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SIR Vaccination Model with Vital Dynamics

Abstract

Various types of vaccination methods have been used as a means to eradicate and/or control communicable diseases worldwide. The purpose of this model will be to analyze the long term behavior of a disease susceptible population and demonstrate the effects vaccinations have. For this model, I wanted to show the best vaccination strategy by comparing whether restricting vaccinations to early age children or allowing vaccinations to be available at any stage of the disease, would be the most effective in eradicating such diseases. This model will show the behavior differences when choosing to vaccinate a population, and highlight the behavior difference for a disease when choosing between strictly early age vaccination and any-age vaccination. The model shows that under necessary constraints and minimal vaccination presence it is possible to arrive at a disease free state through time.

Introduction

The World Health Organization states that vaccinations save around 2.5 million lives per year, and another 1.5 million children under five could be saved every year from preventable/treatable diseases using vaccinations [1]. This model will not be showcasing any one *specific* communicable disease, but any disease that is transferrable through contact with another infected being. The requirements that are encased in this model are as follows,

> In the absence of vaccinations, the disease is a modeled as a SIR-type disease. • Susceptible are infected by the Infected at a rate proportional to the number of susceptible and infected. • The infected group recover at a rate proportional to their population size. • The population size is constant so the number of births and deaths in the population are equal. • Deaths occur at the same rate in each group. (You can relax this assumption to

> better model diseases that increase the death rate for the infected) • There is no vertical transmission of the disease from mother to child at birth. • When there is no vaccinations, all birth are susceptible to the infection. • When a vaccination program is introduced, the births are divided between susceptible and recovered. The fraction of births that are placed in the recovered is the vaccination fraction. • Immunity is lost over time so the recovered become susceptible at a rate proportional to their population size.

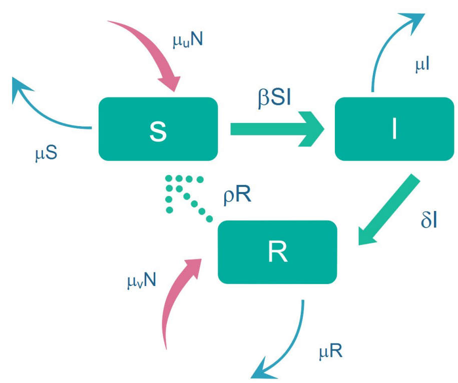
The purpose this model will serve will be to show the behavior of a disease burdened population and determine the most effect vaccination strategies that would help to eradicate the disease. In the remaining report I will show a visual representation of a disease with vaccination, I will use the Law of Mass Action to derive equations governing the disease, then, after finding equations to govern the "reaction", I will apply different techniques to analyze, find results and interpret them to answer the problem in question.

Model

When examining this model, I found there were multiple methods of vaccination. I broke the problem down into three scenarios. Case 1: all vaccinations that occur will do so at birth/early age. Case 2: vaccinations can occur at any point within the SIR model. Case 3: no vaccinations occur. I will ignore the latter case, since (as per requirements) without vaccinations, the problem would model a regular SIR-type model, which was already examined in the course work. So, the focus will be on Case 1 and Case 2.

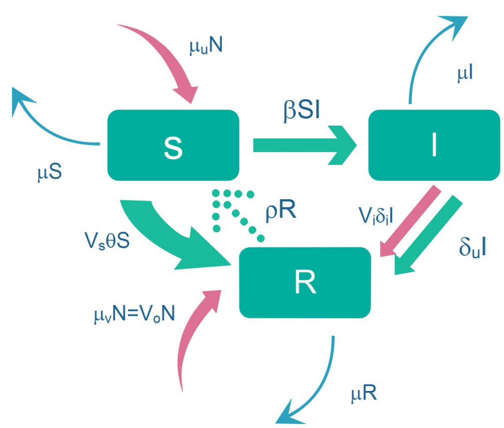
To build both models, I used a typical SIR type model with vital dynamics incorporating given requirements and made assumptions. For both cases, the assumption was made that the total population is split between three groups, susceptible (S), infected (I), and recovered (R). Some other assumptions made were, (1) the only way to become infected is contact with an infected individual, (2) the total population stays constant (birth rate=death rate), (3a) vaccinations occur only at birth/early age (CASE 1), (3b) vaccinations occur in any SIR group (CASE 2). Below (Figure 1 and Figure 2) are the diagrams I derived based on the requirements/assumptions. (Note: the reaction occurring between the recovered population and susceptible population is shown in a dotted arrow to signify that immunity may or may not be lost through time.)

