Simulation of Two-Phase Flow in Porous Media using a Two-Dimensional Network Model

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Contents

Chapter 1

Introduction

1.1 Practical Application

Simulation of two-phase flow in porous media has applications in oil recovery, hydrology, electricity production where pressurized water is passed through heated pipes and is transformed into steam, etc. The model and algorithm presented here is used to find the saturation of a phase with respect to time and model imbibition.

1.2 Other methods of simulating two-phase flows

[NEED to add - all other similar models, what these models have achieved, and how this model is different from them]

1.3 Models already developed

[UNDER-CONSTRUCTION]

Chapter 2

Theory

2.1 Viscosity

[*Bhuvan: please add here the 'diagram-definition-viscosity']

Viscosity is defined as,

$$\frac{F}{A} = \mu \left| \frac{dv}{dr} \right| \tag{2.1}$$

F: shearing force on a surface of the layer, this force acts parallel to the surface plane.
A: the area of the surface.
μ: coefficient of viscosity.
v: velocity of the flow parallel to the plane.
r: coordinate perpendicular to the plane.

Dimension of viscosity,

$$[\mu] = \frac{[F]}{[A]} \frac{[r]}{[v]}$$

$$[\mu] = \frac{[ML/T^2]}{[L^2]} \frac{[L]}{[L/T]}$$

$$[\mu] = \frac{M}{LT} = \frac{kg}{m.s} \tag{2.2}$$

2.2 Volumetric flow rate through a thin tube

[*Bhuvan: please add here the 'diagram-flow-rate-cylinder']

Rearranging 2.1, the viscous force is given by,

$$F_{vis} = A\mu \left| \frac{dv}{dr} \right| \tag{2.3}$$

The force due to pressure gradient, for a cross-sectional area S,

$$F_p = S\Delta P \tag{2.4}$$

For a laminar flow, the force due to pressure gradient is compensated by the viscous resistance in the opposite direction, which also means that the fluid does not acclerate.

$$F_p = F_{vis} \tag{2.5}$$

Substituting 2.3 and 2.4 into 2.5,

$$S\Delta P = A\mu \left| \frac{dv}{dr} \right|$$

The velocity is maximum in the centre, and zero at the boundaries. Which implies, that the velocity decreases with increasing radius. $\left|\frac{dv}{dr}\right|$ can be replaced by $-\frac{dv}{dr}$.

$$S\Delta P = -A\mu \frac{dv}{dr}$$

Substituting these $S = \pi r^2$, and $A = 2\pi r l$,

$$(\pi r^2)\Delta P = -(2\pi r l)\mu \frac{dv}{dr}$$

$$(r)\Delta P = -(2l)\mu \frac{dv}{dr}$$

$$-2dv = \frac{\Delta P}{\mu l} r dr$$

$$-2\int dv = \frac{\Delta P}{\mu l} \int r dr$$

$$-2v + C = \frac{\Delta P}{\mu l} \frac{r^2}{2}$$

At $r = R$, $v = 0$

$$-2(0) + C = \frac{\Delta P}{\mu l} \frac{R^2}{2}$$

$$C = \frac{\Delta P}{\mu l}\frac{R^2}{2}$$

$$-2v + \left(\frac{\Delta P}{\mu l}\frac{R^2}{2}\right) = \frac{\Delta P}{\mu l}\frac{r^2}{2}$$

$$2v = \frac{\Delta P}{\mu l}\frac{R^2}{2} - \frac{\Delta P}{\mu l}\frac{r^2}{2}$$

$$v(r) = \frac{\Delta P}{4\mu l}(R^2 - r^2) \tag{2.6}$$

In order to find the volumetric flow rate Q in a tube, we integrate for each thin strip or ring $2\pi r dr$

