QFT

Ch1: Attempts at relativistic quantum mechanics

Klein-Gordon equation:

$$-\hbar^2 \frac{\partial^2}{\partial t^2} \psi(\vec{x},t) = \left(-\hbar^2 c^2 \vec{\nabla}^2 + m^2 c^4\right) \psi(\vec{x},t)$$

$$x^\mu = (ct, \vec{x})$$

$$g_{\mu\nu} = - + + +$$

$$g^{\mu\nu} g_{\nu\rho} = \delta^\mu{}_\rho$$

$$\bar{x}^\mu = \underbrace{\Lambda^\mu{}_\nu}_{\text{Lorentz transformation}} x^\nu + \underbrace{a^\mu}_{\text{translation}}$$

$$(\bar{x}-\bar{x}')^2 = (\bar{x}-\bar{x}')^\mu (\bar{x}-\bar{x}')_\mu$$

$$= g_{\mu\nu} (\bar{x}-\bar{x}')^\mu (\bar{x}-\bar{x}')^\nu$$

$$= g_{\mu\nu} \Lambda^{\mu}{}_{\rho} (x-x')^{\rho} \Lambda^{\nu}{}_{\sigma} (x-x')^{\sigma}$$

$$= g_{\mu\nu} \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} (x-x')^{\rho} (x-x')^{\sigma}$$

$$= g_{\rho\sigma} (x-x')^{\rho} (x-x')^{\sigma}$$

$$\therefore g_{\mu\nu} \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} = g_{\rho\sigma}$$

Two inertial frames: $\psi(x) = \bar{\psi}(\bar{x})$

$$\partial_{\mu} = \frac{\partial}{\partial x^{\mu}} = \left(\frac{1}{c} \frac{\partial}{\partial t}, \vec{\nabla} \right)$$

$$\partial^{\mu} = \frac{\partial}{\partial x_{\mu}} = \left(-\frac{1}{c} \frac{\partial}{\partial t}, \vec{\nabla} \right)$$

$$\partial^{\mu} x^{\nu} = g^{\mu\nu}$$

$$\bar{\partial}^{\mu} = \Lambda^{\mu}{}_{\nu} \partial^{\nu} : \bar{\partial}^{\mu} \bar{x}^{\nu} = (\Lambda^{\mu}{}_{\rho} \partial^{\rho})(\Lambda^{\nu}{}_{\sigma} x^{\sigma})$$

$$= \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} \underbrace{\partial^{\rho} x^{\sigma}}_{g^{\rho\sigma}}$$

$$= g^{\mu\nu}$$

KG equation:

$$-\hbar^2 \frac{\partial^2}{\partial t^2} \psi(x) = (-\hbar^2 c^2 \vec{\nabla}^2 + m^2 c^4)\psi(x) \quad \Big| \; \frac{1}{\hbar^2 c^2}$$

$$\Big(\underbrace{-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \vec{\nabla}^2}_{\partial_\mu \partial^\mu = \partial^2 = \Box} - \frac{m^2c^2}{\hbar^2}\Big)\psi(x) = 0$$

$$\Big(-\Box + \frac{m^2c^2}{\hbar^2}\Big)\psi(x) = 0$$

$$\Big(-\bar{\Box} + \frac{m^2c^2}{\hbar^2}\Big)\bar{\psi}(\bar{x}) = 0$$

$$\bar{\Box} = \bar{\partial}_\mu \bar{\partial}^\mu = g_{\mu\nu}\bar{\partial}^\mu \bar{\partial}^\nu$$

$$= g_{\mu\nu}\Lambda^\mu{}_\rho \partial^\rho \Lambda^\nu{}_\sigma \partial^\rho$$

$$= g_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \partial^\rho \partial^\sigma$$

$$= g_{\rho\sigma}\partial^\rho \partial^\sigma$$

$$= \partial_\mu \partial^\mu$$

$$= \Box$$

$$\left.\begin{aligned}\bar{\psi}(\bar{x}) &= \psi(x) \\ \bar{\Box} &= \Box\end{aligned}\right\}$$ KG equation is consistent w/ relativity.

