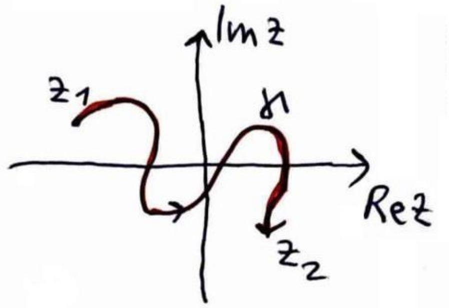
7. CVIKO Integrál v $\mathbb{C}$ (parametrizací křivky)

Křivka v $\mathbb{C}$:

$\gamma : \langle \alpha, \beta \rangle \to \mathbb{C} \quad (\alpha, \beta \in \mathbb{R})$

$\gamma : z(s) = x(s) + j\,y(s)$



$s \in \langle \alpha, \beta \rangle$

$z_1 = z(\alpha) = x_1 + j\,y_1$

$z_2 = z(\beta) = x_2 + j\,y_2$

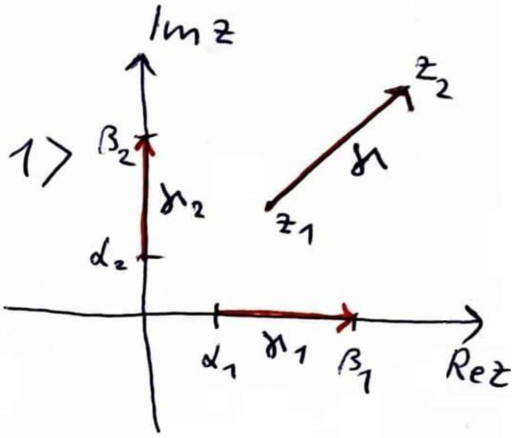
známé křivky

- Úsečka z bodu z_1 do bodu z_2

$\gamma : z(s) = z_1 + (z_2 - z_1)\,s \; ; \; s \in \langle 0, 1 \rangle$

$\left\{ \begin{array}{l} \gamma_1 : z(s) = s ; \; s \in \langle \alpha_1, \beta_1 \rangle \\ \gamma_2 : z(s) = js ; \; s \in \langle \alpha_2, \beta_2 \rangle \end{array} \right.$

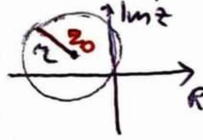
→ Speciální úsečky na osách



- Kružnice o středu z_0 a poloměru r $\Leftrightarrow$ analyticky $\boxed{|z - z_0| = r}$

$\gamma : z(s) = z_0 + r \cdot e^{js}$

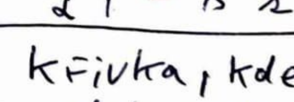
$z(s) = z_0 + r \cdot (\cos s + j \sin s) \; ; \; s \in \langle 0, 2\pi)$



(Oblouk $s \in \langle \alpha, \beta \rangle$) úhly

- Funkce, jejíž reálná složka je s a imaginární $f(s)$

$\gamma : z(s) = s + j \cdot f(s) \; ; \; s \in \langle \alpha, \beta \rangle$



$$\boxed{\int_\gamma f(z)\,dz = \int_\alpha^\beta f(z(s)) \cdot z'(s)\,ds}$$

$\gamma : z(s)$ je křivka, kde $s \in \langle \alpha, \beta \rangle$

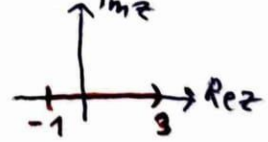
Výsledek je číslo z $\mathbb{C}$

geometricky „něco" jako plocha pod křivkou

→ Používáme v obecných případech, když $f(z)$ není holomorfní na oblasti, ale je (nejlépe všude) spojitá, určitě v bodech křivky tj. $f(z)$ obsahuje $|z|$, $\operatorname{Re} z$, $\operatorname{Im} z$, $\bar{z}$, $\arg z$. Pokud Ne, počítáme jinak!

