The Notes Project

Probability Theory

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Part I

Introduction to Probability Theory

Syllabus

Unit 1: Concepts of Probability & Properties

1. Random Experiment
   - Sample space
   - Event
   - Operation of events
   - Concepts of independent, mutually exclusive and exhaustive events
2. Classical (Mathematical), Empirical, Axiomatic definitions of probability, and their properties
3. Theorems and properties based on Axiomatic definition of probability
4. Conditional probability, Theorems on Addition and Multiplication of probabilities
5. Bayes' theorem and its applications

Unit 2: Random Variable & Distribution

1. Definition of discrete and continuous random variables, probability mass function (pmf), Probability density function (pdf), and their properties, cumulative distribution function and its properties
2. Expectation and variance of a random variable and its properties

Unit 3: Generating Functions

1. Moments and Moment generating function (m.g.f.) and its properties
2. Cumulant generating function (c.g.f.) and its properties
3. Characteristic function and its properties

Basic Definitions

Sample Space (S/Ω) The set of all possible outcomes.

Event A subset of the sample space.

Independent Events Events A & B defined on Ω are independent if they are not affected by each other.

Mutually Exclusive Events Events A & B defined on Ω are mutually exclusive if they cannot occur simultaneously.

Exhaustive Events Events are exhaustive if their union is equal to the sample space.

Types of Probability

Classical Definition of Probability

If a random experiment has n mutually exclusive, equally likely and exhaustive outcomes, and m of them are favourable to event A, then probability of happening of event A is given by:

$$P(A) = \frac{\text{favourable events}}{\text{total events}} = \frac{m}{n}$$

Properties

1. $0 \leq P(A) \leq 1$
2. $P(A') = 1 - P(A)$

Empirical/Relative Definition of Probability

If an experiment is repeated n times, as n tends to ∞ and it produces m outcomes favourable to event A, then probability of happening of event A is given by:

$$P(A) = \lim_{n\to\infty} \frac{m}{n}$$

It is useful for unequally likely events, or countably infinite sample spaces.

Axiomatic Definition of Probability

Let Ω/S be a sample space, A be any event defined on sample space, then function P is said to be probability function on probability measure if it satisfies the following axioms.

- $P(A) \geq 0$
- $P(\Omega) = 1$
- For A & B, any two mutually exclusive events defined on sample space Ω, $P(A \cup B) = P(A) + P(B)$

Various Theorems and Identities

Addition Theorem:

$$P\left(\bigcup_{i=1}^{n} P(A_i)\right) = \sum_{i=1}^{n} P(A_i) - \sum_{0\leq i<j=1}^{n}\sum^{n} P(A_i \cap A_j) + \sum_{0\leq i<j<k=1}^{n}\sum^{n}\sum^{n} P(A_i \cap A_j \cap A_k) \dots$$

Conditional Probability: If A & B are two events, probability of happening of event A if B has already happened is given by:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Similarly,

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Multiplication Theorem:

$$\begin{aligned} P(A \cap B) &= P(A|B) \cdot P(B) \\ &= P(B|A) \cdot P(A) \end{aligned}$$

Bayes' Theorem If $A_1, A_2, \dots A_n$ are n events defined on sample space such that $\bigcup_{i=1}^{n} P(A_i) = 1$ and $\bigcap_{i=1}^{n} P(A_i) = \phi$, B be an event defined on same sample space such that $B \subseteq \cup A_i, P(B) \neq 0$:

$$P(A_i|B) = \frac{P(B|A_i) \cdot P(A_i)}{\sum P(B|A_i) \cdot P(A_i)}$$

More Definitions

Random Variable A real-valued function defined on sample space.

Probability Distribution The set of pairs of values of a random variable and the probability of those values.

Probability Mass Function Let X be a discrete random variable and $P(X = x)$ be a function defined on X. $P(X = x)$ is said to be its p.m.f. if it satisfies the following conditions:
1) $P(x) \geq 0$
2) $\sum P(x) = 1$

Probability Density Function Let X be a continuous random variable and $f(X = x)$ be a function defined on X. $f(X = x)$ is said to be its p.d.f. if it satisfies the following conditions:
1) $f(x) \geq 0$
2) $\int f(x) = 1$

Cumulative Distribution Function Let X be a random variable and $P(X = x)$ be its p.m.f. Distribution function is given by:

$$f(x) = \begin{cases} \sum_{i=1}^{x} P(X = x), & \text{for discrete r.v.} \\ \\ \int_{-\infty}^{x} f(x)dx, & \text{for continuous r.v.} \end{cases}$$

Expectation of a Random Variable Let X be a r.v. with p.m.f. $P(X = x)$ (for discrete) and $f(x)$ (for continuous). Then, expectation is denoted by $E(X)$ and is given by:

$$E(x) = \begin{cases} \sum x \cdot P(X = x), & \text{for discrete r.v.} \\ \\ \int x \cdot f(x)dx, & \text{for continuous r.v.} \end{cases}$$

Variance of a Random Variable Let X be a r.v. with expectation $E(X)$. Then, variance is denoted by $Var(X)$ and is given by:

$$Var(X) = E(X^2) - [E(X)]^2$$

Moments

Definitions:

Central Moments

$$\mu_r = \frac{\sum (x_i - \mu)^r}{n}$$

Raw Moments

$$\mu'_r = \frac{\sum x_i^r}{n}$$

Arbitrary Moments

$$\mu_{r_A} = \frac{\sum (x_i - A)^r}{n}$$

Functions:

Moment Generating Function

$$M_x(t) = E(e^{tx})$$

Cumulant Generating Function

$$K_x(t) = log(M_x(t)) = log(E(e^{tx}))$$

Characteristic Function

$$\phi_x(t) = E(e^{itx})$$

Part II

Discrete Probability Distributions

Syllabus

Unit 1: Standard Univariate Distributions

1. Distributions
   - Uniform
   - Bernoulli
   - Binomial
   - Poisson
   - Geometric
   - Negative Binomial
   - Hypergeometric
2. The following aspects of the above distributions (wherever applicable) to be discussed
   - Mean
   - Mode
   - Standard deviation
   - Moment Generating Function
   - Cumulant Generating Function
   - Additive property
   - Moments, Skewness and Kurtosis
   - Limiting distribution
3. Fitting of Distribution
4. Truncated Binomial and Truncated Poisson Distribution

Unit 2: Bivariate Distributions

1. Joint Probability mass function for discrete random variables, their properties
2. Marginal and conditional Distributions
3. Independence of Random Variables
4. Conditional Expectation & Variance
5. Coefficient of Correlation
6. Transformation of Random Variables
7. Trinomial distribution, Marginal & Conditional distributions, and their means & variances
8. Correlation coefficient
9. Extension to Multinomial distribution

Uniform Distribution:

Definition:

Let X be a discrete r.v. It follows uniform distribution if its p.m.f is:

$$P(X = x) = \frac{1}{n};\ x = 1, 2 \ldots n$$

It is denoted by $X \sim D(n)$.

Applications:

1) Tossing a coin, getting heads or tails.
2) Selection of a student from a class.

Expectation and Variance:

$$\begin{aligned}
E(X) &= \sum x \cdot P(x) \\
&= \sum_{x=1}^{n} x \cdot \frac{1}{n} \\
&= \frac{1}{n} \cdot \frac{n(n+1)}{2} \\
E(X) &= \frac{(n+1)}{2}
\end{aligned}$$

$$\begin{aligned}
E(X^2) &= \sum x^2 \cdot P(x) \\
&= \sum_{x=1}^{n} x^2 \cdot \frac{1}{n} \\
&= \frac{1}{n} \cdot \frac{n(n+1)(2n+1)}{6} \\
E(X^2) &= \frac{(n+1)(2n+1)}{6}
\end{aligned}$$

$$\begin{aligned}
Var(x) &= E(X^2) - [E(X)]^2 \\
&= \frac{(n+1)(2n+1)}{6} - \left[\frac{(n+1)}{2}\right]^2 \\
&= \frac{2n^2+3n+2}{6} - \frac{n^2+2n+1}{4} \\
&= \frac{4n^2+6n+4-3n^2-6n-3}{12} \\
Var(X) &= \frac{n^2-1}{12} = \frac{(n+1)(n-1)}{12}
\end{aligned}$$

Moment Generating Function:

Given $X \sim D(n), f(x) = \dfrac{1}{n}$:

$$\begin{aligned}
M_x(t) &= E(e^{tx}) \\
&= \sum e^{tx} \cdot P(x) \\
&= \sum_{x=1}^{n} e^{tx} \cdot \frac{1}{n} \\
&= \frac{1}{n} \sum_{x=1}^{n} e^{tx} \\
M_x(t) &= \frac{e^t(1-e^{nt})}{n(1-e^t)}
\end{aligned}$$

Bernoulli Distribution:

Definition:

Let X be a discrete r.v. It follows uniform distribution if its p.m.f is:

$$P(x) = \begin{cases} p^x q^{1-x}, & \text{if } x = 0, 1 \\ 0, & \text{otherwise} \end{cases}$$

It is denoted by $X \sim B(p)$.

