n = (n-1)!$$

Expectation and Variance:

$$\begin{aligned} E(X) &= \int_0^\infty x \cdot f(x)dx \\ &= \frac{x\ e^{-x}x^{\lambda-1}}{\Gamma\lambda} \\ &= \frac{e^{-x}x^{\lambda}}{\Gamma\lambda} \\ &= \frac{\Gamma(\lambda+1)}{\Gamma\lambda} \\ &= \frac{\lambda\ \Gamma\lambda}{\Gamma\lambda} \\ E(X) &= \lambda \end{aligned}$$

$$\begin{aligned} E(X^2) &= \int_0^\infty x^2 \cdot f(x)dx \\ &= \frac{x^2\ e^{-x}x^{\lambda-1}}{\Gamma\lambda} \\ &= \frac{e^{-x}x^{\lambda+1}}{\Gamma\lambda} \\ &= \frac{\Gamma(\lambda+2)}{\Gamma\lambda} \\ &= \frac{\lambda(\lambda+1)\Gamma\lambda}{\Gamma\lambda} \\ E(X^2) &= \lambda(\lambda+1) = \lambda^2 + \lambda \end{aligned}$$

$$\begin{aligned} Var(x) &= E(X^2) - [E(X)]^2 \\ &= (\lambda^2 + \lambda) - \lambda^2 \\ Var(X) &= \lambda \end{aligned}$$

Moment Generating Function:

$$M_x(t) = (1-t)^{-\lambda}$$

Cumulant Generating Function: The value of the n^{th} cumulant is $\lambda(n-1)!$.

Raw Moments: The value of the r^{th} raw moment is

$$\begin{aligned}
\mu'_r &= \int_0^\infty x^r\, f(x)dx \\
&= \int_0^\infty x^r \cdot \frac{e^{-x}x^{\lambda-1}}{\Gamma\lambda} \\
&= \frac{e^{-x}x^{\lambda+r-1}}{\Gamma\lambda} \\
&= \frac{\Gamma(\lambda+r)}{\Gamma\lambda} \\
\mu'_r &= \frac{\Gamma(\lambda+r)}{\Gamma\lambda} \\
\mu'_r &= \Pi_{i=0}^{n-1}(\lambda+i)
\end{aligned}$$

Coefficients of Skewness and Kurtosis: The coefficients of skewness and kurtosis for the gamma distribution are:

$$\begin{aligned}
\beta_1 &= \frac{\mu_3^2}{\mu_2^3} \\
&= \frac{(2\lambda)^2}{\lambda^3} \\
\beta_1 &= \frac{4}{\lambda}
\end{aligned}$$

$$\begin{aligned}
\gamma_1 &= \sqrt{\beta_1} \\
&= \sqrt{\frac{4}{\lambda}} \\
\gamma_1 &= \frac{2}{\sqrt{\lambda}}
\end{aligned}$$

$$\begin{aligned}
\beta_2 &= \frac{\mu_4}{\mu_2^2} \\
&= \frac{6\lambda}{\lambda^2} \\
\beta_2 &= \frac{6}{\lambda}
\end{aligned}$$

$$\begin{aligned}
\gamma_2 &= \beta_2 - 3 \\
\gamma_2 &= \frac{6}{\lambda} - 3
\end{aligned}$$

Note: If $X_1, X_2, X_3, \ldots, X_n$ are independent $\gamma(\lambda_i)$ then $\sum_{i=1}^n X_i \sim \gamma(\sum_{i=1}^n \lambda_i)$.

Two Parameter: If c.r.v $X \sim G(\lambda, a)$, then its p.d.f is:

$$f(x) = \begin{cases} \dfrac{a^\lambda e^{-ax}x^{\lambda-1}}{\Gamma\lambda}, & \text{if } 0 < x < \infty,\ \lambda > 0, a > 0 \\ 0, & \text{otherwise} \end{cases}$$

Expectation and Variance:

$$\begin{aligned}
E(X) &= \int_0^\infty x \cdot f(x)dx \\
&= \int_0^\infty \frac{a^\lambda e^{-ax} x^{\lambda-1}}{\Gamma\lambda} dx \\
&= \int_0^\infty \frac{a^\lambda e^{-u} u^{\lambda+1}}{a^{\lambda+1}\,\Gamma\lambda} du && u = ax,\ dx = \frac{du}{a},\ x = \frac{u}{a} \\
&= \int_0^\infty \frac{e^{-u} u^{\lambda+1}}{a^2} du \\
&= \frac{\lambda\,\Gamma\lambda}{a\,\Gamma\lambda} \\
E(X) &= \frac{\lambda}{a}
\end{aligned}$$