$$Q = \int_S v dS = \int_0^R v(r)(2\pi r)dr$$

$$Q = 2\pi \int_0^R v(r) r dr$$

Substituting $v(r)$ from 2.6

$$Q = 2\pi \int_0^R \left(\frac{\Delta P}{4\mu l}(R^2 - r^2)\right) r dr$$

$$Q = \frac{\pi \Delta P}{2\mu l}\int_0^R (R^2 r - r^3) dr$$

$$Q = \frac{\pi \Delta P}{2\mu l}\left[\frac{R^2 r^2}{2} - \frac{r^4}{4}\right]_0^R$$

$$Q = \frac{\pi \Delta P}{2\mu l}\left(\frac{R^4}{2} - \frac{R^4}{4}\right)$$

$$\boxed{Q = \frac{\pi}{8}\frac{\Delta P}{\mu l}R^4} \tag{2.7}$$

2.3 Capillary Action

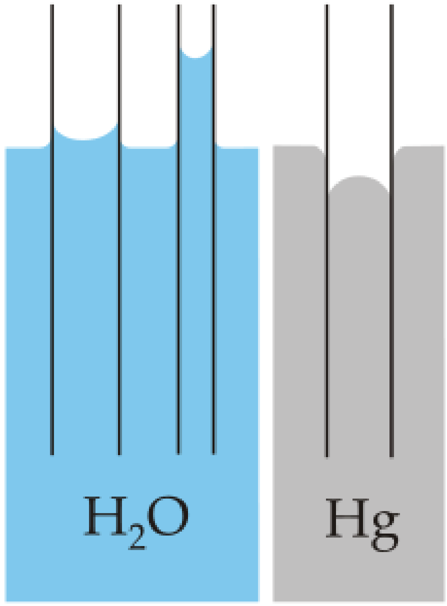


Figure 2.1: Showing capillary action of water (polar) compared to mercury (non-polar), with respect to a polar surface such as glass (Si–OH).

Let us apply this to our case, where the first node is filled with a fluid like water and the second node is filled with a fluid like air, and the our tube is similar to glass. Hence the meniscus will be oriented in a manner shown below.

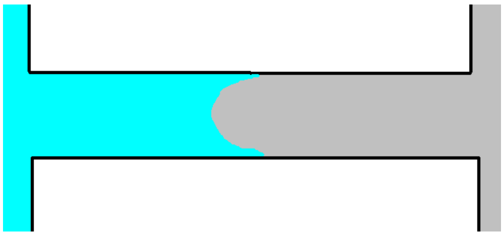


Figure 2.2: Orientation of meniscus, when one of its side is filled with wetting fluid when the other side is filled with a non-wetting fluid.

This causes the water to climb against gravity.

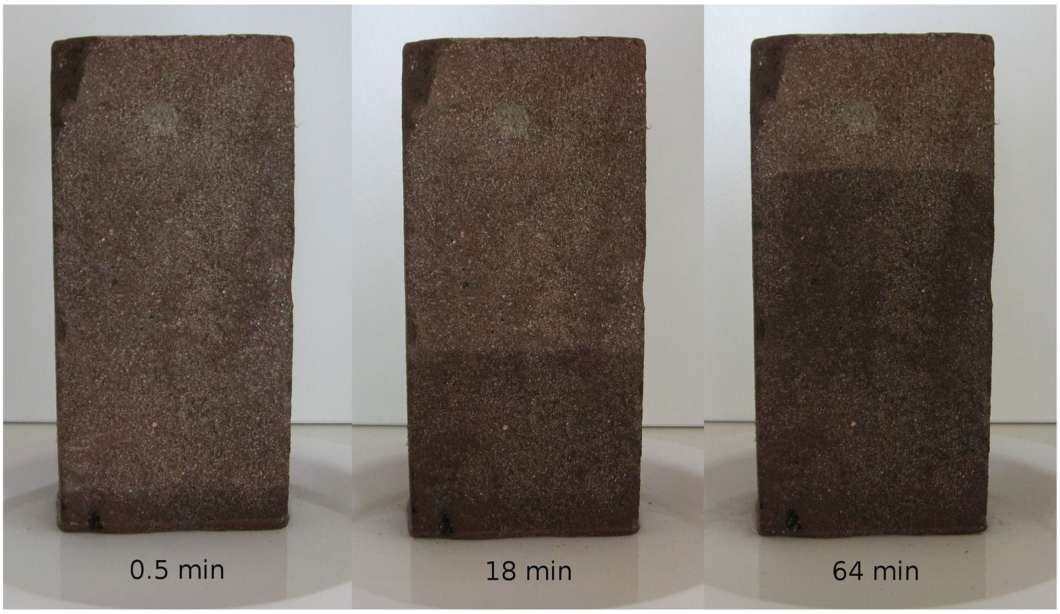


Figure 2.3: Water climbing against gravity through a porous medium.