Applications:

1) Tossing a coin, getting heads or tails.
2) Rolling a die, getting odd or even.
3) Picking a card, getting red or black.
4) Selecting item, defective or not defective.

Expectation and Variance:

$$\begin{aligned} E(X) &= \sum x \cdot P(x) \\ &= \sum_{x=0}^{1} x \cdot p^x q^{1-x} \\ &= 0 \cdot p^0 \cdot q + 1 \cdot p \cdot q^0 \\ E(X) &= p \end{aligned}$$

$$\begin{aligned} E(X^2) &= \sum x^2 \cdot P(x) \\ &= \sum_{x=0}^{1} x^2 \cdot p^x q^{1-x} \\ &= 0 \cdot p^0 \cdot q + 1 \cdot p \cdot q^0 \\ E(X^2) &= p \end{aligned}$$

$$\begin{aligned} Var(x) &= E(X^2) - [E(X)]^2 \\ &= p - p^2 \\ &= p(1-p) \\ Var(X) &= pq \end{aligned}$$

Moment Generating Function:

Given $X \sim B(n), f(x) = p^x q^{1-x}$:

$$\begin{aligned} M_x(t) &= E(e^{tx}) \\ &= \sum e^{tx} \cdot P(x) \\ &= \sum_{x=0}^{1} e^{tx} \cdot p^x q^{1-x} \\ &= e^{0t} \cdot q + e^{1t} \cdot p \\ M_x(t) &= q + pe^t \end{aligned}$$

Binomial Distribution:

Definition:

Let X be a discrete r.v. It follows uniform distribution if its p.m.f is:

$$P(x) = \begin{cases} \binom{n}{x} p^x q^{n-x}, & \text{if } x = 0, 1 \ldots n \\ 0, & \text{otherwise} \end{cases}$$

It is denoted by $X \sim B(n, p)$.

Applications:

1) Tossing multiple coins, getting heads or tails certain number of times.
2) Repeating any Bernoulli event n number of times.

Expectation and Variance:

$$\begin{aligned}
E(X) &= \sum x \cdot P(x) \\
&= \sum_{x=0}^{n} x \cdot \binom{n}{x} p^x q^{1-x} \\
&= 0 + 1 \cdot \binom{n}{1} \cdot p^1 \cdot q^{n-1} + 2 \cdot \binom{n}{2} \cdot p^2 \cdot q^{n-2} + \cdots + n \cdot {}^nC_n \cdot p^n \cdot q^{n-n} \\
&= npq^{n-1} + n(n-1)p^2q^{n-2} + n(n-1)(n-2)p^3q^{n-3} + \cdots + np^n \\
&= np[q^{n-1} + (n-1)pq^{n-2} + \cdots + p^{n-1}] \\
&= np(p+q)^{n-1} && (p+q=1) \\
E(X) &= np
\end{aligned}$$

$$\begin{aligned}
E(X^2) &= \sum x^2 \cdot P(x) \\
&= \sum_{x=0}^{n} x + x(x-1) \cdot \binom{n}{x} p^x q^{n-x} \\
&= \sum_{x=0}^{n} x \cdot \binom{n}{x} p^x q^{n-x} \sum_{x=0}^{n} x(x-1) \cdot \binom{n}{x} p^x q^{n-x} \\
&= np + \sum_{x=0}^{n} x(x-1) \cdot \binom{n}{x} p^x q^{n-x} \\
&= np + (0 + 0 + n(n-1)p^2q^{n-2} + n(n-1)(n-2)p^3q^{n-3} + \cdots + np^n) \\
&= np + n(n-1)p(p+q)^{n-2} \\
E(X^2) &= np + n^2p^2 - np^2
\end{aligned}$$

$$\begin{aligned}
Var(X) &= E(X^2) - [E(X)]^2 \\
&= np + n^2p^2 - np^2 - n^2p^2 \\
&= np - np^2 \\
&= np(1-p) \\
Var(X) &= npq
\end{aligned}$$

Note: If $X_1, X_2, X_3, \ldots, X_k$ are independent B(n, p) then $\sum_{i=1}^{k} X_i \sim B(\sum_{i=1}^{k} n_i, p)$.

Assumptions:

1) The number of trials n is fixed and finite.
2) The probability of success p is the same for every trial.
3) $p + q = 1$, where q is the probability of failure.
4) p is independent for every trial.

Moment Generating Function:

Given $X \sim B(n, p), f(x) = \binom{n}{x} p^x q^{n-x}$:

$$\begin{aligned} M_x(t) &= E(e^{tx}) \\ &= \sum e^{tx} \cdot P(x) \\ &= \sum_{x=0}^{n} e^{tx} \cdot \binom{n}{x} p^x q^{n-x} \\ &= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x \ q^{n-x} \\ M_x(t) &= (q + pe^t)^n \end{aligned}$$

Poisson Distribution:

Let X be a discrete r.v. It follows uniform distribution if its p.m.f is:

$$P(x) = \begin{cases} \dfrac{e^{-\lambda}\lambda^x}{x!} & \text{if } x = 0, 1 \dots \\ 0, & \text{otherwise} \end{cases}$$

It is denoted by $X \sim P(\lambda)$.

Expectation and Variance:

$$\begin{aligned}
E(X) &= \sum_{n=0}^{\infty} x \cdot P(x) \\
&= \sum_{n=0}^{\infty} \frac{x\ e^{-\lambda}\ \lambda^x}{x!} \\
&= e^{-\lambda} \sum_{n=0}^{\infty} \frac{x\ \lambda^x}{x!} \\
&= e^{-\lambda} \left(0 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \dots\right) \\
&= e^{-\lambda} \cdot \lambda \left(1 + \frac{\lambda}{2!} \dots\right) \\
&= e^{-\lambda} \cdot \lambda \cdot e^{\lambda} \\
E(X) &= \lambda
\end{aligned}$$

$$\sum_{n=0}^{\infty} \frac{\lambda^x}{x!} = e^{\lambda}$$

$$\begin{aligned}
E(X^2) &= \sum x^2 \cdot P(x) \\
&= \sum_{n=0}^{\infty} \frac{x^2\ e^{-\lambda}\ \lambda^x}{x!} \\
&= e^{-\lambda} \sum_{n=0}^{\infty} \frac{x + x(x-1)\ \lambda^x}{x!} \\
&= e^{-\lambda} \left[\sum_{n=0}^{\infty} \frac{x\ \lambda^x}{x!} + \sum_{n=0}^{\infty} \frac{x(x-1)\ \lambda^x}{x!}\right] \\
&= e^{-\lambda} \left[\lambda e^{\lambda} + \left(0 + 0 + \frac{2\lambda^2}{2!} + \frac{6\lambda^3}{3!} + \dots\right)\right] \\
&= e^{-\lambda} \left[\lambda e^{\lambda} + \lambda^2 \left(1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} \dots\right)\right] \\
&= e^{-\lambda} \left[\lambda e^{\lambda} + \lambda^2 e^{\lambda}\right] \\
E(X^2) &= \lambda + \lambda^2
\end{aligned}$$

$$\begin{aligned}
Var(X) &= E(X^2) - [E(X)]^2 \\
&= \lambda + \lambda^2 - \lambda^2 \\
Var(X) &= \lambda
\end{aligned}$$

Assumptions

1) Mean = Variance = λ.
2) $\sigma = \sqrt{\lambda}$.

