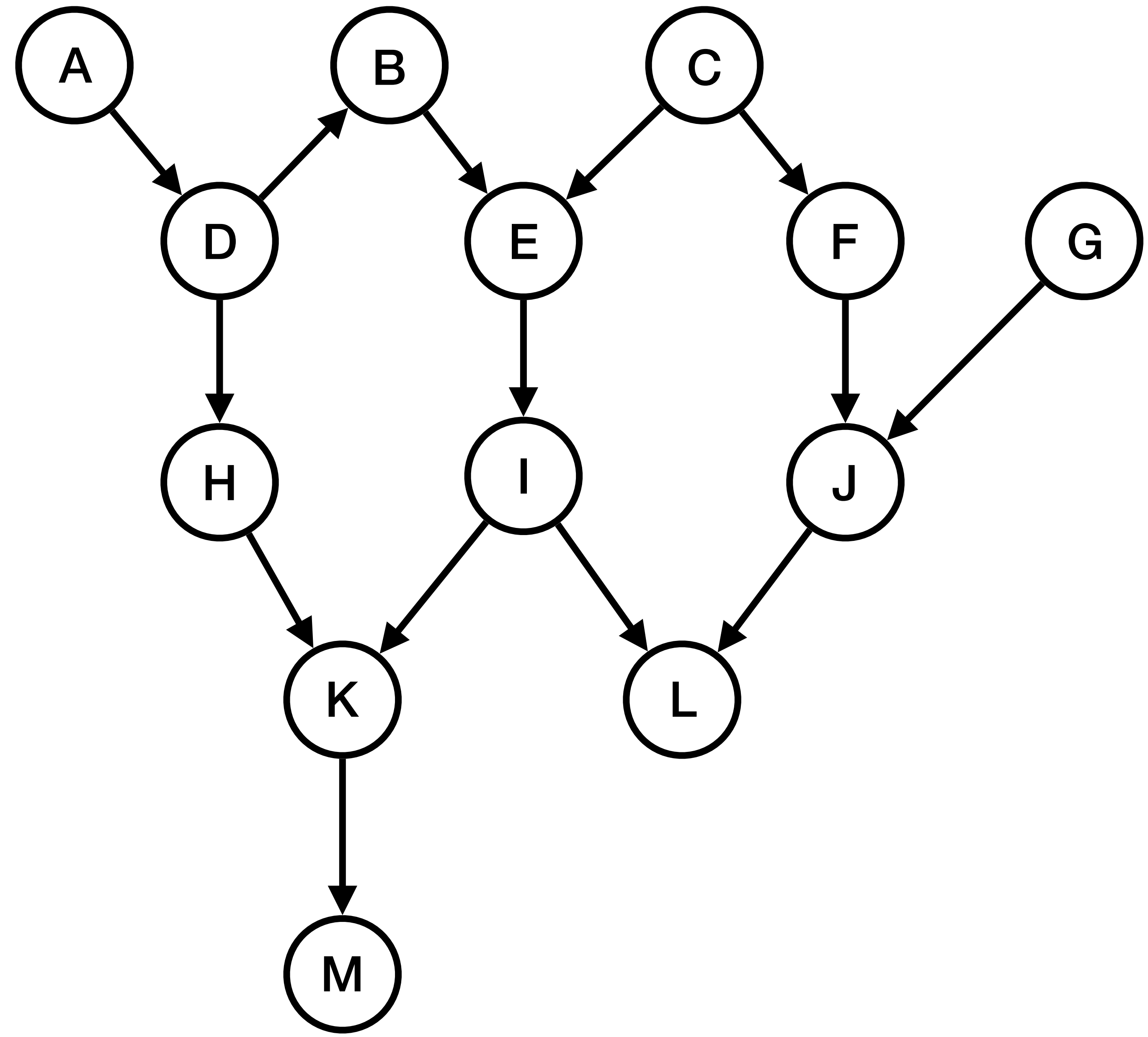
CMT311: Bayesian Networks - Solutions

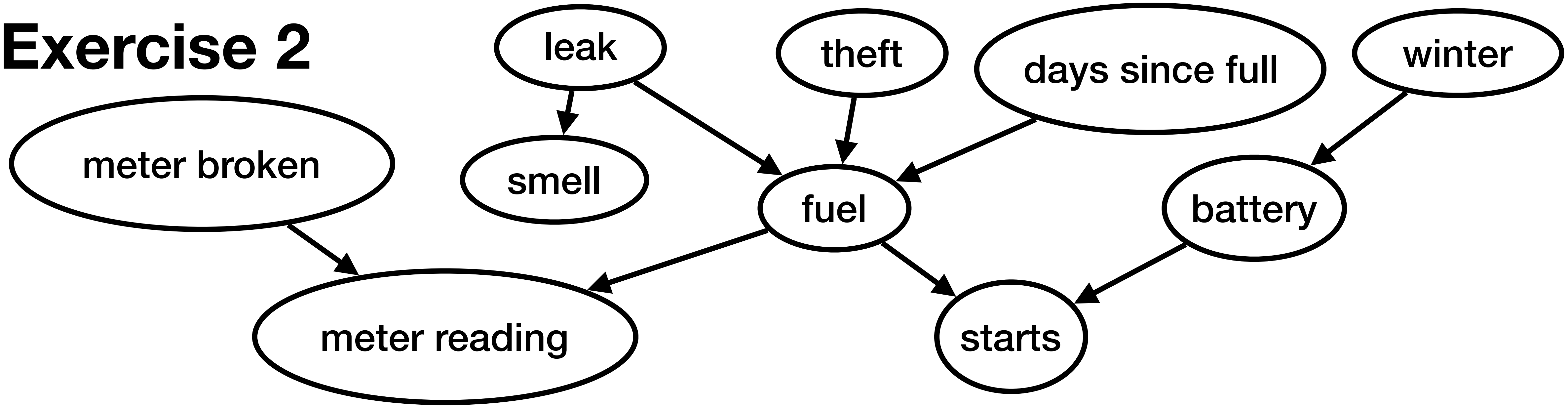
These exercises use material from
Jensen & Nielsen: Bayesian Networks and Decision Graphs

Exercise 1



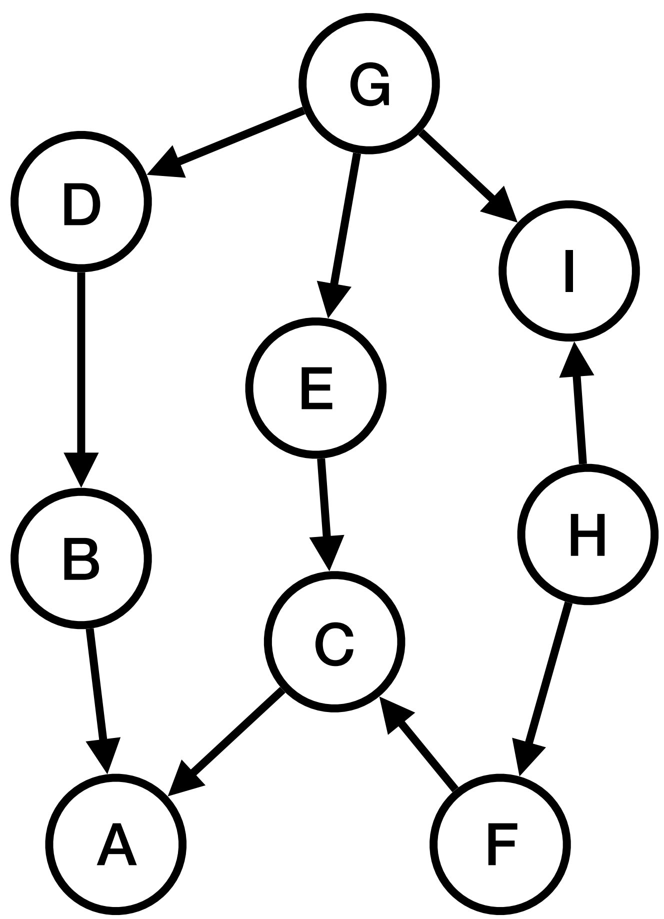
- List all parents of D.
  A
- List all children of D.
  H,B
- List all ancestors of L.
  I,E,B,D,A,C,J,F,G
- List all descendants of B.
  E,I,K,M,L
- List all non-descendants of E.
  A,B,C,D,F,G,H,J
- List all nodes in the Markov blanket of H.
  D,K,I
- List all nodes in the Markov blanket of E.
  B,C,I

