Exact solution of diffusion equation for a single fibre

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1 Solution

1.1 Infinite domain

Starting with Eqn.(8) in *'Exact solution of diffusion equation for a single fibre - Eike Mueller'*[1].

$$\varphi(\rho,\theta) = a_0 + b_0 \log\rho + \sum_{n=1}^{\infty} a_n \rho^n cos(n\theta + \alpha_n) + \sum_{n=1}^{\infty} b_n \rho^{-n} cos(n\theta + \beta_n) \tag{1}$$

With the specific conductivity, there is the implication of

$$\varphi(\rho,\theta) = \begin{cases} \varphi_{in}(\rho,\theta) & \rho \leq R \\ \varphi_{out}(\rho,\theta) & \rho > R \end{cases} \tag{2}$$

with

$$\begin{aligned} \varphi_{in}(\rho,\theta) &= \sum_{n=1}^{\infty} a'_n \rho^n cos(n\theta) \\ \varphi_{out}(\rho,\theta) &= \sum_{n=1}^{\infty} a_n \rho^n cos(n\theta) + \sum_{n=1}^{\infty} b_n \rho^{-n} cos(n\theta) \end{aligned} \tag{3}$$

Dropped terms for a_0 (not needed), b_0 $(= 0)$, and α_n, β_n $(= 0)$. Eqn.(6) in [1] implies that $a_1 = -E_0$.

$$-C_{in} \frac{\partial \varphi_{in}}{\partial \rho}\Big|_{\rho=R} = -C_{out} \frac{\partial \varphi_{out}}{\partial \rho}\Big|_{\rho=R} \tag{5a}$$

$$-\frac{1}{R} \frac{\partial \varphi_{in}}{\partial \theta}\Big|_{\rho=R} = -\frac{1}{R} \frac{\partial \varphi_{out}}{\partial \theta}\Big|_{\rho=R} \tag{5b}$$

Inserting multipole expansions (3) into (5a) leads to

$$\sum_{n=1}^{\infty} -C_{in}a'_n nR^{n-1}cos(n\theta) = \sum_{n=1}^{\infty} -C_{out}a_n nR^{n-1}cos(n\theta) + \sum_{n=1}^{\infty} C_{out}b_n nR^{-(n+1)}cos(n\theta) \tag{6}$$

for $n = 1$:

$$-C_{in}a'_1 = C_{out}\left(E_0 + \frac{b_1}{R^2}\right) \tag{7}$$

Inserting multipole expansions (3) into (5b) leads to

$$\sum_{n=1}^{\infty} a'_n nR^{n-1}sin(n\theta) = \sum_{n=1}^{\infty} a_n nR^{n-1}sin(n\theta) + \sum_{n=1}^{\infty} b_n nR^{-(n+1)}sin(n\theta) \tag{8}$$

for $n = 1$:

$$a'_1 = -E_0 + \frac{b_1}{R^2} \tag{9}$$

Solving (7) and (9) gives me

$$b_1 = E_0R^2\frac{1-\kappa}{1+\kappa} \qquad\qquad a'_1 = -E_0\frac{2\kappa}{1+\kappa} \tag{10}$$

where $\kappa = \frac{C_{out}}{C_{in}}$. This is different to what you got which was

$$b_1 = -E_0R^2\frac{1-\kappa}{1+\kappa} \qquad\qquad a'_1 = -E_0\frac{2}{1+\kappa} \tag{11}$$

The differences in sign for b_1 is resultant in the different a'_1. My working out for b_1 is as follows:

$$-C_{in}a'_1 = C_{out}\left(E_0 + \frac{b_1}{R^2}\right)$$

$$-a'_1 = \frac{C_{out}}{C_{in}}\left(E_0 + \frac{b_1}{R^2}\right) \to -a'_1 = \kappa\left(E_0 + \frac{b_1}{R^2}\right)$$

sub in $a'_1 = -E_0 + \frac{b_1}{R^2}$

$$-\left(-E_0 + \frac{b_1}{R^2}\right) = \kappa\left(E_0 + \frac{b_1}{R^2}\right) \to E_0 - \frac{b_1}{R^2} = \kappa\left(E_0 + \frac{b_1}{R^2}\right)$$

$$E_0 - \frac{b_1}{R^2} = \kappa E_0 + \kappa\frac{b_1}{R^2} \to E_0 - \kappa E_0 = \frac{b_1}{R^2} + \kappa\frac{b_1}{R^2}$$

$$E_0(1-\kappa) = \frac{b_1}{R^2}(1+\kappa) \to b_1 = E_0R^2\frac{1-\kappa}{1+\kappa}$$

