Exact solution of diffusion equation for a single fibre

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1 Physical Problem

The following is adapted from *'J. D. Jackson, Classical electrodynamics, Wiley, 3rd edition, 2021.'*[1] and *'Exact solution of diffusion equation for a single fibre - Eike Mueller'*[2]. The steady-state heat transfer is governed by the following equations:

$$\nabla \cdot \boldsymbol{J}(\boldsymbol{x}) = 0 \tag{1a}$$
$$\boldsymbol{J}(\boldsymbol{x}) = C(\boldsymbol{x}) : \boldsymbol{E}(\boldsymbol{x}) \tag{1b}$$
$$\boldsymbol{E}(\boldsymbol{x}) = -\nabla\varphi(\boldsymbol{x}) \tag{1c}$$

Where: $\varphi(\boldsymbol{x})$ is the scalar temperature field; $\boldsymbol{J}(\boldsymbol{x})$ is the heat flux; $\boldsymbol{E}(\boldsymbol{x})$ is the (negative) temperature gradient; and $\boldsymbol{c}(\boldsymbol{x})$ is the second-order conductivity tensor.

Equation (1a) states that there are no sources of heat and (1b) relates the heat flux $\boldsymbol{J}(\boldsymbol{x})$ to the (negative) temperature gradient $\boldsymbol{E}(\boldsymbol{x})$. Observe that (1c) implies that $\boldsymbol{E}(\boldsymbol{x})$ is irrotational. We assume that the material is isotropic which implies that the thermal conductivity $C(\boldsymbol{x})$ is a scalar field. Equations (1a) and (1c) are solved in the two-dimensional domain $\Omega = [-\frac{L}{2}, +\frac{L}{2}] \times [-\frac{L}{2}, +\frac{L}{2}]$. We impose boundary conditions and

$$\boldsymbol{E}(\boldsymbol{x} + L(\nu_1 \boldsymbol{e}^{(1)} + \nu_2 \boldsymbol{e}^{(2)})) = \boldsymbol{E}(\boldsymbol{x}) \quad \text{for all} \quad \nu_1, \nu_2 \in \mathbb{Z} \tag{2}$$

where $\boldsymbol{e}^{(1)} = (1,0)^\top$, $\boldsymbol{e}^{(2)} = (0,1)^\top$ are the unit vectors in the two coordinate directions and assume an average (negative) temperature gradient $\overline{\boldsymbol{E}} = \langle \boldsymbol{E}(\boldsymbol{x}) \rangle$.

$$\frac{1}{L^2} \int_\Omega \boldsymbol{E}(\boldsymbol{x}) d\boldsymbol{x} = \overline{\boldsymbol{E}} \tag{3}$$

1.1 Conductivity field

We assume conductivity is if the form

$$C(\boldsymbol{x}) = \begin{cases} C_{in} & \|\boldsymbol{x}\| < R \\ C_{out} & \|\boldsymbol{x}\| > R \end{cases} \tag{4}$$

with $R < \frac{L}{2}$ the radius if the fibre and positive constants C_{in}, C_{out}.

2 Solution

2.1 Infinite domain

The probllem is first considered in an infinite domain $\Omega = \mathbb{R}^2$. Instead of (3), we impose a boundary condition

$$\boldsymbol{E}(\boldsymbol{x}) \to \boldsymbol{E}_0 = E_0 \boldsymbol{e}^{(1)} \quad \text{for} \quad \rho \to \infty \tag{5}$$

The following derivation is comparable to the analogous three dimensional case in [1, Section 4.4]. In polar coordinates with $\boldsymbol{x} = (\rho\cos(\theta), \rho\sin(\theta))$ this is equivalent to

$$\varphi(\rho, \theta) \to -E_0 \rho \cos(\theta) \quad \text{for} \quad \rho \to \infty \tag{6}$$

For constant $C(\boldsymbol{x})$, (1) is equivalent to [1, Section 2.11]

$$\Delta\varphi = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial\varphi}{\partial\rho}) + \frac{1}{\rho^2}\frac{\partial^2\varphi}{\partial\theta^2} = 0 \tag{7}$$