$$\begin{aligned}
E(X^2) &= \int_0^\infty x^2 \cdot f(x)dx \\
&= \int_0^\infty \frac{a^\lambda e^{-ax} x^{\lambda+1}}{\Gamma\lambda} dx \\
&= \int_0^\infty \frac{a^\lambda e^{-u} u^{\lambda+1}}{a^{\lambda+2}\,\Gamma\lambda} du && u = ax,\ dx = \frac{du}{a},\ x = \frac{u}{a} \\
&= \int_0^\infty \frac{e^{-u} u^{\lambda+1}}{a^2\,\Gamma\lambda} du \\
&= \frac{\Gamma(\lambda+2)}{a^2\,\Gamma\lambda} \\
&= \frac{\lambda(\lambda+1)\,\Gamma\lambda}{a^2\,\Gamma\lambda} \\
E(X^2) &= \frac{\lambda(\lambda+1)}{a^2}
\end{aligned}$$

$$\begin{aligned}
Var(x) &= E(X^2) - [E(X)]^2 \\
&= \frac{\lambda(\lambda+1)}{a^2} - \frac{\lambda}{a}^2 \\
&= \frac{\lambda^2+\lambda}{a^2} - \frac{\lambda}{a}^2 \\
Var(X) &= \frac{\lambda}{a^2}
\end{aligned}$$

Moment Generating Function:

$$M_x(t) = (1 - {}^t\!/_a)^{-\lambda}$$

Cumulant Generating Function: The value of the n^{th} cumulant is $\frac{\lambda(n-1)!}{a^n}$.

Raw Moments: The value of the r^{th} raw moment is

$$\begin{aligned}
\mu_r' &= \int_0^\infty x^r\ f(x)dx \\
&= \int_0^\infty x^r \cdot a^\lambda \cdot \frac{e^{-ax}x^{\lambda-1}}{\Gamma\lambda} \\
&= \int_0^\infty \frac{a^\lambda \cdot e^{-ax}x^{\lambda+r-1}}{\Gamma\lambda} \\
&= \frac{\Gamma(\lambda+r)}{\Gamma\lambda} \\
\mu_r' &= \frac{\Gamma(\lambda+r)}{\Gamma\lambda} \\
\mu_r' &= \Pi_{i=0}^{n-1}(\lambda+i)
\end{aligned}$$

Coefficients of Skewness and Kurtosis: The coefficients of skewness and kurtosis for the gamma distribution are:

$$\begin{aligned}
\beta_1 &= \frac{\mu_3^2}{\mu_2^3} \\
&= \frac{(2\lambda)^2/\lambda^3}{a^6/a^6} \\
\beta_1 &= \frac{4}{\lambda} \\
\\
\gamma_1 &= \sqrt{\beta_1} \\
&= \sqrt{\frac{4}{\lambda}} \\
\gamma_1 &= \frac{2}{\sqrt{\lambda}} \\
\\
\beta_2 &= \frac{\mu_4}{\mu_2^2} \\
&= \frac{6\lambda/\lambda^2}{a^4/a^4} \\
\beta_2 &= \frac{6}{\lambda} \\
\\
\gamma_2 &= \beta_2 - 3 \\
\gamma_2 &= \frac{6}{\lambda} - 3
\end{aligned}$$

Note: If $X_1, X_2, X_3, \ldots, X_n$ are independent $\gamma(\lambda_i, a)$ then $\sum_{i=1}^n X_i \sim \gamma(\sum_{i=1}^n \lambda_i, a)$.

Beta Distribution:

The p.d.f of the general beta function is given by:

$$\int_a^b \frac{(x-a)^{m-1}(b-x)^{n-1}}{(b-a)^{m+n-1}}, a < x < b$$

Types of β distribution:
If $a \neq 0$ & $b \neq 0$ or $a \neq 0$ & $b \neq \infty$, it is called an incomplete beta distribution.
If $a = 0$ & $b = 1$, it is the first type of beta distribution.
If $a = 0$ & $b = \infty$, it is the second type of beta distribution.

