Ch 2: Problems

2.1) Verify that $\omega_{\rho\sigma} = -\omega_{\sigma\rho}$ follows from $g_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} = g_{\rho\sigma}$.

Solution

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}$$

$$\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}\, g_{\mu\nu} = g_{\rho\sigma}$$

$$(\delta^{\mu}{}_{\rho} + \omega^{\mu}{}_{\rho})(\delta^{\nu}{}_{\sigma} + \omega^{\nu}{}_{\sigma})\, g_{\mu\nu} = g_{\rho\sigma}$$

$$(\delta^{\mu}{}_{\rho}\delta^{\nu}{}_{\sigma} + \delta^{\mu}{}_{\rho}\omega^{\nu}{}_{\sigma} + \omega^{\mu}{}_{\rho}\delta^{\nu}{}_{\sigma} + \omega^{\mu}{}_{\rho}\omega^{\nu}{}_{\sigma})\, g_{\mu\nu}$$

$$= g_{\rho\sigma}$$

$$g_{\rho\sigma} + \omega_{\rho\sigma} + \omega_{\sigma\rho} + O(\omega^2) = g_{\rho\sigma}$$

$$\therefore\ \omega_{\rho\sigma} = -\omega_{\sigma\rho}$$

2.2) Verify that $U(\Lambda)^{-1} M^{\mu\nu} U(\Lambda) = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} M^{\rho\sigma}$ follows from $U(\Lambda)^{-1} U(\Lambda') U(\Lambda) = U(\Lambda^{-1}\Lambda'\Lambda)$.

Solution

$$\mathcal{U}(\Lambda)^{-1}\,\mathcal{U}(\Lambda')\,\mathcal{U}(\Lambda) = \mathcal{U}(\Lambda^{-1}\Lambda'\Lambda)$$

$$\Lambda' = 1 + \omega$$

$$\mathcal{U}(\Lambda)^{-1}\left(1 + \frac{i}{2\hbar}\,\omega_{\mu\nu} M^{\mu\nu}\right)\mathcal{U}(\Lambda) = \mathcal{U}(\Lambda^{-1}\Lambda'\Lambda)$$

$$\begin{aligned}
\mathcal{U}(\Lambda^{-1}\Lambda'\Lambda) &= \mathcal{U}(\Lambda^{-1}(1+\omega)\Lambda) \\
&= \mathcal{U}(1 + \Lambda^{-1}\omega\Lambda) \\
&= 1 + \frac{i}{2\hbar}\,(\Lambda^{-1}\omega\Lambda)_{\mu\nu} M^{\mu\nu} \\
&= 1 + \frac{i}{2\hbar}\,(\Lambda^{-1})_{\mu}{}^{\rho}\,\omega_{\rho\sigma}\,\Lambda^{\sigma}{}_{\nu} M^{\mu\nu} \\
&= 1 + \frac{i}{2\hbar}\,\Lambda^{\rho}{}_{\mu}\,\Lambda^{\sigma}{}_{\nu}\,\omega_{\rho\sigma} M^{\mu\nu}
\end{aligned}$$

$$1 + \frac{i}{2\hbar}\,\omega_{\mu\nu}\,\mathcal{U}(\Lambda)^{-1} M^{\mu\nu}\,\mathcal{U}(\Lambda)$$

$$= 1 + \frac{i}{2\hbar}\,\Lambda^{\rho}{}_{\mu}\,\Lambda^{\sigma}{}_{\nu}\,\omega_{\rho\sigma} M^{\mu\nu}$$

$$\underbrace{\omega_{\mu\nu}\, U(\Lambda)^{-1} M^{\mu\nu} U(\Lambda)}_{\text{antisym. in } \mu,\nu} = \Lambda^{\rho}{}_{\mu} \Lambda^{\sigma}{}_{\nu}\, \omega_{\rho\sigma} M^{\mu\nu}$$

$$= \omega_{\mu\nu} \underbrace{\Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} M^{\rho\sigma}}_{\frac{1}{2}\left(\Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} M^{\rho\sigma} - \underbrace{\Lambda^{\nu}{}_{\rho} \Lambda^{\mu}{}_{\sigma} \underbrace{M^{\rho\sigma}}_{-M^{\sigma\rho}}\right)}_{+\Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} M^{\rho\sigma}}}$$

$$= \omega_{\mu\nu} \underbrace{\Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} M^{\rho\sigma}}_{\text{already antisym.}}$$

$$\therefore U(\Lambda)^{-1} M^{\mu\nu} U(\Lambda) = \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} M^{\rho\sigma}$$

