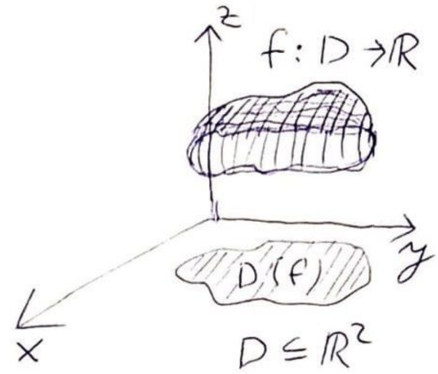
1. CVIKO

Funkce více proměnných

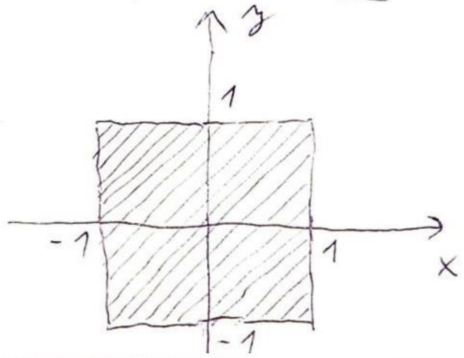


- nejčastěji fce dvou proměnných
- značíme: $f(x,y)$; $z(x,y)$; $u(x,y,z)$

nebo $f(x_1, x_2, \dots, x_m)$

(Př) <u>Určete a v $\mathbb{R}^2$ zobrazte definiční obory funkcí:</u>

① $f(x,y) = \sqrt{1-x^2} + \sqrt{1-y^2}$



$x^2 < 1$
$|x| < 1$
$-1 \leq x \leq 1$

$1 - x^2 \geq 0 \quad \wedge \quad 1 - y^2 \geq 0$

$(1-x)(1+x) \geq 0 \quad \wedge \quad (1-y)(1+y) \geq 0$

$\underline{-1 \leq x \leq 1} \quad \wedge \quad \underline{-1 \leq y \leq 1}$

DÚ $f(x,y) = \sqrt{-\ln x^2} + \sqrt{-\ln y^2}$

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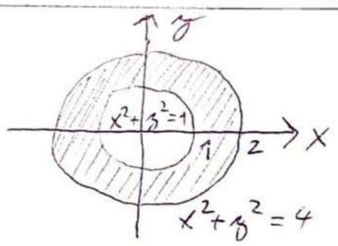
② $f(x,y) = \sqrt{(x^2+y^2-1)(4-x^2-y^2)}$



$x^2 + y^2 \geq 1 \quad \wedge \quad x^2 + y^2 \leq 4 \quad$ nebo $\emptyset$

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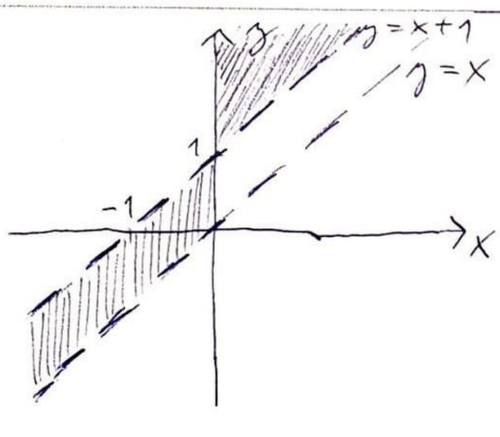
③ $f(x,y) = \ln[x \ln(y-x)]$



$y - x > 0 \quad \wedge \quad x \cdot \ln(y-x) > 0$

$y > x \quad \wedge \quad [x > 0 \ \wedge \ y > x+1]$

nebo

$[x < 0 \ \wedge \ y < x+1]$

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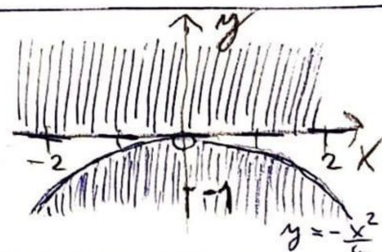
④ $f(x,y) = \sqrt{\frac{x^2}{y} + 4} = \sqrt{\frac{x^2+4y}{y}}$

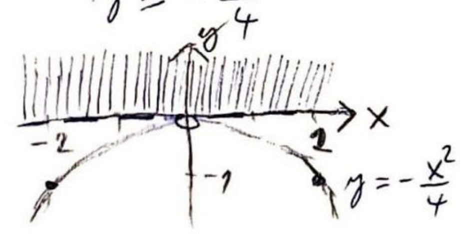
<u>Celkem</u>:

