Bott Periodicity for the Unitary Group

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Abstract

We will present a condensed proof of the Bott Periodicity Theorem for the unitary group U following John Milnor's classic *Morse Theory*. There are many documents on the internet which already purport to do this (and do so very well in my estimation), but I nevertheless will attempt to give a summary of the result.

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1 The Basics

The original proof of the Periodicity Theorem relies on a deep result of Marston Morse's **calculus of variations**, the **(Morse) Index Theorem**. The proof of this theorem, however, goes beyond the scope of this document, the reader is welcome to read the relevant section from Milnor or indeed Morse's own paper titled *The Index Theorem in the Calculus of Variations*.

Perhaps the first thing we should set about doing is introducing the main character of our story; this will be the unitary group. The **unitary group** of degree n (here denoted $\mathrm{U}(n)$) is the set of all **unitary matrices**; that is, the set of all $A \in \mathrm{GL}(n, \mathbb{C})$ such that $AA^* = I$ where A^* is the conjugate of the transpose of A (conjugate transpose for short).

This set, $\mathrm{U}(n)$, it can be shown, is a **smooth manifold** with **(real) dimension** n^2 (viewed as a subset of $\mathbb{C}^{n^2}$, $\mathrm{U}(n)$ is the preimage of the identity I under the map $U \mapsto UU^*$ and, under the **operator norm**, $\|U\| = 1$ for every $U \in \mathrm{U}(n)$ which, by the Heine–Borel Theorem, implies that $\mathrm{U}(n)$ is compact). Moreover, $\mathrm{U}(n)$ is in fact a **Lie group**; that is, a smooth manifold and a group such that the group laws multiplication and inversion are **smooth** (infinitely differentiable, and hence so-called **diffeomorphisms** of itself).

The **Lie algebra** of $\mathrm{U}(n)$, here denoted, $\mathfrak{u}(n)$ is the set of all **skew-Hermitian matrices**; that is, all matrices $A \in M(n, \mathbb{C})$ such that $A + A^* = \mathbf{0}$ (notice that A need no longer be invertible).

It is well established, and we shall not have the time to deal with it here, that any compact Lie group can be equipped with a **bi-invariant (Riemannian) metric**. It is standard, although not at all trivial, that a the so-called **geodesics** of Lie group equipped with a bi-invariant metric correspond to **one-parameter subgroups** $\phi(t) = \exp(tA)$ where A is taken from the Lie algebra of our Lie group. The bi-invariant metric which we shall equip $\mathrm{U}(m)$ with is the standard

$$\langle A, B \rangle := \mathrm{Re}(\mathrm{tr}(AB^*)) \text{ for } A, B \in \mathfrak{u}(m). \tag{1}$$

It can be shown, with little difficulty, that this map really is **positive definite** and hence, describes a Riemannian metric on $\mathrm{U}(m)$

The **exponential map**, denoted exp in the last paragraph, is quite a deal different from the exponential function you known and love. In our particular case, it is enough to say that the exponential map is defined (formally) by its Taylor series expansion as follows

$$\exp(A) = I + A + \tfrac{1}{2}A^2 + \tfrac{1}{6}A^3 + \cdots . \tag{2}$$

2 Fiber Bundles

In elementary topology, just like in abstract algebra, there is a notion of sameness which serves to tell fundamentally different spaces apart. A circle is visibly different from an open interval, but how can we make this rigorous? This is where the notion of **homeomorphism** comes in. A homeomorphism is a continuous map h between topological spaces, say, X and Y with a continuous inverse. After the *correct* formalism has been established, we develop properties (*invariants*) of spaces which are preserved under homeomorphisms so that we may determine when, say, X and Y are distinct as (topological) spaces; for example, X might be compact and Y not. A very powerful invariant are the so-called **connected components** of a space. It is an exercise in elementary topology that the connected components of a manifold coincide with the **path-components** of said manifold. Therefore, it makes sense to ask: Given a manifold M, what are the path-components $\pi_0(M)$ of M? Given a point $p \in M$, let $\pi_0(M, p)$ denote the set of path-components of M with the path-component of p singled out. Also corresponding to said point, let $\Omega_p(M)$ denote the space of maps from the unit circle $S^1 := \{|z| = 1\} \subset \mathbb{C}$ sending 1 to p, made into a topological space with the so-called **compact-open topology**. The path-components of this **loop space** $\Omega_p(M)$ we shall call $\pi_1(M, p)$ (this is precisely the **fundamental group** of M) the set $\pi_0(\Omega_p(M), \bar{p})$ where $\bar{p}$ is the constant map sending S^1 to p. We may similarly define the higher **homotopy groups** recursively as folows