2.3) Verify that $[M^{\mu\nu}, M^{\rho\sigma}] = i\hbar\left[g^{\mu\rho} M^{\nu\sigma} - (\mu \leftrightarrow \nu)\right] - (\rho \leftrightarrow \sigma)$ follows from $U(\Lambda)^{-1} M^{\mu\nu} U(\Lambda) = \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} M^{\rho\sigma}$.

Solution

$$\mathcal{U}(\Lambda)^{-1} M^{\mu\nu} \mathcal{U}(\Lambda) = \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} M^{\rho\sigma}$$

$$\left(1 - \frac{i}{2\hbar} \omega_{\rho\sigma} M^{\rho\sigma}\right) M^{\mu\nu} \left(1 + \frac{i}{2\hbar} \omega_{\rho\sigma} M^{\rho\sigma}\right)$$

$$= (\delta^{\mu}{}_{\rho} + \omega^{\mu}{}_{\rho})(\delta^{\nu}{}_{\sigma} + \omega^{\nu}{}_{\sigma}) M$$

$$\left(M^{\mu\nu} - \frac{i}{2\hbar} \omega_{\rho\sigma} M^{\rho\sigma} M^{\mu\nu}\right)\left(1 + \frac{i}{2\hbar} \omega_{\rho\sigma} M^{\rho\sigma}\right)$$

$$= (\delta^{\mu}{}_{\rho} + \omega^{\mu}{}_{\rho})(M^{\rho\nu} + \omega^{\nu}{}_{\sigma} M^{\rho\sigma})$$

$$M^{\mu\nu} + \frac{i}{2\hbar} \omega_{\rho\sigma} [M^{\mu\nu}, M^{\rho\sigma}] + O(\omega^2)$$

$$= M^{\mu\nu} + \omega^{\nu}{}_{\sigma} M^{\mu\sigma} + \omega^{\mu}{}_{\rho} M^{\rho\nu} + O(\omega^2)$$

$$\frac{i}{2\hbar} \omega_{\rho\sigma} \underbrace{[M^{\mu\nu}, M^{\rho\sigma}]}_{\text{already antisym. in } \rho, \sigma} = \omega^{\nu}{}_{\sigma} M^{\mu\sigma} + \omega^{\mu}{}_{\rho} M^{\rho\nu}$$

$$= \omega_{\rho\sigma} g^{\rho\nu} M^{\mu\sigma} + \underbrace{\omega^{\mu}{}_{\sigma}}_{\omega_{\rho\sigma} g^{\rho\mu}} M^{\sigma\nu}$$

$$= \omega_{\rho\sigma} \underbrace{\left(g^{\rho\nu} M^{\mu\sigma} + g^{\rho\mu} M^{\sigma\nu} \right)}_{\text{antisymmetrize wrt } \rho, \sigma}$$

$$= \omega_{\rho\sigma} \frac{1}{2} \left(g^{\rho\nu} M^{\mu\sigma} + g^{\rho\mu} M^{\sigma\nu} - g^{\sigma\nu} M^{\mu\rho} - g^{\sigma\mu} M^{\rho\nu} \right)$$

$$[M^{\mu\nu}, M^{\rho\sigma}] = -i\hbar \left(g^{\rho\nu} M^{\mu\sigma} + g^{\rho\mu} M^{\sigma\nu} - g^{\sigma\nu} M^{\mu\rho} - g^{\sigma\mu} M^{\rho\nu} \right)$$

