MA 261 Exam 1 Solutions

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Problem 1.1. A line l passes through the points $(-1, 1, 2)$ and is perpendicular to the plane $x - 2y + 2z = 8$. At what point does the line intersect the yz-plane?

Solution. For the line to be perpendicular to the plane $x - 2y + 2z = 8$ its direction vector $\mathbf{v}$ must be $\langle 1, -2, 2\rangle$ (or a multiple of it). Therefore, the line has the form $l(t) = \langle 1, -2, 2\rangle t + (a, b, c)$. We are told that the line passes through the point $(-1, 1, 2)$ so an equation for the line l is

$$l(t) = (t - 1, -2t + 1, 2t + 2).$$

Last, but not least, we need to find the time t when l intersects the yz plane. This happens when $x = 0$, i.e., when $t - 1 = 0$. Therefore, $t = 1$ and the point of intersection must be

$$l(1) = \underline{(0, -1, 4)}.$$

$\diamond$

Problem 1.2. Find the equation of the plane that passes through the point $(1, -1, 2)$ and is perpendicular to both the planes $2x+y-2z =$ and $x+3z = 10$.

Solution. Recall from class that it is enough to find the normal to a vector. That is,

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2$$

where $\mathbf{n}_1 = \langle 2, 1, -2\rangle$ and $\mathbf{n}_2 = \langle 1, 0, 3\rangle$. Therefore,

$$\begin{aligned}\mathbf{v} &= \mathbf{n}_1 \times \mathbf{n}_2 \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -2 \\ 1 & 0 & 3 \end{vmatrix} \\ &= \langle 3, -8, -1\rangle\end{aligned}$$

Now the plane should have the form $3x - 8y - z = C$. Since the plane passes through the point $(1, -1, 2)$, the plane must satisfy $3(1) - 8(-1) - (2) = C$ so $C = 9$ and the equation of the plane must be

$$\underline{3x - 8y - z = 9}.$$

◇

Problem 1.3. Find a vector function that represents the curve of intersection of the cylinder $y^2 + z^2 = 1$ and the plane $x + y + 2z = 3$.

Solution. Assuming the intersection is a curve (i.e., 1-dimensional) we can parametrize the coordinates x, y, and z in terms of a fourth one, say, t. That is, $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$. Now, since $\mathbf{r}$ parametrizes the curve of intersection, its coordinates must satisfy

$$\begin{aligned}y(t)^2 + z(t)^2 &= 1, \\ x(t) + y(t) + 2z(t) &= 3,\end{aligned}$$

so $y(t) = \cos t$, $z(t) = \sin t$ and $x(t) = 3 - \cos t - 2\sin t$. So the desired parametrization is

$$\underline{\mathbf{r}(t) = \langle 3 - \cos t - 2\sin t, \cos t, \sin t\rangle}$$

for $0 \le t \le 2\pi$. ◇

Problem 1.4. Let $\mathbf{r}(t) = \langle t, t^2/2, t^3/3\rangle$, find $\kappa(1)$ (namely, the curvature at $t = 1$).

Solution. Recall that the curvature of a curve $\mathbf{r}$ is defined to be

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}, \tag{1.1}$$

where $\mathbf{T}(t) = \mathbf{r}'(t)/|\mathbf{r}'(t)|$ and is called the unit tangent vector.

To get started, we need to find $\mathbf{r}'(t)$ and $\mathbf{T}'(t)$. These are straightforward calculations, as we now see:

$$\begin{aligned}
\mathbf{r}'(t) &= \langle 1, t, t^2 \rangle, \\
|\mathbf{r}'(t)| &= \sqrt{1+t^2+t^4}, \\
\mathbf{T}(t) &= \langle (1+t^2+t^4)^{-1/2}, t(1+t^2+t^4)^{-1/2}, t^2(1+t^2+t^4)^{-1/2} \rangle, \\
\mathbf{T}'(t) &= \Big\langle -(t+2t^3)(1+t^2+t^4)^{-3/2}, \\
&\qquad -(t+2t^3)(1+t^2+t^4)^{-3/2} + (1+t^2+t^4)^{-1/2}, \\
&\qquad -t^2(t+2t^3)(1+t^2+t^4)^{-3/2} + 2t(1+t^2+t^4)^{-1/2} \Big\rangle
\end{aligned}$$

