Canonical quantization of scalar fields

$$H = \int d^3x \; a^\dagger(\vec{x}) \left(-\frac{1}{2m}\vec{\nabla}^2\right) a(\vec{x}) \; : \text{free particle} \quad (\hbar = 1)$$

$$= \int d^3x \left[\int \frac{d^3p}{\sqrt{2\pi}^3} e^{i\vec{p}\cdot\vec{x}} \tilde{a}(\vec{p})\right]^\dagger \left(-\frac{1}{2m}\vec{\nabla}^2\right)\left[\int \frac{d^3p'}{\sqrt{2\pi}^3} e^{i\vec{p}'\cdot\vec{x}} \tilde{a}(\vec{p}')\right]$$

$$= \int \frac{d^3x\, d^3p\, d^3p'}{(2\pi)^3} e^{-i\vec{p}\cdot\vec{x}} \tilde{a}^\dagger(\vec{p}) \tilde{a}(\vec{p}') \frac{\vec{p}'^2}{2m} e^{i\vec{p}'\cdot\vec{x}}$$

$$= \int d^3p\, d^3p' \; \delta^3(\vec{p}-\vec{p}') \frac{\vec{p}'^2}{2m} \tilde{a}^\dagger(\vec{p}) \tilde{a}(\vec{p}')$$

$$= \int d^3p \; \frac{\vec{p}^2}{2m} \tilde{a}^\dagger(\vec{p}) \tilde{a}(\vec{p})$$

$$[\tilde{a}(\vec{p}), \tilde{a}(\vec{p}')]_\mp = 0$$

$$[\tilde{a}^\dagger(\vec{p}), \tilde{a}^\dagger(\vec{p}')]_\mp = 0$$

$$[\tilde{a}(\vec{p}), \tilde{a}^\dagger(\vec{p}')]_\mp = \delta^3(\vec{p}-\vec{p}')$$

$$\tilde{a}(\vec{p})|0\rangle = 0$$

$$H|0\rangle = 0$$

$$\tilde{a}^\dagger(\vec{p}_1)|0\rangle = |\vec{p}_1\rangle$$

$$H|\vec{p}_1\rangle = \int d^3p \; E(\vec{p}) \tilde{a}^\dagger(\vec{p}) \tilde{a}(\vec{p}) |\vec{p}_1\rangle$$

$$= \int d^3p \; E(\vec{p}) \tilde{a}^\dagger(\vec{p}) \underbrace{\tilde{a}(\vec{p}) \tilde{a}^\dagger(\vec{p}_1)}_{\delta^3(\vec{p}-\vec{p}_1) + \tilde{a}^\dagger(\vec{p}_1)\cancel{\tilde{a}(\vec{p})}^{\,0}} |0\rangle$$

$$= E(\vec{p}_1) \tilde{a}^\dagger(\vec{p}_1)|0\rangle$$

$$= E(\vec{p}_1)|\vec{p}_1\rangle$$

Two-particle state:

$$\tilde{a}^\dagger(\vec{p}_1)\tilde{a}^\dagger(\vec{p}_2)|0\rangle = |\vec{p}_1 \vec{p}_2\rangle$$

$$H|\vec{p}_1\vec{p}_2\rangle = \int d^3p \; E(\vec{p}) \tilde{a}^\dagger(\vec{p}) \underbrace{\tilde{a}(\vec{p}) \tilde{a}^\dagger(\vec{p}_1)}_{\delta^3(\vec{p}-\vec{p}_1') + \tilde{a}^\dagger(\vec{p}_1)\tilde{a}(\vec{p})} \tilde{a}^\dagger(\vec{p}_2)|0\rangle$$