Shorthand notation:

$$[M^{\mu\nu}, M^{\rho\sigma}] = -i\hbar \left(g^{\nu\rho} M^{\mu\sigma} - g^{\mu\rho} M^{\nu\sigma} - g^{\nu\sigma} M^{\mu\rho} + g^{\mu\sigma} M^{\nu\rho} \right)$$

$$= i\hbar \left(g^{\mu\rho} M^{\nu\sigma} - g^{\nu\rho} M^{\mu\sigma} - g^{\mu\sigma} M^{\nu\rho} - g^{\nu\sigma} M^{\mu\rho} \right)$$

$$= i\hbar \left[g^{\mu\rho} M^{\nu\sigma} - (\mu \leftrightarrow \nu) \right] - (\rho \leftrightarrow \sigma)$$

2.4) Verify that

$$[J_i, J_j] = i\hbar \varepsilon_{ijk} J_k$$

$$[J_i, K_j] = i\hbar \varepsilon_{ijk} K_k$$

$$[K_i, K_j] = -i\hbar \varepsilon_{ijk} J_k$$

follows from $[M^{\mu\nu}, M^{\rho\sigma}] = i\hbar[g^{\mu\rho}M^{\nu\sigma} - (\mu \leftrightarrow \nu)] - (\rho \leftrightarrow \sigma)$.

Solution

$$\varepsilon_{ijk}\varepsilon_{lmn} = \delta_{il}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{jn}\delta_{kl} + \delta_{in}\delta_{jl}\delta_{km} - \delta_{im}\delta_{jl}\delta_{kn} - \delta_{il}\delta_{jn}\delta_{km} - \delta_{in}\delta_{jm}\delta_{kl}$$

$$M^{jk} = \varepsilon_{ijk}J_i$$

For the rest see code-1.

2.5) Verify that $[P^\mu, M^{\rho\sigma}] = i\hbar[g^{\mu\sigma}P^\rho - (\rho \leftrightarrow \sigma)]$ follows from $U(\Lambda)^{-1}P^\mu U(\Lambda) = \Lambda^\mu{}_\nu P^\nu$.

Solution

$$U(\Lambda)^{-1}P^\mu U(\Lambda) = \Lambda^\mu{}_\nu P^\nu$$

$$\left(1 - \frac{i}{2\hbar}\omega_{\rho\sigma}M^{\rho\sigma}\right)P^\mu\left(1 + \frac{i}{2\hbar}\omega_{\rho\sigma}M^{\rho\sigma}\right)$$

$$= (\delta^\mu{}_\nu + \omega^\mu{}_\nu)P^\nu$$

$$\left(P^\mu - \frac{i}{2\hbar}\omega_{\rho\sigma}M^{\rho\sigma}P^\mu\right)\left(1 + \frac{i}{2\hbar}\omega_{\rho\sigma}M^{\rho\sigma}\right) = P^\mu + \omega^\mu{}_\nu P^\nu$$

$$p^\mu + \frac{i}{2\hbar}\omega_{\rho\sigma}[p^\mu, M^{\rho\sigma}] + O(\omega^2) = p^\mu + \omega^\mu{}_\nu p^\nu$$

$$\frac{i}{2\hbar}\omega_{\rho\sigma}\underbrace{[p^\mu, M^{\rho\sigma}]}_{\substack{\text{already antisym.}\\ \text{in } \rho,\sigma}} = \omega_{\rho\sigma} g^{\mu\rho}\delta^\sigma{}_\nu p^\nu$$

$$= \omega_{\rho\sigma}\underbrace{g^{\mu\rho} p^\sigma}_{\text{antisymmetrize wrt } \rho,\sigma}$$

$$= \omega_{\rho\sigma}\frac{1}{2}(g^{\mu\rho}p^\sigma - g^{\mu\sigma}p^\rho)$$

$$[p^\mu, M^{\rho\sigma}] = -i\hbar(g^{\mu\rho}p^\sigma - g^{\mu\sigma}p^\rho)$$

Shorthand notation:

$$[p^\mu, M^{\rho\sigma}] = i\hbar[g^{\mu\sigma}p^\rho - (\rho\leftrightarrow\sigma)]$$

2.6) Verify that

$$[J_i, H] = 0$$

$$[J_i, p_j] = i\hbar\varepsilon_{ijk}p_k$$

$$[K_i, H] = i\hbar p_i$$

$$[K_i, p_j] = i\hbar\delta_{ij}H$$

follows from $[p^\mu, M^{\rho\sigma}] = i\hbar[g^{\mu\sigma}p^\rho - (\rho \leftrightarrow \sigma)]$

Solution See code-2.