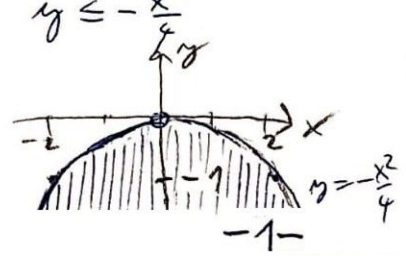


$y > 0$
$y \geq -\frac{x^2}{4}$

nebo

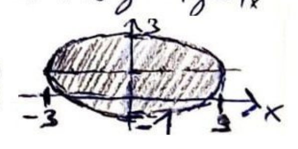
$y < 0$
$y \leq -\frac{x^2}{4}$





⑤ DÚ $f(x,y) = \sqrt{27 + 18y - 9y^2 - 4x^2}$

$\left[\frac{x^2}{9} + \frac{(y-1)^2}{4} \leq 1\right.$



Parciální derivace fce $z(x,y)$

$$\left(z''_{xy} = z''_{yx}\right)$$

$$z'_x = \frac{\partial z}{\partial x};\ z'_y = \frac{\partial z}{\partial y};\ z''_{xx} = \frac{\partial^2 z}{\partial x^2};\ z''_{yy} = \frac{\partial^2 z}{\partial y^2};\ z''_{xy} = \frac{\partial^2 z}{\partial x \partial y};\ z''_{yx} = \frac{\partial^2 z}{\partial y \partial x}$$

Urči všechny derivace fce $\boxed{f(x,y) = x^y}$ až do řádu II.

$$f'_x = y x^{y-1} \quad \Big| \quad f''_{xx} = y(y-1)\cdot x^{y-2} \quad \Big| \quad f''_{xy} = x^{y-1} + y x^{y-1} \ln x = f''_{yx}$$

$$f'_y = x^y \cdot \ln x \quad \Big| \quad f''_{yy} = \ln^2 x \cdot x^y$$

Derivujte (určete z'_x a z'_y):

- $z = 3xy^2 - 2x + \sqrt{y}$ $\left[z'_x = 3y^2 - 2;\ z'_y = 6xy + \frac{1}{2\sqrt{y}} \right]$
- $z = \frac{y}{\sqrt{x^2 - y^2}}$ $\left[z'_x = -\frac{xy}{\sqrt{(x^2-y^2)^3}};\ z'_y = \frac{x^2}{\sqrt{(x^2-y^2)^3}} \right]$
- $z = \ln(xy + \ln y)$ $\left[z'_x = \frac{y}{xy + \ln y};\ z'_y = \frac{1}{xy + \ln y} \cdot \left(x + \frac{1}{y}\right) \right]$
- $z = \arcsin \frac{x}{y}$ $\left[z'_x = \frac{1}{\sqrt{1 - (\frac{x}{y})^2}} \cdot \frac{1}{y};\ z'_y = \frac{1}{\sqrt{1 - (\frac{x}{y})^2}} \cdot \frac{-x}{y^2} \right]$
- $z = \operatorname{arctg} \frac{x-y}{x+y}$ $\left[z'_x = \frac{y}{x^2+y^2};\ z'_y = -\frac{x}{x^2+y^2} \right]$
- $z = \ln\left(\operatorname{tg} \frac{x}{y}\right)$ $\left[z'_x = \frac{1}{\operatorname{tg}\frac{x}{y}} \cdot \frac{1}{\cos^2(\frac{x}{y})} \cdot \frac{1}{y};\ z'_y = \frac{1}{\operatorname{tg}\frac{x}{y}} \cdot \frac{1}{\cos^2(\frac{x}{y})} \cdot \frac{-x}{y^2} \right]$
- $u(x,y,z) = e^{\sin(z+2xy)}$ a určete pouze $u'_x(2,-1,4)$

$$\left[u'_x = e^{\sin(z+2xy)} \cdot \cos(z+2xy) \cdot 2y;\ u'_x(2,-1,4) = -2 \right]$$

DÚ:
- $z = \operatorname{arctg} \sqrt{xy}$
- $z = (x-y)^2$
- $z = \frac{1}{2} \ln(x^2+y^2)$

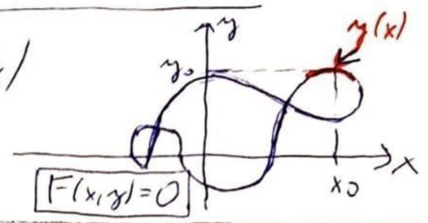
(Derivace až do řádu II.)

DÚ: Dokažte, že $z = e^{\frac{x}{y^2}}$ vyhovuje rovnici $\boxed{2x z'_x + y z'_y = 0}$

Funkce zadaná implicitně - nejprve jedné proměnné

$F(x,y)=0$ zadává v okolí $[x_0,y_0]$ fci $y(x)$

Jak určíme $y'(x_0)$??



