Ch 6: Problems

6.1) a) Find an explicit formula for $\mathcal{D}q$ in

$$\langle q'', t'' | q', t' \rangle = \int \mathcal{D}q \, e^{i\int_{t'}^{t''} dt \, L(\dot{q}, q)}.$$

Your formula should be of the form $\mathcal{D}q = C \prod_{j=1}^{N} dq_j$ where C is a constant that you should compute.

b) For the case of a free particle, $V(Q) = 0$, evaluate the path integral of $\langle q'', t'' | q', t' \rangle$ explicitly. Hint: integrate over q_1, then q_2, etc. and look for a pattern. Express your final answer in terms of q', t', q'', t'', and m. Restore $\hbar$ by dimensional analysis.

c) Compute $\langle q'', t'' | q', t' \rangle = \langle q'' | e^{-iH(t''-t')} | q' \rangle$ by inserting a complete set of momentum eigenstates, and performing the integral over the momentum. Compare w/ your result in part (b).

Solution

a) $\langle q'', t'' | q', t' \rangle = \langle q'' | e^{-iH(t''-t')} | q' \rangle$

$$= \langle q'' | e^{-iHT} | q' \rangle, \quad T := t'' - t'$$

$$= \langle q'' | \underbrace{e^{-iH\delta t} \cdots e^{-iH\delta t}}_{(N+1)\text{-many}} | q'' \rangle, \qquad \delta t = \frac{T}{N+1}$$

$\Rightarrow N$ insertions

$$= \int_{-\infty}^{\infty} dq_N \cdots dq_2 \, dq_1 \, \langle q'' | e^{-iH\delta t} | q_N \rangle \cdots \langle q_2 | e^{-iH\delta t} | q_1 \rangle$$

$$\times \langle q_1 | e^{-iH\delta t} | q' \rangle$$

$$H = \frac{1}{2m} P^2 + V(Q)$$

$$e^{-iH\delta t} = e^{-i\left(\frac{1}{2m} P^2 + V(Q)\right)\delta t}$$

$$= e^{-i\frac{1}{2m} P^2 \delta t} e^{-iV(Q)\delta t} + O(\delta t^2)$$

$$\langle q_2 | e^{-iH\delta t} | q_1 \rangle = \int_{-\infty}^{\infty} dp_1 \, \langle q_2 | e^{-i\frac{1}{2m} P^2 \delta t} | p_1 \rangle$$

$$\times \langle p_1 | e^{-iV(Q)\delta t} | q_1 \rangle$$

$$= \int_{-\infty}^{\infty} dp_1 \, e^{-i\frac{1}{2m}p_1^2 \delta t} \, e^{-iV(q_1)\delta t} \underbrace{\langle q_2 | p_1 \rangle}_{\frac{e^{ip_1 q_2}}{\sqrt{2\pi}}} \underbrace{\langle p_1 | q_1 \rangle}_{\frac{e^{-ip_1 q_1}}{\sqrt{2\pi}}}$$

$$= \int_{-\infty}^{\infty} \frac{dp_1}{2\pi} \, e^{-ip_1^2 \delta t/2m} \, e^{ip_1(q_2 - q_1)} \, e^{-iV(q_1)\delta t}$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{\frac{i\delta t}{m}}} \, e^{\frac{im(q_2-q_1)^2}{2\delta t}} \, e^{-i\delta t \, V(q_1)}$$

using Mathematica.

$$\langle q'', t'' | q', t' \rangle = \int_{-\infty}^{\infty} dq_N \cdots dq_1$$

$$\times \left[\sqrt{\frac{m}{2\pi i \delta t}} \, e^{i\left[\frac{1}{2}m\left(\frac{q''-q_N}{\delta t}\right)^2 - V(q_N)\right]\delta t} \right]$$

$$\times \cdots$$

$$\times \left[\sqrt{\frac{m}{2\pi i \delta t}} \, e^{i\left[\frac{1}{2}m\left(\frac{q_2-q_1}{\delta t}\right)^2 - V(q_1)\right]\delta t} \right]$$

$$\times \left[\sqrt{\frac{m}{2\pi i \delta t}} \, e^{i\left[\frac{1}{2}m\left(\frac{q_1-q'}{\delta t}\right)^2 - V(q')\right]\delta t} \right]$$

