3.1) Derive eq. (3.29) from eqs. (3.21), (3.24), and (3.28).

$$[a(\mathbf{k}), a(\mathbf{k}')] = 0 ,$$
$$[a^\dagger(\mathbf{k}), a^\dagger(\mathbf{k}')] = 0 ,$$
$$[a(\mathbf{k}), a^\dagger(\mathbf{k}')] = (2\pi)^3 2\omega\, \delta^3(\mathbf{k}-\mathbf{k}') . \qquad (3.29)$$

$$a(\mathbf{k}) = \int d^3x\, e^{-ikx}\Big[i\partial_0\varphi(x) + \omega\varphi(x)\Big]$$
$$= i\int d^3x\, e^{-ikx}\, \overleftrightarrow{\partial_0}\varphi(x) , \qquad (3.21)$$

$$\Pi(x) = \dot\varphi(x) \qquad (3.24)$$
$$[\varphi(\mathbf{x},t), \varphi(\mathbf{x}',t)] = 0 ,$$
$$[\Pi(\mathbf{x},t), \Pi(\mathbf{x}',t)] = 0 ,$$
$$[\varphi(\mathbf{x},t), \Pi(\mathbf{x}',t)] = i\delta^3(\mathbf{x}-\mathbf{x}') . \qquad (3.28)$$

See code_2.

3.2) Use the commutation relations, eq. (3.29), to show explicitly that a state of the form

$$|k_1 \ldots k_n\rangle \equiv a^\dagger(\mathbf{k}_1)\ldots a^\dagger(\mathbf{k}_n)|0\rangle \qquad (3.33)$$

is an eigenstate of the hamiltonian, eq. (3.30), with eigenvalue $\omega_1 + \ldots + \omega_n$. The vacuum $|0\rangle$ is annihilated by $a(\mathbf{k})$, $a(\mathbf{k})|0\rangle = 0$, and we take $\Omega_0 = \mathcal{E}_0$ in eq. (3.30).

$$[a(\mathbf{k}), a(\mathbf{k}')] = 0 ,$$
$$[a^\dagger(\mathbf{k}), a^\dagger(\mathbf{k}')] = 0 ,$$
$$[a(\mathbf{k}), a^\dagger(\mathbf{k}')] = (2\pi)^3 2\omega\, \delta^3(\mathbf{k}-\mathbf{k}') . \qquad (3.29)$$

$$H = \int \widetilde{dk}\, \omega\, a^\dagger(\mathbf{k})a(\mathbf{k}) + (\mathcal{E}_0 - \Omega_0)V , \qquad (3.30)$$

See code_3.

3.3) Use $U(\Lambda)^{-1}\varphi(x)U(\Lambda) = \varphi(\Lambda^{-1}x)$ to show that

$$U(\Lambda)^{-1}a(\mathbf{k})U(\Lambda) = a(\Lambda^{-1}\mathbf{k}) ,$$
$$U(\Lambda)^{-1}a^\dagger(\mathbf{k})U(\Lambda) = a^\dagger(\Lambda^{-1}\mathbf{k}) , \qquad (3.34)$$

and hence that

$$U(\Lambda)|k_1 \ldots k_n\rangle = |\Lambda k_1 \ldots \Lambda k_n\rangle , \qquad (3.35)$$

where $|k_1 \ldots k_n\rangle = a^\dagger(\mathbf{k}_1)\ldots a^\dagger(\mathbf{k}_n)|0\rangle$ is a state of n particles with momenta $k_1, \ldots, k_n$.

$$\varphi(x) = \int \widetilde{dk}\, [a(\vec k)e^{ikx} + a^\dagger(\vec k)e^{-ikx}]$$

$$U(\Lambda)^{-1}\varphi(x)U(\Lambda) = \int \widetilde{dk}\, [U(\Lambda)^{-1}a(\vec k)U(\Lambda)e^{ikx} + U(\Lambda)^{-1}a^\dagger(\vec k)U(\Lambda)e^{-ikx}]$$

$$= \varphi(\Lambda^{-1}x)$$

$$= \int \widetilde{dk}\, [a(\vec k)\underbrace{e^{ik\Lambda^{-1}x}}_{e^{i\Lambda kx}} + a^\dagger(\vec k)\underbrace{e^{-ik\Lambda^{-1}x}}_{e^{-i\Lambda kx}}] \quad \Big| \; k \to \Lambda^{-1}k$$

$$= \int \widetilde{dk}\, [a(\Lambda^{-1}\vec k)e^{ikx} + a^\dagger(\Lambda^{-1}\vec k)e^{-ikx}]$$