The general solution of (7) is given by two dimensional multipole expansion

$$\varphi(\rho, \theta) = a_0 + b_0 \text{log}\rho + \sum_{n=1}^{\infty} a_n \rho^n cos(n\theta + \alpha_n) + \sum_{n=1}^{\infty} b_n \rho^{-n} cos(n\theta + \beta_n) \tag{8}$$

With the specific conductivity (4), there is the implication of

$$\varphi(\rho, \theta) = \begin{cases} \varphi_{in}(\rho, \theta) & \rho \leq R \\ \varphi_{out}(\rho, \theta) & \rho > R \end{cases} \tag{9}$$

with

$$\begin{aligned} \varphi_{in}(\rho, \theta) &= \sum_{n=1}^{\infty} a'_n \rho^n cos(n\theta) \\ \varphi_{out}(\rho, \theta) &= \sum_{n=1}^{\infty} a_n \rho^n cos(n\theta) + \sum_{n=1}^{\infty} b_n \rho^{-n} cos(n\theta) \end{aligned} \tag{10}$$

Dropped terms for a_0 (not needed), B.C (6) implies b_0 $(= 0)$, and α_n, β_n $(= 0)$ for $n > 1$, and $a_1 = -E_0$.

At interface $\rho = a$, the normal component of $\boldsymbol{J}$ and the tangential component of $\boldsymbol{E}$ are continuous (see [1, Section 1.5]), which implies that

$$-C_{in} \frac{\partial\varphi_{in}}{\partial\rho}\Big|_{\rho=R} = -C_{out} \frac{\partial\varphi_{out}}{\partial\rho}\Big|_{\rho=R} \tag{11a}$$

$$-\frac{1}{R}\frac{\partial\varphi_{in}}{\partial\theta}\Big|_{\rho=R} = -\frac{1}{R}\frac{\partial\varphi_{out}}{\partial\theta}\Big|_{\rho=R} \tag{11b}$$

Inserting multipole expansions (10) into (11a) leads to

$$\sum_{n=1}^{\infty} -C_{in}a'_n nR^{n-1}cos(n\theta) = \sum_{n=1}^{\infty} -C_{out}a_n nR^{n-1}cos(n\theta) + \sum_{n=1}^{\infty} C_{out}b_n nR^{-(n+1)}cos(n\theta) \quad (12)$$

for $n = 1$:

$$-C_{in}a'_1 = C_{out}\left(E_0 + \frac{b_1}{R^2}\right) \quad (13)$$

and for $n > 1$:

$$-C_{in}a'_n = C_{out}\frac{b_n}{R^{2n}} \quad (14)$$

Inserting multipole expansions (10) into (11b) leads to

$$\sum_{n=1}^{\infty} a'_n nR^{n-1}sin(n\theta) = \sum_{n=1}^{\infty} a_n nR^{n-1}sin(n\theta) + \sum_{n=1}^{\infty} b_n nR^{-(n+1)}sin(n\theta) \quad (15)$$

for $n = 1$:

$$a'_1 = -E_0 + \frac{b_1}{R^2} \quad (16)$$

and for $n > 1$:

$$a'_n = \frac{b_n}{R^{2n}}\,, \qquad \alpha'_n = \alpha_n = \beta_n \quad (17a)$$

Solving (13) and (16) gives me

$$b_1 = E_0R^2\frac{1-\kappa}{1+\kappa} \qquad a'_1 = -E_0\frac{2\kappa}{1+\kappa} \quad (18a)$$

where $\kappa = \frac{C_{out}}{C_{in}}$.
Working out for b_1 is as follows:

$$-C_{in}a'_1 = C_{out}\left(E_0 + \frac{b_1}{R^2}\right)$$

$$-a'_1 = \frac{C_{out}}{C_{in}}\left(E_0 + \frac{b_1}{R^2}\right) \rightarrow -a'_1 = \kappa\left(E_0 + \frac{b_1}{R^2}\right)$$