3) If $X_1, X_2, X_3, \ldots, X_k$ are independent P(λ_i) then $\sum_{i=1}^{k} X_i \sim P(\sum_{i=1}^{k} \lambda_i)$.

Applications:

1) Probability of rain in many summers.
2) Probability of a misprint in a page across a library.
3) Probability of an accident in a large parking lot.

Moment Generating Function:

Given $X \sim P(\lambda), f(x) = \dfrac{e^{-\lambda}\lambda^x}{x!}$:

$$\begin{aligned}
M_x(t) &= E(e^{tx}) \\
&= \sum e^{tx} \cdot P(x) \\
&= \sum_{x=0}^{\infty} e^{tx} \cdot \frac{e^{-\lambda}\lambda^x}{x!} \\
&= e^{-\lambda} \sum_{x=0}^{\infty} \frac{e^{tx}\lambda^x}{x!} \\
&= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} \\
&= e^{-\lambda} \cdot e^{\lambda e^t} \qquad\qquad \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{\lambda} \\
M_x(t) &= e^{\lambda(e^t - 1)}
\end{aligned}$$

Geometric Distribution:

Definition:

Let X be a discrete r.v. It follows uniform distribution if its p.m.f is:

Type 1:

$$P(x) = \begin{cases} pq^{x-1}, & \text{if } x = 1, 2, \dots \\ 0, & \text{otherwise} \end{cases}$$

It is denoted by $X \sim G(p)$. In this case, x is the number of trials.

Expectation and Variance:

$$\begin{aligned}
E(X) &= \sum x \cdot P(x) \\
&= \sum x \cdot pq^{x-1} \\
&= p \sum xq^{x-1} \\
&= p \left[0 + 1 + 2q + 3q^2 \dots\right] \\
&= p \cdot \frac{1}{p^2} \\
E(X) &= \frac{1}{p}
\end{aligned}$$

$$\begin{aligned}
E(X^2) &= \sum x^2 \cdot P(x) \\
&= \sum [x + x(x-1)] \cdot pq^{x-1} \\
&= \sum x \cdot pq^{x-1} + \sum x(x-1) \cdot pq^{x-1} \\
&= \frac{1}{p} + p \sum x(x-1) \cdot q^{x-1} \\
&= \frac{1}{p} + p(0 + 0 + 2q + 6q^2 + 12q^3 \dots) \\
&= \frac{1}{p} + 2pq(1 + 3q + 6q^2 \dots) \\
&= \frac{1}{p} + p \cdot q \cdot \frac{1}{p^3} \\
&= \frac{p + 2q}{p^2} \\
&= \frac{p + q + q}{p^2} \\
E(X^2) &= \frac{1 + q}{p^2}
\end{aligned}$$

$$\begin{aligned}
Var(X) &= E(X^2) - [E(X)]^2 \\
&= \frac{q+1}{p^2} - \frac{1}{p^2} \\
&= \frac{q}{p^2} + \frac{1}{p^2} - \frac{1}{p^2} \\
Var(X) &= \frac{q}{p^2}
\end{aligned}$$

Type 2:

$$P(x) = \begin{cases} pq^x, & \text{if } x = 0, 1, 2, \dots \\ 0, & \text{otherwise} \end{cases}$$

It is denoted by $X \sim G(p)$. In this case, x is the number of failures.

Expectation and Variance:

$$\begin{aligned}
E(X) &= \sum x \cdot P(x) \\
&= \sum x \cdot pq^x \\
&= p \sum xq^x \\
&= p[q + 2q^2 + 3q^3 \dots] \\
&= pq \left[1 + 2q + 3q^2 \dots\right] \\
&= pq \cdot \frac{1}{p^2} \\
E(X) &= \frac{q}{p}
\end{aligned}$$

$$\begin{aligned}
E(X^2) &= \sum x^2 \cdot P(x) \\
&= \sum [x + x(x-1)] \cdot pq^x \\
&= \sum x \cdot pq^x + \sum x(x-1) \cdot pq^x \\
&= \frac{q}{p} + p \sum x(x-1) \cdot q^x \\
&= \frac{q}{p} + p(0 + 0 + 2q^2 + 6q^3 + 12q^4 \dots) \\
&= \frac{q}{p} + 2pq^2(1 + 3q + 6q^2 \dots) \\
&= \frac{q}{p} + 2p \cdot q^2 \cdot \frac{1}{p^3} \\
E(X^2) &= \frac{q}{p} + \frac{2p^2}{q^2}
\end{aligned}$$

$$\begin{aligned}
Var(X) &= E(X^2) - [E(X)]^2 \\
&= \frac{q}{p} + \frac{2p^2}{q^2} - \frac{q^2}{p^2} \\
&= \frac{q}{p} + \frac{q^2}{p^2} \\
&= \frac{pq + q^2}{p^2} \\
&= \frac{q(p+q)}{p^2} \\
Var(X) &= \frac{q}{p^2}
\end{aligned}$$

Moment Generating Function: Given $X \sim G(p), f(x) = pq^x$:

$$\begin{aligned}
M_x(t) &= E(e^{tx}) \\
&= \sum e^{tx} \cdot P(x) \\
&= \sum_{x=0}^{\infty} e^{tx} \cdot pq^x \\
&= p \sum_{x=0}^{\infty} (qe^t)^x \\
M_x(t) &= \frac{p}{1 - qe^t}
\end{aligned}$$

Negative Binomial Distribution:

Definition:

Let X be a discrete r.v. It follows uniform distribution if its p.m.f is:

$$P(x) = \begin{cases} \binom{k+r-1}{r-1} p^r q^x, & \text{if } x = 0, 1, 2, \ldots \\ 0, & \text{otherwise} \end{cases}$$

It is denoted by $X \sim G(p)$.

Part III

Continuous Probability Distributions

Syllabus

Unit 1: Standard Univariate Distributions

1. Distributions
   - Rectangular
   - Triangular
   - Exponential
   - Cauchy (with Single & Double parameters)
   - Gamma (with Single & Double parameters)
   - Beta (Type I & Type II)
2. The following aspects of the above distributions (wherever applicable) to be discussed
   - Mean
   - Median
   - Mode
   - Standard deviation
   - Moment Generating Function
   - Additive property
   - Cumulant Generating Function
   - Skewness and Kurtosis
   - Fitting of Distribution
   - Interrelation between the distributions.

   Normal Distribution
   - Mean, Median, Mode
   - Standard deviation
   - Moment Generating function
   - Cumulant Generating function
   - Moments & Cumulants (up to fourth order)
   - Skewness & kurtosis
   - Mean absolute deviation
   - Distribution of linear function of independent normal variables
   - Fitting of Normal Distribution, q-q plot.
3. Log Normal Distribution: Derivation of mean & variance.

Unit 2: Bivariate Distributions

1. Joint Probability density function for Continuous random variables, their properties
2. Marginal and conditional distributions
3. Independence of Random Variables
4. Conditional Expectation & Variance
5. Regression Function
6. Coefficient of Correlation
7. Transformation of Random Variables, Jacobian of transformation
8. Bivariate Normal distribution, Marginal & Conditional distributions, their means & variances.