Type 1(β_1) If c.r.v $X \sim \beta_1(m, n)$, then its p.d.f is:

$$f(x) = \begin{cases} \dfrac{x^{m-1}(1-x)^{n-1}}{\beta(m,n)}, & \text{if } 0 \leq x \leq 1; m, n > 0 \\ \\ 0, & \text{otherwise} \end{cases}$$

Properties:

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

$$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)}$$

Expectation and Variance:

$$
\begin{aligned}
E(X) &= \int_0^1 x \cdot f(x)dx \\
&= \int_0^1 \frac{x\ x^{m-1}(1-x)^{n-1}}{\beta(m,n)} \\
&= \int_0^1 \frac{x^{m}(1-x)^{n-1}}{\beta(m,n)} \\
&= \frac{\beta(m+1,n)}{\beta(m,n)} \\
&= \frac{\Gamma(m+1)\ \Gamma n}{\Gamma(m+n+1)} \frac{\Gamma(m+n)}{\Gamma m\ \Gamma n} \\
&= \frac{m\Gamma m\ \Gamma n}{(m+n)\Gamma(m+n)} \frac{\Gamma(m+n)}{\Gamma m\ \Gamma n} \\
E(X) &= \frac{m}{m+n}
\end{aligned}
$$

$$
\begin{aligned}
E(X^2) &= \int_0^1 x^2 \cdot f(x)dx \\
&= \int_0^1 \frac{x^2\ x^{m-1}(1-x)^{n-1}}{\beta(m,n)} \\
&= \int_0^1 \frac{x^{m+1}(1-x)^{n-1}}{\beta(m,n)} \\
&= \frac{\beta(m+2,n)}{\beta(m,n)} \\
&= \frac{\Gamma(m+2)\ \Gamma n}{\Gamma(m+n+2)} \frac{\Gamma(m+n)}{\Gamma m\ \Gamma n} \\
&= \frac{(m+1)m\Gamma m\ \Gamma n}{(m+n+1)(m+n)\Gamma(m+n)} \frac{\Gamma(m+n)}{\Gamma m\ \Gamma n} \\
E(X^2) &= \frac{m(m+1)}{(m+n)(m+n+1)}
\end{aligned}
$$

$$
\begin{aligned}
Var(x) &= E(X^2) - [E(X)]^2 \\
&= \frac{m(m+1)}{(m+n)(m+n+1)} - \frac{m}{m+n} \\
&= \frac{m}{m+n}\left[\frac{m+1}{m+n+1} - \frac{m}{m+n}\right] \\
&= \frac{m}{m+n}\left[\frac{(m+1)(m+n) - m(m+n+1)}{(m+n)(m+n+1)}\right] \\
&= \frac{m}{m+n}\left[\frac{m^2+mn+m+n-m^2-mn-m^2}{(m+n)(m+n+1)}\right] \\
&= \frac{m}{m+n}\left[\frac{n}{(m+n)(m+n+1)}\right] \\
Var(X) &= \frac{mn}{(m+n)^2(m+n+1)}
\end{aligned}
$$

Harmonic Mean:

$$\frac{1}{HM} = \int \frac{1}{x} f(x)dx$$
$$\frac{1}{HM} = \int \frac{x^{m-2}(1-x)^{n-1}dx}{\beta(m,n)}$$
$$\frac{1}{HM} = \frac{\beta(m-1,n)}{\beta(m,n)}$$
$$HM = \frac{\beta(m,n)}{\beta(m-1,n)}$$
$$HM = \frac{(m-1)\Gamma(m-1)\Gamma(m+n-1)}{(m+n-1)\Gamma(m+n-1)\Gamma(m-1)}$$
$$HM = \frac{m-1}{m+n-1}$$

Raw Moments: The value of the r^{th} raw moment is

$$\begin{aligned}
\mu'_r &= \int_0^1 x^r \, f(x)dx \\
&= \int_0^1 \frac{x^r \, x^{m-1}(1-x)^{n-1}}{\beta(m,n)} \\
&= \int_0^1 \frac{x^{m+r-1}(1-x)^{n-1}}{\beta(m,n)} \\
&= \frac{\beta(m+r,n)}{\beta(m,n)} \\
&= \frac{\Gamma(m+r)\ \Gamma n}{\Gamma(m+n+r)} \frac{\Gamma(m+n)}{\Gamma m \ \Gamma n} \\
\mu'_r &= \frac{\Gamma(m+r)\Gamma(m+n)}{\Gamma(m+n+r)\Gamma m} \\
\mu'_r &= \frac{\Pi_{i=0}^{n-1}(m+i)}{\Pi_{i=0}^{n-1}(m+n+i)}
\end{aligned}$$

Type 2(β_2) If c.r.v $X \sim \beta_2(m,n)$, then its p.d.f is:

$$f(x) = \begin{cases} \dfrac{1}{\beta(m,n)} \dfrac{x^{m-1}}{(1+x)^{m+n}}, & \text{if } 0 < x < \infty \\ \\ 0, & \text{otherwise} \end{cases}$$

Properties:

$$\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$$
$$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)}$$