sub in $a'_1 = -E_0 + \frac{b_1}{R^2}$

$$-\left(-E_0 + \frac{b_1}{R^2}\right) = \kappa\left(E_0 + \frac{b_1}{R^2}\right) \rightarrow E_0 - \frac{b_1}{R^2} = \kappa\left(E_0 + \frac{b_1}{R^2}\right)$$

$$E_0 - \frac{b_1}{R^2} = \kappa E_0 + \kappa \frac{b_1}{R^2} \rightarrow E_0 - \kappa E_0 = \frac{b_1}{R^2} + \kappa \frac{b_1}{R^2}$$

$$E_0(1-\kappa) = \frac{b_1}{R^2}(1+\kappa) \rightarrow b_1 = E_0 R^2 \frac{1-\kappa}{1+\kappa}$$

Putting everything together, this gives

$$\begin{aligned} \varphi_{in}(\rho,\theta) &= -\frac{2\kappa}{1+\kappa} E_0 \rho cos(\theta) = -\frac{2\kappa}{1+\kappa} E_0 x \\ \varphi_{out}(\rho,\theta) &= \left(-1 + \frac{1-\kappa}{1+\kappa}\frac{R^2}{\rho^2}\right) E_0 \rho cos(\theta) = \left(-1 + \frac{1-\kappa}{1+\kappa}\frac{R^2}{\rho^2}\right) E_0 x \end{aligned} \tag{19}$$

The corresponding gradient for $\mathbf{E}_{in}$ is as follows:

$$\mathbf{E}_{in}(\mathbf{x}) = -\nabla \varphi_{in}(\mathbf{x}) \rightarrow \mathbf{E}_{in}(\rho,\theta) = -\nabla \varphi_{in}(\rho,\theta)$$

$$\mathbf{E}_{in}(\rho,\theta) = -\nabla \varphi_{in}(\rho,\theta) = -\left(\frac{\partial \varphi_{in}(\rho,\theta)}{\partial \rho} + \frac{1}{\rho}\frac{\partial \varphi_{in}(\rho,\theta)}{\partial \theta}\right)$$

$$\mathbf{E}_{in}(\rho,\theta) = -\left(-\frac{2\kappa}{1+\kappa}\mathbf{E}_0 cos(\theta)\hat{\rho} + \frac{1}{\rho}\frac{2\kappa}{1+\kappa}\mathbf{E}_0 \rho sin(\theta)\hat{\varphi}\right)$$

$$\mathbf{E}_{in}(\rho,\theta) = \frac{2\kappa}{1+\kappa}\mathbf{E}_0 cos(\theta)\hat{\rho} - \frac{2\kappa}{1+\kappa}\mathbf{E}_0 sin(\theta)\hat{\varphi} \rightarrow \left(\frac{2\kappa}{1+\kappa}\mathbf{E}_0\right)(cos(\theta)\hat{\rho} - sin(\theta)\hat{\varphi})$$

$$\mathbf{E}_{in}(\rho,\theta) = \frac{2\kappa}{1+\kappa}\mathbf{E}_0(\mathbf{x})$$

The corresponding gradient for $\mathbf{E}_{out}$ is as follows:

$$\mathbf{E}_{out}(\mathbf{x}) = -\nabla \varphi_{out}(\mathbf{x}) \rightarrow \mathbf{E}_{out}(\rho,\theta) = -\nabla \varphi_{out}(\rho,\theta)$$

$$\mathbf{E}_{out}(\rho,\theta) = -\nabla \varphi_{out}(\rho,\theta) = -\left(\frac{\partial \varphi_{out}(\rho,\theta)}{\partial \rho} + \frac{1}{\rho}\frac{\partial \varphi_{out}(\rho,\theta)}{\partial \theta}\right)$$

$$\mathbf{E}_{out}(\rho,\theta) = -\left(-\mathbf{E}_0 cos(\theta)\hat{\rho} - \frac{1-\kappa}{1+\kappa}\frac{R^2}{\rho^2}\mathbf{E}_0 cos(\theta)\hat{\rho} + \frac{1}{\rho}\mathbf{E}_0 \rho sin(\theta)\hat{\varphi} - \frac{1}{\rho}\frac{1-\kappa}{1+\kappa}\frac{R^2}{\rho}\mathbf{E}_0 sin(\theta)\hat{\varphi}\right)$$