Příklad $\int_\gamma \bar{z}\,dz$; γ je úsečka z bodu -1 do bodu 3

$$= \int_{-1}^{3} \bar{s} \cdot 1\,ds = \int_{-1}^{3} s\,ds = \left[\frac{s^2}{2}\right]_{-1}^{3} = \frac{9}{2} - \frac{1}{2} = \underline{\underline{4}}$$



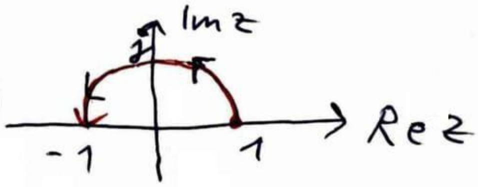
$\gamma : z(s) = s$

$z'(s) = 1$

$s \in \langle -1, 3 \rangle$

(1. Př) $\int\limits_{\gamma} |z|\, dz$; γ: půlkružnice se středem v 0 a oblouk z bodu 1 do bodu −1

$\gamma: z(s) = e^{js}$; $s \in \langle 0, \pi \rangle$
$\quad z'(s) = je^{js}$



$$= \int_0^{\pi} |e^{js}| \cdot je^{js}\, ds = \int_0^{\pi} \sqrt{\cos^2 s + \sin^2 s}\; j\, e^{js}\, ds =$$

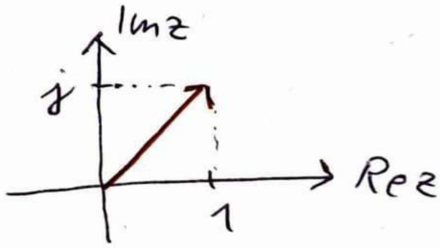
$$= \int_0^{\pi} 1 \cdot je^{js}\, ds = \left[\cancel{j} \frac{e^{js}}{\cancel{j}} \right]_0^{\pi} = e^{j\pi} - 1 = \cos\pi + j\sin\pi - 1 =$$

$$= -1 + 0 - 1 = \underline{\underline{-2}}$$

(vzorec: $|e^{js}| = |\cos s + j\sin s| = \sqrt{\cos^2 s + \sin^2 s} = 1$)

(2 Př) $\int\limits_{\gamma} \operatorname{Im} z\, dz$; γ: úsečka z bodu 0 do bodu $1+j$

$\gamma: z(s) = 0 + (1+j)s$; $s \in \langle 0, 1 \rangle$
$\quad z'(s) = 1+j$

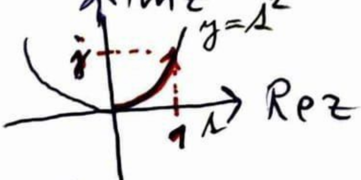


$$= \int_0^1 \operatorname{Im}\{\overbrace{(1+j)s}^{s+js}\} \cdot (1+j)\, ds = \int_0^1 s \cdot (1+j)\, ds =$$

$$= \left[(1+j)\frac{s^2}{2} \right]_0^1 = \frac{1+j}{2} = \underline{\underline{\frac{1}{2} + \frac{1}{2}j}}$$

DÚ $\int\limits_{\gamma} \arg z\, dz$; $\gamma: z(s) = e^{js}$, $s \in \langle 0, \pi \rangle$
$[-\pi - 2j]$ (P.P.)

(3 Př) $\int\limits_{\gamma} \operatorname{Re} z\, dz$; γ: Oblouk paraboly z bodu 0 do bodu $1+j$

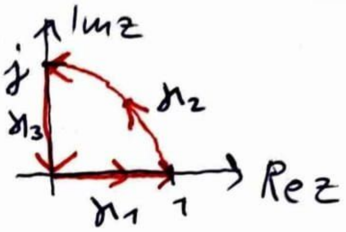


$\gamma: z(s) = s + js^2$; $s \in \langle 0, 1 \rangle$
$\quad z'(s) = 1 + 2js$

$$= \int_0^1 \operatorname{Re}(s + js^2)(1 + 2js)\, ds = \int_0^1 s(1 + 2js)\, ds = \int_0^1 (s + 2js^2)\, ds$$

$$= \left[\frac{s^2}{2} + j\,\frac{2}{3}s^3 \right]_0^1 = \underline{\underline{\frac{1}{2} + \frac{2}{3}j}}$$

(4. Př) $\int_{\gamma} |z|\cdot\bar{z}\, dz$;



$\boxed{\gamma=\gamma_1+\gamma_2+\gamma_3}$

$\gamma_1: z_1(s)=s$, $z_1'(s)=1$, $s\in\langle 0,1\rangle$

$\gamma_2: z_2(s)=e^{js}$, $z_2'(s)=j e^{js}$, $s\in\langle 0,\frac{\pi}{2}\rangle$