Raw Moments: The value of the r^{th} raw moment is

$$\begin{aligned} \mu_r' &= \int_0^\infty x^r \, f(x) dx \\ &= \int_0^\infty x^r \, \frac{1}{\beta(m,n)} \frac{x^{m-1}}{(1+x)^{m+n}} \\ &= \frac{1}{\beta(m,n} \int_0^\infty \frac{x^{m+r-1}}{(1+x)^n} \\ &= \frac{\beta(m+r, n-r)}{\beta(m,n)} \\ &= \frac{\Gamma(m+r) \, \Gamma(n-r)}{\Gamma(m+n)} \frac{\Gamma(m+n)}{\Gamma m \, \Gamma n} \\ \mu_r' &= \frac{\Gamma(m+r)\Gamma(n-r)}{\Gamma m \, \Gamma n} \\ \mu_r' &= \frac{\Pi_{i=0}^{r-1}(m+i)}{\Pi_{i=0}^{r-1}(n-i-1)} \end{aligned}$$

Expectation and Variance:

$$\begin{aligned}
E(X) &= \mu_1' \\
&= \frac{\Gamma(m+1)\Gamma(n-1)}{\Gamma m\,\Gamma n} \\
&= \frac{m\,\Gamma m\,\Gamma(n-1)}{\Gamma m\,(n-1)\,\Gamma(n-1)} \\
E(X) &= \frac{m}{n-1}
\end{aligned}$$

$$\begin{aligned}
E(X^2) &= \mu_2' \\
&= \frac{\Gamma(m+2)\Gamma(n-2)}{\Gamma m\,\Gamma n} \\
&= \frac{m(m+1)\Gamma(m)\Gamma(n-2)}{\Gamma m(n-1)(n-2)\Gamma(n-2)} \\
E(X^2) &= \frac{m(m+1)}{(n-1)(n-2)}
\end{aligned}$$

$$\begin{aligned}
Var(x) &= E(X^2) - [E(X)]^2 \\
&= \frac{m(m+1)}{(n-1)(n-2)} - \left(\frac{m}{n-1}\right)^2 \\
&= \frac{m}{n-1}\left(\frac{m+1}{(n-2)} - \frac{m}{(n-1)}\right) \\
&= \frac{m}{n-1}\left(\frac{(m+1)(n-1)-m(n-2)}{(n-1)(n-2)}\right) \\
Var(X) &= \frac{m(m+n-1)}{(n-1)^2(n-2)}
\end{aligned}$$

Harmonic Mean:

$$\begin{aligned}
\frac{1}{HM} &= \int_0^\infty \frac{1}{x} f(x)dx \\
\frac{1}{HM} &= \int_0^\infty \frac{1}{\beta(m,n)} \frac{x^{m-2}}{(1+x)^{m+n}} \\
\frac{1}{HM} &= \frac{\beta(m-1,n+1)}{\beta(m,n)} \\
HM &= \frac{\beta(m,n)}{\beta(m-1,n+1)} \\
HM &= \frac{\Gamma m\Gamma n}{\Gamma(m+n)} \frac{\Gamma(m+n)}{\Gamma(m-1)\Gamma(n+1)} \\
HM &= \frac{(m-1)\Gamma(m-1)\Gamma(n)}{\Gamma(m-1)n\Gamma n} \\
HM &= \frac{m-1}{n}
\end{aligned}$$

If $X \sim \beta_1(1,1), X \sim U(0,1)$.

Exponential Distribution:

If c.r.v $X \sim \exp(\theta)$, then its p.d.f is:

$$f(x) = \begin{cases} \theta e^{-\theta x} & \theta > 0,\ 0 < x < \infty \\ 0, & \text{otherwise} \end{cases}$$

Expectation and Variance:

$$\begin{aligned} E(X) &= \int_0^\infty x \cdot f(x)dx \\ &= \int_0^\infty x\theta e^{-\theta x} \\ &= \int_0^\infty \frac{ue^{-u}}{\theta} du && \theta x = u,\ dx = \frac{du}{\theta} \\ &= \frac{1}{\theta}\int_0^\infty u^{2-1}e^{-u}du \\ &= \frac{\Gamma 2}{\theta} \\ &= \frac{1!}{\theta} \\ E(X) &= \frac{1}{\theta} \end{aligned}$$

$$\begin{aligned} E(X^2) &= \int_0^1 x^2 \cdot f(x)dx \\ &= \int_0^\infty x^2\theta e^{-\theta x} \\ &= \int_0^\infty x \cdot \theta x e^{-\theta x} \\ &= \int_0^\infty \frac{u}{\theta}\frac{ue^{-u}}{\theta} du && \theta x = u,\ dx = \frac{du}{\theta} \\ &= \frac{1}{\theta^2}\int_0^\infty u^{3-1}e^{-u}du \\ &= \frac{\Gamma 3}{\theta^2} \\ &= \frac{2!}{\theta^2} \\ E(X^2) &= \frac{2}{\theta^2} \end{aligned}$$

$$\begin{aligned} Var(X) &= E(X^2) - [E(X)]^2 \\ &= \frac{2}{\theta^2} - \frac{1}{\theta^2} \\ Var(X) &= \frac{1}{\theta^2} \end{aligned}$$

