

Relaxation Model for a Porous Medium Saturated with Two Fluids

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INTRODUCTION

The behavior of a porous medium saturated with immiscible fluids exhibits a number of features. Examples of these macroscopic effects are relaxation of the capillary pressure, hysteresis of the capillary-pressure-saturation curve, dependence of the phase permeabilities and of the capillary pinching on the rate and the variation direction of the saturation. The Muskat–Leverett model [1, 2] cannot describe these phenomena associated with a nonequilibrium character of the combined motion of fluids [3, 4]. In the absence of equilibrium, the variation of saturation of a material element (or any other action) results in the fact that the state of the element varies even in the absence of the mass, momentum, or energy exchange with surrounding parts of the body.

This is related to a redistribution of fluids inside the element of the medium. As a result, the nonwetting and wetting fluids occupy larger and smaller capillaries, respectively. This is caused by a decrease in the surface energy of the element of the porous medium.

THE KINEMATICS AND CONSERVATION LAWS

We assume that the saturated porous medium can be considered as consisting of mutually penetrating and interacting continua occupying the same volume under consideration. In this case, we consider three continua: the skeleton (denoted by the subscript s) and two fluids (with the corresponding subscripts $\alpha = 1, 2$).

These continua are characterized by the volume fractions ϕ_A ($A = s, \alpha$), which obey the relation $\sum_A \phi_A = 1$.

The sum of the volume fractions of the fluids is equal to

the porosity $\phi = \phi_1 + \phi_2$. The saturations of the porous medium with the first and the second fluid are $S = \phi_1/\phi$ and $1 - S = \phi_2/\phi$, respectively.

Let χ be a fixed domain in the Euclidean space. At each point $\mathbf{x} \in \chi$ of this domain, elements of the skeleton and of the fluids are superimposed. The domains $\kappa_A(t)$ originally occupied by the particles of continuum A , which reside in the domain χ at the current time t , are called initial configurations. The domains $\kappa_A(t)$ are time-dependent insofar as different combinations of particles of the skeleton and of the fluids are present in the domain χ at different instants of time. The mappings $\kappa_A(t) \rightarrow \chi$ are assumed to be piecewise injective (one-to-one) and differentiable. In this case, the following laws of motion, which possess the same properties, are valid:

$$\mathbf{X}_A = \mathbf{X}_A(\mathbf{x}, t), \quad \mathbf{x} \in \chi, \quad \mathbf{X}_A \in \kappa_A(t), \quad (1)$$

$$t \geq 0, \quad A = s, \alpha.$$

Let $\mathbf{G}_A = [\nabla \otimes \mathbf{X}_A(\mathbf{x}, t)]^T$ be the distortion of continuum A so that $d\mathbf{X}_A = \mathbf{G}_A \cdot d\mathbf{x}$, $J_A = \det \mathbf{G}_A \neq 0$. For the nondegenerate tensor, there exists the polar expansion

$$\mathbf{G}_A = \mathbf{R}_A \cdot \mathbf{U}_A = \mathbf{V}_A \cdot \mathbf{R}_A, \quad (2)$$

where \mathbf{U}_A , \mathbf{V}_A are the symmetric tension (contraction) tensors and \mathbf{R}_A is the orthogonal rotation tensor. Expansions (2) exhibit a uniqueness property.

Further, let \mathbf{v}_A be the mass velocity the particles of continuum A . The distortion and the velocity vector are related by the compatibility equation for strains and velocities [5, 6]

$$\frac{d_A \mathbf{G}_A}{dt} = -\mathbf{G}_A \cdot (\nabla \otimes \mathbf{v}_A)^T, \quad (3)$$

where $\frac{d_A f}{dt} = \frac{\partial f}{\partial t} + \mathbf{v}_A \cdot \nabla f$ is the derivative along the trajectory of a particle of continuum A .

Let us consider a situation where the mass exchange between the continua is absent, all continua are nonpolar, the skeleton and the fluids are characterized by the same absolute temperature $\theta(\mathbf{x}, t) > 0$, and we can ignore the contribution of pulsations to the kinetic energy [7] and the effect of tortuosity [3]. In this case, the set of differential conservation laws for mass, momentum, and energy can be written in the following form:

$$\frac{d_A r_A}{dt} + r_A \nabla \cdot \mathbf{v}_A = 0, \quad (4)$$

$$r_A \frac{d_A \mathbf{v}_A}{dt} - \nabla \cdot \mathbf{T}_A = r_A \mathbf{g} - \mathbf{b}_A^{\text{int}}, \quad (5)$$

$$\begin{aligned} \sum_A r_A \frac{d_A u_A}{dt} &= \sum_A \mathbf{T}_A : (\nabla \otimes \mathbf{v}_A) \\ &+ \sum_\alpha \mathbf{b}_\alpha^{\text{int}} \cdot \mathbf{w}_\alpha + \nabla \cdot \mathbf{q} + rQ, \end{aligned} \quad (6)$$

