Ch7: Problems

7.1) Starting w/

$$G(t-t') = \int_{-\infty}^{\infty} \frac{dE}{2\pi} \frac{e^{-iE(t-t')}}{-E^2+\omega^2-i\varepsilon}$$

do the contour integral to verify

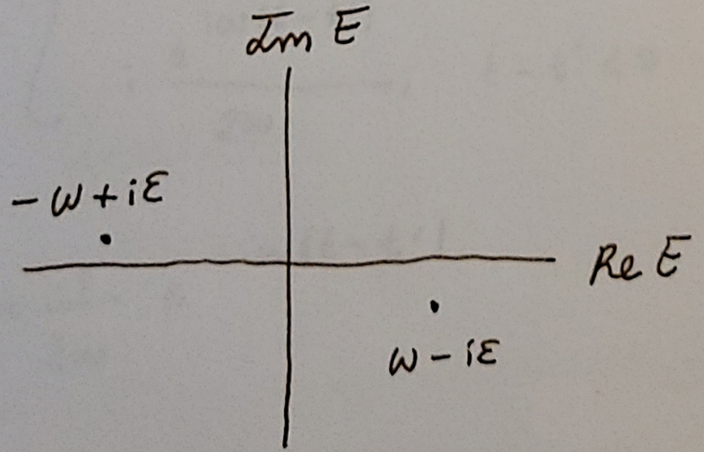
$$G(t-t') = \frac{i}{2\omega} e^{-i\omega|t-t'|}$$

Solution

$$E^2-\omega^2+i\varepsilon = 0 \Rightarrow E = \pm\sqrt{\omega^2-i\varepsilon}$$

$$= \pm(\omega - i\varepsilon)$$

$$= \begin{cases} \omega - i\varepsilon \\ -\omega + i\varepsilon \end{cases}$$



$$e^{-iE(t-t')} = e^{-iR[\cos(\theta)+i\sin(\theta)](t-t')}$$
$$= e^{R(t-t')\sin(\theta)}$$

$t-t'>0 \Rightarrow$ close in the lower half plane

$t-t'<0 \Rightarrow$ close in the upper half plane

$$G(t-t') = -\int_{-\infty}^{\infty} \frac{dE}{2\pi} \frac{e^{-iE(t-t')}}{(E-\omega+i\varepsilon)(E+\omega-i\varepsilon)}$$

direction

$$= \begin{cases} -(-1)i \left. \dfrac{e^{-iE(t-t')}}{E+\omega-i\varepsilon} \right|_{E=\omega-i\varepsilon}, & t-t'>0 \\ -i \left. \dfrac{e^{-iE(t-t')}}{E-\omega+i\varepsilon} \right|_{E=-\omega+i\varepsilon}, & t-t'<0 \end{cases}$$

$$= \begin{cases} i\dfrac{e^{-i\omega(t-t')}}{2\omega}, & t-t'>0 \\ i\dfrac{e^{i\omega(t-t')}}{2\omega}, & t-t'<0 \end{cases}$$

$$= \frac{i}{2\omega} e^{-i\omega|t-t'|}$$

7.2) Starting w/

$$G(t-t') = \frac{i}{2\omega} e^{-i\omega|t-t'|}$$

verify

$$\left(\frac{\partial^2}{\partial t^2} + \omega^2\right) G(t-t') = \delta(t-t')$$

Solution

$t > t'$:

$$G(t-t') = \frac{i}{2\omega} e^{-i\omega(t-t')}$$

$$\left(\frac{\partial^2}{\partial t^2} + \omega^2\right) G(t-t') \overset{?}{=} 0$$

$$\frac{i}{2\omega} e^{-i\omega(t-t')} (-\omega^2) + \frac{i}{2\omega} e^{-i\omega(t-t')} \omega^2 \overset{?}{=} 0$$

$0 = 0$ identically

Simile for $t < t'$.