$\gamma_3: z_3(s)=js$, $z_3'(s)=j$, $s\in\langle 1,0\rangle$!!

$$=\int_0^1 |s|\cdot\bar{s}\cdot 1\, ds+\int_0^{\frac{\pi}{2}} |e^{js}|\cdot\overline{e^{js}}\cdot j e^{js}\, ds+\int_1^0 |js|\cdot\overline{js}\cdot j\, ds$$

$$=\int_0^1 s\cdot s\, ds+\int_0^{\frac{\pi}{2}} 1\cdot\cancel{e^{-js}}\cdot j\,\cancel{e^{js}}\, ds-\int_0^1 s(-js)\cdot j\, ds=$$

$$=\cancel{\int_0^1 s^2\, ds}+\int_0^{\frac{\pi}{2}} j\, ds-\cancel{\int_0^1 s^2\, ds}=\left[js\right]_0^{\frac{\pi}{2}}=\underline{\underline{\frac{\pi}{2}j}}$$

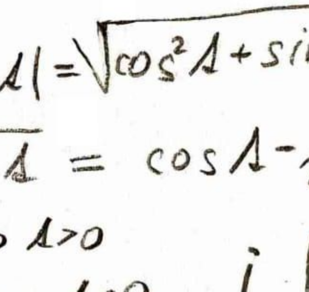
<u>Vzorce odvození</u>:

$\underline{\underline{|e^{js}|}}=|\cos s+j\sin s|=\sqrt{\cos^2 s+\sin^2 s}=\underline{\underline{1}}$

$\underline{\underline{\overline{e^{js}}}}=\overline{\cos s+j\sin s}=\cos s-j\sin s=\cos(-s)+j\sin(-s)=\underline{\underline{e^{-js}}}$

<u>Pozor</u>: $|s|=\begin{cases} s & \text{pro } s>0\\ -s & \text{pro } s<0\end{cases}$ i $|js|=\begin{cases} s & \text{pro } s>0\ \left(\sqrt{0^2+s^2}\right)\\ -s & \text{pro } s<0\ \left(\sqrt{0+(-s)^2}\right)\end{cases}$

(5. Př) $\int_{\gamma} |z|^2\, dz$;



$\boxed{\gamma=\gamma_1+\gamma_2}$

$\gamma_1: z(s)=js$, $z'(s)=j$, $s\in\langle 1,-1\rangle$

$\gamma_2: z(s)=e^{js}$, $z'(s)=j e^{js}$, $s\in\langle -\frac{\pi}{2},\frac{\pi}{2}\rangle$

$$=\int_1^{-1} |js|^2 j\, ds+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |e^{js}|^2 j e^{js}\, ds=$$

$$=\int_1^0 (s)^2 j\, ds+\int_0^{-1} (-s)^2 j\, ds+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1^2\cdot j e^{js}\, ds=$$

$$=\int_1^0 j s^2\, ds+\int_0^{-1} j s^2\, ds+\left[\cancel{j}\frac{e^{js}}{\cancel{j}}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=\left[j\frac{s^3}{3}\right]_1^0+\left[j\frac{s^3}{3}\right]_0^{-1}+\left[e^{js}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

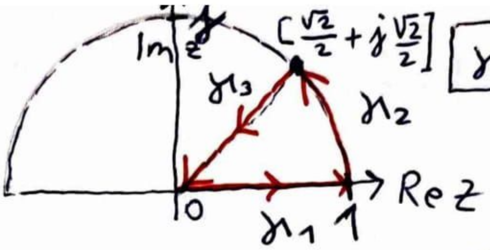
$$=-\frac{1}{3}j-\frac{1}{3}j+e^{\frac{j\pi}{2}}-e^{-\frac{j\pi}{2}}=-\frac{2}{3}j+\cos\frac{\pi}{2}+j\sin\frac{\pi}{2}-\left[\cos\left(-\frac{\pi}{2}\right)+j\sin\left(-\frac{\pi}{2}\right)\right]$$

$$=-\frac{2}{3}j+j-(-j)=\boxed{\frac{4}{3}j}$$

6. Př - DÚ

$\int_\gamma \frac{\bar{z}}{z}\,dz$;