2.7) What property should be attributed to the translation operator, $T(a)$, that could be used to prove

$$[p_i, p_j] = 0 = [p_i, H]$$

Solution These two can be summarized as

$$[p_\mu, p_\nu] = 0$$

This can be obtained if successive translations commute:

$$[T(a), T(b)] = 0$$

∞-simal translation:

$$\left[1 - \frac{i}{\hbar} p a, 1 - \frac{i}{\hbar} p b\right] = 0$$

$$\underbrace{\left(-\frac{i}{\hbar}\right)^2 a^\mu b^\nu}_{\neq 0} \underbrace{[p_\mu, p_\nu]}_{\therefore = 0} = 0$$

2.8) a) Let $\Lambda = 1 + \omega$ in $U(\Lambda)^{-1} \varphi(x) U(\Lambda) = \varphi(\Lambda^{-1} x)$ and show that

$$[\varphi(x), M^{\mu\nu}] = L^{\mu\nu} \varphi(x)$$

Where

$$L^{\mu\nu} := \frac{\hbar}{i} (x^\mu \partial^\nu - x^\nu \partial^\mu)$$

b) Show that $[[\varphi(x), M^{\mu\nu}], M^{\rho\sigma}] = L^{\mu\nu} L^{\rho\sigma} \varphi(x)$.

c) Prove the Jacobi identity,

$$[[A, B], C] + [[B, C], A] + [[C, A], B] = 0$$

Hint: Write out all the commutators.

d) Use your results from parts (b) and (c) to show that

$$[\varphi(x), [M^{\mu\nu}, M^{\rho\sigma}]] = [L^{\mu\nu}, L^{\rho\sigma}] \varphi(x)$$

e) Simplify the right-hand side of the above result as much as possible.

f) Use your results from part (e) to verify

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\hbar\left[g^{\mu\rho} M^{\nu\sigma} - (\mu \leftrightarrow \nu)\right] - (\rho \leftrightarrow \sigma)$$

up to the possibility of a term on the right-hand side that commutes w/ $\varphi(x)$ and its derivatives. (Such a term, called a central charge, in fact does not arise for the Lorentz algebra.)

Solution

a) $\mathcal{U}(\Lambda)^{-1} \varphi(x)\, \mathcal{U}(\Lambda) = \varphi(\Lambda^{-1} x)$

$$\left(1 - \frac{i}{2\hbar} \omega_{\mu\nu} M^{\mu\nu}\right) \varphi(x) \left(1 + \frac{i}{2\hbar} \omega_{\mu\nu} M^{\mu\nu}\right)$$

$$= \varphi(x - \omega x)$$

$$= \varphi(x) - (\omega x)^{\mu} \partial_\mu \varphi(x)$$

$$= \varphi(x) - \omega^{\mu}{}_{\nu}\, x^{\nu} \partial_\mu \varphi(x)$$

$$\left(\varphi - \frac{i}{2\hbar} \omega_{\mu\nu} M^{\mu\nu} \varphi\right)\left(1 + \frac{i}{2\hbar} \omega_{\mu\nu} M^{\mu\nu}\right)$$

$$= \varphi - \omega_{\mu\nu}\, x^{\nu} \partial^{\mu} \varphi$$

$$\varphi + \frac{i}{2\hbar}\omega_{\mu\nu}[\varphi, M^{\mu\nu}] + O(\omega^2) = \varphi - \omega_{\mu\nu} x^\nu \partial^\mu \varphi$$

