MA 261 Quiz 6
February 27, 2018

If you do not know how to do any one of these problems, circle **"(E) I don't know"** as your answer choice. You will receive **two points** for doing that. **Each problem** is worth **five points**. You get **two points** for writing your **full name** and **three points** for writing your **section number**.

Problem 6.1. Find the maximum and minimum of $f(x, y, z) = xyz$ subject to the constraint $x + y + z = 1$ for $x, y, z \geq 0$.
(*Hint*: Use Lagrange multipliers.)
(A) maximum $1/27$, minimum 0
(B) maximum 0, minimum 0
(C) maximum $1/27$, minimum $-1/27$
(D) maximum $1/3$, minimum 0
(E) I don't know how to do this

Solution. Let $g(x, y, z) = x + y + z$. By the method of Lagrange multipliers,

$$\nabla f(x, y, z) = \langle yz, xz, xy \rangle = \langle \lambda, \lambda, \lambda \rangle = \lambda \nabla g(x, y, z).$$

Therefore, we have the following system of equations

$$\begin{aligned} yz &= \lambda, \\ xz &= \lambda, \\ xy &= \lambda. \end{aligned}$$

From here, it is not difficult to see that we must have the following sequence of equalities: $x = y = z$.

Feeding this back into the constraint, $g(x, y, z) = 3x = 1$ we see that $x = 1/3$ so $y = z = 1/3$ and the maximum value must be $1/27$, subject to these constraints.

The minimum is trickier to come up with. Note that $x, y, z \geq 0$ so, in particular, they can equal 0. Let $x = 0$. Then, since $x + y + z = 1$, $y + z = 1$ so let $y = z = 1/2$ (or whatever else you can think of that adds up to 1). At any rate, the product $xyz = 0$ for any choice of y and z since $x = 0$. Therefore, the minimum of f subject to the constraint g is 0.

Answer: (A) $\diamond$

Problem 6.2. Find the absolute maximum of $f(x, y) = 2x^2 - y^2 + 6y$ on $x^2 + y^2 \leq 16$.

(A) 8
(B) 9
(C) 35
(D) 40
(E) I don't know how to do this

Solution. To find the absolute maximum of this function, subject to the constraints, we systematically apply the conclusion to the Extreme Value Theorem (EVT). Recall that the EVT says that a continuous function takes its extremal values at critical points within its domain or on the boundary of said domain.

In this case, $f(x, y) = 2x^2 - y^2 + 6y$ has $x^2 + y^2 \leq 16$ as its domain so we need to check the points at which $\nabla f(x, y) = \langle 0, 0 \rangle$ or at points in the boundary, i.e., (x, y) with $x^2 + y^2 = 16$.

Let's parametrize in terms of y to avoid leaving a square root dangling. Then

$$\tilde{f}(y) = f\left(\pm\sqrt{16 - y^2}, y\right) = 2(16 - y^2) - y^2 + 6y = 32 - 3y^2 + 6y.$$

Thus,

$$\tilde{f}'(y) = -6y + 6 = 0$$

for $y = 1$. It is easy to check, by the second derivative test (or simply observing that $\tilde{f}'(y)$ is positive to the left of 1 and negative to the right) It follows that $\tilde{f}(1) = f(\pm\sqrt{15}, 1) = 35$ is a max. Moreover, it is the largest value we have encountered, so it must be the global max in this domain.
Answer: (C). ◇

Problem 6.3. Find the value of the iterated integral

$$\iint_R 2 - x \, dA, \quad R = \{(x, y) \colon 0 \leq x \leq 2, 0 \leq y \leq 3\}.$$

(A) 6
(B) 8
(C) 10
(D) 12
(E) I don't know how to do this

Solution. A straightforward calculation yields the answer:

$$\begin{aligned}
\iint_R 2 - x \, dA &= \int_0^2 \int_0^3 2 - x \, dy dx \\
&= 3 \int_0^2 2 - x \, dx \\
&= 3(2x - x^2/2)\Big|_0^2 \\
&= 3(4 - 2^2/2) \\
&= 6.
\end{aligned}$$

Answer: (A) ◇