LSZ reduction formula

Free theory:

$$|k\rangle = a^\dagger(\vec{k})|0\rangle$$

$$a^\dagger(\vec{k}) = \left[\int d^3x\, e^{-ikx}(\omega\varphi + i\dot{\varphi})\right]^\dagger$$

$$= \int d^3x\, e^{ikx}(\omega\varphi - i\dot{\varphi})$$

$$= \int d^3x\, (\varphi\omega e^{ikx} - i\dot{\varphi} e^{ikx})$$

$$= -i\int d^3x\, (\varphi i\omega e^{ikx} + \dot{\varphi} e^{ikx})$$

$$= -i\int d^3x\, \left(-\varphi (e^{ikx})^{\cdot} + \dot{\varphi} e^{ikx}\right)$$

$$= -i\int d^3x\, e^{ikx} \overleftrightarrow{\partial_0} \varphi$$

$$a(\vec{k})|0\rangle = 0$$

$$\langle 0|0\rangle = 1$$

$$\langle k|k'\rangle = (2\pi)^3 2\omega\, \delta^3(\vec{k}-\vec{k}'), \quad \omega = \sqrt{\vec{k}^2+m^2}$$

Define a time-independent operator that (in the free theory) creates a particle localized in momentum space near $\vec{k}_1$ and localized in position space near the origin:

$$a_1^\dagger := \int d^3k\, f_1(\vec{k})\, a^\dagger(\vec{k})$$

$$f_1(\vec{k}) \propto e^{-(\vec{k}-\vec{k}_1)^2/4\sigma^2}$$

Consider the state $a_1^\dagger|0\rangle$. If we time-evolve this state in Schrödinger pic, wave packet will propagate (and spread out). The particle is thus localized far from the origin as $t \to \pm\infty$. If we consider $a_1^\dagger a_2^\dagger|0\rangle$ w/ $\vec{k}_1 \neq \vec{k}_2$, then the two particles are widely separated in the far past.

Suppose this still works in the interacting theory. $a^\dagger(\vec{k})$ will no longer be time-independent ($a^\dagger$ may not commute w/ H in an interacting theory), so will $a_1^\dagger$. Our guess for a suitable initial state of a scattering experiment is then

$$|i\rangle = \lim_{t\to-\infty} a_1^\dagger(t)\, a_2^\dagger(t)|0\rangle$$

$$\langle i|i\rangle = 1$$

Simile, for a final state,

$$|f\rangle = \lim_{t\to+\infty} a_{1'}^\dagger(t)\, a_{2'}^\dagger(t)|0\rangle$$

w/ $\vec{k}_1' \neq \vec{k}_2'$ and

$$\langle f|f\rangle = 1$$

This describes two widely separated particles in the far future.

$$T_{if} = \langle f|i\rangle$$

Note that

$$a_1^\dagger(\infty) - a_1^\dagger(-\infty) = \int_{-\infty}^{\infty} dt\, \partial_0 a_1^\dagger(t)$$

$$= \int_{-\infty}^{\infty} dt\, \partial_0 \int d^3k\, f_1(\vec{k})\, a^\dagger(\vec{k})$$

$$= \int_{-\infty}^{\infty} dt\, \partial_0 \int d^3k\, f_1(\vec{k}) \int d^3x\, e^{ikx}(\omega\varphi - i\dot{\varphi})$$

$$= \int d^3k\, f_1(\vec{k}) \int_{-\infty}^{\infty} dt \int d^3x\, \underbrace{\left[(-i\omega)e^{ikx}(\omega\varphi - i\dot{\varphi}) + e^{ikx}(\omega\dot{\varphi} - i\ddot{\varphi})\right]}$$

$$= e^{ikx}(-i\omega^2\varphi - \omega\dot{\varphi} + \omega\dot{\varphi} - i\ddot{\varphi})$$

$$= e^{ikx}(-i)(\partial_0^2 + \omega^2)\varphi$$

$$= \underbrace{e^{ikx}(-i)(\partial_0^2 + \vec{k}^2}_{-\vec{\nabla}^2 e^{ikx}} + m^2)\varphi \quad | \text{ ibp}$$

$$= e^{ikx}(-i)(\partial_0^2 - \vec{\nabla}^2 + m^2)\varphi$$

$$= -ie^{ikx}(-\partial^2 + m^2)\varphi$$

$$= -i\int d^3k\, f_1(\vec{k}) \int d^4x\, e^{ikx} \underbrace{(-\partial^2+m^2)\varphi}_{\substack{=0 \text{ in free theory} \\ \neq 0 \text{ in interacting theory}}}$$

