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Enumeration of spanning trees on contact graphs of disk packings

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HIGHLIGHTS

- We investigate the number of spanning trees on contact graph of disk packings.
- An exact analytical expression for this quantity is determined.
- Some electrically equivalent transformations are adopted.
- The new network has small-world scale-free topology with the maximum entropy.

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ABSTRACT

Obtaining the number of spanning trees of complex networks is an outstanding challenge, since traditional approaches, such as calculating the eigenvalues of the matrix and decomposing of spanning subgraphs, are awkward or even infeasible for a large scale network. The foundation and importance of this quantity relating to some topological and dynamic properties prompt us to explore the role of determinant identities for Laplace matrices. We introduce the basic electrically equivalent technique to determine an exact analytical expression for the quantity on the contact graph of disk packings, which is proposed by Zhang et al. (2009). Our theoretical results shed light on the relationship between the microscopic change of the quantity and topological iteration of the network. In particular, we compare the entropy of spanning trees on the network with the other two-dimensional and three-dimensional lattices. We show that the new model is a small-world scale-free network with the maximum entropy so far found. In addition, our method for employing the electrically equivalent technique to enumerate spanning trees is general and can be easily extended to other complex networks.

1. Introduction

With the rapid development of computer technology, there is a considerable progress in acquiring massive data from many real-world systems and even processing them. Recently, many scientists and engineers from various disciplines, such as mathematics, physics, engineering, sociology, and biology, have proposed and investigated a huge variety of network models and inherent mechanisms to mimic topological structures and evolving processes of these networks [1,2]. Despite these research works have led to a significant improvement in the understanding of complexity and characteristics of complex networks, there has been little work dealing explicitly with the geographical effects of all nodes. For most previous studies focusing on topological structures and dynamics of complex networks, the information of spatial or planar location

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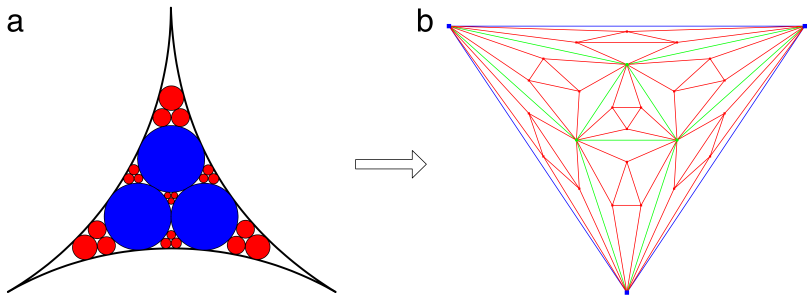

Fig. 1. (a) A variant of Apollonian packing that three disks are added to fill each interstice at arbitrary generation. (b) The corresponding Apollonian network.

of a node is simplified or ignored. However, bus stations in urban transport networks [3], the routers of the Internet [4], and the airports of airline networks [5], are all examples whose nodes have well-defined node positions. It has been shown that such networks associated with geography or spatiality have special topological structures and dynamics [6,7]. In particular, based on space-filling disk packings, Apollonian network is proposed by means of the contact relationships of all disks [8]. Meanwhile, it not only has the features of complex networks such as small-world [9] and scale-free [10], but also has the geographical characteristics of Euclidean and space filling. Accordingly, some research works have been pursued in topological characteristics of high-dimensional Apollonian networks [11,12], multifractal energy spectrum in Apollonian networks [13], correlations in random Apollonian networks [14], as well as contact graphs of disk packings [15].

Unique topological characteristics of Apollonian networks lead a series of interesting research topics. The enumerating spanning trees on a graph or network is one of them, and is also a fundamental issue in mathematics [16–19] and physics [20–22]. The number of spanning trees in a network is generally very large but invariant, even in a small-scale network, which characterizes the reliability of some scale-free unweighted [23,24] and weighted networks [25]. It also determines the mean first-passage time between two nodes in terms of random walks [26]. In addition, it is closed relevant to other interesting problems of graphs or networks, such as synchronization [27] and percolation [28].

The traditional methods to determine this quantity are involved with the calculation of the determinant or the eigenvalues of the Laplacian matrix [29], which is very difficult even intractable for a large-scale network. Recently, it is of considerable current interest to enumerate spanning trees on a scale-free small-world network, since most real networks in nature and society are small-world [9] and scale-free [10]. Based on the self-similarity of network, the spanning trees problem on a pseudofractal scale-free web [30] is firstly considered [31], then the enumeration of spanning trees on an Apollonian network is also investigated [32–34]. This new method associated with the decomposition of spanning subgraphs is effective but complicated for some complex structural networks, since the characteristic subgraphs must be classified with non-repetition and non-omission for all possible subgraphs [34].