Moment Generating Function:

$$M_x(t) = (1 - {}^{t}/_{\theta})^{-1}$$

Cumulant Generating Function: The value of the n^{th} cumulant is $\dfrac{(n-1)!}{\theta^n}$.

Raw Moments: The value of the r^{th} raw moment is

$$
\begin{aligned}
\mu'_r &= \int_0^\infty x^r\, f(x)dx \\
&= \int_0^\infty x^r \cdot \theta e^{-\theta x} dx \\
&= \int_0^\infty \left(\frac{u}{\theta}\right)^r \cdot \theta e^{-u} \cdot \frac{du}{\theta} && u = \theta x \\
&= \frac{1}{\theta^r} \int_0^\infty u^r e^{-u} du \\
&= \frac{\Gamma(r+1)}{\theta^r} \\
\mu'_r &= \frac{r!}{\theta^r}
\end{aligned}
$$

Coefficients of Skewness and Kurtosis: The coefficients of skewness and kurtosis for the exponential distribution are:

$$
\begin{aligned}
\beta_1 &= \frac{\mu_3^2}{\mu_2^3} \\
&= \frac{(2/\theta^3)^2}{(1/\theta^2)^3} \\
\beta_1 &= 4
\end{aligned}
$$

$$
\begin{aligned}
\gamma_1 &= \sqrt{\beta_1} \\
&= \sqrt{4} \\
\gamma_1 &= 2
\end{aligned}
$$

$$
\begin{aligned}
\beta_2 &= \frac{\mu_4}{\mu_2^2} \\
&= \frac{6/\theta^4}{(1/\theta^2)^2} \\
\beta_2 &= 6
\end{aligned}
$$

$$
\begin{aligned}
\gamma_2 &= \beta_2 - 3 \\
&= 6 - 3 \\
\gamma_2 &= 3
\end{aligned}
$$

Normal Distribution:

If c.r.v $X \sim N$ with mean μ and variance σ^2, then its p.d.f is:

$$f(x) = \begin{cases} \frac{1}{\sigma\sqrt{2\pi}} \text{ exp}\left[\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] & -\infty < x < \infty,\ -\infty < \mu < \infty,\ 0 < \sigma < \infty \\ 0, & \text{otherwise} \end{cases}$$

It is denoted as $X \sim N(\mu, \sigma^2)$

Properties: 1) The normal distribution follows a bell-shaped curve.
2) In a normal distribution, mean=median=mode.
3) The normal distribution is symmetric.
4) The scale parameter σ distributes the curve in the following percentage:
i) $\mu \pm \sigma$ contains $\sim 68\%$ data
ii) $\mu \pm 2\sigma$ contains $\sim 95\%$ data
iii) $\mu \pm 3\sigma$ contains $\sim 99.73\%$ data
5) $Z = \frac{x-\mu}{\sigma} \sim$ s.n.d. i.e. $N(0,1)$.

$$P(Z = z) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}z^2}$$

Median and Mode: Assuming median$> \mu$,

$$\implies \int_{-\infty}^{\mu} f(x)dx + \int_{\mu}^{M} f(x)dx = \frac{1}{2}$$
$$\implies 1/2 + \int_{\mu}^{M} f(x)dx = \frac{1}{2}$$
$$\implies \int_{\mu}^{M} f(x)dx = 0$$
$$\implies \int_{\mu}^{M} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} = 0$$
$$\implies \frac{1}{\sigma\sqrt{2\pi}} \int_{\mu}^{M} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} = 0$$
$$\implies (x-\mu)\Big|_{\mu}^{M} = 0$$
$$\implies M - \mu = 0$$
$$\implies M = \mu$$

To find mode, we will use the first derivative test, i.e. $f'(x) = 0$.

$$log(f(x)) = -log(\sigma\sqrt{2\pi}) - \frac{(x-\mu)^2}{2\sigma^2}$$

Differentiating, we get

$$\frac{f'(x)}{f(x)} = -\frac{x-\mu}{\sigma^2}$$
$$\implies f(x)(x-\mu) = 0$$

As $f(x)$ cannot be 0 at the mode, $x - \mu = 0$

$$\implies x = \mu.$$

Hence, the second property (mean = mode = median) is proved.