We need $a_1^\dagger(-\infty)$ for $|i\rangle$:

$$a_1^\dagger(-\infty) = a_1^\dagger(\infty) + i\int d^3k\, f_1(\vec{k}) \int d^4x\, e^{ikx}(-\partial^2+m^2)\varphi$$

We need $a_1(\infty)$ for $|f\rangle$:

$$a_1(-\infty) = a_1(\infty) - i\int d^3k\, f_1(\vec{k}) \int d^4x\, e^{-ikx}(-\partial^2+m^2)\varphi$$

$$\therefore a_1(\infty) = a_1(-\infty) + i\int d^3k\, f_1(\vec{k}) \int d^4x\, e^{-ikx}(-\partial^2+m^2)\varphi$$

Scattering amplitude:

$$\langle f|i\rangle = \langle 0|\, a_{1'}(\infty)\, a_{2'}(\infty)\, a_1^\dagger(-\infty)\, a_2^\dagger(-\infty)|0\rangle$$

The operators are already in time order but we can put a time-ordering symbol:

$$\langle f|i\rangle = \langle 0|\, T\{a_{1'}(\infty)\, a_{2'}(\infty)\, a_1^\dagger(-\infty)\, a_2^\dagger(-\infty)\}|0\rangle$$

Let us expand $a_{i'}(\infty)$ and $a_i^\dagger(-\infty)$. $a_i^\dagger(-\infty)$ contains $a_i^\dagger(\infty)$, which is send to the leftmost position by T, which then annihilates $\langle 0|$. Simile, $a_{i'}(\infty)$ contains $a_{i'}(-\infty)$, which is sent to the rightmost position by T, which then annihilates $|0\rangle$. Thus, we can effectively write

$$a_i^\dagger(-\infty) \equiv i\int d^3k\, f_i(\vec{k}) \int d^4x\, e^{ikx}(-\partial^2+m^2)\varphi$$

$$a_{i'}(+\infty) \equiv i\int d^3k\, f_{i'}(\vec{k}) \int d^4x\, e^{-ikx}(-\partial^2+m^2)\varphi$$

The wave packets no longer play a key role so we can take $\sigma \to 0$ to write

$$f_i(\vec{k}) = \delta^3(\vec{k}-\vec{k}_i)$$

and hence

$$a_i^\dagger(-\infty) \equiv i\int d^4x_i\, e^{ik_ix_i}(-\partial_i^2+m^2)\varphi(x_i)$$

$$a_{i}(+\infty) \equiv i\int d^4x_{i'}\, e^{-ik_{i'}x_{i'}}(-\partial_{i'}^2+m^2)\varphi(x_{i'})$$

and therefore

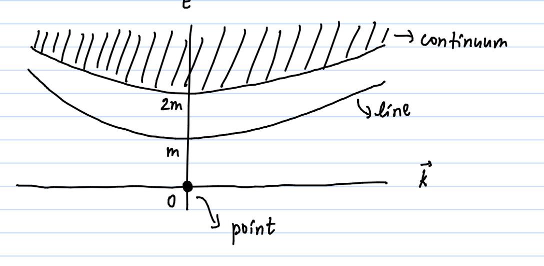
$$\langle f|i\rangle = \langle 0|\, T\{a_{1'}(+\infty)\ldots a_{n'}(+\infty)\, a_1^\dagger(-\infty)\ldots a_n^\dagger(-\infty)\}|0\rangle$$

$$= \langle 0|\, T\Big\{ i\int d^4x_{1'}\, e^{-ik_{1'}x_{1'}}(-\partial_{1'}^2+m^2)\varphi(x_{1'}) \ldots i\int d^4x_{n'}\, e^{-ik_{n'}x_{n'}}(-\partial_{n'}^2+m^2)\varphi(x_{n'})\; i\int d^4x_1\, e^{ik_1x_1}(-\partial_1^2+m^2)\varphi(x_1) \ldots i\int d^4x_n\, e^{ik_nx_n}(-\partial_n^2+m^2)\varphi(x_n)\Big\}|0\rangle$$

$$= i^{n+n'}\int d^4x_{1'}\ldots d^4x_{n'}\, d^4x_1\ldots d^4x_n\, e^{-ik_{1'}x_{1'}}(-\partial_{1'}^2+m^2)\ldots e^{-ik_{n'}x_{n'}}(-\partial_{n'}^2+m^2)\, e^{ik_1x_1}(-\partial_1^2+m^2)\ldots e^{ik_nx_n}(-\partial_n^2+m^2)\, \langle 0|\, T\{\varphi(x_{1'})\ldots\varphi(x_{n'})\,\varphi(x_1)\ldots\varphi(x_n)\}|0\rangle$$

This is the Lehmann-Symanzik-Zimmermann reduction formula. It relies on the supposition that the creation operators of free theory would work comparably in interacting theory. This is a suspicious assumption, which should be reviewed.

