Quantum Computing

3. Robustness

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[1]:
```
from qutip import *
import numpy as np
import matplotlib.pyplot as plt
import matplotlib.animation as animation
from IPython.display import HTML
```

Decoherence

- Real quantum devices must deal with **decoherence**
  - Loss of coherence / quantum information due to **interactions with the environment**
  - Also, **fabrication unceratinties** - we cannot reliably build a device atom by atom
  - Note, steering a quantum device to perform a desired operation (rather than its natural evolution), always requries interaction with the environment
    - So also **noise and uncertainty in controls**
    - Perfectly isolated systems can remain coherent for a long time, but likely do not compute anything interesting

Simulating a Coherent Quantum System

- Follows the Schroedinger equation

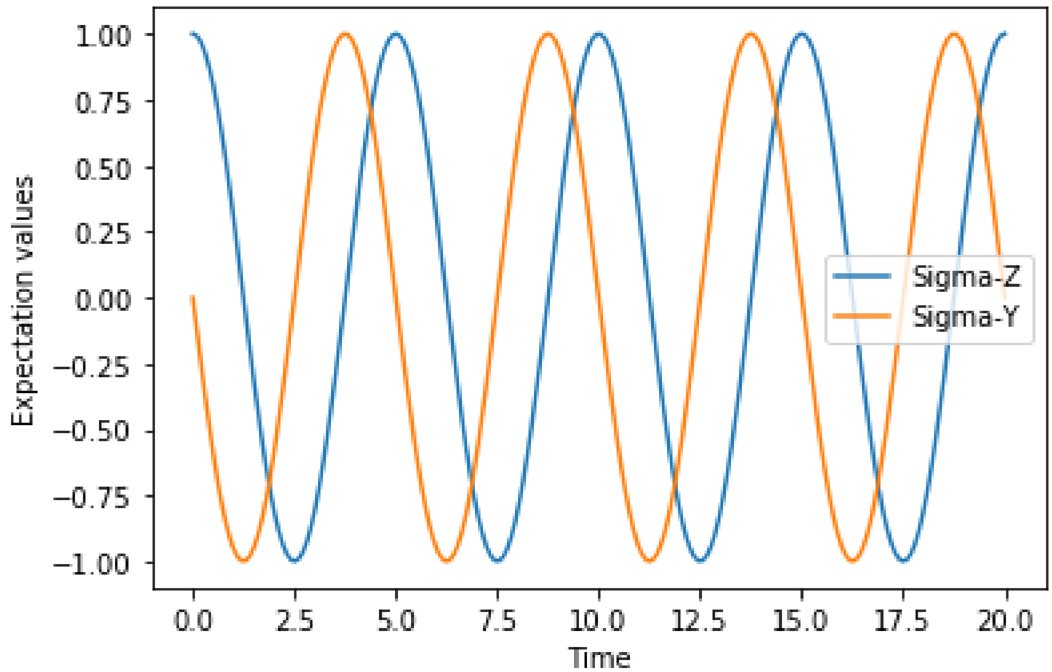
$$i\hbar \frac{\partial}{\partial t}|\Psi\rangle = H|\Psi\rangle$$

  - H here is the Hamiltonian of the system describing its dynamics
  - The propagator (for constant Hamiltonian, otherwise matrix exponential is more complex)

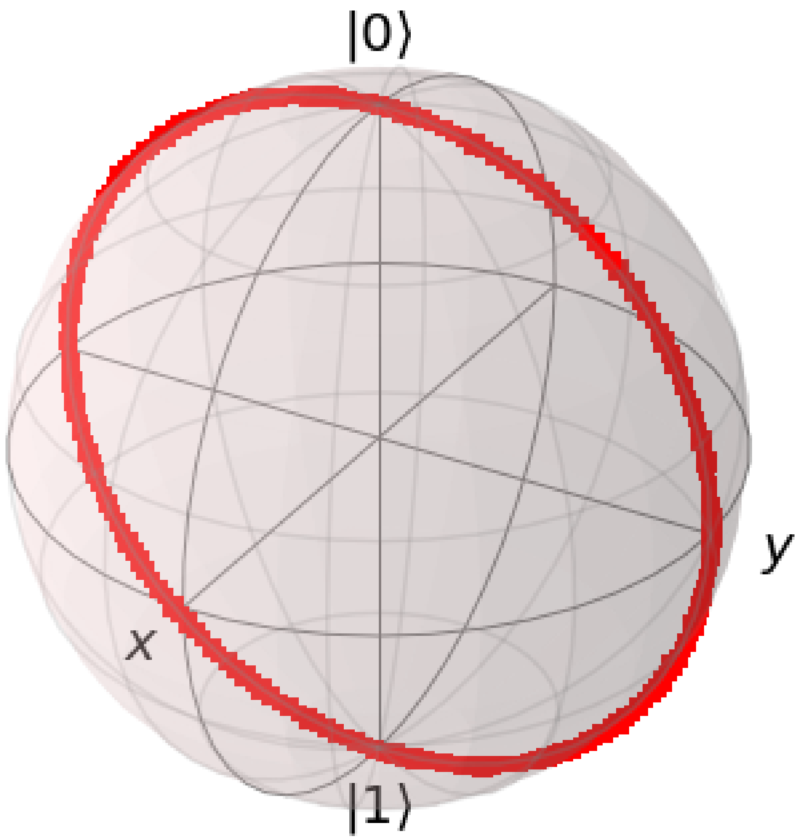
$$U(t) = \text{expm}\left(-iH\frac{t}{\hbar}\right)$$

