Ch 3: Problems

3.1) Derive

$$[a(\vec{k}), a(\vec{k}')] = 0$$

$$[a^\dagger(\vec{k}), a^\dagger(\vec{k}')] = 0$$

$$[a(\vec{k}), a^\dagger(\vec{k}')] = (2\pi)^3 2\omega \, \delta^3(\vec{k}-\vec{k}')$$

from

$$a(\vec{k}) = i\int d^3x \, e^{-ikx} \overleftrightarrow{\partial_0} \varphi(x)$$

$$\pi(x) = \dot{\varphi}(x)$$

$$[\varphi(\vec{x},t), \varphi(\vec{x}',t)] = 0$$

$$[\pi(\vec{x},t), \pi(\vec{x}',t)] = 0$$

$$[\varphi(\vec{x},t), \pi(\vec{x}',t)] = i\delta^3(\vec{x}-\vec{x}')$$

Solution see code_2.

3.2) Use the commutation relations of a and $a^\dagger$ to show explicitly that a state of the form

$$|k_1 \dots k_n\rangle = a^\dagger(\vec{k}_1) \dots a^\dagger(\vec{k}_n)\,|0\rangle$$

is an eigenstate of the hamiltonian

$$H = \int \widetilde{dk}\; \omega\, a^\dagger(\vec{k})\, a(\vec{k})$$

w/ eigenvalue $\omega_1 + \dots + \omega_n$. The vacuum is annihilated by $a(\vec{k})$, $a(\vec{k})|0\rangle = 0$, and we take $\Omega_0 = \mathcal{E}_0$.

Solution See code-3.

3.3) use $\mathcal{U}(\Lambda)^{-1} \varphi(x) \mathcal{U}(\Lambda) = \varphi(\Lambda^{-1}x)$ to show that

$$\mathcal{U}(\Lambda)^{-1} a(\vec{k})\, \mathcal{U}(\Lambda) = a(\Lambda^{-1}\vec{k})$$

$$\mathcal{U}(\Lambda)^{-1} a^\dagger(\vec{k})\, \mathcal{U}(\Lambda) = a^\dagger(\Lambda^{-1}\vec{k})$$

and hence that

$$\mathcal{U}(\Lambda)\, |k_1 \dots k_n\rangle = |\Lambda k_1 \dots \Lambda k_n\rangle$$

where $|k_1 \dots k_n\rangle = a^\dagger(\vec{k}_1) \dots a^\dagger(\vec{k}_n)|0\rangle$ is a state of n particles w/ momenta $k_1, \dots, k_n$.

<u>Solution</u>

$$\varphi(x) = \int \widetilde{dk}\ [a(\vec{k})e^{ikx} + a^{+}(\vec{k})e^{-ikx}]$$

$$\mathcal{U}(\Lambda)^{-1}\varphi(x)\mathcal{U}(\Lambda) = \int \widetilde{dk}\Big[\ \mathcal{U}(\Lambda)^{-1}a(\vec{k})\mathcal{U}(\Lambda)e^{ikx} + \mathcal{U}(\Lambda)^{-1}a^{+}(\vec{k})\mathcal{U}(\Lambda)e^{-ikx}\Big]$$

$$= \varphi(\Lambda^{-1}x)$$

$$= \int \widetilde{dk}\Big[a(\vec{k})\underbrace{e^{ik\Lambda^{-1}x}}_{e^{i\Lambda kx}} + a^{+}(\vec{k})\underbrace{e^{-ik\Lambda^{-1}x}}_{e^{-i\Lambda kx}}\Big]\Big|\ k\to\Lambda^{-1}k$$

$$= \int \widetilde{dk}\Big[a(\Lambda^{-1}\vec{k})e^{ikx} + a^{+}(\Lambda^{-1}\vec{k})e^{-ikx}\Big]$$

$$\therefore\ \mathcal{U}(\Lambda)^{-1}a(\vec{k})\mathcal{U}(\Lambda) = a(\Lambda^{-1}\vec{k})$$

$$\mathcal{U}(\Lambda)^{-1}a^{+}(\vec{k})\mathcal{U}(\Lambda) = a^{+}(\Lambda^{-1}\vec{k})$$

$$\mathcal{U}(\Lambda)|k_1 \dots k_n\rangle = \mathcal{U}(\Lambda)a^{+}(\vec{k}_1)\dots a^{+}(\vec{k}_n)|0\rangle$$

$$= \mathcal{U}(\Lambda)a^{+}(\vec{k}_1)\mathcal{U}(\Lambda)^{-1}\mathcal{U}(\Lambda)\dots \qquad \mathcal{U}(\Lambda)^{-1}\mathcal{U}(\Lambda)$$

$$\times\, a^{+}(\vec{k}_n)\mathcal{U}(\Lambda)^{-1}\mathcal{U}(\Lambda)|0\rangle$$

claim : $\mathcal{U}(\Lambda)|0\rangle = |0\rangle$.