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$$= \int_{-\infty}^{\infty} \left(\prod_{j=1}^{N} dq_j \right) \underbrace{\sqrt{\frac{m}{2\pi i \delta t}}^{\,N+1}}_{C} e^{i \underbrace{\sum_{k=0}^{N} \left[\frac{1}{2} m \left(\frac{q_{k+1} - q_k}{\delta t} \right)^2 - V(q_k) \right] \delta t}_{\text{classical action when } \delta t \to 0.}}$$

$$C = \sqrt{\frac{m}{2\pi i \delta t}}^{\,N+1}, \quad q_{N+1} = q'', \quad q_0 = q'$$

b) $V = 0$

$$\langle q'', t'' | q', t' \rangle = \int_{-\infty}^{\infty} \left(\prod_{j=1}^{N} dq_j \right) \sqrt{\frac{m}{2\pi i \delta t}}^{\,N+1} e^{i \sum_{k=0}^{N} \frac{1}{2} m \left(\frac{q_{k+1} - q_k}{\delta t} \right)^2 \delta t}$$

Integrate one by one using Mathematica:

$$dq_1 \Rightarrow -\frac{i e^{\frac{1}{4} i m \delta t (q_0 - q_2)^2} m}{2\sqrt{\pi}\, \delta t \sqrt{-i m \delta t}}$$

$$dq_2 \Rightarrow \frac{e^{\frac{1}{6} i m \delta t (q_0 - q_3)^2} \sqrt{-\frac{im}{\delta t}}}{\sqrt{6\pi}\, \delta t^2}$$

$$dq_3 \Rightarrow \frac{e^{\frac{1}{8} i m \delta t (q_0 - q_4)^2} \sqrt{-i m \delta t}}{2\sqrt{2\pi}\, \delta t^4}$$

$$dq_4 \Rightarrow \frac{e^{\frac{1}{10} i m \delta t (q_0 - q_5)^2} \sqrt{-\frac{im}{\delta t}}}{\sqrt{10\pi}\, \delta t^4}$$

Guess (see code-1):

$$\langle q_{N+1}, t'' | q_0, t' \rangle = e^{\frac{1}{2(N+1)} im\delta t^{-1}(q_{N+1}-q_0)^2} \sqrt{-im\delta t^{-1}}$$

$$\times \frac{1}{\sqrt{2(N+1)\pi}}$$

Put $\delta t = T/(N+1)$ and simplify:

$$\langle q'', t'' | q', t' \rangle = \sqrt{-\frac{im}{2\pi(t''-t')}}\, e^{\frac{im(q''-q')^2}{2(t''-t')}}$$

Restore $\hbar$ by dimensional analysis:

$$[\hbar] = [px] = [mvx] = \left[\frac{mx^2}{t}\right]$$

$$[\langle q | q' \rangle] = ?$$

$$|\psi\rangle = \int dq \, |q\rangle\langle q|\psi\rangle \text{ in 1D } \therefore [|q\rangle] = \left[\frac{1}{\sqrt{x}}\right]$$

$$\therefore [\langle q|q'\rangle] = \left[\frac{1}{x}\right] \text{ in 1D}$$

$$\left[\sqrt{\frac{m}{t}}\,\hbar^\alpha\right] = [m^{\frac{1}{2}} t^{-\frac{1}{2}} m^\alpha x^{2\alpha} t^{-\alpha}] = [m^0 x^{-1} t^0]$$

$$= [m^{\frac{1}{2}+\alpha} x^{2\alpha} t^{-(\frac{1}{2}+\alpha)}] \quad \therefore \alpha = -\frac{1}{2}$$

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$$\left[\frac{mq^2}{t} \hbar^{\beta} \right] = \left[m x^2 t^{-1} m^{\beta} x^{2\beta} t^{-\beta} \right]$$

$$= \left[m^{1+\beta} x^{2(1+\beta)} t^{-(1+\beta)} \right]$$

$$= 1 = \left[m^0 x^0 t^0 \right]$$

$\therefore \beta = -1$

$$\therefore \langle q'', t'' | q', t' \rangle = \sqrt{\frac{m}{2\pi i \hbar (t''-t')}} \, e^{\frac{im(q''-q')^2}{2\hbar(t''-t')}}$$

c) $\langle q'', t'' | q', t' \rangle = \langle q'' | e^{-iHT} | q' \rangle$

$$= \langle q'' | e^{-i\frac{1}{2m}p^2 T} | q' \rangle$$

$$= \int_{-\infty}^{\infty} dp \, \langle q'' | e^{-i\frac{1}{2m}p^2 T} | p \rangle \langle p | q' \rangle$$

$$= \int_{-\infty}^{\infty} dp \, e^{-i\frac{1}{2m}p^2 T} \langle q'' | p \rangle \langle p | q' \rangle$$

$$= \int_{-\infty}^{\infty} dp \, e^{-i\frac{1}{2m}p^2 T} \frac{e^{ipq''}}{\sqrt{2\pi}} \frac{e^{-ipq'}}{\sqrt{2\pi}}$$

$$= \sqrt{\frac{m}{2\pi i T}} \, e^{im(q''-q')^2/2T}$$

using Mathematica — same result.