12. CVIKO — $\mathcal{Z}$-transformace

$f(n): \mathbb{N}_0 \to \mathbb{R}$ je reálná posloupnost. Potom

$$\boxed{F(z) = \mathcal{Z}\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n} = f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \dots + \frac{f(n)}{z^n} + \dots}$$

je $\mathcal{Z}$-transformace posloupnosti $\{f(n)\}_{n=0}^{\infty}$.

$F(z): \mathbb{C} \to \mathbb{C}$ je komplexní fce a $\exists\, R \in \mathbb{R}$ tak, že:

$F(z)$ existuje a je holomorfní pro všechna $|z| > R$.

$\mathcal{Z}$-transformace je lineární: $\boxed{\mathcal{Z}\{a\, f(n) + b \cdot g(n)\} = a \cdot \mathcal{Z}\{f(n)\} + b \cdot \mathcal{Z}\{g(n)\}}$ $(a, b \in \mathbb{R})$

Č.	Předmět, vzor: $f(n)$, $n = 0, 1, 2, \dots$	Obraz: $\mathcal{Z}\{f(n)\} = F(z) = \sum_{n=0}^{\infty} f(n)\, z^{-n}$
1.	$c \quad (c \in \mathbb{R})$	$\frac{cz}{z-1}$
2.	$a^n \quad (a \in \mathbb{R})$	$\frac{z}{z-a}$
3.	n	$\frac{z}{(z-1)^2}$
4.	n^2	$\frac{z(z+1)}{(z-1)^3}$
5.	$n \cdot a^n$	$\frac{az}{(z-a)^2}$
6.	$n^2 a^n$	$\frac{az(z+a)}{(z-a)^3}$
7.	$f(n+1)$	$z \cdot F(z) - z \cdot f(0)$
8.	$f(n+2)$	$z^2 \cdot F(z) - z^2 f(0) - z \cdot f(1)$
9.	$f(n+3)$	$z^3 F(z) - z^3 f(0) - z^2 f(1) - z f(2)$
10.	$f(n+k)$	$z^k F(z) - z^k f(0) - z^{k-1} f(1) - \dots - z f(k-1)$
11.	$f(n-k)$ $\left(\begin{smallmatrix} f(n)=0 \\ n<k \end{smallmatrix}\right)$	$z^{-k} F(z)$

ZÁKLADNÍ SLOVNÍK $\mathcal{Z}$-TRANSFORM.

(Př - Já) Odvoďte vzorec $\boxed{1.\ \mathcal{Z}\{c\}}$ a $\boxed{2.\ \mathcal{Z}\{a^n\},\ a>0}$.

1. $$F(z) = \mathcal{Z}\{c\} = \sum_{n=0}^{\infty} c\cdot z^{-n} = \sum_{n=0}^{\infty} \frac{c}{z^n} = c + \frac{c}{z} + \dots + \frac{c}{z^n} + \dots$$

$$= \frac{c}{1-\frac{1}{z}} = \frac{c}{\frac{z-1}{z}} = \boxed{\frac{cz}{z-1}} \qquad \left(\left|\frac{1}{z}\right| < 1 \Rightarrow |z| > 1\ \forall z\right)$$

2. $$F(z) = \mathcal{Z}\{a^n\} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n = 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \dots =$$

$$= \frac{1}{1-\frac{a}{z}} = \frac{1}{\frac{z-a}{z}} = \boxed{\frac{z}{z-a}} \qquad \left(\left|\frac{a}{z}\right| < 1 \Rightarrow |z| > a\ \forall z\right)$$

Zpětná $\mathcal{Z}$-transformace Nechť $F(z)$ je holomorfní až na póly $z = z_k$ $(k = 1, 2, \dots, m)$ a konečná (tj. $\lim_{z\to\infty} F(z) = c$).
Tj. $F(z)$ je ryze lomená fce nebo podíl polynomů stejného stup.