where r_A is the averaged mass density of the continuum A in the current state, \mathbf{T}_A is the tensor of averaged partial stresses, \mathbf{g} is the mass force, $\mathbf{b}_A^{\text{int}}$ is the interaction force between the continuum A and other components of the medium, u_A is the specific internal energy, \mathbf{q} is the thermal flux, Q is the heat source density, $r = \sum_A r_A$ is the total-mass density, and $\mathbf{w}_\alpha = \mathbf{v}_\alpha - \mathbf{v}_s$ is the relative velocity of the fluid α . The sum of the interaction forces for the continua is zero: $\sum_A \mathbf{b}_A^{\text{int}} = 0$ [7].

The interaction forces, partial stresses, and the volume fractions of fluids can be represented as the sums of equilibrium and dissipative terms

$$\begin{aligned} \mathbf{b}_\alpha^{\text{int}} &= \mathbf{b}_\alpha^0 + \mathbf{b}_\alpha^{\text{dis}}, \quad \mathbf{T}_\alpha = \mathbf{T}_\alpha^0 + \mathbf{T}_\alpha^{\text{dis}}, \\ \phi_\alpha &= \phi_\alpha^0 + \phi_\alpha^{\text{dis}}. \end{aligned} \quad (7)$$

In the equilibrium state, the dissipative components $\mathbf{b}_\alpha^{\text{dis}}$, $\mathbf{T}_\alpha^{\text{dis}}$, and ϕ_α^{dis} vanish (the temperature gradient and the mass velocities are zero). The equilibrium interaction force of the fluid α with other continua is $\mathbf{b}_\alpha^0 = \boldsymbol{\sigma}_\alpha^0 \cdot \nabla \phi_\alpha^0$. This expression is valid for a saturating fluid with an arbitrary rheology in contrast to the dissipative term $\mathbf{b}_\alpha^{\text{dis}}$ for which it is necessary to formulate the constitutive relationship resulting in the generalized Darcy law.

The second principle of thermodynamics for smooth motions of the medium under study is formu-

lated in the form of a local inequality for the entropy production rate

$$\sum_A r_A \frac{d_A \eta_A}{dt} - \nabla \cdot \left(\frac{\mathbf{q}}{\theta} \right) - \frac{rQ}{\theta} \geq 0, \quad (8)$$

where η_A is the entropy density for continuum A . Inequality (8) is the condition to be satisfied by the constitutive relationships for arbitrary history of variation of the state of the medium under consideration [8, 9]. In order to derive constraints for the constitutive relationships, it is convenient to use inequality (8) in the following form:

$$\begin{aligned} & - \sum_A \left(r_A \frac{d_A \psi_A}{dt} + r_A \eta_A \frac{d_A \theta}{dt} \right) - (\mathbf{T} \cdot \mathbf{G}_s^{-1}) : \frac{d_s \mathbf{G}_s^T}{dt} \\ & + \sum_\alpha \boldsymbol{\sigma}_\alpha^0 : (\nabla \otimes (\phi_\alpha^0 \mathbf{w}_\alpha)) + \delta_f + \delta_T \geq 0, \end{aligned} \quad (9)$$

where $\mathbf{T} = \sum_A \mathbf{T}_A$ is the total stress tensor, $\delta_f = \sum (\mathbf{T}_\alpha^{\text{dis}} : (\nabla \otimes \mathbf{w}_\alpha) + \mathbf{b}_\alpha^{\text{dis}} \cdot \mathbf{w}_\alpha)$ is the filtration dissipation, and $\delta_T = \theta^{-1} \mathbf{q} \cdot \nabla \theta$ is the thermal dissipation.

CONSTITUTIVE EQUATIONS

For the medium under consideration, the state of the point \mathbf{x} at the time t is given by the set of the quantities

$$\begin{aligned} \Lambda(\mathbf{x}, t) &= \{ \mathbf{G}_A, \theta, \xi, \boldsymbol{\gamma}, \mathbf{w}_\alpha \}, \\ A &= s, \alpha, \quad \alpha = 1, 2, \end{aligned} \quad (10)$$

where ξ is the scalar parameter of nonequilibrium, which is proportional to the surface area of the contact between the nonwetting fluid and the skeleton, and $\boldsymbol{\gamma} \equiv \nabla \theta$ is the temperature gradient. The response of an element of the medium at the point (\mathbf{x}, t) is characterized by the set of the quantities

$$\begin{aligned} \Upsilon(\mathbf{x}, t) &= \{ \psi_A(\mathbf{x}, t), \mathbf{T}_A(\mathbf{x}, t), \eta_A(\mathbf{x}, t), \phi_\alpha(\mathbf{x}, t), \\ & \mathbf{q}(\mathbf{x}, t), \mathbf{b}_\alpha^{\text{dis}}(\mathbf{x}, t) \}, \quad A = s, \alpha. \end{aligned}$$

