Ch 3 : Canonical Quantization of scalar fields

$$H = \int d^3x \; a^\dagger(\vec{x}) \underbrace{\left(-\frac{1}{2m} \vec{\nabla}^2 \right)}_{\substack{\text{free particle,} \\ \hbar = 1}} a(\vec{x})$$

$$= \int d^3x \left[\int \frac{d^3p}{\sqrt{2\pi}^3} \, e^{i\vec{p}\cdot\vec{x}} \, \tilde{a}(\vec{p}) \right]^\dagger$$

$$\times \left(-\frac{1}{2m} \vec{\nabla}^2 \right) \left[\int \frac{d^3p'}{\sqrt{2\pi}^3} \, e^{i\vec{p}'\cdot\vec{x}} \, \tilde{a}(\vec{p}') \right]$$

$$= \int \frac{d^3x \, d^3p \, d^3p'}{(2\pi)^3} \, e^{-i\vec{p}\cdot x} \, \tilde{a}^\dagger(\vec{p}) \, \tilde{a}(\vec{p}') \, \frac{\vec{p}'^2}{2m} \, e^{i\vec{p}'\cdot\vec{x}}$$

$$= \int d^3p \, d^3p' \; \delta^3(\vec{p} - \vec{p}') \, \frac{\vec{p}'^2}{2m} \, \tilde{a}^\dagger(\vec{p}) \, \tilde{a}(\vec{p}')$$

$$= \int d^3p \, \frac{\vec{p}^2}{2m} \, \tilde{a}^\dagger(\vec{p}) \, \tilde{a}(\vec{p})$$

$$[\tilde{a}(\vec{p}), \tilde{a}(\vec{p}')]_\mp = 0$$

$$[\tilde{a}^\dagger(\vec{p}), \tilde{a}^\dagger(\vec{p}')]_\mp = 0$$

$$[\tilde{a}(\vec{p}), \tilde{a}^\dagger(\vec{p}')]_\mp = 0$$

$$\tilde{a}(\vec{p})|0\rangle = 0$$

$$H|0\rangle = 0$$

$$\tilde{a}^\dagger(\vec{p}_1)|0\rangle =: |\vec{p}_1\rangle$$

$$H|\vec{p}_1\rangle = \int d^3p \; E(\vec{p})\,\tilde{a}^\dagger(\vec{p})\,\tilde{a}(\vec{p}) \quad |\vec{p}_1\rangle$$

$$= \int d^3p \; E(\vec{p})\,\tilde{a}^\dagger(\vec{p})\underbrace{\tilde{a}(\vec{p})\,\tilde{a}^\dagger(\vec{p}_1)}_{\delta^3(\vec{p}-\vec{p}_1) + \tilde{a}^\dagger(\vec{p}_1)\tilde{a}(\vec{p})}|0\rangle \quad =0$$

$$= E(\vec{p}_1)\,\tilde{a}^\dagger(\vec{p}_1)|0\rangle$$

$$= E(\vec{p}_1)|\vec{p}_1\rangle$$

Two-particle state:

$$\tilde{a}^\dagger(\vec{p}_1)\tilde{a}^\dagger(\vec{p}_2)|0\rangle = |\vec{p}_1\vec{p}_2\rangle$$

$$H|\vec{p}_1\vec{p}_2\rangle = \int d^3p \; E(\vec{p})\,\tilde{a}^\dagger(\vec{p})\underbrace{\tilde{a}(\vec{p})\,\tilde{a}^\dagger(\vec{p}_1)}_{\delta^3(\vec{p}-\vec{p}_1)+\tilde{a}^\dagger(\vec{p}_1)\tilde{a}(\vec{p})}\tilde{a}^\dagger(\vec{p}_2)|0\rangle$$

$$= \int d^3p \; E(\vec{p})\Big[\delta^3(\vec{p}-\vec{p}_1)\,\tilde{a}^\dagger(\vec{p})\,\tilde{a}^\dagger(\vec{p}_2)$$

$$+ \tilde{a}^\dagger(\vec{p})\,\tilde{a}^\dagger(\vec{p}_1)\underbrace{\tilde{a}(\vec{p})\,\tilde{a}^\dagger(\vec{p}_2)}_{\delta(\vec{p}-\vec{p}_2)+\tilde{a}^\dagger(\vec{p}_2)\tilde{a}(\vec{p})}\Big]|0\rangle \quad =0$$

$$= E(\vec{p}_1)\,\tilde{a}^\dagger(\vec{p}_1)\,\tilde{a}^\dagger(\vec{p}_2)\,|0\rangle$$

$$+ E(\vec{p}_2)\,\tilde{a}^\dagger(\vec{p}_1)\,\tilde{a}^\dagger(\vec{p}_2)\,|0\rangle$$

$$= \left[E(\vec{p}_1) + E(\vec{p}_2)\right] \tilde{a}^\dagger(\vec{p}_1)\,\tilde{a}^\dagger(\vec{p}_2)\,|0\rangle$$

$$= \left[E(\vec{p}_1) + E(\vec{p}_2)\right] |\vec{p}_1 \vec{p}_2\rangle$$

$\therefore\ \tilde{a}^\dagger(\vec{p}_1) \cdots \tilde{a}^\dagger(\vec{p}_n)|0\rangle$ has eigenvalue $E(\vec{p}_1) + \cdots + E(\vec{p}_n)$ where $E(\vec{p}) = \vec{p}^{\,2}/2m$.