$$= \int d^3p \; E(\vec{p}) \Big[\delta^3(\vec{p}-\vec{p}_1)\tilde{a}^\dagger(\vec{p}_1)\tilde{a}^\dagger(\vec{p}_2) + \tilde{a}^\dagger(\vec{p})\tilde{a}^\dagger(\vec{p}_1) \underbrace{\tilde{a}(\vec{p})\tilde{a}^\dagger(\vec{p}_2)}_{\delta^3(\vec{p}-\vec{p}_2) + \tilde{a}^\dagger(\vec{p}_2)\cancel{\tilde{a}(\vec{p})}^{\,0}}\Big]|0\rangle$$

$$= E(\vec{p}_1)\tilde{a}^\dagger(\vec{p}_1)\tilde{a}^\dagger(\vec{p}_2)|0\rangle + E(\vec{p}_2)\tilde{a}^\dagger(\vec{p}_1)\tilde{a}^\dagger(\vec{p}_2)|0\rangle$$

$$= \left[E(\vec{p}_1) + E(\vec{p}_2)\right]\tilde{a}^\dagger(\vec{p}_1)\tilde{a}^\dagger(\vec{p}_2)|0\rangle$$

$$= \left[E(\vec{p}_1) + E(\vec{p}_2)\right]|\vec{p}_1\vec{p}_2\rangle$$

$\therefore$ $\tilde{a}^\dagger(\vec{p}_1)\dots\tilde{a}^\dagger(\vec{p}_n)|0\rangle$ has energy eigenvalue $\Sigma_{i=1}^n E(\vec{p}_i)$ where $E(\vec{p}) = \vec{p}^2/2m$.

Relativistic generalization:

$$E(\vec{p}) = \sqrt{\vec{p}^2c^2 + m^2c^4}$$

$H = \int d^3p \; E(\vec{p}) \tilde{a}^\dagger(\vec{p})\tilde{a}(\vec{p})$: theory of free relativistic spinless particles (bosons or fermions)

Is this theory Lorentz invariant? Construct this theory again from a different point of view that emphasizes Lorentz invariance from the beginning

Classical physics of a real scalar $\varphi(x)$. Two frames: $\bar{\varphi}(x) = \varphi(x)$, $\bar{x}^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$ $\therefore$ $\varphi(x)$ and $\bar{\varphi}(\bar{x})$ satisfy the same equation. One example: KG.

$$(-\partial^2 + m^2)\varphi(x) = 0 \qquad (\hbar = c = 1)$$

Adopt this as the classical EOM for $\varphi(x)$. EOM can be derived from action, $S = \int dt\, L = \int dt\, d^3x\, \mathcal{L} = \int d^4x\, \mathcal{L}$. d^4x is Lorentz inv, $d^4\bar{x} = |\det(\Lambda)|\, d^4x = d^4x$ $\therefore$ so must be $\mathcal{L}$.

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\varphi)^2 - \frac{1}{2}m^2\varphi^2 + \Omega_0$$

Ω_0: arbitrary constant

Hamilton principle:

$$0 = \delta S$$

$$= \delta\int d^4x\, \mathcal{L}$$

$$= \delta\int d^4x \left(-\frac{1}{2}(\partial_\mu\varphi)^2 - \frac{1}{2}m^2\varphi^2 + \Omega_0\right)$$

$$= \int d^4x \left(\underbrace{-\frac{1}{2}\, 2\, \partial^\mu\varphi\, \partial_\mu\delta\varphi}_{\text{ibp}} - \frac{1}{2}m^2\, 2\varphi\delta\varphi\right)$$

$$= \int d^4x \left(\partial_\mu\partial^\mu\varphi\, \delta\varphi - m^2\varphi\,\delta\varphi\right)$$

$$= \int d^4x \underbrace{\left[(\Box - m^2)\varphi\right]}_{=0 \;\Leftarrow\; \text{arbitrary}} \delta\varphi$$

Solutions to KG are plane waves: $e^{i(\vec{k}\cdot\vec{x} \pm \omega t)}$, $\omega = \sqrt{\vec{k}^2 + m^2}$.

