6.1) a) Find an explicit formula for $\mathcal{D}q$ in eq. (6.9). Your formula should be of the form $\mathcal{D}q = C\prod_{j=1}^{N} dq_j$, where C is a constant that you should compute.

b) For the case of a free particle, $V(Q) = 0$, evaluate the path integral of eq. (6.9) explicitly. Hint: integrate over q_1, then q_2, etc, and look for a pattern. Express you final answer in terms of q', t', q'', t'', and m. Restore $\hbar$ by dimensional analysis.

c) Compute $\langle q'', t''|q', t'\rangle = \langle q''|e^{-iH(t''-t')}|q'\rangle$ by inserting a complete set of momentum eigenstates, and performing the integral over the momentum. Compare with your result in part (b).

$$\langle q'', t''|q', t'\rangle = \int \mathcal{D}q \, \exp\left[i\int_{t'}^{t''} dt \; L(\dot{q}(t), q(t))\right], \qquad (6.9)$$

a) $$\langle q'',t''|q',t'\rangle = \langle q''|e^{-iH(t''-t')}|q'\rangle$$

$$= \langle q''|e^{-iHT}|q'\rangle, \qquad T := t''-t'$$

$$= \langle q''|\underbrace{e^{-iH\delta t}\cdots e^{-iH\delta t}}_{(N+1)\text{-many} \Rightarrow N \text{ insertions}}|q'\rangle, \qquad \delta t := \frac{T}{N+1}$$

$$= \int_{-\infty}^{\infty} dq_N \cdots dq_1 \, \langle q''|e^{-iH\delta t}|q_N\rangle \cdots \langle q_2|e^{-iH\delta t}|q_1\rangle\langle q_1|e^{-iH\delta t}|q'\rangle$$

$$H = \frac{1}{2m}P^2 + V(Q)$$

$$e^{-iH\delta t} = e^{-i\left(\frac{1}{2m}P^2 + V(Q)\right)\delta t}$$

$$= e^{-i\frac{1}{2m}P^2\delta t}\, e^{-iV(Q)\delta t} + O(\delta t^2)$$

$$\langle q_2|e^{-iH\delta t}|q_1\rangle = \int_{-\infty}^{\infty} dp_1 \, \langle q_2|e^{-i\frac{1}{2m}P^2\delta t}|p_1\rangle\langle p_1|e^{-iV(Q)\delta t}|q_1\rangle$$

$$= \int_{-\infty}^{\infty} dp_1 \, e^{-i\frac{1}{2m}p_1^2\delta t}\, e^{-iV(q_1)\delta t}\, \underbrace{\langle q_2|p_1\rangle}_{\frac{e^{ip_1q_2}}{\sqrt{2\pi}}}\underbrace{\langle p_1|q_1\rangle}_{\frac{e^{-ip_1q_1}}{\sqrt{2\pi}}}$$

$$= \int_{-\infty}^{\infty} \frac{dp_1}{2\pi}\, e^{-ip_1^2\delta t/2m}\, e^{ip_1(q_2-q_1)}\, e^{-iV(q_1)\delta t}$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{\frac{i\delta t}{m}}}\, e^{\frac{im(q_2-q_1)^2}{2\delta t}}\, e^{-i\delta t\, V(q_1)}$$

$$\langle q'',t''|q',t'\rangle = \int_{-\infty}^{\infty} dq_N\cdots dq_1 \left\{\sqrt{\frac{m}{2\pi i\delta t}}\, e^{i\left[\frac{1}{2}m\left(\frac{q''-q_N}{\delta t}\right)^2 - V(q_N)\right]\delta t}\right\}\cdots\left\{\sqrt{\frac{m}{2\pi i\delta t}}\, e^{i\left[\frac{1}{2}m\left(\frac{q_2-q_1}{\delta t}\right)^2 - V(q_1)\right]\delta t}\right\}\left\{\sqrt{\frac{m}{2\pi i\delta t}}\, e^{i\left[\frac{1}{2}m\left(\frac{q_1-q'}{\delta t}\right)^2 - V(q')\right]\delta t}\right\}$$

$$= \int_{-\infty}^{\infty} \left(\prod_{j=1}^{N} dq_j\right) \underbrace{\sqrt{\frac{m}{2\pi i\delta t}}^{\,N+1}}_{C}\; e^{\underbrace{i\sum_{k=0}^{N}\left[\frac{1}{2}m\left(\frac{q_{k+1}-q_k}{\delta t}\right)^2 - V(q_k)\right]\delta t}_{\text{classical action when } \delta t\to 0}}$$

$$C = \sqrt{\frac{m}{2\pi i\delta t}}^{\,N+1}, \qquad q_{N+1} = q'', \quad q_0 = q'$$

b) $V = 0$

$$\langle q'',t''|q',t'\rangle = \int_{-\infty}^{\infty}\left(\prod_{j=1}^{N} dq_j\right)\sqrt{\frac{m}{2\pi i\delta t}}^{\,N+1}\, e^{i\sum_{k=0}^{N}\frac{1}{2}m\left(\frac{q_{k+1}-q_k}{\delta t}\right)^2\delta t}$$