Continuous Probability Distributions

Rectangular Distribution:

Definition: If c.r.v $X \sim U(a,b)$, then its p.d.f is:

$$f(x) = \begin{cases} \dfrac{1}{b-a}, & \text{if } a \leq x \leq b \\ \\ 0, & \text{otherwise} \end{cases}$$

Expectation, Median, Mode and Variance:

$$
\begin{aligned}
E(X) &= \int_a^b x \cdot f(x)dx \\
&= \int_a^b \frac{x}{b-a}dx \\
&= \left.\frac{x^2}{2(b-a)}\right|_a^b \\
&= \frac{b^2-a^2}{2(b-a)} \\
&= \frac{(b+a)(b-a)}{2(b-a)} \\
E(X) &= \frac{b+a}{2}
\end{aligned}
$$

$$
\begin{aligned}
E(X^2) &= \int_a^b x^2 \cdot f(x)dx \\
&= \int_a^b \frac{x^2}{b-a}dx \\
&= \left.\frac{x^3}{3(b-a)}\right|_a^b \\
&= \frac{b^3-a^3}{2(b-a)} \\
&= \frac{(b^2+ba+a^2)(b-a)}{3(b-a)} \\
E(X^2) &= \frac{b^2+ba+a^2}{3}
\end{aligned}
$$

$$
\begin{aligned}
Var(x) &= E(X^2) - [E(X)]^2 \\
&= \frac{b^2+ba+a^2}{3} - \left[\frac{b+a}{2}\right]^2 \\
&= \frac{4a^2+4ab+4b^2-3a^2-6ab-3b^2}{12} \\
&= \frac{b^2-2ab+a^2}{12} \\
Var(X) &= \frac{(b-a)^2}{12}
\end{aligned}
$$

Standard Deviation σ:

$$
\begin{aligned}
\sigma &= \sqrt{Var(x)} \\
\sigma &= \frac{(b-a)}{\sqrt{12}}
\end{aligned}
$$

To find median M:

$$
\begin{aligned}
\int_a^M f(x)dx &= \frac{1}{2} \\
\frac{M-a}{b-a} &= \frac{1}{2} \\
2M-2a &= b-a \\
M &= \frac{b+a}{2}
\end{aligned}
$$

Mode of Rectangular Distribution is every x such that $a \leq x \leq b$ as:

$$\frac{d}{dx}\,\frac{1}{b-a} = 0$$

Therefore, all points are its maxima and minima.

Moment Generating Function:

$$\begin{aligned} M_x(t) &= E(e^{tx}) \\ &= \int_a^b e^{tx} \cdot f(x)dx \\ &= \frac{1}{b-a}\,\frac{e^{tx}}{t}\Bigg|_a^b \\ &= \frac{1}{t(b-a)} \cdot (e^{bt} - e^{at}) \\ M_x(t) &= \frac{e^{bt} - e^{at}}{t(b-a)} \end{aligned}$$

First Raw Moment:

$$\begin{aligned} \mu_r' &= \int_a^b x^r f(x)dx \\ \mu_r' &= \frac{1}{b-a}\left[\frac{b^{r+1} - a^{r+1}}{r+1}\right] \end{aligned}$$

Cumulant Generating Function:

$$K_x(t) = log\left[\frac{e^{bt} - e^{at}}{t(b-a)}\right]$$

C.G.F for $t = 1$:

$$K_x(1) = log\left[\frac{e^{b} - e^{a}}{b-a}\right]$$

Characteristic Function:

$$\begin{aligned} \phi_x(t) &= E(e^{itx}) \\ \phi_x(t) &= \frac{e^{ibt} - e^{iat}}{it(b-a)} \end{aligned}$$

Triangular Distribution:

Definition: If c.r.v $X \sim T(a, b)$ with mode c, then its p.d.f is:

$$f(x) = \begin{cases} \dfrac{2(x-a)}{(b-a)(c-a)}, & \text{if } a \leq x \leq c \\[2ex] \dfrac{2(b-x)}{(b-a)(b-c)}, & \text{if } c \leq x \leq b \\[2ex] 0, & \text{otherwise} \end{cases}$$

$$f(c) = \frac{2}{b-a}$$

Moment Generating Function: Moment Generating Function of $X \sim T(a,b)$ with mode c is:

$$M_x(t) = \frac{2}{t^2}\left[\frac{e^{at}}{(a-b)(a-c)}\frac{e^{ct}}{(c-a)(c-b)}\frac{e^{bt}}{(b-a)(b-c)}\right]$$

If X & Y are i.i.d. $U(-a,a)$, then addition of X & Y, i.e. $X+Y \sim T(-2a,2a)$, with mode 0. Additionally, $X-Y \sim T(-2a,2a)$, with mode 0.

Properties: 1) $-\infty < a < b < \infty, c \in [a,b]$
2) if $C < E(X)$, distribution is positively skewed.
if $C > E(X)$, distribution is negatively skewed.
if $C = E(X)$, distribution is symmetric.

Gamma Distribution:

One Parameter: If c.r.v $X \sim \gamma(\lambda)$, then its p.d.f is:

$$f(x) = \begin{cases} \dfrac{e^{-x}x^{\lambda-1}}{\Gamma\lambda}, & \text{if } 0 < x < \infty,\ \lambda > 0 \\ \\ 0, & \text{otherwise} \end{cases}$$

Properties:

$$\Gamma n = \int_0^\infty e^{-x}x^{n-1}dx$$
$$\Gamma(n+1) = n\Gamma n$$
$$\Gamma n = (n-1)!$$

Expectation and Variance:

$$\begin{aligned} E(X) &= \int_0^\infty x \cdot f(x)dx \\ &= \frac{x\ e^{-x}x^{\lambda-1}}{\Gamma\lambda} \\ &= \frac{e^{-x}x^{\lambda}}{\Gamma\lambda} \\ &= \frac{\Gamma(\lambda+1)}{\Gamma\lambda} \\ &= \frac{\lambda\ \Gamma\lambda}{\Gamma\lambda} \\ E(X) &= \lambda \end{aligned}$$

$$\begin{aligned} E(X^2) &= \int_0^\infty x^2 \cdot f(x)dx \\ &= \frac{x^2\ e^{-x}x^{\lambda-1}}{\Gamma\lambda} \\ &= \frac{e^{-x}x^{\lambda+1}}{\Gamma\lambda} \\ &= \frac{\Gamma(\lambda+2)}{\Gamma\lambda} \\ &= \frac{\lambda(\lambda+1)\Gamma\lambda}{\Gamma\lambda} \\ E(X^2) &= \lambda(\lambda+1) = \lambda^2 + \lambda \end{aligned}$$

$$\begin{aligned} Var(x) &= E(X^2) - [E(X)]^2 \\ &= (\lambda^2 + \lambda) - \lambda^2 \\ Var(X) &= \lambda \end{aligned}$$

Moment Generating Function:

$$M_x(t) = (1-t)^{-\lambda}$$

Cumulant Generating Function: The value of the n^{th} cumulant is $\lambda(n-1)!$.

Raw Moments: The value of the r^{th} raw moment is

$$
\begin{aligned}
\mu'_r &= \int_0^\infty x^r\, f(x)dx \\
&= \int_0^\infty x^r \cdot \frac{e^{-x}x^{\lambda-1}}{\Gamma\lambda} \\
&= \frac{e^{-x}x^{\lambda+r-1}}{\Gamma\lambda} \\
&= \frac{\Gamma(\lambda+r)}{\Gamma\lambda} \\
\mu'_r &= \frac{\Gamma(\lambda+r)}{\Gamma\lambda} \\
\mu'_r &= \Pi_{i=0}^{n-1}(\lambda+i)
\end{aligned}
$$

Coefficients of Skewness and Kurtosis: The coefficients of skewness and kurtosis for the gamma distribution are:

$$
\begin{aligned}
\beta_1 &= \frac{\mu_3^2}{\mu_2^3} \\
&= \frac{(2\lambda)^2}{\lambda^3} \\
\beta_1 &= \frac{4}{\lambda}
\end{aligned}
$$

$$
\begin{aligned}
\gamma_1 &= \sqrt{\beta_1} \\
&= \sqrt{\frac{4}{\lambda}} \\
\gamma_1 &= \frac{2}{\sqrt{\lambda}}
\end{aligned}
$$

$$
\begin{aligned}
\beta_2 &= \frac{\mu_4}{\mu_2^2} \\
&= \frac{6\lambda}{\lambda^2} \\
\beta_2 &= \frac{6}{\lambda}
\end{aligned}
$$

$$
\begin{aligned}
\gamma_2 &= \beta_2 - 3 \\
\gamma_2 &= \frac{6}{\lambda} - 3
\end{aligned}
$$

Note: If $X_1, X_2, X_3, \ldots, X_n$ are independent $\gamma(\lambda_i)$ then $\sum_{i=1}^n X_i \sim \gamma(\sum_{i=1}^n \lambda_i)$.

