Dynamic Study of the Lorenz System

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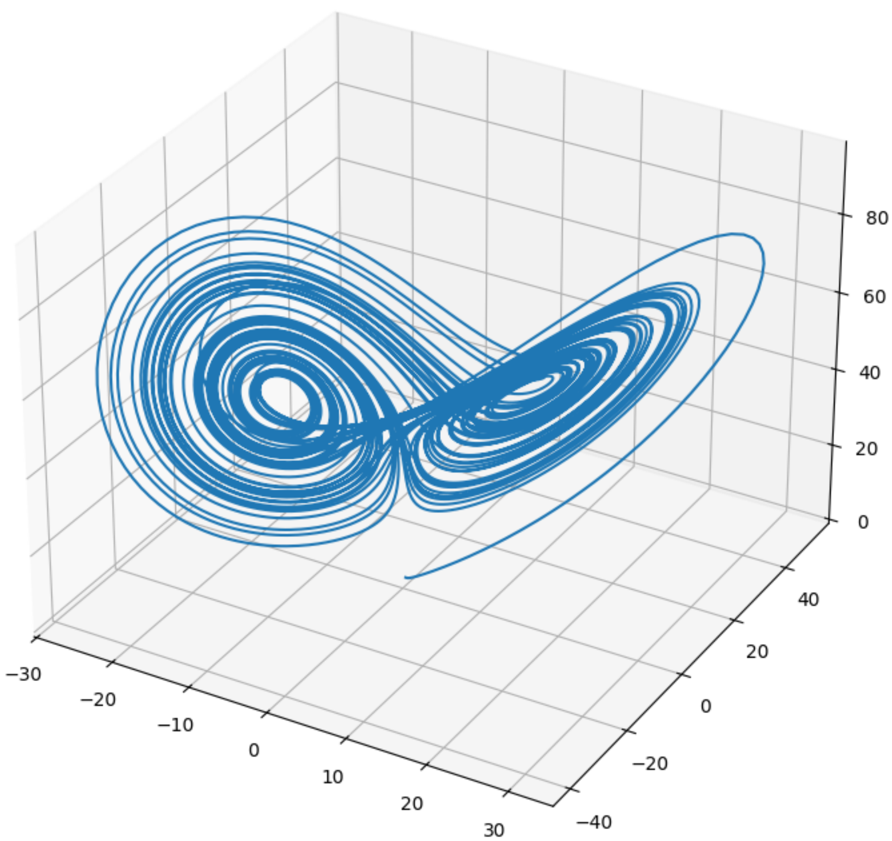

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1 Introduction

In 1963, the mathematician and meteorologist Edward Norton Lorenz was studying the dynamics of a fluid in a two-dimensional cell. The fluid was heated from below and cooled from above, resulting in convection movements also known as *Rayleigh-Bénard convection*. As the solutions of this system were described by differential equations with many parameters, notably the Navier-Stokes equation for Newtonian fluids, Lorenz decided to simplify the system as much as possible. This almost exaggerated simplification (since it has almost no link left with reality) created a system that defied logic. Indeed, all the system parameters are known, so according to the principle of Laplacian determinism at the time, all solutions should be predictable. However, the Lorenz system yielded totally unpredictable solutions, and the representation of its solutions has the shape of butterfly wings (or a figure-eight). Moreover, the solutions differ even when the starting parameters are close. This is when chaos theory was born, which is the very character of the Lorenz system.
The purpose of this study is more exploratory than innovative, and we will analyze the system, its equilibrium and stability, and we will finally talk about this notion of the *Strange Attractor*, quite strange itself!

2 Mathematical Modeling of the System

2.1 Mathematical formulation

The Lorenz system is a system of 3 first-order ordinary differential equations, defined by:

$$\begin{cases} \frac{dx}{dt} = \sigma(y - x) \\ \frac{dy}{dt} = x(\rho - z) - y \\ \frac{dz}{dt} = xy - \beta z \end{cases}$$

with variables:

- x the intensity of convection movement
- y the horizontal temperature difference between ascending and descending currents
- z the vertical temperature deviation

and parameters:

- σ the *Prandtl number* (representing the viscosity or thermal diffusivity of the system)
- ρ the *normalized Rayleigh number*
- β the geometric factor of the convection cell

The parameters are all considered positive, with additionally: $\sigma > b + 1$. We will denote this system as $X' = L(X)$.