2.4 Flow rate in a tube containing meniscus

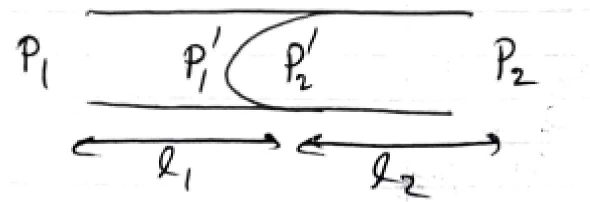


Figure 2.4: Pressures in a tube with two phases.

Let there be a higher pressure in P_1 than P_2, the fluid in node which has P_1 produces a meniscus which tends to move towards the second node. We can break it down into two separate tubes of lengths l_1 and l_2, containing fluid of viscosites μ_1 and μ_2. Then the flow rates for each of the tubes are given by:

$$Q_1 = \frac{\pi}{8\mu_1} \frac{P_1 - P_1^{'}}{l_1} R_1^4 \tag{2.8}$$

$$Q_2 = \frac{\pi}{8\mu_2} \frac{P_2^{'} - P_2}{l_2} R_2^4 \tag{2.9}$$

Multiplying equations 2.8 and 2.9 by $\mu_i l_i$

$$Q_1 \mu_1 l_1 = \frac{\pi}{8}(P_1 - P_1^{'}) R_1^4 \tag{2.10}$$

$$Q_2\mu_2 l_2 = \frac{\pi}{8}(P_2^{'} - P_2)R_2^4 \tag{2.11}$$

Due to continuity, which means no vacuum or fluid can be created, $Q_1 = Q_2$. Since it is the same tube, $R_1 = R_2$. Adding equation 2.10 and 2.11, we get:

$$Q(\mu_1 l_1 + \mu_2 l_2) = \frac{\pi}{8}R^4(P_1 - P_2 + P_2^{'} - P_1^{'}) \tag{2.12}$$

In figure 2.1 the water rises because there is a pressure jump at the meniscus, the pressure is lower on the side of the water. Therefore in our case $P_2^{'} - P_1^{'}$ will have a positive value. Equation 2.12 becomes:

$$Q = \frac{\pi R^4}{8(\mu_1 l_1 + \mu_2 l_2)}\left(\Delta P + \frac{2\sigma}{R}\right) \tag{2.13}$$

It is clear that $Q > Q'$, where Q' is the flow without the meniscius.

2.5 The sign of capillary pressure

Let the node on which we are generating linear equations be N_i and the node connected by a tube be N_j, if the concave side of the meniscus points towards N_j from N_i, then let us say that the meniscus points away from N_i or simply points away and in the case of opposite orientation points towards. Let the sign due to the orientation of meniscus be decided by a function called $s(d, n_{mns})$, where d is the direction or orientation, and n_{mns} is the number of meniscus in the tube:

$$s(d, n_{mns}) = \begin{cases} -1, & \text{points towards, } n_{mns} = 1 \\ 0, & n_{mns} = 0,\, 2 \\ +1, & \text{points away, } n_{mns} = 1 \end{cases} \tag{2.14}$$

Equation 2.13 can be written as:

$$Q = \frac{\pi R^4}{8(\mu_1 l_1 + \mu_2 l_2)}\left(\Delta P + \frac{2s\sigma}{R}\right) \tag{2.15}$$