$$\mathbf{E}_{out}(\rho,\theta) = \mathbf{E}_0 cos(\theta)\hat{\rho} + \frac{1-\kappa}{1+\kappa}\frac{R^2}{\rho^2}\mathbf{E}_0 cos(\theta)\hat{\rho} - \mathbf{E}_0 sin(\theta)\hat{\varphi} + \frac{1-\kappa}{1+\kappa}\frac{R^2}{\rho^2}\mathbf{E}_0 sin(\theta)\hat{\varphi}$$

$$\mathbf{E}_{out}(\rho,\theta) = \left(\mathbf{E}_0 + \frac{1-\kappa}{1+\kappa}\frac{R^2}{\rho^2}\mathbf{E}_0\right) cos(\theta)\hat{\rho} - \left(\mathbf{E}_0 - \frac{1-\kappa}{1+\kappa}\frac{R^2}{\rho^2}\mathbf{E}_0\right) sin(\theta)\hat{\varphi}$$

$$\mathbf{E}_{out}(\rho,\theta) = \left(\mathbf{E}_0 + \frac{1-\kappa}{1+\kappa}\frac{R^2}{\rho^2}\mathbf{E}_0\right)(cos(\theta)\hat{\rho} - sin(\theta)\hat{\varphi})$$

$$\mathbf{E}_{out}(\rho,\theta) = \left(1 + \frac{1-\kappa}{1+\kappa}\frac{R^2}{\rho^2}\right)\mathbf{E}_0(\mathbf{x})$$

Where $\mathbf{E}_0^{\perp} := E_0\boldsymbol{e}^{(2)}$ such that $\mathbf{E}_0$ and $\mathbf{E}_0^{\perp}$ span the right-handed coordinate system. Note that for $\kappa = 1$ we have $\mathbf{E}_{in}(\rho,\theta) = \mathbf{E}_{out}(\rho,\theta) = \mathbf{E}_0$, as expected.

2.2 Finite domain

Similar to [2, Section 2.2] the terms I have that violate the boundary conditions $\Omega = \left[-\frac{L}{2}, +\frac{L}{2}\right] \times \left[-\frac{L}{2}, +\frac{L}{2}\right]$ are of the form:

$$\mathcal{O}\left(\frac{1-\kappa}{1+\kappa}\frac{R^2}{\rho^2}\right) \tag{13}$$

We can compute the average value of $\mathbf{E}(\mathbf{x})$ over the finite domain Ω by using Eqn.(23) and Eqn.(24) from [1] Implies

$$\bar{\mathbf{E}} = \mathbf{E}_0 + \mathcal{O}\left(\frac{1-\kappa}{1+\kappa}\frac{R^2}{\rho^2}\right) \tag{14}$$

2.3 Result

This test case covers the exact solution of the diffusion equation for a single fibre extrapolated from a single dipole in an electrical field by *'J. D. Jackson, Classical electrodynamics, Wiley, 3rd edition, 2021.'*[2] This is done in an unbound, infinitely large domain, therefore there will be some differences to the FFT result as the FFT solution is in a finite, periodic domain.

This solution is derived using multipole expansions and the understanding that for specific conductivities inside the fibre (C_{in}) and outside (C_{out}) the fibre, there is a contrast of $\kappa = \frac{C_{out}}{C_{in}}$. Other parameters are: L=2m, radial distance ρ, and a fibre radius R=25cm.

For the FFT solution the parameters L = 2m, voxels = 500, $\overline{\boldsymbol{E}} = [2, 0]$, convergence criterion of $\varepsilon = 1{\times}10^{-12}$, fibre and matrix conductivity of $\boldsymbol{c}_f(\boldsymbol{x})$=2 and $\boldsymbol{c}_m(\boldsymbol{x})$=1 respectively in keeping with κ, and $C_0 = \frac{(\boldsymbol{c}_f(\boldsymbol{x})+\boldsymbol{c}_m(\boldsymbol{x}))}{2}$.

The proximity in result between the FFT and the exact solution over the domain is shown in Fig.1. With both components showing the largest differences in gradient at the fibre boundary.