2.2 The Lorenz Attractor

When Lorenz first studied this system, he considered the parameters:

- $\sigma = 10$
- $b = \frac{8}{3}$
- $r = 28$

These parameters will be the ones used throughout this section. Considering the initial conditions $P_0 = (0, 1, 1.05)$, we obtain figure 1.

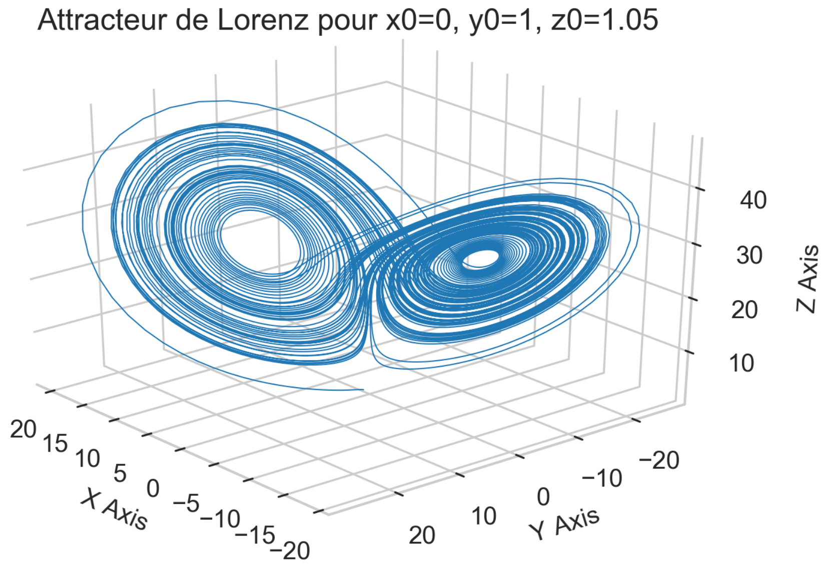

Figure 1: Lorenz system for $P_0 = (0, 1, 1.05)$

It is observed that the solutions seem to rotate around a point alternately, in concentric circles without ever touching. The solutions therefore tend to remain in this same set by encircling two unknown points, which is called the **Lorenz Attractor**. This is the first characteristic property of this system.

2.3 Sensitivity to initial conditions

As previously mentioned, all solutions of this system tend to revolve around an attractor, regardless of the initial conditions (see figures 2 and 3).

