Spin-statistics theorem

$$H_0 = \int \widetilde{dk}\ \omega\ a^\dagger(\vec{k})\, a(\vec{k})$$

$$\omega = \sqrt{\vec{k}^2 + m^2}$$

$$[a(\vec{k}), a(\vec{k}')]_\mp = 0$$

$$[a^\dagger(\vec{k}), a^\dagger(\vec{k}')]_\mp = 0$$

$$[a(\vec{k}), a^\dagger(\vec{k}')]_\mp = (2\pi)^3\, 2\omega\, \delta^3(\vec{k} - \vec{k}')$$

Consider adding terms to the hamiltonian that will result in local, Lorentz-inv interactions. Let

$$\varphi^+(\vec{x}, 0) := \int \widetilde{dk}\ e^{i\vec{k}\cdot\vec{x}}\ a(\vec{k})$$

$$\varphi^-(\vec{x}, 0) := \int \widetilde{dk}\ e^{-i\vec{k}\cdot\vec{x}}\ a^\dagger(\vec{k})$$

Time-evolve w/ H_0:

$$\varphi^+(\vec{x}, t) = e^{iH_0t}\varphi^+(\vec{x},0)e^{-iH_0t} = \int \widetilde{dk}\ e^{ikx}\ a(\vec{k})$$

$$\varphi^-(\vec{x}, t) = e^{iH_0t}\varphi^-(\vec{x},0)\, e^{-iH_0t} = \int \widetilde{dk}\ e^{-ikx}\ a^\dagger(\vec{k})$$

$\varphi(x) = \varphi^+(x) + \varphi^-(x)$: usual hermitian free field

Under proper orthochronous Lorentz transf:

$$\mathcal{U}(\Lambda)^{-1}\varphi(x)\,\mathcal{U}(\Lambda) = \varphi(\Lambda^{-1}x)$$

We have shown that

$$\mathcal{U}(\Lambda)^{-1}\, a(\vec{k})\,\mathcal{U}(\Lambda) = a(\Lambda^{-1}\vec{k})$$

$$\mathcal{U}(\Lambda)^{-1}\, a^\dagger(\vec{k})\,\mathcal{U}(\Lambda) = a^\dagger(\Lambda^{-1}\vec{k})$$

$$\therefore\ \mathcal{U}(\Lambda)^{-1}\varphi^\pm(x)\,\mathcal{U}(\Lambda) = \varphi^\pm(\Lambda^{-1}x)$$

$\therefore$ φ^+ and φ^- are Lorentz scalars. We will then have local, Lorentz-inv interactions if we take the interaction lagrangian $\mathcal{L}_1$ to be a hermitian function of φ^+ and φ^-.

Transition amplitude T_{if} from $|i\rangle$ at $t = -\infty$ to $|f\rangle$ at $t = +\infty$:

$$T_{if} = \langle f|\, T\, e^{-i\int_{-\infty}^{\infty} dt\ H_I(t)}\ |i\rangle$$

$H_I(t) := e^{iH_0t}\ H_1\, e^{-iH_0t}$: perturbing hamiltonian in the interaction pic

H_1 : interaction hamiltonian in Schrödinger pic

T : time-ordering symbol

Key point: For T_{if} to be Lorentz inv, time-ordering must be frame-independent. Time-ordering of two spacetime points x and x' is frame-independent if their separation is timelike,

$$(x - x')^2 < 0$$

Spacelike-separated x and x' can have different frames $\therefore$ we require

$$[\mathcal{H}_I(x), \mathcal{H}_I(x')] = 0 \quad \text{whenever} \quad (x - x')^2 > 0$$

