Path integrals in quantum mechanics

$$H(P,Q) = \frac{1}{2m}P^2 + V(Q)$$

$$[Q,P] = i \qquad (\hbar = 1)$$

Probability amp for particle to start at (q',t') and end up at (q'',t''):

$$\langle q'',t''|e^{-iH(t''-t')}|q',t'\rangle$$

$|q'\rangle, |q''\rangle$: eigenstates of Q

Heisenberg pic: $Q(t) = e^{iHt} Q e^{-iHt}$

$Q(t)|q,t\rangle = q|q,t\rangle$: instantaneous eigenstate

$$|q,t\rangle = e^{iHt}|q\rangle$$

$$Q|q\rangle = q|q\rangle$$

Transition amp: $\langle q'',t''|q',t'\rangle$

Divide time interval $T := t''-t'$ into $N+1$ equal pieces, introduce N complete sets of position eigenkets:

$$\langle q'',t''|q',t'\rangle = \int_{-\infty}^{\infty} \prod_{j=1}^{N} dq_j \, \langle q''|e^{-iH\delta t}|q_N\rangle\langle q_N|e^{-iH\delta t}|q_{N-1}\rangle \cdots \langle q_1|e^{-iH\delta t}|q'\rangle$$

Campbell-Baker-Hausdorf formula:

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A+B]+\cdots}$$

$$\therefore e^{-iH\delta t} = e^{-i\frac{1}{2m}P^2\delta t} e^{-iV(Q)\delta t} e^{O(\delta t^2)}$$

$$\langle q_2|e^{-iH\delta t}|q_1\rangle = \langle q_2|e^{-i\frac{1}{2m}P^2\delta t} e^{-iV(Q)\delta t}|q_1\rangle$$

$$= \int_{-\infty}^{\infty} dp_1 \, \langle q_2|e^{-i\frac{1}{2m}P^2\delta t}|p_1\rangle\langle p_1|e^{-iV(Q)\delta t}|q_1\rangle$$

$$= \int_{-\infty}^{\infty} dp_1 \, \underbrace{\langle q_2|p_1\rangle}_{\frac{e^{ip_1q_2}}{\sqrt{2\pi}}} \underbrace{\langle p_1|q_1\rangle}_{\frac{e^{-ip_1q_1}}{\sqrt{2\pi}}} e^{-i\frac{1}{2m}p_1^2\delta t} e^{-iV(q_1)\delta t}$$

$$= \int_{-\infty}^{\infty} \frac{dp_1}{2\pi} e^{ip_1(q_2-q_1)} e^{-ip_1^2\delta t/2m} e^{-iV(q_1)\delta t}$$

$$= \int_{-\infty}^{\infty} \frac{dp_1}{2\pi} e^{-iH(p_1,q_1)\delta t} e^{ip_1(q_2-q_1)}$$

More general hamiltonian: worry about ordering of P and Q in any term that contains both. Weyl-ordering:

$$H(P,Q) := \int \frac{dx}{2\pi}\frac{dk}{2\pi} e^{ixP+ikQ} \int dp\,dq\, e^{-ixp-ikq} H(p,q)$$

Then, (this part is not clear to me)

$$H(p_1,q_1) \longrightarrow H(p_1,\bar{q}_1), \quad \bar{q}_1 := \frac{q_1+q_2}{2}$$

Our hamiltonian is Weyl-ordered $\therefore$ this replacement makes no difference in the limit $\delta t \to 0$.

Adopting Weyl-ordering for the general case:

$$\langle q'',t''|q',t'\rangle = \int \prod_{k=1}^{N} dq_k \prod_{j=0}^{N} \frac{dp_j}{2\pi} e^{ip_j(q_{j+1}-q_j)} e^{-iH(p_j,\bar{q}_j)\delta t}, \quad \bar{q}_j := \frac{1}{2}(q_j+q_{j+1}), \quad q_0 = q', \quad q_{N+1} = q''$$

Let $\dot{q}_j := \frac{q_{j+1}-q_j}{\delta t}$ and take formal limit of $\delta t \to 0$:

$$\langle q'',t''|q',t'\rangle = \int \mathcal{D}q\,\mathcal{D}p\, e^{i\int_{t'}^{t''} dt\,(p\dot{q} - H(p,q))}$$

If $H(p,q)$ is no more than quadratic in p, then p integral is gaussian. If the term that is quadratic in p is indep. of q, then the prefactors generated by the gaussian integrals are all constants and can be absorbed into the definition of $\mathcal{D}q$. The result of integrating out p is then

$$\langle q'',t''|q',t'\rangle = \int \mathcal{D}q\, e^{i\int_{t'}^{t''} dt\, L(\dot{q},q)}$$

where $L(\dot{q},q)$ is computed by first finding the stationary point of the p integral by solving

$$0 = \frac{\partial}{\partial p}(p\dot{q} - H) = \dot{q} - \frac{\partial H}{\partial q}$$

for p in terms of $\dot{q}$ and q and then plugging this solution back into $p\dot{q}-H$ to get L.