These quantities are considered to be the functions of state

$$\Upsilon(\mathbf{x}, t) = \Upsilon^+ \{ \Lambda(\mathbf{x}, t) \}, \quad (11)$$

where $\Upsilon^+ = \{ \psi_A^+, \mathbf{T}_A^+, \eta_A^+, \phi_\alpha^+, \mathbf{q}^+, \mathbf{b}_\alpha^+ \}$ is the set of the functions defining the constitutive equations. We now formulate the following kinetic equation for the parameter of nonequilibrium:

$$\frac{d_s \xi}{dt} = \Omega \{ \Lambda(\mathbf{x}, t) \}. \quad (12)$$

The solution to Eq. (12) at the point \mathbf{x} is the functional defined for the prehistory $\{ \mathbf{G}_A(\tau), \theta(\tau), \boldsymbol{\gamma}(\tau), \mathbf{w}_\alpha(\tau) \}$,

$\tau \leq t$. Therefore, the material possesses a long-term memory.

In the initial state, the temperature and stresses are considered to be constant over the space. The material is assumed to be homogeneous, therefore the radii vectors \mathbf{X}_A are absent among the parameters of state.

The porous medium saturated with two fluids is composed of three continua with their own initial states. Therefore, the symmetry of the medium is characterized by a combination of three groups $g = \{g_{\kappa_s}, g_{\kappa_\alpha}\}$. It is assumed that the skeleton is solid, which implies that the skeleton has an undistorted configuration κ_0 such that, for an arbitrary reference configurations $\kappa_\alpha(t)$ of the fluids, the constitutive equations have the symmetry group $g = \{g_0, u, u\}$. Here, $g_0 \in o$ is the subgroup of an orthogonal group, which corresponds to the given skeleton symmetry, and u is the unimodular group of second-rank tensors.

In addition to the requirement of the invariance for the transformations of the initial-state symmetry, further consideration is based on the objectiveness principle (i.e., on the independence of the constitutive equations on the choice of the reference system) and on the principle of thermodynamic consistency [8, 9] (i.e., on the validity of inequality (9) for arbitrary history of the state). In order to satisfy these principles in the case of a compressible rigid skeleton, it is necessary and sufficient for constitutive equations (10)–(12) with an arbitrary type of symmetry to have the following form:

$$\begin{aligned} \psi_\alpha &= \psi_\alpha^\times(\rho_\alpha, \theta), \quad \mathbf{T}_\alpha = -\phi_\alpha p_\alpha \mathbf{I}, \quad p_\alpha = \rho_\alpha^2 \frac{\partial \psi_\alpha^\times}{\partial \rho_\alpha}, \\ \eta_\alpha &= -\frac{\partial \psi_\alpha^\times}{\partial \theta}, \\ \psi_s &= F(\mathbf{V}, \phi_\beta, \theta, \xi), \quad \mathbf{T}_s \cdot \mathbf{G}_s^{-1} = -r_s \frac{\partial F}{\partial \mathbf{G}_s^T}, \\ p_\alpha &= r_\alpha \frac{\partial F}{\partial \phi_\alpha}, \quad \eta_s = -\frac{\partial F}{\partial \theta}, \\ \Pi_\alpha(\mathbf{V}, \phi_\beta, \rho_\beta, \theta, \xi) &\equiv \rho_\alpha^2 \frac{\partial \psi_\alpha^\times(\rho_\alpha, \theta)}{\partial \rho_\alpha} - r_s \frac{\partial F(\mathbf{V}, \phi_\beta, \theta, \xi)}{\partial \phi_\alpha} = 0, \\ \alpha, \beta &= 1, 2, \\ \mathbf{q} &= \mathbf{R}_s^T \cdot \mathbf{q}^\times(\mathbf{V}, \phi_\beta, \theta, \xi, \boldsymbol{\gamma}^R, \mathbf{w}_\beta^R), \\ \mathbf{b}_\alpha^{\text{dis}} &= \mathbf{R}_s^T \cdot \mathbf{b}_\alpha^\times(\mathbf{V}, \phi_\beta, \theta, \xi, \boldsymbol{\gamma}^R, \mathbf{w}_\beta^R); \\ \frac{d_s \xi}{dt} &= \Omega^\times(\mathbf{V}, \phi_\beta, \theta, \xi, \boldsymbol{\gamma}^R, \mathbf{w}_\beta^R), \end{aligned} \quad (13)$$

$$\boldsymbol{\gamma}^R = \mathbf{R}_s \cdot \boldsymbol{\gamma}, \quad \mathbf{w}_\alpha^R = \mathbf{R}_s \cdot \mathbf{w}_\alpha, \quad (14)$$

$$\delta_c + \delta_f + \delta_T \geq 0, \quad \delta_c = -r_s \frac{\partial F}{\partial \xi} \Omega^\times,$$

$$\delta_f = \sum_\alpha \mathbf{b}_\alpha^\times \cdot \mathbf{w}_\alpha^R, \quad \delta_T = \theta^{-1} \mathbf{q}^\times \cdot \boldsymbol{\gamma}^R.$$

Hereinafter, $\mathbf{V} = \mathbf{V}_s$ is the skeleton tension tensor, and ρ_α is the true mass density of fluid α . The equations $\Pi_\alpha = 0$ correspond to the set of equations connecting the volume fractions and the true densities of the fluids with each other. The quantity δ_c corresponds to the dissipation of the capillary forces.