Exercise 2



- **Meter reading** has **three** possible values {empty, half, full}, **days since full** has **eight** possible values {0,1,2,3,4,5,6,7} (0 meaning it is Saturday and the tank freshly filled, 7 meaning it is Saturday and the tank not yet filled), and **all other variables** have **two** possible values each.
- we need CPTs for P(leak), P(theft), P(winter), P(meter broken), P(days since full), P(smell|leak), P(battery|winter), P(starts|fuel,battery), P(fuel|leak,theft,days since full), P(meter reading|meter broken,fuel)
- The BN needs $4 \cdot 1 + 7 + 2 \cdot 2 + 4 + 32 + 8 = 59$ parameters:
  - 1 parameter each for P(leak), P(theft), P(winter), and P(meter broken),
  - 7 parameters for P(days since full),
  - 2 parameters each for P(smell|leak) and P(battery|winter),
  - 4 parameters for P(starts|fuel,battery),
  - 32 parameters for P(fuel|leak,theft,days since full),
  - 8 parameters for P(meter reading|meter broken,fuel)
- The full joint distribution is over $3 \cdot 8 \cdot 2^8 = 6144$ possible worlds and thus needs 6143 parameters specified.

Exercise 3



- On the path D - G - I - H, which nodes are colliders? I
- On the path D - G - E - C - F - H, which nodes are colliders? C
- On the path D - B - A - C - F - H, which nodes are colliders? A
- Which set(s) of evidence variables $\mathscr{Z}$ block the path D - G - I - H? Any subset of {A,B,C,E,F,G,I} that contains G or that does not contain I.
- Which set(s) of evidence variables $\mathscr{Z}$ block the path D - G - E - C - F - H? Any subset of {A,B,C,E,F,G,I} that contains at least one of G,E,F, or that contains neither C nor A.
- Which set(s) of evidence variables $\mathscr{Z}$ block the path D - B - A - C - F - H? Any subset of {A,B,C,E,F,G,I} that contains at least one of B,C,F, or that does not contain A.

Exercise 4

a) no variables are d-separated from A
(B and I block any path going through them, but all non-observed nodes can still be reached from A through a path that is not blocked)

b) C and F are d-separated from A
(the observed J blocks no paths; the collider G on paths A-G-D-… blocks no paths because its descendant J is observed; all paths through H are blocked, but there is always an un-blocked alternative through B; all paths through I (i.e., all paths to F or C) are blocked)

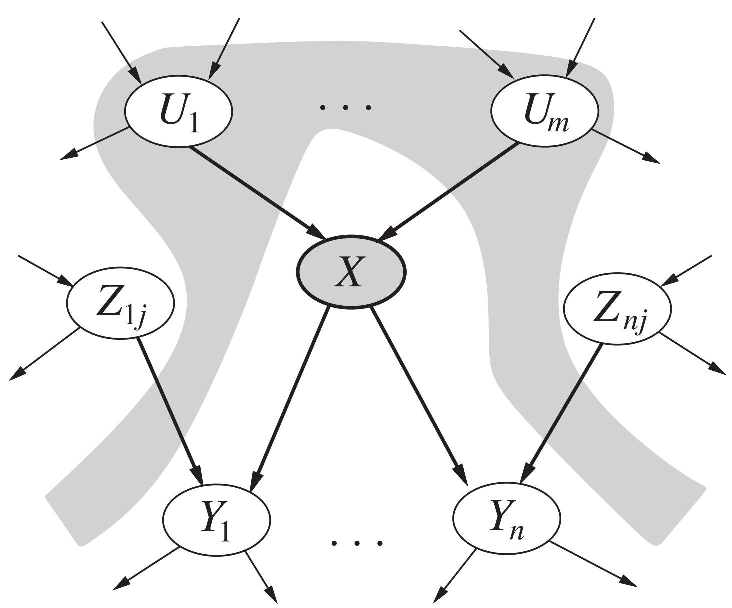
Exercise 5

Minimal sets d-separating C and E: {B, D, F} and {A, B, D}
(we need B to block C-B-E and D to block C-D-E, which also blocks C-A-D-E; we need A or F to block C-A-F-E, but using both would not be minimal)
Minimal sets d-separating A and B: ∅
(all paths from B to A go through one of the colliders C, D or E and are thus blocked if no nodes are observed)
Maximal set d-separating C and E: {A, B, D, F}
(follows from previous explanation)
Maximal set d-separating A and B: {F}
(adding F keeps the previous blocking; adding any other node un-blocks B-C-A)

Exercise 6

Show that each of the two characterisations below follows from the more general notion of d-separation.