$$\pi_{k+1}(M, p) := \pi_k(\Omega_p M, \bar{p}).$$

A **fiber bundle**, which we shall denote in the following style

$$F \longrightarrow E \longrightarrow B,$$

(in the style of Wikipedia) is a continuous surjection p from the *total space* E to the *base space* B such that every point $b \in B$ has a neighborhood U such that $p^{-1}(U)$ is **homeomorphic** $U \times F$. That is, a fiber bundles is *locally* a product space. A typical example of this where the bundle is *nontrivial* is the **Möbius strip**.

A basic result from the theory of fiber bundles is that if the base space B is path connected, such a fiber bundle gives rise to a long **exact sequence** of homotopy groups

$$\cdots \longrightarrow \pi_n(F) \longrightarrow \pi_n(E) \longrightarrow \pi_n(B) \longrightarrow \pi_{n-1}(F) \longrightarrow \cdots. \tag{3}$$

(For a proof see the following Wikipedia page: **Long exact sequence of a fibration**.)

For the proof of the theorem, it will be necessary to look at the **stable unitary group** U which is the **direct limit** of the finite-dimensional unitary groups $\mathrm{U}(n)$ under the inclusion $\mathrm{U}(n) \hookrightarrow \mathrm{U}(m)$ given by the mapping $A \mapsto \left(\begin{smallmatrix} \mathbf{0} & \mathbf{1} \\ A & \mathbf{0} \end{smallmatrix}\right)$; that is,

$$\mathrm{U} := \operatorname{dir}\lim_n \mathrm{U}(n) := \bigsqcup_n \mathrm{U}(n)/\sim,$$

where for $A_j \in \mathrm{U}(j), A_k \in \mathrm{U}(k)$, $A_j \sim A_k$ if there is some $l \in \mathbb{N}$ such that $\iota_{jl}(A_j) = \iota_{kl}(A_k)$ (the ι are compositions of inclusions). You can think of this space as the irredundant union of all the $\mathrm{U}(n)$.

2.1 First fiber bundle

The inclusion $i\colon\ \mathrm{U}(n) \to \mathrm{U}(n+1)$ given by $A \mapsto \left(\begin{smallmatrix} 1 & 0 \\ \mathbf{0} & A \end{smallmatrix}\right)$ and the map $j\colon\ \mathrm{U}(n+1) \to S^{2n+1} \subset \mathbb{C}^{n+1}$ which takes a matrix to its first column, gives a fiber bundle

$$\cdots \longrightarrow \pi_{i+1}S^{2n+1} \longrightarrow \pi_i(\mathrm{U}(n)) \longrightarrow \pi_i(\mathrm{U}(n+1)) \longrightarrow \pi_i(S^{2n+1}) \longrightarrow \cdots$$

which, by Equation (3) gives

$$\pi_i(\mathrm{U}(n)) \cong \pi_i(\mathrm{U}(n+1)) \cong \pi_i(\mathrm{U}(n+2)) \cong \cdots \tag{4}$$

since $\pi_i(S^{2n+1}) = 0$ for $2n+1 > i$.

2.2 Second Fiber Bundle

We define the **(complex) Stiefel manifold**, which we shall denote $V(m, \mathbb{C}^n)$, as the space of m-tuples of orthonormal vectors in $\mathbb{C}^n$ which called frames.

Then $\mathrm{U}(n)$ acts **transitively** on $V(m, \mathbb{C}^n)$ by matrix multiplication (any m frame can be taken to any other m frame by an appropriate choice of unitary matrix). Moreover, an m-tuple of $\mathbb{C}^n$ is fixed by an element of $\mathrm{U}(n)$ which acts non-trivially only on the complementary $n-m$ frame; that is, the action has a **stabilizer** isomorphic to $\mathrm{U}(n-m)$. Hence we have a fiber bundle $\mathrm{U}(n-m) \to \mathrm{U}(n) \to V(m, \mathbb{C}^n)$ which, by Equation (3) yields

$$\pi_i(V(m, \mathbb{C}^n)) = 0 \text{ for } i < 2(m-n). \tag{5}$$

2.3 Third Fiber Bundle

We define the **(complex) Grassmannian**, denoted $\mathrm{Gr}(m, \mathbb{C}^n)$ as the collection of m dimensional subspaces of $\mathbb{C}^n$ where $m \leq n$.