Potom $$\boxed{\mathcal{Z}^{-1}\{F(z)\} = f(n) = \sum_{k=1}^{m} \operatorname*{res}_{z=z_k} \left[F(z)\cdot z^{n-1}\right];\ n = 0, 1, 2, \dots}$$
(někdy je nutné pro $\boxed{n=0}$ dělat zvlášť).

(Př. Já) $F(z) = \frac{1}{(z-2)^2}$; $f(n) = ?$ $\qquad \boxed{F(z)\cdot z^{n-1} = \frac{z^{n-1}}{(z-2)^2}}$

$\boxed{n=0}$ $\quad F(z)\cdot z^{-1} = \frac{1}{z(z-2)^2}$ Póly $\boxed{z=0}$ 1. řádu, $\boxed{z=2}$ 2. řádu

$$\operatorname*{res}_{z=0} F(z)\cdot z^{-1} = \lim_{z\to 0}\left\{\cancel{z}\cdot\frac{1}{\cancel{z}(z-2)^2}\right\} = \boxed{\frac{1}{4}}$$

$$\operatorname*{res}_{z=2} F(z)\cdot z^{-1} = \frac{1}{1!}\lim_{z\to 2}\left\{\left[\cancel{(z-2)^2}\cdot\frac{1}{z\cancel{(z-2)^2}}\right]'\right\} = \lim_{z\to 2}\left\{-\frac{1}{z^2}\right\} = \boxed{-\frac{1}{4}}$$

$\boxed{f(0) = \frac{1}{4} - \frac{1}{4} = 0}$

$\boxed{n = 1, 2, \dots}$ $\quad F(z)\cdot z^{n-1} = \frac{z^{n-1}}{(z-2)^2}$ Pól $\boxed{z=2}$ 2. řádu

$$\operatorname*{res}_{z=2} F(z)\cdot z^{n-1} = \frac{1}{1!}\lim_{z\to 2}\left\{\left[\cancel{(z-2)^2}\cdot\frac{z^{n-1}}{\cancel{(z-2)^2}}\right]'\right\} = \lim_{z\to 2}\left\{(n-1)z^{n-2}\right\} = \boxed{(n-1)2^{n-2}}$$

$$\boxed{f(n) = \begin{cases} 0 & \text{pro } n=0 \\ (n-1)\cdot 2^{n-2} & \text{pro } n = 1, 2, \dots \end{cases}}$$ nebo $\boxed{f(n) = \{0, 0, 1, 4, 12, 32, 80, \dots\}}$

Př $F(z) = \dfrac{1}{(z+2)(z+1)}$; $f(n) = ?$ $\boxed{F(z)\cdot z^{n-1} = \dfrac{z^{n-1}}{(z+2)(z+1)}}$

$\boxed{n=0}$ $F(z)\cdot z^{-1} = \dfrac{1}{z(z+2)(z+1)}$ Póly $\left.\begin{matrix}\boxed{z=0}\\ \boxed{z=-2}\\ \boxed{z=-1}\end{matrix}\right\}$ 1. řádu

$$\operatorname*{res}_{z=0} F(z)\cdot z^{-1} = \lim_{z\to 0}\left\{\cancel{z}\,\frac{1}{\cancel{z}(z+2)(z+1)}\right\} = \frac{1}{2}$$

$$\operatorname*{res}_{z=-2} F(z)\cdot z^{-1} = \lim_{z\to -2}\left\{\cancel{(z+2)}\cdot\frac{1}{z\cancel{(z+2)}(z+1)}\right\} = \frac{1}{2}$$

$$\operatorname*{res}_{z=-1} F(z)\cdot z^{-1} = \lim_{z\to -1}\left\{\cancel{(z+1)}\cdot\frac{1}{z(z+2)\cancel{(z+1)}}\right\} = -1$$