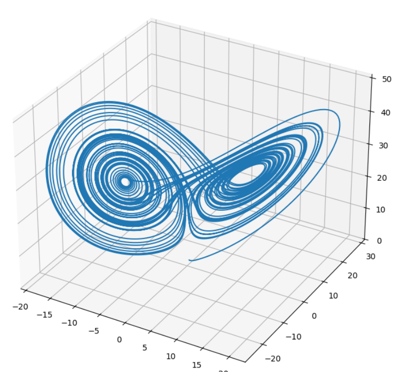

Figure 2: Initial Conditions 1: $P_0 = (0, 1, 1.05)$

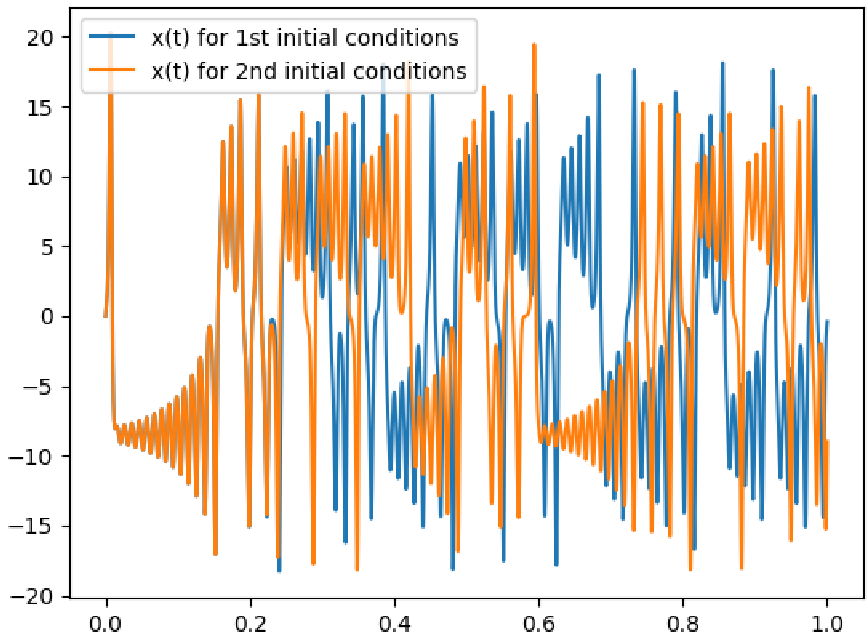

Figure 4: Comparison of $x(t)$ for $P_1 = (0, 1, 1.05)$ and $P_0 = (0, 1.001, 1.05)$

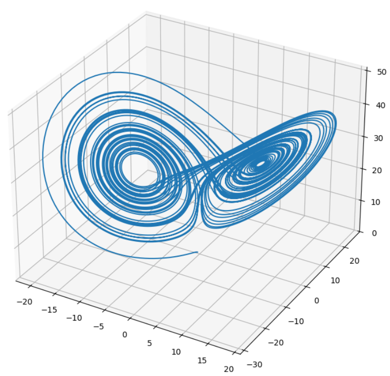

Figure 3: Initial Conditions 2: $P_0 = (0, -1, 1.05)$

Where is the sensitivity to initial conditions then? The sight of these two sets, although visibly different, seems to have a certain homogeneity, so much so that one is tempted to say that, intuitively, with close initial conditions, one should have solutions very close to each other. However, the analysis of the system demonstrates the contrary, as shown in figure 4.

The only difference in initial conditions is a gap of 0.001 in the y-coordinate, and yet extremely different values for the solutions of $x(t)$ (the abscissa) are observed over time. This is how we can highlight the phenomenon that we will describe here as *hypersensitivity* to initial conditions.

3 Equilibrium and Stability

Recall the equations of the Lorenz system:

$$\begin{cases} \frac{dx}{dt} = \sigma(y - x) \\ \frac{dy}{dt} = x(\rho - z) - y \\ \frac{dz}{dt} = xy - \beta z \end{cases}$$

with variables:

- $x > 0$ the intensity of convection movement
- $y > 0$ the horizontal temperature difference between ascending and descending currents
- $z > 0$ the vertical temperature deviation

and parameters:

- σ the *Prandtl number* (representing the viscosity or thermal diffusivity of the system), such that $\sigma > b + 1$
- ρ the *normalized Rayleigh number*
- β the geometric factor of the convection cell

3.1 Equilibrium points

The equilibrium points or fixed points of the system are the points where the system does not vary, i.e., where we have:

$$\begin{cases} \frac{dx}{dt} = 0 \\ \frac{dy}{dt} = 0 \\ \frac{dz}{dt} = 0 \end{cases}$$

Solving this system: For simplicity, we will denote the equations of the system as $\dot{x}$, $\dot{y}$ and $\dot{z}$ respectively.

$$\begin{aligned} \dot{x} = 0 &\implies y = x \\ \dot{y} = 0 &\implies x(\rho - z) - y = 0 \\ &\implies x(\rho - z) - x = 0 \\ &\implies x(\rho - z - 1) = 0 \\ &\implies x = 0 \text{ or } z = \rho - 1 \\ \dot{z} = 0 &\implies xy - \beta z = 0 \\ &\implies x^2 - \beta z = 0 \end{aligned}$$

If $x = 0$, then $y = 0$ and $z = 0$. So an equilibrium point is $O = (0, 0, 0)$.
Otherwise, if $z = \rho - 1$, we have:

$$\begin{aligned} x^2 - \beta(\rho - 1) = 0 &\implies x^2 = \beta(\rho - 1) \\ &\implies x = \pm\sqrt{\beta(\rho - 1)} \end{aligned}$$

So two other equilibrium points are:

$$C_+ = \left(\sqrt{\beta(\rho - 1)}, \sqrt{\beta(\rho - 1)}, \rho - 1\right)$$

$$C_- = \left(-\sqrt{\beta(\rho - 1)}, -\sqrt{\beta(\rho - 1)}, \rho - 1\right)$$

We notice that points C_+ and C_- only exist if $\rho > 1$.