$$\frac{i}{2\hbar}\omega_{\mu\nu}\underbrace{[\varphi, M^{\mu\nu}]}_{\text{already antisym in } \mu,\nu} = -\omega_{\mu\nu}\underbrace{x^\nu \partial^\mu \varphi}_{\text{antisymmetrize wrt } \mu,\nu}$$

$$= -\omega_{\mu\nu}\frac{1}{2}(x^\nu\partial^\mu - x^\mu\partial^\nu)\varphi$$

$$= \omega_{\mu\nu}\frac{1}{2}(x^\mu\partial^\nu - x^\nu\partial^\mu)\varphi$$

$$[\varphi, M^{\mu\nu}] = \frac{\hbar}{i}(x^\mu\partial^\nu - x^\nu\partial^\mu)\varphi$$

$$L^{\mu\nu} := \frac{\hbar}{i}(x^\mu\partial^\nu - x^\nu\partial^\mu)$$

cf. $[x,p] = i\hbar = -\frac{\hbar}{i} \Rightarrow [f(x), p] = \underbrace{-\frac{\hbar}{i}\frac{\partial}{\partial x}}_{\text{representation of } p \text{ in position space}} f(x)$

$\therefore -L^{\mu\nu}$ is the representation of $M^{\mu\nu}$ when acting on a scalar.

b) $L^{\mu\nu}$ and $M^{\mu\nu}$ can be concluded to commute because one is the representation of the other in a certain space.

$$[[\varphi(x), M^{\mu\nu}], M^{\rho\sigma}] = [L^{\mu\nu}\varphi(x), M^{\rho\sigma}]$$

$$= L^{\mu\nu}[\varphi(x), M^{\rho\sigma}]$$

$$= L^{\mu\nu}L^{\rho\sigma}\varphi(x)$$

c) $$[[A,B],C] + [[B,C],A] + [[C,A],B] \stackrel{?}{=} 0$$

$$[AB-BA,C] + [BC-CB,A] + [CA-AC,B] \stackrel{?}{=} 0$$

$$ABC - CAB - BAC + CBA + BCA - ABC - CBA$$
$$+ ACB + CAB - BCA - ACB + BAC \stackrel{?}{=} 0$$

$0 = 0$ identically

d) $$[\varphi(x), [M^{\mu\nu}, M^{\rho\sigma}]] = [\varphi(x), M^{\mu\nu}M^{\rho\sigma} - M^{\rho\sigma}M^{\mu\nu}]$$

$$= M^{\mu\nu}[\varphi(x), M^{\rho\sigma}] + [\varphi(x), M^{\mu\nu}]M^{\rho\sigma}$$
$$- M^{\rho\sigma}[\varphi(x), M^{\mu\nu}] - [\varphi(x), M^{\rho\sigma}]M^{\mu\nu}$$

$$= M^{\mu\nu}L^{\rho\sigma}\varphi(x) + L^{\mu\nu}\varphi(x)M^{\rho\sigma}$$
$$- M^{\rho\sigma}L^{\mu\nu}\varphi(x) - L^{\rho\sigma}\varphi(x)M^{\mu\nu}$$

$$= M^{\mu\nu} L^{\rho\sigma} \varphi(x) + L^{\mu\nu}\left[L^{\rho\sigma}\varphi(x) + M^{\rho\sigma}\varphi(x)\right]$$

$$- M^{\rho\sigma} L^{\mu\nu} \varphi(x) - L^{\rho\sigma}\left[L^{\mu\nu}\varphi(x) + M^{\mu\nu}\varphi(x)\right]$$