$f(0) = \frac{1}{2}+\frac{1}{2}-1$

$\boxed{f(0)=0}$

$n = 1, 2, \dots$ $F(z)\cdot z^{n-1} = \dfrac{z^{n-1}}{(z+2)(z+1)}$ Póly $\left.\begin{matrix}\boxed{z=-2}\\ \boxed{z=-1}\end{matrix}\right\}$ 1. řádu

$$\operatorname*{res}_{z=-2} F(z)\cdot z^{n-1} = \lim_{z\to -2}\left\{\cancel{(z+2)}\cdot\frac{z^{n-1}}{\cancel{(z+2)}(z+1)}\right\} = \frac{(-2)^{n-1}}{-1}$$

$$\operatorname*{res}_{z=-1} F(z)\cdot z^{n-1} = \lim_{z\to -1}\left\{\cancel{(z+1)}\cdot\frac{z^{n-1}}{(z+2)\cancel{(z+1)}}\right\} = (-1)^{n-1}$$

$$\boxed{f(n) = (-1)^{n-1} - (-2)^{n-1} \quad \text{pro } n = 1, 2, \dots}$$

Celkem $\boxed{f(n) = \{\underset{0}{0}, \underset{1}{0}, \underset{2}{1}, \underset{3}{-3}, \underset{4}{7}, \underset{5}{-15}, \underset{6}{31}, \underset{7}{-63}, \dots\}}$

Řešení diferenční rovnice k-tého řádu

$$\boxed{y(n+k) + a_1 y(n+k-1) + \dots + a_k y(n) = f(n) \; ; \; a_1, \dots, a_k \in \mathbb{R}}$$

Postup: analogie Laplaceovy transformace. Převedeme $y(n+k), \dots, y(n)$ na $\mathbf{Y(z)}$ a $f(n)$ na $F(z)$ a vyjádříme neznámou $Y(z)$. Poté provedeme zpětnou $\mathcal{Z}$-transformaci a dostaneme hledané řešení, tj. $\boxed{\text{posloupnost } y(n)}$

(Př. 3á) $y(n+1) - y(n) = 2^n (n-1); \quad y(0) = 0$

$$\underset{(7.)}{y(n+1)} - y(n) = \underset{(5.)}{2^n \cdot n} - \underset{(2.)}{2^n} \qquad / \mathcal{Z}$$

$$zY(z) - \cancel{z \cdot 0} - Y(z) = \frac{2z}{(z-2)^2} - \frac{z}{z-2}$$

$$Y(z)(z-1) = \frac{2z - z(z-2)}{(z-2)^2} \qquad /:(z-1)$$

$$Y(z) = \frac{2z - z^2 + 2z}{(z-1)(z-2)^2} = \frac{4z - z^2}{(z-1)(z-2)^2}$$

$$\boxed{Y(z) = \frac{z(4-z)}{(z-1)(z-2)^2}}$$

$$Y(z) \cdot z^{n-1} = \frac{z^n (4-z)}{(z-1)(z-2)^2}$$

Póly: $\boxed{z=1}$ 1. řádu, $\boxed{z=2}$ 2. řádu (pro $n = 0, 1, \dots$)

$$\operatorname*{res}_{z=1} Y(z) \cdot z^{n-1} = \lim_{z \to 1} \left\{ \cancel{(z-1)} \cdot \frac{z^n (4-z)}{\cancel{(z-1)}(z-2)^2} \right\} = \boxed{3}$$

$$\operatorname*{res}_{z=2} Y(z) \cdot z^{n-1} = \frac{1}{1!} \lim_{z \to 2} \left\{ \left[\cancel{(z-2)^2} \cdot \frac{z^n (4-z)}{(z-1)\cancel{(z-2)^2}} \right]' \right\} =$$

$$= \lim_{z \to 2} \left\{ \frac{[n \cdot z^{n-1} (4-z) + z^n (-1)] \cdot (z-1) - z^n (4-z) \cdot 1}{(z-1)^2} \right\}$$