    gives the "gate" realised at time t, $|\Psi(t)\rangle = U|\Psi(0)\rangle$
- E.g. $H = 2\pi 0.1 X$ (note, X here is the Pauli X operator, but represent a time-dependent rotation)

[2]:
```
H = 2.0 * np.pi * 0.1 * sigmax()
psi0 = ket("0")
times = np.linspace(0.0, 20.0, 200)
# Simulate and produce result.expect arrays of measuremnet operators Z and Y
result1 = mesolve (H, psi0, times, [], [sigmaz(), sigmay()])
# Plot
fig, ax = plt.subplots()
ax.plot(result1.times, result1.expect[0]); ax.plot(result1.times, result1.expect[1])
ax.set_xlabel('Time'); ax.set_ylabel('Expectation values'); ax.
↪legend(("Sigma-Z","Sigma-Y"));
```



[3]:
```
b = Bloch(); b.point_color = ['r']; b.point_marker = ['s']; b.clear()
for l in range(len(result1.expect[0])):
  b.add_points([0,result1.expect[1][l],result1.expect[0][l]])
b.show()
```



Pure vs Mixed States

- We have only studied **pure states** $|\Psi\rangle$, i.e. $\|\Psi\|^2 = \langle\Psi|\Psi\rangle = 1$
- Decoherence causes the state to become a **mixed state**, described by a **density matrix**

$$\rho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|$$

  where the $|\Psi_k\rangle$ are pure states and p_k is the probability of the system being in $|\Psi_k\rangle$
  - Can be thought of as an ensemble state (mixture state in a statistical ensemble of independent systems)
  - Their Bloch vector has a length < 1
  - Some of the information of the system is in an inaccessible (Hilbert) space K of the total (Hilbert) space $H \otimes K$
- Pure sate: $\mathrm{tr}(\rho^2) = 1$; mixed state $\mathrm{tr}(\rho^2) < 1$

Lindblad Master Equation

- To describe non-unitary evolution, we need to generalise the Schroedinger equation:

$$\rho(0) = \sum p_k |\Psi_k\rangle\langle\Psi_k|; U_t|\Psi_k\rangle = \mathrm{expm}(-iHt/\hbar)|\Psi_k\rangle$$

$$\rho(t) = \sum p_k U_t |\Psi_k\rangle\langle\Psi_k| U_t^\dagger = U_t \rho U_t^\dagger$$

- This gives the Liouville-von Neumann equation

$$\frac{d}{dt}\rho(t) = -\frac{i}{\hbar}[H, \rho]$$

- The inaccessible Hilbert space K is also in H here. As we do not know it, we replace it with the most general trace-preserving and completely positive form of that part of the evolution
- This gives the Lindblad equation