Proof: The ground state is the state of null* momentum, which transform trivially. Moreover, from elementary quantum mechanics, we know that the ground state should be unique. If $p^\mu = 0$ transforms to $\bar{p}^\mu = 0$ in all frames, and if the ground state is unique, then it means that the ground state is invariant under Lorentz transformation. qed.

$$\therefore \mathcal{U}(\Lambda)|k_1 \dots k_n\rangle = a^\dagger(\Lambda\vec{k}_1)\dots a^\dagger(\Lambda\vec{k}_n)|0\rangle$$

$$= |\Lambda k_1 \dots \Lambda k_n\rangle$$

3.4) Recall that $T(a)^{-1}\varphi(x)T(a) = \varphi(x-a)$ where $T(a) := e^{-ipa}$ is the spacetime translation operator and p^0 is identified as the hamiltonian, H.

a) Let a^μ be ∞simal. Derive an expression for $[\varphi(x), p^\mu]$.

b) Show that the time component of your result is equivalent to the Heisenberg equation of motion, $i\dot{\varphi} = [\varphi, H]$.

c) For a field, use the Heisenberg EOM to derive the Klein-Gordon equation.

d) Define a spatial momentum operator

$$\vec{p} := -\int d^3x \; \pi(x) \vec{\nabla} \varphi(x)$$

Use the canonical commutation relations to show that $\vec{p}$ obeys the relation you derived in part (a).

e) Express $\vec{p}$ in terms of $a(\vec{k})$ and $a^\dagger(\vec{k})$.

Solution

a) $T(a)^{-1}\varphi(x)T(a) = \varphi(x-a)$

$$(1+ip^\mu a_\mu)\varphi(x)(1-ip^\mu a_\mu)$$

$$= \varphi(x) - a_\mu \partial^\mu \varphi(x)$$

$$\cancel{\varphi(x)} - ia_\mu \varphi(x)p^\mu + ia_\mu p^\mu \varphi(x) + \mathcal{O}(a^2)$$

$$= \cancel{\varphi(x)} - a_\mu \partial^\mu \varphi(x)$$

$$\cancel{-}\, i\cancel{a_\mu}[\varphi(x), p^\mu] = \cancel{-}\,\cancel{a_\mu}\partial^\mu \varphi(x)$$

$$[\varphi(x), p^\mu] = -i\partial^\mu \varphi(x)$$

b) $[\varphi(x), p^0] = -i\underbrace{\partial^0 \varphi(x)}_{-\dot{\varphi}}$

$\therefore [\varphi(x), H] = i\dot{\varphi}(x)$

c) $\dot{\varphi} = \pi$

$i\dot{\varphi} = i\pi = [\varphi, H]$

$i\dot{\pi} = i\ddot{\varphi} = [\pi, H] = i\nabla^2\varphi - im^2\varphi$ (see code_4)

$$\underbrace{\ddot{\varphi} - \nabla^2\varphi}_{-\partial^2\varphi} + m^2\varphi = 0$$

$$(-\partial^2 + m^2)\varphi = 0$$

d) $[\varphi(x), \vec{P}] = -i\vec{\nabla}\varphi(x)$ (see code_5)

From part (a),

$$[\varphi(x), P^\mu] = -i\partial^\mu\varphi(x)$$

$$\therefore [\varphi(x), P^i] = -i\partial^i\varphi(x) = \left(-i\vec{\nabla}\varphi(x)\right)_i$$

e) $$\vec{P} = \underbrace{V\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\vec{k}}_{\text{total zero-point momentum of all oscillators}} + \int \widetilde{dk}\,\vec{k}\,a^\dagger(\vec{k})\,a(\vec{k})$$

cf. $H = V\frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\,\omega + \int \widetilde{dk}\;\omega\, a^\dagger(\vec{k})\, a(\vec{k})$ (p. 13)

3.5) consider a complex (that is, nonhermitian) scalar field φ w/ lagrangian density

$$\mathcal{L} = -\partial^\mu\varphi^\dagger\partial_\mu\varphi - m^2\varphi^\dagger\varphi + \Omega_0$$

a) Show that φ obeys the KG equation.

b) Treat φ and $\varphi^\dagger$ as independent fields and find the conjugate momentum for each. Compute the hamiltonian density in terms of these conjugate momenta and the fields themselves (but not their time derivatives).