2.6 Flow Rate Equation used in our network model

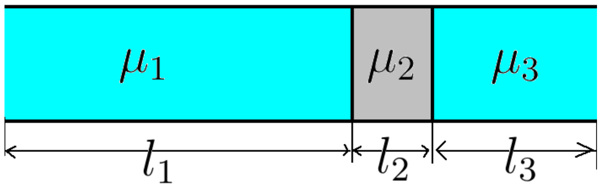


Figure 2.5: designation of viscosity and corresponding lengths of phases in a tube

Q_{ij} is the flow from N_i to N_j and $\Delta P_{ij} = P_i - P_j$.

$$\boxed{Q_{ij} = \frac{\pi R_{ij}^4}{8M_{ij}l}\left(\Delta P_{ij} + \frac{2s_{ij}\sigma}{R_{ij}}\right)} \tag{2.16}$$

Here M_{ij} is:

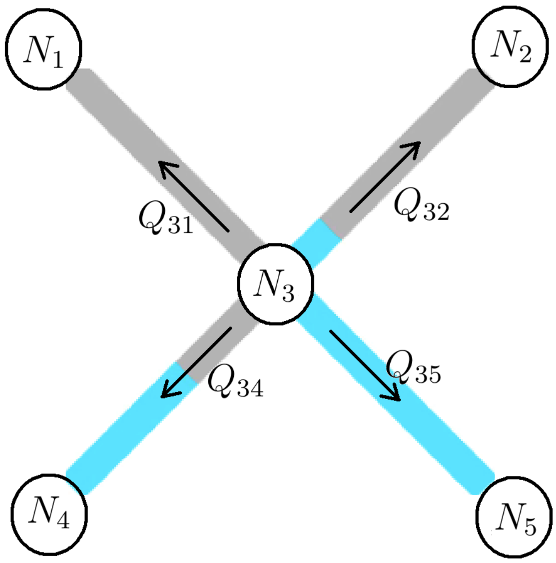
$$M_{ij} = \sum_k \mu_{ijk}\frac{l_{ijk}}{l}$$

It is clear that

$$Q_{ij} = -Q_{ji} \tag{2.17}$$

2.7 Flow rate linear equations for a node

Let us apply our method on a simple system consisting of only 5 nodes.



Since there are 4 tubes, we can write 4 equations according to 2.16

$$\begin{aligned}
Q_{31} &= \frac{\pi R_{31}^3}{8lM_{31}}(R_{31}\Delta P_{31} + 2s_{31}\sigma) \\
Q_{32} &= \frac{\pi R_{32}^3}{8lM_{32}}(R_{32}\Delta P_{32} + 2s_{32}\sigma) \\
Q_{34} &= \frac{\pi R_{34}^3}{8lM_{34}}(R_{34}\Delta P_{34} + 2s_{34}\sigma) \\
Q_{35} &= \frac{\pi R_{35}^3}{8lM_{35}}(R_{35}\Delta P_{35} + 2s_{35}\sigma)
\end{aligned}$$

Due to the conservation of volume, we have:

$$\sum_k Q_{3k} = 0$$

Where $k = 1, 2, 4, 5$.

When generating the linear equations. For each row it is necessary to do the following for each direction:

$$\begin{aligned}
[P_i]+ &= R_{ij}K_{ij} \\
[P_j]- &= R_{ij}K_{ij} \\
[const]- &= 2s_{ij}\sigma K_{ij}
\end{aligned}$$

Here let $K_{ij} = R_{ij}^3/M_{ij}$.

For simplicity we rewrite the system of four euqations as ... [UNDER Construction]

In case of 5 nodes in our system, where the pressure of the bottom and top nodes are given and fixed, the matrix for Gaussian elimination will be:

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & P_{up} \\
0 & 1 & 0 & 0 & 0 & P_{up} \\
-R_{31}K_{31} & -R_{32}K_{32} & (R_{3k}K_{3k} + ...) & -R_{34}K_{34} & -R_{35}K_{35} & -2\sigma(s_{3k}K_{3k} + ...) \\
0 & 0 & 0 & 1 & 0 & P_{down} \\
0 & 0 & 0 & 0 & 1 & P_{down}
\end{pmatrix}$$

