11. CVIKO

Fourierovy řady = Jistá analogie řad Taylorových

① Fce $f(x)$ je 2π - periodická na intervalu $\langle -\pi; \pi\rangle$ či $\langle a, a+2\pi\rangle$

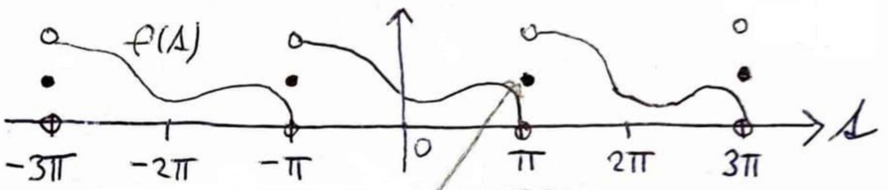
a (po částech) spojitá na $(-\pi, \pi)$ či $(a, a+2\pi)$. Potom

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx)\right],}$$ kde

a_n, b_n jsou tzv. Fourierovy koeficienty, splňující:

$$\boxed{a_n = \frac{1}{\pi}\int_{-\pi\,(a)}^{\pi\,(a+2\pi)} f(x)\cos(nx)\,dx} \implies \boxed{a_0 = \frac{1}{\pi}\int_{-\pi\,(a)}^{\pi\,(a+2\pi)} f(x)\,dx} \quad \boxed{\begin{aligned}T &= 2\pi \\ \omega &= \frac{2\pi}{2\pi} = 1\end{aligned}}$$

$$\boxed{b_n = \frac{1}{\pi}\int_{-\pi\,(a)}^{\pi\,(a+2\pi)} f(x)\sin(nx)\,dx}$$



$f(\pi) = \frac{1}{2}\left(f(\pi^-) + f(\pi^+)\right)$

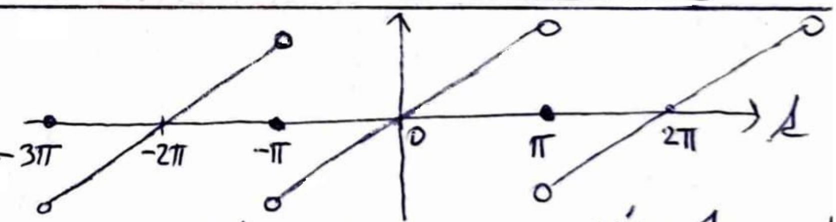
$f(x)$ lichá $\implies \boxed{a_n = 0}$

$f(x)$ sudá $\implies \boxed{b_n = 0}$

$\implies$ zbylé integrály : $\boxed{2\cdot\int_0^{\pi}(\cdots)\,dx}$

Př-Já $f(x) = x$ na $(-\pi, \pi)$

$a_n = 0$ ($f(x)$ je lichá)



$$b_n = \frac{1}{\pi}\int f(x)\sin(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin(nx)\,dx \quad \left|\begin{array}{ll} u = x & u' = 1 \\ v' = \sin(nx) & v = -\frac{\cos(nx)}{n}\end{array}\right|$$

$$= \frac{2}{\pi}\left[-x\frac{\cos(nx)}{n} + \int\frac{\cos(nx)}{n}\right]_0^{\pi} = \frac{2}{\pi}\left[-x\frac{\cos(nx)}{n} + \frac{\sin(nx)}{n^2}\right]_0^{\pi} =$$

$$= \frac{2}{\pi}\left[-\pi\cdot\frac{\cos(n\pi)}{n} + 0 - 0\right] = -\frac{2}{n}(-1)^n = \boxed{\frac{2}{n}(-1)^{n+1} \quad n = 1, 2, \ldots}$$