Moment Generating Function and Cumulant Generating Function:

$$\begin{aligned}
M_x(t) &= E(e^{tx}) \\
&= \int_{-\infty}^{\infty} e^{tx} \cdot f(x)dx \\
&= \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} dx \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} dx \\
\text{Let } z &= \frac{x-\mu}{\sigma} \\
\implies x &= \sigma z + \mu \\
\&\ dx &= \sigma dz \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\sigma z+\mu)-\frac{z^2}{2}} \sigma dz \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tz\sigma} \cdot e^{t\mu} \cdot e^{-\frac{z^2}{2}} dz \\
&= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2-2tz\sigma)} dz \\
&= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2-2tz\sigma+t^2\sigma^2-t^2\sigma^2)} dz \\
&= \frac{e^{t\mu+\frac{t^2\sigma^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-t\sigma)^2} dz \\
\text{Let } \theta &= z - t\sigma \\
\implies dz &= d\theta \\
&= \frac{e^{t\mu+\frac{t^2\sigma^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{\theta^2}{2}} d\theta \\
&= \frac{e^{t\mu+\frac{t^2\sigma^2}{2}}}{\sqrt{2\pi}} \cdot \sqrt{2\pi} \\
M_x(t) &= e^{t\mu+\frac{t^2\sigma^2}{2}}
\end{aligned}$$

$$\begin{aligned}
K_x(t) &= log(M_x(t)) \\
&= log(e^{t\mu+\frac{t^2\sigma^2}{2}}) \\
&= t\mu + \frac{t^2\sigma^2}{2}
\end{aligned}$$

Cumulants:

$$K_1 = \text{coefficient of } t/1! = \mu$$

$$K_2 = \text{coefficient of } t^2/2! = \sigma^2$$

$$K_3 = \text{coefficient of } t^3/3! = 0$$

All further cumulants are 0.

Mean Deviation: M.D. from $\bar{X}$ is calculated as:

$$\int_{-\infty}^{\infty} |X - \bar{X}| = \int_{-\infty}^{\infty} |X - \mu|$$

$$
\begin{aligned}
E(|X-\mu|) &= \int_{-\infty}^{\infty} |x-\mu| f(x) dx \\
&= \int_{-\infty}^{\infty} |x-\mu| \frac{1}{\sigma\sqrt{2\pi}}\ e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} dx \\
\text{Let } z &= \frac{x-\mu}{\sigma} \\
\implies x-\mu &= \sigma z \\
\&\ dx &= \sigma dz \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} |z\sigma| e^{-\frac{z^2}{2}} \sigma dz \\
&= \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |z| e^{-\frac{z^2}{2}} dz \\
&= \frac{2\sigma}{\sqrt{2\pi}} \int_{0}^{\infty} z e^{-\frac{z^2}{2}} dz \\
\text{Let } t &= \frac{z^2}{2} \\
\implies \quad dz &= dt/z \\
&= \sigma\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} z e^{-t} \frac{dt}{z} \\
&= \sigma\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-t} dt \\
&= \sigma\sqrt{\frac{2}{\pi}}\ \ -e^{-t}\Big|_0^{\infty} \\
&= \sigma\sqrt{\frac{2}{\pi}} (e^{-0} - e^{-\infty}) \\
E(|X-\mu|) &= \sigma\sqrt{\frac{2}{\pi}} \approx 0.8\,\sigma
\end{aligned}
$$

Central Moments of Normal Distribution: Odd Moments:

$$
\begin{aligned}
\mu_{2n+1} &= E(X-\mu)^{2n+1} \\
&= \int_{-\infty}^{\infty} (x-\mu)^{2n+1} \frac{1}{\sigma\sqrt{2\pi}}\ e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} dx \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)^{2n+1} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} dx \\
\text{Let } z &= \frac{x-\mu}{\sigma} \\
\implies x-\mu &= \sigma z \\
\&\ dx &= \sigma dz \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (z\sigma)^{2n+1} e^{-\frac{z^2}{2}} dz \\
\mu_{2n+1} &= \frac{\sigma^{2n+1}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^{2n+1} e^{-\frac{z^2}{2}} dz
\end{aligned}
$$

As z^{2n+1} is an odd function, and $e^{-\frac{z^2}{2}}$ is an even function, its product is an odd function. Integrals of odd functions from $-\infty$ to ∞ are 0. Hence, the integral given equals 0, and $\mu_{2n+1} = 0\ \forall\ n \in \mathbb{N}$.

