Ch 7 : Path integral for harmonic oscillator

$$H(P,Q) = \frac{1}{2m} P^2 + \frac{1}{2} m\omega^2 Q^2$$

$$\langle 0|0\rangle_f = \int \mathcal{D}p\, \mathcal{D}q\; e^{i\int_{-\infty}^{\infty} dt\,(p\dot{q} - (1-i\varepsilon)H + fq)}$$

$$\{H \to (1-i\varepsilon)H\} \equiv \begin{Bmatrix} m \to (1+i\varepsilon)\,m \\ m\omega^2 \to (1-i\varepsilon)\,m\omega^2 \end{Bmatrix}$$

Lagrangian formulation:

$$\langle 0|0\rangle_f = \int \mathcal{D}q\; e^{i\int_{-\infty}^{\infty} dt\left[\frac{1}{2}(1+i\varepsilon)m\dot{q}^2 - \frac{1}{2}(1-i\varepsilon)m\omega^2 q^2 + fq\right]}$$

To simplify notation, set $m=1$.

Fourier-transformed variables:

$$\tilde{q}(E) = \int_{-\infty}^{\infty} dt\; e^{iEt}\, q(t)$$

$$q(t) = \int_{-\infty}^{\infty} \frac{dE}{2\pi}\, e^{-iEt}\, \tilde{q}(E)$$

$$\frac{1}{2}(1+i\varepsilon)\ \dot{q}^2 - \frac{1}{2}(1-i\varepsilon)\ \omega^2 q^2 + fq$$

$$= \frac{1}{2}(1+i\varepsilon) \int_{-\infty}^{\infty} \frac{dE}{2\pi} (-iE) e^{-iEt}\ \tilde{q}(E)$$

$$\times \int_{-\infty}^{\infty} \frac{dE'}{2\pi} (-iE') e^{-iE't}\ \tilde{q}(E')$$

$$- \frac{1}{2}(1-i\varepsilon)\ \omega^2 \int_{-\infty}^{\infty} \frac{dE}{2\pi}\ e^{-iEt}\ \tilde{q}(E)$$

$$\times \int_{-\infty}^{\infty} \frac{dE'}{2\pi}\ e^{-iE't}\ \tilde{q}(E)$$

$$+ \frac{1}{2}\Bigg[\int_{-\infty}^{\infty} \frac{dE}{2\pi}\ e^{-iEt}\ \tilde{f}(E) \int_{-\infty}^{\infty} \frac{dE'}{2\pi}\ e^{-iE't}\ \tilde{q}(E')$$

$$+ \int_{-\infty}^{\infty} \frac{dE}{2\pi}\ e^{-iEt}\ \tilde{q}(E) \int_{-\infty}^{\infty} \frac{dE'}{2\pi}\ e^{-iE't}\ \tilde{f}(E')$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{dE}{2\pi}\frac{dE'}{2\pi}\ e^{-i(E+E')t}\Big(\left[-(1+i\varepsilon)EE' - (1-i\varepsilon)\omega^2\right]\tilde{q}(E)\tilde{q}(E')$$

$$+ \tilde{f}(E)\tilde{q}(E') + \tilde{f}(E')\tilde{q}(E)\Big)$$

The only t dependence is in $e^{-i(E+E')t}$, which gives $2\pi\delta(E+E')$ when integrated:

3

$$S = \int_{-\infty}^{\infty} dt \left[\frac{1}{2}(1+i\varepsilon) q^2 - \frac{1}{2}(1-i\varepsilon)\omega^2 q^2 + fq \right]$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \frac{dE}{2\pi} \Big\{ \left[-(1+i\varepsilon) E(-E) - (1-i\varepsilon)\omega^2 \right] \tilde{q}(E)\tilde{q}(-E)$$
$$+ \tilde{f}(E)\tilde{q}(-E) + \tilde{f}(-E)\tilde{q}(E) \Big\}$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \frac{dE}{2\pi} \Big\{ \left[(1+i\varepsilon) E^2 - (1-i\varepsilon)\omega^2 \right] \tilde{q}(E)\tilde{q}(-E)$$
$$+ \tilde{f}(E)\tilde{q}(-E) + \tilde{f}(-E)\tilde{q}(E) \Big\}$$