3.2 Stability analysis around equilibrium points

To analyze the stability of the equilibrium points, we will calculate the Jacobian matrix of the system, then evaluate its eigenvalues at each equilibrium point. The Jacobian matrix J of the system is given by:

$$J = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho - z & -1 & -x \\ y & x & -\beta \end{pmatrix}$$

Reminder on the stability of a dynamical system: An equilibrium point is said to be:

- **Stable** if all eigenvalues of the Jacobian matrix evaluated at that point have **negative real parts**.
- **Asymptotically stable** if all eigenvalues of the Jacobian matrix evaluated at that point have **strictly negative real parts**.
- **Unstable** if at least one eigenvalue of the Jacobian matrix evaluated at that point has a **positive real part**.

3.2.1 Stability at point $O = (0, 0, 0)$

Evaluating the Jacobian matrix J at O, we obtain:

$$J_O = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -\beta \end{pmatrix}$$

The eigenvalues of J_0 are the solutions of the characteristic equation:

$$\det(J_0 - \lambda I) = 0$$

Let $P_0(\lambda)$ be this characteristic equation:

$$P_0(\lambda) = -(\beta + \lambda)\left(\lambda^2 + (\sigma + 1)\lambda + \sigma(1 - \rho)\right)$$

A first root of P_0 is therefore: $\lambda_1 = -\beta$. (Note that $\lambda_1 < 0$)
The two other roots are the solutions of the second factor, which we will denote as $Q_0(\lambda)$:

$$Q_0(\lambda) = \lambda^2 + (\sigma + 1)\lambda + \sigma(1 - \rho)$$

What matters most here is not so much the value of the roots (eigenvalues) as their sign. To determine the signs of the roots of this polynomial, we will note the remarkable form of $Q_0(\lambda)$:

$$Q_0(\lambda) = \lambda^2 + S\lambda + P$$

with:

- $S = \sigma + 1 > 0$ which represents the sum of the roots, and
- $P = \sigma(1 - \rho)$ which represents the product of the roots.

The sign of the roots therefore depends on the sign of P:

- If $P > 0 \iff \rho < 1$, then both roots are negative[1].
- If $P = 0 \iff \rho = 1$, then at least one of the roots is zero.
- If $P < 0 \iff \rho > 1$, then the roots have opposite signs.

Consequently, the equilibrium point O is:

- **Asymptotically stable** if $\rho < 1$,
- **Stable** if $\rho = 1$,
- **Unstable** if $\rho > 1$.

Justification The sign of P depends on ρ. If $\rho < 1$, then $P > 0$ and both roots of $Q_0(\lambda)$ are negative, so all eigenvalues of J_O have strictly negative real parts, making point O asymptotically stable. From $\rho = 1$, P becomes zero, so at least one of the roots of $Q_0(\lambda)$ is zero, making point O only stable. Finally, if $\rho > 1$, then $P < 0$ and the roots of $Q_0(\lambda)$ have opposite signs, so at least one eigenvalue of J_O has a positive real part, making point O unstable. For $\rho = 1$, the only existing equilibrium point is the origin. Furthermore, we notice that it is only for $\rho > 1$ that the equilibrium points C_+ and C_- appear. This is therefore a bifurcation point of the system.

[1] Logically, we should say that both roots have the same sign. However, if a 2nd degree polynomial has roots, at least one of them is negative. Consequently, if the product of the two roots is positive, it is obvious that the second root is also negative.

3.2.2 Stability at points C_+ and C_-

Evaluating the Jacobian matrix J at C_+, we obtain:

$$J_{C_+} = \begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -\sqrt{\beta(\rho-1)} \\ \sqrt{\beta(\rho-1)} & \sqrt{\beta(\rho-1)} & -\beta \end{pmatrix}$$

The eigenvalues of J_{C_+} are the solutions of the characteristic equation:

$$\det(J_{C_+} - \lambda I) = 0$$

Let $P_+(\lambda)$ be this characteristic equation:

$$P_+(\lambda) = +\lambda^3 + (\sigma + \beta + 1)\lambda^2 + \beta(\sigma + \rho)\lambda + 2\sigma\beta(\rho - 1)$$

To analyze the sign of the roots of this polynomial, we will use the Routh-Hurwitz criterion.