$$= \frac{2n \cdot 2^{n-1} - 2^n - 2 \cdot 2^n}{1} =$$

$$= n 2^n - 2^n - 2 \cdot 2^n = 2^n (n-1-2) = \boxed{(n-3) 2^n}$$

$$\boxed{y(n) = 3 + (n-3) 2^n \qquad n = 0, 1, 2, \dots}$$ nebo

$$\boxed{y(n) = \{0, -1, -1, 3, 19, 67, 195, \dots\}}$$

$\boxed{\text{Př} \quad y(n+2) - 5y(n+1) + 6y(n) = 1; \quad y(0)=0,\ y(1)=0}$

$$z^2 Y(z) - z^2 \cdot 0 - z \cdot 0 - 5\left(zY(z) - z \cdot 0\right) + 6Y(z) = \frac{z}{z-1}$$

$$Y(z)\left(z^2 - 5z + 6\right) = \frac{z}{z-1} \qquad /:\left(z^2-5z+6\right)$$

$$Y(z) = \frac{z}{(z-1)(z^2-5z+6)} = \boxed{\frac{z}{(z-1)(z-2)(z-3)}}$$

$$\boxed{Y(z) \cdot z^{n-1} = \frac{z^n}{(z-1)(z-2)(z-3)}}$$

<u>Póly</u>: $\left.\begin{array}{l}\boxed{z=1}\\ \boxed{z=2}\\ \boxed{z=3}\end{array}\right\}$ 1. řádu $(n=0,1,2,\dots)$

$$\operatorname*{res}_{z=1} Y(z) \cdot z^{n-1} = \lim_{z \to 1} \left\{ \cancel{(z-1)} \cdot \frac{z^n}{\cancel{(z-1)}(z-2)(z-3)} \right\} = \boxed{\frac{1}{2}}$$

$$\operatorname*{res}_{z=2} Y(z) \cdot z^{n-1} = \lim_{z \to 2} \left\{ \cancel{(z-2)} \cdot \frac{z^n}{(z-1)\cancel{(z-2)}(z-3)} \right\} = \boxed{\frac{2^n}{-1}}$$

$$\operatorname*{res}_{z=3} Y(z) \cdot z^{n-1} = \lim_{z \to 3} \left\{ \cancel{(z-3)} \cdot \frac{z^n}{(z-1)(z-2)\cancel{(z-3)}} \right\} = \boxed{\frac{3^n}{2}}$$

$$\boxed{y(n) = \frac{1}{2} - 2^n + \frac{1}{2} \cdot 3^n \qquad n=0,1,2,\dots}$$

Př $\quad y(n+2) - 9y(n) = 0; \quad y(0) = 0,\ y(1) = 1$

$$z^2 Y(z) - z^2 \cdot 0 - z \cdot 1 - 9Y(z) = 0$$

$$Y(z) \cdot (z^2 - 9) = z$$

$$\boxed{Y(z) = \frac{z}{z^2-9}} \quad \Longrightarrow \quad \boxed{Y(z) \cdot z^{n-1} = \frac{z^n}{(z-3)(z+3)}}$$

<u>Póly</u>: $\left.\begin{array}{l}\boxed{z=3}\\ \boxed{z=-3}\end{array}\right\}$ 1. řádu

$$\operatorname*{res}_{z=3} Y(z) \cdot z^{n-1} = \lim_{z \to 3} \left\{ \cancel{(z-3)} \cdot \frac{z^n}{\cancel{(z-3)}(z+3)} \right\} = \boxed{\frac{3^n}{6}}$$

$$\operatorname*{res}_{z=-3} Y(z) \cdot z^{n-1} = \lim_{z \to -3} \left\{ \cancel{(z+3)} \cdot \frac{z^n}{(z-3)\cancel{(z+3)}} \right\} = \boxed{\frac{(-3)^n}{-6}}$$

$$\boxed{y(n) = \frac{3^n}{6} - \frac{(-3)^n}{6} \qquad n=0,1,2,\dots}$$

$\textcircled{Př}$ $y(n+2) - 5y(n+1) + 4y(n) = 2^n; \quad y(0)=0, \ y(1)=0$

$$z^2 Y(z) - z^2 \cdot 0 - z \cdot 0 - 5\left(zY(z) - z \cdot 0\right) + 4Y(z) = \frac{z}{z-2}$$