It can be proven that this matrix always has a solution. Once the solution is determined the flow rate can be calculated using equation 2.16, and the velocity of flow in each tube is given by

$$\boxed{v_{ij} = \frac{R_{ij}}{8lM_{ij}}(R_{ij}\Delta P_{ij} + 2s_{ij}\sigma)} \tag{2.18}$$

2.8 Numbering tubes and other parameters in the network model

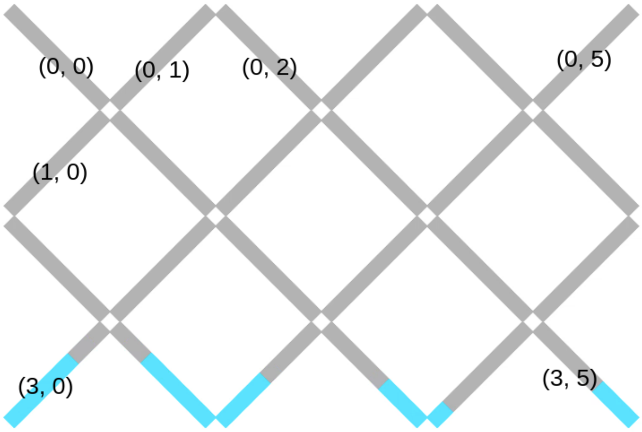


Figure 2.6: Numbering of tubes for a (rows, cols) = (4, 6) network model

If X is a parameter then X_{ij} is on the i th row and j th column, note that counting begins from zero. This type of counting is suitable for std::vector¡ data structure in C++.

Chapter 3

Model

3.1 Difficulty Modeling Imhibition

3.1.1 Main problem now

When we solve the system of linear equations, more than one row can be zeros.

This problem rarely occurs when there is a larger pressure difference between the top and bottom row of nodes. Rarely because it is possible to have a subsystem even when there is a significant pressure difference between the top and bottom nodes, where the system of linear equations have infinitely many solutions.

3.1.2 The flow does not start in the nodes

The flow did not start when all the meniscus were located inside the nodes. Because in our model we assumed that there is not capillary pressure in the nodes. To overcome this, the meniscus were made to be situated inside the tubes.

3.1.3 The flow does not start, all meniscus midway in thick tubes

The solution of linear equation were such that, the capillary force balanced out the pressure gradient. The pressure was much higher outside in the thick tubes than in the thin tubes. [NEED TO VERIFY] whether this was caused by error in the process of solving the linear equation or due to the initial setup, needs to be checked again. Error is, that it is impossible to determine whether the coefficient during the process of gauss elimination is zero or not. Because of the way how floating point numbers are handled by the CPU, 0 is often seen as a small number, for example

3.2 Description of the Model

The algorithms and methods used to simulate two-phase flow in porous media has many practical applications in oil recovery, hydrology, electricity production where pressurized water is passed through heated pipes and is transformed into steam, etc. Our algorithm presented here is used to find the saturation of a phase with respect to time, the hysteresis curve when the pressure across the porous body is reversed, total capillary pressure as a function of saturation[4], and determination of permeability which appears in Darcy's law.

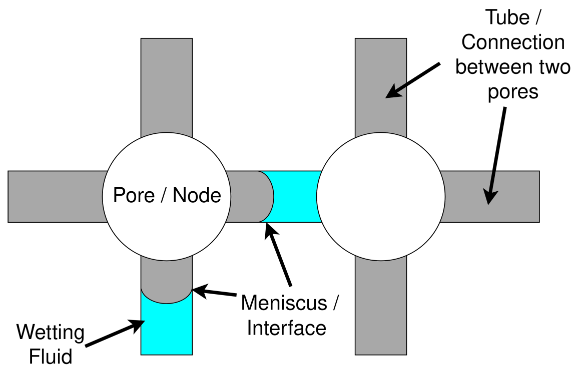


Figure showing two nodes from the network where the size of the node is much larger than the radius such that the capillary force tends to zero when the meniscus enters a node.

