Ch 4: Problems

4.1) Verify $[\varphi^+(x), \varphi^-(x')]_\mp = \frac{m}{4\pi^2 r} K_1(mr)$.
Verify its limit as $m \to 0$.

Solution

$$[\varphi^+(x), \varphi^-(x')]_\mp = \int \widetilde{dk}\, \widetilde{dk'}\, e^{ikx} e^{-ik'x'} [a(\vec{k}), a^\dagger(\vec{k}')]_\mp$$

$$= \int \frac{d^3k}{\cancel{(2\pi)^3 2\omega}} \frac{d^3k'}{(2\pi)^3 2\omega'} e^{ikx - ik'x'} \cancel{(2\pi)^3 2\omega}\, \delta^3(\vec{k}-\vec{k}')$$

$$= \int \frac{d^3k\, d^3k'}{(2\pi)^3 2\omega'} e^{i\vec{k}\cdot\vec{x} - i\vec{k}'\cdot\vec{x}'} e^{-i\omega t + i\omega' t'} \delta^3(\vec{k}-\vec{k}')$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} e^{-i\omega(t-t')} \quad =$$

Go to a frame where $t - t' = 0$ so that

$$(x-x')^2 = -(t-t')^2 + (\vec{x}-\vec{x}')^2 = r^2 > 0.$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega} e^{i\vec{k}\cdot\vec{r}}$$

$$= \frac{1}{16\pi^3} \int_0^\infty dq\, q^2 \int_{-1}^{1} d\psi \underbrace{\int_0^{2\pi} d\varphi}_{2\pi}\; e^{iqr\psi} \frac{1}{\sqrt{q^2+m^2}}$$

$$= \frac{1}{iqr}\left(e^{iqr} - e^{-iqr}\right)$$

$$= \frac{1}{iqr}\, 2i \sin(qr)$$

$$= \frac{2}{qr} \sin(qr)$$

$$= \frac{1}{16\pi^3} \int_0^\infty dq\, q^2 \frac{2}{qr} \sin(qr)\, 2\pi \frac{1}{\sqrt{q^2+m^2}}$$

$$= \frac{1}{4\pi^2 r} \int_0^\infty dq \frac{q \sin(qr)}{\sqrt{q^2+m^2}}, \quad p := \frac{q}{m}$$

$$= \frac{1}{4\pi^2 r} \int_0^\infty dp\, m \frac{mp \sin(pmr)}{\sqrt{m^2p^2+m^2}}$$

$$= \frac{m}{4\pi^2 r} \int_0^\infty dt \frac{t \sin(mrt)}{\sqrt{t^2+1}}, \quad t = \sinh(u)$$

$$dt = \cosh(u)\, du$$

$$\int_0^\infty \to \int_0^\infty$$

$$= \frac{m}{4\pi^2 r} \int_0^\infty \cosh(u)du \; \frac{\sinh(u)\sin(mr\sinh(u))}{\sqrt{\sinh(u)^2+1}}$$

$$= \frac{m}{4\pi^2 r} \int_0^\infty du \, \sinh(u) \sin(mr\sinh(u))$$

See functions.wolfram.com/Bessel-TypeFunctions/BesselK/07/01/01/0005:

$$K_\nu(x) = \operatorname{cosec}\left(\frac{\pi\nu}{2}\right) \int_0^\infty dt \, \sin(x\sinh(t)) \sinh(\nu t)$$

$$\therefore \int_0^\infty du \, \sinh(u) \sin(mr\sinh(u)) = \frac{K_1(mr)}{\operatorname{cosec}\left(\frac{\pi}{2}\right)}$$

$$= K_1(mr)$$

$$\therefore [\varphi^+(x), \varphi^-(x^-)]_\mp = \frac{m}{4\pi^2 r} K_1(mr)$$

For small m, $K_1(mr) = \frac{1}{mr} + O(mr)^0$, so there is a fine limit of $mK_1(mr)$ as $m \to 0$, which is 1. Therefore,

$$[\varphi^+(x), \varphi^-(x^-)]_\mp \Big|_{m\to 0} = \frac{1}{4\pi^2 r^2}$$