Most general solution:

$$\varphi(\vec{x}, t) = \int \frac{d^3k}{f(k)}\left[a(\vec{k})e^{i\vec{k}\cdot\vec{x} - i\omega t} + b(\vec{k})e^{i\vec{k}\cdot\vec{x} + i\omega t}\right]$$

$f(k) := f(|\vec{k}|)$: for later convenience

$$e^{\mp i\omega t} \sim e^{\mp iEt}$$

The plane-wave solution of Schrödinger equation $\sim e^{i\vec{p}\cdot\vec{x} - iE(\vec{p})t}$ so the second term in $\varphi \Rightarrow$ negative energy.

Impose realness:

$$\varphi^*(x) = \int \frac{d^3k}{f(k)}\left[a^*(\vec{k})e^{-i\vec{k}\cdot\vec{x} + i\omega t} + b^*(\vec{k})e^{-i\vec{k}\cdot\vec{x} - i\omega t}\right]$$

$$= \int \frac{d^3k}{f(k)}\left[a^*(\vec{k})e^{-i\vec{k}\cdot\vec{x} + i\omega t} + b^*(-\vec{k})e^{i\vec{k}\cdot\vec{x} - i\omega t}\right]$$

$$= \varphi(x)$$

$$= \int \frac{d^3k}{f(k)}\left[a(\vec{k})e^{i\vec{k}\cdot\vec{x} - i\omega t} + b(\vec{k})e^{i\vec{k}\cdot\vec{x} + i\omega t}\right]$$

$$= \int \frac{d^3k}{f(k)}\left[a(\vec{k})e^{i\vec{k}\cdot\vec{x} - i\omega t} + b(-\vec{k})e^{-i\vec{k}\cdot\vec{x} + i\omega t}\right]$$

$$\therefore b^*(-\vec{k}) = a(\vec{k}) \Rightarrow b(\vec{k}) = a^*(-\vec{k})$$

$$\therefore \varphi(x) = \int \frac{d^3k}{f(k)}\left[a(\vec{k})e^{i\vec{k}\cdot\vec{x} - i\omega t} + a^*(-\vec{k})e^{i\vec{k}\cdot\vec{x} + i\omega t}\right]$$

$$= \int \frac{d^3k}{f(k)}\left[a(\vec{k})e^{i\vec{k}\cdot\vec{x} - i\omega t} + a^*(\vec{k})e^{-i\vec{k}\cdot\vec{x} + i\omega t}\right]$$

$$= \int \frac{d^3k}{f(k)}\left[a(\vec{k})e^{ikx} + a^*(\vec{k})e^{-ikx}\right]$$

$$k^\mu = (\omega, \vec{k}), \quad x = (t, \vec{x}), \quad k^2 = \vec{k}^2 - \omega^2 = -m^2 : \text{on-shell}$$

e^{ikx} is Lorentz-inv. Pick $f(k)$ s.t integral measure is also Lorentz-inv.

Consider the object $d^4k\, \delta(k^2+m^2)\theta(k^0)$, which is a really meaningful object — it enforces on-shell and +energy; moreover every factor is Lorentz-inv.

$$\int_{-\infty}^{\infty} dk^0 \int d^3k\; \delta(k^2+m^2)\theta(k^0) = \int_0^\infty dk^0 \int d^3k\; \delta(\underbrace{\vec{k}^2 - (k^0)^2 + m^2}_{\omega^2 - (k^0)^2})$$

$$\delta(\omega^2 - (k^0)^2) = \frac{\delta(k^0 - \omega)}{2\omega} + \cancel{\frac{\delta(k^0+\omega)}{2\omega}}^{\,0}$$

$$= \int d^3k\, \frac{1}{2\omega}$$

$$\therefore f(k) \propto \omega$$

Take $f(k) = (2\pi)^3 2\omega$ and let

$$\widetilde{dk} := \frac{d^3k}{(2\pi)^3 2\omega}$$

$$\therefore \varphi(x) = \int \widetilde{dk}\left[a(\vec{k})e^{ikx} + a^*(\vec{k})e^{-ikx}\right]$$