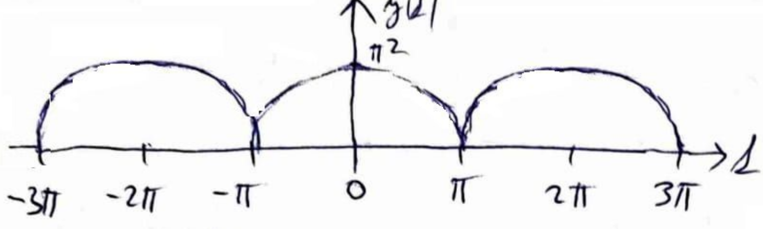
Pamatuj!
$\cos(n\pi) = (-1)^n \quad n = 1, 2, \ldots$

$$\boxed{f(x) = \sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin(nx)} = 2\sin x - \sin 2x + \frac{2}{3}\sin 3x - \frac{2}{4}\sin 4x +$$

$$+ \frac{2}{5}\sin 5x - \frac{2}{6}\sin 6x + \cdots +$$

$(x \in \mathbb{R})$

(Př) Najdi F.Ř. fce $\boxed{f(t)=\pi^2-t^2}$; $t\in(-\pi,\pi)$ a pomocí ní urči $\boxed{\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{1}{n^2}}$ a $\boxed{\sum_{n=1}^{\infty}\frac{1}{n^2}}$



Sudá Funkce $\Rightarrow$ $\boxed{b_n=0}$

$$a_0=\frac{2}{\pi}\int_0^{\pi}(\pi^2-t^2)\,dt=\frac{2}{\pi}\left[\pi^2 t-\frac{t^3}{3}\right]_0^{\pi}=\frac{2}{\pi}\left[\pi^3-\frac{\pi^3}{3}\right]=\boxed{\frac{4}{3}\pi^2}\Rightarrow\boxed{\frac{a_0}{2}=\frac{2}{3}\pi^2}$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}(\pi^2-t^2)\cos(nt)\,dt\left|\begin{matrix}\text{P.P.}\\ u=\pi^2-t^2 & u'=-2t\\ v'=\cos(nt) & v=\frac{\sin(nt)}{n}\end{matrix}\right|=$$

$$=\frac{2}{\pi}\left[(\pi^2-t^2)\cdot\frac{\sin(nt)}{n}+\int 2t\cdot\frac{\sin(nt)}{n}\,dt\left|\begin{matrix}\text{P.P.}\\ u=2t & u'=2\\ v'=\frac{\sin(nt)}{n} & v=-\frac{\cos(nt)}{n^2}\end{matrix}\right|\right]_0^{\pi}$$

$$=\frac{2}{\pi}\left[(\pi^2-t^2)\cdot\frac{\sin(nt)}{n}+2t\left(-\frac{\cos(nt)}{n^2}\right)+2\int\frac{\cos(nt)}{n^2}\,dt\right]_0^{\pi}=$$

$$=\frac{2}{\pi}\left[(\pi^2-t^2)\cdot\frac{\sin(nt)}{n}-2t\cdot\frac{\cos(nt)}{n^2}+2\frac{\sin(nt)}{n^3}\right]_0^{\pi}=$$

$$=\frac{2}{\pi}\left[-2\pi\cdot\frac{\cos(n\pi)}{n^2}\right]=\frac{-4(-1)^n}{n^2}=\boxed{\frac{4}{n^2}(-1)^{n+1}\quad n=1,2,\dots}$$

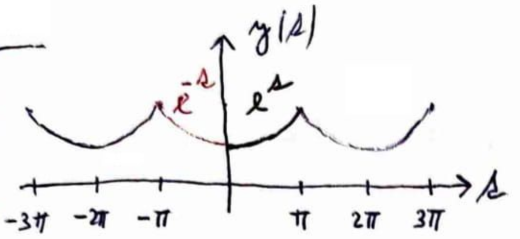
$$\boxed{f(t)=\frac{2}{3}\pi^2+\sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^{n+1}\cos(nt);\quad t\in\mathbb{R}}$$

$$\boxed{\pi^2-t^2=\frac{2}{3}\pi^2+\sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^{n+1}\cos(nt);\quad t\in\langle-\pi,\pi\rangle}$$