The form of Eqs. (13), (14), which is necessary and sufficient for the validity of the principles of objectiveness and thermodynamic consistency, is not unique. If we pass from the variables ϕ_α to the variables (p, S) , where $p = S p_1 + (1 - S) p_2$ is the mean porous pressure and $S = \phi_1 / \phi$, then Eqs. (13) and (14) written in terms of these variables acquire the following form:

$$\begin{aligned} \Psi &= \Psi(\mathbf{E}, p, S, \theta, \xi), \quad \mathbf{G}_s \cdot \mathbf{T} \cdot \mathbf{G}_s^T = r_s \frac{\partial \Psi}{\partial \mathbf{E}}, \\ \eta_s &= -\frac{\partial \Psi}{\partial \theta}, \\ \phi &= -r_s \frac{\partial \Psi}{\partial p}, \quad \phi p_{\text{cap}} = -r_s \frac{\partial \Psi}{\partial S}, \\ \rho_1^2 \frac{\partial \psi_1(\rho_1, \theta)}{\partial \rho_1} &= p - (1 - S) \frac{\partial \Psi / \partial S}{\partial \Psi / \partial p}, \\ \rho_2^2 \frac{\partial \psi_2(\rho_2, \theta)}{\partial \rho_2} &= p + S \frac{\partial \Psi / \partial S}{\partial \Psi / \partial p}, \\ \mathbf{q} &= \mathbf{R}_s^T \cdot \hat{\mathbf{q}}(\mathbf{E}, p, S, \theta, \xi, \boldsymbol{\gamma}^R, \mathbf{w}_\beta^R), \\ \mathbf{b}_\alpha^{\text{dis}} &= \mathbf{R}_s^T \cdot \hat{\mathbf{b}}_\alpha(\mathbf{E}, p, S, \theta, \xi, \boldsymbol{\gamma}^R, \mathbf{w}_\beta^R), \\ \frac{d_s \xi}{dt} &= \Omega(\mathbf{E}, p, S, \theta, \xi, \boldsymbol{\gamma}^R, \mathbf{w}_\beta^R), \\ \delta_c &= -\rho_s \left(\frac{\partial \Psi}{\partial \xi} \right) \Omega, \quad \delta_c + \delta_f + \delta_T \geq 0, \end{aligned} \quad (15)$$

where ϕ is the porosity, $p_{\text{cap}} = p_2 - p_1$ is the capillary pressure, $\mathbf{T} = \mathbf{T}_s - \phi p \mathbf{I}$ is the total stress tensor, and $\mathbf{E} = (\mathbf{V}^2 - \mathbf{I})/2$ is the Lagrange tensor of finite strains.

Equations (15) for the equilibrium case ($d_s \xi / dt = 0$) were considered by Coussy [3] as a form of the constitutive equations that was sufficient to reduce the inequality for the entropy to the condition $\delta_f + \delta_T \geq 0$. The constraints for the vector functions of both the heat flux and the dissipative forces were apparently not formulated until now.

The forms obtained for the constitutive equations lead to the new concept of the thermodynamics of the multiphase filtration, i.e., the dissipation of capillary forces. Below, we analyze a particular case where all dissipation forms are nonnegative. This is sufficient for the inequality involving the total dissipation to be valid. The rate of variation of the nonequilibrium parameter is assumed to be proportional to the derivative of the potential with respect to this parameter. Thus,

$$\frac{d_s \xi}{dt} = -(\tau\beta)^{-1} r_s \frac{\partial \Psi(\lambda)}{\partial \xi}, \quad \lambda \equiv (\mathbf{G}_s, p, S, \theta, \xi), \quad (16)$$

where $\tau > 0$ is the relaxation time and $\beta = \text{const} > 0$ is the constant parameter having the dimension of pressure (specific energy)

$$\delta_f = \sum \mathbf{b}_\alpha^{\text{dis}} \cdot \mathbf{w}_\alpha > 0, \quad \delta_T = \theta^{-1} \mathbf{q} \cdot \boldsymbol{\gamma} > 0 \quad (17)$$

for all $\mathbf{w}_\alpha \neq 0, \quad \boldsymbol{\gamma} \neq 0$.