Consider the map from $V(m, \mathbb{C}^n)$ to $\mathrm{Gr}(m, \mathbb{C}^n)$ which sends an m-frame to the m-dimensional subspace spanned by it. The fibers of this map is the collection of m-frames spanning the same m-dimensional subspace, which, can be shown, is isomorphic to $\mathrm{U}(m)$. Hence, we have a fiber bundle $\mathrm{U}(m) \to V(m, \mathbb{C}^n) \to \mathrm{Gr}(m, \mathbb{C}^n)$. By Equation (3), this yields

$$\pi_{i-1}(\mathrm{U}(m)) \cong \pi_i(\mathrm{Gr}(m, \mathbb{C}^n)) \text{ for } i < 2(n-m). \tag{6}$$

2.4 Fourth Fiber Bundle

The last bundle we are going to need comes from the **special unitary group** $\mathrm{SU}(m)$ which is a subgroup of $\mathrm{U}(m)$ consisting of matrices with determinant 1.

The determinant map $\det\colon \mathrm{U}(m) \to S^1 \subset \mathbb{C}$ gives a fiber bundle $\mathrm{SU}(m) \to \mathrm{U}(m) \to S^1$ which by Equation (3) gives

$$\pi_i(\mathrm{SU}(m)) \cong \pi_i(\mathrm{U}(m)) \text{ for } i > 1$$

and hence

$$\pi_i(\mathrm{SU}) \cong \pi_i(\mathrm{U}) \text{ for } i > 1. \tag{7}$$

3 Proof of the Periodicity Theorem

We have done most of the set up, now it is time to state our theorem

Theorem 1 (Bott's Periodicity Theorem for U)**.** *For sufficiently large n, $\pi_{i-1}(\mathrm{U}(n)) \cong \pi_{i+1}(\mathrm{U}(2n))$ for $i \geq 1$. In other words,*

$$\pi_{i-1}(\mathrm{U}) \cong \pi_{i+1}(\mathrm{U}).$$

Everything will now magically come together. Recall from Equation (6) and Equation (7) that

$$\pi_{i-1}(\mathrm{U}(m)) \cong \pi_i(\mathrm{Gr}(m, \mathbb{C}^n)) \text{ and } \pi_{i+1}(\mathrm{SU}(2m)) \cong \pi_{i+1}(\mathrm{U}(2m)).$$

All we need to do to prove Theorem 1 is show that

$$\pi_i(\mathrm{Gr}(m, \mathbb{C}^{2m})) \cong \pi_{i+1}(\mathrm{SU}(2m)).$$

This is where the voodoo magic from calculus of variations will play a big role.

Rather than prove this equality directly, we can reduce to the *space of minimal geodesics* in $\mathrm{SU}(2m)$ (this is a reduced version of the **path-space** more commonly called loop space) from I to $-I$. The customary notation for this space is $\Omega(\mathrm{SU}(2m); I, -I)$, but we shall simply denoted by $\Omega(I, -I)$.

We will argue, in two steps, that

$$\pi_i(\mathrm{Gr}(m, \mathbb{C}^{2m})) \cong \pi_i(\Omega(I, -I))$$

and

$$\pi_i(\Omega(I, -I)) \cong \pi_{i+1}(\mathrm{SU}(2m)).$$

3.1 The first equivalence

We will prove something much stronger in fact,

Lemma 2. *The spaces* $\mathrm{Gr}(m, \mathbb{C}^{2m})$ *and* $\Omega(I, -I)$ *are homeomorphic.*

Proof. We more or less follow Milnor's discussion. The Lie algebra $\mathfrak{su}(2m)$ of $\mathrm{SU}(2m)$ is comprised of $2m$-by-$2m$ matrices A such that $A + A^* = 0$ and $\mathrm{tr}(A) = 0$.

As we have already discussed, the exponential map exp takes vectors in the Lie algebra to elements of the Lie group, and moreover geodesics on the Lie group correspond to one-parameter subgroups, i.e., they are the maps $\gamma(t) = \exp(tA)$ for $A \in \mathfrak{su}(2m)$.