Two Parameter: If c.r.v $X \sim G(\lambda, a)$, then its p.d.f is:

$$
f(x) = \begin{cases} \dfrac{a^\lambda e^{-ax}x^{\lambda-1}}{\Gamma\lambda}, & \text{if } 0 < x < \infty,\ \lambda > 0, a > 0 \\[2ex] 0, & \text{otherwise} \end{cases}
$$

Expectation and Variance:

$$
\begin{aligned}
E(X) &= \int_0^\infty x \cdot f(x)dx \\
&= \int_0^\infty \frac{a^\lambda e^{-ax} x^{\lambda-1}}{\Gamma\lambda} dx \\
&= \int_0^\infty \frac{a^\lambda e^{-u} u^{\lambda+1}}{a^{\lambda+1}\,\Gamma\lambda} du && u = ax,\ dx = \frac{du}{a},\ x = \frac{u}{a} \\
&= \int_0^\infty \frac{e^{-u} u^{\lambda+1}}{a^2} du \\
&= \frac{\lambda\,\Gamma\lambda}{a\,\Gamma\lambda} \\
E(X) &= \frac{\lambda}{a}
\end{aligned}
$$

$$
\begin{aligned}
E(X^2) &= \int_0^\infty x^2 \cdot f(x)dx \\
&= \int_0^\infty \frac{a^\lambda e^{-ax} x^{\lambda+1}}{\Gamma\lambda} dx \\
&= \int_0^\infty \frac{a^\lambda e^{-u} u^{\lambda+1}}{a^{\lambda+2}\,\Gamma\lambda} du && u = ax,\ dx = \frac{du}{a},\ x = \frac{u}{a} \\
&= \int_0^\infty \frac{e^{-u} u^{\lambda+1}}{a^2\,\Gamma\lambda} du \\
&= \frac{\Gamma(\lambda+2)}{a^2\,\Gamma\lambda} \\
&= \frac{\lambda(\lambda+1)\,\Gamma\lambda}{a^2\,\Gamma\lambda} \\
E(X^2) &= \frac{\lambda(\lambda+1)}{a^2}
\end{aligned}
$$

$$
\begin{aligned}
Var(x) &= E(X^2) - [E(X)]^2 \\
&= \frac{\lambda(\lambda+1)}{a^2} - \frac{\lambda}{a}^2 \\
&= \frac{\lambda^2+\lambda}{a^2} - \frac{\lambda}{a}^2 \\
Var(X) &= \frac{\lambda}{a^2}
\end{aligned}
$$

Moment Generating Function:

$$M_x(t) = (1 - {}^t/_a)^{-\lambda}$$

Cumulant Generating Function: The value of the n^{th} cumulant is $\frac{\lambda(n-1)!}{a^n}$.

Raw Moments: The value of the r^{th} raw moment is

$$\begin{aligned}
\mu_r' &= \int_0^\infty x^r\, f(x)dx \\
&= \int_0^\infty x^r \cdot a^\lambda \cdot \frac{e^{-ax}x^{\lambda-1}}{\Gamma\lambda} \\
&= \int_0^\infty \frac{a^\lambda \cdot e^{-ax}x^{\lambda+r-1}}{\Gamma\lambda} \\
&= \frac{\Gamma(\lambda+r)}{\Gamma\lambda} \\
\mu_r' &= \frac{\Gamma(\lambda+r)}{\Gamma\lambda} \\
\mu_r' &= \Pi_{i=0}^{n-1}(\lambda+i)
\end{aligned}$$

Coefficients of Skewness and Kurtosis: The coefficients of skewness and kurtosis for the gamma distribution are:

$$\begin{aligned}
\beta_1 &= \frac{\mu_3^2}{\mu_2^3} \\
&= \frac{(2\lambda)^2/\lambda^3}{a^6/a^6} \\
\beta_1 &= \frac{4}{\lambda} \\
\\
\gamma_1 &= \sqrt{\beta_1} \\
&= \sqrt{\frac{4}{\lambda}} \\
\gamma_1 &= \frac{2}{\sqrt{\lambda}} \\
\\
\beta_2 &= \frac{\mu_4}{\mu_2^2} \\
&= \frac{6\lambda/\lambda^2}{a^4/a^4} \\
\beta_2 &= \frac{6}{\lambda} \\
\\
\gamma_2 &= \beta_2 - 3 \\
\gamma_2 &= \frac{6}{\lambda} - 3
\end{aligned}$$

Note: If $X_1, X_2, X_3, \ldots, X_n$ are independent $\gamma(\lambda_i, a)$ then $\sum_{i=1}^n X_i \sim \gamma(\sum_{i=1}^n \lambda_i, a)$.

Beta Distribution:

The p.d.f of the general beta function is given by:

$$\int_a^b \frac{(x-a)^{m-1}(b-x)^{n-1}}{(b-a)^{m+n-1}}, a < x < b$$

Types of β distribution:
If $a \neq 0$ & $b \neq 0$ or $a \neq 0$ & $b \neq \infty$, it is called an incomplete beta distribution.
If $a = 0$ & $b = 1$, it is the first type of beta distribution.
If $a = 0$ & $b = \infty$, it is the second type of beta distribution.

Type 1(β_1) If c.r.v $X \sim \beta_1(m,n)$, then its p.d.f is:

$$f(x) = \begin{cases} \dfrac{x^{m-1}(1-x)^{n-1}}{\beta(m,n)}, & \text{if } 0 \leq x \leq 1; m,n > 0 \\ \\ 0, & \text{otherwise} \end{cases}$$

Properties:

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

$$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)}$$

Expectation and Variance:

$$\begin{aligned}
E(X) &= \int_0^1 x \cdot f(x)dx \\
&= \int_0^1 \frac{x\ x^{m-1}(1-x)^{n-1}}{\beta(m,n)} \\
&= \int_0^1 \frac{x^m(1-x)^{n-1}}{\beta(m,n)} \\
&= \frac{\beta(m+1,n)}{\beta(m,n)} \\
&= \frac{\Gamma(m+1)\ \Gamma n}{\Gamma(m+n+1)} \frac{\Gamma(m+n)}{\Gamma m\ \Gamma n} \\
&= \frac{m\Gamma m\ \Gamma n}{(m+n)\Gamma(m+n)} \frac{\Gamma(m+n)}{\Gamma m\ \Gamma n} \\
E(X) &= \frac{m}{m+n}
\end{aligned}$$

$$\begin{aligned}
E(X^2) &= \int_0^1 x^2 \cdot f(x)dx \\
&= \int_0^1 \frac{x^2\ x^{m-1}(1-x)^{n-1}}{\beta(m,n)} \\
&= \int_0^1 \frac{x^{m+1}(1-x)^{n-1}}{\beta(m,n)} \\
&= \frac{\beta(m+2,n)}{\beta(m,n)} \\
&= \frac{\Gamma(m+2)\ \Gamma n}{\Gamma(m+n+2)} \frac{\Gamma(m+n)}{\Gamma m\ \Gamma n} \\
&= \frac{(m+1)m\Gamma m\ \Gamma n}{(m+n+1)(m+n)\Gamma(m+n)} \frac{\Gamma(m+n)}{\Gamma m\ \Gamma n} \\
E(X^2) &= \frac{m(m+1)}{(m+n)(m+n+1)}
\end{aligned}$$

$$\begin{aligned}
Var(x) &= E(X^2) - [E(X)]^2 \\
&= \frac{m(m+1)}{(m+n)(m+n+1)} - \frac{m}{m+n} \\
&= \frac{m}{m+n}\left[\frac{m+1}{m+n+1} - \frac{m}{m+n}\right] \\
&= \frac{m}{m+n}\left[\frac{(m+1)(m+n) - m(m+n+1)}{(m+n)(m+n+1)}\right] \\
&= \frac{m}{m+n}\left[\frac{m^2+mn+m+n-m^2-mn-m^2}{(m+n)(m+n+1)}\right] \\
&= \frac{m}{m+n}\left[\frac{n}{(m+n)(m+n+1)}\right] \\
Var(X) &= \frac{mn}{(m+n)^2(m+n+1)}
\end{aligned}$$

Harmonic Mean:

$$
\begin{aligned}
\frac{1}{HM} &= \int \frac{1}{x} f(x)dx \\
\frac{1}{HM} &= \int \frac{x^{m-2}(1-x)^{n-1}dx}{\beta(m,n)} \\
\frac{1}{HM} &= \frac{\beta(m-1,n)}{\beta(m,n)} \\
HM &= \frac{\beta(m,n)}{\beta(m-1,n)} \\
HM &= \frac{(m-1)\Gamma(m-1)\Gamma(m+n-1)}{(m+n-1)\Gamma(m+n-1)\Gamma(m-1)} \\
HM &= \frac{m-1}{m+n-1}
\end{aligned}
$$