$$Y(z) \cdot (z^2 - 5z + 4) = \frac{z}{z-2}$$

$$Y(z) = \frac{z}{(z-2)(z^2-5z+4)} = \boxed{\frac{z}{(z-2)(z-4)(z-1)}}$$

$$\boxed{Y(z) \cdot z^{n-1} = \frac{z^n}{(z-2)(z-4)(z-1)}}$$

Póly: $\boxed{z=2}$, $\boxed{z=4}$, $\boxed{z=1}$ } póly 1. řádu pro $n = 0, 1, 2, \dots$

$$\operatorname*{res}_{z=2} Y(z) \cdot z^{n-1} = \lim_{z \to 2} \left\{ \cancel{(z-2)} \cdot \frac{z^n}{\cancel{(z-2)}(z-4)(z-1)} \right\} = \boxed{\frac{2^n}{-2}}$$

$$\operatorname*{res}_{z=4} Y(z) \cdot z^{n-1} = \lim_{z \to 4} \left\{ \cancel{(z-4)} \cdot \frac{z^n}{(z-2)\cancel{(z-4)}(z-1)} \right\} = \boxed{\frac{4^n}{6}}$$

$$\operatorname*{res}_{z=1} Y(z) \cdot z^{n-1} = \lim_{z \to 1} \left\{ \cancel{(z-1)} \cdot \frac{z^n}{(z-2)(z-4)\cancel{(z-1)}} \right\} = \boxed{\frac{1}{3}}$$

$$\boxed{\underset{n=0,1,2,\dots}{y(n)} = \frac{1}{3} + \frac{4^n}{6} - \frac{2^n}{2}}$$

$\textcircled{Př DÚ}$ $y(n+1) - 3y(n) = n(-1)^n; \quad y(0) = 1$

$$zY(z) - z - 3Y(z) = -\frac{z}{(z+1)^2}$$

$$Y(z) \cdot (z-3) = z - \frac{z}{(z+1)^2} = \frac{z(z^2+2z+1) - z}{(z+1)^2} = \frac{z^3 + 2z^2 + \cancel{z} - \cancel{z}}{(z+1)^2}$$

$$\boxed{Y(z) = \frac{z^2(z+2)}{(z-3)(z+1)^2}} \Rightarrow \boxed{Y(z) \cdot z^{n-1} = \frac{z^{n+1}(z+2)}{(z-3)(z+1)^2}}$$

Póly: $\boxed{z=3}$ 1ř. $\boxed{z=-1}$ 2ř.

$$\operatorname*{res}_{z=3} Y(z) \cdot z^{n-1} = \lim_{z \to 3} \left\{ \cancel{(z-3)} \frac{z^{n+1}(z+2)}{\cancel{(z-3)}(z+1)^2} \right\} = \boxed{\frac{5 \cdot 3^{n+1}}{16}}$$

$$\operatorname*{res}_{z=-1} Y(z) \cdot z^{n-1} = \frac{1}{1!} \lim_{z \to -1} \left\{ \left[\cancel{(z+1)^2} \cdot \frac{z^{n+1} \cdot (z+2)}{(z-3)\cancel{(z+1)^2}} \right]' \right\} = \lim_{z \to -1} \left\{ \frac{\left[(n+1) z^n \cdot (z+2) + z^{n+1}\right] \cdot (z-3) - z^{n+1} \cdot (z+2) \cdot 1}{(z-3)^2} \right\}$$

$$= \frac{-4\left[(n+1)(-1)^n + (-1)^{n+1}\right] - (-1)^{n+1}}{16} = \frac{(-1)^{n+1} \cdot (4n+4-4-1)}{16} = \boxed{\frac{(4n-1) \cdot (-1)^{n+1}}{16}}$$

$$\boxed{y(n) = \frac{1}{16} \cdot \left[5 \cdot 3^{n+1} + (4n-1) \cdot (-1)^{n+1}\right] \quad n = 0, 1, 2, \dots}$$