Noting that M and L commute, we have

$$[\varphi(x), [M^{\mu\nu}, M^{\rho\sigma}]] = L^{\mu\nu} L^{\rho\sigma}\varphi(x) - L^{\rho\sigma} L^{\mu\nu}\varphi(x)$$

$$= [L^{\mu\nu}, L^{\rho\sigma}]\varphi(x)$$

e) $$L^{\mu\nu} = \frac{\hbar}{i}(x^\mu \partial^\nu - x^\nu \partial^\mu)$$

$$= x^\mu p^\nu - x^\nu p^\mu$$

$$[L^{\mu\nu}, L^{\rho\sigma}] = [x^\mu p^\nu - x^\nu p^\mu, x^\rho p^\sigma - x^\sigma p^\rho]$$

$$= [x^\mu p^\nu, x^\rho p^\sigma] - [x^\mu p^\nu, x^\sigma p^\rho]$$

$$- [x^\nu p^\mu, x^\rho p^\sigma] + [x^\nu p^\mu, x^\sigma p^\rho]$$

$$=: A^{\mu\nu\rho\sigma} - A^{\mu\nu\sigma\rho} - A^{\nu\mu\rho\sigma} + A^{\nu\mu\sigma\rho}$$

$$A^{\mu\nu\rho\sigma} := [x^\mu p^\nu, x^\rho p^\sigma]$$

$$= x^\mu [p^\nu, x^\rho p^\sigma] + [x^\mu, x^\rho p^\sigma] p^\nu$$

$$= x^\mu x^\rho [p^\nu, p^\sigma] + x^\mu [p^\nu, x^\rho] p^\sigma$$

$$+ x^\rho [x^\mu, p^\sigma] p^\nu + [x^\mu, x^\rho] p^\sigma p^\nu$$

$$[x^\mu, p^\nu] = x^\mu p^\nu - p^\nu x^\mu$$

$$= x^\mu p^\nu - (p^\nu x^\mu) - x^\mu p^\nu$$

$$= -(p^\nu x^\mu)$$

$$= -\frac{\hbar}{i} \partial^\nu x^\mu$$

$$= i\hbar g^{\mu\nu}$$

$$A^{\mu\nu\rho\sigma} = x^\mu (-i\hbar g^{\nu\rho}) p^\sigma + x^\rho (i\hbar g^{\mu\sigma}) p^\nu$$

$$= -i\hbar g^{\nu\rho} x^\mu p^\sigma + i\hbar g^{\mu\sigma} x^\rho p^\nu$$

$$[L^{\mu\nu}, L^{\rho\sigma}] = -i\hbar g^{\nu\rho} x^\mu p^\sigma + i\hbar g^{\mu\sigma} x^\rho p^\nu$$

$$+ i\hbar g^{\nu\sigma} x^\mu p^\rho - i\hbar g^{\mu\rho} x^\sigma p^\nu$$

$$+ i\hbar g^{\mu\rho} x^\nu p^\sigma - i\hbar g^{\nu\sigma} x^\rho p^\mu$$

$$- i\hbar g^{\mu\sigma} x^\nu p^\rho + i\hbar g^{\nu\rho} x^\sigma p^\mu$$

$$= -i\hbar g^{\nu\rho} (x^\mu p^\sigma - x^\sigma p^\mu) - i\hbar g^{\mu\sigma} (-x^\rho p^\nu + x^\nu p^\rho)$$

$$+ i\hbar g^{\nu\sigma} (x^\mu p^\rho - x^\rho p^\mu) + i\hbar g^{\mu\rho} (-x^\sigma p^\nu + x^\nu p^\sigma)$$

$$= i\hbar\left[g^{\mu\rho} L^{\nu\sigma} - (\mu \leftrightarrow \nu)\right] - (\rho \leftrightarrow \sigma)$$

$\therefore L$ satisfies the same commutation as M, as expected — expected because L is just a representation of M on scalars.

f) L and M satisfy the same self-commutation. $\exists$ no central-change term.