Raw Moments: The value of the r^{th} raw moment is

$$
\begin{aligned}
\mu'_r &= \int_0^1 x^r \, f(x)dx \\
&= \int_0^1 \frac{x^r \, x^{m-1}(1-x)^{n-1}}{\beta(m,n)} \\
&= \int_0^1 \frac{x^{m+r-1}(1-x)^{n-1}}{\beta(m,n)} \\
&= \frac{\beta(m+r,n)}{\beta(m,n)} \\
&= \frac{\Gamma(m+r)\,\Gamma n}{\Gamma(m+n+r)} \frac{\Gamma(m+n)}{\Gamma m\,\Gamma n} \\
\mu'_r &= \frac{\Gamma(m+r)\Gamma(m+n)}{\Gamma(m+n+r)\Gamma m} \\
\mu'_r &= \frac{\Pi_{i=0}^{n-1}(m+i)}{\Pi_{i=0}^{n-1}(m+n+i)}
\end{aligned}
$$

Type 2(β_2) If c.r.v $X \sim \beta_2(m,n)$, then its p.d.f is:

$$f(x) = \begin{cases} \dfrac{1}{\beta(m,n)} \dfrac{x^{m-1}}{(1+x)^{m+n}}, & \text{if } 0 < x < \infty \\ \\ 0, & \text{otherwise} \end{cases}$$

Properties:

$$\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

$$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)}$$

Raw Moments: The value of the r^{th} raw moment is

$$\begin{aligned} \mu_r' &= \int_0^\infty x^r \, f(x) dx \\ &= \int_0^\infty x^r \, \frac{1}{\beta(m,n)} \frac{x^{m-1}}{(1+x)^{m+n}} \\ &= \frac{1}{\beta(m,n} \int_0^\infty \frac{x^{m+r-1}}{(1+x)^n} \\ &= \frac{\beta(m+r, n-r)}{\beta(m,n)} \\ &= \frac{\Gamma(m+r) \, \Gamma(n-r)}{\Gamma(m+n)} \frac{\Gamma(m+n)}{\Gamma m \, \Gamma n} \\ \mu_r' &= \frac{\Gamma(m+r)\Gamma(n-r)}{\Gamma m \, \Gamma n} \\ \mu_r' &= \frac{\Pi_{i=0}^{r-1}(m+i)}{\Pi_{i=0}^{r-1}(n-i-1)} \end{aligned}$$

Expectation and Variance:

$$
\begin{aligned}
E(X) &= \mu_1' \\
&= \frac{\Gamma(m+1)\Gamma(n-1)}{\Gamma m \, \Gamma n} \\
&= \frac{m \, \Gamma m \, \Gamma(n-1)}{\Gamma m \, (n-1) \, \Gamma(n-1)} \\
E(X) &= \frac{m}{n-1}
\end{aligned}
$$

$$
\begin{aligned}
E(X^2) &= \mu_2' \\
&= \frac{\Gamma(m+2)\Gamma(n-2)}{\Gamma m \, \Gamma n} \\
&= \frac{m(m+1)\Gamma(m)\Gamma(n-2)}{\Gamma m(n-1)(n-2)\Gamma(n-2)} \\
E(X^2) &= \frac{m(m+1)}{(n-1)(n-2)}
\end{aligned}
$$

$$
\begin{aligned}
Var(x) &= E(X^2) - [E(X)]^2 \\
&= \frac{m(m+1)}{(n-1)(n-2)} - \left(\frac{m}{n-1}\right)^2 \\
&= \frac{m}{n-1}\left(\frac{m+1}{(n-2)} - \frac{m}{(n-1)}\right) \\
&= \frac{m}{n-1}\left(\frac{(m+1)(n-1) - m(n-2)}{(n-1)(n-2)}\right) \\
Var(X) &= \frac{m(m+n-1)}{(n-1)^2(n-2)}
\end{aligned}
$$

Harmonic Mean:

$$
\begin{aligned}
\frac{1}{HM} &= \int_0^\infty \frac{1}{x} f(x)dx \\
\frac{1}{HM} &= \int_0^\infty \frac{1}{\beta(m,n)} \frac{x^{m-2}}{(1+x)^{m+n}} \\
\frac{1}{HM} &= \frac{\beta(m-1,n+1)}{\beta(m,n)} \\
HM &= \frac{\beta(m,n)}{\beta(m-1,n+1)} \\
HM &= \frac{\Gamma m \Gamma n}{\Gamma(m+n)} \frac{\Gamma(m+n)}{\Gamma(m-1)\Gamma(n+1)} \\
HM &= \frac{(m-1)\Gamma(m-1)\Gamma(n)}{\Gamma(m-1)n\Gamma n} \\
HM &= \frac{m-1}{n}
\end{aligned}
$$

If $X \sim \beta_1(1,1), X \sim U(0,1)$.

Exponential Distribution:

If c.r.v $X \sim \exp(\theta)$, then its p.d.f is:

$$f(x) = \begin{cases} \theta e^{-\theta x} & \theta > 0,\ 0 < x < \infty \\ 0, & \text{otherwise} \end{cases}$$

Expectation and Variance:

$$\begin{aligned}
E(X) &= \int_0^\infty x \cdot f(x) dx \\
&= \int_0^\infty x\theta e^{-\theta x} \\
&= \int_0^\infty \frac{ue^{-u}}{\theta} du && \theta x = u,\ dx = \frac{du}{\theta} \\
&= \frac{1}{\theta} \int_0^\infty u^{2-1} e^{-u} du \\
&= \frac{\Gamma 2}{\theta} \\
&= \frac{1!}{\theta} \\
E(X) &= \frac{1}{\theta}
\end{aligned}$$

$$\begin{aligned}
E(X^2) &= \int_0^1 x^2 \cdot f(x) dx \\
&= \int_0^\infty x^2 \theta e^{-\theta x} \\
&= \int_0^\infty x \cdot \theta x e^{-\theta x} \\
&= \int_0^\infty \frac{u}{\theta} \frac{ue^{-u}}{\theta} du && \theta x = u,\ dx = \frac{du}{\theta} \\
&= \frac{1}{\theta^2} \int_0^\infty u^{3-1} e^{-u} du \\
&= \frac{\Gamma 3}{\theta^2} \\
&= \frac{2!}{\theta^2} \\
E(X^2) &= \frac{2}{\theta^2}
\end{aligned}$$

$$\begin{aligned}
Var(X) &= E(X^2) - [E(X)]^2 \\
&= \frac{2}{\theta^2} - \frac{1}{\theta^2} \\
Var(X) &= \frac{1}{\theta^2}
\end{aligned}$$

Moment Generating Function:

$$M_x(t) = (1 - t/\theta)^{-1}$$

Cumulant Generating Function: The value of the n^{th} cumulant is $\dfrac{(n-1)!}{\theta^n}$.