$$\int d^3x\, e^{-ik'x}\varphi(x) = \int d^3x\, e^{-ik'x}\int \frac{d^3k}{(2\pi)^3 2\omega}\left[a(\vec{k})e^{ikx} + a^*(\vec{k})e^{-ikx}\right]$$

$$= \int \frac{d^3x\, d^3k}{(2\pi)^3 2\omega} a(\vec{k})e^{i(k-k')x} + \int \frac{d^3x\, d^3k}{(2\pi)^3 2\omega} a^*(\vec{k})e^{-i(k+k')x}$$

$$= \int \frac{d^3k}{2\omega} a(\vec{k})\delta^3(\vec{k}-\vec{k}')e^{-i(\omega-\omega')t} + \int \frac{d^3k}{2\omega} a^*(\vec{k})\delta^3(\vec{k}+\vec{k}')e^{i(\omega+\omega')t}$$

$$= \frac{1}{2\omega'}a(\vec{k}') + \frac{1}{2\omega'}a^*(-\vec{k}')e^{2i\omega' t}$$

$$\dot{\varphi}(x) = \int \widetilde{dk}\; i\omega\left[-a(\vec{k})e^{ikx} + a^*(\vec{k})e^{-ikx}\right]$$

$$\int d^3x\, e^{-ik'x}\dot{\varphi}(x) = \int d^3x\, e^{-ik'x}\int \frac{d^3k}{(2\pi)^3 2\omega}\, i\omega\left[-a(\vec{k})e^{ikx} + a^*(\vec{k})e^{-ikx}\right]$$

$$= -i\int \frac{d^3x\, d^3k}{(2\pi)^3 2} a(\vec{k})e^{i(k-k')x} + i\int \frac{d^3x\, d^3k}{(2\pi)^3 2} a^*(\vec{k})e^{-i(k+k')x}$$

$$= -\frac{i}{2}\int d^3k\, a(\vec{k})\delta^3(\vec{k}-\vec{k}')e^{-i(\omega-\omega')t} + \frac{i}{2}\int d^3k\, a^*(\vec{k})\delta^3(\vec{k}+\vec{k}')e^{i(\omega+\omega')t}$$

$$= -\frac{i}{2}a(\vec{k}') + \frac{i}{2}e^{2i\omega' t}a^*(-\vec{k}')$$

$$\frac{1}{2\omega}a(\vec{k}) + \frac{1}{2\omega}e^{2i\omega t}a^*(-\vec{k}) = \int d^3x\, e^{-ikx}\varphi(x) \quad \Big|\; i$$

$$-\frac{i}{2}a(\vec{k}) + \frac{i}{2}e^{2i\omega t}a^*(-\vec{k}) = \int d^3x\, e^{-ikx}\varphi(x) \quad \Big|\; -\frac{1}{\omega}$$

$$\frac{i}{2\omega}a(\vec{k}) + \frac{i}{2\omega}a(\vec{k}) = \int d^3x\, e^{-ikx}\left(i\varphi - \frac{1}{\omega}\dot{\varphi}\right) \quad \Big|\; \frac{\omega}{i}$$

$$a(\vec{k}) = \int d^3x\, e^{-ikx}(\omega\varphi + i\dot{\varphi})$$

$$= i\int d^3x\left(\dot{\varphi}e^{-ikx} - \varphi\, i\omega\, e^{-ikx}\right)$$

$$= i\int d^3x\left(\dot{\varphi}e^{-ikx} - \varphi\,(e^{-ikx})^{\cdot}\right)$$

$$= i\int d^3x\; e^{-ikx}\overleftrightarrow{\partial_0}\varphi(x)$$

$$f\overleftrightarrow{\partial_\mu}g := f\partial_\mu g - \partial_\mu f\, g$$

Note that $a(\vec{k})$ is time independent.