$$\therefore U(\Lambda)^{-1}a(\vec k)U(\Lambda) = a(\Lambda^{-1}\vec k)$$

$$U(\Lambda)^{-1}a^\dagger(\vec k)U(\Lambda) = a^\dagger(\Lambda^{-1}\vec k)$$

$$U(\Lambda)|k_1 \ldots k_n\rangle = U(\Lambda)a^\dagger(\vec k_1)\ldots a^\dagger(\vec k_n)|0\rangle$$

$$= U(\Lambda)a^\dagger(\vec k_1)U(\Lambda)^{-1}U(\Lambda)\ldots U(\Lambda)^{-1}U(\Lambda)a^\dagger(\vec k_n)U(\Lambda)^{-1}U(\Lambda)|0\rangle$$

$$= a^\dagger(\Lambda\vec k_1)\ldots a^\dagger(\Lambda\vec k_n)U(\Lambda)|0\rangle$$

Claim: $U(\Lambda)|0\rangle = |0\rangle$.

Proof: The ground state is the state of zero momentum, which transforms trivially. Moreover, from elementary quantum mechanics, we know that the ground state should be unique. If $p^\mu = 0$ transforms into $\bar p^\mu = 0$ in all frames, and if the ground state is unique, then it means that the ground state is inv under Lorentz.
qed

$$\therefore U(\Lambda)|k_1 \ldots k_n\rangle = a^\dagger(\Lambda\vec k_1)\ldots a^\dagger(\Lambda\vec k_n)|0\rangle$$

$$= |\Lambda k_1 \ldots \Lambda k_n\rangle$$

3.4) Recall that $T(a)^{-1}\varphi(x)T(a) = \varphi(x-a)$, where $T(a) \equiv \exp(-iP^\mu a_\mu)$ is the spacetime translation operator, and P^0 is identified as the hamiltonian H.

a) Let a^μ be infinitesimal, and derive an expression for $[\varphi(x), P^\mu]$.

b) Show that the time component of your result is equivalent to the Heisenberg equation of motion $i\dot\varphi = [\varphi, H]$.

c) For a free field, use the Heisenberg equation to derive the Klein-Gordon equation.

d) Define a spatial momentum operator

$$\mathbf{P} \equiv -\int d^3x\, \Pi(x)\nabla\varphi(x) . \qquad (3.36)$$

Use the canonical commutation relations to show that $\mathbf{P}$ obeys the relation you derived in part (a).

e) Express $\mathbf{P}$ in terms of $a(\mathbf{k})$ and $a^\dagger(\mathbf{k})$.

a) $$T(a)^{-1}\varphi(x)T(a) = \varphi(x-a)$$

$$(1+ip^\mu a_\mu)\varphi(x)(1-ip^\mu a_\mu) = \varphi(x) - a_\mu\partial^\mu\varphi(x)$$

$$\varphi(x) - ia_\mu\varphi(x)p^\mu + ia_\mu p^\mu\varphi(x) + O(a^2) = \varphi(x) - a_\mu\partial^\mu\varphi(x)$$

$$-ia_\mu[\varphi(x), p^\mu] = -a_\mu\partial^\mu\varphi(x)$$

$$[\varphi(x), p^\mu] = -i\partial^\mu\varphi(x)$$

b) $$[\varphi(x), p^0] = -i\underbrace{\partial^0\varphi(x)}_{\dot\varphi}$$

$$\therefore [\varphi(x), H] = i\dot\varphi(x)$$

c) $$\dot\varphi = \pi$$

$$i\dot\varphi = i\pi = [\varphi, H]$$

$$i\dot\pi = i\ddot\varphi = [\pi, H] = i\nabla^2\varphi - im^2\varphi \quad \text{(see code\_4)}$$

$$\underbrace{\ddot\varphi - \nabla^2\varphi}_{-\partial^2\varphi} + m^2\varphi = 0$$

$$(-\partial^2 + m^2)\varphi = 0$$

d) $$[\varphi(x), \vec p] = -i\vec\nabla\varphi(x) \quad \text{(see code\_5)}$$

From part (a),

$$[\varphi(x), p^\mu] = -i\partial^\mu\varphi(x)$$

$$\therefore [\varphi(x), p^i] = -i\partial^i\varphi(x) = (-i\vec\nabla\varphi(x))_i$$

e) $$\vec p = \underbrace{V\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\vec k}_{\text{total zero-point momentum of all oscillators}} + \int\widetilde{dk}\,\vec k\, a^\dagger(\vec k)a(\vec k)$$

cf. $$H = V\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\omega + \int\widetilde{dk}\,\omega\, a^\dagger(\vec k)a(\vec k)$$