Raw Moments: The value of the r^{th} raw moment is

$$\begin{aligned}
\mu_r' &= \int_0^\infty x^r\, f(x)dx \\
&= \int_0^\infty x^r \cdot \theta e^{-\theta x} dx \\
&= \int_0^\infty \left(\frac{u}{\theta}\right)^r \cdot \theta e^{-u} \cdot \frac{du}{\theta} && u = \theta x \\
&= \frac{1}{\theta^r} \int_0^\infty u^r e^{-u} du \\
&= \frac{\Gamma(r+1)}{\theta^r} \\
\mu_r' &= \frac{r!}{\theta^r}
\end{aligned}$$

Coefficients of Skewness and Kurtosis: The coefficients of skewness and kurtosis for the exponential distribution are:

$$\begin{aligned}
\beta_1 &= \frac{\mu_3^2}{\mu_2^3} \\
&= \frac{(^2/_{\theta^3})^2}{(^1/_{\theta^2})^3} \\
\beta_1 &= 4
\end{aligned}$$

$$\begin{aligned}
\gamma_1 &= \sqrt{\beta_1} \\
&= \sqrt{4} \\
\gamma_1 &= 2
\end{aligned}$$

$$\begin{aligned}
\beta_2 &= \frac{\mu_4}{\mu_2^2} \\
&= \frac{^6/_{\theta^4}}{(^1/_{\theta^2})^2} \\
\beta_2 &= 6
\end{aligned}$$

$$\begin{aligned}
\gamma_2 &= \beta_2 - 3 \\
&= 6 - 3 \\
\gamma_2 &= 3
\end{aligned}$$

Normal Distribution:

If c.r.v $X \sim N$ with mean μ and variance σ^2, then its p.d.f is:

$$f(x) = \begin{cases} \dfrac{1}{\sigma\sqrt{2\pi}} \exp\left[\dfrac{-1}{2}\left(\dfrac{x-\mu}{\sigma}\right)^2\right] & -\infty < x < \infty,\ -\infty < \mu < \infty,\ 0 < \sigma < \infty \\ 0, & \text{otherwise} \end{cases}$$

It is denoted as $X \sim N(\mu, \sigma^2)$

Properties: 1) The normal distribution follows a bell-shaped curve.
2) In a normal distribution, mean=median=mode.
3) The normal distribution is symmetric.
4) The scale parameter σ distributes the curve in the following percentage:
i) $\mu \pm \sigma$ contains $\sim 68\%$ data
ii) $\mu \pm 2\sigma$ contains $\sim 95\%$ data
iii) $\mu \pm 3\sigma$ contains $\sim 99.73\%$ data
5) $Z = \frac{x-\mu}{\sigma} \sim$ s.n.d. i.e. $N(0,1)$.

$$P(Z = z) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}z^2}$$

Median and Mode: Assuming median$> \mu$,

$$\begin{aligned}
\implies \int_{-\infty}^{\mu} f(x)dx + \int_{\mu}^{M} f(x)dx &= \frac{1}{2} \\
\implies 1/2 + \int_{\mu}^{M} f(x)dx &= \frac{1}{2} \\
\implies \int_{\mu}^{M} f(x)dx &= 0 \\
\implies \int_{\mu}^{M} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} &= 0 \\
\implies \frac{1}{\sigma\sqrt{2\pi}} \int_{\mu}^{M} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} &= 0 \\
\implies (x-\mu)\big|_{\mu}^{M} &= 0 \\
\implies M - \mu &= 0 \\
\implies M &= \mu
\end{aligned}$$

To find mode, we will use the first derivative test, i.e. $f'(x) = 0$.

$$log(f(x)) = -log(\sigma\sqrt{2\pi}) - \frac{(x-\mu)^2}{2\sigma^2}$$

Differentiating, we get

$$\frac{f'(x)}{f(x)} = -\frac{x-\mu}{\sigma^2}$$

$$\implies f(x)(x-\mu) = 0$$

As $f(x)$ cannot be 0 at the mode, $x - \mu = 0$

$$\implies x = \mu.$$

Hence, the second property (mean = mode = median) is proved.

Moment Generating Function and Cumulant Generating Function:

$$\begin{aligned}
M_x(t) &= E(e^{tx}) \\
&= \int_{-\infty}^{\infty} e^{tx} \cdot f(x)dx \\
&= \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\sigma\sqrt{2\pi}}\ e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} dx \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} dx \\
\text{Let } z &= \frac{x-\mu}{\sigma} \\
\implies x &= \sigma z + \mu \\
\&\ dx &= \sigma dz \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\sigma z+\mu)-\frac{z^2}{2}} \sigma dz \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tz\sigma} \cdot e^{t\mu} \cdot e^{-\frac{z^2}{2}} dz \\
&= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2-2tz\sigma)} dz \\
&= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2-2tz\sigma+t^2\sigma^2-t^2\sigma^2)} dz \\
&= \frac{e^{t\mu+\frac{t^2\sigma^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-t\sigma)^2} dz \\
\text{Let } \theta &= z - t\sigma \\
\implies dz &= d\theta \\
&= \frac{e^{t\mu+\frac{t^2\sigma^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{\theta^2}{2}} d\theta \\
&= \frac{e^{t\mu+\frac{t^2\sigma^2}{2}}}{\sqrt{2\pi}} \cdot \sqrt{2\pi} \\
M_x(t) &= e^{t\mu+\frac{t^2\sigma^2}{2}}
\end{aligned}$$

$$\begin{aligned}
K_x(t) &= log(M_x(t)) \\
&= log(e^{t\mu+\frac{t^2\sigma^2}{2}}) \\
&= t\mu + \frac{t^2\sigma^2}{2}
\end{aligned}$$

Cumulants:

$$K_1 = \text{coefficient of } t/1! = \mu$$

$$K_2 = \text{coefficient of } t^2/2! = \sigma^2$$

$$K_3 = \text{coefficient of } t^3/3! = 0$$

All further cumulants are 0.

Mean Deviation: M.D. from $\bar{X}$ is calculated as:

$$\int_{-\infty}^{\infty} |X - \bar{X}| = \int_{-\infty}^{\infty} |X - \mu|$$

$$
\begin{aligned}
E(|X-\mu|) &= \int_{-\infty}^{\infty} |x-\mu| f(x) dx \\
&= \int_{-\infty}^{\infty} |x-\mu| \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} dx \\
\text{Let } z &= \frac{x-\mu}{\sigma} \\
\implies x-\mu &= \sigma z \\
\&\ dx &= \sigma dz \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} |z\sigma| e^{-\frac{z^2}{2}} \sigma dz \\
&= \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |z| e^{-\frac{z^2}{2}} dz \\
&= \frac{2\sigma}{\sqrt{2\pi}} \int_{0}^{\infty} z e^{-\frac{z^2}{2}} dz \\
\text{Let } t &= \frac{z^2}{2} \\
\implies \quad dz &= dt/z \\
&= \sigma\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} z e^{-t} \frac{dt}{z} \\
&= \sigma\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-t} dt \\
&= \sigma\sqrt{\frac{2}{\pi}} \ -e^{-t}\Big|_0^{\infty} \\
&= \sigma\sqrt{\frac{2}{\pi}} (e^{-0} - e^{-\infty}) \\
E(|X-\mu|) &= \sigma\sqrt{\frac{2}{\pi}} \approx 0.8\,\sigma
\end{aligned}
$$

Central Moments of Normal Distribution: Odd Moments:

$$
\begin{aligned}
\mu_{2n+1} &= E(X-\mu)^{2n+1} \\
&= \int_{-\infty}^{\infty} (x-\mu)^{2n+1} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} dx \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)^{2n+1} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} dx \\
\text{Let } z &= \frac{x-\mu}{\sigma} \\
\implies x-\mu &= \sigma z \\
\&\ dx &= \sigma dz \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (z\sigma)^{2n+1} e^{-\frac{z^2}{2}} dz \\
\mu_{2n+1} &= \frac{\sigma^{2n+1}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^{2n+1} e^{-\frac{z^2}{2}} dz
\end{aligned}
$$

As z^{2n+1} is an odd function, and $e^{-\frac{z^2}{2}}$ is an even function, its product is an odd function. Integrals of odd functions from $-\infty$ to ∞ are 0. Hence, the integral given equals 0, and $\mu_{2n+1} = 0\ \forall\ n \in \mathbb{N}$.