2.9) Let us write

$$\Lambda^{\rho}{}_{\tau} = \delta^{\rho}{}_{\tau} + \frac{i}{2\hbar}\, \omega_{\mu\nu} (S_V^{\mu\nu})^{\rho}{}_{\tau}$$

where

$$(S_V^{\mu\nu})^{\rho}{}_{\tau} := \frac{\hbar}{i}\left(g^{\mu\rho}\delta^{\nu}{}_{\tau} - g^{\nu\rho}\delta^{\mu}{}_{\tau}\right)$$

are matrices which constitute the vector representation of the Lorentz generators.

a) Let $\Lambda = 1 + \omega$ in

$$\mathcal{U}(\Lambda)^{-1} \partial^{\mu} \varphi(x)\, \mathcal{U}(\Lambda) = \Lambda^{\mu}{}_{\rho}\, \bar{\partial}^{\rho}\, \varphi(\Lambda^{-1}x)$$

and show that

$$[\partial^{\rho}\varphi(x), M^{\mu\nu}] = L^{\mu\nu}\partial^{\rho}\varphi(x) + (S_V^{\mu\nu})^{\rho}{}_{\tau}\, \partial^{\tau}\varphi(x)$$

b) Show that the matrices $S_V^{\mu\nu}$ must have the same commutation relations as the operators $M^{\mu\nu}$. Hint: See the previous problem.

c) For a rotation by an angle θ about the z axis, we have

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) & 0 \\ 0 & \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Show that

$$\Lambda = e^{-i\theta S_V^{12}/\hbar}$$

d) For a boost by rapidity in the z direction, we have

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \cosh(\eta) & 0 & 0 & \sinh(\eta) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh(\eta) & 0 & 0 & \cosh(\eta) \end{pmatrix}$$

Show that

$$\Lambda = e^{i\eta S_V^{30}/\hbar}$$

Solution

a) $\mathcal{U}(\Lambda)^{-1} \partial^\mu \varphi(x) \mathcal{U}(\Lambda) = \Lambda^\mu{}_\rho \, \bar{\partial}^\rho \varphi(\Lambda^{-1} x)$

$$\left(1 - \frac{i}{2\hbar} \omega_{\rho\sigma} M^{\rho\sigma}\right) \partial^\mu \varphi \left(1 + \frac{i}{2\hbar} \omega_{\rho\sigma} M^{\rho\sigma}\right)$$

$$= \Lambda^\mu{}_\rho (\Lambda^{-1})^\rho{}_\sigma \; \partial^\sigma \underbrace{\varphi(x - \omega x)}_{\varphi - (\omega x)\cdot\partial\varphi}$$

$$\partial^\mu \varphi + \frac{i}{2\hbar} \omega_{\rho\sigma} \left[\partial^\mu \varphi, M^{\rho\sigma}\right] + O(\omega^2)$$

$$= \partial^\mu (\varphi - \omega^\rho{}_\sigma x^\sigma \partial_\rho \varphi)$$

$$\frac{i}{2\hbar} \omega_{\rho\sigma} \underbrace{\left[\partial^\mu \varphi, M^{\rho\sigma}\right]}_{\text{already antisym. in } \rho, \sigma} = -\omega_{\rho\sigma} \partial^\mu (x^\sigma \partial^\rho \varphi)$$

$$= -\omega_{\rho\sigma} \underbrace{(g^{\mu\sigma} \partial^\rho \varphi + x^\sigma \partial^\mu \partial^\rho \varphi)}_{\text{antisymmetrize wrt } \rho, \sigma}$$

$$= -\omega_{\rho\sigma}\frac{1}{2}\Big(g^{\mu\sigma}\partial^\rho\varphi - g^{\mu\rho}\partial^\sigma\varphi + x^\sigma\partial^\mu\partial^\rho\varphi - x^\rho\partial^\mu\partial^\sigma\varphi\Big)$$

$$= -\omega_{\rho\sigma}\frac{1}{2}\Big(\overset{\rho\nu}{g^{\mu\sigma}}\,\overset{\mu\tau}{\delta^\rho{}_\tau} - \overset{\mu\rho}{g^{\mu\rho}}\,\overset{\nu\tau}{\delta^\sigma{}_\tau} + x^\sigma\partial^\rho\delta^\mu{}_\tau - x^\rho\partial^\sigma\delta^\mu{}_\tau\Big)\partial^\tau\varphi$$

$$= -\omega_{\rho\sigma}\frac{i}{2\hbar}\Big[-(S_V^{\rho\sigma})^\mu{}_\tau - L^{\rho\sigma}\delta^\mu{}_\tau\Big]\partial^\tau\varphi$$