In this paper, we investigate the number of spanning trees on a variant of the Apollonian network proposed by Zhang et al. [15], which is devised from the disk packing and the construction method of the Apollonian network, called a contact graph of disk packings. It is constructed whose nodes are described by the circles (generally those nodes contract to the center of the circles) and edges represent contact relation, see Fig. 1. Similar to the Apollonian network, the contact graph of disk packings has the special characteristics in nature and society: large clustering coefficients, small-world effect and a power-law degree distribution[15]. We employ techniques from the theory of electrical networks [19] – such as Kirchhoff's current law and some simple transformations – to determine the number of spanning trees of this variant, and compute the entropy of its spanning trees.

2. Preliminaries

Let G be a simple connected network with node set $V(G) = \{v_1, \dots, v_n\}$. A spanning tree of the network is a minimal set of edges that connect all nodes. The degree of node v, denoted by $d(v)$, is the number of edges attached to it. $D(G)$ denotes the diagonal matrix $(d_{ij})_{n\times n}$ whose elements are $d_{ii} = d(v_i)$. The adjacency matrix $A(G)$ of G is a matrix $(a_{ij})_{n\times n}$ whose elements are $a_{ij} = 1$ if nodes v_i and v_j are adjacent, and $a_{ij} = 0$ otherwise. Then the Laplacian matrix $L(G)$ of G is the matrix $L(G) := D(G) - A(G)$. The reader is referred to Biggs [29] for these and further related definitions.

It should be pointed out that two typical methods related to the Laplacian matrix for enumerating spanning trees on G. One method is "the Matrix-Theorem" [29], which expresses the number of spanning trees $t(G)$ of G as a determinant: $t(G)$ equals the determinant of the submatrix obtained by deleting row v_r and column v_r from $L(G)$ for any $1 \le r \le n$. The other is "the eigenvalue method" [29]. Let $\lambda_1(G) \ge \lambda_2(G) \ge \cdots \ge \lambda_n(G)$ denote the eigenvalues of $L(G)$. Then it is easily shown that $\lambda_n(G) = 0$. Furthermore, $t(G) = \frac{1}{n}\prod_{i=1}^{n-1}\lambda_i(G)$. If $t(G)$ grows exponentially with $|V(G)| = n$ as $n \to \infty$, there exists a constant $E(G)$, called the entropy of spanning trees of G, describing this exponential growth [17,18]:

$$E(G) := \lim_{n\to\infty} \frac{\ln t(G)}{n}. \tag{1}$$

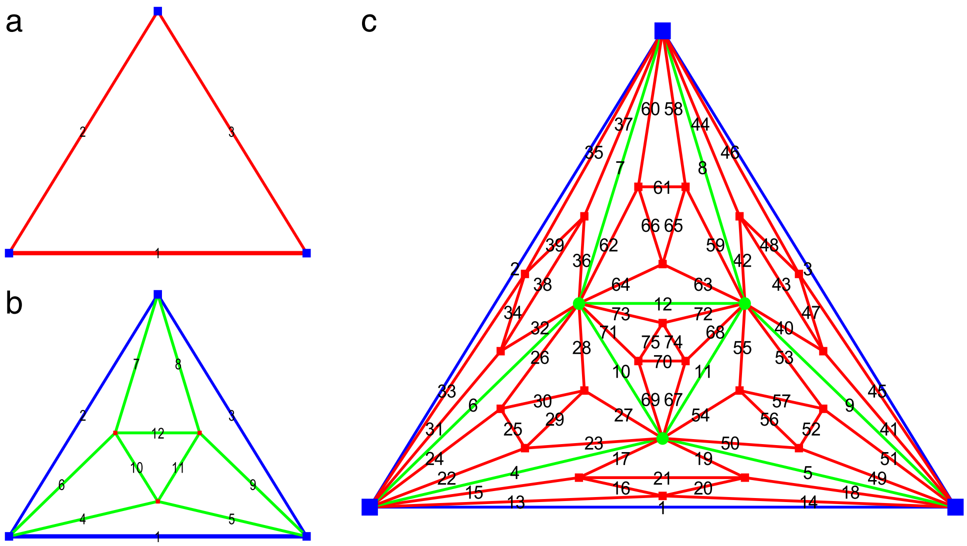

Fig. 2. The spatial planar network with disk packing of the original triangle (a), first generation (b), and second generation (c).

3. Spanning trees on a contact graph of disk packings

3.1. The contact graph of disk packings

By definition, disk packing is an arrangement of circles inside a given boundary such that no two overlap and all of them are mutually internally tangent. When the number of circles degenerates into one and the specific boundary is given by the hole between three circles that contact with each other, an *Apollonian packing* is formed. Apollonian packing is a typical representative of space-filling packing of spheres and derives from the more general osculatory packing [35,36]. In its iterative algorithm, three pairwise externally tangent circles of equal size are all internally tangent to an initial circle, and then the hole between those three circles is filled by the circle of maximal size that is internally tangent to all three, repeating the packing process for all the new smaller holes in the same way [8].