Even Moments:

$$
\begin{aligned}
\mu_{2n} &= \int_{-\infty}^{\infty} (x-\mu)^{2n} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} dx \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)^{2n} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} dx \\
\text{Let } z &= \frac{x-\mu}{\sigma} \\
\implies x-\mu &= \sigma z \\
\&\ dx &= \sigma dz \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (z\sigma)^{2n} e^{-\frac{z^2}{2}} dz \\
&= \frac{\sigma^{2n}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^{2n} e^{-\frac{z^2}{2}} dz \\
\text{Let } t &= \frac{z^2}{2} \\
\implies dz &= dt/\sqrt{2t} \\
&= \frac{\sigma^{2n}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (2t)^n e^{-t} \frac{dt}{\sqrt{2t}} \\
&= \frac{2^n\ \sigma^{2n}}{2\sqrt{\pi}} \int_{-\infty}^{\infty} t^{n-1/2}\ e^{-t} dt \\
&= \frac{2^n\ \sigma^{2n}}{2\sqrt{\pi}} 2 \int_{0}^{\infty} t^{n-1/2}\ e^{-t} dt \\
\mu_{2n} &= \frac{2^n\ \sigma^{2n}}{\sqrt{\pi}} \Gamma(n+1/2)
\end{aligned}
$$

Recursive Relation of Even Moments:

$$
\begin{aligned}
\mu_{2n} &= \frac{2^n\ \sigma^{2n}}{\sqrt{\pi}} \Gamma(n+1/2) \\
\mu_{2n-2} &= \frac{2^{n-1}\ \sigma^{2n-2}}{\sqrt{\pi}} \Gamma(n-1/2) \\
\implies \frac{\mu_{2n}}{\mu_{2n-2}} &= \frac{2^n\ \sigma^{2n}\Gamma(n+1/2)}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2^{n-1}\ \sigma^{2n-2}\Gamma(n-1/2)} \\
&= \frac{2\sigma^2\Gamma(n+1/2)}{\Gamma(n-1/2)} \\
&= \frac{2\sigma^2(n-1/2)\Gamma(n-1/2)}{\Gamma(n-1/2)} \\
\frac{\mu_{2n}}{\mu_{2n-2}} &= 2\sigma^2(n-1/2)
\end{aligned}
$$

Or,

$$\mu_{2n} = \sigma^2(2n-1)\cdot \mu_{2n-2}$$

Addition Property: Theorem: If $X_1, X_2, X_3, \ldots, X_n$ are independent $N(\mu_i, \sigma_i^2)$ then $\sum_{i=1}^n a_i X_i \sim N(\sum_{i=1}^n a_i\mu_i, \sum_{i=1}^n a_i^2\sigma_i^2)$.
Proof: For Normal Distribution,

$$M_x(t) = e^{t\mu + \frac{t^2\sigma^2}{2}}$$

$$
\begin{aligned}
M_{X_1+X_2+\dots+X_n} &= M_{X_1}(t)\cdot M_{X_2}(t)\dots M_{X_n}(t)\\
&= e^{t\mu_1+\frac{t^2\sigma_1^2}{2}}\cdot e^{t\mu_2+\frac{t^2\sigma_2^2}{2}}\cdot\dots\cdot e^{t\mu_n+\frac{t^2\sigma_n^2}{2}}\\
M_{\sum_{i=1}^n X_i} &= e^{t(\mu_1+\mu_2+\dots+\mu_n)+\frac{t^2}{2}(\sigma_1^2+\sigma_2^2+\dots+\sigma_n^2)}
\end{aligned}
$$

By Uniqueness Theorem of m.g.f., the m.g.f. of any linear combination of n.r.v's follows normal distribution with mean $\sum_{i=1}^n a_i\mu_i$ and variance $\sum_{i=1}^n a_i^2\sigma_i^2)$.

Skewness and Kurtosis: We know the first four central moments of the normal distribution are:

$$
\begin{aligned}
\mu_1 &= 0\\
\mu_2 &= \sigma^2\\
\mu_3 &= 0\\
\mu_4 &= K_4+K_2^2\\
&= 3\sigma^4
\end{aligned}
$$

This gives us values of skewness and kurtosis:

$$
\begin{aligned}
\beta_1 &= \frac{\mu_3^2}{\mu_2^3}\\
&= \frac{0}{\sigma^6}\\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\beta_2 &= \frac{\mu^4}{\mu_2^2}\\
&= \frac{3\sigma^4}{\sigma^4}\\
&= 3
\end{aligned}
$$

As $\beta_1 = 0$ and $\beta_2 = 3$, we can definitively say that the normal distribution is always symmetric, irrespective of its parameters.

Log-Normal Distribution:

If $Y = logX \sim N(\mu,\sigma^2), X$ follows lognormal distribution.

$$
\begin{aligned}
F_X(X) &= f(X\le x)\\
&= f(logX \le logx)\\
&= f(Y\le logx)\\
&= \int_{-\infty}^{logx}\frac{1}{\sigma\sqrt{2\pi}}\text{exp}\left(-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2\right)dy
\end{aligned}
$$

$$
\begin{aligned}
\mu_r' &= E(x^r)\\
&= E(e^{yr})\\
&= \text{exp}\left(\mu r+\frac{\sigma^2 r^2}{2}\right)
\end{aligned}
$$

Hence,

$$\begin{aligned}E(X) &= \mu_1' \\ &= \exp\left(\mu + \frac{\sigma^2}{2}\right)\end{aligned}$$

$$\begin{aligned}E(X^2) &= \mu_2' \\ &= \exp[2(\mu + \sigma^2)]\end{aligned}$$

$$\begin{aligned}Var(X) &= \mu_2' - \mu_1'^2 \\ &= \exp[2(\mu + \sigma^2)] - \exp\left(\mu + \frac{\sigma^2}{2}\right)^2\end{aligned}$$

Cauchy Distribution:

If c.r.v $X \sim C$, then its p.d.f is:

$$f(x) = \begin{cases} \dfrac{1}{\pi}\dfrac{1}{1+x^2} & ; -\infty < x < \infty \\ \\ 0, & ; \text{otherwise} \end{cases}$$

$$X = \frac{Y - \mu}{\sigma}$$

Then, if Y $\sim C(\lambda, \mu)$, $X \sim C(1, 0)$. And, X is the standard Cauchy distribution.

$$G_Y(Y) = \frac{\lambda}{\pi(\lambda^2 + (Y - \mu)^2)}$$

Hence,

$$\begin{aligned}E(Y) &= \int_{-\infty}^{\infty} y\ g(y)dy \\ &= \frac{\lambda}{\pi}\int_{-\infty}^{\infty} \frac{y}{(\lambda^2 + (y-\mu)^2)} \\ &= \frac{\mu\lambda}{\pi}\int_{-\infty}^{\infty} \frac{dy}{(\lambda^2 + (y-\mu)^2)} + \frac{\lambda}{\pi}\int_{-\infty}^{\infty} \frac{y-\mu}{(\lambda^2 + (y-\mu)^2)}dy \\ &= \mu + \frac{\lambda}{\pi}\int_{-\infty}^{\infty} \frac{z}{(\lambda^2 + z^2)}dz\end{aligned}$$

This integral is undefined, hence the first moment i.e. the mean does not exist.

Bivariate Distributions: If X and Y are c.r.v.s, then $F_{XY}(X,Y)$ is the joint p.d.f. iff

$$0 < f_{XY}(x,y) < 1$$

If $f(X,Y)$ is the joint density function then,

$$\int_X \int_Y f(X,Y)\; dy\; dx = 1$$

Marginal p.d.f's are:

$$f_X(x) = \int_Y f(X,Y)\; dy,$$
$$f_Y(y) = \int_X f(X,Y)\; dx$$

Conditional p.d.f's are:

$$f(Y|X=x) = \frac{f(X,Y)}{f(X=x)}$$

$$P(x_1 < X < x_2, y_1 < Y < y_2) = F(x_2,y_2) - f(x_1,y_1)$$

If X and Y are independent,

$$f_{XY}(x,y) = f_X(x) \cdot f_Y(y)\; \forall\; x,y$$

Bivariate Normal Distribution:

If X & Y are jointly distributed with bivariate normal distribution with parameters $(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$, then their j.p.d.f. is:

$$f(X,Y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - \frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]\right\}$$

The j.p.d.f. of s.n.b.d. is:

$$f(X,Y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[x^2 - 2\rho xy + y^2\right]\right\}$$

If X & Y are jointly distributed with bivariate normal distribution with joint probability distribution function $f(X,Y)$, then conditional probability distribution $P(X|Y)$ is given as:

$$\frac{f(X,Y)}{f(Y)} = \frac{1}{\sigma_1\sqrt{2\pi}\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2\sigma_1^2(1-\rho^2)}\left[x - \left(\mu_1 + \rho\frac{\sigma_1}{\sigma_2}(y-\mu_2)\right)\right]^2\right\}$$

Then,

$$E(X|Y) = \mu_1 + \rho\frac{\sigma_1}{\sigma_2}(y-\mu_2)$$

And,

$$Var(X|Y) = \sigma_1^2(1-\rho^2)$$
$$S.D.(X|Y) = \sigma_1\sqrt{1-\rho^2}$$