$$(1+i\varepsilon)E^2 - (1-i\varepsilon)\omega^2 = E^2 - \omega^2 + i(E^2+\omega^2)\varepsilon$$
$$\to E^2 - \omega^2 + i\varepsilon$$

$$S = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dE}{2\pi} \Big\{ (E^2 - \omega^2 + i\varepsilon)\tilde{q}(E)\tilde{q}(-E)$$
$$+ \tilde{f}(E)\tilde{q}(-E) + \tilde{f}(-E)\tilde{q}(E) \Big\}$$

$$\tilde{q}(E) = \tilde{x}(E) + B_1$$
$$\tilde{q}(-E) = \tilde{x}(-E) + B_2$$
$$\{\cdots\} = \tilde{x}(E)(E^2-\omega^2+i\varepsilon)\tilde{x}(-E) + C_1 \tilde{x}(E) + C_2 \tilde{x}(-E)$$
$$+ D$$

$$c_1 = 0 = c_2 \Rightarrow B_1 = -\frac{\tilde{f}(E)}{E^2-\omega^2+i\varepsilon}, \quad B_2 = -\frac{\tilde{f}(-E)}{E^2-\omega^2+i\varepsilon}$$

using Mathematica.

$$D = -\frac{\tilde{f}(E)\tilde{f}(-E)}{E^2-\omega^2+i\varepsilon}$$

$$\therefore S = \frac{1}{2}\int_{-\infty}^{\infty}\frac{dE}{2\pi}\left[\tilde{x}(E)(E^2-\omega^2+i\varepsilon)\tilde{x}(-E) - \frac{\tilde{f}(E)\tilde{f}(-E)}{E^2-\omega^2+i\varepsilon}\right]$$

$$Dq = Dx$$

$$\therefore \langle 0|0\rangle_f = \int Dq\, e^{iS}$$

$$= e^{\frac{i}{2}\int_{-\infty}^{\infty}\frac{dE}{2\pi}\frac{\tilde{f}(E)\tilde{f}(-E)}{-E^2+\omega^2-i\varepsilon}}$$

$$\times \int Dx\, e^{\frac{i}{2}\int_{-\infty}^{\infty}\frac{dE}{2\pi}\tilde{x}(E)(E^2-\omega^2+i\varepsilon)\tilde{x}(-E)}$$

Key point: The path integral is $\langle 0|0\rangle_{f=0}$. But if $\exists$ no external force, a system in its ground state will remain so, $\therefore \langle 0|0\rangle_{f=0} = 1$.

$$\therefore \langle 0|0\rangle_f = e^{\frac{i}{2}\int_{-\infty}^{\infty}\frac{dE}{2\pi}\frac{\tilde{f}(E)\tilde{f}(-E)}{-E^2+\omega^2-i\varepsilon}}$$

Time-domain variables:

$$\int_{-\infty}^{\infty}\frac{dE}{2\pi}\left[\int_{-\infty}^{\infty}dt\, e^{iEt}f(t)\right]\frac{1}{-E^2+\omega^2-i\varepsilon}\left[\int_{-\infty}^{\infty}dt'\, e^{-iEt'}f(t')\right]$$

$$= \int_{-\infty}^{\infty}dt\, dt'\, f(t)\left[\int_{-\infty}^{\infty}\frac{dE}{2\pi}\frac{e^{iE(t-t')}}{-E^2+\omega^2-i\varepsilon}\right]f(t'),\ t\leftrightarrow t'$$