Each node is conditionally independent of its **non-descendants given** its **parents**.



Let W be a non-descendant of X. Any path from X to W has to leave X either via a parent of X or via a child of X. If a path leaves X via a parent P, then P is a non-collider (the edge between P and X points to X and thus away from P) on that path and observed, and thus blocks the path.

If a path to a non-descendant leaves X through a child C of X, there has to be a descendant D of X (either C itself or a descendant of C) that is a collider on this path, i.e., that is left through another parent (otherwise, we'd reach a descendant as final node). As only the parents of X are observed, D and all its descendants are unobserved, and thus D blocks the path.

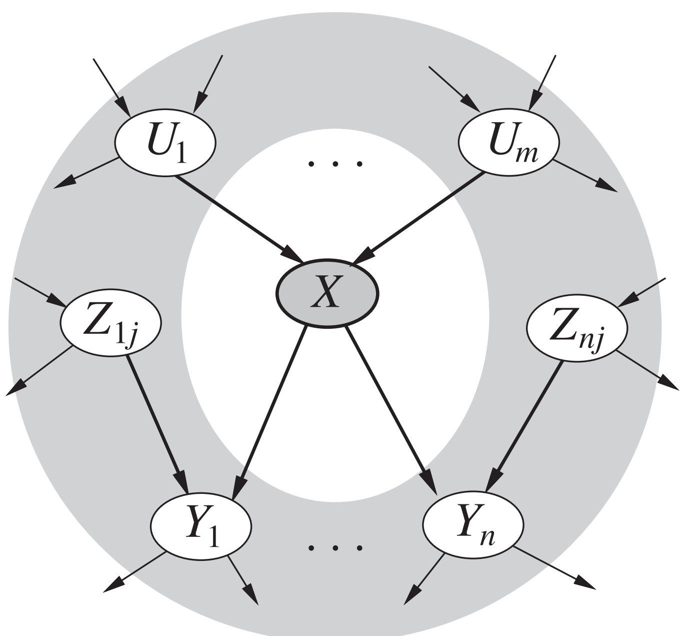
Thus, all paths between X and its non-descendants are blocked by X's parents. This means that X is d-separated from every of its non-descendants given its parents, which in turn means X is conditionally independent of its non-descendants given its parents.

[figures: Russell& Norvig]

Exercise 6

Show that each of the two characterisations below follows from the more general notion of d-separation.

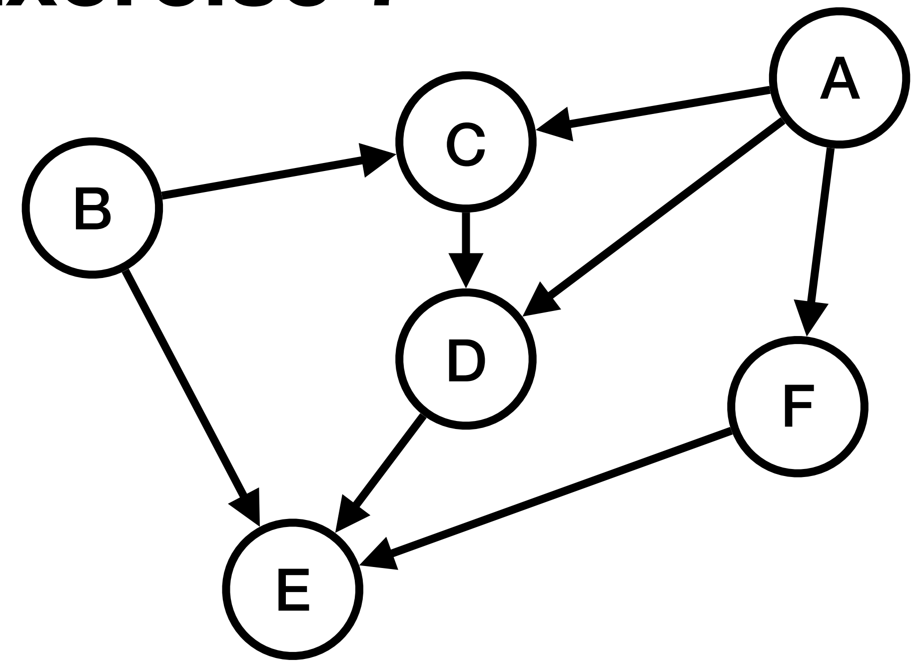
Each node is conditionally independent of **all other nodes** in the network **given** its **Markov blanket**.