Even Moments:

$$\begin{aligned}
\mu_{2n} &= \int_{-\infty}^{\infty} (x-\mu)^{2n} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} dx \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)^{2n} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} dx \\
\text{Let } z &= \frac{x-\mu}{\sigma} \\
\implies\ x-\mu &= \sigma z \\
\&\ dx &= \sigma dz \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (z\sigma)^{2n} e^{-\frac{z^2}{2}} dz \\
&= \frac{\sigma^{2n}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^{2n} e^{-\frac{z^2}{2}} dz \\
\text{Let } t &= \frac{z^2}{2} \\
\implies\ dz &= dt/\sqrt{2t} \\
&= \frac{\sigma^{2n}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (2t)^n e^{-t} \frac{dt}{\sqrt{2t}} \\
&= \frac{2^n\ \sigma^{2n}}{2\sqrt{\pi}} \int_{-\infty}^{\infty} t^{n-1/2}\ e^{-t} dt \\
&= \frac{2^n\ \sigma^{2n}}{2\sqrt{\pi}} 2 \int_{0}^{\infty} t^{n-1/2}\ e^{-t} dt \\
\mu_{2n} &= \frac{2^n\ \sigma^{2n}}{\sqrt{\pi}} \Gamma(n+1/2)
\end{aligned}$$

Recursive Relation of Even Moments:

$$\begin{aligned}
\mu_{2n} &= \frac{2^n\ \sigma^{2n}}{\sqrt{\pi}} \Gamma(n+1/2) \\
\mu_{2n-2} &= \frac{2^{n-1}\ \sigma^{2n-2}}{\sqrt{\pi}} \Gamma(n-1/2) \\
\implies\ \frac{\mu_{2n}}{\mu_{2n-2}} &= \frac{2^n\ \sigma^{2n}\Gamma(n+1/2)}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2^{n-1}\ \sigma^{2n-2}\Gamma(n-1/2)} \\
&= \frac{2\sigma^2\Gamma(n+1/2)}{\Gamma(n-1/2)} \\
&= \frac{2\sigma^2(n-1/2)\Gamma(n-1/2)}{\Gamma(n-1/2)} \\
\frac{\mu_{2n}}{\mu_{2n-2}} &= 2\sigma^2(n-1/2)
\end{aligned}$$

Or,

$$\mu_{2n} = \sigma^2(2n-1)\cdot\mu_{2n-2}$$

Addition Property: Theorem: If $X_1, X_2, X_3, \ldots, X_n$ are independent $N(\mu_i, \sigma_i^2)$ then $\sum_{i=1}^n a_i X_i \sim N(\sum_{i=1}^n a_i\mu_i, \sum_{i=1}^n a_i^2\sigma_i^2)$.
Proof: For Normal Distribution,

$$M_x(t) = e^{t\mu + \frac{t^2\sigma^2}{2}}$$

$$\begin{aligned}
M_{X_1+X_2+\cdots+X_n} &= M_{X_1}(t) \cdot M_{X_2}(t) \ldots M_{X_n}(t) \\
&= e^{t\mu_1 + \frac{t^2\sigma_1^2}{2}} \cdot e^{t\mu_2 + \frac{t^2\sigma_2^2}{2}} \cdot \cdots \cdot e^{t\mu_n + \frac{t^2\sigma_n^2}{2}} \\
M_{\sum_{i=1}^{n} X_i} &= e^{t(\mu_1+\mu_2+\cdots+\mu_n) + \frac{t^2}{2}(\sigma_1^2+\sigma_2^2+\cdots+\sigma_n^2)}
\end{aligned}$$

By Uniqueness Theorem of m.g.f., the m.g.f. of any linear combination of n.r.v's follows normal distribution with mean $\sum_{i=1}^{n} a_i\mu_i$ and variance $\sum_{i=1}^{n} a_i^2\sigma_i^2$).

Skewness and Kurtosis: We know the first four central moments of the normal distribution are:

$$\begin{aligned}
\mu_1 &= 0 \\
\mu_2 &= \sigma^2 \\
\mu_3 &= 0 \\
\mu_4 &= K_4 + K_2^2 \\
&= 3\sigma^4
\end{aligned}$$

This gives us values of skewness and kurtosis:

$$\begin{aligned}
\beta_1 &= \frac{\mu_3^2}{\mu_2^3} \\
&= \frac{0}{\sigma^6} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\beta_2 &= \frac{\mu^4}{\mu_2^2} \\
&= \frac{3\sigma^4}{\sigma^4} \\
&= 3
\end{aligned}$$

As $\beta_1 = 0$ and $\beta_2 = 3$, we can definitively say that the normal distribution is always symmetric, irrespective of its parameters.