KG: not first order in time derivative
∴ not compatible w/ Schrödinger equation.
$|\psi(x)|^2$ is in general not time-indep.

$$\begin{cases}\left(\Box - \dfrac{m^2c^2}{\hbar^2}\right)\psi = 0 \\ \psi^*\left(\overleftarrow{\Box} - \dfrac{m^2c^2}{\hbar^2}\right) = 0\end{cases}$$

$$\psi^*\left(\Box - \frac{m^2c^2}{\hbar^2}\right)\psi = 0$$

$$-\ \psi^*\left(\overleftarrow{\Box} - \frac{m^2c^2}{\hbar^2}\right)\psi = 0$$

$$\psi^*\Box\psi - (\Box\psi^*)\psi = 0$$

$$\psi^*\partial_\mu \partial^\mu\psi - \partial_\mu\partial^\mu\psi^*\psi = 0$$

$$\partial_\mu(\psi^*\partial^\mu\psi) - \cancel{\partial_\mu\psi^*\partial^\mu\psi} - \partial_\mu(\partial^\mu\psi^*\psi)$$

$$+\ \cancel{\partial^\mu\psi^*\partial_\mu\psi} = 0$$

$$\partial_\mu \left(\psi^* \partial^\mu \psi - \partial^\mu \psi^* \psi \right) = 0$$

$$j^\mu = \psi^* \partial^\mu \psi - \partial^\mu \psi^* \psi : \quad \partial_\mu j^\mu = 0$$

$$c\rho = j^0$$

$$= -\psi^* \frac{1}{c} \frac{\partial \psi}{\partial t} + \frac{1}{c} \frac{\partial \psi^*}{\partial t} \psi$$: not positive definite ∴ problem w/ probability interpretation

Dirac : spin 1/2

$$i\hbar \frac{\partial}{\partial t} \psi_a(x) = \left(-i\hbar c \vec{\alpha}_{ab} \cdot \vec{\nabla} + mc^2 \beta_{ab} \right) \psi_b(x)$$

: Dirac equation

$a, b = 1, 2$: spin indices

Compatible w/ Schrödinger equation ∵ first order in time derivative. $\vec{\alpha}, \beta$ are matrices in spin space.

$$H_{ab} = c\vec{p} \cdot \vec{\alpha}_{ab} + mc^2 \beta_{ab}$$

$$(H^2)_{ab} = \left[(c\vec{p}\cdot\vec{\alpha} + mc^2\beta)(c\vec{p}\cdot\vec{\alpha} + mc^2\beta) \right]_{ab}$$

$$= \left[c^2 p_i p_j \alpha_i \alpha_j + mc^3 p_i (\alpha_i \beta + \beta \alpha_i) + m^2 c^4 \beta^2 \right]_{ab}$$

$$= c^2 p_i p_j \frac{\{\alpha_i, \alpha_j\}_{ab}}{2} + mc^3 \{\alpha_i, \beta\}_{ab} + m^2 c^4 (\beta^2)_{ab}$$

$$= (c^2 \vec{p}^{\,2} + m^2 c^4)\delta_{ab}$$

$$\frac{\{\alpha_i, \alpha_j\}_{ab}}{2} = \delta_{ij}\delta_{ab} \Rightarrow \{\alpha_i, \alpha_j\} = 2\delta_{ij}$$

$$\{\alpha_i, \beta\} = 0$$

$$(\beta^2)_{ab} = \delta_{ab} \Rightarrow \beta^2 = 1$$

Problems w/ Dirac equation: We want 2×2 matrices for the two spin states. We have

$\{\sigma_i, \sigma_j\} = 2\delta_{ij}$ but there is no fourth matrix that anticommutes w/ $\vec{\sigma}$. (2×2 complex matrices are spanned by 1 and $\vec{\sigma}$ but 1 commutes w/ $\vec{\sigma}$.) Thus, $\vec{\alpha}$ and β must be higher-dimensional.