CASE 1: Figure 1 - All Vaccinations at Birth



CASE 1:	symbol	definition
Variables	S I R N=S+I+R	Susceptible population (people) Infected population (people) Recovered population (people) Total population (people)
Parameters	β δ ρ μ_u μ_v μ	Rate of susceptible becoming ill Rate of recovery Rate of immunity loss Birth rate unvaccinated Birth rate vaccinated Birth rate = death rate

CASE 2: Figure 2 - Vaccinations at any age



CASE 2:	symbol	definition
Variables	S I R N=S+I+R	Susceptible population (people) Infected population (people) Recovered population (people) Total population (people)
Parameters	β δ_u δ_i Θ ρ μ_u $\mu_v = V_o$ μ V_s, V_i	Rate of susceptible becoming ill Rate of recovery unvaccinated Rate of recovery vaccinated Rate of susceptible vaccination Rate of immunity loss Birth rate unvaccinated Birth rate vaccinated= initial vaccination rate Birth rate = death rate Vaccination rate (susceptible, infected)

Using the Law of Mass action to transform Figure 1 and Figure 2 into equations governing the system we get,

CASE 1:

Equations: $dS/dt = -\beta SI + \rho R + \mu_u N - \mu S$

$dI/dt = \beta SI - \delta I - \mu I$

$dR/dt = \delta I - \rho R + \mu_v N - \mu R$

CASE 2:

Equations: $dS/dt = -\beta SI + \rho R + \mu_u N - \mu S - V_s \Theta S$

$dI/dt = \beta SI - (V_i \delta_i + \delta_u) I - \mu I$

$dR/dt = (V_i \delta_i + \delta u) I - \rho R + \mu_v N - \mu R + V_s \Theta S$

Furthermore, using mathematical methods such as conservation laws and non-dimensionalization, I was able to rewrite each system of three equations and turn them into two systems of two simplified, non-dimensional equations, leaving the resulting systems to be:

CASE 1:

a) $dx/d\tau = -\beta' xy + \rho'(1-x-y) + \mu' - x$

b) $dy/d\tau = \beta' xy - y(\delta + 1)$

CASE 2:

a) $dx/d\tau = -\beta' xy + \rho'(1-x-y) + \mu' - (1 + V_s \Theta')x$

b) $dy/d\tau = \beta' xy - (1 + V_i \delta_i' + \delta_u')y$

Analysis

After the building and simplification of the model are complete, the last steps in analyzing this reaction are finding the fixed points to each system and linearizing those fixed points using the Jacobian. Once the systems have been linearized, we will solve each Jacobian (one Jacobian matrix per fixed point, resulting in four matrices) for its trace and determinant to understand the behavior of the model through time.

CASE 1: Fixed Points: (x*,y*)=

$$\left(\frac{\mu+\rho}{\rho+1}, 0\right), \quad \left(\frac{\delta'+1}{\beta'}, \frac{-\rho'\delta'+\rho'\beta'-\rho'+\mu'\beta'-\delta'-1}{(\delta'+\rho'+1)\beta'}\right)$$

Jacobian =

$$\begin{pmatrix} -\beta' y - \rho' - 1 & -\beta' x - \rho' \\ \beta' y & \beta' x - \delta' - 1 \end{pmatrix}$$

CASE 2: Fixed Points: (x*,y*)=

$$\left(\frac{\mu'+\rho'}{\rho'+V s\Theta'+1} , 0\right) , \left(\frac{1+V i\delta i'+\delta u'}{\beta'} , \frac{\rho'\beta'+\mu'\beta'-\rho'(V i\delta i'+\delta u')-\rho'-1-(V i\delta i'+\delta u')-V s\theta'(V i\delta i'+\delta u')}{(1+V i\delta i'+\delta u'+\rho')\beta'} \right)$$

Jacobian=

$$\begin{pmatrix} -\beta' y - \rho' - V s\Theta' - 1 & -\beta' x - \rho' \\ \beta' y & \beta' x - V i\delta i' - \delta u' - 1 \end{pmatrix}$$