In Fig.2 below the result can be seen for the x and y components for the exact (analytical) solution (2c and 2b) and the FFT solution (2a and 2b). They appear to have a high degree of similarity especially for the x components as shown through the minimal differences seen in the line profile taken at $Y = 0$ of the gradient shown in (2e). The y component (2f) shows some differences with the exact solution showing some gradient at the edges of the fibre where the FFT shows none, this is likely due to differences in the domains.

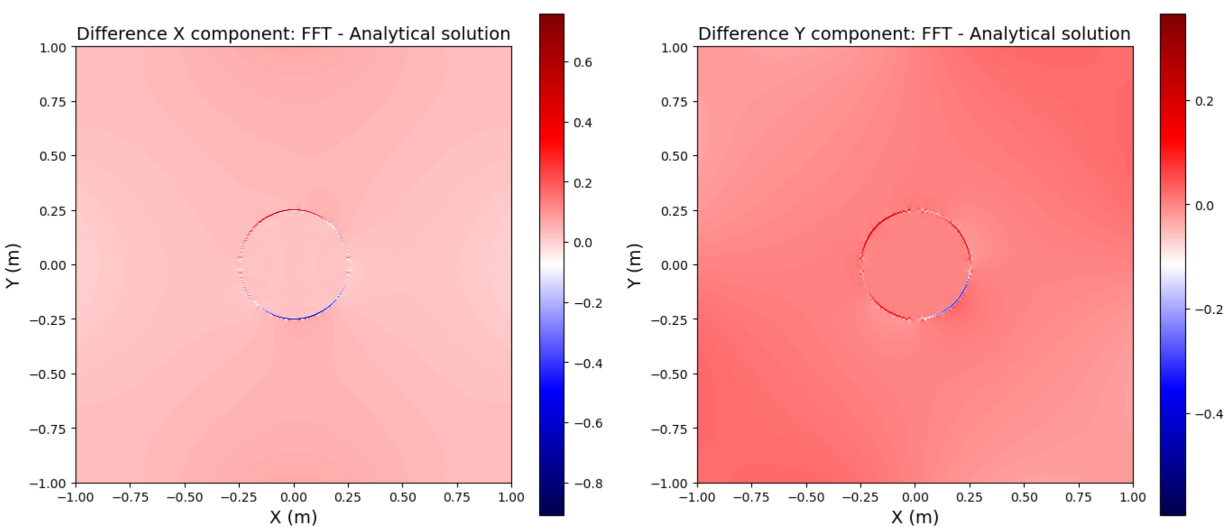
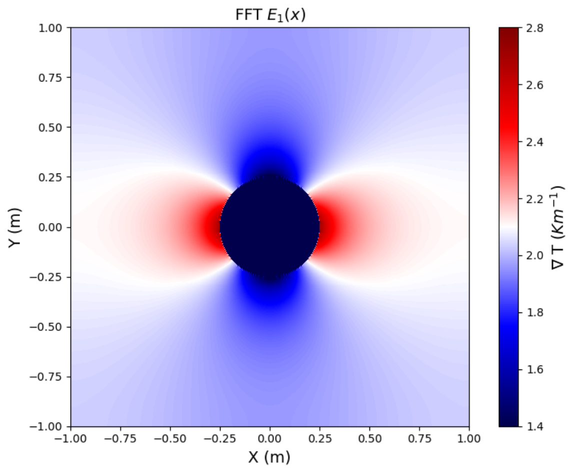
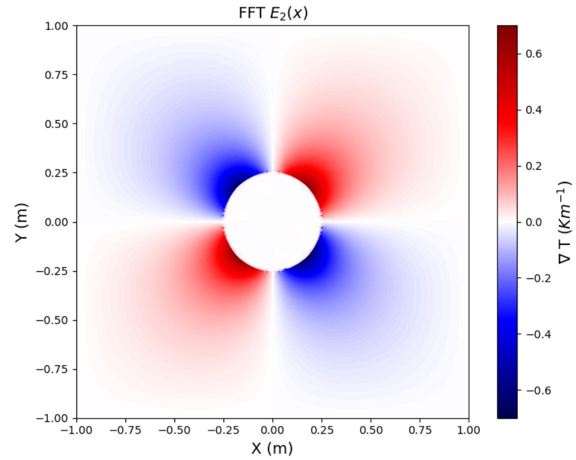


