Path integral for harmonic oscillator

$$H(P,Q) = \frac{1}{2m} P^2 + \frac{1}{2} m\omega^2 Q^2$$

$$\langle 0|0\rangle_f = \int Dp\, Dq\; e^{i\int_{-\infty}^{\infty} dt\, \left(p\dot{q} - (1-i\varepsilon)H + fq\right)}$$

$$\{H \to (1-i\varepsilon)H\} \equiv \left\{ \begin{array}{l} m \to (1-i\varepsilon)m \\ m\omega^2 \to (1-i\varepsilon)m\omega^2 \end{array} \right\}$$

Lagrangian formulation:

$$\langle 0|0\rangle_f = \int Dq\; e^{i\int_{-\infty}^{\infty} dt \left[\frac{1}{2}(1+i\varepsilon)m\dot{q}^2 - \frac{1}{2}(1-i\varepsilon)m\omega^2 q^2 + fq\right]}$$

To simplify notation: $m = 1$

Fourier-transformed variables:

$$\tilde{q}(E) = \int_{-\infty}^{\infty} dt\, e^{iEt} q(t)$$

$$q(t) = \int_{-\infty}^{\infty} \frac{dE}{2\pi} e^{-iEt} \tilde{q}(E)$$

$$\frac{1}{2}(1+i\varepsilon)\dot{q}^2 - \frac{1}{2}(1-i\varepsilon)\omega^2 q^2 + fq = \frac{1}{2}(1+i\varepsilon)\int_{-\infty}^{\infty} \frac{dE}{2\pi}(-iE)e^{-iEt}\tilde{q}(t)\int_{-\infty}^{\infty}\frac{dE'}{2\pi}(-iE')e^{-iE't}\tilde{q}(E') - \frac{1}{2}(1-i\varepsilon)\omega^2\int_{-\infty}^{\infty}\frac{dE}{2\pi}e^{-iEt}\tilde{q}(E)\int_{-\infty}^{\infty}\frac{dE'}{2\pi}e^{-iE't}\tilde{q}(E') + \frac{1}{2}\left[\int_{-\infty}^{\infty}\frac{dE}{2\pi}e^{-iEt}\tilde{f}(E)\int_{-\infty}^{\infty}\frac{dE'}{2\pi}e^{-iE't}\tilde{q}(E') + \int_{-\infty}^{\infty}\frac{dE}{2\pi}e^{-iEt}\tilde{q}(E)\int_{-\infty}^{\infty}\frac{dE'}{2\pi}e^{-iE't}f(E')\right]$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\frac{dE}{2\pi}\frac{dE'}{2\pi}e^{-i(E+E')t}\left\{\left[-(1+i\varepsilon)EE' - (1-i\varepsilon)\omega^2\right]\tilde{q}(E)\tilde{q}(E') + \tilde{f}(E)\tilde{q}(E') + \tilde{f}(E')\tilde{q}(E)\right\}$$

The only t dependence is in $e^{-i(E+E')t}$, which gives $2\pi\,\delta(E+E')$ when integrated:

$$S = \int_{-\infty}^{\infty} dt\left[\frac{1}{2}(1+i\varepsilon)\dot{q}^2 - \frac{1}{2}(1-i\varepsilon)\omega^2 q^2 + fq\right]$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\frac{dE}{2\pi}\left\{\left[-(1+i\varepsilon)E(-E) - (1-i\varepsilon)\omega^2\right]\tilde{q}(E)\tilde{q}(-E) + \tilde{f}(E)\tilde{q}(-E) + \tilde{f}(-E)\tilde{q}(E)\right\}$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\frac{dE}{2\pi}\left\{\left[(1+i\varepsilon)E^2 - (1-i\varepsilon)\omega^2\right]\tilde{q}(E)\tilde{q}(-E) + \tilde{f}(E)\tilde{q}(-E) + \tilde{f}(-E)\tilde{q}(E)\right\}$$

$$(1+i\varepsilon)E^2 - (1-i\varepsilon)\omega^2 = E^2 - \omega^2 + i(E^2+\omega^2)\varepsilon$$

$$\to E^2 - \omega^2 + i\varepsilon$$

$$S = \frac{1}{2}\int_{-\infty}^{\infty}\frac{dE}{2\pi}\left[(E^2-\omega^2+i\varepsilon)\tilde{q}(E)\tilde{q}(-E) + \tilde{f}(E)\tilde{q}(-E) + \tilde{f}(-E)\tilde{q}(E)\right]$$

$$\tilde{q}(E) = \tilde{x}(E) + B_1$$

$$\tilde{q}(-E) = \tilde{x}(-E) + B_2$$

$$[\cdots] = \tilde{x}(E)(E^2-\omega^2+i\varepsilon)\tilde{x}(-E) + C_1\tilde{x}(E) + C_2\tilde{x}(-E) + D$$

$$C_1 = C_2 = 0 \Rightarrow B_1 = -\frac{\tilde{f}(E)}{E^2-\omega^2+i\varepsilon}, \quad B_2 = -\frac{\tilde{f}(-E)}{E^2-\omega^2+i\varepsilon}, \quad D = -\frac{\tilde{f}(E)\tilde{f}(-E)}{E^2-\omega^2+i\varepsilon}$$

using Mathematica.