We are interested in finding *minimal* geodesics from I to $-I$, i.e., γ starting at I such that $\gamma(1) = -I$ with γ having the shortest length possible (with respect to the metric given by Equation (1)). From linear algebra, we know that for any $A \in \mathfrak{su}(2m)$ we can find a unitary matrix T such that TAT^{-1} is diagonal and for A such that $\exp(A) = -I$, since $\exp(TAT^{-1}) = T\exp(A)T^{-1} = T(-I)T^{-1} = -I$, it suffices to consider diagonal matrices.

Since any diagonal $A \in \mathfrak{su}(2m)$ is a skew-symmetric Hermitian matrix, it must have entries with real part equal to 0. Since $\exp(A) = -I$ we must also have that each entry on the diagonal, say $\mathrm{i}a_i$ must be of the form $\mathrm{i}k_i\pi$ for k_i odd and $\sum_{i=1}^{2m} k_i = 0$.

Therefore, from (1), we deduce that any such A determines a geodesic of length $\pi\sqrt{k_1^2 + \cdots + k_{2m}^2}$. To minimize this, we must have $k_i = \pm 1$ for $i = 1, \ldots, 2m$ and at least half of these must be positive and half negative.

We conclude that the minimal geodesics from I to $-I$ in $\mathrm{SU}(2m)$ are of the form $\exp(tA)$ where A is a diagonal matrices with half of the entries equal to $i\pi$ and the other half $-i\pi$. Such a matrix is uniquely determined by the eigenspace for just one of these, say, $i\pi$ which is an m-dimensional subspace of $\mathbb{C}^{2m}$.

The homeomorphism is the map which sends such a geodesic to its corresponding eigenspace in $\mathbb{C}^{2m}$. This gives a well-defined map to $\mathrm{Gr}(m, \mathbb{C}^{2m})$. $\square$

3.2 The second equality

The previous equality followed very quickly from simple observation. The next equality requires some very deep results, in particular, we will use a corollary of Morse's Index Theorem, which states that

Theorem 3. *For a Riemannian manifold M, if the space of minimal geodesics from p to q is a topological manifold, denoted $\Omega(p,q)$, and if every non-minimal geodesic from p to q has index $\geq \lambda_0$, then $\pi_i(\Omega(p,q)) \cong \pi_{i+1}(M)$ for $i = 0, \ldots, \lambda_0 - 2$.*

The *index* referenced here is the so-called **Morse index** and is defined as follows. We say that two points p and q along a curve γ are *conjugate points* if there is a non-zero **Jacobi field** J that vanishes at p and q. The multiplicity of the conjugate points is the dimension of the vector space of all such J.

The Jacobi field J is a solution to the second-order ODE

$$\tfrac{D^2}{dt}J + R(\tfrac{d}{dt}\gamma, J)\tfrac{d}{dt}\gamma = 0,$$

where D/dt denotes the **covariant derivative** and R the **curvature tensor**.

To prove that $\pi_i(\Omega(I, -I) \cong \pi_{i+1}(\mathrm{SU}(2m))$, all we need to show is that every non-minimal geodesic from I to $-I$ has index greater than or equal to $2m+2$, so that, for sufficiently large m, $\pi_i(\Omega(I,-i)) \cong \pi_{i+1}(\mathrm{SU}(2m))$ concluding the proof of Bott's Periodicity Theorem for U. This however is very laborious and it makes it a good place for us to stop.

4 The Homotopy Groups of U

We finish off by listing the homotopy groups of U as obtained using Bott Periodicity. From Theorem (1), $\pi_i(\mathrm{U})$ is completely determined by $\pi_0(\mathrm{U})$ and $\pi_1(\mathrm{U})$. From Equation (4),

$$\pi_0(\mathrm{U}) \cong \pi_0(\mathrm{U}(1)) \text{ and } \pi_1(\mathrm{U}) \cong \pi_1(\mathrm{U}(1)).$$

Since $\mathrm{U}(1) \approx S^1$, $\pi_0(\mathrm{U}(1)) = 0$ (since S^1 is path-connected) and $\pi_1(\mathrm{U}(1)) = \mathbb{Z}$ (from elementary theory of covering spaces). Therefore,

$$\pi_n(\mathrm{U}) \cong \begin{cases} 0 & \text{for } n \text{ even}, \\ \mathbb{Z} & \text{for } n \text{ odd}. \end{cases}$$