Routh-Hurwitz Criterion

The Routh-Hurwitz criterion allows determining the stability of a linear system by examining the coefficients of its characteristic polynomial without having to explicitly calculate the eigenvalues. For a characteristic polynomial of degree n given by:

$$P(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + ... + a_1\lambda + a_0$$

The system is stable if and only if all the following conditions are met:

- All coefficients a_i are positive.
- The determinants of the Routh-Hurwitz matrices are positive.

For a 3rd degree polynomial, the specific conditions are:

- $a_2 > 0$
- $a_0 > 0$
- $a_2a_1 > a_3$

Application to the polynomial $P_+(\lambda)$:

By comparing $P_+(\lambda)$ with the general form, we identify the coefficients:

- $a_0 = 1$
- $a_1 = \sigma + \beta + 1$
- $a_2 = \beta(\sigma + \rho)$

- $a_3 = 2\sigma\beta(\rho - 1)$

Recall that the parameters σ, β and ρ are all positive, with additionally: $\sigma > b + 1$. Consequently, the first two conditions of the Routh-Hurwitz criterion are always satisfied:

- $a_2 = \beta(\sigma + \rho) > 0$
- $a_0 = 2\sigma\beta(\rho - 1) > 0$ (since $\rho > 1$ for the existence of points C_+ and C_-)

It therefore remains to analyze the third condition:

$$\begin{aligned}
a_2 a_1 > a_0 &\iff \beta(\sigma + \rho)(\sigma + \beta + 1) > 2\sigma\beta(\rho - 1) \\
&\iff (\sigma + \rho)(\sigma + \beta + 1) > 2\sigma(\rho - 1) \\
&\iff \rho(\sigma + \beta + 1) + \sigma(\sigma + \beta + 1) > 2\sigma\rho - 2\sigma \\
&\iff \rho(\sigma + \beta + 1 - 2\sigma) > -\sigma(\sigma + \beta + 1 + 2) \\
&\iff \rho(\beta + 1 - \sigma) > -\sigma(\sigma + \beta + 3) \\
&\iff \rho < \frac{-\sigma(\sigma + \beta + 3)}{\beta + 1 - \sigma}
\end{aligned}$$

Setting:

$$\rho_h = \frac{\sigma(\sigma + \beta + 3)}{\sigma - (\beta + 1)}$$

, we have that the equilibrium point C_+ is:

- **Asymptotically stable** if $1 < \rho < \rho_h$,
- **Unstable** if $\rho > \rho_h$.

By symmetry, the equilibrium point C_- has the same stability as C_+.
The point $\rho = \rho_h$ is therefore another critical bifurcation point of the system: it is a Hopf bifurcation.

Justification For $1 < \rho < \rho_h$, all conditions of the Routh-Hurwitz criterion are satisfied, so all eigenvalues of J_{C_+} have strictly negative real parts, making points C_+ and C_- asymptotically stable. However, if $\rho > \rho_h$, the third condition of the Routh-Hurwitz criterion is no longer satisfied, so at least one eigenvalue of J_{C_+} has a positive real part, making points C_+ and C_- unstable.

4 Global Analysis and Characterization of the Attractor

The analysis of the Lorenz system around equilibrium points has revealed great instability, so one might intuitively think that the solutions diverge to infinity unpredictably as time passes. However, a more global analysis of the system shows that this is not the case, and that the solutions actually remain confined within a bounded region of the phase space, forming a complex structure known as a **Strange Attractor**.

4.1 Phase space contraction

Recall the equations of the Lorenz system:

$$\begin{cases} \frac{dx}{dt} = \sigma(y - x) \\ \frac{dy}{dt} = x(\rho - z) - y \\ \frac{dz}{dt} = xy - \beta z \end{cases}$$

with variables:

- x the intensity of convection movement
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- z the vertical temperature deviation

and parameters:

- σ the *Prandtl number* (representing the viscosity or thermal diffusivity of the system)
- ρ the *normalized Rayleigh number*
- β the geometric factor of the convection cell

The parameters are all considered positive, with additionally: $\sigma > b + 1$. We will denote this system as $X' = L(X)$ (as mentioned in the first part of this document).