Let W be a node different from X that is not in the Markov blanket of X. Any path from X to W has to leave X either via a parent of X or via a child of X.
Remember the Markov blanket contains all parents of X, all children of X, and the other parents of these children.
If a path leaves X via a parent P, then P is a non-collider (the edge between P and X points to X and thus away from P) on that path and observed, and thus blocks the path.
If a path leaves X through a child C of X, it next has to visit either a child of C or another parent of C. If the next node visited is a child of C, C is a non-collider that is observed, and thus blocks the path. If the next node visited is a parent Q of C, Q is a non-collider (as the edge to C points away from Q) and observed, and thus blocks the path.
Thus, the nodes in the Markov blanket block all paths between X and other nodes, and the claimed conditional independence follows.

[figures: Russell& Norvig]

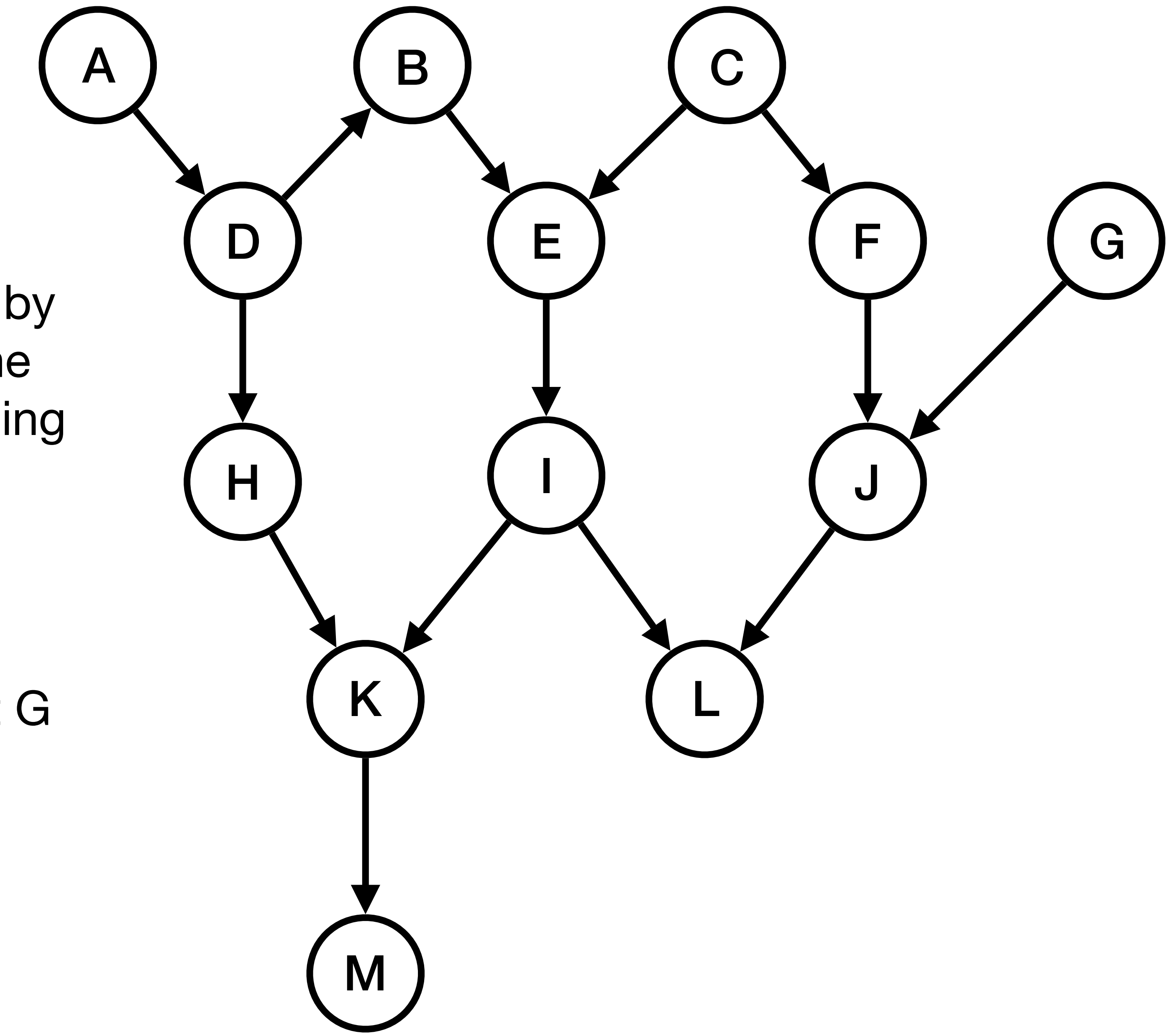
Exercise 7



- Each node is conditionally independent of its **non-descendants given** its **parents:**
  - $A \perp\!\!\!\perp \{B\} | \{\}$
  - $B \perp\!\!\!\perp \{A,F\} | \{\}$
  - $C \perp\!\!\!\perp \{F\} | \{A,B\}$
  - $D \perp\!\!\!\perp \{B,F\} | \{A,C\}$
  - $E \perp\!\!\!\perp \{A,C\} | \{B,D,F\}$
