7.1) Starting with eq. (7.12), do the contour integral to verify eq. (7.14).

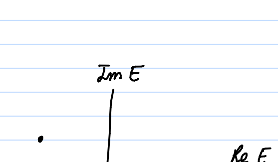
$$G(t-t') = \int_{-\infty}^{+\infty} \frac{dE}{2\pi} \frac{e^{-iE(t-t')}}{-E^2+\omega^2-i\epsilon} . \qquad (7.12)$$

$$G(t-t') = \frac{i}{2\omega} \exp\left(-i\omega|t-t'|\right) . \qquad (7.14)$$

$$E^2 - \omega^2 + i\varepsilon = 0 \Rightarrow E = \pm\sqrt{\omega^2 - i\varepsilon}$$

$$= \pm(\omega - i\varepsilon)$$

$$= \begin{cases} \omega - i\varepsilon \\ -\omega + i\varepsilon \end{cases}$$



$$e^{-iE(t-t')} = e^{-iR(\cos(\theta)+i\sin(\theta))(t-t')}$$

$$= e^{R\sin(\theta)(t-t') + i(\dots)}$$

$t-t'>0 \Rightarrow$ close in the upper half plane

$t-t'<0 \Rightarrow$ close in the lower half plane

$$G(t-t') = -\int_{-\infty}^{\infty} \frac{dE}{2\pi} \frac{e^{-iE(t-t')}}{(E-\omega+i\varepsilon)(E+\omega-i\varepsilon)}$$

$$= \begin{cases} \underset{\substack{\downarrow \\ \text{direction}}}{-(-1)i} \left.\dfrac{e^{-iE(t-t')}}{E+\omega-i\varepsilon}\right|_{E=\omega-i\varepsilon}, & t-t'>0 \\ -i \left.\dfrac{e^{-iE(t-t')}}{E-\omega+i\varepsilon}\right|_{E=-\omega+i\varepsilon}, & t-t'<0 \end{cases}$$

$$= \begin{cases} i\dfrac{e^{-i\omega(t-t')}}{2\omega}, & t-t'>0 \\ i\dfrac{e^{i\omega(t-t')}}{2\omega}, & t-t'<0 \end{cases}$$

$$= \frac{i}{2\omega} e^{-i\omega|t-t'|}$$

7.2) Starting with eq. (7.14), verify eq. (7.13).

$$G(t-t') = \frac{i}{2\omega} \exp\left(-i\omega|t-t'|\right) . \qquad (7.14)$$

$$\left(\frac{\partial^2}{\partial t^2} + \omega^2\right) G(t-t') = \delta(t-t') . \qquad (7.13)$$

$t>t'$:

$$G(t-t') = \frac{i}{2\omega} e^{-i\omega(t-t')}$$

$$\left(\frac{\partial^2}{\partial t^2} + \omega^2\right) G(t-t') \stackrel{?}{=} 0$$

$$\frac{i}{2\omega} e^{-i\omega(t-t')}(-\omega^2) + \frac{i}{2\omega} e^{-i\omega(t-t')}\omega^2 \stackrel{?}{=} 0$$

$0 = 0$ identically

Simile for $t-t'<0$.

At $t=t'$: G is continuous. What about derivative?

$$\left(\frac{\partial^2}{\partial t^2} + \omega^2\right) G(t-t') = \delta(t-t') \quad \Big\backslash \int_{t'-\varepsilon}^{t'+\varepsilon} dt$$

$$\left.\frac{\partial G(t-t')}{\partial t}\right|_{t'-\varepsilon}^{t'+\varepsilon} \stackrel{?}{=} 1$$

$$\left[\frac{\partial}{\partial t}\frac{i}{2\omega}e^{-i\omega(t-t')}\right]_{t'} - \left[\frac{\partial}{\partial t}\frac{i}{2\omega}e^{-i\omega(t'-t)}\right]_{t'} \stackrel{?}{=} 1$$

$$\frac{i}{2\omega}(-i\omega) - \frac{i}{2\omega}(+i\omega) \stackrel{?}{=} 1$$

$1 = 1$ identically

$\therefore G(t-t')$ satisfies the harmonic equation w/ δ source.