The derivative $-\partial \Psi / \partial \xi$ is equal to the sum of the absorption rate for the energy spent to increase the contact surface area of the nonwetting fluid and the skeleton, as well as the energy-liberation rate due to enhancing the state of nonequilibrium. The condition $-\partial \Psi / \partial \xi > 0$ implies that approaching the equilibrium state is accompanied by energy dissipation; i.e., the energy release exceeds the absorbed energy.

As follows from (17), the dissipative forces and the heat flux in the equilibrium state are zero

$$\mathbf{q}(\lambda, 0, 0) = 0, \quad \mathbf{b}_\alpha^{\text{dis}}(\lambda, 0, 0) = 0, \quad (18)$$

and the symmetric tensors of heat conduction and resistance

$$\mathbf{k}(\lambda) = \left. \frac{\partial \mathbf{q}}{\partial \boldsymbol{\gamma}} \right|_0, \quad \mathbf{Y}_{\alpha\beta}(\lambda) = \left. \frac{1}{2} \left(\frac{\partial \mathbf{b}_\alpha^{\text{dis}}}{\partial \mathbf{w}_\beta} + \frac{\partial \mathbf{b}_\beta^{\text{dis}}}{\partial \mathbf{w}_\alpha} \right) \right|_0 \quad (19)$$

possess the properties of the positive definiteness

$$\mathbf{k} : (\mathbf{a} \otimes \mathbf{a}) > 0, \quad \sum \sum \mathbf{Y}_{\alpha\beta} : (\mathbf{b}_\alpha \otimes \mathbf{b}_\beta) > 0, \\ \mathbf{a} \neq 0, \quad \mathbf{b}_\alpha \neq 0.$$

For smooth motions of the porous medium with nonequilibrium kinetics, which is described by formula (16), the entropy production rate at the point (\mathbf{x}, t) is determined by the equation

$$\theta \sum_A r_A \frac{d_A \eta_A}{dt} = (\tau\beta)^{-1} \left(r_s \frac{\partial \Psi}{\partial \xi} \right)^2 \\ + \sum_\alpha \mathbf{b}_\alpha^{\text{dis}} \cdot \mathbf{w}_\alpha + \nabla \cdot \mathbf{q} + rQ.$$

Hence, it follows that the adiabatic process ($\mathbf{q} = 0$, $Q = 0$) for the medium under consideration is not isentropic (the Joule–Thompson effect). In the medium under study, in contrast to the equilibrium porous

medium, the entropy production rate in the adiabatic process is not only determined by the filtration dissipation, but also depends on the capillary forces and can be nonzero even in the absence of the macroscopic motion of fluids.

A POROUS MEDIUM WITH AN INCOMPRESSIBLE SKELETON

For a porous medium with an incompressible skeleton whose true density is $\rho_s = \text{const}$, it follows from the formula $\phi_1 + \phi_2 = \phi$ and from the mass-conservation law (4) that

$$1 - \phi = (1 - \phi_\kappa) J_s, \quad J_s = \det \mathbf{G}_s, \quad (20)$$

where ϕ and ϕ_κ are the current and the initial porosity. Therefore, the constitutive equations of the medium contain only one independent relationship for the volume fractions. For this relationship, we can use the saturation equation.

The constitutive relationships for the nonequilibrium porous medium having an incompressible skeleton saturated with two immiscible fluids are independent of the frame of reference and are thermodynamically consistent if and only if they have the following form:

$$\Psi_\alpha = \Psi_\alpha(\rho_\alpha, \theta), \quad p_\alpha = \rho_\alpha^2 \frac{\partial \Psi_\alpha}{\partial \rho_\alpha}, \quad \eta_\alpha = -\frac{\partial \Psi_\alpha}{\partial \theta}, \\ \alpha = 1, 2, \\ \Psi_s = \Psi(\mathbf{E}, S, \theta, \xi), \quad \mathbf{T}^{\text{eff}} \cdot \mathbf{G}_s^{-1} = -r_s \frac{\partial \Psi}{\partial \mathbf{G}_s}, \\ \eta_s = -\frac{\partial \Psi}{\partial \theta}, \quad \phi p_{\text{cap}} = -r_s \frac{\partial \Psi}{\partial S}, \\ \mathbf{q} = \mathbf{R}_s^T \cdot \mathbf{q}^\times(\mathbf{E}, p, S, \theta, \xi, \boldsymbol{\gamma}^R, \mathbf{w}_\alpha^R), \\ \mathbf{b}_\alpha^{\text{dis}} = \mathbf{R}_s^T \cdot \mathbf{b}_\alpha^\times(\mathbf{E}, p, S, \theta, \xi, \boldsymbol{\gamma}^R, \mathbf{w}_\alpha^R), \\ \frac{d_s \xi}{dt} = \Omega(\mathbf{E}, p, S, \theta, \xi, \boldsymbol{\gamma}^R, \mathbf{w}_\beta^R), \quad \boldsymbol{\gamma}^R = \mathbf{R}_s \cdot \boldsymbol{\gamma}, \\ \mathbf{w}_\alpha^R = \mathbf{R}_s \cdot \mathbf{w}_\alpha, \\ \delta_c + \delta_f + \delta_T \geq 0, \quad \delta_c = -r_s \frac{\partial \Psi}{\partial \xi} \Omega, \\ \delta_f = \sum_\alpha \mathbf{b}_\alpha^\times \cdot \mathbf{w}_\alpha^R, \quad \delta_T = \theta^{-1} \mathbf{q}^\times \cdot \boldsymbol{\gamma}^R,$$