$t=0$: $\pi^2=\frac{2}{3}\pi^2+4\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{1}{n^2}\Rightarrow\boxed{\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{1}{n^2}=\frac{\pi^2}{12}}$

$t=\pi$: $0=\frac{2}{3}\pi^2+4\sum_{n=1}^{\infty}\frac{1}{n^2}\cdot\underbrace{(-1)^{n+1}\cdot(-1)^n}_{-1}\Rightarrow\boxed{\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}}$

DÚ Rozviň fci $f(t) = \begin{cases} e^{-t} & t \in \langle -\pi, 0) \\ e^{t} & t \in \langle 0, \pi) \end{cases}$ ve F.Ř.



sudá fce $\Rightarrow$ $\boxed{b_n = 0}$

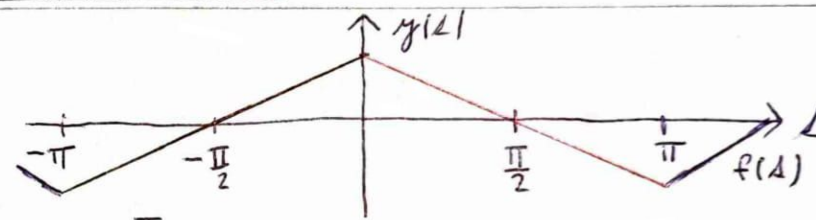
2× p.p. a pak vyřešit rovnici

$$\boxed{\frac{a_0}{2} = \frac{1}{\pi}(e^{\pi} - 1)} \; ; \; \boxed{a_n = \frac{2}{\pi} \cdot \frac{1}{n^2+1} \cdot \left[e^{\pi}(-1)^n - 1\right]}$$

$$\boxed{f(t) = \frac{1}{\pi}(e^{\pi} - 1) + \sum_{n=1}^{\infty} \frac{2}{\pi(n^2+1)} \cdot \left[e^{\pi}(-1)^n - 1\right] \cos(nt)}$$

Př. Najděte kosinovu řadu (tj. udělejte sudé rozšíření) funkce $\boxed{f(t) = \frac{\pi}{4} - \frac{t}{2} \quad \text{na } \langle 0, \pi)}$

$$f(t) = \begin{cases} \frac{t}{2} + \frac{\pi}{4} & t \in \langle -\pi, 0) \\ \frac{\pi}{4} - \frac{t}{2} & t \in \langle 0, \pi) \end{cases}$$



$\boxed{b_n = 0}$ (fce je sudá); $a_0 = \frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi}{4} - \frac{t}{2}\right) dt = \frac{2}{\pi}\left[\frac{\pi}{4}t - \frac{t^2}{4}\right]_0^{\pi} = \underline{\underline{0}}$

$$a_n = \frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi}{4} - \frac{t}{2}\right)\cos(nt)\,dt \left| \begin{matrix} \text{p.p.} \\ u = \frac{\pi}{4} - \frac{t}{2} & u' = -\frac{1}{2} \\ v' = \cos(nt) & v = \frac{\sin(nt)}{n} \end{matrix} \right| =$$

$$= \frac{2}{\pi}\left[\left(\frac{\pi}{4} - \frac{t}{2}\right) \cdot \frac{\sin(nt)}{n} + \int \frac{1}{2} \cdot \frac{\sin(nt)}{n}\,dt\right]_0^{\pi} =$$

$$= \frac{2}{\pi}\left[\left(\frac{\pi}{4} - \frac{t}{2}\right) \cdot \frac{\sin(nt)}{n} - \frac{1}{2} \cdot \frac{\cos(nt)}{n^2}\right]_0^{\pi} =$$

$$= \frac{2}{\pi}\left[\left(\frac{\pi}{4} - \frac{t}{2}\right) \cdot \frac{\sin(nt)}{n} - \frac{1}{2} \cdot \frac{\cos(nt)}{n^2}\right]_0^{\pi} =$$