(Př) Urči $y'(x)$ fce zadané rovnicí $\boxed{x^2y-\sin(xy)-\pi=0}$ např. v okolí bodu $[1,\pi]$

Buď derivací rovnice

$$\frac{d}{dx}:\quad 2xy+x^2y'-\cos(xy)\cdot(1y+xy')=0$$

$$y'\left(x^2-x\cos(xy)\right)=-2xy+y\cos(xy)$$

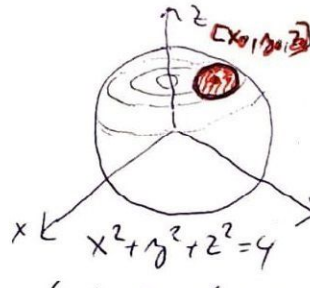
$$\boxed{y'=-\frac{2xy-y\cos(xy)}{x^2-x\cos(xy)}}$$

Nebo vzorec $\boxed{y'=-\frac{F'_x}{F'_y}}$: $\boxed{y'=-\frac{2xy-\cos(xy)\cdot y}{x^2-\cos(xy)\cdot x}}$

Funkce zadaná implicitně - dvou proměnných

$F(x,y,z)=0$ zadává v okolí $[x_0,y_0,z_0]$ fci $z(x,y)$

(Např. $x^2+y^2+z^2=4$ zadává kulovou plochu.
(Ne vždy (jako tady) lze z rovnice $F(x,y,z)=0$ vyjádřit „z".)



Není fce, ale v okolí $[x_0,y_0,z_0]$ ANO!!)

Jak určíme z'_x a z'_y takto zadaných fcí ??

a) Derivace rovnice b) Pomocí vzorců: $\boxed{z'_x=-\frac{F'_x}{F'_z}}$ a $\boxed{z'_y=-\frac{F'_y}{F'_z}}$

(Př) Urči z'_x a z'_y fce $z(x,y)$ zadané implicitně rovnicí $\boxed{x^2-\sin y+\cos z-1=0}$ a hodnoty $z'_x(1,0,\frac{\pi}{2})$ a $z'_y(1,0,\frac{\pi}{2})$.

a) $\frac{\partial z}{\partial x}$: $\quad 2x-\sin z\cdot z'_x=0$

$$\boxed{z'_x=\frac{2x}{\sin z}} \Rightarrow z'_x(1,0,\tfrac{\pi}{2})=2$$

$\frac{\partial z}{\partial y}$: $\quad -\cos y-\sin z\cdot z'_y=0$

$$\boxed{z'_y=-\frac{\cos y}{\sin z}} \Rightarrow z'_y(1,0,\tfrac{\pi}{2})=-1$$

b) vzorce

$$z'_x=-\frac{2x}{-\sin z}$$

$$z'_y=-\frac{-\cos y}{-\sin z}$$

(Př) Urči z'_x a z'_y pro $z(x,y)$ zadané implicitně jako

$$\boxed{\ln(x^2 z - \sin(zy)) = 0}$$

$$F'_x = \frac{2xz}{x^2 z - \sin(zy)} \qquad \boxed{z'_x = -\frac{2xz}{x^2 - \cos(zy)\cdot y}}$$

$$F'_y = \frac{-\cos(zy)\cdot z}{x^2 z - \sin(zy)}$$

$$F'_z = \frac{x^2 - \cos(zy)\cdot y}{x^2 z - \sin(zy)} \qquad \boxed{z'_y = \frac{\cos(zy)\cdot z}{x^2 - \cos(zy)\cdot y}}$$

Aplikace : Tečná rovina k fci $f(x,y)$ v bodě $T = [x_0, y_0, ?]$:

$$\boxed{\alpha_T: z - z_0 = f'_x(x_0,y_0)\cdot(x - x_0) + f'_y(x_0,y_0)\cdot(y - y_0)}$$

(Př)
- $f(x,y) = \frac{x^2}{2} - y^2$; $T = [2, -1, ?]$ $[2x + 2y - z - 1 = 0]$
- $f(x,y) = \sqrt{x^2 + y^2} - xy$; $T = [3, 4, ?]$ $[17x + 11y + 5z - 60 = 0]$
- $x^3 + y^3 + z^3 + xyz - 6 = 0$; $T = [1, 2, -1]$ $[x + 11y + 5z - 18 = 0]$

Pro implicitně zadanou fci lze nejrychleji určit pomocí vzorce:
Nechť $F(x,y,z) = 0$ zadává v okolí $T = [x_0, y_0, z_0]$ fci $z(x,y)$. PAK

$$\boxed{\alpha_T: F'_x(x_0,y_0,z_0)(x - x_0) + F'_y(x_0,y_0,z_0)(y - y_0) + F'_z(x_0,y_0,z_0)\cdot(z - z_0) = 0}$$

[Čas] Gradient = směr, ve kterém funkce nejrychleji roste.

$$\boxed{\operatorname{Grad} f(x_0,y_0) = \left(f'_x(x_0,y_0),\ f'_y(x_0,y_0)\right)}$$

(Př) Urči:
- $\operatorname{Grad} f(2,3)$ pro $f(x,y) = e^{x-y}$ $\left[\operatorname{Grad} f(2,3) = \left(\frac{1}{e}, -\frac{1}{e}\right)\right]$
- $\operatorname{Grad} f(1,\frac{\pi}{2})$ pro $f(x,y) = \frac{e^x}{\sin y}$ $\left[\operatorname{Grad} f(1,\frac{\pi}{2}) = (e, 0)\right]$