where $\mathbf{T}^{\text{eff}} = \mathbf{T} + p\mathbf{I}$ is the effective stress tensor and $p = Sp_1 + (1 - S)p_2$ is the mean porous pressure.

The given definition for the effective stress tensor extends the concept of the effective stress tensor [10] to the case of a porous medium saturated with two immiscible fluids. This concept is generated exclusively by

the requirement of the thermodynamic consistency. In contrast to the situation that takes place, e.g., in the classical mechanics of an immiscible fluid, in the given model, no kinematic constraints arise for which the work of external forces is zero. The cause for the appearance of the effective stresses is the connection between the skeleton strains and the porosity of the medium. Therefore, in contrast to the case of Eqs. (13), it follows from Eqs. (21) that there is only one constitutive equation for the saturation, namely,

$$\begin{aligned} & \rho_2 \frac{\partial \Psi_2(\rho_2, \theta)}{\partial \rho_2} - \rho_1 \frac{\partial \Psi_1(\rho_1, \theta)}{\partial \rho_1} \\ &= -\phi^{-1} r_s \frac{\partial \Psi(\mathbf{E}, S, \theta, \xi)}{\partial S}. \end{aligned}$$

THE QUASI-LINEAR APPROXIMATION

Let the displacement of skeleton particles be small compared to the characteristic size of the body. The displacement gradient, mass-flow velocities, and variations of temperature and of the mean porous pressure are also considered to be small. Then, the strain tensor \mathbf{E} and the orthogonal rotation tensor \mathbf{R}_s are

$$\mathbf{E} = \mathbf{e} + O(\delta^2), \quad \mathbf{R}_s = \mathbf{I} - \boldsymbol{\omega} + O(\delta^2), \quad \delta \ll 1,$$

where \mathbf{e} is the small-strain tensor and $\boldsymbol{\omega}$ is the small-rotation tensor of a skeleton element.

Let \mathbf{T}_0 , p_0 , θ_0 , S_0 , ξ_0 be the total stress, the mean porous pressure, temperature, the saturation, and the nonequilibrium parameter, respectively, related to the initial state. An external action causes small displacements, as well as small variations of the mean porous pressure and temperature. The variations of the saturation and of the parameter ξ are finite. In order to allow for this feature, we introduce a reference state in which $\mathbf{E} = 0$, $p = p_0$, and $\theta = \theta_0$. The reference state differs from the initial state by the fact that, in the former, the saturation and the nonequilibrium-state parameter are equal to their current values. Upon expanding the potential in the Taylor series, we obtain

$$\begin{aligned} r_s^0 \Psi(\mathbf{E}, \Delta p, \vartheta, S, \xi) &= r_s^0 \Psi_* + \mathbf{T}_* : \mathbf{E} - \phi_* \Delta p \\ &- r_s^0 \eta_* \vartheta + \frac{1}{2} \mathbf{e} : \mathbf{L} : \mathbf{e} - \frac{1}{2} N^{-1} (\Delta p)^2 - \mathbf{B} : \mathbf{e} \Delta p \\ &- \mathbf{A}_\theta : \mathbf{e} \vartheta + \alpha_\phi \vartheta \Delta p - \frac{1}{2} \theta_0^{-1} r_s^0 c_s \vartheta^2 + O(\delta^3), \end{aligned} \quad (22)$$

where asterisks indicate that the values are taken in the reference state and the expansion coefficients are functions of (S, ξ) . Here, we took into account that $\mathbf{T}_* = r_s^0 \partial \Psi / \partial \mathbf{E}|_*$ are stresses, $\phi_* = -r_s^0 \partial \Psi / \partial p|_*$ is the porosity, $\eta_*^* = -\partial \Psi / \partial \theta|_*$ is the entropy in the reference state, $\mathbf{L} = r_0^2 \partial^2 \Psi / \partial \mathbf{E} \otimes \partial \mathbf{E}|_*$ is the elastic mod-

ulus tensor, $c_s = -\theta_0 \partial^2 \Psi / \partial \theta^2|_*$ is the heat capacity, $\mathbf{A}_\theta = -r_s^0 \partial^2 \Psi / \partial \theta \partial \mathbf{E}|_*$ is the tensor of thermoelastic coefficients, $\alpha_\phi = r_s^0 \partial^2 \Psi / \partial p \partial \theta|_*$ is the thermal coefficient of porosity, $\mathbf{B} = -r_s^0 \frac{\partial^2 \Psi}{\partial p \partial \mathbf{E}}|_*$ is the tensor of Biot moduli, and $N^{-1} = -r_s^0 \partial^2 \Psi / \partial p^2|_*$ is the Biot scalar coefficient.