Based on a combination of the disk packing and the generation method of Apollonian networks, a contact graph of disk packings is proposed by Zhang et al. [15], in which network three circles are added to fill each hole at arbitrary iterative generation. The iterative algorithm for the new network of generation n is denoted by G_n as follows: for $n = 0$, G_0 is a triangle with three nodes and three edges (see Fig. 2(a)). A new triangle is then inserted into all original triangles at generation n, and both nodes of each edge of this new triangle are linked to a node of the original triangle without edges crossing. That is, each original triangle in G_{n-1} will be replaced by seven small new triangles in G_n. Therefore, for $n \geq 1$, G_n is obtained from G_{n-1} by filling each triangle of G_{n-1} with the connected cluster with three nodes and nine edges (see Fig. 2(b)). According to the iterative algorithm, one can find that the total number of nodes V_n and edges E_n at generation n is $V(G_n) = (7^n + 5)/2$ and $E(G_n) = (3 \times 7^n + 3)/2$, respectively [15]. Fig. 2(c) shows the second generation of the new network with 27 nodes and 75 edges.

3.2. Counting spanning trees

The selection of the appropriate method for determining the number of spanning trees is a key factor in a given network. For this variant of Apollonian network, the number of edges is exponentially growing, so the traditional method associated with the determinant or the eigenvalues of the Laplacian matrix is infeasible due to high computing time complexity. Meanwhile, the special evolving process of this network indicates that the decomposition of spanning subgraphs is intractable. Luckily, Teufl and Wagner have shown that determinant identity of Laplace matrices can be effectively applied to counting spanning trees in self-similar graphs according to electrically equivalent technique of electrical network [19,37]. When an edge-weighted network is regarded as an electrical network, those weights are the conductances of the respective edges. The number of spanning trees in a network only differs by a factor if its any subnetwork is replaced by an electrical equivalent network, which is very important and convenient to determine the number of spanning trees in the network.

Firstly, we introduce the core theorem of the elegant method and four basic electrically equivalent transformations.

Theorem 1 *([19]). Suppose that a network G can be partitioned into two "edge-disjoint" subnetworks S_1 and S_2, and the node set of G satisfies $V(S_1) \bigcup V(S_2) = V(G)$ and $V(S_1) \bigcap V(S_2) = S_*$. Suppose that S_2' is a network with $E(S_1) \bigcap E(S_2') = \emptyset$ and*

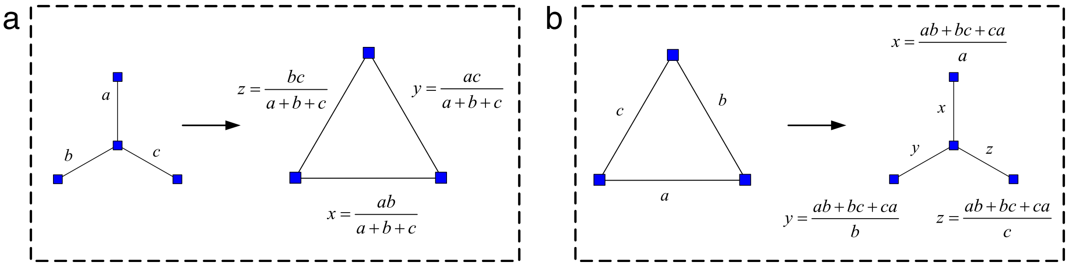

Fig. 3. (a) Wye–Delta transform: To transform a Y network to the equivalent Δ network, each resistor in the Δ network is the sum of all possible products of the Y resistors taken two at a time, divided by the opposite Y resistor. Since the conductance of an object is the inverse of its resistor, one can obtain $\frac{1}{x}=[\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac}]/(\frac{1}{a})$, i.e., $x=\frac{bc}{a+b+c}$. Similarly, $y=\frac{ac}{a+b+c}$, and $z=\frac{ab}{a+b+c}$. (b) Delta–Wye transform: According to the corresponding rule of electrically equivalent method for transforming a Y network to the equivalent Δ network, one can find that $\frac{1}{x}=[\frac{1}{b}\cdot\frac{1}{c}]/[\frac{1}{a}+\frac{1}{b}+\frac{1}{c}]$, then $x=\frac{ab+bc+ac}{a}$, and $y=\frac{ab+bc+ac}{b}$, $z=\frac{ab+bc+ac}{c}$ by analogy.

$V(S_1)\bigcap V(S_2')=S_*$. Furthermore, S_2' is an electrically equivalent network of S_2 with respect to S_*. Let $G'=S_1\bigcup S_2'$. If $t(G)\neq 0$ and $t(S_2)\neq 0$, the following ratio holds

$$\frac{t(G')}{t(G)}=\frac{t(S_2')}{t(S_2)}. \tag{2}$$

The above technique of electrically equivalent network is actually a transformation of the structure of the network. Here, we consider the effect of four simple transformations on the number of spanning trees: parallel edges, serial edges, Wye–Delta transform, and Delta–Wye transform [19].