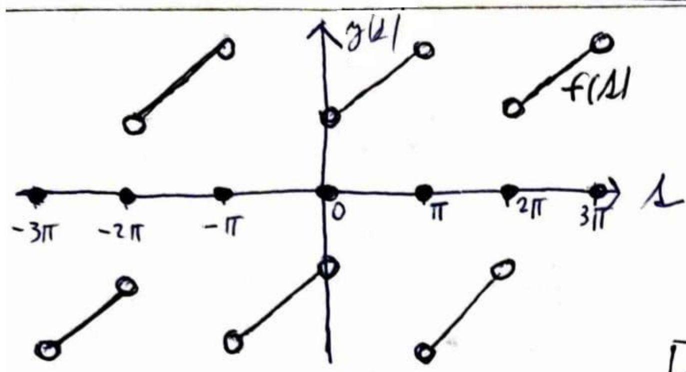
$$= \frac{2}{\pi}\left[\left(0 - \frac{1}{2} \cdot \frac{(-1)^n}{n^2}\right) - \left(0 - \frac{1}{2} \cdot \frac{1}{n^2}\right)\right] = \frac{2}{\pi} \cdot \frac{1}{2} \cdot \frac{1}{n^2}\left[1 - (-1)^n\right] =$$

$$= \frac{1}{\pi n^2}\left[1 - (-1)^n\right] = \boxed{\begin{cases} 0 & n\text{-}\underline{\text{sudé}} \\ \frac{2}{\pi n^2} & n\text{-}\underline{\text{liché}} \end{cases}}$$

(Liché $\Leftrightarrow n \rightsquigarrow 2n-1, \; n = 1, 2, \dots$)

$$\boxed{f(t) = \sum_{n=1}^{\infty} \frac{2}{\pi(2n-1)^2} \cos\left[(2n-1)t\right]} = \frac{2}{\pi}\left[\cos t + \frac{1}{9}\cos 3t + \frac{1}{25}\cos 5t + \dots\right]$$

(Př) Najděte sinovu řadu (tj. udělejte liché rozšíření funkce $\boxed{f(x) = x+1;\ x \in (0,\pi)}$.



$$f(x) = \begin{cases} x-1 & x \in (-\pi, 0) \\ 0 & x = 0 \\ x+1 & x \in (0,\pi) \end{cases}$$

$\boxed{a_n = 0}$ (fce je lichá)

$$b_n = \frac{2}{\pi}\int_0^\pi (x+1)\sin(nx)\,dx \quad \left| \begin{array}{ll} \text{P.P} & \\ u = x+1 & u' = 1 \\ v' = \sin(nx) & v = -\dfrac{\cos(nx)}{n} \end{array} \right|$$

$$= \frac{2}{\pi}\left[-(x+1)\cdot\frac{\cos(nx)}{n} + \int \frac{\cos(nx)}{n}\right]_0^\pi =$$

$$= \frac{2}{\pi}\left[-(x+1)\cdot\frac{\cos(nx)}{n} + \frac{\sin(nx)}{n^2}\right]_0^\pi = \frac{2}{\pi}\left[-(\pi+1)\cdot\frac{(-1)^n}{n} + \frac{1}{n}\right] =$$

$$= \boxed{\frac{2}{\pi n}\left[1-(\pi+1)\cdot(-1)^n\right]}$$

$$\boxed{f(x) = \sum_{n=1}^{\infty} \frac{2}{\pi n}\left[1-(\pi+1)\cdot(-1)^n\right]\sin(nx)}$$

(Př - Čas) Rozviň fci $\boxed{f(x) = \sin^4 x}$, $x \in (-\pi,\pi)$ do F.Ř.

[Integrací složité! Fce je ale sudá $\Rightarrow$ $\boxed{b_n = 0}$]; (Nebude tam $\sin(nx)$)

$$\sin^4 x = (\sin^2 x)^2 = \left(\frac{1-\cos 2x}{2}\right)^2 = \frac{1-2\cos 2x + \cos^2 2x}{4} =$$

$$= \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{4}\left(\frac{1+\cos 4x}{2}\right) = \boxed{\frac{1}{4} + \frac{1}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x}$$

$$\boxed{f(x) = \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x}$$

součet pouze 3 členů !!