Integrate one by one using Mathematica:

$$dq_1 \Rightarrow -\frac{i\, e^{\frac{1}{4}im\delta t^{-1}(q_0-q_2)^2}\, m}{2\sqrt{\pi}\,\delta t\sqrt{-im\delta t}}$$

$$dq_2 \Rightarrow \frac{e^{\frac{1}{6}im\delta t^{-1}(q_0-q_3)^2}\sqrt{-\frac{im}{\delta t}}}{\sqrt{6\pi}\,\delta t^2}$$

$$dq_3 \Rightarrow \frac{e^{\frac{1}{8}im\delta t^{-1}(q_0-q_4)^2}\sqrt{-im\delta t}}{2\sqrt{2\pi}\,\delta t^4}$$

$$dq_4 \Rightarrow \frac{e^{\frac{1}{10}im\delta t^{-1}(q_0-q_5)^2}\sqrt{-\frac{im}{\delta t}}}{\sqrt{10\pi}\,\delta t^4}$$

Guess (see code_1):

$$\langle q_{N+1}, t''|q_0, t'\rangle = e^{\frac{1}{2(N+1)}im\delta t^{-1}(q_{N+1}-q_0)^2}\sqrt{-im\delta t^{-1}}\,\frac{1}{\sqrt{2(N+1)\pi}}$$

Put $\delta t = \frac{T}{N+1}$ and simplify:

$$\langle q'', t''|q', t'\rangle = \sqrt{-\frac{im}{2\pi(t''-t')}}\; e^{\frac{im(q''-q')^2}{2(t''-t')}}$$

Restore $\hbar$ by dimensional analysis:

$$[\hbar] = [px] = [mvx] = \left[\frac{mx^2}{t}\right]$$

$$[\langle q|q'\rangle] = ?$$

$$\left[\int dq\, |q\rangle\langle q|\right] = 1 \quad \therefore \quad [\langle q|q'\rangle] = \left[\frac{1}{q}\right] \text{ in 1D}$$

$$\left[\sqrt{\frac{m}{t}}\,\hbar^{\alpha}\right] = [m^{\frac{1}{2}}t^{-\frac{1}{2}}m^{\alpha}x^{2\alpha}t^{-\alpha}]$$

$$= [m^0x^{-1}t^0]$$

$$= [m^{\frac{1}{2}+\alpha}x^{2\alpha}t^{-(\frac{1}{2}+\alpha)}]$$

$$\therefore \alpha = -\frac{1}{2}$$

$$\left[\frac{mq^2}{t}\hbar^{\beta}\right] = [mx^2t^{-1}m^{\beta}x^{2\beta}t^{-\beta}]$$

$$= [m^{1+\beta}x^{2(1+\beta)}t^{-(1+\beta)}]$$

$$= 1$$

$$= [m^0x^0t^0]$$

$$\therefore \beta = -1$$

$$\therefore \langle q'', t''|q', t'\rangle = \sqrt{\frac{m}{2\pi i\hbar(t''-t')}}\; e^{\frac{im(q''-q')^2}{2\hbar(t''-t')}}$$

c) $$\langle q'',t''|q',t'\rangle = \langle q''|e^{-iHT}|q'\rangle$$

$$= \langle q''|e^{-i\frac{1}{2m}P^2T}|q'\rangle$$

$$= \int_{-\infty}^{\infty} dp\, \langle q''|e^{-i\frac{1}{2m}P^2T}|p\rangle\langle p|q'\rangle$$

$$= \int_{-\infty}^{\infty} dp\, e^{-i\frac{1}{2m}p^2T}\langle q''|p\rangle\langle p|q'\rangle$$

$$= \int_{-\infty}^{\infty} dp\, e^{-i\frac{1}{2m}p^2T}\,\frac{e^{ipq''}}{\sqrt{2\pi}}\,\frac{e^{-ipq'}}{\sqrt{2\pi}}$$

$$= \sqrt{\frac{m}{2\pi iT}}\; e^{im(q''-q')^2/2T} \qquad \therefore \text{same result}$$