  - $F \perp\!\!\!\perp \{B,C,D\} | \{A\}$

- Each node is conditionally independent of **all other nodes** in the network **given** its **Markov blanket:**
  - $A \perp\!\!\!\perp \{E\} | \{B,C,D,F\}$
  - $B \perp\!\!\!\perp \{\} | \{A,C,D,E,F\}$
  - $C \perp\!\!\!\perp \{E,F\} | \{A,B,D\}$
  - $D \perp\!\!\!\perp \{\} | \{A,B,C,E,F\}$
  - $E \perp\!\!\!\perp \{A,C\} | \{B,D,F\}$
  - $F \perp\!\!\!\perp \{C\} | \{A,E,B,D\}$

Exercise 8



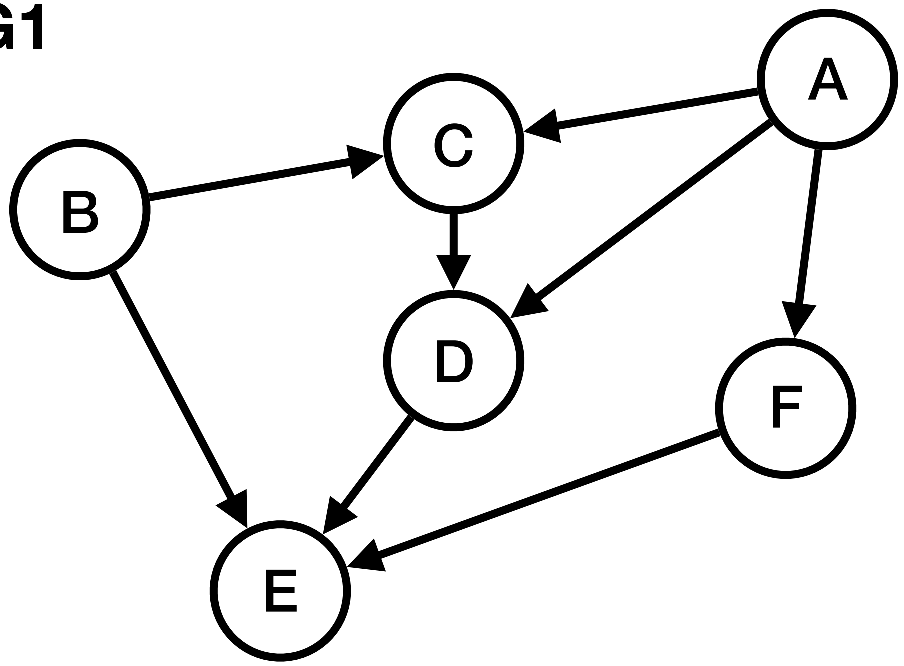
- List all nodes that are conditionally independent of A given B and M.
  G
  (paths can be blocked in two ways only: by containing D-B-E, which is blocked as the non-collider B is observed, or by containing F-J-G, which is blocked as neither the collider J nor its only descendant L is observed; note that all other potential colliders have the observed M as descendant; this leaves all nodes except G reachable through a non-blocked path)

- List all nodes that are conditionally independent of A given B.
  E,I,C,L,F,J,G
  (all paths from A to any of these nodes have to go either through B, which is blocked by being observed, or through K, which is blocked as an unobserved collider with only unobserved descendants; the shortest paths from A to D, H, K and M all contain no colliders and no observed nodes and are thus not blocked)