(VZORCE:
$\sin^2 x = \dfrac{1-\cos 2x}{2}$
$\cos^2 x = \dfrac{1+\cos 2x}{2}$)

Př - DÚ
čas - nic

Rozviň fci $f(t) = \sin^3 t$, $t \in (-\pi, \pi)$ do F.Ř.
[Integrací složité! Fce je ale lichá $\Rightarrow \boxed{a_n = 0}$ (Nebude tam $\cos(n\omega t)$)]

Moiavrova Věta: $(\cos t + j\sin t)^3 = \cos 3t + j\sin 3t$

$$\cos^3 t + 3j\cos^2 t \sin t - 3\cos t \sin^2 t - j\sin^3 t = \cos 3t + j\sin 3t$$

j:
$$3\cos^2 t \sin t - \sin^3 t = \sin 3t$$
$$3(1-\sin^2 t)\sin t - \sin^3 t = \sin 3t$$
$$(3 - 3\sin^2 t)\sin t - \sin^3 t = \sin 3t$$
$$3\sin t - 3\sin^3 t - \sin^3 t = \sin 3t$$
$$-4\sin^3 t = -3\sin t + \sin 3t$$
$$\boxed{\sin^3 t = \frac{3}{4}\sin t - \frac{1}{4}\sin 3t}$$

Pouze 2 členy:

$$\boxed{f(t) = \frac{3}{4}\sin t - \frac{1}{4}\sin 3t}$$

---

② Fce $f(t)$ je periodická na $\langle -\frac{T}{2}, \frac{T}{2} \rangle$ či $\langle a, a+T \rangle$ a (po částech) spojitá na $(-\frac{T}{2}, \frac{T}{2})$ či $(a, a+T)$. Potom:

$$\boxed{f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(n\omega t) + b_n \sin(n\omega t)\right]}$$, kde $\boxed{\omega = \frac{2\pi}{T}}$ je ÚHLOVÁ FREKVENCE

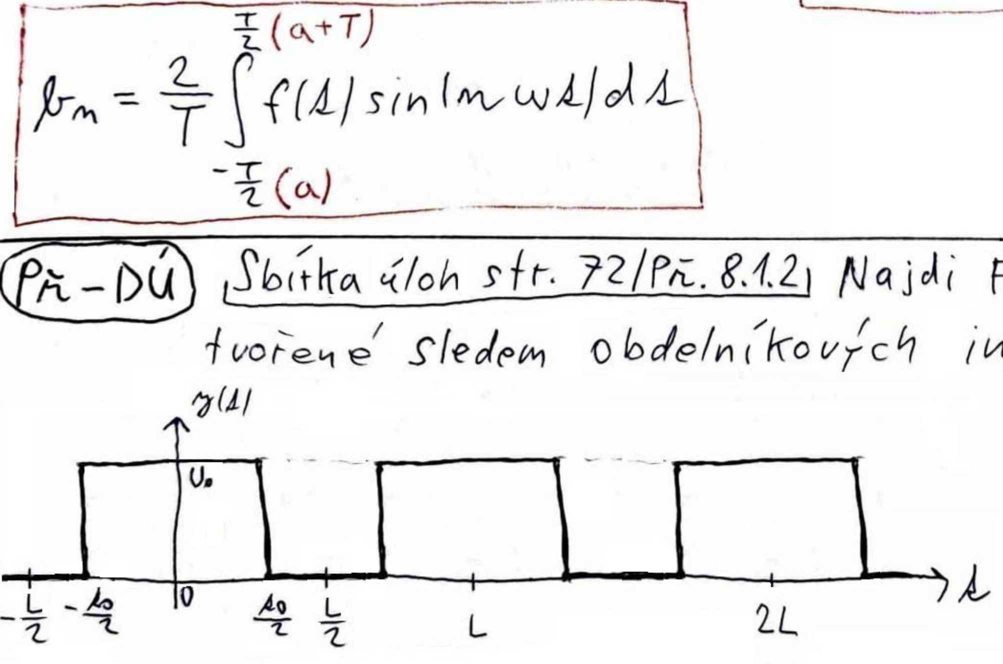
T je perioda

a a_n, b_n splňují:

$$\boxed{a_n = \frac{2}{T}\int_{-\frac{T}{2}\,(a)}^{\frac{T}{2}\,(a+T)} f(t)\cos(n\omega t)\,dt} \Rightarrow \boxed{a_0 = \frac{2}{T}\int_{-\frac{T}{2}\,(a)}^{\frac{T}{2}\,(a+T)} f(t)\,dt}$$

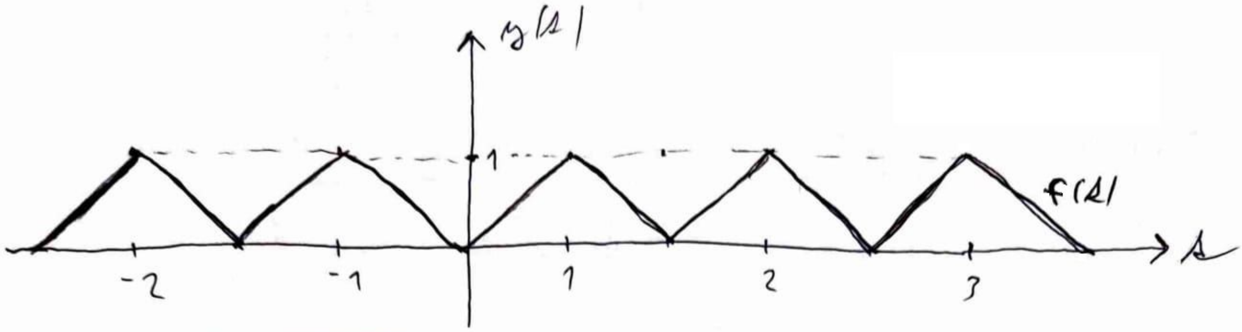
$$\boxed{b_n = \frac{2}{T}\int_{-\frac{T}{2}\,(a)}^{\frac{T}{2}\,(a+T)} f(t)\sin(n\omega t)\,dt}$$

Př - DÚ Sbírka úloh str. 72/Př. 8.1.2, Najdi F.Ř. na $(-\frac{L}{2}, \frac{L}{2})$ pro napětí tvořené sledem obdelníkových impulsů:



$$\left[f(t) = \frac{U_0 t_0}{L} + \sum_{n=1}^{\infty} \frac{2U_0}{n\pi}\sin\left(\frac{n\pi t_0}{L}\right)\cos\left(\frac{2n\pi t}{L}\right)\right]$$

(Př) Rozviň fci $\boxed{f(t)=|t|}$, $t\in\langle -1, 1\rangle$ do F.Ř.



$\boxed{T=2}$ $\boxed{\omega=\frac{2\pi}{2}=\pi}$; $\boxed{b_n=0}$ (fce je sudá)

$$a_0=\frac{2}{2}\int_{-1}^{1}|t|\,dt=2\int_0^1 t\,dt=\left[t^2\right]_0^1=1 \Rightarrow \boxed{\frac{a_0}{2}=\frac{1}{2}}$$

$$a_n=2\int_0^1 t\cos(n\pi t)\,dt\left|\begin{matrix}u=t & u'=1\\ v'=\cos(n\pi t) & v=\frac{\sin(n\pi t)}{n\pi}\end{matrix}\right|=$$

$$=2\left[t\,\frac{\sin(n\pi t)}{n\pi}-\int\frac{\sin(n\pi t)}{n\pi}\right]_0^1=2\left[\frac{t\sin(n\pi t)}{n\pi}+\frac{\cos(n\pi t)}{n^2\pi^2}\right]_0^1=$$