$$\mathcal{L} = -\frac{1}{2}\underbrace{(\partial_\mu\varphi)^2}_{\partial^0\varphi\partial_0\varphi + \partial^i\varphi\partial_i\varphi = -\dot{\varphi}^2 + (\vec{\nabla}\varphi)^2} - \frac{1}{2}m^2\varphi^2 + \Omega_0$$

$$= \frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}(\vec{\nabla}\varphi)^2 - \frac{1}{2}m^2\varphi^2 + \Omega_0$$

Conjugate momentum:

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = \dot{\varphi}(x)$$

Hamiltonian density:

$$\mathcal{H}(x) = \pi(x)\dot{\varphi}(x) - \mathcal{L}$$

$$= \pi^2 - \frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{1}{2}m^2\varphi^2 - \Omega_0$$

$$= \frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{1}{2}m^2\varphi^2 - \Omega_0$$

Hamiltonian: See code_1.

$$H = \int d^3x\; \mathcal{H}(x)$$

$$= -\Omega_0 V + \frac{1}{2}\int \widetilde{dk}\; \omega\left[a^*(\vec{k})a(\vec{k}) + a(\vec{k})a^*(\vec{k})\right]$$

So far, it's been all classical.

Canonical quantization:

$$[\varphi(\vec{x},t), \varphi(\vec{x}',t)] = 0$$

$$[\pi(\vec{x},t), \pi(\vec{x}',t)] = 0$$

$$[\varphi(\vec{x},t), \pi(\vec{x}',t)] = i\delta^3(\vec{x}-\vec{x}')$$

See code_2:

$$[a(\vec{k}), a(\vec{k}')] = 0$$

$$[a^\dagger(\vec{k}), a^\dagger(\vec{k}')] = 0$$

$$[a(\vec{k}), a^\dagger(\vec{k}')] = (2\pi)^3 2\omega\, \delta^3(\vec{k}-\vec{k}')$$

Back to hamiltonian:

$$H = -\Omega_0 V + \frac{1}{2}\int \widetilde{dk}\; \omega\Big[a^\dagger(\vec{k})a(\vec{k}) + \underbrace{a(\vec{k})a^\dagger(\vec{k})}_{(2\pi)^3 2\omega\,\delta^3(0) + a^\dagger(\vec{k})a(\vec{k})}\Big]$$

$$= -\Omega_0 V + \int \widetilde{dk}\; \omega\, a^\dagger(\vec{k})a(\vec{k}) + \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2\omega}\, (2\pi)^3 2\omega\, \underbrace{\delta^3(0)}_{V}\, \omega$$

$$= -\Omega_0 V + V\underbrace{\frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\omega}_{=:\, \mathcal{E}_0 \,:\, \text{total zero-point energy of all oscillators}} + \int \widetilde{dk}\; \omega\, a^\dagger(\vec{k})a(\vec{k})$$

$$= (\mathcal{E}_0 - \Omega_0)V + \int \widetilde{dk}\; \omega\, a^\dagger(\vec{k})a(\vec{k})$$

$\mathcal{E}_0$ is ∞, so we can integrate it up to some UV cut-off, $\Lambda \gg m$:

$$\mathcal{E}_0 = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\omega$$

$$\to \frac{1}{2}\int_0^\Lambda \frac{q^2 dq}{(2\pi)^3}\int_0^{4\pi} d\Omega\, \sqrt{q^2+m^2}, \qquad q := |\vec{k}|$$

$$\approx \frac{1}{2}\, 4\pi\, \frac{q^4}{4}\Big|_0^\Lambda \frac{1}{(2\pi)^3}$$

$$= \frac{1}{2}\frac{\Lambda^4}{8\pi^2}$$

$$= \frac{\Lambda^4}{16\pi^2}$$

This is physically justified if the formalism of quantum field theory breaks down at some large energy scale. For now, since Ω_0 is arbitrary, set $\Omega_0 = \mathcal{E}_0$. W/ this choice, the ground state has zero energy.