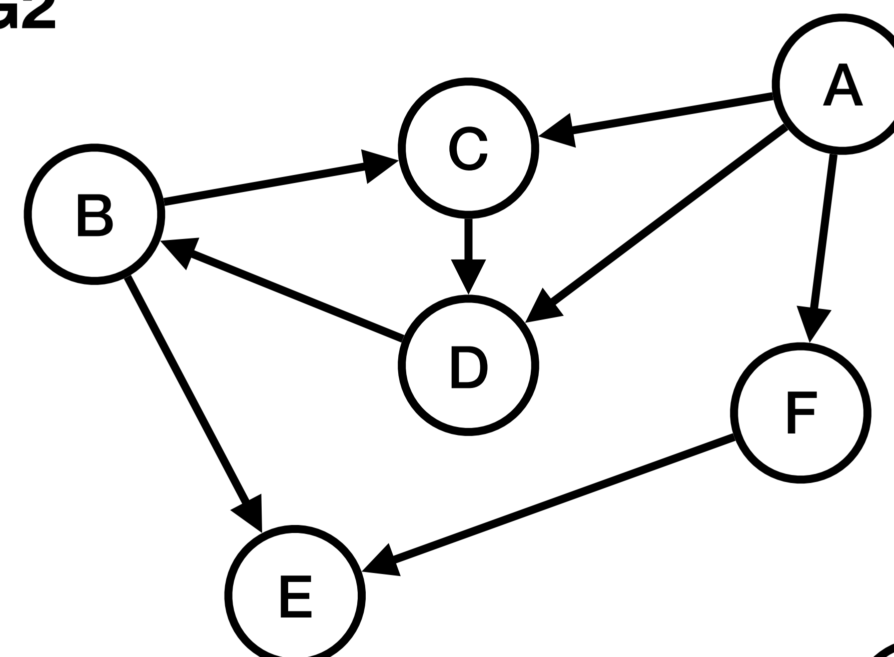
Exercise 9

The immoralities in G1 are (B,C,A), (B,E,D), (B,E,F), (D,E,F)
G2's skeleton is different from all other graphs, so it is not Markov equivalent to any of them.
G1 and G3 are Markov equivalent (same skeleton and same immoralities).
G4 is not Markov equivalent to G1 and G3 despite having the same skeleton, as it misses the immoralities (B,E,D), (B,E,F).