(1) Parallel edges: the number of spanning trees remains unchanged (the corresponding factor is equal to 1), when two parallel edges with conductances a and b are changed into an electrically equivalent edges with conductance $a+b$.
(2) Serial edges: When a single edge is merged by two serial edges with conductances a and b, its conductance is $ab/(a+b)$, and the number of spanning trees will vary as follows:

$$t(G')=\frac{1}{a+b}\cdot t(G). \tag{3}$$

(3) Wye–Delta transform: when a Wye subnetwork that contains four nodes and three edges with conductances a, b, and c is replaced by an electrically equivalent triangle with conductances x, y, and z (see Fig. 3(a)), the number of spanning trees will vary as follows:

$$t(G')=\frac{1}{a+b+c}\cdot t(G). \tag{4}$$

(4) Delta–Wye transform: when a triangle subnetwork with conductances a, b, and c is replaced by an electrically equivalent Wye subnetwork with conductances x, y, and z (see Fig. 3(b)), the corresponding conversion formula of the number of spanning trees is:

$$t(G')=\frac{(ab+bc+ca)^2}{abc}\cdot t(G). \tag{5}$$

For example, we consider the network that is shown in Fig. 4. A few applications of the transformations suffice to count spanning trees of this network. The evolution of five electrically equivalent topologies and the corresponding calculations of the conductances are as follows: (1) Since the conductance of each edge of the original network is 1, the corresponding conductance of the resulting graph for Delta–Wye transformation is $\frac{1+1+1}{1\cdot1\cdot1}=3$. (2) When two serial edges with conductances 1 and 3 are merged into a new edges, the conductance is $\frac{1\cdot3}{1+3}=\frac{3}{4}$. (3) Replace the Delta subgraph by the Wye subgraph to further simply the network, we can obtain three conductances $(1\cdot\frac{3}{4}+1\cdot3+3\cdot\frac{3}{4})/3=2$, $(1\cdot\frac{3}{4}+1\cdot3+3\cdot\frac{3}{4})/\frac{3}{4}=8$, and $(1\cdot\frac{3}{4}+1\cdot3+3\cdot\frac{3}{4})/1=6$. (4) The conductances of serial edges are updated by $\frac{3}{4}\cdot6/(\frac{3}{4}+6)=\frac{2}{3}$ and $\frac{1\cdot8}{1+8}=\frac{8}{9}$. (5) For two parallel edges, the conductance of the new edge is the sum of two original conductances, i.e., $\frac{2}{3}+\frac{8}{9}=\frac{14}{9}$. Obviously, the weighted number of spanning trees in the final network is $\frac{14}{9}\cdot2=\frac{28}{9}$. So the original network has

$$\frac{1}{9}\cdot4^2\cdot\frac{1}{16}\cdot\left(\frac{27}{4}\cdot9\right)\cdot1\cdot\frac{28}{9}=21$$

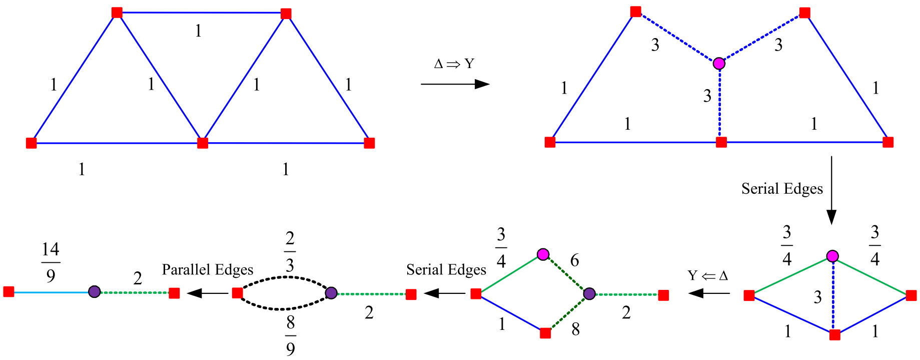

Fig. 4. A simple example using the technique of those transformations to calculate the number of spanning trees.

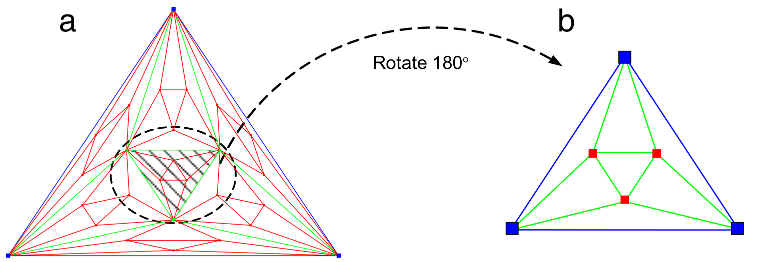

Fig. 5. Illustration of the extraction of a subnetwork (b) (shaded subnetwork) from the original network (a). Actually, there are seven subnetworks in this network like the extracted subnetwork. However, three subnetworks among them are involved in the boundary edges. The remaining four subnetworks can be extracted. Generally, for generation n, G_n contains 7^n subnetworks and $7^n - 3$ subnetworks can be an example to analyze the change of the corresponding conductances.

spanning trees according to the aforementioned theorem and the factors of those transformations. Furthermore, it is shown that the determinant of submatrix obtained by deleting row 5 and column 5 from $L(G)$ is just 21, verifying the correctness of this result by means of the method of the Matrix-Theorem.