$$= \int_{-\infty}^{\infty} dt\, dt'\, f(t) \left[\int_{-\infty}^{\infty} \frac{dE}{2\pi} \frac{e^{-iE(t-t')}}{-E^2+\omega^2-i\varepsilon} \right] f(t')$$

$$= \int_{-\infty}^{\infty} dt\, dt'\, f(t)\, G(t-t')\, f(t')$$

$$\langle 0|0\rangle_f = e^{\frac{i}{2}\int_{-\infty}^{\infty} dt\, dt'\, f(t)\, G(t-t')\, f(t')}$$

$$G(t-t') := \int_{-\infty}^{\infty} \frac{dE}{2\pi} \frac{e^{-iE(t-t')}}{-E^2+\omega^2-i\varepsilon}$$: Green function for oscillator

$$\left(\frac{\partial^2}{\partial t^2} + \omega^2\right) G(t-t') = \delta(t-t')$$

$$= \int_{-\infty}^{\infty} \frac{dE}{2\pi} \frac{-E^2 e^{-iE(t-t')}}{-E^2+\omega^2-i\varepsilon} + \omega^2 \int_{-\infty}^{\infty} \frac{dE}{2\pi} \frac{e^{-iE(t-t')}}{-E^2+\omega^2-i\varepsilon}$$

$$= \int_{-\infty}^{\infty} \frac{dE}{2\pi} \frac{(-E^2+\omega^2)\, e^{-iE(t-t')}}{-E^2+\omega^2-i\varepsilon} \Bigg|_{\varepsilon\to 0}$$

$$= \int_{-\infty}^{\infty} \frac{dE}{2\pi} e^{-iE(t-t')}$$

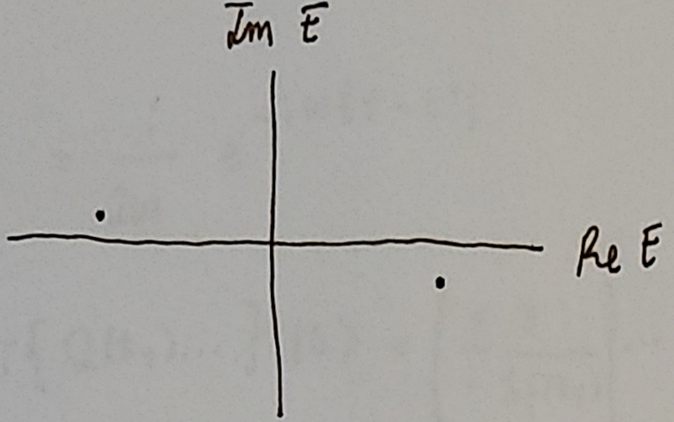
$$= \delta(t-t') \checkmark$$

As a contour integral:

$$E^2-\omega^2+i\varepsilon = 0 \Rightarrow E = \pm\sqrt{\omega^2-i\varepsilon}$$

$$= \pm(\omega-i\varepsilon)$$

$$= \begin{cases} \omega-i\varepsilon \\ -\omega+i\varepsilon \end{cases}$$



$$e^{-iE(t-t')} = e^{-iR(i\sin(\theta)+\cos(\theta))(t-t')}$$

$$= e^{R(t-t')\sin(\theta)+i(\cdots)}$$

$t-t'>0 \Rightarrow$ close in the lower half plane

$t-t'<0 \Rightarrow$ close in the upper half plane

$$G(t-t') = -\int_{-\infty}^{\infty}\frac{dE}{2\pi}\,\frac{e^{-iE(t-t')}}{(E-\omega+i\varepsilon)(E+\omega-i\varepsilon)}$$

$$= \begin{cases} -(-1)\, i\, \dfrac{e^{-iE(t-t')}}{E+\omega-i\varepsilon}\Bigg|_{E=\omega-i\varepsilon}, & t-t'>0 \\ -\, i\, \dfrac{e^{-iE(t-t')}}{E-\omega+i\varepsilon}\Bigg|_{E=-\omega+i\varepsilon}, & t-t'<0 \end{cases}$$

