1.1) Show that the Dirac matrices must be even dimensional. Hint: show that the eigenvalues of β are all ± 1, and that $\mathrm{Tr}\,\beta = 0$. To show that $\mathrm{Tr}\,\beta = 0$, consider, e.g., $\mathrm{Tr}\,\alpha_1^2\beta$. Similarly, show that $\mathrm{Tr}\,\alpha_i = 0$.

$$\alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij} \Rightarrow \alpha_i^2 = 1 \quad \forall i$$

$$\alpha_i\beta + \beta\alpha_i = 0$$

$$\beta\alpha_i\beta + \beta^2\alpha_i = 0, \qquad \beta^2 = 1$$

$$\underbrace{tr\ \beta\alpha_i\beta}_{tr\ \alpha_i} + tr\ \alpha_i = 0 \Rightarrow tr\ \alpha_i = 0 \quad \forall i$$

$$M^2 = 1$$

$$= (\mathcal{U}^\dagger M_* \mathcal{U})^2$$

$$= \mathcal{U}^\dagger M_*^2 \mathcal{U}$$

$$\therefore \mathcal{U}\mathcal{U}^\dagger = 1$$

$$= M_*^2$$

$$= diag(\lambda_1^2, \lambda_2^2, \ldots) \Rightarrow \lambda_i^2 = 1 \quad \forall i$$

$$\therefore \alpha_{i*} = diag(\underbrace{1, \ldots, 1}_{N_+}, \underbrace{-1, \ldots, -1}_{N_-})$$

$$tr\ \alpha_i = 0 \Rightarrow N_+ = N_- \quad \therefore D(\alpha) = 2N_+ : \text{even-dimensional}$$

$$\underbrace{\alpha_i\alpha_i}_{1}\beta + \alpha_i\beta\alpha_i = 0$$

$$tr\ \beta + \underbrace{tr\ \alpha_i\beta\alpha_i}_{tr\ \beta} = 0 \Rightarrow tr\ \beta = 0$$

$$\left.\begin{array}{l} tr\ \beta = 0 \\ \beta^2 = 1 \end{array}\right\} D(\beta) = 2N_+ : \text{even-dimensional similarly to } \alpha_i$$

Since we have already eliminated the case of 2x2, the minimum dimension should be 4x4.

1.2) With the hamiltonian of eq. (1.32), show that the state defined in eq. (1.33) obeys the abstract Schrödinger equation, eq. (1.1), if and only if the wave function obeys eq. (1.30). Your demonstration should apply both to the case of bosons, where the particle creation and annihilation operators obey the commutation relations of eq. (1.31), and to fermions, where the particle creation and annihilation operators obey the anticommutation relations of eq. (1.38).

$$\begin{aligned} H = &\int d^3x\, a^\dagger(\mathbf{x})\Big(-\tfrac{\hbar^2}{2m}\nabla^2 + U(\mathbf{x})\Big)a(\mathbf{x}) \\ &+ \tfrac{1}{2}\int d^3x\, d^3y\, V(\mathbf{x}-\mathbf{y})a^\dagger(\mathbf{x})a^\dagger(\mathbf{y})a(\mathbf{y})a(\mathbf{x})\,. \end{aligned} \tag{1.32}$$

$$|\psi,t\rangle = \int d^3x_1 \ldots d^3x_n\, \psi(\mathbf{x}_1,\ldots,\mathbf{x}_n;t)a^\dagger(\mathbf{x}_1)\ldots a^\dagger(\mathbf{x}_n)|0\rangle\,, \tag{1.33}$$

$$i\hbar\frac{\partial}{\partial t}|\psi,t\rangle = H|\psi,t\rangle\,, \tag{1.1}$$

$$i\hbar\frac{\partial}{\partial t}\psi = \left[\sum_{j=1}^{n}\left(-\frac{\hbar^2}{2m}\nabla_j^2 + U(\mathbf{x}_j)\right) + \sum_{j=1}^{n}\sum_{k=1}^{j-1}V(\mathbf{x}_j-\mathbf{x}_k)\right]\psi\,, \tag{1.30}$$

$$\begin{aligned} [a(\mathbf{x}),a(\mathbf{x}')] &= 0\,, \\ [a^\dagger(\mathbf{x}),a^\dagger(\mathbf{x}')] &= 0\,, \\ [a(\mathbf{x}),a^\dagger(\mathbf{x}')] &= \delta^3(\mathbf{x}-\mathbf{x}')\,, \end{aligned} \tag{1.31}$$

$$\begin{aligned} \{a(\mathbf{x}),a(\mathbf{x}')\} &= 0\,, \\ \{a^\dagger(\mathbf{x}),a^\dagger(\mathbf{x}')\} &= 0\,, \\ \{a(\mathbf{x}),a^\dagger(\mathbf{x}')\} &= \delta^3(\mathbf{x}-\mathbf{x}')\,, \end{aligned} \tag{1.38}$$

See code_1 and code_3.

1.3) Show explicitly that $[N,H]=0$, where H is given by eq. (1.32) and N by eq. (1.35).

$$\begin{aligned} H = &\int d^3x\, a^\dagger(\mathbf{x})\Big(-\tfrac{\hbar^2}{2m}\nabla^2 + U(\mathbf{x})\Big)a(\mathbf{x}) \\ &+ \tfrac{1}{2}\int d^3x\, d^3y\, V(\mathbf{x}-\mathbf{y})a^\dagger(\mathbf{x})a^\dagger(\mathbf{y})a(\mathbf{y})a(\mathbf{x})\,. \end{aligned} \tag{1.32}$$

$$N = \int d^3x\, a^\dagger(\mathbf{x})a(\mathbf{x}) \tag{1.35}$$

See code_3.