Next, we focus on the simplification of the structure of G_n so as to use the above-mentioned conversion formulas. Based on the recursive construction of the network, its internal structure has a self-similar characteristic. Obviously, three edges of the outermost triangle are different from other edges of an arbitrary internal triangle in the whole evolving process. So we classify the conductance on each edge of G_n into two categories: the conductance of the outermost triangle is a_n; while the other edges have the conductance of b_n. Meanwhile, considering the self-similar structure of the network, we extract a subnetwork of G_n that does not contain any boundary edge of the original network, as shown in Fig. 5, to obtain the change of conductances of all edges of the $n-1$ generation from G_n of generation n.

Denote $G_n^{(0)}$ as the extracted subnetwork from G_n. Fig. 6 shows the electrically equivalent evolving process from $G_n^{(0)}$ to $G_{n-1}^{(0)}$. Five transformations are used in sequence and the corresponding conductances of the resulting edges are calculated as follows:

(1) *Delta–Wye transformation*: Replace the small triangle nested inside a large triangle (Δ network) with its electrically equivalent component (Y network). Denote $G_n^{(1)}$ as the resulting graph with three new edges and conductance $\frac{(b_n^2+b_n^2+b_n^2)}{b_n} = 3b_n$ (those edges are denoted by three dashed black lines in $G_n^{(1)}$ in Fig. 6).

(2) *Wye–Delta transformation*: Replace three Y subgraphs (three red squares and their attached edges) by three triangles (their edges denoted by nine dashed black curves) to obtain a new subgraph $G_n^{(2)}$. Noting that the conductance of one edge is $3b_n$, while that of the other two edges is b_n for each Y network, each parallel edge within the triangle has conductance $\frac{b_n \cdot 3b_n}{b_n+b_n+3b_n} = \frac{3}{5}b_n$ and all parallel edges outside the triangle have conductance $\frac{b_n \cdot b_n}{b_n+b_n+3b_n} = \frac{1}{5}b_n$.

(3) *Parallel edges*: Merge those parallel edges of $G_n^{(2)}$ we can obtain $G_n^{(3)}$. The corresponding conductances of all parallel edges are also merged. Obviously, $G_n^{(3)}$ is a classical Apollonian network and three edges inside the triangle have conductance $\frac{3}{5}b_n + \frac{3}{5}b_n = \frac{6}{5}b_n$. Noting that $G_n^{(0)}$ is only an intermediate stage of the electrically equivalent simplification and it belongs to an internal graph of the network of generation $n+1$. Therefore, there should be **two** additional parallel edges on the boundary of each current network. That is to say, each edge of the border on the network has conductance $b_n + \frac{1}{5}b_n + \boldsymbol{\frac{1}{5}b_n} = \frac{7}{5}b_n$.

(4) *Wye–Delta transformation*: Replace the internal Y component by its electrically equivalent Δ networks (dashed green curves) with conductances $\frac{6}{5}b_n \cdot \frac{6}{5}b_n/(3 \cdot \frac{6}{5}b_n) = \frac{2}{5}b_n$.

