Ch 2: Lorentz invariance

Lorentz transformation:

$$\bar{x}^\mu = \Lambda^\mu{}_\nu \, x^\nu$$

Λ includes ordinary spatial rotations:

$$\Lambda^0{}_0 = 1, \quad \Lambda^0{}_i = \Lambda^i{}_0 = 0, \quad \Lambda^i{}_j = R_{ij} : R^T R = 1$$

Set of all Lorentz transformations form a group.

$$\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \, g_{\mu\nu} = g_{\rho\sigma}$$

$$\Lambda_{\nu\rho} \Lambda^\nu{}_\sigma = g_{\rho\sigma}$$

$$\Lambda_\nu{}^\rho \Lambda^\nu{}_\sigma = g^\rho{}_\sigma = \delta^\rho{}_\sigma$$

$$\therefore \Lambda_\nu{}^\rho = (\Lambda^{-1})^\rho{}_\nu$$

∞simal Lorentz transformations:

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$$

$$\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \, g_{\mu\nu} = g_{\rho\sigma}$$

$$(\delta^\mu{}_\rho + \omega^\mu{}_\rho)(\delta^\nu{}_\sigma + \omega^\nu{}_\sigma)\, g_{\mu\nu} = g_{\rho\sigma}$$

$$(\delta^\mu{}_\rho \delta^\nu{}_\sigma + \delta^\mu{}_\rho \omega^\nu{}_\sigma + \omega^\mu{}_\rho \delta^\nu{}_\sigma + \omega^\mu{}_\rho \omega^\nu{}_\sigma)\, g_{\mu\nu}$$

$$= g_{\rho\sigma}$$

$$g_{\rho\sigma} + \omega_{\rho\sigma} + \omega_{\sigma\rho} + O(\omega^2) = g_{\rho\sigma}$$

$$\therefore \omega_{\rho\sigma} = -\omega_{\sigma\rho}$$

$$\omega \doteq \begin{pmatrix} 0 & \cdot & \cdot & \cdot \\ & 0 & \cdot & \cdot \\ & & 0 & \cdot \\ & & & 0 \end{pmatrix}$$: $\exists$ only 6 independent ∞simal Lorentz transformation

3 rotations and 3 boosts:

$$\omega_{ij} = -\varepsilon_{ijk}\, \hat{n}_k\, \delta\theta$$

$$\omega_{i0} = \hat{n}_i\, \delta\eta$$

$\delta\theta$: ∞simal rotation

$\delta\eta$: ∞simal rapidity

Only proper Lorentz transformations can be reached by compounding ∞simal ones:

$$(\Lambda^{-1})^{\rho}{}_{\nu} = \Lambda_{\nu}{}^{\rho} \Rightarrow \det(\Lambda)^{-1} = \det(\Lambda)$$

$$\Rightarrow \det(\Lambda)^2 = 1$$

$$\Rightarrow \det(\Lambda) = \pm 1$$

$\det(\Lambda) = 1$: proper

$\det(\Lambda) = -1$: improper

Proper Lorentz transformations form a subgroup of the Lorentz group.

Another subgroup: orthochronous, i.e. $\Lambda^0{}_0 \geq +1$.

$$g_{\mu\nu} \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} = g_{\rho\sigma}$$

$$g_{\mu\nu} \Lambda^{\mu}{}_{0} \Lambda^{\nu}{}_{0} = -1$$

$$g_{00} \Lambda^{0}{}_{0} \Lambda^{0}{}_{0} + g_{ij} \Lambda^{i}{}_{0} \Lambda^{j}{}_{0} = -1$$

$$-(\Lambda^0{}_0)^2 = 1 + \Lambda^i{}_0 \Lambda^i{}_0$$

$$(\Lambda^0{}_0)^2 = 1 + \Lambda^i{}_0 \Lambda^i{}_0$$

$$\Lambda^0{}_0 = \pm\sqrt{1 + \Lambda^i{}_0 \Lambda^i{}_0}$$

$$\therefore \Lambda^0{}_0 \geq 1 \quad \vee \quad \Lambda^0{}_0 \leq -1$$

Since $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$ for ∞simal transformation and ω is antisymmetric, it is orthochronous. Product of two orthochronous transformations is also orthochronous $\therefore$ Lorentz transformations that can be reached by compounding ∞simal ones are both proper and orthochronous and they form a subgroup.

Two discrete transformations that take us out of this subgroup: parity and time-reversal.

$$P^\mu{}_\nu = (P^\mu{}_\nu)^{-1} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} : \text{orthochronous improper}$$

$$T^\mu{}_\nu = (T^{-1})^\mu{}_\nu = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$: non-orthochronous improper

When we say Lorentz invariance, we mean under the proper orthochronous subgroup.