c) Write the mod expansion of φ as

$$\varphi(x) = \int \widetilde{dk}\,[a(\vec{k})e^{ikx} + b^\dagger(\vec{k})e^{-ikx}]$$

Express $a(\vec{k})$ and $b(\vec{k})$ in terms of φ and $\varphi^\dagger$ and their time derivatives.

d) Assuming canonical commutation relations for the fields and their conjugate momenta, find the commutation relations obeyed by $a(\vec{k})$ and $b(\vec{k})$ and their hermitian conjugates.

e) Express the hamiltonian in terms of $a(\vec{k})$ and $b(\vec{k})$ and their hermitian conjugates. What value must Ω_0 have in order for the ground state to have zero energy?

<u>Solution</u>

a) Euler–Lagrange EOM for $\varphi^\dagger$:

$$\frac{\partial \mathcal{L}}{\partial \varphi^\dagger} - \partial^\mu \frac{\partial \mathcal{L}}{\partial(\partial^\mu \varphi^\dagger)} = 0$$

$$-m^2\varphi - \partial^\mu(-\partial_\mu \varphi) = 0$$

$$(-\partial^2 + m^2)\varphi = 0$$

b) $\mathcal{L} = -\partial^\mu \varphi^\dagger \partial_\mu \varphi - m^2 \varphi^\dagger \varphi + \Omega_0$

$= -\partial^0 \varphi^\dagger \partial_0 \varphi - \partial^i \varphi^\dagger \partial_i \varphi - m^2 \varphi^\dagger \varphi + \Omega_0$

$= \dot{\varphi}^\dagger \dot{\varphi} - \vec{\nabla}\varphi^\dagger \cdot \vec{\nabla}\varphi - m^2 \varphi^\dagger \varphi + \Omega_0$

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \dot{\varphi}^\dagger$$

$$\pi^\dagger = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}^\dagger} = \dot{\varphi}$$

$$\mathcal{H} = \pi \dot{\varphi} + \pi^\dagger \dot{\varphi}^\dagger - \mathcal{L}$$

$$= \cancel{\pi \pi^\dagger} + \pi^\dagger \pi - \cancel{\pi \pi^\dagger} + \vec{\nabla}\varphi^\dagger \cdot \vec{\nabla}\varphi$$
$$+ m^2 \varphi^\dagger \varphi - \Omega_0$$

$$= \pi^\dagger \pi + \vec{\nabla}\varphi^\dagger \cdot \vec{\nabla}\varphi + m^2 \varphi^\dagger \varphi - \Omega_0$$

c) $\varphi(x) = \int \widetilde{dk} \, [a(\vec{k}) e^{ikx} + b^\dagger(\vec{k}) e^{-ikx}]$

This is not so much different from what we have done in the chapter. We can directly quote our results for a and φ:

$$a(\vec{k}) = \int d^3x\, e^{-ikx}(\omega\varphi + i\dot{\varphi})$$

The only difference between a and b is $\varphi \to \varphi^\dagger$:

$$b(\vec{k}) = \int d^3x\, e^{-ikx}(\omega\varphi^\dagger + i\dot{\varphi}^\dagger)$$

d) Assumptions:

$$[\varphi(\vec{x},t), \pi(\vec{y},t)] = i\delta^3(\vec{x}-\vec{y})$$

$$[\varphi^\dagger(\vec{x},t), \pi^\dagger(\vec{y},t)] = i\delta^3(\vec{x}-\vec{y})$$

w/ all other commutators being zero.

See code_7:

$$[a(\vec{k}), a^\dagger(\vec{k}')] = (2\pi)^3\, 2\omega\, \delta^3(\vec{k}-\vec{k}')$$

$$[b(\vec{k}), b^\dagger(\vec{k}')] = (2\pi)^3\, 2\omega\, \delta^3(\vec{k}-\vec{k}')$$

and all other commutators are zero.

e) $H = \int d^3x\, \mathcal{H}(x)$

$$= -\Omega_0 V + 2V\int \widetilde{dk}\, \omega^2 + \int \widetilde{dk}\, [a^\dagger(\vec{k})a(\vec{k}) + b^\dagger(\vec{k})b(\vec{k})]$$

(see code_8) $\therefore\ \Omega_0 = 2\int \widetilde{dk}\, \omega^2 = \cancel{2}\int \frac{d^3k}{(2\pi)^3 \cancel{2\omega}}\, \omega^{\cancel{2}} = \int \frac{d^3k}{(2\pi)^3}\, \omega$