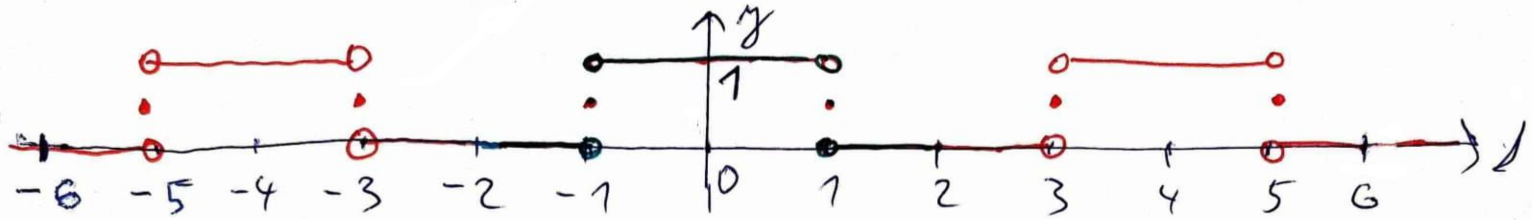
$$=2\left[\frac{(-1)^n}{n^2\pi^2}-\frac{1}{n^2\pi^2}\right]=\frac{2}{n^2\pi^2}\left[(-1)^n-1\right]=\boxed{\begin{cases}0 & n\text{-sudé}\\ -\frac{4}{n^2\pi^2} & n\text{-liché}\end{cases}}$$

↳ n-liché lze psát jako $\underline{2n-1}$

$$\boxed{f(t)=\frac{1}{2}+\sum_{n=1}^{\infty}\frac{-4}{(2n-1)^2\pi^2}\cos\left[(2n-1)\pi t\right]}$$

$$\boxed{f(t)=\frac{1}{2}-\frac{4}{\pi^2}\left[\cos(\pi t)+\frac{1}{9}\cos(3\pi t)+\frac{1}{25}\cos(5\pi t)+\cdots+\right]}$$

(PF) Rozviň fci $f(t)$ danou obrázkem do F.Ř.:



$\boxed{T=4}$ $\quad \omega = \frac{2\pi}{T}$

$\boxed{\omega = \frac{2\pi}{4} = \frac{\pi}{2}}$

$$\boxed{a_0} = \frac{2}{4}\int_{-2}^{2} f(t)\,dt = \frac{1}{2}\int_{-1}^{1} 1\,dt = \frac{1}{2}\Big[t\Big]_{-1}^{1}$$

$$= \frac{1}{2}\cdot 2 = \underline{\underline{1}} \Rightarrow \boxed{\frac{a_0}{2} = \frac{1}{2}}$$

$$a_n = \frac{2}{4}\int_{-2}^{2} f(t)\cos\left(n\frac{\pi}{2}t\right)dt = \frac{\overset{2}{\cancel{1}}}{2}\int_{\cancel{-1}\,0}^{1} 1\cdot\cos\left(n\frac{\pi}{2}t\right)dt$$

$$= \frac{2}{2}\int_{0}^{1}\cos\left(\frac{n\pi}{2}t\right)dt = \left[\sin\left(\frac{n\pi}{2}t\right)\cdot\frac{2}{n\pi}\right]_{0}^{1}$$

$$= \boxed{\sin\left(\frac{n\pi}{2}\right)\cdot\frac{2}{n\pi}} \quad ; \quad \boxed{b_n = 0} \text{ (f je sudá)}$$

$$\boxed{f(t) = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\cdot\cos\left(\frac{n\pi t}{2}\right)}$$

$$= \frac{1}{2} + \frac{2}{\pi}\left[\cos\left(\frac{\pi t}{2}\right) + 0 - \frac{1}{3}\cos\left(\frac{3\pi t}{2}\right) + 0 + \cdots\right]$$

$$\boxed{f(t) = \frac{1}{2} + \frac{2}{\pi}\left[\cos\left(\frac{\pi t}{2}\right) - \frac{1}{3}\cos\left(\frac{3\pi t}{2}\right) + \frac{1}{5}\cos\left(\frac{5\pi t}{2}\right) - \cdots\right]}$$