Consider what we can deduce about energy and momentum eigenstates of the interacting theory on physical grounds. First, we assume $\exists$ unique ground state $|0\rangle$, w/ $p^\mu = 0$. The first excited state of a single particle w/ mass m. This state can have arbitrary momentum $\vec{k}$; its energy is $E = \sqrt{\vec{k}^2+m^2}$. The next excited state is that of two particles. These two particles could form a bound state w/ energy $< 2m$. For simplicity, assume $\nexists$ no such bound states. Then the lowest possible energy of a two-particle state is $2m$. However, a two-particle state w/ $\vec{k}_{tot} = 0$ can have any energy above $\because$ the two particles could have some relative momentum that contributes to their total energy.



What happens when we act on the ground state w/ field operator $\varphi(x)$?

$$\varphi(x) = e^{-ipx}\varphi(0)\, e^{ipx}$$

$$\langle 0|\varphi(x)|0\rangle = \langle 0|\overset{0}{e^{-ipx}}\varphi(0)\, \overset{0}{e^{ipx}}|0\rangle$$

$$= \langle 0|\varphi(0)|0\rangle : \text{a Lorentz-inv. number}$$

$|0\rangle$ is the ground state of the interacting theory ($=$ difficult to obtain) $\therefore$ we have in general no idea what this number is.

We want $\langle 0|\varphi(0)|0\rangle = 0$ $\because$ we want $a_1^\dagger(\pm\infty)|0\rangle$ to create a single-particle state, not a linear combo of a single-particle state and the ground state.

$\therefore$ If $v := \langle 0|\varphi(0)|0\rangle \neq 0$, let $\varphi \to \varphi + v$. This does not change the physics but leads to $\langle 0|\varphi(x)|0\rangle = 0$.

Consider $\langle p|\varphi(x)|0\rangle$ w/ $|p\rangle$: $\langle p|p'\rangle = (2\pi)^3 2\omega\, \delta^3(\vec{p}-\vec{p}')$.

$$\langle p|\varphi(x)|0\rangle = \langle p|e^{-ipx}\varphi(0)\, \overset{0}{e^{ipx}}|0\rangle$$

$$= e^{-ipx}\underbrace{\langle p|\varphi(0)|0\rangle}_{\text{Lorentz-inv. number}}$$

$\langle p|\varphi(0)|0\rangle$ is a function of p but the only Lorentz-inv functions of p are functions of $p^2 = -m^2$ $\therefore$ $\langle p|\varphi(0)|0\rangle$ is just a number that depends on m and other parameters in Lagrangian.

We want $\langle p|\varphi(0)|0\rangle = 1$. That is what it is in free theory: $a_1^\dagger(\pm\infty)$ creates a correctly normalized one-particle state.

If $\langle p|\varphi(0)|0\rangle \neq 1$, we will rescale (or renormalize) $\varphi(x)$.

Finally consider $\langle p,n|\varphi(x)|0\rangle$, where $|p,n\rangle$ is a multiparticle state w/ total momentum p^μ, and n is short for all other labels.

$$\langle p,n|\varphi(x)|0\rangle = \langle p,n|e^{-ipx}\varphi(0)\, e^{ipx}|0\rangle$$

$$= e^{-ipx}\langle p,n|\varphi(0)|0\rangle$$

$$= e^{-ipx}A_n(p)$$

$$p^0 = \sqrt{\vec{p}^2+M^2}$$

M: invariant mass $\in \{n\}$

$M \geq 2n$

We want $\langle p,n|\varphi(x)|0\rangle = 0$ $\because$ we want $a_1^\dagger(\pm\infty)|0\rangle$ to create a single-particle state, not a multiparticle state.