At $t = t'$: G is continuous. What about derivative?

$$\left(\frac{\partial^2}{\partial t^2} + \omega^2\right) G(t-t') = \delta(t-t') \quad \Big| \int_{t'-\varepsilon}^{t'+\varepsilon} dt$$

$$\left.\frac{\partial G(t-t')}{\partial t}\right|_{t'-\varepsilon}^{t'+\varepsilon} \stackrel{?}{=} 1$$

$$\left[\frac{\partial}{\partial t}\frac{i}{2\omega}e^{-i\omega(t-t')}\right]_{t'} - \left[\frac{\partial}{\partial t}\frac{i}{2\omega}e^{-i\omega(t'-t)}\right]_{t'} \stackrel{?}{=} 1$$

$$\frac{i}{2\cancel{\omega}}(-i\cancel{\omega}) - \frac{i}{2\cancel{\omega}}(+i\cancel{\omega}) \stackrel{?}{=} 1$$

$$\frac{1}{2} + \frac{1}{2} \stackrel{?}{=} 1$$

$1 = 1$ identically

$\therefore G(t-t')$ satisfies the harmonic equation w/ δ source.

7.3) a) Use the Heisenberg EOM, $i\dot{A} = [A, H]$, to find explicit expressions for $\dot{Q}$ and $\dot{P}$. Solve these to get the Heisenberg-pic operators $Q(t)$ and $P(t)$ in terms of the Schr.-pic operators Q and P.

b) Write the Schr.-pic operators Q and P in terms of creation and annihilation operators a and $a^\dagger$, where $H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$.
Then, using your result from part (a), write the Heisenberg-pic operators $Q(t)$ and $P(t)$ in terms of a and $a^\dagger$.

c) using your result from part (b), and $a|0\rangle = 0 = \langle 0|a^\dagger$, verify

$$\langle 0| T\{Q(t_1)Q(t_2)\}|0\rangle = \frac{1}{i} G(t_2 - t_1)$$

and

$$\langle 0| T\{Q(t_1)Q(t_2)Q(t_3)Q(t_4)\}|0\rangle = \frac{1}{i^2}\left(G_{12}G_{34} + G_{13}G_{24} + G_{14}G_{23}\right)$$

where $G_{ij} := G(t_i - t_j)$.

<u>Solution</u>

a) $$H = \frac{1}{2m}P^2 + \frac{1}{2}m\omega^2 Q^2$$

$$i\hbar\dot{Q} = [Q, H] = \left[Q, \frac{1}{2m}P^2\right] = \frac{1}{2m}P[Q,P] + \frac{1}{2m}[Q,P]P$$

$$= \frac{1}{2m}(P i\hbar + i\hbar P) = \frac{i\hbar}{m}P$$

$$i\hbar \dot{P} = [P, H] = \left[P, \frac{1}{2} m\omega^2 Q^2\right]$$

$$= \frac{1}{2} m\omega^2 \left(Q[P,Q] + [P,Q]Q \right)$$

$$= \frac{1}{2} m\omega^2 (-2 i\hbar Q)$$

$$= -i\hbar m\omega^2 Q$$

$$\left.\begin{aligned} \dot{Q} &= \frac{1}{m} P \\ \dot{P} &= -m\omega^2 Q \end{aligned}\right\} \quad \begin{aligned} \ddot{Q} &= -\omega^2 Q \\ \ddot{P} &= -\omega^2 P \end{aligned}$$

$$Q(t) = A \sin(\omega t) + \underset{\displaystyle Q(0) =: Q}{\underset{\downarrow}{B}} \cos(\omega t)$$

$$\dot{Q}(t) = \underbrace{A\omega}_{\frac{P(0)}{m} =: \frac{P}{m}} \cos(\omega t) - \omega Q \sin(\omega t) = \frac{1}{m} P(t)$$

$$Q(t) = \frac{P}{m\omega} \sin(\omega t) + Q \cos(\omega t)$$

$$P(t) = -m\omega Q \sin(\omega t) + P \cos(\omega t)$$

b) $$a = \frac{1}{\sqrt{\hbar\omega}}\left(\sqrt{\frac{1}{2}m\omega^2}\,Q + i\sqrt{\frac{1}{2m}}\,P\right)$$

$$Q = \sqrt{\hbar\omega}\,\frac{1}{\sqrt{\frac{1}{2}m\omega^2}}\,\mathrm{Re}\,a$$

$$= \sqrt{\frac{\cancel{2}\hbar\cancel{\omega}}{m\omega^{\cancel{2}}\,2}}\,\frac{a+a^\dagger}{\cancel{2}}$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\,(a+a^\dagger)$$

$$P = \sqrt{\hbar\omega}\,\frac{1}{\sqrt{\frac{1}{2m}}}\,\mathrm{Im}\,a$$

$$= \sqrt{2m\hbar\omega}\,\frac{a-a^\dagger}{2i}$$

$$= -i\sqrt{\frac{m\hbar\omega}{2}}\,(a-a^\dagger)$$

using Mathematica,

$$Q(t) = \sqrt{\frac{\hbar}{2m\omega}}\left(a e^{-i\omega t} + a^\dagger e^{i\omega t}\right)$$

$$P(t) = -i\sqrt{\frac{m\omega\hbar}{2}}\left(a e^{-i\omega t} - a^\dagger e^{i\omega t}\right)$$

c) See code-2.