Obviously, $[\varphi^+(x), \varphi^+(x')]_\mp = 0 = [\varphi^-(x), \varphi^-(x')]_\mp$. However,

$$[\varphi^+(x), \varphi^-(x')]_\mp = \int \widetilde{dk}\,\widetilde{dk}'\ e^{ikx}\, e^{-ik'x'}\ [a(\vec{k}), a^\dagger(\vec{k}')]_\mp$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega}\frac{d^3k'}{(2\pi)^3 2\omega'}\ e^{ikx - ik'x'}\ (2\pi)^3 2\omega\, \delta^3(\vec{k} - \vec{k}')$$

$$= \int \frac{d^3k\, d^3k'}{(2\pi)^3 2\omega'}\ e^{i\vec{k}\cdot\vec{x} - i\vec{k}'\cdot\vec{x}'}\ e^{-i\omega t + i\omega' t'}\ \delta^3(\vec{k} - \vec{k}')$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega}\ e^{i\vec{k}\cdot(\vec{x} - \vec{x}')}\ e^{-i\omega(t - t')} \ \ominus$$

Go to a frame w/ $t - t' = 0$ s.t $(x - x')^2 = -(t - t')^2 + (\vec{x} - \vec{x}')^2 = r^2 > 0$.

$$\ominus \int \frac{d^3k}{(2\pi)^3 2\omega}\ e^{i\vec{k}\cdot\vec{r}}$$

$$= \frac{1}{16\pi^3}\int_0^\infty dq\ q^2 \underbrace{\int_{-1}^{1} d\psi \underbrace{\int_0^{2\pi} d\varphi}_{2\pi}\ e^{iqr\psi}}\ \frac{1}{\sqrt{q^2 + m^2}}$$

$$= \frac{1}{iqr}\left(e^{iqr} - e^{-iqr}\right) = \frac{1}{iqr}\, 2i\sin(qr) = \frac{2}{qr}\sin(qr)$$

$$= \frac{1}{16\pi^3}\int_0^\infty dq\ q^2\, \frac{2}{qr}\sin(qr)\ 2\pi\ \frac{1}{\sqrt{q^2 + m^2}}$$

$$= \frac{1}{4\pi^2 r}\int_0^\infty dq\ \frac{q\sin(qr)}{\sqrt{q^2 + m^2}}, \qquad p := \frac{q}{m}$$

$$= \frac{1}{4\pi^2 r}\int_0^\infty dp\ m\ \frac{mp\sin(pmr)}{\sqrt{m^2p^2 + m^2}}$$

$$= \frac{m}{4\pi^2 r}\int_0^\infty dt\ \frac{t\sin(mrt)}{\sqrt{t^2 + 1}}, \quad t = \sinh(u),\ dt = \cosh(u)du,\ \int_0^\infty \to \int_0^\infty$$

$$= \frac{m}{4\pi^2 r}\int_0^\infty \cosh(u)du\ \frac{\sinh(u)\sin(mr\sinh(u))}{\sqrt{\sinh(u)^2 + 1}}$$

$$= \frac{m}{4\pi^2 r}\int_0^\infty du\ \sinh(u)\ \sin(mr\sinh(u))$$

See functions.wolfram.com/Bessel-TypeFunctions/BesselK/07/01/01/0005 :

$$K_\nu(x) = \operatorname{cosec}\left(\frac{\pi\nu}{2}\right)\int_0^\infty dt\ \sin(x\sinh(t))\ \sinh(\nu t)$$

$$\therefore \int_0^\infty du\ \sinh(u)\ \sin(mr\sinh(u)) = \frac{K_1(mr)}{\operatorname{cosec}\left(\frac{\pi}{2}\right)}$$

$$= K_1(mr)$$

$$\therefore\ [\varphi^+(x), \varphi^-(x')]_\mp = \frac{m}{4\pi^2 r}K_1(mr)$$

$$=: C(r)$$

Note that $C(r) > 0$ $\forall r > 0$. For small m, $K_1(mr) = \frac{1}{mr} + O(mr)^0$, so even for $m = 0$, we have $mr K_1(mr) = 1$ and hence

$$[\varphi^+(x), \varphi^-(x')]_\mp = \frac{1}{4\pi^2 r^2} \qquad (m = 0)$$

which is never 0 $\therefore$ $\mathcal{H}_I(x)$, involving both φ^+ and φ^-, will not satisfy

$$[\mathcal{H}_I(x), \mathcal{H}_I(x')] = 0 \quad \text{for} \quad (x - x')^2 > 0$$

generically.