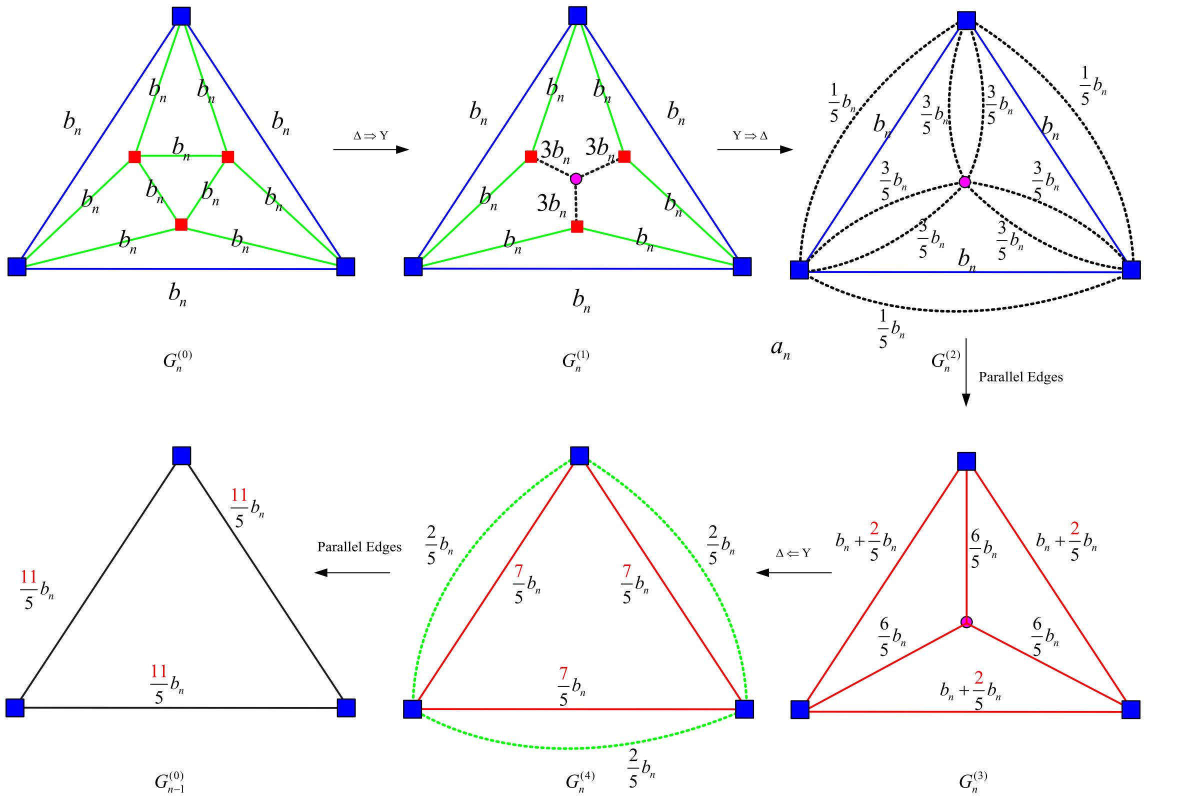

Fig. 6. The electrically equivalent evolution of the new network from $G_n^{(0)}$ to $G_{n-1}^{(0)}$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

(5) *Parallel edges*: Merge the corresponding parallel edges, similarly, we can obtain a simple triangle $G_{n-1}^{(0)}$ with conductance $\frac{7}{5}b_n + \frac{2}{5}b_n + \frac{2}{5}b_n = \frac{11}{5}b_n$ for each edge.

In addition, if one edge of the outermost triangle is involved, there is only one additional parallel edge, which is merged into the original edge, which indicates the conductances of the edge after merged in the third and final steps of the evolving process are $a_n + \frac{1}{5}b_n$ and $a_n + \frac{1}{5}b_n + \frac{2}{5}b_n = a_n + \frac{3}{5}b_n$, respectively.

Now, the values a_n, b_n, a_{n-1} and b_{n-1} satisfy the following two recurrences:

$$b_{n-1} = b_n + \frac{6}{5}b_n = \frac{11}{5}b_n, \tag{6}$$

$$a_{n-1} = a_n + \frac{3}{5}b_n. \tag{7}$$

We can solve these two recursive equations to obtain

$$b_1 = \frac{11}{5}b_2 = \left(\frac{11}{5}\right)^2 b_3 = \cdots = \left(\frac{11}{5}\right)^{n-1} b_n, \tag{8}$$

and

$$a_0 = a_n + \frac{3}{5}\left[1 + \frac{11}{5} + \cdots + \left(\frac{11}{5}\right)^{n-1}\right] b_n = a_n + \frac{1}{2}\left[\left(\frac{11}{5}\right)^n - 1\right] b_n. \tag{9}$$

Finally, we deduce the exact analytical expression on the number of spanning trees. Let $t_e(G_n)$ be the number of spanning trees of the extracted subnetwork. Noting that five applications of the aforementioned transformations suffice to determine the conversion formula of spanning trees of the network (see Fig. 6). Now we obtain the factors from the results of Eqs. (2)–(5) that $t_e(G_n)$ is given by

$$t_e(G_n) = t_e(G_n^{(0)}) = \frac{b_n^3}{(3b_n^2)^2} \cdot t_e(G_n^{(1)}) = \frac{1}{9b_n} \cdot t_e(G_n^{(1)}) \qquad (\Delta \Rightarrow \mathrm{Y}) \tag{10}$$

$$= \frac{1}{9b_n} \cdot (b_n + b_n + 3b_n)^3 \cdot t_e(G_n^{(2)}) = \frac{125}{9} b_n^2 \cdot t_e(G_n^{(2)}) \qquad (\mathrm{Y} \Rightarrow \Delta) \tag{11}$$

$$= \frac{125}{9} b_n^2 \cdot 1^6 \cdot t_e(G_n^{(3)}) \qquad \text{(Parallel Edges)} \tag{12}$$

$$= \frac{125}{9} b_n^2 \cdot (3 \cdot \frac{6}{5} b_n) \cdot t_e(G_n^{(4)}) = 5^2 \cdot 2 \cdot b_n^3 \cdot t_e(G_n^{(4)}) \qquad (\mathrm{Y} \Rightarrow \Delta) \tag{13}$$

$$= 5^2 \cdot 2 \cdot b_n^3 \cdot 1^3 \cdot t_e(G_n^{(5)}) \qquad \text{(Parallel Edges)} \tag{14}$$