Consider

$$\langle q'',t''|Q(t''>t_1>t'')|q',t'\rangle = \langle q''|e^{-iH(t''-t_1)} Q e^{-iH(t_1-t')}|q'\rangle$$

$$= \int \mathcal{D}p\,\mathcal{D}q\, q(t_1) e^{iS}$$

$$S := \int_{t'}^{t''} dt\,(p\dot{q} - H)$$

Consider $\int \mathcal{D}p\,\mathcal{D}q\, q(t_1)q(t_2)e^{iS}$. This requires $Q(t_1)$ and $Q(t_2)$ but their order depends on the order of t_1 and t_2:

$$\int \mathcal{D}p\,\mathcal{D}q\, q(t_1)q(t_2)e^{iS} = \langle q'',t''|T\{Q(t_1)Q(t_2)\}|q',t'\rangle$$

Functional derivatives:

$$\frac{\delta}{\delta f(t_1)} f(t_2) = \delta(t_1 - t_2)$$

$$H(p,q) \to H(p,q) - fq - hp$$

$$\langle q'',t''|q',t'\rangle_{f,h} = \int \mathcal{D}p\,\mathcal{D}q\, e^{i\int_{t'}^{t''} dt\,(p\dot{q} - H + fq + hp)}$$

$$\frac{1}{i}\frac{\delta}{\delta f(t_1)} \langle q'',t''|q',t'\rangle_{f,h} = \int \mathcal{D}p\,\mathcal{D}q\, q(t_1) e^{iS_{f,h}}$$

$$\frac{1}{i}\frac{\delta}{\delta f(t_1)} \frac{1}{i}\frac{\delta}{\delta f(t_2)} \langle q'',t''|q',t'\rangle_{f,h} = \int \mathcal{D}p\,\mathcal{D}q\, q(t_1)q(t_2) e^{iS_{f,h}}$$

$$\frac{1}{i}\frac{\delta}{\delta h(t_1)} \langle q'',t''|q',t'\rangle_{f,h} = \int \mathcal{D}p\,\mathcal{D}q\, p(t_1) e^{iS_{f,h}}$$

...

At the end, set $f = h = 0$.

$$\langle q'',t''|T\{Q(t_1)\ldots P(t_2)\ldots\}|q',t'\rangle = \left[\frac{1}{i}\frac{\delta}{\delta f(t_1)}\right]\cdots\left[\frac{1}{i}\frac{\delta}{\delta h(t_2)}\right]\cdots \langle q'',t''|q',t'\rangle\Big|_{f=h=0}$$

Suppose we are also interested in initial and final states other than position eigenstates. Then, we must multiply by the wave functions for these states and integrate. We will be interested, in particular, in the ground state as both initial and final states. Also, we will take limits $t' \to -\infty$ and $t'' \to +\infty$:

$$\langle 0|0\rangle_{f,h} = \lim_{\substack{t'\to-\infty \\ t''\to+\infty}} \int_{-\infty}^{\infty} dq'\,dq''\, \psi_0^*(q'') \langle q'',t''|q',t'\rangle_{f,h} \psi_0(q')$$

Too cumbersome. Need to simplify.

$$H|n\rangle = E_n|n\rangle$$

$$E_0 = 0$$

$$|q',t'\rangle = e^{iHt'}|q'\rangle$$

$$= \sum_{n\geq 0} e^{iHt'}|n\rangle\langle n|q\rangle$$

$$= \sum_{n\geq 0} \psi_n^*(q') e^{iE_nt'}|n\rangle$$

$$H \to (1-i\varepsilon)H$$

$$|q',t'\rangle = \sum_{n\geq 0} \psi_n^*(q') e^{i(1-i\varepsilon)E_nt'}|n\rangle$$

$$= \sum_{n\geq 0} \psi_n^*(q') e^{iE_nt'}|n\rangle e^{\varepsilon E_nt'}$$

As $t' \to -\infty$, only the ground state contributes:

$$\lim_{t'\to-\infty} |q',t'\rangle\Big|_{H\to(1-i\varepsilon)H} = \psi_0^*(q')|0\rangle$$

Next, multiply by an arbitrary function $\chi(q')$ and integrate over q'. The only requirement is that $\langle 0|\chi\rangle \neq 0$. We then have a constant times $|0\rangle$ and this constant can be absorbed into the normalization of the path integral.

Simile, $H \to (1-i\varepsilon)H$ picks the ground state in $\langle q'',t''| = \langle q''|e^{-iHt''}$ in limit $t'' \to +\infty$.

$\therefore$ if we use $(1-i\varepsilon)H$ instead of H, we can be cavalier (= careless) about the boundary conditions on the endpoints of the path. Any reasonable boundary conditions will result in the ground state as both initial and final states.

$$\langle 0|0\rangle_{f,h} = \int \mathcal{D}p\,\mathcal{D}q\, e^{i\int_{-\infty}^{\infty} dt\,(p\dot{q} - (1-i\varepsilon)H + fq + hp)}$$

Suppose now $H = H_0 + H_1$ and we know the solution of H_0, treating H_1 as perturbation. Suppress $i\varepsilon$:

$$\langle 0|0\rangle_{f,h} = \int \mathcal{D}p\,\mathcal{D}q\, e^{i\int_{-\infty}^{\infty} dt\,(p\dot{q} - H_0(p,q) - H_1(p,q) + fq + hp)}$$

$$= e^{-i\int_{-\infty}^{\infty} dt\, H_1\left(\frac{1}{i}\frac{\delta}{\delta h(t)}, \frac{1}{i}\frac{\delta}{\delta f(t)}\right)} \int \mathcal{D}p\,\mathcal{D}q\, e^{i\int_{-\infty}^{\infty} dt\,(p\dot{q} - H_0(p,q) + fq + hp)}$$

The trick was as follows: We can extract each power of q w/ an f-derivative. Why not expand H_1 in a power series about q, replace each q w/ an f-derivative, and resum to obtain the original form of H_1. Simile for p.

If
- H_1 depends only on q
- we are interested only in time-ordered products of Q
- H is no more than quadratic in P
- term quadratic in P doesn't involve Q

then

$$\langle 0|0\rangle_f = e^{i\int_{-\infty}^{\infty} dt\, L_1\left(\frac{1}{i}\frac{\delta}{\delta f(t)}\right)} \int \mathcal{D}q\, e^{i\int_{-\infty}^{\infty} dt\,(L_0(\dot{q},q) + fq)}$$

where $L_1(q) = -H_1(q)$.