Attempts at relativistic quantum mechanics

Klein-Gordon:

$$-\hbar^2 \frac{\partial^2}{\partial t^2} \psi(\vec{x},t) = \left(-\hbar^2 c^2 \vec{\nabla}^2 + m^2 c^4\right) \psi(\vec{x},t)$$

$$x^\mu = (ct, \vec{x})$$

$$g_{\mu\nu} = -+++$$

$$g^{\mu\nu} g_{\nu\rho} = \delta^\mu{}_\rho$$

$$\bar{x}^\mu = \underbrace{\Lambda^\mu{}_\nu}_{\text{Lorentz transformation}} x^\nu + \underbrace{a^\mu}_{\text{translation}}$$

$$\begin{aligned}
(\bar{x}-\bar{x}')^2 &= (\bar{x}-\bar{x}')^\mu (\bar{x}-\bar{x}')_\mu \\
&= g_{\mu\nu} (\bar{x}-\bar{x}')^\mu (\bar{x}-\bar{x}')^\nu \\
&= g_{\mu\nu} \Lambda^\mu{}_\rho (x-x')^\rho \Lambda^\nu{}_\sigma (x-x')^\sigma \\
&= g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma (x-x')^\rho (x-x')^\sigma \\
&= g_{\rho\sigma} (x-x')^\rho (x-x')^\sigma
\end{aligned}$$

$$\therefore g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = g_{\rho\sigma}$$

Two inertial frames: $\psi(x) = \bar{\psi}(\bar{x})$

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right)$$

$$\partial^\mu = \frac{\partial}{\partial x_\mu} = \left(-\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right)$$

$$\partial^\mu x^\nu = g^{\mu\nu}$$

$$\begin{aligned}
\bar{\partial}^\mu = \Lambda^\mu{}_\nu \partial^\nu : \quad \bar{\partial}^\mu \bar{x}^\nu &= (\Lambda^\mu{}_\rho \partial^\rho)(\Lambda^\nu{}_\sigma x^\sigma) \\
&= \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \underbrace{\partial^\rho x^\sigma}_{g^{\rho\sigma}} \\
&= g^{\mu\nu}
\end{aligned}$$

KG:

$$-\hbar^2 \frac{\partial^2}{\partial t^2} \psi(x) = \left(-\hbar^2 c^2 \vec{\nabla}^2 + m^2 c^4\right) \psi(x) \quad \Big| \ \frac{1}{\hbar^2 c^2}$$

$$\Bigg(\underbrace{-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \vec{\nabla}^2}_{\partial_\mu \partial^\mu = \partial^2 =: \Box} - \frac{m^2 c^2}{\hbar^2}\Bigg) \psi(x) = 0$$

$$\left(-\Box + \frac{m^2c^2}{\hbar^2}\right) \psi(x) = 0$$

$$\left(-\bar{\Box} + \frac{m^2c^2}{\hbar^2}\right) \bar{\psi}(\bar{x}) = 0$$

$$\begin{aligned}
\bar{\Box} &= \bar{\partial}_\mu \bar{\partial}^\mu \\
&= g_{\mu\nu} \bar{\partial}^\mu \bar{\partial}^\nu \\
&= g_{\mu\nu} \Lambda^\mu{}_\rho \partial^\rho \Lambda^\nu{}_\sigma \partial^\sigma \\
&= g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \partial^\rho \partial^\sigma \\
&= g_{\rho\sigma} \partial^\rho \partial^\sigma \\
&= \partial_\mu \partial^\mu \\
&= \Box
\end{aligned}$$

$$\left.\begin{aligned} \bar{\psi}(x) &= \psi(x) \\ \bar{\Box} &= \Box \end{aligned}\right\} \text{KG is consistent w/ relativity.}$$

KG: not first order in time derivative $\therefore$ not compatible w/ Schrödinger. $|\psi(x)|^2$ is in general not time-indep.

$$\begin{cases} \left(\Box - \dfrac{m^2c^2}{\hbar^2}\right)\psi = 0 \\ \psi^*\left(\overleftarrow{\Box} - \dfrac{m^2c^2}{\hbar^2}\right) = 0 \end{cases}$$

$$\psi^*\left(\Box - \frac{m^2c^2}{\hbar^2}\right)\psi = 0$$

$$\psi^*\left(\overleftarrow{\Box} - \frac{m^2c^2}{\hbar^2}\right)\psi = 0$$

$-$

$$\psi^* \Box \psi - (\Box \psi^*)\psi = 0$$

$$\psi^* \partial_\mu \partial^\mu \psi - \partial_\mu \partial^\mu \psi^* \psi = 0$$

$$\partial_\mu(\psi^* \partial^\mu \psi) - \partial_\mu \psi^* \partial^\mu \psi - \partial_\mu(\partial^\mu \psi^* \psi) + \partial^\mu \psi^* \partial_\mu \psi = 0$$

$$\partial_\mu(\psi^* \partial^\mu \psi - \partial^\mu \psi^* \psi) = 0$$

$$j^\mu = \psi^* \partial^\mu \psi - \partial^\mu \psi^* \psi : \quad \partial_\mu j^\mu = 0$$

$$\begin{aligned}
c\rho &= j^0 \\
&= -\psi^* \frac{1}{c}\frac{\partial \psi}{\partial t} + \frac{1}{c}\frac{\partial \psi^*}{\partial t}\psi \ : \text{not positive definite}
\end{aligned}$$