If the stresses in the reference state are small compared to the elastic moduli, then in this case, the total stress, porosity, entropy, and the capillary pressure are of the form

$$\begin{aligned} \mathbf{T}(\mathbf{e}, \Delta p, \vartheta, S, \xi) &= \mathbf{T}_*(S, \xi) + \mathbf{L}(S, \xi) : \mathbf{e} \\ &- \mathbf{A}_\theta(S, \xi) \vartheta - \mathbf{B}(S, \xi) \Delta p, \end{aligned} \quad (23)$$

$$\begin{aligned} \phi(\mathbf{e}, \Delta p, \vartheta, S, \xi) &= \phi_*(S, \xi) + \mathbf{B}(S, \xi) : \mathbf{e} \\ &+ \frac{\Delta p}{N(S, \xi)} - \alpha_\phi(S, \xi) \vartheta, \end{aligned}$$

$$\begin{aligned} r_s^0 \eta_s(\mathbf{e}, \Delta p, \vartheta, S, \xi) &= r_s^0 \eta_*(S, \xi) + \mathbf{A}_\theta(S, \xi) : \mathbf{e} \\ &- \alpha_\phi(S, \xi) \Delta p + \frac{r_s^0 c_s(S, \xi) \vartheta}{\theta_0}, \end{aligned} \quad (24)$$

$$-\phi p_{\text{cap}} = r_s^0 \frac{\partial \Psi_*}{\partial S} + \mathbf{e} : \frac{\partial \mathbf{T}_*}{\partial S} - \Delta p \frac{\partial \phi_*}{\partial S} - r_s^0 \vartheta \frac{\partial \eta_*}{\partial S}.$$

In the case under consideration, kinetics equation (16) can be rewritten as

$$\begin{aligned} \frac{\partial \xi}{\partial t} &= \frac{1}{\tau \beta} \frac{\partial}{\partial \xi} [r_s^0 \Psi_*(S, \xi) + \mathbf{T}_*(S, \xi) : \mathbf{e} \\ &- \phi_*(S, \xi) \Delta p - r_s^0 \eta_*(S, \xi) \vartheta]. \end{aligned} \quad (25)$$

If the time of the state variation is large compared to the relaxation time τ , then the equilibrium variation of the parameter ξ occurs so that the right-hand side of Eq. (25) vanishes. This means that, in this process, the nonequilibrium-state parameter is a function of the current strain, pressure, temperature, and the saturation.

In the linear approximation, the heat flow and the dissipative interaction forces with allowance for (18) and (19) are determined by the formulas

$$\mathbf{q} = \mathbf{k}(S, \xi) \cdot \nabla \theta, \quad \mathbf{b}_\alpha^{\text{dis}} = \sum \mathbf{Y}_{\alpha\beta}(S, \xi) \cdot \mathbf{w}_\beta, \quad (26)$$

where \mathbf{k} , $\mathbf{Y}_{\alpha\beta}$ are the tensors of the heat conduction and of the resistance. In applications, it is more convenient to use the tensors of the true permeability

$$\mathbf{K}_{\alpha\beta} = \phi_\alpha \phi_\beta (\mu_\alpha \mu_\beta)^{1/2} \mathbf{Y}_{\alpha\beta}^{-1}(S, \xi), \quad (27)$$

rather than the resistance tensors, where μ_α are the dynamic viscosities of the fluids.

Relationship (27) allows for the most important feature of the combined filtration, namely, vanishing the permeability of one fluid when the medium is completely saturated with the other fluid.

Using representation (7) for the interaction forces and relationships (26) and (27), we can rewrite Eq. (5) of motion in the form of the generalized Darcy law for the multiphase filtration

$$\mathbf{W}_\alpha = - \sum_{\beta} (\mu_\alpha \mu_\beta)^{-1/2} \mathbf{K}_{\alpha\beta} \cdot (\nabla p_\beta - \rho_\beta (\mathbf{g} - \mathbf{a}_\beta)), \quad (28)$$

$$\mathbf{W}_\alpha = \phi_\alpha \mathbf{w}_\alpha,$$

which contains crossover terms corresponding to the interaction of the fluids occurring in the same capillary channel.