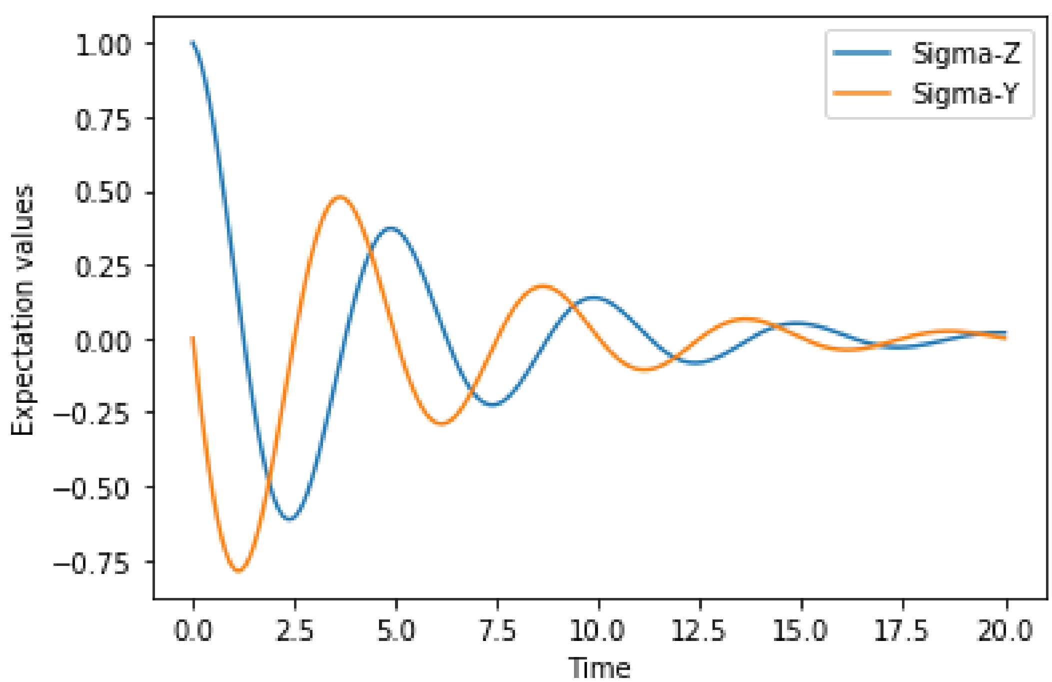
$$\frac{d}{dt}\rho(t) = -\frac{i}{\hbar}[H(t), \rho(t)] + \sum_n \frac{1}{2}\left(2C_n\rho(t)C_n^\dagger - \rho(t)C_n^\dagger C_n - C_n^\dagger C_n \rho(t)\right)$$

  - where $C_n = \sqrt{\gamma_n}A_n$ are the collapse operators, A_n are the operators coupling the environment to the system, γ_n are the corresponding rates

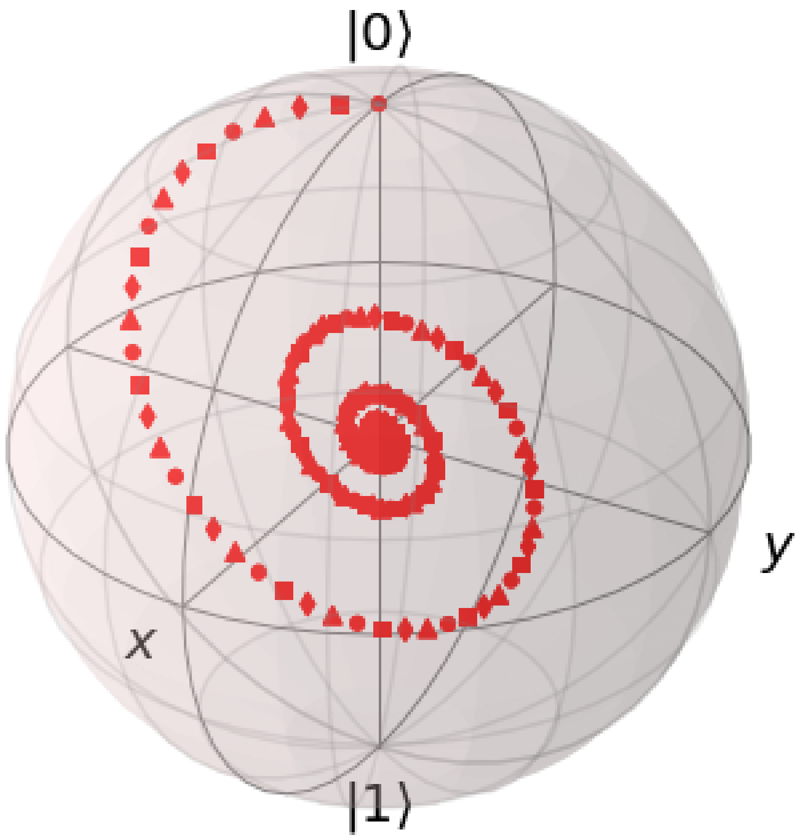
Energy Relaxation (T_1)

- An excited state decays toward the ground state
  - T_1 is the time constant indicating the speed of this
- E.g. add $\sqrt{0.1}X$ as relaxation process
  - Dissipation of energy of the qubit to its environment at rate 0.1 coupled via X

[4]:
```
# Simulate now with energy relaxation process
result2 = mesolve (H, psi0, times, [np.sqrt(0.1) * sigmax()], [sigmaz(), sigmay()])
# Plot
fig, ax = plt.subplots()
ax.plot(result2.times, result2.expect[0]); ax.plot(result2.times, result2.expect[1])
ax.set_xlabel('Time'); ax.set_ylabel('Expectation values'); ax.
↪legend(("Sigma-Z","Sigma-Y"));
```



[5]:
```
b = Bloch(); b.point_color = ['r']; b.point_maker = ['s']; b.clear()
for l in range(len(result2.expect[0])):
  b.add_points([0,result2.expect[1][l],result2.expect[0][l]])
b.show()
```