3.3 Algorithm steps !UNDER CONSTRUCTION

1. **Input Files:** read input files, radius.txt and mns.txt, mns.txt contains the initial setup of the meniscus
2. **Random radius:** add very small random values to the radius, this is done in order to remove the case of two equal radius for simplicity, can be removed later
3. **Loop time:** do until a certain proportion of invading fluid is reached for example 0.90:
   (a) **Pressure:** determine the pressure at each node using the linear equations given in section 2.7.
   (b) **Velocity:** Calculate the velocity using equation 2.18
   (c) **time step:** determine the time step, the time step is calculated such that, it is the minimum of the time taken for a meniscus to reach a node it is heading towards. It is calculated by iterating through all the tubes, for a tube the time is determined using the $t_{ij} = x_{ij}/v_{ij}$, here x_{ij} is the distance between the node and and the meniscus closest to it, if the fluid is traveling upwards then it is the node located on the top of the tube, if the velocity is downwards then the node which is at the bottom is used. In case there is no meniscus present, $x_{ij} = l_{ij}/4$ is used, it is because a meniscus can enter the tube during the time step and this meniscus

must not reach the next node, it can happen if the velocity is high and the radius is small. Also if $x_{ij} > l_{ij}/4$, then $x_{ij} = l_{ij}/4$ is used. This is done in order to smoothen the integration. In the video $l_{ij}/4$ is used, the number can be increased.

(d) **volume:** The volume displaced in each tube is determined by iterating through all the tubes, $V_{ij} = v_{ij} * t_{min}$.

(e) **integration:**

i. **Store insertion:** create a matrix to store how much of which fluid to insert in each of these tubes.

ii. **Loop nodes:** Iterate through all the nodes, and for each of the nodes.

A. divide the tubes into two categories, flow-in-tube - here the fluid from these tubes flow into the nodes, flow-out-tubes here we insert the fluid into the tube from the node

B. Find out the total of fluid1, fluid2, which is the total of each fliud from all flow-in-tubes.

C. Start filling the each of the flow-out-tubes where the flow will go into in ascending order of the radius of the tube. This will be done simply be adding the quantity of fluid1 and fluid2 to the matrix created above.

D. while filling fist use fluid1, once fluid1 is used up then start using fluid2, which means if in a tube we have to insert two fluids, then fluid1 will go in first.

iii. **Fluid addition:** For each of the tubes, add the volume of fluid determined to be added. After addition if there are more than 2 meniscus, then merge them retaining their center of masses.

(f) **Picture:** Save a picture of the current configuration.

4. Video: Make a video file from the pictures.

Chapter 4

Result

4.1 New Results 2023-06-12

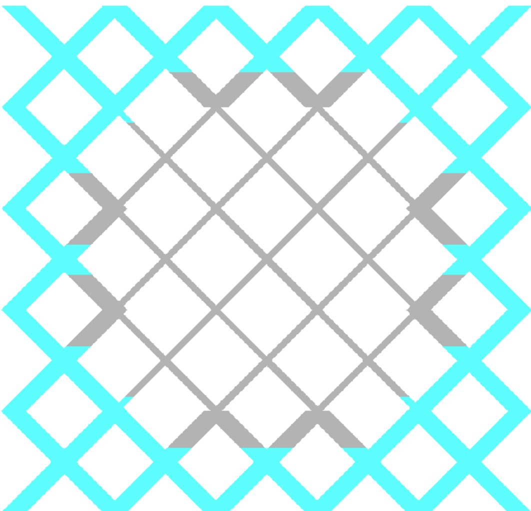


Figure 4.1: Initial setup, outer radius is 3 times larger than inner.