1/8 kružnice



$\boxed{\gamma=\gamma_1+\gamma_2+\gamma_3}$

Lépe vzít s opačnou orientací a prohodit 0 a 1

$\gamma_1: z_1(s)=s \qquad z_1'(s)=1 \qquad s\in\langle 0,1\rangle$

$\gamma_2: z_2(s)=e^{js} \qquad z_2'(s)=je^{js} \qquad s\in\langle 0,\frac{\pi}{4}\rangle$

$\gamma_3: z_3(s)=\left(\frac{\sqrt2}{2}+j\frac{\sqrt2}{2}\right)s \qquad z_3'(s)=\frac{\sqrt2}{2}+j\frac{\sqrt2}{2} \qquad s\in\langle 1,0\rangle$!!

$$=\int_0^1 \frac{\bar{s}}{s}\cdot 1\,ds+\int_0^{\frac{\pi}{4}}\frac{\overline{e^{js}}}{e^{js}}\,j e^{js}\,ds+\int_1^0 \frac{\overline{\left(\frac{\sqrt2}{2}+j\frac{\sqrt2}{2}\right)s}}{\cancel{\left(\frac{\sqrt2}{2}+j\frac{\sqrt2}{2}\right)}s}\cdot\cancel{\left(\frac{\sqrt2}{2}+j\frac{\sqrt2}{2}\right)}ds$$

$$=\int_0^1 1\,ds+\int_0^{\pi/4}\frac{e^{-js}}{\cancel{e^{js}}}\cdot j\cancel{e^{js}}\,ds+\int_1^0\frac{\left(\frac{\sqrt2}{2}-j\frac{\sqrt2}{2}\right)\cancel{s}}{\cancel{s}}\,ds=$$

$$[s]_0^1+\left[\frac{e^{-js}}{-\cancel{j}}\cancel{j}\right]_0^{\frac{\pi}{4}}+\left[\left(\frac{\sqrt2}{2}-j\frac{\sqrt2}{2}\right)s\right]_1^0=$$

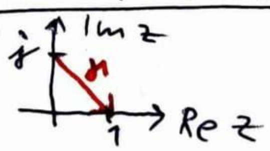
$$1+\left[-\left(\cos(-s)+j\sin(-s)\right)\right]_0^{\frac{\pi}{4}}+0-\left(\frac{\sqrt2}{2}-j\frac{\sqrt2}{2}\right)=$$

$$1\underbrace{-\cos\left(\frac{\pi}{4}\right)}_{-\frac{\sqrt2}{2}}+j\underbrace{\sin\frac{\pi}{4}}_{\frac{\sqrt2}{2}}-[-1]+0-\frac{\sqrt2}{2}+j\frac{\sqrt2}{2}=\underline{\underline{2-\sqrt2+\sqrt2\,j}}$$

[Pozor: v $z=0$ není $\frac{\bar z}{z}$ spojitá, ale má lim. po jednotlivých úsečkách ⇒ integrál exist.]

7. Př - DÚ

$\int_\gamma \frac{1}{z}\,dz$;



$z(s)=z_1+(z_2-z_1)s$

$z(s)=j+(1-j)s$

$z'(s)=1-j \;;\quad s\in\langle 0,1\rangle$

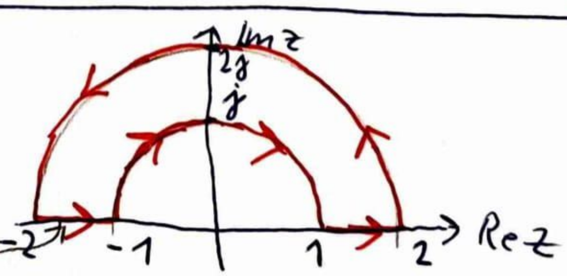
$$=\int_0^1\frac{1-j}{j+(1-j)s}\,ds=\Big[\ln\big(j+(1-j)s\big)\Big]_0^1=\ln 1-\ln j=0-\left(\ln|1|+j\frac{\pi}{2}\right)$$

stačí hlavní větev

$$=\underline{\underline{-j\frac{\pi}{2}}}$$

Př - DÚ

$\int_\gamma |z|\,\bar z\,dz$;



$[7\pi j]$

Pozor! $|s|=-s$ $(s\in\langle -2,-1\rangle)$