FILTRATION OF FLUIDS IN A RIGID SKELETON

Below, we will ignore the skeleton strain and temperature variations and consider the case of incompressible fluids and the homogeneous and isotropic skeleton. We will consider the porosity ϕ and the permeability k having constant values. We also assume the constant viscosity of the fluids ($\mu_\alpha = \text{const}$). We take the potential Ψ_* entering into expansion (22) in the form

$$r_s^0 \Psi_*(S, \xi) = \frac{1}{2} \beta \xi^2 + \frac{1}{2} M s^2 - \alpha \xi s + \gamma s, \quad (29)$$

$$s = 1 - S,$$

where r_s^0 , α , β , M , and $\gamma = \text{const} > 0$. The capillary pressure (24) is determined from the relation

$$\phi p_{\text{cap}}(s, \xi) = \gamma + M s - \alpha \xi. \quad (30)$$

Here, γ is the input capillary pressure and M is the instantaneous modulus of the capillary pressure. Equation (25) corresponding to potential (29) has the following form

$$\dot{\xi} = \frac{\alpha s - \beta \xi}{\tau \beta}. \quad (31)$$

The parameter ξ and the capillary pressure in the equilibrium state are determined from the following relations

$$\xi_{\text{eq}}(s) = \frac{\alpha s}{\beta}, \quad \phi p_{\text{cap}}(s) = \gamma + M_{\text{eq}} s, \quad (32)$$

$$M_{\text{eq}} = M - \frac{\alpha^2}{\beta},$$

where M_{eq} is the equilibrium modulus of the capillary pressure.

As follows from expressions (30)–(32), the maximum width of the hysteresis loop that envelops the equilibrium dependence $p_{\text{cap}}(s)$ in the plane (p_{cap}, s) is proportional to the variation of the saturation Δs with

the coefficient $\alpha^2/\beta = M - M_{\text{eq}}$. The maximum slope of the hysteresis curve is determined by the instantaneous modulus M .

The other important corollary of formulas (30), (31) is the existence of the residual saturations s_0 and s_1 . This follows from the conditions that the capillary pressure cannot be lower than the input one and must not exceed the equilibrium value for $s = 1$. The constraints for the residual saturations can be written out in the form

$$s_1 \leq \frac{M_{\text{eq}}}{M \left(1 + \frac{\alpha^4}{\beta^2 M^2} \right)}, \quad s_0 \geq \frac{\alpha^2 s_1}{\beta M}.$$

Let $\mathbf{W} = \mathbf{W}_1 + \mathbf{W}_2$ be the vector of the total filtration, where $\mathbf{W}_\alpha = \phi_\alpha \mathbf{w}_\alpha$ are the partial-filtration vectors. The field $\mathbf{W}(\mathbf{x}, t)$ is solenoidal, i.e.,

$$\nabla \cdot \mathbf{W} = 0, \quad (33)$$

and the saturation satisfies the parabolic equation

$$\frac{\partial \phi S}{\partial t} + \nabla \cdot [B(S, \xi) \mathbf{W} + n(S, \xi) \nabla p_{\text{cap}}(S, \xi) + \mathbf{g} G(S, \xi)] = 0,$$

where

$$B = n(m_{22} - m_{21}), \quad n = (m_{11} - 2m_{21} + m_{22})^{-1},$$

$$G = kn \sum_{\beta} \sum_{\lambda} \frac{(m_{1\beta} - m_{2\beta}) f_{\beta\lambda} \rho_\lambda}{(\mu_\beta \mu_\lambda)^{1/2}},$$

$m_{\alpha\beta}$ is the matrix inverse to the conductance matrix $k_{\alpha\beta} = k f_{\alpha\beta} / (\mu_\alpha \mu_\beta)^{1/2}$, k is the absolute permeability, and $f_{\alpha\beta}(S, \xi)$ is the relative phase permeability.

If the relative phase permeabilities and the functions B , n , and G depend on the effective saturation $\omega = S + \alpha \xi / M$, then in this case,

$$\frac{\partial \phi \omega}{\partial t} + (B'(\omega) \mathbf{W} + G'(\omega) \mathbf{g}) \cdot \nabla \omega - \nabla \cdot (A(\omega) \nabla(\omega)) = \frac{\alpha \phi \Lambda(\omega, \xi)}{\tau M}, \quad (34)$$

$$\frac{\partial \xi}{\partial t} = \tau^{-1} \Lambda(\omega, \xi),$$

$$\Lambda(\omega, \xi) = \frac{\alpha(1-\omega)}{\beta} - \frac{M_{\text{eq}} \xi}{M}, \quad A(\omega) = \phi^{-1} M n(\omega).$$

The system of equations (34) has three basic differences from the Muscat–Leverett model. Instead of the saturation S , the effective saturation ω is used that corresponds to the shift of the permeability towards greater saturations. The right-hand side of the equations for the saturation and kinetics, which by virtue of the condition

$\xi \leq \xi_0 = \alpha(1 - S)/\beta$ are nonnegative, provide the relaxation of the capillary pressure for the fixed saturation S . The functions $B(\omega)$, $A(\omega)$, $G(\omega)$ are determined by three rather than two relative phase permeabilities, which allows for the interactions between the phases. We call Eqs. (33) and (34) the modified Muscat–Leverett model.

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