1. (a) Treat each dimension separately. First, consider row with global index $m = 17$ and block size $k = 3$. Global block index is $B = m/k = 17/3 = 5$ (in integer arithmetic). Number of processes over row direction is $N = 2$, so row position of process is $p = B \bmod N = 5 \bmod 2 = 1$.

   Now consider the column direction with global index $m = 72$ and block size $k = 4$. Global block index is $B = m/k = 72/4 = 18$ (in integer arithmetic). Number of processes over column direction is $N = 6$, so column position of process is $p = B \bmod N = 18 \bmod 6 = 0$.

   Thus, global index $(17, 72)$ is stored in the process at position $(1, 0)$ in the process mesh.

   (b) Treat each dimension separately. First, consider the row direction for which the local array index is $j = 2$. The block size in this direction is $k = 3$, so the local block index is $b = j/k = 2/3 = 0$ (in integer arithmetic). The local index within the block is $i = j \bmod k = 2 \bmod 3 = 2$. The global index in the row direction for $N = 2$, $p = 1$, $b = 0$ and $i = 2$ is:

$$m = (N * b + p) * k + i = (2 * 0 + 1) * 3 + 2 = 5$$

   Now, consider the column direction for which the local array index is $j = 4$. The block size in this direction is $k = 4$, so the local block index is $b = j/k = 4/4 = 1$ (in integer arithmetic). The local index within the block is $i = j \bmod k = 4 \bmod 4 = 0$. The global index in the row direction for $N = 6$, $p = 3$, $b = 1$ and $i = 0$ is:

$$m = (N * b + p) * k + i = (6 * 1 + 3) * 4 + 0 = 36$$

   Thus, the item at local index $(2, 4)$ in the process at position $(1, 3)$ has global index $(5, 36)$.

2. (a) Treat each dimension separately. First, consider row with global index $m = 101$ and block size $k = 5$. Global index is $B = m/k = 101/5 = 20$ (in integer arithmetic). Number of processes over row direction is $N = 3$, so row position of process is $p = B \bmod N = 20 \bmod 3 = 2$.

   Now consider the column direction with global index $m = 97$ and block size $k = 6$. Global index is $B = m/k = 97/6 = 16$ (in integer arithmetic). Number of processes over column direction is $N = 4$, so column position of process is $p = B \bmod N = 16 \bmod 4 = 0$.

   Thus, global index $(101, 97)$ is stored in the process at position $(2, 0)$ in the process mesh.

   (b) Treat each dimension separately. First, consider the row direction for which the local array index is $j = 7$. The block size in this direction is $k = 5$, so the local block index is $b = j/k = 7/5 = 1$ (in integer arithmetic). The local index within the block is $i = j \bmod k = 7 \bmod 5 = 2$. The global index in the row direction for $N = 3$, $p = 2$, $b = 1$ and $i = 2$ is:

$$m = (N * b + p) * k + i = (3 * 1 + 2) * 5 + 2 = 27$$

Now, consider the column direction for which the local array index is $j = 5$. The block size in this direction is $k = 6$, so the local block index is $b = j/k = 5/6 = 0$ (in integer arithmetic). The local index within the block is $i = j \bmod k = 5 \bmod 6 = 5$. The global index in the row direction for $N = 4$, $p = 3$, $b = 0$ and $i = 5$ is:

$$m = (N * b + p) * k + i = (4 * 0 + 3) * 6 + 5 = 23$$

Thus, the item at local index $(7, 5)$ in the process at position $(2, 3)$ has global index $(27, 23)$.

3. The answer to the these questions uses Table 1.

   (a) $d0 = 2$ and $d1 = 3$. 23 in binary is 10111. Partition as $(101, 11) = (5, 3)$ and apply inverse Gray code to each set of bits, i.e., find numbers in column 1 of table corresponding to number 5 and 3. This gives 6 and 2, so node 23 is at position $(6, 2)$ of the $8 \times 4$ process mesh.

   (b) $d0 = 2$ and $d1 = 2$ and location is $(3, 2)$. 2-bit Gray codes for 3 and 2 are 2 and 3. Write as 2-bit binary numbers, $(10, 11)$ and remove the comma to give 1011 which is 11 in decimal. So node 11 is at location $(3, 2)$.

   (c) $d0 = 5$ and $d1 = 4$ and location is $(10, 22)$. 4-bit Gray codes for 10 is 15, and 5-bit Gray code for 22 is 29. Write as binary numbers, $(1111, 11101)$ and remove the comma to give 111111101 which is 509 in decimal. So node 509 is at location $(10, 22)$.