$$\therefore S = \frac{1}{2}\int_{-\infty}^{\infty}\frac{dE}{2\pi}\left[\tilde{x}(E)(E^2-\omega^2+i\varepsilon)\tilde{x}(-E) - \frac{\tilde{f}(E)\tilde{f}(-E)}{E^2-\omega^2+i\varepsilon}\right]$$

$$Dq = Dx$$

$$\therefore \langle 0|0\rangle_f = \int Dq\, e^{iS}$$

$$= e^{\frac{i}{2}\int_{-\infty}^{\infty}\frac{dE}{2\pi}\frac{\tilde{f}(E)\tilde{f}(-E)}{-E^2+\omega^2-i\varepsilon}} \int Dx\; e^{\frac{i}{2}\int_{-\infty}^{\infty}\frac{dE}{2\pi}\tilde{x}(E)(E^2-\omega^2+i\varepsilon)\tilde{x}(-E)}$$

Key point: The path integral here is $\langle 0|0\rangle_{f=0}$. But if $\exists$ no external force, a system in its ground state will remain so $\therefore \langle 0|0\rangle_{f=0} = 1$.

$$\therefore \langle 0|0\rangle_f = e^{\frac{i}{2}\int_{-\infty}^{\infty}\frac{dE}{2\pi}\frac{\tilde{f}(E)\tilde{f}(-E)}{-E^2+\omega^2-i\varepsilon}}$$

Time-domain variables:

$$\int_{-\infty}^{\infty}\frac{dE}{2\pi}\left[\int_{-\infty}^{\infty}dt\, e^{iEt}f(t)\right]\frac{1}{-E^2+\omega^2-i\varepsilon}\left[\int_{-\infty}^{\infty}dt'\, e^{-iEt'}f(t')\right] = \int_{-\infty}^{\infty}dt\,dt'\, f(t)\left[\int_{-\infty}^{\infty}\frac{dE}{2\pi}\frac{e^{iE(t-t')}}{-E^2+\omega^2-i\varepsilon}\right]f(t'), \quad t \leftrightarrow t'$$

$$= \int_{-\infty}^{\infty}dt\,dt'\, f(t)\left[\int_{-\infty}^{\infty}\frac{dE}{2\pi}\frac{e^{-iE(t-t')}}{-E^2+\omega^2-i\varepsilon}\right]f(t')$$

$$= \int_{-\infty}^{\infty}dt\,dt'\, f(t)G(t-t')f(t')$$

$$\langle 0|0\rangle_f = e^{\frac{i}{2}\int_{-\infty}^{\infty}dt\,dt'\, f(t)G(t-t')f(t')}$$

$$G(t-t') := \int_{-\infty}^{\infty}\frac{dE}{2\pi}\frac{e^{-iE(t-t')}}{-E^2+\omega^2-i\varepsilon}$$: Green function for oscillator

$$\left(\frac{\partial^2}{\partial t^2}+\omega^2\right)G(t-t') = \delta(t-t')$$

$$= \int_{-\infty}^{\infty}\frac{dE}{2\pi}\frac{-E^2e^{-iE(t-t')}}{-E^2+\omega^2-i\varepsilon} + \omega^2\int_{-\infty}^{\infty}\frac{dE}{2\pi}\frac{e^{-iE(t-t')}}{-E^2+\omega^2-i\varepsilon}$$

$$= \int_{-\infty}^{\infty}\frac{dE}{2\pi}\frac{(-E^2+\omega^2)e^{-iE(t-t')}}{-E^2+\omega^2-i\varepsilon}\Bigg|_{\varepsilon\to 0}$$

$$= \int_{-\infty}^{\infty}\frac{dE}{2\pi}e^{-iE(t-t')}$$

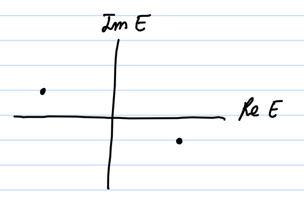
$$= \delta(t-t') \checkmark$$

As a contour integral:

$$E^2 - \omega^2 + i\varepsilon = 0 \Rightarrow E = \pm\sqrt{\omega^2 - i\varepsilon}$$

$$= \pm(\omega - i\varepsilon)$$

$$= \left\{\begin{array}{l}\omega - i\varepsilon \\ -\omega + i\varepsilon\end{array}\right.$$