direction of loop (contour)

$$= \begin{cases} i\,\dfrac{e^{-i\omega(t-t')}}{2\omega}, & t-t'>0 \\ i\,\dfrac{e^{+i\omega(t-t')}}{2\omega}, & t-t'<0 \end{cases}$$

$$= \frac{i}{2\omega} e^{-i\omega|t-t'|}$$

$$\langle 0|T\{Q(t_1)\ldots\}|0\rangle = \left[\frac{1}{i}\frac{\delta}{\delta f(t_1)}\right]\cdots \langle 0|0\rangle_f \Big|_{f=0}$$

$$\langle 0|T\{Q(t_1)Q(t_2)\}|0\rangle = \left[\frac{1}{i}\frac{\delta}{\delta f(t_1)}\right]\left[\frac{1}{i}\frac{\delta}{\delta f(t_2)}\right]\langle 0|0\rangle_f \Big|_{f=0}$$

$$\langle 0|0\rangle_f = e^{\frac{i}{2}\int_{-\infty}^{\infty} dt\, dt'\, f(t)G(t-t')f(t')}$$

$$\frac{\delta}{\delta f(t_2)}\langle 0|0\rangle_f = \frac{i}{2}\int_{-\infty}^{\infty} dt\, dt' \Big[\delta(t-t_2)G(t-t')f(t') + f(t)G(t-t')\delta(t'-t_2)\Big]$$

$$\times e^{\frac{i}{2}\int_{-\infty}^{\infty} dt\, dt'\, f(t)G(t-t')f(t')}$$

$$= \frac{i}{2}\left[\int_{-\infty}^{\infty} dt'\, G(t_2-t')f(t') + \int_{-\infty}^{\infty} dt\, f(t)G(t-t_2)\right]$$

$$\times e^{\frac{i}{2}\int_{-\infty}^{\infty} dt\, dt'\, f(t)G(t-t')f(t')}$$

$$= i\int_{-\infty}^{\infty} dt\, G(t_2-t)f(t)\,\langle 0|0\rangle_f$$

$$\frac{\delta}{\delta f(t_1)} \frac{\delta}{\delta f(t_2)} \langle 0|0\rangle_f = \left[\frac{\delta}{\delta f(t_1)}\, i\int_{-\infty}^{\infty} dt\, G(t_2-t) f(t)\right] \langle 0|0\rangle_f$$

$$+ \underbrace{i\int_{-\infty}^{\infty} dt\, G(t_2-t) f(t) \frac{\delta}{\delta f(t_1)} \langle 0|0\rangle_f}$$

this piece doesn't matter because $\exists$ an explicit f here, set to zero at the end.

$$= i\int_{-\infty}^{\infty} dt\, G(t_2-t)\, \delta(t-t_1) \langle 0|0\rangle_f + \cdots$$

$$= i\, G(t_2-t_1) \langle 0|0\rangle_f + \cdots$$

$$\therefore \langle 0| T\{Q(t_1)Q(t_2)\}|0\rangle = \frac{1}{i^2}\Big[\, i\, G(t_2-t_1)\langle 0|0\rangle_f + \cdots \Big]_{f=0}$$

$$= \frac{1}{i} G(t_2-t_1)$$

Odd number of Qs $\Rightarrow 0$.

See code_1 :

$$\langle 0| T\{Q(t_1)Q(t_2)Q(t_3)Q(t_4)\}|0\rangle = \frac{1}{i^2}\left[G_{12}G_{34} + G_{13}G_{24} + G_{14}G_{23}\right]$$

$$(G_{ij} := G(t_i - t_j))$$

More generally:

$$\langle 0| T\{Q(t_1)\dots Q(t_{2n})\}|0\rangle = \frac{1}{i^n}\sum_{\text{pairings}} G(t_{i_1}-t_{i_2})\dots G(t_{i_{2n-1}}-t_{i_{2n}})$$