Therefore,

$$\begin{aligned}
|\mathbf{r}'(1)| &= \sqrt{3}, \\
|\mathbf{T}'(1)| &= \sqrt{2/3},
\end{aligned}$$

so

$$\kappa(1) = \sqrt{2}/3.$$

$\diamond$

Problem 1.5. A particle travels with position vector $\mathbf{r}(t) = \langle 3t, 4\sin t, 4\cos t \rangle$, $t \geq 0$. Find $\alpha \geq 0$ such that during the interval of the time from 0 to α the particle has traveled a distance 20.

Solution. This is an arclength problem in disguise. We need to find $\alpha \geq 0$ such that

$$s(\alpha) = \int_0^\alpha |\mathbf{r}'(t)|\,dt = 20.$$

That is, first we find $\mathbf{r}'$ which is

$$\mathbf{r}'(t) = \langle 3, 4\cos t, -4\sin t \rangle,$$

so

$$\begin{aligned} s(\alpha) &= \int_0^\alpha |9 + 16\cos^2 t + 16\sin^2 t|\,dt \\ &= \int_0^\alpha 5\,dt \\ &= 5\alpha \\ &= 20. \end{aligned}$$

Therefore $\underline{\alpha = 4}$. $\diamond$

Problem 1.6. A particle has acceleration $\mathbf{a} = \langle 6t-2, -1/t^2, 0\rangle$. It is known that the velocity at the time $t = 1$ is $\mathbf{v}(1) = \langle 1, 1, 1\rangle$ and that the position vector at time $t = 1$ is $\mathbf{r}(1) = \langle 0, 0, 3\rangle$. Find the magnitude of the position vector at time $t = 2$.

Solution. This is an initial value problem (IVP). We are trying to find $|\mathbf{r}(2)|$; for that we need to find the equation for $\mathbf{r}(t)$.

First we integrate $\mathbf{a}$ to find $\mathbf{v}$:

$$\mathbf{v}(t) = \langle 3t^2 - 2t, 1/t, 0\rangle + \langle v_1, v_2, v_3\rangle.$$

Using the initial condition, i.e., $\mathbf{v}(1) = \langle 1, 1, 1\rangle$, we see that $v_1 = v_2 = 0$ and $v_3 = 1$ so

$$\mathbf{v}(t) = \langle 3t^2 - 2t, 1/t, 1\rangle.$$

Next we integrate $\mathbf{v}$ to get $\mathbf{r}$:

$$\mathbf{r}(t) = \langle t^3 - t^2, \ln t, t\rangle + \langle r_1, r_2, r_3\rangle.$$

Again, using the initial condition, we see that $r_1 = r_2 = 0$ and $r_3 = 2$. Therefore,

$$\mathbf{r}(t) = \langle t^3 - t^2, \ln t, t + 2\rangle.$$

Lastly, $\mathbf{r}(2) = \langle 4, \ln 2, 4\rangle$ so $|\mathbf{r}(2)| = \underline{\sqrt{32 + (\ln 2)^2}}$. $\diamond$

Problem 1.7. The level curves of $f(x, y) = \sqrt{x^2 + 1} - 2y$ are

Solution. Fix a real number k and let

$$k = \sqrt{x^2 + 1} - 2y. \tag{1.2}$$

Then, after some algebraic manipulations on Equation (1.2), we get

$$(k + 2y)^2 - x^2 = 1.$$

This is the equation of a hyperbola (whose asymptotes have been shifted from their usual position at the origin). ◇

Problem 1.8. If $f(x, y, z) = xz/\sqrt{y^2 - z}$, then $f_{xyz}(1, 2, 3)$ is equal to