   (d) $d0 = 5$ and $d1 = 4$. 410 in binary is 110011010. Partition as $(1100, 11010) = (12, 26)$ and apply inverse Gray code to each set of bits, i.e., find numbers in column 1 of table corresponding to number 12 and 26. This gives 8 and 19, so node 410 is at position $(8, 19)$ of the $16 \times 32$ process mesh.

4. (a) In the parallel algorithm, $\alpha n$ of the operations must be done serially, and $(1 - \alpha)n$ are done in parallel. So the parallel execution time is,

$$\begin{aligned} T_{par} &= \alpha n t + \left(\frac{(1-\alpha)n}{N}\right) t + C \\ &= \left(\alpha + \frac{(1-\alpha)}{N}\right) n t + C \end{aligned}$$

where $C \geq 0$ is the overhead. The execution speed of the parallel algorithm is,

$$\begin{aligned} E &= n/T_{par} \\ &= \frac{1}{(\alpha + (1-\alpha)/N)t + C/n} \end{aligned}$$

Ignoring the $C/n$ term makes the righthand side bigger so,

$$\begin{aligned} E &\leq \frac{1}{(\alpha + (1-\alpha)/N)t} \\ &= \frac{5}{(\alpha + (1-\alpha)/4096)} \text{ Gflop/s} \end{aligned}$$

since $1/t = 5$ Gflop/s and $N = 4096$.

(b) If the maximum execution speed is 1000 Gflop/s we must find the value of $\alpha$ that satisfies,

$$1000 = \frac{5}{\alpha + (1-\alpha)/4096}$$

Thus,

$$\alpha + \frac{(1-\alpha)}{4096} = \frac{5}{1000} \Rightarrow \alpha\left(1 - \frac{1}{4096}\right) = \frac{5}{1000} - \frac{1}{4096}$$

which gives $\alpha = 0.004757$.

5. (a) The number of sequential floating-point operations is 5mM, so the sequential execution time is:

$$T_{seq} = 5mMt_{calc} = 5m\log_2 m\, t_{calc}$$

(b) In the parallel algorithm there are $n$ iterations in which each process exchanges $M/N$ data items with another process, so the time for communication is:

$$n\frac{M}{N}t_{comm}$$

(c) In the parallel algorithm the $5mM$ floating-point operations are evenly spread over $N$ processes so each process does $5mM/N$ operations, and the parallel execution time is:

$$T_{par} = \frac{5mM}{N}t_{calc} + n\frac{M}{N}t_{comm}$$

(d) The speed-up is given by:

$$S = \frac{T_{seq}}{T_{par}} = \frac{5mMt_{calc}}{(5mM/N)t_{calc} + n(M/N)t_{comm}}$$

$$
\begin{aligned}
&= \frac{N}{1 + \left(\frac{n(M/N)}{5mM/N}\right)\frac{t_{comm}}{t_{calc}}} \\
&= \frac{N}{1 + \left(\frac{n}{5m}\right)\tau} \\
&= \frac{N}{1 + \frac{1}{5}\left(\frac{\log_2 N}{\log_2 M}\right)\tau}
\end{aligned}
$$

| $L$ | $L_2$ | $N_2$ | $N$ |
|---|---|---|---|
| 0 | 00000 | 00000 | 0 |
| 1 | 00001 | 00001 | 1 |
| 2 | 00010 | 00011 | 3 |
| 3 | 00011 | 00010 | 2 |
| 4 | 00100 | 00110 | 6 |
| 5 | 00101 | 00111 | 7 |
| 6 | 00110 | 00101 | 5 |
| 7 | 00111 | 00100 | 4 |
| 8 | 01000 | 01100 | 12 |
| 9 | 01001 | 01101 | 13 |
| 10 | 01010 | 01111 | 15 |
| 11 | 01011 | 01110 | 14 |
| 12 | 01100 | 01010 | 10 |
| 13 | 01101 | 01011 | 11 |
| 14 | 01110 | 01001 | 9 |
| 15 | 01111 | 01000 | 8 |
| 16 | 10000 | 11000 | 24 |
| 17 | 10001 | 11001 | 25 |
| 18 | 10010 | 11011 | 27 |
| 19 | 10011 | 11010 | 26 |
| 20 | 10100 | 11110 | 30 |
| 21 | 10101 | 11111 | 31 |
| 22 | 10110 | 11101 | 29 |
| 23 | 10111 | 11100 | 28 |
| 24 | 11000 | 10100 | 20 |
| 25 | 11001 | 10101 | 21 |
| 26 | 11010 | 10111 | 23 |
| 27 | 11011 | 10110 | 22 |
| 28 | 11100 | 10010 | 18 |
| 29 | 11101 | 10011 | 19 |
| 30 | 11110 | 10001 | 17 |
| 31 | 11111 | 10000 | 16 |

Table 1: 5-bit Gray code table giving node number $N$ corresponding to location $L$.