7.3) a) Use the Heisenberg equation of motion, $\dot{A} = i[H, A]$, to find explicit expressions for $\dot{Q}$ and $\dot{P}$. Solve these to get the Heisenberg-picture operators $Q(t)$ and $P(t)$ in terms of the Schrödinger picture operators Q and P.

b) Write the Schrödinger picture operators Q and P in terms of the creation and annihilation operators a and $a^\dagger$, where $H = \hbar\omega(a^\dagger a+\frac{1}{2})$. Then, using your result from part (a), write the Heisenberg-picture operators $Q(t)$ and $P(t)$ in terms of a and $a^\dagger$.

c) Using your result from part (b), and $a|0\rangle = \langle 0|a^\dagger = 0$, verify eqs. (7.16) and (7.17).

$$\begin{aligned} \langle 0|\mathrm{T}Q(t_1)Q(t_2)|0\rangle &= \frac{1}{i}\frac{\delta}{\delta f(t_1)}\frac{1}{i}\frac{\delta}{\delta f(t_2)}\langle 0|0\rangle_f\Big|_{f=0} \\ &= \frac{1}{i}\frac{\delta}{\delta f(t_1)}\left[\int_{-\infty}^{+\infty} dt'\, G(t_2-t')f(t')\right]\langle 0|0\rangle_f\Big|_{f=0} \\ &= \left[\tfrac{1}{i}G(t_2-t_1) + (\text{term with } f\text{'s})\right]\langle 0|0\rangle_f\Big|_{f=0} \\ &= \tfrac{1}{i}G(t_2-t_1) . \end{aligned} \qquad (7.16)$$

$$\begin{aligned} \langle 0|\mathrm{T}Q(t_1)Q(t_2)Q(t_3)Q(t_4)|0\rangle = \frac{1}{i^2}\Big[&G(t_1-t_2)G(t_3-t_4) \\ &+ G(t_1-t_3)G(t_2-t_4) \\ &+ G(t_1-t_4)G(t_2-t_3)\Big]. \end{aligned} \qquad (7.17)$$

a) $$H = \frac{1}{2m}P^2 + \frac{1}{2}m\omega^2 Q^2$$

$$i\dot{Q} = [Q, H]$$

$$= \left[Q, \frac{1}{2m}P^2\right]$$

$$= \frac{1}{2m}P[Q,P] + \frac{1}{2m}[Q,P]P$$

$$= \frac{1}{2m}(Pi + iP)$$

$$= \frac{i}{m}P$$

$$i\dot{P} = [P, H]$$

$$= \left[P, \frac{1}{2}m\omega^2 Q^2\right]$$

$$= \frac{1}{2}m\omega^2 (Q[P,Q] + [P,Q]Q)$$

$$= \frac{1}{2}m\omega^2(-2iQ)$$

$$= -im\omega^2 Q$$

$$\left.\begin{aligned} \dot{Q} &= \frac{1}{m}P \\ \dot{P} &= -m\omega^2 Q \end{aligned}\right\} \quad \begin{aligned} \ddot{Q} &= -\omega^2 Q \\ \ddot{P} &= -\omega^2 P \end{aligned}$$

$$Q(t) = A\sin(\omega t) + \underset{\substack{\downarrow \\ Q(0) =: Q}}{B}\cos(\omega t)$$

$$\dot{Q}(t) = \underbrace{A\omega}_{\frac{P(0)}{m} =: \frac{P}{m}}\cos(\omega t) - \omega Q\sin(\omega t)$$

$$= \frac{P(t)}{m}$$

$$Q(t) = \frac{P}{m\omega}\sin(\omega t) + Q\cos(\omega t)$$

$$P(t) = -m\omega Q\sin(\omega t) + P\cos(\omega t)$$

b) $$a = \frac{1}{\sqrt{\hbar\omega}}\left(\sqrt{\frac{1}{2}m\omega^2}\,Q + i\sqrt{\frac{1}{2m}}\,P\right)$$

$$Q = \sqrt{\hbar\omega}\frac{1}{\sqrt{\frac{1}{2}m\omega^2}}\,\mathrm{Re}\,a$$

$$= \sqrt{\frac{2\hbar\omega}{m\omega^2}}\frac{a+a^\dagger}{2}$$

$$= \sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger)$$

$$P = \sqrt{\hbar\omega}\frac{1}{\sqrt{\frac{1}{2m}}}\,\mathrm{Im}\,a$$

$$= \sqrt{2m\hbar\omega}\,\frac{a-a^\dagger}{2i}$$

$$= -i\sqrt{\frac{m\hbar\omega}{2}}(a-a^\dagger)$$

Using Mathematica,

$$Q(t) = \sqrt{\frac{\hbar}{2m\omega}}\left(a e^{-i\omega t} + a^\dagger e^{i\omega t}\right)$$

$$P(t) = -i\sqrt{\frac{m\omega\hbar}{2}}\left(a e^{-i\omega t} - a^\dagger e^{i\omega t}\right)$$

c) See code_2.