Phase space

To better understand the dynamic behavior of the Lorenz system, it is useful to examine the phase space, which is a three-dimensional space where each point represents a possible state of the system, defined by the variables x, y, and z. By plotting trajectories in this space, we can visualize how solutions evolve over time. Thus, each point in this space represents the complete state of the system at a given moment, and the trajectory followed by this point illustrates the temporal evolution of the system variables.

Divergence of a field

The notion of divergence (borrowed from fluid mechanics) measures the local variation of the flux of a vector field. Denoting the divergence of field L as $\nabla \cdot L$, we have:

- If $\nabla \cdot L > 0$, the flux flows out of this point (source). The volume expands or diverges.
- If $\nabla \cdot L = 0$, the flux is conserved; the system is said to be conservative.
- If $\nabla \cdot L < 0$, the volume contracts or converges (sink). The system is said to be dissipative.

Divergence of the Lorenz system

Let's calculate the divergence of the Lorenz system:

$$\begin{aligned}\nabla \cdot L &= \frac{\partial}{\partial x}(\sigma(y-x)) + \frac{\partial}{\partial y}(x(\rho - z) - y) + \frac{\partial}{\partial z}(xy - \beta z) \\ &= -\sigma + (-1) + (-\beta) \\ &= -(\sigma + \beta + 1)\end{aligned}$$

Since σ, β and 1 are all positive, we have:

$$\nabla \cdot L = -(\sigma + \beta + 1) < 0 \tag{1}$$

Application of Liouville's theorem

Liouville's theorem stipulates that if we consider a volume $V(t)$ in the phase space that evolves with the flow of the dynamic system, then the rate of variation of this volume is given by:

$$\frac{dV}{dt} = (\nabla \cdot L)V$$

Using equation **??**, we obtain:

$$\frac{dV}{dt} = -(\sigma + \beta + 1)V$$

This is a linear differential equation whose solution is:

$$V(t) = V(0)e^{-(\sigma+\beta+1)t}$$

Physical interpretation This expression shows that the volume $V(t)$ in the phase space decreases exponentially over time, which confirms that the Lorenz system is dissipative. Since $\lim_{t\to+\infty} V(t) = 0$, all system trajectories will eventually be trapped on a set of zero measure in the phase space: **this is the Strange Attractor**.

4.2 Trapping region

For this, we will start by choosing a control function for the energy of the system, or a *Lyapunov candidate function*.

4.2.1 Essential criteria

This function must be positive definite and its derivative along the trajectories negative or semi-negative.

- Positive definite: $V(x, y, z) > 0$ for all $(x, y, z) \neq (0, 0, 0)$ and $V(0, 0, 0) = 0$, and continuous with continuous partial derivatives.
- Negative or semi-negative derivative along the trajectories:

$$\frac{dV}{dt} = \nabla V \cdot L(X) \leq 0$$

- Unbounded: $V(x, y, z) \to +\infty$ as $\|(x, y, z)\| \to +\infty$.

4.2.2 Lyapunov candidate function V

We consider the following function:

$$V(x, y, z) = \rho x^2 + \sigma y^2 + \sigma(z - 2\rho)^2 \tag{2}$$

This function respects the criteria stated above.

4.2.3 Temporal variation of V

To understand the behavior of this function along the trajectories of the system, let's calculate its temporal derivative using the differential equations of the system. By definition, we have:

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} + \frac{\partial V}{\partial z}\frac{dz}{dt}$$

Let's calculate the partial derivatives of V:

$$\frac{\partial V}{\partial x} = 2\rho x$$
$$\frac{\partial V}{\partial y} = 2\sigma y$$
$$\frac{\partial V}{\partial z} = 2\sigma(z - 2\rho)$$

By substituting the equations of the Lorenz system into the expression for $\frac{dV}{dt}$, we obtain:

$$\begin{aligned}\frac{dV}{dt} &= 2\rho x(\sigma(y - x)) + 2\sigma y(x(\rho - z) - y) + 2\sigma(z - 2\rho)(xy - \beta z)\\ &= 2\rho\sigma xy - 2\rho\sigma x^2 + 2\sigma\rho xy - 2\sigma yz - 2\sigma y^2 + 2\sigma xyz - 4\sigma\rho xy - 2\sigma\beta z^2 + 4\sigma\beta\rho z\end{aligned}$$