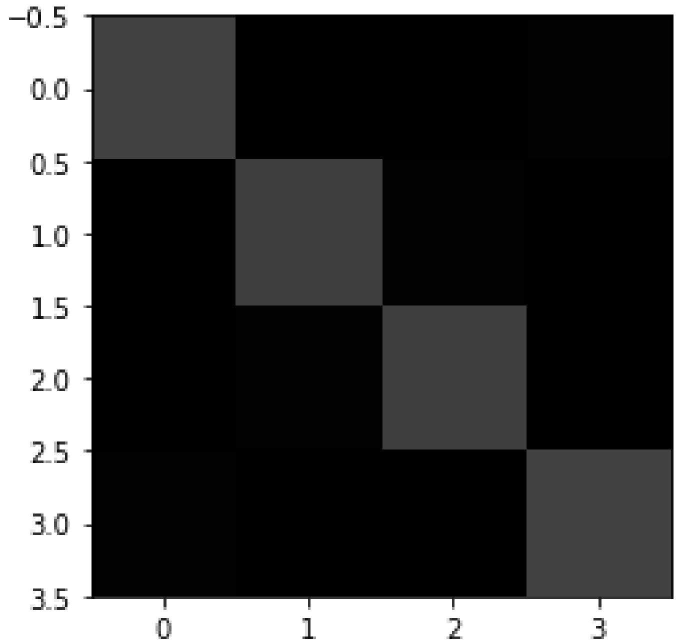
Dephasing (T_2)

- Loss of coherence between states, rather than loss of energy
  - Affects superposition of states
  - T_2 is the time constant indicating speed of this effect
- Hahn Echo with Decay (Wikipedia)

[6]:
```
# Simulate two XX-coupled qubits with dephasing and energy relaxation
psi1 = (ket("00")+ket("11"))/np.sqrt(2)
H1 = tensor(qeye(2),sigmax()) * tensor(sigmax(),qeye(2)) + tensor(qeye(2),sigmay()) *␣
↪tensor(sigmay(),qeye(2))
result3 = mesolve (H1, psi1, times, [np.sqrt(0.1)*tensor(qeye(2),sigmax()),np.sqrt(0.
↪1)*tensor(qeye(2),sigmaz())], [])
# Plot
fig = plt.figure(); ims = [];
```

```
for l in range(len(result3.states)):
    ims.append([plt.imshow(np.abs(result3.states[l].
↪full()),cmap='gray',vmin=0,vmax=1)]);
anim = animation.ArtistAnimation(fig, ims, interval=50, blit=True);
HTML(anim.to_html5_video())
```

[6]: <IPython.core.display.HTML object>



Repetition Code

- Simplest example of a **quantum error correction code**
- Encode a 1 bit message by copying it several times (say 3)
  - 00 encodes to 000000 and 11 encodes to 111111
- Store 000000 or 111111 for some time and assume an error occurs
  - Assume each bit may flip with probability $p < 1/2$, independently
  - Original message is the value indicated by a majority of the bits

$$\mathrm{maj}(a, b, c) = ab \oplus bc \oplus ca$$

  - This only works for a single error
    * Reduces the failure probability as the probability of more than one error is

$$3p^2(1-p) + p^3 < p, \text{ if } p < 1/2$$

    * For $p = 0.1$, failure after encoding is less than 0.028

Quantum Repetition Code - Bit-Flips

- Repetition code is classical error correction
  - 3-qubit bit-flip code example to encode, decode and detect errors
- Encode 1-qubit message: $|\Psi\rangle = a|0\rangle + b|1\rangle$ in 3-qubit bit-flip code:

$$a|000\rangle + b|111\rangle$$

  - Note, we cannot copy the state, as in the classical case, but can still create an entangled state
- Decode the message (after some time where decoherence may have caused an error):
  - Reversible majority voter to decode the bit-flip code
- bit-flip encoder/decoder

Phase-Flip Code

- Error operators arising from independent decoherence processes on each qubit can be written as linear combinations of I, X, $Y = -iZX$, Z (this is not trivial to show; Shor 1995)
- As quantum mechanics is linear, only bit-flip X and phase-flip Z errors must be corrected
  - Correcting a discrete set of errors means the sums of those errors, and hence the continuum of errors, are corrected as well [Shor 1995]
- Bit flip code can only correct bit-flip errors, though
  - A phase-flip changes the message to $a|000\rangle - b|111\rangle$
  - But this is an encoding of $a|0\rangle - b|1\rangle$, which is a valid code, too
  - So the bit-flip code is not sufficient to correct realistic errors
- phase-flip encoder/decoder (bit-flip encoder in the superposition basis)

Shor Code

- phase-flip and bit-flip encoder/decoder (combine both circuits; Shor code)
- Note there are more recent codes than this (see **stabiliser codes**, **topological codes**)