After solving each Jacobian case for its trace and determinant;

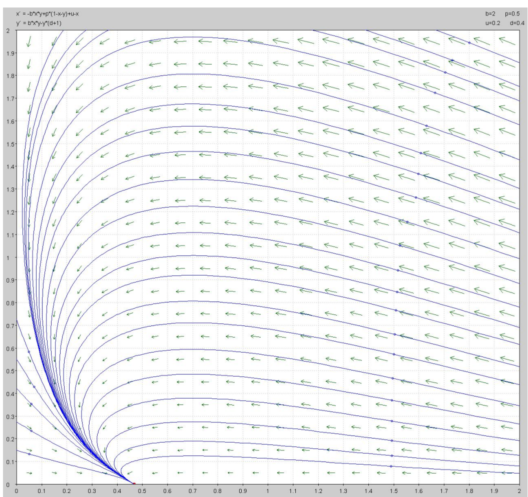
For Case 1, the disease free equilibrium (where y*=0) is stable when $\beta' < \frac{1+\rho'+\delta'+\rho'\delta'}{\mu'+\rho'}$ and for those same constraints it results in the endemic situation (where y>0, meaning there exists a group of infected individuals) being unstable through time. Meaning, if those constraints are met, through time, there will be zero sick people within the population.

For Case 2, the disease free equilibrium (y*=0) is stable when $\beta' < (V s\Theta' + V i\delta i' + \delta u' + \rho' + 1)(\frac{\rho'+V s\theta'+1}{\mu'+\rho'})$ and for those constraints when evaluating the endemic fixed point it too is unstable (as with case 1). The overall conclusion being, when we can control the transmission rate under the given constraints and we assume that the ratio of unvaccinated birth rate to the total birth rate is strictly less than one (meaning there exists at least one birth that is vaccinated), as time goes on, the disease will settle into a state where there is zero sick individuals.

Results

To briefly summarize the conclusion, with the presence of vaccinations (even very minimal presence) the communicable disease, through time, will end up at a state with the infected population equaling zero for all time. The figures below are a few diagrams to visually show what the disease would look like through time, under different initial conditions;

CASE 1: Figure 3



CASE 2: Figure 4

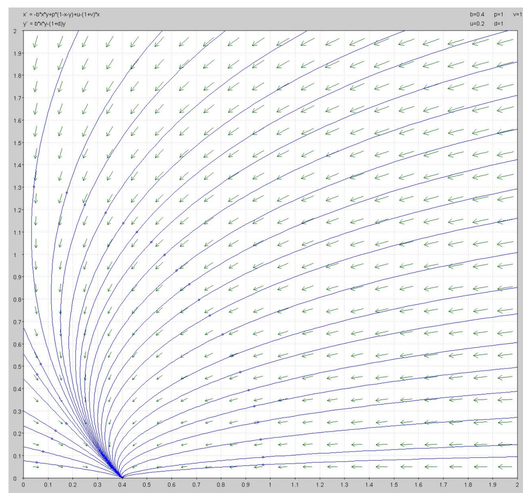


Figure 3 is showing the behavior of the disease when all vaccinations occur at birth, under the aforementioned constraints. On the x-axis the susceptible population is shown, and on the y-axis the infected population. As the susceptible population decreases, the infected population increases, until enough time passes, and the infection wears off, leaving both the susceptible and infected populations decreasing (in turn the recovered population is increasing) until the disease free equilibrium is met, resulting in zero sick people, a few susceptibles and mostly recovered. It clearly shows that under any initials conditions, as time goes on, the disease will be eradicated, leaving the infected population equal to zero.

Figure 4 is representing a disease where all vaccinations can occur at any stage in a person's life. This graph shows the susceptible population on the x-axis and the infected population on the y-axis. Because the susceptible population is decreasing due to transmission of the disease and later-in-life vaccinations, the susceptible population decreases and in turn the infected population also decreases (under certain initial conditions). They both decrease to the stable disease free equilibrium. Again, resulting in an infected population equal to zero. Although both cases are able to achieve a disease free equilibrium, as shown by Figure 4, having vaccinations at any stage of the disease's life is the most effective strategy. Having any-stage vaccinations will result in a decrease of the population that is susceptible to the infection, resulting in a decrease in the total amount of people that get infected. If ever was there a situation where there were very limited supply of vaccinations, based on Figure 4, giving those vaccines to those susceptible will ultimately decrease the amount of individuals that become infected. Thus, not only does this case achieve a disease free solution, but it also will have fewer individuals that are infected overall through the course of the disease.