3.5) Consider a complex (that is, nonhermitian) scalar field φ with lagrangian density

$$\mathcal{L} = -\partial^\mu\varphi^\dagger\partial_\mu\varphi - m^2\varphi^\dagger\varphi + \Omega_0 . \qquad (3.37)$$

a) Show that φ obeys the Klein-Gordon equation.

b) Treat φ and $\varphi^\dagger$ as independent fields, and find the conjugate momentum for each. Compute the hamiltonian density in terms of these conjugate momenta and the fields themselves (but not their time derivatives).

c) Write the mode expansion of φ as

$$\varphi(x) = \int\widetilde{dk}\Big[a(\mathbf{k})e^{ikx} + b^\dagger(\mathbf{k})e^{-ikx}\Big] . \qquad (3.38)$$

Express $a(\mathbf{k})$ and $b(\mathbf{k})$ in terms of φ and $\varphi^\dagger$ and their time derivatives.

d) Assuming canonical commutation relations for the fields and their conjugate momenta, find the commutation relations obeyed by $a(\mathbf{k})$ and $b(\mathbf{k})$ and their hermitian conjugates.

e) Express the hamiltonian in terms of $a(\mathbf{k})$ and $b(\mathbf{k})$ and their hermitian conjugates. What value must Ω_0 have in order for the ground state to have zero energy?

a) Euler-Lagrange EOM for $\varphi^\dagger$:

$$\frac{\partial\mathcal{L}}{\partial\varphi^\dagger} - \partial^\mu\frac{\partial\mathcal{L}}{\partial(\partial^\mu\varphi^\dagger)} = 0$$

$$-m^2\varphi - \partial^\mu(-\partial_\mu\varphi) = 0$$

$$(-\partial^2 + m^2)\varphi = 0$$

b) $$\mathcal{L} = -|\partial_\mu\varphi|^2 - m^2|\varphi|^2 + \Omega_0$$

$$= -|\partial_0\varphi|^2 - |\partial_i\varphi|^2 - m^2|\varphi|^2 + \Omega_0$$

$$= |\dot\varphi|^2 - |\vec\nabla\varphi|^2 - m^2|\varphi|^2 + \Omega_0$$

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot\varphi} = \dot\varphi^\dagger$$

$$\pi^\dagger = \dot\varphi$$

$$\mathcal{H} = \pi\dot\varphi + \pi^\dagger\dot\varphi^\dagger - \mathcal{L}$$

$$= \pi\pi^\dagger + \pi^\dagger\pi - \pi\pi^\dagger + \vec\nabla\varphi^\dagger\cdot\vec\nabla\varphi + m^2\varphi^\dagger\varphi - \Omega_0$$

$$= \pi^\dagger\pi + \vec\nabla\varphi^\dagger\cdot\vec\nabla\varphi + m^2\varphi^\dagger\varphi - \Omega_0$$

c) $$\varphi(x) = \int\widetilde{dk}\,[a(\vec k)e^{ikx} + b^\dagger(\vec k)e^{-ikx}]$$

This is not so much different from what we have done in the section. We can directly quote our results for a in terms of φ:

$$a(\vec k) = \int d^3x\, e^{-ikx}(\omega\varphi + i\dot\varphi)$$

The only difference b/w a and b is $\varphi \to \varphi^\dagger$:

$$b(\vec k) = \int d^3x\, e^{-ikx}(\omega\varphi^\dagger + i\dot\varphi)$$

d) Assumptions:

$$[\varphi(\vec x,t), \pi(\vec y,t)] = i\delta^3(\vec x - \vec y)$$

$$[\varphi^\dagger(\vec x,t), \pi^\dagger(\vec y,t)] = i\delta^3(\vec x - \vec y)$$

w/ all other commutators being zero. See code_7:

$$[a(\vec k), a^\dagger(\vec k')] = (2\pi)^3 2\omega\,\delta^3(\vec k - \vec k')$$

$$[b(\vec k), b^\dagger(\vec k')] = (2\pi)^3 2\omega\,\delta^3(\vec k - \vec k')$$

and all other commutators are zero.

e) $$H = \int d^3x\,\mathcal{H}(x)$$

$$= -\Omega_0 V + 2V\int\widetilde{dk}\,\omega^2 + \int\widetilde{dk}\,[a^\dagger(\vec k)a(\vec k) + b^\dagger(\vec k)b(\vec k)]$$

See code_8:

$$\Omega_0 = 2\int\widetilde{dk}\,\omega^2$$

$$= 2\int\frac{d^3k}{(2\pi)^3 2\omega}\omega^2$$

$$= \int\frac{d^3k}{(2\pi)^3}\omega$$