$$\begin{cases} \alpha_i \alpha_j + \alpha_j \alpha_i = 2\delta_{ij} \Rightarrow \alpha_i^2 = 1 \ \forall i \\ \alpha_i \beta + \beta \alpha_i = 0 \end{cases}$$

$$\beta \alpha_i \beta + \underbrace{\beta^2}_{1} \alpha_i = 0$$

$$\underbrace{\mathrm{tr}\, \beta \alpha_i \beta}_{\mathrm{tr}\, \alpha_i} + \mathrm{tr}\, \alpha_i = 0 \Rightarrow \mathrm{tr}\, \alpha_i = 0 \ \forall i$$

$$\alpha_i^2 = 1 \Rightarrow \alpha_i \to \mathrm{diag}(\underbrace{1, \dots, 1}_{N_+}, \underbrace{-1, \dots, -1}_{N_-})$$

$$\mathrm{tr}\, \alpha_i = 0 \Rightarrow N_+ = N_-$$

$$\Rightarrow D(\alpha) = 2N_+ : \text{even-dimensional}$$

Thus, the minimum size for $\vec{\alpha}$ and β is 4x4. What to do w/ these extra states?

$$H = c\vec{p}\cdot\vec{\alpha} + mc^2\beta$$

$$\mathrm{tr}\, H = 0 \Rightarrow H \to \mathrm{diag}(E(\vec{p}), E(\vec{p}), -E(\vec{p}), -E(\vec{p}))$$

$$E(\vec{p}) = \sqrt{\vec{p}^{\,2}c^2 + m^2c^4}$$

Negative-energy eigenvalues imply $\exists$ no ground state. A +energy electron cloud could emit a photon and drop down into a -energy state. This downward cascade could continue forever.

Dirac's solution: Pauli principle + all -energy states are already occupied. Then, a +energy electron can drop into a -energy state.

Questions: Why don't we see this sea of -energy particles? They are uniform ∴

no force on daily life. However, if a -energy particle is exited to a +energy state via radiation, then $\exists$ a hole of +charge left behind. This is the antiparticle of electron.

We started w/ a single relativistic particle but now we have ∞-many of them. Moreover, we haven't solved the problem of particles that do not obey Pauli principle.

Think about what's going on: Why it is so hard to find an acceptable theory/relativistic wave equation for a single quantum particle. Is there something wrong w/ our basic assumptions?

Yes! Recall the axiom "observables are

represented by hermitian operators."
But time is not. t is just a label in a state, not an eigenvalue of any time operator, cf. $\vec{x}$ ∴ space and time are not treated equally.

Two solutions: Promote time to an operator or demote position to a label. The former gives string theory. The latter promotes operators to fields: quantum fields.

The two solutions turn out to be equivalent. There is another useful equivalence: Ordinary non-relativistic QM, for a fixed number of particles, can be rewritten as a quantum field theory.

n particles, all w/ mass m, under external potential, $\mathcal{U}(\vec{x})$, and interacting via an interparticle potential, $V(\vec{x}_i - \vec{x}_j)$.

$$i\hbar \frac{\partial}{\partial t} \psi = \left\{ \sum_{j=1}^{n} \left[-\frac{\hbar^2}{2m} \vec{\nabla}_j^2 + \mathcal{U}(\vec{x}_j) \right] + \sum_{j=1}^{n} \sum_{k=1}^{j-1} V(\vec{x}_j - \vec{x}_k) \right\} \psi$$

$$\psi = \psi(\vec{x}_1, \dots, \vec{x}_n, t)$$

Introduce $a(\vec{x})$ and $a^\dagger(\vec{x})$ quantum fields in the Schrödinger picture.