In quantum theory, symmetries are represented by unitary (or antiunitary) operators ∴ we associate a unitary operator $\mathcal{U}(\Lambda)$ to each proper orthochronous Λ.

$$\mathcal{U}(\Lambda'\Lambda) = \mathcal{U}(\Lambda')\mathcal{U}(\Lambda)$$

Recall the general structure from quantum mechanics:

$$\mathcal{U} = 1 + iG\varepsilon$$

where $G^\dagger = G$ is the generator of the symmetry.

Here:

$$\mathcal{U} = 1 + \frac{i}{2\hbar}\,\underbrace{\omega_{\mu\nu}}_{\text{antisymmetric}}\,\overbrace{\underbrace{M^{\mu\nu}}_{\text{antisymmetric}}}^{\text{generator of Lorentz group}}$$

$$\mathcal{U}(\Lambda)^{-1}\,\mathcal{U}(\Lambda')\,\mathcal{U}(\Lambda) = \mathcal{U}(\Lambda^{-1}\Lambda'\Lambda)$$

$$\Lambda' = 1 + \omega$$

$$\mathcal{U}(\Lambda)^{-1}\left(1 + \frac{i}{2\hbar}\,\omega_{\mu\nu} M^{\mu\nu}\right)\mathcal{U}(\Lambda) = \mathcal{U}(\Lambda^{-1}\Lambda'\Lambda)$$

$$\begin{aligned}
\mathcal{U}(\Lambda^{-1}\Lambda'\Lambda) &= \mathcal{U}(\Lambda^{-1}(1+\omega)\Lambda) \\
&= \mathcal{U}(1 + \Lambda^{-1}\omega\Lambda) \\
&= 1 + \frac{i}{2\hbar}(\Lambda^{-1}\omega\ \Lambda)_{\mu\nu} M^{\mu\nu} \\
&= 1 + \frac{i}{2\hbar}(\Lambda^{-1})_{\mu}{}^{\rho}\,\omega_{\rho\sigma}\Lambda^{\sigma}{}_{\nu} M^{\mu\nu} \\
&= 1 + \frac{i}{2\hbar}\Lambda^{\rho}{}_{\mu}\Lambda^{\sigma}{}_{\nu}\,\omega_{\rho\sigma} M^{\mu\nu}
\end{aligned}$$

$$1 + \frac{i}{2\hbar}\,\omega_{\mu\nu}\,\mathcal{U}(\Lambda)^{-1} M^{\mu\nu}\,\mathcal{U}(\Lambda) = 1 + \frac{i}{2\hbar}\Lambda^{\rho}{}_{\mu}\Lambda^{\sigma}{}_{\nu}\,\omega_{\rho\sigma} M^{\mu\nu}$$

$$\omega_{\mu\nu}\underbrace{\mathcal{U}(\Lambda)^{-1} M^{\mu\nu}\,\mathcal{U}(\Lambda)}_{\text{already antisym. in } \mu,\nu} = \Lambda^{\rho}{}_{\mu}\Lambda^{\sigma}{}_{\nu}\,\omega_{\rho\sigma} M^{\mu\nu}$$

$$= \omega_{\mu\nu} \underbrace{\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma M^{\rho\sigma}}_{\frac{1}{2}\left(\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma M^{\rho\sigma} \underbrace{- \Lambda^\nu{}_\rho \Lambda^\mu{}_\sigma \underbrace{M^{\rho\sigma}}_{-M^{\sigma\rho}}}_{+\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma M^{\rho\sigma}}\right)}$$

$$= \omega_{\mu\nu} \underbrace{\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma M^{\rho\sigma}}_{\text{already antisymmetric}}$$

$$\therefore \mathcal{U}(\Lambda)^{-1} M^{\mu\nu} \mathcal{U}(\Lambda) = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma M^{\rho\sigma}$$

$\therefore$ each index gets its own Lorentz transformation.