7.4) Consider a harmonic oscillator in its ground state at $t = -\infty$. It is then subjected to an internal force $f(t)$. Compute the probability $|\langle 0|0\rangle_f|^2$ that the oscillator is still in its ground state at $t = +\infty$. Write your answer as a manifestly real expression, and in terms of the Fourier transform $\tilde{f}(E) = \int_{-\infty}^{\infty} dt\, e^{iEt} f(t)$. Your answer should not involve any other unevaluated integrals.

Solution

$$\langle 0|0\rangle_f = e^{\frac{i}{2}\int_{-\infty}^{\infty} \frac{dE}{2\pi} \frac{\tilde{f}(E)\tilde{f}(-E)}{-E^2+\omega^2-i\varepsilon}}$$

$|\langle 0|0\rangle_f|^2$ would be 1 if it wasn't for $i\varepsilon$.

There is a nice formula for small ε here:

See math.stackexchange.com/questions/1696809.

$$\frac{1}{x+i\varepsilon} = \text{P.V}\,\frac{1}{x} - i\pi\delta(x)$$

$$\therefore \frac{1}{E^2-\omega^2+i\varepsilon} = \text{P.V}\,\frac{1}{E^2-\omega^2} - i\pi\delta(E^2-\omega^2)$$

$$\langle 0|0\rangle_f = e^{-\frac{i}{2}\int_{-\infty}^{\infty}\frac{dE}{2\pi}\tilde{f}(E)\tilde{f}(-E)\left[\text{P.V}\frac{1}{E^2-\omega^2} - i\pi\delta(E^2-\omega^2)\right]}$$

$$= e^{i(\cdots)}\, e^{-\frac{\pi}{2}\int_{-\infty}^{\infty}\frac{dE}{2\pi}\tilde{f}(E)\tilde{f}(-E)\delta(E^2-\omega^2)}$$

$$= e^{i(\cdots)}\, e^{-\frac{\pi}{2}\cancel{2}\int_{0}^{\infty}\frac{dE}{\cancel{2\pi}}\tilde{f}(E)\tilde{f}(-E)\delta(E^2-\omega^2)} \quad (=)$$

$$\delta(E^2-\omega^2): \quad E=\pm\omega, \quad (E^2-\omega^2)' = 2E$$

$$\therefore \delta(E^2-\omega^2) = \frac{\delta(E-\omega)}{2\omega}$$

$$(=)\; e^{i(\cdots)}\, e^{-\frac{1}{2}\int_0^{\infty} dE\,\tilde{f}(E)\tilde{f}(-E)\frac{\delta(E-\omega)}{2\omega}}$$

$$= e^{i(\cdots)}\, e^{-\frac{1}{4\omega}\tilde{f}(\omega)\tilde{f}(-\omega)}$$

Assume the force is real-valued.

$$\tilde{f}(E) = \int_{-\infty}^{\infty} dt\, e^{iEt} f(t)$$

$$\tilde{f}(-E) = \int_{-\infty}^{\infty} dt\, e^{-iEt} f(t) = \tilde{f}(E)^*$$

$$\therefore \tilde{f}(\omega)\tilde{f}(-\omega) = |\tilde{f}(\omega)|^2$$

$$\therefore \langle 0|0\rangle_f = e^{i(\cdots)} e^{-|\tilde{f}(\omega)|^2/4\omega}$$

$$\therefore |\langle 0|0\rangle_f|^2 = e^{-|\tilde{f}(\omega)|^2/4\omega}$$

The terms in $(\ldots)$ goes like $\tilde{f}(E)\tilde{f}(-E)$ times some real stuff, so $|\tilde{f}(E)|^2$ times some real stuff, which is then manifestly real and hence becomes just a phase, which vanishes when we take modulus square.