By grouping similar terms, we get:

$$\frac{dV}{dt} = -2\rho\sigma x^2 - 2\sigma y^2 - 2\sigma\beta z^2 + 2\sigma xyz + 4\sigma\beta\rho z \tag{3}$$

4.2.4 Interpretations

Referring to equation **??**, let us denote by $\Phi(x, y, z)$ the expression in brackets, which represents the equation of an ellipsoid that we will call E. The objective is to determine the sign of $\frac{dV}{dt}$, which will always be the opposite of that of Φ (because of the $-$ sign).
Let's analyze $\Phi(x, y, z) = \rho x^2 + y^2 + \beta(z - \rho)^2 - \beta\rho^2$. Let $Q(x, y, z) = \rho x^2 + y^2 + \beta(z - \rho)^2$. This equation represents a family of ellipsoids centered at $(0, 0, \rho)$, and $\beta\rho^2$ is a fixed positive constant.

Exterior and Interior of ellipsoid E The exterior of this ellipsoid corresponds to points (x, y, z) such that $Q(x, y, z) > \beta\rho^2$, which means that $\Phi(x, y, z) = Q(x, y, z) - \beta\rho^2 > 0$. Consequently, outside the ellipsoid, the sign of $\frac{dV}{dt}$ is always negative: the system is indeed dissipative, the control function decreases, and the trajectories are *sucked* towards the interior. The same reasoning shows that inside the ellipsoid ($Q(x, y, z) < \beta\rho^2$), $\frac{dV}{dt}$ becomes positive: the system is no longer dissipative and its energy can then increase (the source of chaos).

The ellipsoid E thus serves as a bound for the system's instabilities: as soon as a trajectory approaches the boundary of ellipsoid E, it enters the zone where the system becomes dissipative and is automatically sent back towards the interior. And inside, as the energy increases, the solutions move away from the equilibrium points (C_+ and C_-) and start to spiral.

4.3 Synthesis of the Global Analysis

The analysis conducted in this section allows us to resolve the apparent paradox of the Lorenz system. By combining the results of local stability (Section 3) and global dynamics (Sections 4.1 and 4.2), we can draw a complete portrait of the system:

1. **The engine of agitation:** Section 3 showed that for $\rho > \rho_h$, all equilibrium points are unstable. The system can therefore never come to rest; it is condemned to perpetual motion, repelled from points O, C_+, and C_-.
2. **The confinement constraint:** Section 4.2 proved the existence of a trapping region (the Lorenz ellipsoid). Trajectories, although repelled from the centers, are systematically brought back towards the interior as soon as they stray too far from the origin.
3. **Geometric condensation:** Section 4.1 demonstrated that the volume occupied by these trajectories in the phase space contracts exponentially until it becomes zero.

Since the system is **bounded** (it does not diverge), **restless** (no stable fixed point), and its final volume is **zero**, the trajectories are forced to condense onto a geometric object of a singular nature. This object must possess an infinite surface to allow trajectories that never cross, while occupying zero volume. This is the very definition of the **Strange Lorenz Attractor**.

5 Conclusion

In conclusion, this study has demonstrated that the dynamics of the Lorenz system is governed by a succession of equilibrium ruptures called bifurcations, triggered by the evolution of the Rayleigh parameter ρ. We have established that the origin loses its stability in favor of two new equilibrium points as soon as $\rho > 1$, before these in turn become unstable during the passage of the Hopf critical threshold, ρ_h. Beyond this threshold, the system switches to a chaotic regime characterized by hypersensitivity to initial conditions. However, the originality of this work has been to demonstrate that this chaos is not synonymous with infinite divergence. Through the study of **volume contraction** and the existence of an **ellipsoidal trapping region**, we have mathematically proven that the trajectories remain confined in a bounded space. It is precisely this duality between local instability and global confinement that gives rise to the **strange attractor**, thus illustrating the existence of a deterministic order and a lasting geometric structure within chaos itself.

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