$\therefore$ problem w/ probability interpretation

Dirac: spin-1/2

$$i\hbar \frac{\partial}{\partial t} \psi_a(x) = \left(-i\hbar c\, \vec{\alpha}_{ab} \cdot \vec{\nabla} + mc^2 \beta_{ab}\right) \psi_b(x) \ : \text{Dirac equation}$$

$a, b = 1, 2$: spin indices

Compatible w/ Schrödinger $\because$ first order in time derivative.
$\vec{\alpha}$, β are matrices in spin space.

$$H_{ab} = c\vec{p}\cdot\vec{\alpha}_{ab} + mc^2 \beta_{ab}$$

$$\begin{aligned}
(H^2)_{ab} &= \left[(c\vec{p}\cdot\vec{\alpha} + mc^2\beta)(c\vec{p}\cdot\vec{\alpha} + mc^2\beta)\right]_{ab} \\
&= \left[c^2 p_i p_j \alpha_i \alpha_j + mc^3 p_i (\alpha_i \beta + \beta \alpha_i) + m^2c^4 \beta^2\right]_{ab} \\
&= c^2 p_i p_j \frac{\{\alpha_i, \alpha_j\}_{ab}}{2} + mc^3 \{\alpha_i, \beta\}_{ab} + m^2c^4 (\beta^2)_{ab} \\
&= (c^2 \vec{p}^2 + m^2c^4)\delta_{ab}
\end{aligned}$$

$$\frac{\{\alpha_i, \alpha_j\}_{ab}}{2} = \delta_{ij}\delta_{ab} \Rightarrow \{\alpha_i, \alpha_j\} = 2\delta_{ij}$$

$$\{\alpha_i, \beta\} = 0$$

$$(\beta^2)_{ab} = \delta_{ab} \Rightarrow \beta^2 = 1$$

Problems w/ Dirac equation: We want 2×2 matrices for the two spin states. We have $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$ but there is no fourth matrix that anticommutes w/ $\vec{\sigma}$. (2×2 complex matrices are spanned by 1 and $\vec{\sigma}$ but 1 commutes w/ $\vec{\sigma}$.) $\therefore$ $\vec{\alpha}$ and β must be higher-dimensional.

$$\begin{cases} \alpha_i \alpha_j + \alpha_j \alpha_i = 2\delta_{ij} \Rightarrow \alpha_i^2 = 1 \ \forall i \\ \alpha_i \beta + \beta \alpha_i = 0 \end{cases}$$

$$\beta \alpha_i \beta + \underbrace{\beta^2}_{1} \alpha_i = 0$$

$$\underbrace{\mathrm{tr}\, \beta \alpha_i \beta}_{\mathrm{tr}\, \alpha_i} + \mathrm{tr}\, \alpha_i = 0 \Rightarrow \mathrm{tr}\, \alpha_i = 0 \ \forall i$$

$$\alpha_i^2 = 1 \Rightarrow \alpha_i \to \mathrm{diag}(\underbrace{1, \dots, 1}_{N_+}, \underbrace{-1, \dots, -1}_{N_-})$$

$$\begin{aligned}
\mathrm{tr}\, \alpha_i = 0 &\Rightarrow N_+ = N_- \\
&\Rightarrow D(\alpha) = 2N_+ \ : \text{even-dimensional}
\end{aligned}$$

Simile for β. $\therefore$ minimum size is 4×4. What to do w/ these extra states?

$$H = c\vec{p}\cdot\vec{\alpha} + mc^2\beta$$

$$\mathrm{tr}\, H = 0 \Rightarrow H \to \mathrm{diag}\left(E(\vec{p}), E(\vec{p}), -E(\vec{p}), -E(\vec{p})\right)$$

$$E(\vec{p}) = \sqrt{\vec{p}^2 c^2 + m^2 c^4}$$

Negative energies $\Rightarrow$ no ground state. A +energy e^- could could emit a γ and drop down into a $-$energy state. This downward cascade could continue forever.

Dirac's solution: Pauli principle + all $-$energy states are already occupied. Then, a +energy e^- cannot drop into a $-$energy state.

Questions: Why don't we see this sea of $-$energy particles? They are uniform $\therefore$ no force on daily life. However, if a $-$energy particle is excited to a +energy state via radiation, then $\exists$ a hole of +charge left behind. This is the antiparticle of electron.

We started w/ a single relativistic particle but now we have ∞-many of them. Moreover, we haven't solved the problem of particles that do not obey Pauli principle.

Think about what's going on: Why it is so hard to find an acceptable theory/relativistic wave equation for a single quantum particle. Is there something wrong w/ our basic assumptions?

Yes! Recall the axiom "observables are represented by hermitian operators." But time is not. t is just a label in a state, not an eigenvalue of any time operator, cf $\vec{x}$ $\therefore$ space and time are not treated equally.