$$= 5^2 \cdot 2 \cdot b_n^3 \cdot t_e(G_{n-1}^{(0)}). \tag{15}$$

To calculate the number of spanning trees of G_n, we analyze the number of the elementary triangles, which does not contain any node or edge inside the triangle. Denote the number of the elementary triangles by $F(G_n)$. For generation n, $F(G_n) = 7^n$. Using the result of $t_e(G_n)$ and Eq. (8), we obtain

$$\begin{aligned} t(G_n) &= [5^2 \cdot 2 \cdot b_n^3]^{7^{(n-1)}} \cdot t(G_{n-1}) \\ &= [5^2 \cdot 2 \cdot b_n^3]^{7^{(n-1)}} \cdot [5^2 \cdot 2 \cdot b_{n-1}^3]^{7^{(n-2)}} \cdot t(G_{n-2}) \\ &= 50^{7^{n-1}+7^{n-2}} b_n^{3 \cdot 7^{n-1}} \cdot b_{n-1}^{3 \cdot 7^{n-2}} \cdot t(G_{n-2}) \\ &= 50^{7^{n-1}+7^{n-2}+\cdots+7^1+7^0} \cdot [b_n^{3 \cdot 7^{n-1}} \cdot b_{n-1}^{3 \cdot 7^{n-2}} \cdot \cdots \cdot b_1^{3 \cdot 7^0}] \cdot t(G_0). \end{aligned} \tag{16}$$

Here initial value $t(G_0) = 3 \cdot a_0^2$.

Combining with Eq. (16), the above-mentioned initial value, and Eq. (9), we obtain

$$\begin{aligned} t(G_n) &= 50^{7^{n-1}+7^{n-2}+\cdots+7^1+7^0} \cdot [b_n^{3 \cdot 7^{n-1}} \cdot b_{n-1}^{3 \cdot 7^{n-2}} \cdot \cdots \cdot b_1^{3 \cdot 7^0}] \cdot t(G_0) \\ &= 50^{\frac{7^n-1}{6}} \cdot \left(\frac{11}{5}\right)^{\frac{7^n-6n-1}{12}} \cdot b_n^{\frac{7^n-1}{2}} \cdot 3\left[a_n + \frac{1}{2}\left[\left(\frac{11}{5}\right)^n - 1\right] b_n\right]^2. \end{aligned} \tag{17}$$

In particular, when $a_0 = 1, t(G_0) = 3$; when $a_1 = b_1 = 1, t(G_1) = 384$; when $a_2 = b_2 = 1, t(G_2) = 10\,639\,348\,500\,000\,000$, which are consistent with numerical values of $t(G_n)$ using "the matrix-tree theorem" [29] and exhibit an exponentially growth trend similar to other lattices [33].

3.3. Entropy of the contact graph of disk packings

Let $E(G_n)$ be the entropy of spanning trees for G_n [17,18]. It is an interesting quantity characterizing the network structure and is closely relevant to the number of spanning trees. With the same average degree of the nodes $\langle k \rangle$ for a network, the bigger the entropy value, the more the number of spanning trees compared with regular lattice.

From Eq. (17) ($a_n = b_n = 1$) and $V(G_n) = (7^n + 5)/2$, we obtain

$$E(G_n) = 2\left(\frac{1}{6}\ln 50 + \frac{1}{12}\ln\frac{11}{5}\right) \approx 1.4354. \tag{18}$$

It has been proved that the degree distribution of the new network satisfies a power-law decay with the exponent $\gamma = 1+\frac{\ln 7}{\ln 3}$ [15]. Comparing with other types of complex networks, the entropy of disk packing network is the highest among all two-dimensional networks (see Table 1), indicating its structural topology has stronger heterogeneous than regular lattice and Apollonian network. Table 1 also shows the comparison of the limiting average degrees of nodes for those networks. It implies that the entropies of three-dimensional network, such as 3-dimensional Sierpinski network and 3-dimensional hypercubic lattice, are bigger than the disk packing network with the same limiting average degree $\langle k \rangle = 6$, which means the entropy of high dimensional network is higher than that of low dimensional network.

Table 1
Comparison of limiting average degree and entropy of several networks.

Type of network	$\langle k \rangle$	$E(G)$
Pseudofractal web [31]	4	0.8959
Fractal scale-free lattice [38]	4	1.0397
Two-dimensional Sierpinski gasket [22]	4	1.0486
Square lattice [21]	4	1.1662
Apollonian network [32–34]	6	1.3540
Our studied network	6	**1.4354**
Three-dimensional Sierpinski graph [22]	6	1.5694
Three-hypercubic lattice [39]	6	1.6734

4. Conclusions

In summary, by means of the electrically equivalent technique to simplify the network structure, we have explicitly studied and enumerated the number of spanning trees on a variant of Apollonian networks. Depending on the formula for the conductance of edges and the conversion factor of the number of spanning trees from generation n to generation $n-1$, the exact analytical results are obtained. Our method on deriving the recursive expression for the number of spanning trees is not directly analysis of iterative structure of the new network, but the seeking of electrically equivalent networks to obtain the evolving formula between the current and the previous generation. So this method is instructive to solve the relevant spanning trees problem. In addition, the entropy of spanning trees on the new network is obtained and compared with the other two-dimensional and three-dimensional lattices, indicating that it has more number of spanning trees with the same average degree. Furthermore, our results show that its structural topology has stronger heterogeneous than regular lattice and Apollonian network.