To resolve the problem, try using a particular linear combo of φ^+ and φ^-:

$$\varphi_\lambda(x) := \varphi^+(x) + \lambda\varphi^-(x)$$

$$\varphi_\lambda^\dagger(x) := \varphi^-(x) + \lambda^*\varphi^+(x)$$

where $\lambda \in \mathbb{C}$. Then,

$$[\varphi_\lambda(x), \varphi_\lambda^\dagger(x')]_\mp = [\varphi^+(x), \lambda\varphi^-(x), \varphi^-(x') + \lambda^*\varphi^+(x')]_\mp$$

$$= [\varphi^+(x), \varphi^-(x')]_\mp + \lambda^*[\varphi^+(x), \varphi^+(x')]_\mp + \lambda[\varphi^-(x), \varphi^-(x')]_\mp + |\lambda|^2[\varphi^-(x), \varphi^+(x')]_\mp$$

$$= C(r) + |\lambda|^2(\mp C(r))$$

$$= (1 \mp |\lambda|^2)\, C(r)$$

$$[\varphi_\lambda(x), \varphi_\lambda(x')]_\mp = [\varphi^+(x) + \lambda\varphi^-(x), \varphi^+(x') + \lambda\varphi^-(x')]_\mp$$

$$= \lambda[\varphi^+(x), \varphi^-(x')]_\mp + \lambda[\varphi^-(x), \varphi^+(x')]_\mp$$

$$= \lambda[C(r) \mp C(r)]$$

$$= \lambda(1 \mp 1)\, C(r)$$

If we want $\varphi_\lambda(x)$ to either commute or anticommute w/ both $\varphi_\lambda(x')$ and $\varphi_\lambda^\dagger(x')$ at spacelike separations, then we must choose $|\lambda| = 1$ and commutators. Only then, we can find a suitable $\mathcal{H}_I(x)$ by making it a hermitian function of $\varphi_\lambda(x)$.

But this has simply returned us to the theory of a real scalar $\because$ for $\lambda = e^{i\alpha}$, $e^{-i\alpha/2}\varphi_\lambda(x)$ is hermitian. In fact, if we make the replacements $a(\vec{k}) \to e^{i\alpha/2}a(\vec{k})$ and $a^\dagger(\vec{k}) \to e^{-i\alpha/2}a^\dagger(\vec{k})$, then the commutation relations of a and $a^\dagger$ are unchanged, and $e^{-i\alpha/2}\varphi_\lambda(x) = \varphi(x) = \varphi^+(x) + \varphi^-(x)$ $\therefore$ our attempt to start w/ a and $a^\dagger$ as fundamental objects has simply led us back to the real, commuting, scalar field $\varphi(x)$ as the fundamental object.

Consider again $\varphi(x)$ as fundamental, w/ a Lagrangian given by some function of Lorentz scalars $\varphi(x)$ and $\partial^\mu\varphi\,\partial_\mu\varphi$. Then, quantization will result in $[\varphi(x), \varphi(x')]_\mp = 0$ for $t = t'$. If we choose anticommutators, then $\varphi^2 = 0 = (\partial_\mu\varphi)^2$, resulting in a trivial $\mathcal{L}$ that is at most linear in φ and independent of $\dot\varphi$. This does not lead to the correct physics.

The situation generalizes to fields of higher spin, in any number of spacetime dimensions. One choice of quantization always leads to a trivial $\mathcal{L}$, so this choice is not allowed. The allowed choice is always commutators for fields of integer spin and anticommutators for fields of half-integer spin.