Putting everything together, this gives

$$\begin{aligned} \varphi_{in}(\rho,\theta) &= -\frac{2\kappa}{1+\kappa}E_0\rho cos(\theta) = -\frac{2\kappa}{1+\kappa}E_0 x \\ \varphi_{out}(\rho,\theta) &= \left(-1+\frac{1-\kappa}{1+\kappa}\frac{R^2}{\rho^2}\right)E_0\rho cos(\theta) = \left(-1+\frac{1-\kappa}{1+\kappa}\frac{R^2}{\rho^2}\right)E_0 x \end{aligned} \tag{12}$$

The corresponding gradient for $\mathbf{E}_{in}$ is as follows:

$$\mathbf{E}_{in}(\mathbf{x}) = -\nabla\varphi_{in}(\mathbf{x}) \rightarrow \mathbf{E}_{in}(\rho,\theta) = -\nabla\varphi_{in}(\rho,\theta)$$

$$\mathbf{E}_{in}(\rho,\theta) = -\nabla\varphi_{in}(\rho,\theta) = -\left(\frac{\partial\varphi_{in}(\rho,\theta)}{\partial\rho} + \frac{1}{\rho}\frac{\partial\varphi_{in}(\rho,\theta)}{\partial\theta}\right)$$

$$\mathbf{E}_{in}(\rho,\theta) = -\left(-\frac{2\kappa}{1+\kappa}\mathbf{E}_0 cos(\theta)\hat{\rho} + \frac{1}{\rho}\frac{2\kappa}{1+\kappa}\mathbf{E}_0\rho sin(\theta)\hat{\varphi}\right)$$

$$\mathbf{E}_{in}(\rho,\theta) = \frac{2\kappa}{1+\kappa}\mathbf{E}_0 cos(\theta)\hat{\rho} - \frac{2\kappa}{1+\kappa}\mathbf{E}_0 sin(\theta)\hat{\varphi} \rightarrow \left(\frac{2\kappa}{1+\kappa}\mathbf{E}_0\right)(cos(\theta)\hat{\rho} - sin(\theta)\hat{\varphi})$$

$$\mathbf{E}_{in}(\rho,\theta) = \frac{2\kappa}{1+\kappa}\mathbf{E}_0(\mathbf{x})$$

The corresponding gradient for $\mathbf{E}_{out}$ is as follows:

$$\mathbf{E}_{out}(\mathbf{x}) = -\nabla\varphi_{out}(\mathbf{x}) \rightarrow \mathbf{E}_{out}(\rho,\theta) = -\nabla\varphi_{out}(\rho,\theta)$$

$$\mathbf{E}_{out}(\rho,\theta) = -\nabla\varphi_{out}(\rho,\theta) = -\left(\frac{\partial\varphi_{out}(\rho,\theta)}{\partial\rho} + \frac{1}{\rho}\frac{\partial\varphi_{out}(\rho,\theta)}{\partial\theta}\right)$$

$$\mathbf{E}_{out}(\rho,\theta) = -\left(-\mathbf{E}_0 cos(\theta)\hat{\rho} - \frac{1-\kappa}{1+\kappa}\frac{R^2}{\rho^2}\mathbf{E}_0 cos(\theta)\hat{\rho} + \frac{1}{\rho}\mathbf{E}_0\rho sin(\theta)\hat{\varphi} - \frac{1}{\rho}\frac{1-\kappa}{1+\kappa}\frac{R^2}{\rho}\mathbf{E}_0 sin(\theta)\hat{\varphi}\right)$$

$$\mathbf{E}_{out}(\rho,\theta) = \mathbf{E}_0 cos(\theta)\hat{\rho} + \frac{1-\kappa}{1+\kappa}\frac{R^2}{\rho^2}\mathbf{E}_0 cos(\theta)\hat{\rho} - \mathbf{E}_0 sin(\theta)\hat{\varphi} + \frac{1-\kappa}{1+\kappa}\frac{R^2}{\rho^2}\mathbf{E}_0 sin(\theta)\hat{\varphi}$$