$$[a(\vec{x}), a(\vec{x}')] = 0$$

$$[a^\dagger(\vec{x}), a^\dagger(\vec{x}')] = 0$$

$$[a(\vec{x}), a^\dagger(\vec{x}')] = \delta^3(\vec{x} - \vec{x}')$$

$$H = \int d^3x \, a^\dagger(\vec{x}) \left[-\frac{\hbar^2}{2m} \vec{\nabla}^2 + \mathcal{U}(\vec{x}) \right] a(\vec{x})$$

$$+ \frac{1}{2} \int d^3x \, d^3y \; V(\vec{x} - \vec{y}) \, a^\dagger(\vec{x}) a^\dagger(\vec{y}) a(\vec{y}) a(\vec{x})$$

$$|\psi(t)\rangle = \int d^3x_1 \cdots d^3x_n \; \psi(\vec{x}_1, \ldots, \vec{x}_n, t)$$

$$\times \; a^\dagger(\vec{x}_1) \ldots a^\dagger(\vec{x}_n) |0\rangle$$

$a(\vec{x})|0\rangle = 0$

$|0\rangle$: vacuum, no-particle state

$a^\dagger(\vec{x}_1)|0\rangle$: one particle at $\vec{x}_1$

$a^\dagger(\vec{x}_1) a^\dagger(\vec{x}_2)|0\rangle$: one particle at $\vec{x}_1$ and another at $\vec{x}_2$

$$H|\psi(t)\rangle = \left\{ \int d^3x \; a^\dagger(\vec{x}) \left[-\frac{\hbar^2}{2m} \vec{\nabla}^2 + \mathcal{U}(\vec{x}) \right] a(\vec{x}) \right.$$

$$\left. + \frac{1}{2} \int d^3x \, d^3y \; V(\vec{x}-\vec{y}) \, a^\dagger(\vec{x}) a^\dagger(\vec{y}) a(\vec{y}) a(\vec{x}) \right\}$$

$$\times \left[\int d^3x_1 \cdots d^3x_n \; \psi(\vec{x}_1, \ldots, \vec{x}_n, t) \right.$$

$$\left. \times \; a^\dagger(\vec{x}_1) \ldots a^\dagger(\vec{x}_n) |0\rangle \right]$$

See code-1.

$$H|\psi(t)\rangle = \int d^3x_1 \dots d^3x_n \left[\sum_{j=1}^{n} Q(\vec{x}_j) + \sum_{j<k}^{n} V(\vec{x}_j - \vec{x}_k) \right]$$

$$\times \psi(\vec{x}_1, \dots, \vec{x}_n, t)\, a^\dagger(\vec{x}_1) \dots a^\dagger(\vec{x}_n)|0\rangle$$

$$Q(\vec{x}) := -\frac{\hbar^2}{2m} \vec{\nabla}^2 + U(\vec{x})$$

$\sum_{i=1}^{n} Q(\vec{x}_i) + \sum_{i<j}^{n} V(\vec{x}_i - \vec{x}_j)$: Schrödinger operator

$\therefore$ Schrödinger equation is satisfied.

Number operator:

$$N = \int d^3x\, a^\dagger(\vec{x})\, a(\vec{x})$$

$[H, N] = 0$: See code.2.

Another important aspect:

$$|\psi(t)\rangle = \int d^3x_1 \dots d^3x_n\, \psi(\vec{x}_1, \dots, \vec{x}_n, t) a^\dagger(\vec{x}_1) \dots a^\dagger(\vec{x}_n)|0\rangle$$

w/ $[a^\dagger(\vec{x}), a^\dagger(\vec{x}')] = 0$. Then, the wavefunction must be symmetric:

$$\psi(\vec{x}_i, \vec{x}_j, t) = +\psi(\vec{x}_j, \vec{x}_i, t): \text{bosons}$$

If we impose $\{a^\dagger(\vec{x}), a^\dagger(\vec{x}')\} = 0$, then the wavefunction must be antisymmetric:

$$\psi(\vec{x}_i, \vec{x}_j, t) = -\psi(\vec{x}_j, \vec{x}_i, t) \; : \text{ fermions}$$

Fermions obey the abstract Schrödinger equation, as well. See code_3. The output is the same as that of code_1.