$$e^{-iE(t-t')} = e^{-iR(\cos(\theta)+i\sin(\theta))(t-t')}$$

$$= e^{R\sin(\theta)(t-t') + i(\dots)}$$

$t - t' > 0 \Rightarrow$ close in the upper half plane

$t - t' < 0 \Rightarrow$ close in the lower half plane

$$G(t-t') = -\int_{-\infty}^{\infty}\frac{dE}{2\pi}\frac{e^{-iE(t-t')}}{(E-\omega+i\varepsilon)(E+\omega-i\varepsilon)}$$

$$= \left\{\begin{array}{ll} \underbrace{-(-1)}_{\text{direction}}\, i\,\dfrac{e^{-iE(t-t')}}{E+\omega-i\varepsilon}\Bigg|_{E=\omega-i\varepsilon}, & t-t'>0 \\ -i\,\dfrac{e^{-iE(t-t')}}{E-\omega+i\varepsilon}\Bigg|_{E=-\omega+i\varepsilon}, & t-t'<0 \end{array}\right.$$

$$= \left\{\begin{array}{ll} i\,\dfrac{e^{-i\omega(t-t')}}{2\omega}, & t-t'>0 \\ i\,\dfrac{e^{i\omega(t-t')}}{2\omega}, & t-t'<0 \end{array}\right.$$

$$= \frac{i}{2\omega}e^{-i\omega|t-t'|}$$

$$\langle 0|T\{Q(t_1)\dots\}|0\rangle = \left[\frac{1}{i}\frac{\delta}{\delta f(t_1)}\right]\cdots\langle 0|0\rangle_f\Bigg|_{f=0}$$

$$\langle 0|T\{Q(t_1)Q(t_2)\}|0\rangle = \left[\frac{1}{i}\frac{\delta}{\delta f(t_1)}\right]\left[\frac{1}{i}\frac{\delta}{\delta f(t_2)}\right]\langle 0|0\rangle_f\Bigg|_{f=0}$$

$$\langle 0|0\rangle_f = e^{\frac{i}{2}\int_{-\infty}^{\infty}dt\,dt'\, f(t)G(t-t')f(t')}$$

$$\frac{\delta}{\delta f(t_2)}\langle 0|0\rangle_f = \frac{i}{2}\int_{-\infty}^{\infty}dt\,dt'\left[\delta(t-t_2)G(t-t')f(t') + f(t)G(t-t')\delta(t'-t_2)\right]e^{\frac{i}{2}\int_{-\infty}^{\infty}dt\,dt'\, f(t)G(t-t')f(t')}$$

$$= \frac{i}{2}\left[\int_{-\infty}^{\infty}dt'\, G(t_2-t')f(t') + \int_{-\infty}^{\infty}dt\, f(t)G(t-t_2)\right]e^{\frac{i}{2}\int_{-\infty}^{\infty}dt\,dt'\, f(t)G(t-t')f(t')}$$

$$= i\int_{-\infty}^{\infty}dt\, G(t_2-t)f(t)\,\langle 0|0\rangle_f$$

$$\frac{\delta}{\delta f(t_1)}\frac{\delta}{\delta f(t_2)}\langle 0|0\rangle_f = \left[\frac{\delta}{\delta f(t_1)}\, i\int_{-\infty}^{\infty}dt\, G(t_2-t)f(t)\right]\langle 0|0\rangle_f + \underbrace{i\int_{-\infty}^{\infty}dt\, G(t_2-t)f(t)\frac{\delta}{\delta f(t_1)}\langle 0|0\rangle_f}_{\text{this piece doesn't matter} \because \exists \text{ explicit } f \text{ here, set to zero at the end.}}$$

$$= i\int_{-\infty}^{\infty}dt\, G(t_2-t)\,\delta(t-t_1)\,\langle 0|0\rangle_f + \cdots$$

$$= i\,G(t_2-t_1) + \cdots$$

$$\langle 0|T\{Q(t_1)Q(t_2)\}|0\rangle = \frac{1}{i^2}\left[i\,G(t_2-t_1) + \cdots\right]_{f=0}$$

$$= \frac{1}{i}G(t_2-t_1)$$

Odd number of Qs $\Rightarrow 0$.

See code_1:

$$\langle 0|T\{Q(t_1)Q(t_2)Q(t_3)Q(t_4)\}|0\rangle = \frac{1}{i^2}\left(G_{12}G_{34} + G_{13}G_{24} + G_{14}G_{23}\right), \quad G_{ij} := G(t_i - t_j)$$

More generally:

$$\langle 0|T\{Q(t_1)\dots Q(t_{2n})\}|0\rangle = \frac{1}{i^n}\sum_{\text{pairings}} G(t_{i_1}-t_{i_2})\cdots G(t_{i_{2n-1}}-t_{i_{2n}})$$