Solution. This problem is straight forward; we will find the partial derivatives in steps:

$$\begin{aligned} f_x(x, y, z) &= \frac{z}{\sqrt{y^2 - z}}, \\ f_{xy}(x, y, z) &= -\frac{yz}{(y^2 - z)^{3/2}}, \\ f_{xyz}(x, y, z) &= -\frac{y(y^2 - z)^{3/2} + \frac{3}{2}yz(y^2 - z)^{1/2}}{(y^2 - z)^3}. \end{aligned}$$

Therefore,

$$f_{xyz}(1, 2, 3) = \underline{-11}.$$

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Problem 1.9. Let $z = e^r \cos\theta$, $r = 12st$, $\theta = \sqrt{s^2 + t^2}$. The partial derivative $\partial z/\partial s$ is

Solution. For this problem we require the use of the Chain Rule. By the Chain Rule,

$$\begin{aligned} \partial z/\partial s &= e^r(\partial r/\partial s)\cos\theta - e^r \sin\theta(\partial\theta/\partial s), \\ &= e^r[\partial r/\partial s \cos\theta - \partial\theta/\partial s \sin\theta], \end{aligned}$$

where

$$\partial r/\partial s = 12t, \quad \partial\theta/\partial s = s/\sqrt{s^2 + t^2}.$$

Thus,

$$\partial z/\partial s = \underline{e^{12st}\left(12t\cos\left(\sqrt{s^2 + t^2}\right) - \frac{s\sin\left(\sqrt{s^2 + t^2}\right)}{\sqrt{s^2 + t^2}}\right)}.$$

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Problem 1.10. The direction in which $f(x, y) = x^2y + e^{xy}\sin y + 15$ increases most rapidly at $(1, 0)$ is

(*Note*: Give your answer in the form of a unit vector.)

Solution. Recall that the direction in which a function increases the most rapidly is along its gradient. Therefore, we must find $\nabla f(x, y)$ and, especially, the unit direction vector $\mathbf{u}$ of f at $(1, 0)$, i.e., $\mathbf{u} = \nabla f(1, 0)/|\nabla f(1, 0)|$. First,

$$\nabla f(x, y) = \langle 2xy + ye^{xy} \sin y, x^2 + xe^{xy} \sin y + e^{xy} \cos y\rangle.$$

Therefore,

$$\nabla f(1, 0) = \langle 0, 2\rangle$$

so $\mathbf{u} = \langle 0, 1\rangle$. $\diamond$

Problem 1.11. The equation of the tangent plane to the graph of the function $f(x, y) = x - y^2/2$ at $(2, 4, -6)$ is:

Solution. First we need to find the gradient of the function, which is

$$\nabla f(x, y) = \langle 1, -y\rangle.$$

Therefore, $\nabla f(2, 4) = \langle 1, -4\rangle$ so the equation for the tangent plane is

$$z + 6 = (x - 2) - 4(y - 4)$$

so the equation for the plane is $x - 4y - z = -8$ or (as is in the answer choices) $-x + 4y + z = 8$. $\diamond$

Problem 1.12. The function $f(x, y) = 6x^2 + 3y^2 - 16$ attains its local minimum at:

Solution. To find the local minimum of this function we first need to find its critical points. These happen when $\nabla f(x, y) = \langle 0, 0\rangle$ and can easily be solved for:

$$\begin{aligned} \nabla f(x, y) &= \langle 12x, 6y\rangle, \\ 12x &= 0, \\ 6y &= 0. \end{aligned}$$

So f has the unique critical point $(0, 0)$. Now we can check, by the Second Derivative Test, whether this is a minimum or not

$$f_{xx}(x, y) = 12, \qquad f_{xy}(x, y) = f_{yx}(x, y) = 0, \qquad f_{yy}(x, y) = 6,$$

so $D = 72 > 0$ and $f_{xx}(0, 0) = 12 > 0$, so this is indeed the local minimum. $\diamond$