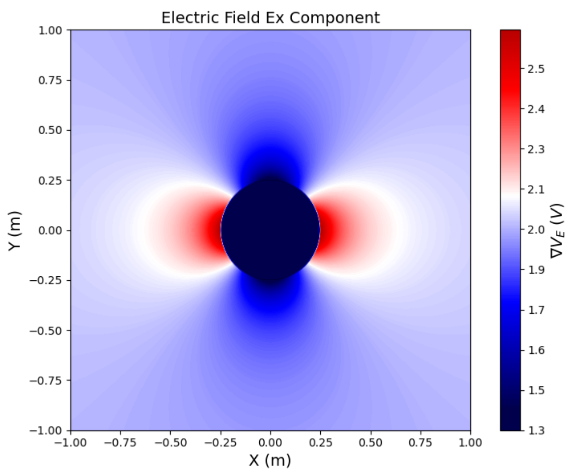
Figure 1: Difference between FFT and analytical solution.



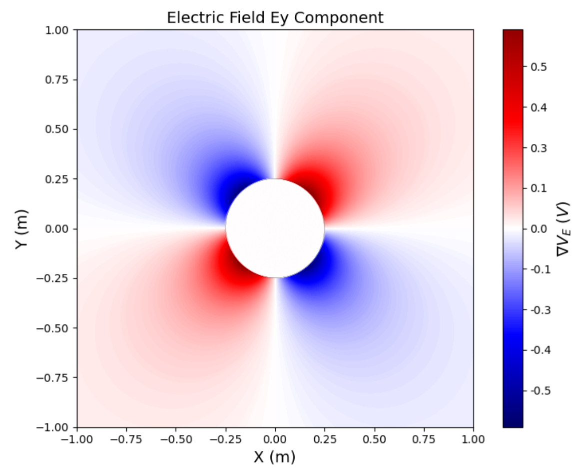
(a) FFT x-component



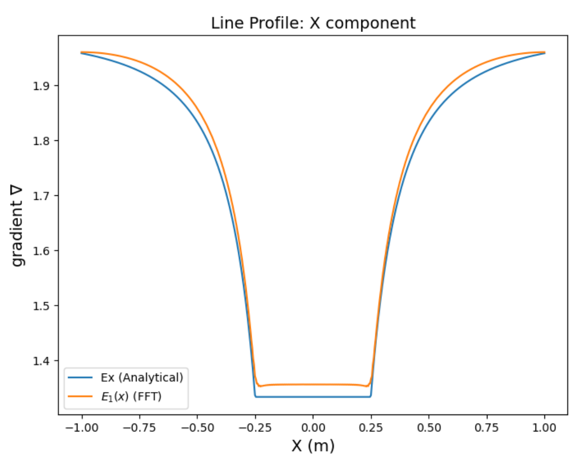
(b) FFT y-component



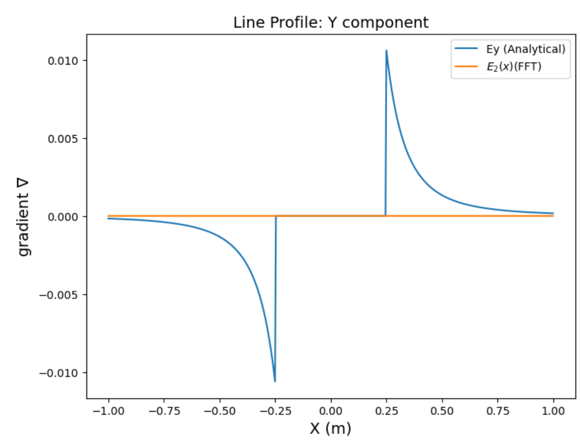
(c) Analytical x-component



(d) Analytical y-component



(e) line profile x-component



(f) line profile y-component

Figure 2: *Comparison of FFT and exact solution*