$$\mathbf{E}_{out}(\rho,\theta) = \left(\mathbf{E}_0 + \frac{1-\kappa}{1+\kappa}\frac{R^2}{\rho^2}\mathbf{E}_0\right)cos(\theta)\hat{\rho} - \left(\mathbf{E}_0 - \frac{1-\kappa}{1+\kappa}\frac{R^2}{\rho^2}\mathbf{E}_0\right)sin(\theta)\hat{\varphi}$$

$$\mathbf{E}_{out}(\rho,\theta) = \left(\mathbf{E}_0 + \frac{1-\kappa}{1+\kappa}\frac{R^2}{\rho^2}\mathbf{E}_0\right)(cos(\theta)\hat{\rho} - sin(\theta)\hat{\varphi})$$

$$\mathbf{E}_{out}(\rho,\theta) = \left(1 + \frac{1-\kappa}{1+\kappa}\frac{R^2}{\rho^2}\right)\mathbf{E}_0(\mathbf{x})$$

Note that for $\kappa = 1$ we have $\mathbf{E}_{in}(\rho,\theta) = \mathbf{E}_{out}(\rho,\theta) = \mathbf{E}_0$, as expected.

1.2 Finite domain

Similar to Section 2.2 in *'Exact solution of diffusion equation for a single fibre - Eike Mueller'*[1] the terms I have that violate the boundary conditions $\Omega = \left[-\frac{L}{2}, +\frac{L}{2}\right] \times \left[-\frac{L}{2}, +\frac{L}{2}\right]$ are of the form:

$$\mathcal{O}\left(\frac{1-\kappa}{1+\kappa}\frac{R^2}{\rho^2}\right) \tag{13}$$

We can compute the average value of $\mathbf{E}(\mathbf{x})$ over the finite domain Ω by using Eqn.(23) and Eqn.(24) from [1] Implies

$$\bar{\mathbf{E}} = \mathbf{E}_0 + \mathcal{O}\left(\frac{1-\kappa}{1+\kappa}\frac{R^2}{\rho^2}\right) \tag{14}$$

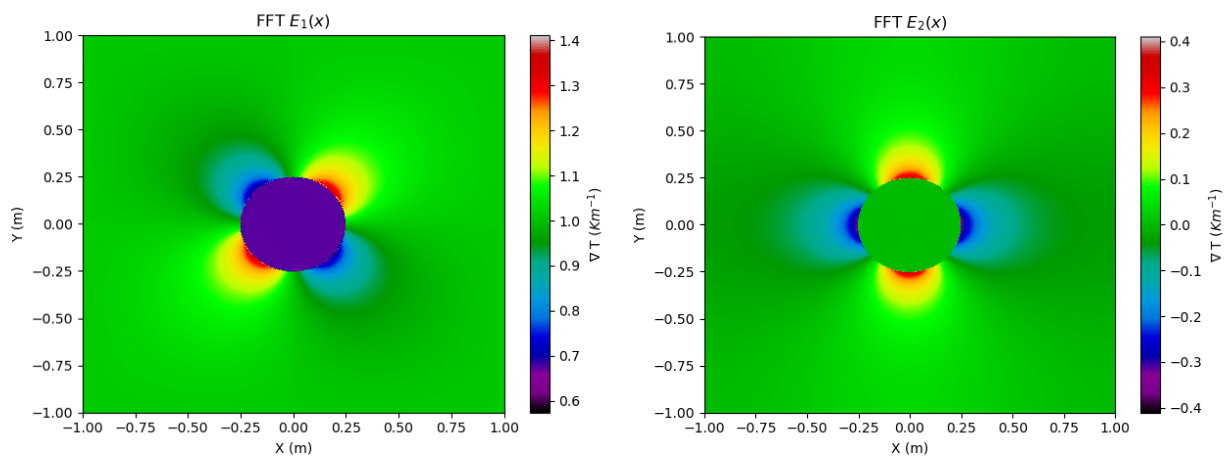

Figure 1: Temperature gradient field components E_1 (left) and E_2 (right) for a domain of size $\Omega = [-1, +1] \times [-1, +1]$ and contrast $\kappa = \frac{1}{2}$.

Field	Mean	Max	Min	Std Dev
Rotation	-6.4027e-10	3.4673e+02	-3.2737e+02	5.1156e+00
Divergence	1.4635e-10	3.3386e+02	-3.4544e+02	5.7490e+00