$$= \omega_{\rho\sigma}\frac{i}{2\hbar}\Big[L^{\rho\sigma}\delta^\mu{}_\tau + (S_V^{\rho\sigma})^\mu{}_\tau\Big]\partial^\tau\varphi$$

$$[\partial^\mu\varphi, M^{\rho\sigma}] = L^{\rho\sigma}\partial^\mu\varphi + (S_V^{\rho\sigma})^\mu{}_\tau\,\partial^\tau\varphi$$

b) By virtue of the representation argument, we expect for S_V to be a representation of M when acting on vectors $\therefore$ S should satisfy the same self-anticommutation, just like L does.

c) Mathematica code for the matrices S_V^{12} and S_V^{30} : See code_3_

$$S_V^{12} = i\hbar \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = i\hbar\sigma^{12}$$

$$e^{\lambda S_V^{12}} = \sum_{k \geq 0} \frac{\lambda^k}{k!} (S_V^{12})^k$$

$$= \sum_{k \geq 0} \frac{(i\hbar\lambda)^k}{k!} (\sigma^{12})^k, \quad \lambda = -\frac{i\theta}{\hbar}$$

$$(\sigma^{12})^2 = \mathrm{diag}(0, -1, -1, 0) =: -\hat{1}$$

$$(\sigma^{12})^3 = -\sigma^{12}$$

$$(\sigma^{12})^4 = \hat{1}$$

$$(\sigma^{12})^5 = \sigma^{12}$$

$$(\sigma^{12})^6 = -\hat{1}$$

$$(\sigma^{12})^7 = -\sigma^{12}$$

...

$$(\sigma^{12})^k = \begin{cases} i^k \hat{1}, & k \text{ even}, \ k > 0 \\ -i^{k+1} \sigma^{12}, & k \text{ odd} \end{cases}$$

$$e^{\lambda S_V^{12}} = 1 + \sum_{k\geq 1} \frac{1}{k!} \underbrace{\left[i\hbar\left(-\frac{i\theta}{\hbar}\right)\right]^k}_{\theta^k} (\sigma^{12})^k$$

$$= 1 + \sum_{k\geq 1} \frac{\theta^k}{k!} (\sigma^{12})^k$$

$$= 1 + \sum_{k\geq 1} \frac{\theta^{2k}}{(2k)!} \underbrace{(\sigma^{12})^{2k}}_{i^{2k}\hat{1}} + \sum_{k\geq 1} \frac{\theta^{2k-1}}{(2k-1)!} \underbrace{(\sigma^{12})^{2k-1}}_{-i^{2k}\sigma^{12}}$$

Do the sums using Mathematica:

$$e^{-iS_V^{12}\theta/\hbar} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) & 0 \\ 0 & \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

d)

$$S_V^{30} = -i\hbar \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} =: -i\hbar\sigma^{30}$$

$$(\sigma^{30})^2 = \mathrm{diag}(1,0,0,1) =: \hat{1}$$

$$(\sigma^{30})^3 = \sigma^{30}$$

$$(\sigma^{30})^4 = \hat{1}$$

...

$$(\sigma^{30})^k = \begin{cases} \sigma^{30}, & k \text{ odd} \\ \hat{1}, & k \text{ even}, \quad k > 0 \end{cases}$$

$$e^{\lambda S_V^{30}} = 1 + \sum_{k \geq 1} \frac{(-i\hbar\lambda)^k}{k!} (\sigma^{30})^k, \quad \lambda = \frac{i\eta}{\hbar}$$

$$= 1 + \sum_{k \geq 1} \frac{\eta^{2k}}{(2k)!} \hat{1} + \sum_{k \geq 1} \frac{\eta^{2k-1}}{(2k-1)!} \sigma^{30}$$

Do the sum using mathematica:

$$e^{i\eta S_V^{30}/\hbar} = \begin{pmatrix} \cosh(\eta) & 0 & 0 & \sinh(\eta) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh(\eta) & 0 & 0 & \cosh(\eta) \end{pmatrix}$$