$$\mathcal{U}(\Lambda)^{-1} p^\mu \mathcal{U}(\Lambda) = \Lambda^\mu{}_\nu p^\nu$$

$$\mathcal{U}(\Lambda)^{-1} T^{\mu\nu\rho\ldots} \mathcal{U}(\Lambda) = \Lambda^\mu{}_{\mu'} \Lambda^\nu{}_{\nu'} \Lambda^\rho{}_{\rho'} \ldots T^{\mu'\nu'\rho'\ldots}$$

$$\mathcal{U}(\Lambda)^{-1} M^{\mu\nu} \mathcal{U}(\Lambda) = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma M^{\rho\sigma}$$

$$\left(1 - \frac{i}{2\hbar} \omega_{\rho\sigma} M^{\rho\sigma}\right) M^{\mu\nu} \left(1 + \frac{i}{2\hbar} \omega_{\rho\sigma} M^{\rho\sigma}\right)$$

$$= (\delta^\mu{}_\rho + \omega^\mu{}_\rho)(\delta^\nu{}_\sigma + \omega^\nu{}_\sigma) M^{\rho\sigma}$$

$$\left(M^{\mu\nu} - \frac{i}{2\hbar}\omega_{\rho\sigma} M^{\rho\sigma} M^{\mu\nu}\right)\left(1 + \frac{i}{2\hbar}\omega_{\rho\sigma} M^{\rho\sigma}\right)$$

$$= (\delta^\mu{}_\rho + \omega^\mu{}_\rho)(M^{\rho\nu} + \omega^\nu{}_\sigma M^{\rho\sigma})$$

$$M^{\mu\nu} + \frac{i}{2\hbar}\omega_{\rho\sigma}[M^{\mu\nu}, M^{\rho\sigma}] + O(\omega^2)$$

$$= M^{\mu\nu} + \omega^\nu{}_\sigma M^{\mu\sigma} + \omega^\mu{}_\rho M^{\rho\nu} + O(\omega^2)$$

$$\frac{i}{2\hbar}\omega_{\rho\sigma}\underbrace{[M^{\mu\nu}, M^{\rho\sigma}]}_{\substack{\text{already antisym.}\\ \text{in } \rho, \sigma}} = \omega^\nu{}_\sigma M^{\mu\sigma} + \omega^\mu{}_\rho M^{\rho\nu}$$

$$= \omega_{\rho\sigma} g^{\rho\nu} M^{\mu\sigma} + \underbrace{\omega^\mu{}_\sigma}_{\omega_{\rho\sigma} g^{\rho\mu}} M^{\sigma\nu}$$

$$= \omega_{\rho\sigma}\underbrace{(g^{\rho\nu} M^{\mu\sigma} + g^{\rho\mu} M^{\sigma\nu})}_{\text{antisym. in } \rho, \sigma}$$

$$= \omega_{\rho\sigma} \frac{1}{2} \left(g^{\rho\nu} M^{\mu\sigma} + g^{\rho\mu} M^{\sigma\nu} - g^{\sigma\nu} M^{\mu\rho} - g^{\sigma\mu} M^{\rho\nu} \right)$$

$$[M^{\mu\nu}, M^{\rho\sigma}] = -i\hbar \left(g^{\rho\nu} M^{\mu\sigma} + g^{\rho\mu} M^{\sigma\nu} - g^{\sigma\nu} M^{\mu\rho} - g^{\sigma\mu} M^{\rho\nu} \right)$$

Shorthand notation:

$$[M^{\mu\nu}, M^{\rho\sigma}] = -i\hbar \left(g^{\nu\rho} M^{\mu\sigma} - g^{\mu\rho} M^{\nu\sigma} - g^{\nu\sigma} M^{\mu\rho} + g^{\mu\sigma} M^{\nu\rho} \right)$$

$$= i\hbar \left(g^{\mu\rho} M^{\nu\sigma} - g^{\nu\rho} M^{\mu\sigma} - g^{\mu\sigma} M^{\nu\rho} - g^{\nu\sigma} M^{\mu\rho} \right)$$

$$= i\hbar \left[g^{\mu\rho} M^{\nu\sigma} - (\mu \leftrightarrow \nu) \right] - (\rho \leftrightarrow \sigma)$$

Useful identity:

$$\varepsilon_{ijk} \varepsilon_{lmn} = \delta_{il}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{jn}\delta_{kl} + \delta_{in}\delta_{jl}\delta_{km} - \delta_{im}\delta_{jl}\delta_{kn} - \delta_{il}\delta_{jn}\delta_{km} - \delta_{in}\delta_{jm}\delta_{kl}$$

Define:

$$J_i = \frac{1}{2}\varepsilon_{ijk} M^{jk} \longleftrightarrow M^{jk} = \varepsilon_{ijk} J_i$$: angular momentum

$$K_i = M^{i0}$$: boost

See code_1:

$$[J_i, J_j] = i\hbar \varepsilon_{ijk} J_k$$: usual angular momentum commutator

$$[J_i, K_j] = i\hbar \varepsilon_{ijk} K_k$$: says boost is a vector.