Actually, we want $\langle p,n|a_1^\dagger(\pm\infty)|0\rangle = 0$, it may be zero even when $\langle p,n|\varphi(x)|0\rangle \neq 0$. Also, we should test $a_1^\dagger(\pm\infty)|0\rangle$ only against normalizable states.

$$|\psi\rangle = \underbrace{\sum_n}_{\text{schematic sum}} \int d^3p\, \psi_n(\vec{p})\, |p,n\rangle$$

$$\langle\psi|a_1^\dagger(\infty)|0\rangle = \sum_n \int d^3p\, \psi_n^*(\vec{p})\, \langle p,n| \int d^3k\, f_1(\vec{k}) \int d^3x\, e^{ikx}(\omega\varphi - i\dot{\varphi})|0\rangle$$

$$= \sum_n \int d^3p\, d^3k\, d^3x\, \psi_n^*(\vec{p})\, f_1(\vec{k})\, e^{ikx}\Big[\omega\underbrace{\langle p,n|\varphi(x)|0\rangle}_{e^{-ipx}A_n(\vec{p})} - i\partial_0\underbrace{\langle p,n|\varphi(x)|0\rangle}_{e^{-ipx}A_n(\vec{p})}\Big]$$

$$= \sum_n \int d^3p\, d^3k\, d^3x\, \psi_n^*(\vec{p})\, f_1(\vec{k})\, e^{ikx}\left[\omega e^{-ipx}A_n(\vec{p}) - i(ip^0)e^{-ipx}A_n(\vec{p})\right]$$

$$= \sum_n \int d^3p\, d^3k\, d^3x\, \psi_n^*(\vec{p})\, f_1(\vec{k})\, e^{i(k-p)x}(p^0+k^0)A_n(\vec{p})$$

$$= \sum_n \int d^3p\, d^3k\, \psi_n^*(\vec{p})\, f_1(\vec{k})\, (2\pi)^3\delta^3(\vec{k}-\vec{p})\, e^{-i(k^0-p^0)t}(p^0+k^0)A_n(\vec{p})$$

$$= \sum_n \int d^3p\, \psi_n^*(\vec{p})\, f_1(\vec{p})\, (2\pi)^3 e^{i(p^0-k^0)t}(p^0+k^0)A_n(\vec{p})$$

where $p^0 = \sqrt{\vec{p}^2+M^2}$, $k^0 = \sqrt{\vec{p}^2+m^2}$.

Key point: $p^0 > k^0$ $\because$ $M \geq 2m > m$ $\therefore$ integrand contains a phase factor that oscillates more and more rapidly as $t \to \pm\infty$ $\therefore$ by Riemann-Lebesque lemma, the RHS vanishes as $t \to \pm\infty$.

Physically, this means a one-particle wave packet spreads out differently than a multiparticle wave packet, and the overlap b/w them $\to 0$ as $t \to \infty$. Thus, through $a_1^\dagger(t)$ creates some multiparticle states that we don't want, we can follow the one-particle state that we do want by using an appropriate wave packet. By waiting long enough, we can make the multiparticle contribution to the scattering amplitude as small as we like.

Summary:

$$\langle f|i\rangle = i^{n+n'}\int d^3x_{1'}\ldots d^3x_{n'}\, d^3x_1\ldots d^3x_n\, e^{-ik_{1'}x_{1'}}(-\partial_{1'}^2+m^2)\ldots e^{-ik_{n'}x_{n'}}(-\partial_{n'}^2+m^2)\, e^{ik_1x_1}(-\partial_1^2+m^2)\ldots e^{ik_nx_n}(-\partial_n^2+m^2)\, \langle 0|\, T\{\varphi(x_{1'})\ldots\varphi(x_{n'})\,\varphi(x_1)\ldots\varphi(x_n)\}|0\rangle$$

Valid if

$$\langle 0|\varphi(x)|0\rangle = 0$$

$$\langle k|\varphi(x)|0\rangle = e^{-ikx}$$

These normalization conditions may conflict w/ our original choice of field and parameter normalization in the Lagrangian.

Consider

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\varphi)^2 - \frac{1}{2}m^2\varphi^2 + \frac{1}{3!}g\varphi^3$$

After shifting and rescaling,

$$\mathcal{L} = -\frac{1}{2}Z_\varphi(\partial_\mu\varphi)^2 - \frac{1}{2}Z_m m^2\varphi^2 + \frac{1}{3!}Z_g g\varphi^3 + Y\varphi$$

Zs and Y are yet unknown constants. They must be chosen to ensure the validity of the normalization conditions above. This gives two conditions in four unknowns. We fix m by requiring it to be equal to the mass and g by requiring some particular scattering cross section to depend on g in some particular way $\therefore$ it is possible to compute Zs and Y order by order in g.