Log-Normal Distribution:

If $Y = logX \sim N(\mu, \sigma^2), X$ follows lognormal distribution.

$$\begin{aligned}
F_X(X) &= f(X \leq x) \\
&= f(logX \leq logx) \\
&= f(Y \leq logx) \\
&= \int_{-\infty}^{logx} \frac{1}{\sigma\sqrt{2\pi}} \text{exp}\left(-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2\right) dy
\end{aligned}$$

$$\begin{aligned}
\mu_r' &= E(x^r) \\
&= E(e^{yr}) \\
&= \text{exp}\left(\mu r + \frac{\sigma^2 r^2}{2}\right)
\end{aligned}$$

Hence,

$$\begin{aligned} E(X) &= \mu_1' \\ &= \text{exp}\left(\mu + \frac{\sigma^2}{2}\right) \end{aligned}$$

$$\begin{aligned} E(X^2) &= \mu_2' \\ &= \text{exp}[2(\mu + \sigma^2)] \end{aligned}$$

$$\begin{aligned} Var(X) &= \mu_2' - \mu_1'^2 \\ &= \text{exp}[2(\mu + \sigma^2)] - \text{exp}\left(\mu + \frac{\sigma^2}{2}\right)^2 \end{aligned}$$

Cauchy Distribution:

If c.r.v $X \sim C$, then its p.d.f is:

$$f(x) = \begin{cases} \frac{1}{\pi}\frac{1}{1+x^2} & ; -\infty < x < \infty \\ \\ 0, & ; \text{otherwise} \end{cases}$$

$$X = \frac{Y - \mu}{\sigma}$$

Then, if Y $\sim C(\lambda, \mu)$, $X \sim C(1, 0)$. And, X is the standard Cauchy distribution.

$$G_Y(Y) = \frac{\lambda}{\pi(\lambda^2 + (Y - \mu)^2)}$$

Hence,

$$\begin{aligned} E(Y) &= \int_{-\infty}^{\infty} y\ g(y) dy \\ &= \frac{\lambda}{\pi}\int_{-\infty}^{\infty} \frac{y}{(\lambda^2 + (y-\mu)^2)} \\ &= \frac{\mu\lambda}{\pi}\int_{-\infty}^{\infty} \frac{dy}{(\lambda^2 + (y-\mu)^2)} + \frac{\lambda}{\pi}\int_{-\infty}^{\infty} \frac{y-\mu}{(\lambda^2 + (y-\mu)^2)} dy \\ &= \mu + \frac{\lambda}{\pi}\int_{-\infty}^{\infty} \frac{z}{(\lambda^2 + z^2)} dz \end{aligned}$$

This integral is undefined, hence the first moment i.e. the mean does not exist.

Bivariate Distributions: If X and Y are c.r.v.s, then $F_{XY}(X, Y)$ is the joint p.d.f. iff

$$0 < f_{XY}(x, y) < 1$$

If $f(X, Y)$ is the joint density function then,

$$\int_X \int_Y f(X, Y)\ dy\ dx = 1$$

Marginal p.d.f's are:

$$f_X(x) = \int_Y f(X, Y)\ dy,$$

$$f_Y(y) = \int_X f(X, Y)\ dx$$

Conditional p.d.f's are:

$$f(Y|X = x) = \frac{f(X, Y)}{f(X = x)}$$

$$P(x_1 < X < x_2, y_1 < Y < y_2) = F(x_2, y_2) - f(x_1, y_1)$$

If X and Y are independent,

$$f_{XY}(x, y) = f_X(x) \cdot f_Y(y)\ \forall\ x, y$$

Bivariate Normal Distribution:

If X & Y are jointly distributed with bivariate normal distribution with parameters $(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$, then their j.p.d.f. is:

$$f(X, Y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - \frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]\right\}$$

The j.p.d.f. of s.n.b.d. is:

$$f(X, Y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[x^2 - 2\rho xy + y^2\right]\right\}$$

If X & Y are jointly distributed with bivariate normal distribution with joint probability distribution function $f(X, Y)$, then conditional probability distribution $P(X|Y)$ is given as:

$$\frac{f(X, Y)}{f(Y)} = \frac{1}{\sigma_1\sqrt{2\pi}\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2\sigma_1^2(1-\rho^2)}\left[x - \left(\mu_1 + \rho\frac{\sigma_1}{\sigma_2}(y-\mu_2)\right)\right]^2\right\}$$

Then,

$$E(X|Y) = \mu_1 + \rho\frac{\sigma_1}{\sigma_2}(y - \mu_2)$$

And,

$$Var(X|Y) = \sigma_1^2(1-\rho^2)$$

$$S.D.(X|Y) = \sigma_1\sqrt{1-\rho^2}$$