4.1) Verify eq. (4.12). Verify its limit as $m \to 0$.

$$
\begin{aligned}
[\varphi^+(x), \varphi^-(x')]_\mp &= \int \widetilde{dk}\, \widetilde{dk}'\, e^{i(kx-k'x')}[a(\mathbf{k}), a^\dagger(\mathbf{k}')]_\mp \\
&= \int \widetilde{dk}\, e^{ik(x-x')} \\
&= \frac{m}{4\pi^2 r} K_1(mr) \\
&\equiv C(r)\,. \qquad (4.12)
\end{aligned}
$$

$$[\varphi^+(x), \varphi^-(x')]_\mp = \int \widetilde{dk}\, \widetilde{dk}'\; e^{ikx}\, e^{-ik'x'}\, [a(\vec{k}), a^\dagger(\vec{k}')]_\mp$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega} \frac{d^3k'}{(2\pi)^3 2\omega'}\; e^{ikx - ik'x'}\, (2\pi)^3 2\omega\, \delta^3(\vec{k}-\vec{k}')$$

$$= \int \frac{d^3k\, d^3k'}{(2\pi)^3 2\omega'}\; e^{i\vec{k}\cdot\vec{x} - i\vec{k}'\cdot\vec{x}'}\; e^{-i\omega t + i\omega' t'}\; \delta^3(\vec{k}-\vec{k}')$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega}\; e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\; e^{-i\omega(t-t')} \quad \ominus$$

Go to a frame w/ $t - t' = 0$ s.t $(x-x')^2 = -(t-t')^2 + (\vec{x}-\vec{x}')^2 = r^2 > 0$.

$$\ominus \int \frac{d^3k}{(2\pi)^3 2\omega}\; e^{i\vec{k}\cdot\vec{r}}$$

$$= \frac{1}{16\pi^3} \int_0^\infty dq\, q^2 \underbrace{\int_{-1}^{1} d\psi\; \underbrace{\int_0^{2\pi} d\varphi}_{2\pi}\; e^{iqr\psi}}_{= \frac{1}{iqr}(e^{iqr} - e^{-iqr}) = \frac{1}{iqr} 2i \sin(qr) = \frac{2}{qr}\sin(qr)} \frac{1}{\sqrt{q^2+m^2}}$$

$$= \frac{1}{16\pi^3} \int_0^\infty dq\, q^2\, \frac{2}{qr} \sin(qr)\, 2\pi\, \frac{1}{\sqrt{q^2+m^2}}$$

$$= \frac{1}{4\pi^2 r} \int_0^\infty dq\, \frac{q \sin(qr)}{\sqrt{q^2+m^2}}, \qquad p := \frac{q}{m}$$

$$= \frac{1}{4\pi^2 r} \int_0^\infty dp\, m\, \frac{mp \sin(pmr)}{\sqrt{m^2p^2+m^2}}$$

$$= \frac{m}{4\pi^2 r} \int_0^\infty dt\, \frac{t \sin(mrt)}{\sqrt{t^2+1}}, \qquad t = \sinh(u),\ dt = \cosh(u)\,du,\ \int_0^\infty \to \int_0^\infty$$

$$= \frac{m}{4\pi^2 r} \int_0^\infty \cosh(u)\,du\, \frac{\sinh(u) \sin(mr \sinh(u))}{\sqrt{\sinh(u)^2 + 1}}$$

$$= \frac{m}{4\pi^2 r} \int_0^\infty du\, \sinh(u)\, \sin(mr \sinh(u))$$

See functions.wolfram.com/Bessel-TypeFunctions/BesselK/07/01/01/0005:

$$K_\nu(x) = \operatorname{cosec}\left(\frac{\pi\nu}{2}\right) \int_0^\infty dt\, \sin(x \sinh(t))\, \sinh(\nu t)$$

$$\therefore \int_0^\infty du\, \sinh(u)\, \sin(mr \sinh(u)) = \frac{K_1(mr)}{\operatorname{cosec}\left(\frac{\pi}{2}\right)}$$

$$= K_1(mr)$$

$$\therefore [\varphi^+(x), \varphi^-(x')]_\mp = \frac{m}{4\pi^2 r} K_1(mr)$$