$$[K_i, K_j] = -i\hbar \varepsilon_{ijk} J_k$$: two successive boosts give rotation.

$$\mathcal{U}(\Lambda)^{-1} p^\mu \mathcal{U}(\Lambda) = \Lambda^\mu{}_\nu p^\nu$$

$$\left(1 - \frac{i}{2\hbar}\omega_{\rho\sigma} M^{\rho\sigma}\right) p^\mu \left(1 + \frac{i}{2\hbar}\omega_{\rho\sigma} M^{\rho\sigma}\right)$$

$$= (\delta^\mu{}_\nu + \omega^\mu{}_\nu) p^\nu$$

$$\left(p^\mu - \frac{i}{2\hbar}\omega_{\rho\sigma} M^{\rho\sigma} p^\mu\right)\left(1 + \frac{i}{2\hbar}\omega_{\rho\sigma} M^{\rho\sigma}\right) = p^\mu + \omega^\mu{}_\nu p^\nu$$

$$p^\mu + \frac{i}{2\hbar}\omega_{\rho\sigma}[p^\mu, M^{\rho\sigma}] + O(\omega^2) = p^\mu + \omega^\mu{}_\nu p^\nu$$

$$\frac{i}{2\hbar}\omega_{\rho\sigma}\underbrace{[p^\mu, M^{\rho\sigma}]}_{\text{already antisym. in } \rho,\sigma} = \omega_{\rho\sigma} g^{\mu\rho}\delta^\sigma{}_\nu p^\nu$$

$$= \omega_{\rho\sigma}\underbrace{g^{\mu\rho}p^\sigma}_{\text{antisymmetrize wrt. } \rho,\sigma}$$

$$= \omega_{\rho\sigma}\frac{1}{2}\left(g^{\mu\rho}p^\sigma - g^{\mu\sigma}p^\rho\right)$$

$$[p^\mu, M^{\rho\sigma}] = -i\hbar\left(g^{\mu\rho}p^\sigma - g^{\mu\sigma}p^\rho\right)$$

Shorthand notation:

$$[p^\mu, M^{\rho\sigma}] = i\hbar\left[g^{\mu\sigma}p^\rho - (\rho\leftrightarrow\sigma)\right]$$

This gives (see code-2)

$$[J_i, H] = 0$$

$$[J_i, p_j] = i\hbar\varepsilon_{ijk}p_k$$

$$[K_i, H] = i\hbar p_i$$

$$[K_i, p_j] = i\hbar\delta_{ij}H$$

Obviously,

$$[p_i, p_j] = 0$$

$$[p_i, H] = 0$$

All these commutators of $\vec{J}$, $\vec{K}$, $\vec{p}$, and H form the Lie algebra of the Poincaré group.

What happens to a quantum scalar field, $\varphi(x)$, under Lorentz transformation?

$$e^{iHt/\hbar}\,\varphi(\vec{x},0)\,e^{-iHt/\hbar} = \varphi(\vec{x},t)$$

Generalize:

$$e^{-ipx/\hbar}\,\varphi(0)\,e^{ipx/\hbar} = \varphi(x)$$

$T(a) := e^{ipa/\hbar}$: spacetime translation operator

$$T(x)\varphi(0)T(x)^{-1} = \varphi(x) = \varphi(0+x)$$

$$T(a)\varphi(x)\;T(a)^{-1} = \varphi(x+a) \;\Big|\; a \to -a$$

$$T(-a)\varphi(x)T(-a)^{-1} = \varphi(x-a)$$

$$T(a)^{-1}\varphi(x)T(a) = \varphi(x-a)$$

∞simal translation:

$$T(a) = 1 - \frac{i}{\hbar} p a$$

Expectation:

$$\mathcal{U}(\Lambda)^{-1}\varphi(x)\mathcal{U}(\Lambda) = \varphi(\underbrace{\Lambda^{-1}x}_{\bar{x}})$$

Derivatives:

$$\mathcal{U}(\Lambda)^{-1}\partial^{\mu}\varphi(x)\mathcal{U}(\Lambda) = \Lambda^{\mu}{}_{\rho}\bar{\partial}^{\rho}\varphi(\bar{x})$$

$\bar{\partial}$: derivative w.r.t. $\bar{x}$

$$\mathcal{U}(\Lambda)^{-1}\partial^{2}\varphi(x)\mathcal{U}(\Lambda) = \bar{\partial}^{2}\varphi(\bar{x})$$

so that the KG equation, $\left(-\partial^2 + \frac{m^2c^2}{\hbar^2}\right)\varphi(x) = 0$, is Lorentz invariant.