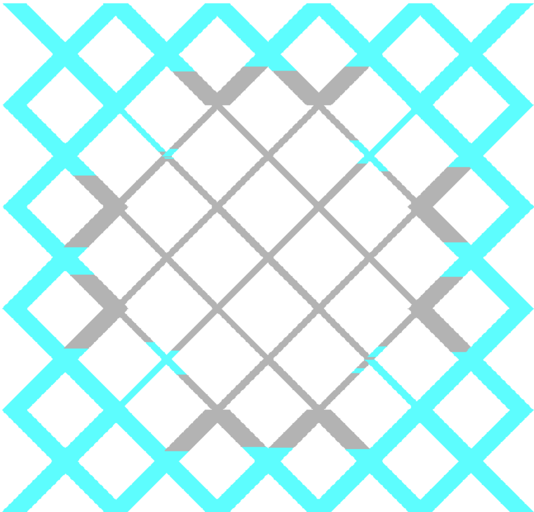


Figure 4.2: Showing invasion of wetting(blue) fluid into the region which contains thinner radius. The flow acclerates because, for a corner initially there are 3 meniscus, it multiplies into 3 when the meniscus reaches the node. The corner where the meniscus reaches the node late is pushed back because of the excessive pressures from the other corners.

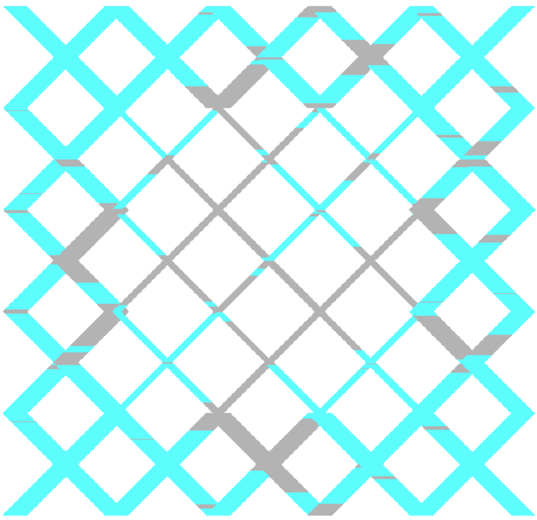


Figure 4.3: The invasion slows down and possibly ossicilates, it is due to the meniscus in the inner region being ineffective to suck more blue fluid as most tubes have two meniscus. In our algorithm, tubes with two meniscus have a zero net pressure.

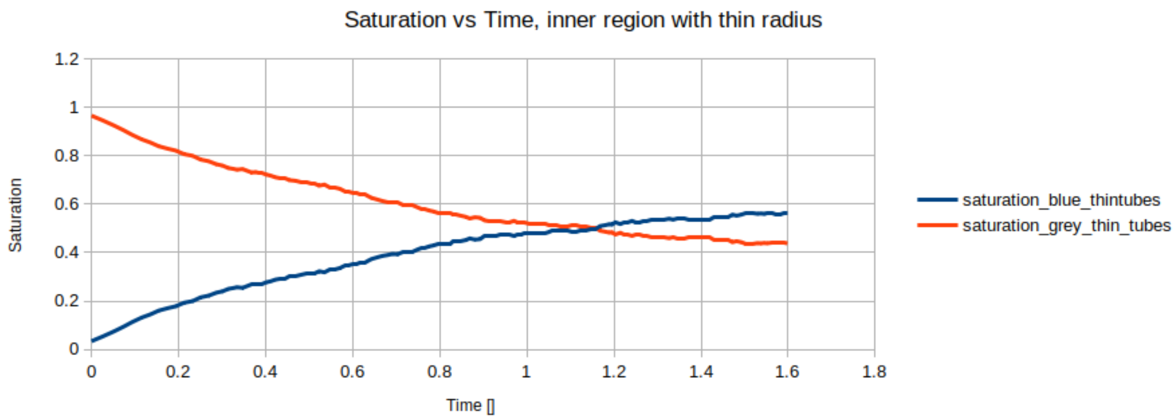


Figure 4.4: Plot of saturation of blue fluid in the region of thinner radius with respect to time, here the dimension of time is arbitary.

4.1.1 Discussion

There may be minor inaccuracies in the code, mostly in the part of distributing phases in the nodes. Even if there are minor mistakes.