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References

[1] M.E.J. Newman, SIAM Rev. 45 (2003) 167.
[2] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, D.U. Hwang, Phys. Rep. 424 (2006) 175.
[3] P. Zhang, B.S. Cheng, Z. Zhao, D.Q. Li, G.Q. Lu, Y.P. Wang, J.H. Xiao, Europhys. Lett. 103 (2013) 68005.
[4] F. Ouedraogo, C. Magnien, Comput. Comm. 34 (2011) 670.
[5] B. Mirzasoleiman, M. Babaei, M. Jalili, M. Safari, Phys. Rev. E 84 (2011) 046114.
[6] C.P. Warren, L.M. Sander, I.M. Sokolov, Phys. Rev. E 66 (2002) 056105.
[7] C.Y. Yin, B.H. Wang, W.X. Wang, G.R. Chen, Phys. Rev. E 77 (2008) 027102.
[8] J.S.J. Andrade, H.J. Herrmann, R.F.S. Andrade, L.R. da Silva, Phys. Rev. Lett. 94 (2005) 018702.
[9] D.J. Watts, S.H. Strogatz, Nature 393 (1998) 440.
[10] A.L. Barabási, R. Albert, Science 286 (1999) 509.
[11] Z.Z. Zhang, F. Comellas, G. Fertin, L.L. Rong, J. Phys. A 39 (2006) 1811.
[12] Z.Z. Zhang, L.C. Chen, S.G. Zhou, L.J. Fang, J.H. Guan, T. Zou, Phys. Rev. E 77 (2008) 017102.
[13] I.N. de Oliveira, F.A.B.F. de Moura, M.L. Lyra, J.S. Andrade, E.L. Albuquerque, Phys. Rev. E 79 (2009) 016104.
[14] Z.Z. Zhang, S. Zhou, Physica A 380 (2007) 621.
[15] Z.Z. Zhang, J.H. Guan, B.L. Ding, L.C. Chen, S.G. Zhou, New J. Phys. 11 (2009) 083007.
[16] A. Cayley, Quart. J. Pure Appl. Math. 23 (1889) 376.
[17] R. Lyons, Combin. Probab. Comput. 14 (2005) 491.
[18] R. Burton, R. Pemantle, Ann. Probab. 21 (1993) 1329.
[19] E. Teufl, S. Wagner, Linear Algebra Appl. 432 (2010) 441.
[20] G. Kirchhoff, Annu. Rev. Phys. Chem. 72 (1847) 497.
[21] F.Y. Wu, J. Phys. A 10 (1977) L113.
[22] S.C. Chang, L.C. Chen, W.S. Yang, J. Stat. Phys. 126 (2007) 649.
[23] F.T. Beosch, J. Graph Theory 3 (1986) 339.
[24] G.J. Szab, M. Alava, J. Kertsz, Physica A 330 (2003) 31.
[25] P.J. Macdonald, E. Almaas, A.L. Barabási, Europhys. Lett. 72 (2005) 308.
[26] P. Marchal, Electron. Comm. Probab. 5 (2000) 39.
[27] T. Nishikawa, A.E. Motter, Phys. Rev. E 73 (2006) 065106.
[28] D. Dhar, Physica A 369 (2006) 29.
[29] N.L. Biggs, Algebraic Graph Theory, second ed., Cambridge Univ. Press, Cambridge, 1993.
[30] S.N. Dorogovtsev, A.V. Goltsev, J.F.F. Mendes, Phys. Rev. E 65 (2002) 066122.
[31] Z.Z. Zhang, H.X. Liu, B. Wu, S.G. Zhou, Europhys. Lett. 90 (2010) 68002.
[32] Y. Lin, B. Wu, Z.Z. Zhang, G.R. Chen, J. Math. Phys. 52 (2011) 113303.
[33] J.Y. Zhang, W.G. Sun, G.H. Xu, J. Stat. Mech. 2013 (2013) 09015.
[34] Z.Z. Zhang, B. Wu, F. Comellas, Discrete Appl. Math. 169 (2014) 206.
[35] D.W. Boyd, J. Math. 25 (1973) 303.
[36] M. Borkovec, W. De Paris, R. Peikert, Fractals 2 (1994) 521.
[37] E. Teufl, S. Wagner, J. Stat. Phys. 142 (2011) 879.
[38] Z.Z. Zhang, H.X. Liu, B. Wu, T. Zou, Phys. Rev. E 83 (2011) 016116.
[39] J.H. Felker, R. Lyons, J. Phys. A 36 (2003) 8361.