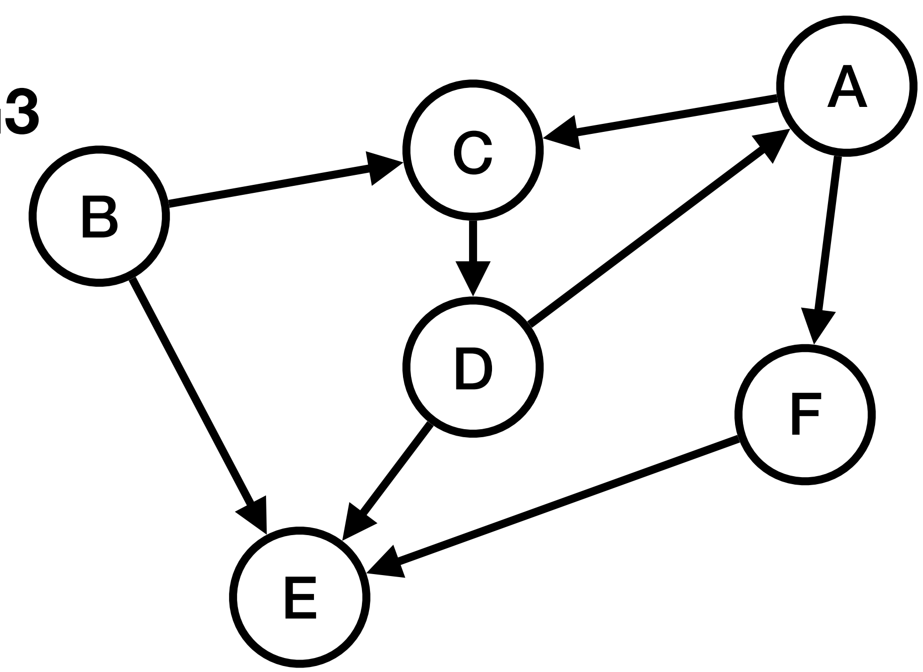
G1



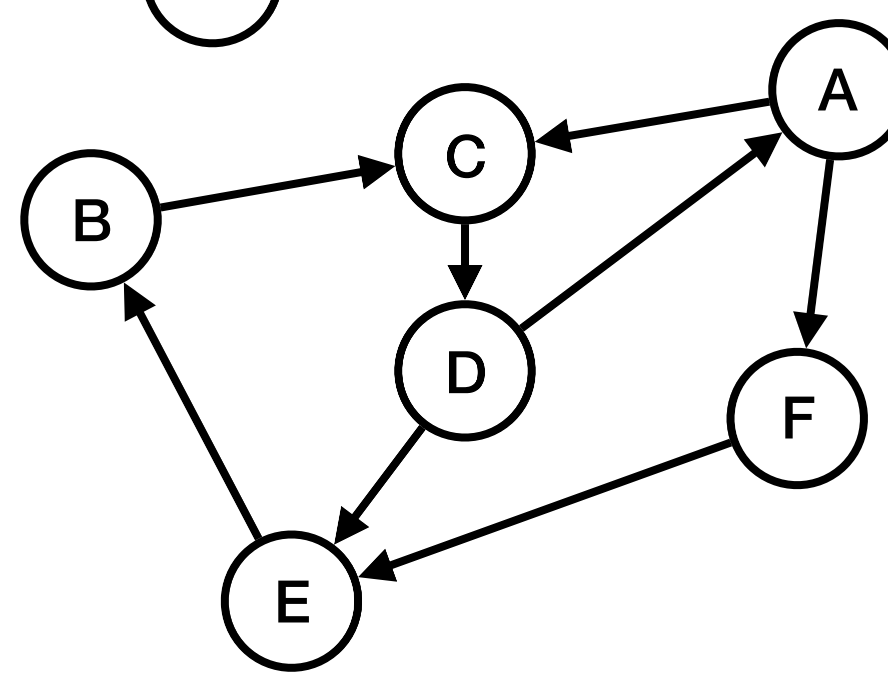
G2



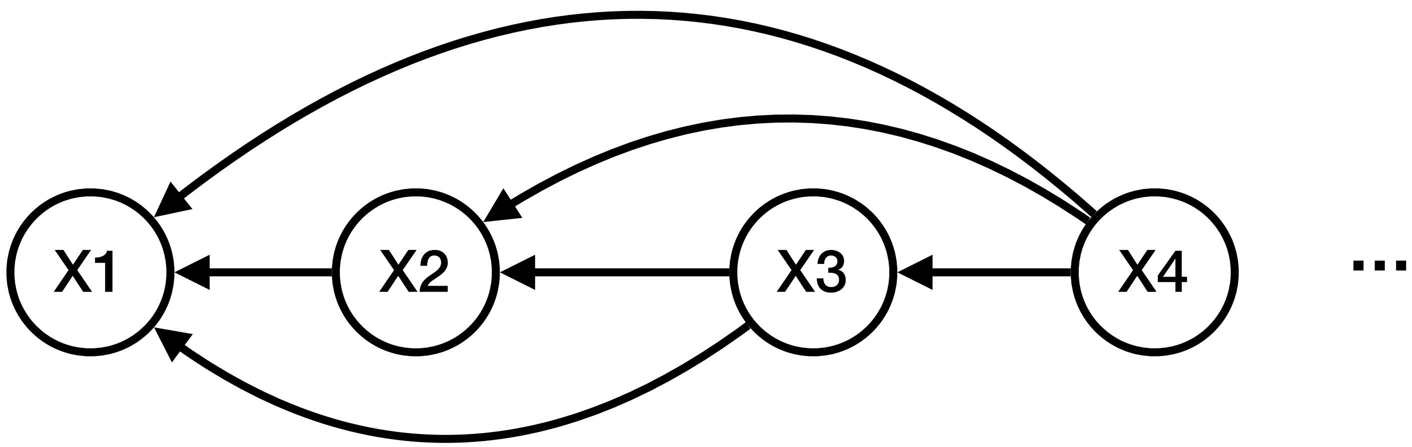
G3



G4



Exercise 10



Two DAGs are Markov equivalent if they have the same skeleton and the same sets of immoralities.

In our DAGs, each node is a child of all nodes that appear later in the order, and a parent of all nodes that appear earlier in the order, i.e., has an edge to (or from) all other nodes (edge directions vary based on the order, but the presence of an edge does not). Thus, the skeleton of the graph is the same for all orders, and there are no immoralities (as these require two parents without an edge between them, and we have edges between all nodes).