7.4) Consider a harmonic oscillator in its ground state at $t=-\infty$. It is then then subjected to an external force $f(t)$. Compute the probability $|\langle 0|0\rangle_f|^2$ that the oscillator is still in its ground state at $t=+\infty$. Write your answer as a manifestly real expression, and in terms of the Fourier transform $\widetilde{f}(E) = \int_{-\infty}^{+\infty} dt\, e^{iEt} f(t)$. Your answer should not involve any other unevaluated integrals.

$$\langle 0|0\rangle_f = e^{\frac{i}{2}\int_{-\infty}^{\infty}\frac{dE}{2\pi}\frac{\tilde{f}(E)\tilde{f}(-E)}{-E^2+\omega^2-i\varepsilon}}$$

$|\langle 0|0\rangle_f|^2$ would be 1 if it wasn't for $i\varepsilon$. $\exists$ a nice formula for small ε here. See math.stackexchange.com/questions/1696809.

$$\frac{1}{x\pm i\varepsilon} = \mathrm{P.V}\frac{1}{x} \mp i\pi\delta(x)$$

$$\therefore \frac{1}{E^2-\omega^2+i\varepsilon} = \mathrm{P.V}\frac{1}{E^2-\omega^2} - i\pi\delta(E^2-\omega^2)$$

$$\langle 0|0\rangle_f = e^{-\frac{i}{2}\int_{-\infty}^{\infty}\frac{dE}{2\pi}\tilde{f}(E)\tilde{f}(-E)\left[\mathrm{P.V}\frac{1}{E^2-\omega^2} - i\pi\delta(E^2-\omega^2)\right]}$$

$$= e^{i(\dots)}e^{-\frac{\pi}{2}\int_{-\infty}^{\infty}\frac{dE}{2\pi}\tilde{f}(E)\tilde{f}(-E)\delta(E^2-\omega^2)}$$

$$= e^{i(\dots)}e^{-\frac{\pi}{2}2\int_{0}^{\infty}\frac{dE}{2\pi}\tilde{f}(E)\tilde{f}(-E)\delta(E^2-\omega^2)}$$

$$\delta(E^2-\omega^2) \to \frac{\delta(E-\omega)}{2\omega} + \cancel{\frac{\delta(E+\omega)}{2\omega}}^{\,0}$$

$$= e^{i(\dots)}e^{-\frac{1}{2}\int_{0}^{\infty}dE\,\tilde{f}(E)\tilde{f}(-E)\frac{\delta(E-\omega)}{2\omega}}$$

$$= e^{i(\dots)}e^{-\frac{1}{4\omega}\tilde{f}(\omega)\tilde{f}(-\omega)}$$

Assume the force is real-valued.

$$\tilde{f}(E) = \int_{-\infty}^{\infty} dt\, e^{iEt} f(t)$$

$$\tilde{f}(-E) = \int_{-\infty}^{\infty} dt\, e^{-iEt} f(t) = \tilde{f}(E)^*$$

$$\therefore \tilde{f}(\omega)\tilde{f}(-\omega) = |\tilde{f}(\omega)|^2$$

$$\langle 0|0\rangle_f = e^{i(\dots)}e^{-|\tilde{f}(\omega)|^2/4\omega}$$

$$\therefore |\langle 0|0\rangle_f|^2 = e^{-|\tilde{f}(\omega)|^2/4\omega}$$

The terms in $(\dots)$ go like $\tilde{f}(E)\tilde{f}(-E)$ time some real stuff $\therefore |\tilde{f}(E)|^2$ times some real stuff, which is then manifestly real $\therefore$ becomes just a phase, which vanishes when we take modulus square.