Two solutions: Promote time to an operator or demote position to a label. The former gives string theory. The latter promotes operators to fields: quantum fields.

The two solutions turn out to be equivalent. There is another useful equivalence: Ordinary non-relativistic QM, for a fixed number of particles, can be written as a quantum field theory.

n particles, all w/ mass m, under external potential, $U(\vec{x})$, and interacting via an interparticle potential, $V(\vec{x}_i - \vec{x}_j)$.

$$i\hbar \frac{\partial}{\partial t}\psi = \left\{ \sum_{j=1}^n \left[-\frac{\hbar^2}{2m}\vec{\nabla}_j^2 + U(\vec{x}_j)\right] + \sum_{j=1}^n \sum_{k=1}^{j-1} V(\vec{x}_j - \vec{x}_k) \right\} \psi$$

$$\psi = \psi(\vec{x}_1, \dots, \vec{x}_n, t)$$

Introduce $a(\vec{x})$ and $a^\dagger(\vec{x})$ quantum fields in the Schrödinger pic.

$$[a(\vec{x}), a^\dagger(\vec{x}')] = 0$$

$$[a^\dagger(\vec{x}), a^\dagger(\vec{x}')] = 0$$

$$[a(\vec{x}), a^\dagger(\vec{x}')] = \delta^3(\vec{x} - \vec{x}')$$

$$H = \int d^3x\, a^\dagger(\vec{x})\left[-\frac{\hbar^2}{2m}\vec{\nabla}^2 + U(\vec{x})\right] a(\vec{x}) + \frac{1}{2}\int d^3x\, d^3y\, V(\vec{x} - \vec{y})\, a^\dagger(\vec{x}) a^\dagger(\vec{y}) a(\vec{y}) a(\vec{x})$$

$$|\psi(t)\rangle = \int d^3x_1 \dots d^3x_n\, \psi(\vec{x}_1, \dots, \vec{x}_n, t)\, a^\dagger(\vec{x}_1) \dots a^\dagger(\vec{x}_n)|0\rangle$$

$$a(\vec{x})|0\rangle = 0$$

$|0\rangle$: vacuum, no-particle state

$a^\dagger(\vec{x}_1)|0\rangle$: one particle at $\vec{x}_1$

$a^\dagger(\vec{x}_1) a^\dagger(\vec{x}_2)|0\rangle$: one particle at $\vec{x}_1$, another at $\vec{x}_2$

$$H|\psi(t)\rangle = \left\{\int d^3x\, a^\dagger(\vec{x})\left[-\frac{\hbar^2}{2m}\vec{\nabla}^2 + U(\vec{x})\right] a(\vec{x}) + \frac{1}{2}\int d^3x\, d^3y\, V(\vec{x} - \vec{y})\, a^\dagger(\vec{x}) a^\dagger(\vec{y}) a(\vec{y}) a(\vec{x})\right\} \int d^3x_1 \dots d^3x_n\, \psi(\vec{x}_1, \dots, \vec{x}_n, t)\, a^\dagger(\vec{x}_1) \dots a^\dagger(\vec{x}_n)|0\rangle$$

See code_1.

$$H|\psi(t)\rangle = \int d^3x_1 \dots d^3x_n \left[\sum_{j=1}^n Q(\vec{x}_j) + \sum_{j<k}^n V(\vec{x}_j - \vec{x}_k)\right] \psi(\vec{x}_1, \dots, \vec{x}_n, t)\, a^\dagger(\vec{x}_1) \dots a^\dagger(\vec{x}_n)|0\rangle$$

$$Q(\vec{x}) := -\frac{\hbar^2}{2m}\vec{\nabla}^2 + U(\vec{x})$$

$$\sum_{i=1}^n Q(\vec{x}_i) + \sum_{i<j}^n V(\vec{x}_i - \vec{x}_j) \ : \text{Schrödinger operator}$$

$\therefore$ Schrödinger equation is satisfied.

Number operator:

$$N = \int d^3x\, a^\dagger(\vec{x}) a(\vec{x})$$

$[H, N] = 0$: See code_2.

Another important aspect:

$$|\psi(t)\rangle = \int d^3x_1 \dots d^3x_n\, \psi(\vec{x}_1, \dots, \vec{x}_n, t)\, a^\dagger(\vec{x}_1) \dots a^\dagger(\vec{x}_n)|0\rangle$$

w/ $[a^\dagger(\vec{x}), a^\dagger(\vec{x}')] = 0$. $\therefore$ Wave function must be symmetric.

$$\psi(\vec{x}_i, \vec{x}_j, t) = +\psi(\vec{x}_j, \vec{x}_i, t) \ : \text{bosons}$$

If we impose $\{a^\dagger(\vec{x}), a^\dagger(\vec{x}')\} = 0$, then the wavefunction must be antisymmetric.

$$\psi(\vec{x}_i, \vec{x}_j, t) = -\psi(\vec{x}_j, \vec{x}_i, t) \ : \text{fermions}$$

Fermions obey the abstract Schrödinger equation, as well. See code_3. The output is the same as that of code_1.