1. The total volume for each phase for the whole system remains the same with the accuracy of 10^-9.
2. The blue fluid has approximately logarithmic dependence with time, the invasion rapidly rises and slows down, until it becomes almost constant. The calculation was stopped after 150 steps, becasue there was very small progress after it. Note that the time step for each step is different.
3. The blue fluid enters upto 0.56 of the saturation.
4. The saturation vs time appears similar to the ones in the reference [1], [2].

4.1.2 Initial conditon

1. All calculations are done in double.
2. For $K = 0.2$ the size of time-step. Or, if the velocity of all tubes were equal for all moment of time, then it would take 5 steps for all initial fluid to flow out of a tube.
3. To save time for now, it was done on a 10x10 tubes.
4. Allowing 2 tubes on the outer, the saturation of blue was 0.77 while saturation of grey was 0.22 for the whole system, if we make the size bigger like 30x30 then we can achieve a number closer to 0.50, 0.50 which we ilitially wanted.
5. Random numbers were added to the radius, the inner radius are approximately 3 times smaller than the outer radius. The random value changed a maximum

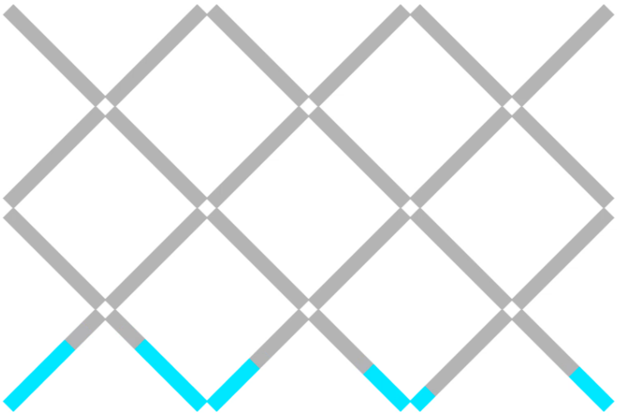
of 0.01.

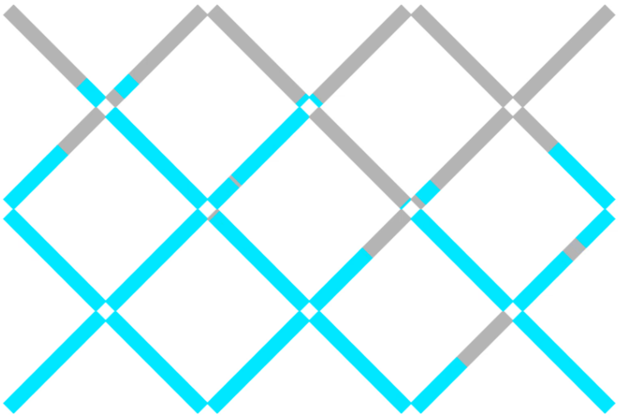
6. The node at the centre was chosen to have 0 pressure. Which is necessary for the linear equations to have a solution.

7. blue,grey,blue,grey was combined to blue,grey,blue.

8. viscosities of both fluids are equal

4.2 Old Results, shown at the 65 mipt conference





Our model is initially set up such that the wetting fluid is low in saturation and is confined to the bottom of our network. A higher pressure is fixed for all nodes at the bottom layer, while a low pressure is fixed for the top row. In all nodes, law of conservation of volume is applied, since mass is conserved and the phases are non-compressible. However for the bottom layer of nodes, the wetting fluid is injected as much required according to the sum of flow rates determined in the tubes connected to those nodes, while from the top layer of nodes a fluid is removed.

Chapter 5

Conclusion

This algorithm can be extended to the case where there are more than 4 tube connections to a node, since for two phase flow into a node case, we distribute in an ascending order of radii, in our model it is distributed to a maximum number to two tubes, but for hexagonal model it can be 4. We only need to update the function which produces the connections. The same model can be used for a 3-dimensional case[2], where one surface has higher pressure than the opposite surface which has a lower pressure, it is to be used in order to more accurately represent the porous body.

Chapter 6

Reference

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Chapter 7

Appendix-sample

This is a sample appendix.