After analyzing both cases, and achieving a stable disease free situation, I wanted to further analyze what would happen if the necessary constraints were not met. Because both cases are very similar to one another, I will only be further analyzing case 1. For the first "what-if" scenario instead of the original constraints, I set $\beta' > \frac{1+\rho'+\delta'+\rho'\delta'}{\mu'+\rho'}$, and kept all other parameters the same. For both Figure 5 and Figure 6 the susceptible population is modeled on the x-axis and the infected population will be modeled on the y-axis. Figure 5 shows the resulting graph; in this case, there is a stable endemic solution, which means that if

$\beta' > \frac{1+\rho'+\delta'+\rho'\delta'}{\mu'+\rho'}$ as time goes on, the disease will settle into a solution where there will always be a group of infected individuals present.

Figure 6 shows what the behavior of the disease would look like if all births were unvaccinated ($\mu' = 1$) . All parameters have remained the same as the previous disease free solution (Figure 3), with the only adjustment being no vaccinations. Without the presence of vaccination, there would be a stable point with a constant group of infected individuals through time. Again showing the importance of the original constraints and vaccinations.

Figure 5:

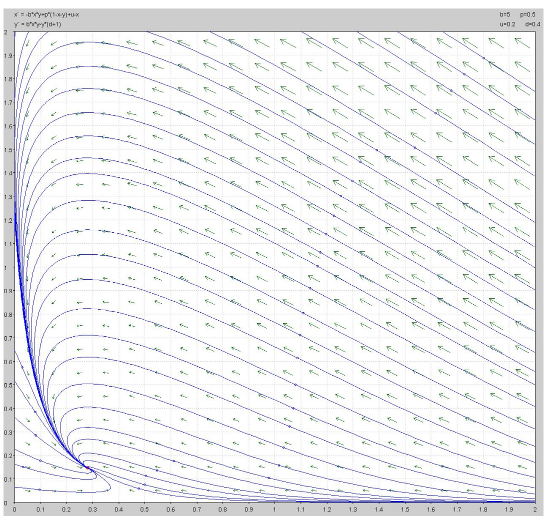
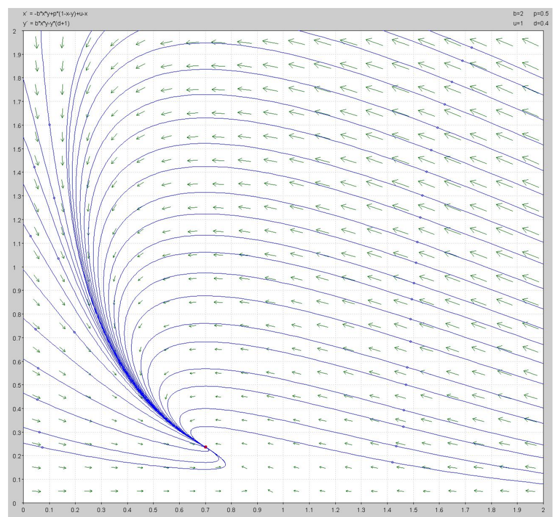


Figure 6:



Conclusion

Within a susceptible population, with the introduction of communicable disease into the environment, a disease free solution is achievable under given conditions. The rate of transmission (β') must be restricted in some form such that the constraints are met above. Restricting the transmission rate can be done in several ways; whether that be through quarantine, or mass media coverage to warn individuals not to be in close proximity to outbreak areas. The second important condition is there must be vaccinations present in the population. As shown in the figures above, with even a minute amount of vaccinations, the disease free solution is met. This model ultimately shows the importance of vaccinations in a disease burdened community, and with easily achievable constraints, in a short amount of time, the disease will be eradicated and the population will be burden free.

References

[1] WHO. (2011). *Global Immunization Data.* Retrieved April 2017. http://www.who.int/mediacentre/factsheets/fs178/en/index.html