Relativistic generalization:

$$E(\vec{p}) = \sqrt{\vec{p}^{\,2}c^2 + m^2c^4}$$

$H = \int d^3p\ E(\vec{p})\,\tilde{a}^\dagger(\vec{p})\,\tilde{a}(\vec{p})$: theory of free relativistic spin-0 particles (bosons or fermions)

Is this theory Lorentz invariant? Construct this theory again from a different point of view that emphasizes Lorentz invariance from the beginning.

Classical physics of a real scalar $\varphi(x)$.

Two frames: $\bar{\varphi}(\bar{x}) = \varphi(x)$, $\bar{x}^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$.

$\therefore$ $\varphi(x)$ and $\bar{\varphi}(\bar{x})$ satisfy the same equation.

One example: KG equation:

$$(-\partial^2 + m^2)\varphi(x) = 0 \quad (\hbar = c = 1)$$

Adopt this as the classical EOM for $\varphi(x)$.

EOM can be derived from action, $S = \int dt\, L$

$= \int dt\, d^3x\, \mathcal{L} = \int d^4x\, \mathcal{L}$. d^4x is Lorentz-inv:

$$d^4\bar{x} = |\det(\Lambda)|\, d^4x = d^4x$$

$\therefore$ $\mathcal{L}$ must be Lorentz-inv, as well.

$$\mathcal{L} = -\frac{1}{2}\partial^\mu\varphi\,\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2 + \Omega_0$$

Ω_0: arbitrary constant

Hamilton principle to get EOM:

$$0 = \delta S$$
$$= \delta \int d^4x\, \mathcal{L}$$

$$= \delta \int d^4x \left(-\frac{1}{2} \partial^\mu \varphi \, \partial_\mu \varphi - \frac{1}{2} m^2 \varphi^2 + \Omega_0 \right)$$

$$= \int d^4x \left(-\frac{1}{\cancel{2}} \cancel{2} \partial^\mu \varphi \, \partial_\mu \delta\varphi - \frac{1}{\cancel{2}} m^2 \cancel{2} \varphi \, \delta\varphi \right)$$

$$= \int d^4x \left(+ \partial_\mu \partial^\mu \varphi \, \delta\varphi - m^2 \varphi \, \delta\varphi \right)$$

$$= \int d^4x \underbrace{\left[(\Box - m^2) \varphi \right]}_{=0 \;\Longleftarrow\; \text{arbitrary}} \delta\varphi$$

Solutions to KG equation are plane waves, $e^{i(\vec{k}\cdot\vec{x} \pm \omega t)}$, $\omega = \sqrt{\vec{k}^2 + m^2}$. Most general solution:

$$\varphi(\vec{x}, t) = \int \frac{d^3k}{f(k)} \left[a(\vec{k}) e^{i\vec{k}\cdot\vec{x} - i\omega t} + b(\vec{k}) e^{i\vec{k}\cdot\vec{x} + i\omega t} \right]$$

$f(k) = f(|\vec{k}|)$: inserted for later convenience

$$e^{\mp i\omega t} \sim e^{\mp iEt}$$

The plane-wave solution of Schr. eqn. $\sim \exp(i\vec{p}\cdot\vec{x} - iE(\vec{p})t)$, so the second term in φ appears to have negative energy.

Impose the reality condition:

$$\varphi^*(x) = \int \frac{d^3k}{f(k)} \left[a^*(\vec{k}) e^{-i\vec{k}\cdot\vec{x} + i\omega t} + b^*(\vec{k}) e^{-i\vec{k}\cdot\vec{x} - i\omega t} \right]$$

$$= \int \frac{d^3k}{f(k)} \left[a^*(\vec{k}) e^{-i\vec{k}\cdot\vec{x} + i\omega t} + b^*(-\vec{k}) e^{i\vec{k}\cdot\vec{x} - i\omega t} \right]$$

$$= \varphi(x)$$

$$= \int \frac{d^3k}{f(k)} \left[a(\vec{k}) e^{i\vec{k}\cdot\vec{x} - i\omega t} + b(\vec{k}) e^{i\vec{k}\cdot\vec{x} + i\omega t} \right]$$

$$= \int \frac{d^3k}{f(k)} \left[a(\vec{k}) e^{i\vec{k}\cdot\vec{x} - i\omega t} + b(-\vec{k}) e^{-i\vec{k}\cdot\vec{x} + i\omega t} \right]$$

$$\therefore\ b^*(-\vec{k}) = a(\vec{k}) \implies b(\vec{k}) = a^*(-\vec{k})$$

$$\varphi(\vec{x}, t) = \int \frac{d^3k}{f(k)} \left[a(\vec{k}) e^{i\vec{k}\cdot\vec{x} - i\omega t} + a^*(-\vec{k}) e^{i\vec{k}\vec{x} + i\omega t} \right]$$

$$= \int \frac{d^3k}{f(k)} \left[a(\vec{k}) e^{i\vec{k}\cdot\vec{x} - i\omega t} + a^*(\vec{k}) e^{-i\vec{k}\cdot\vec{x} + i\omega t} \right]$$

$$= \int \frac{d^3k}{f(k)} \left[a(\vec{k}) e^{ikx} + a^*(\vec{k}) e^{-ikx} \right]$$

$$k^\mu = (\omega, \vec{k}), \quad x = (t, \vec{x}), \quad k^2 = \vec{k}^2 - \omega^2 = -m^2$$

on-shell

e^{ikx} is Lorentz invariant. Pick $f(k)$ such that the integral measure is also Lorentz invariant.

Consider the object $d^4k\, \delta(k^2+m^2)\, \theta(k^0)$, which is a really meaningful object — it enforces on-shell and +energy; plus, d^4k is Lorentz inv.

$$\int_{-\infty}^{\infty} dk^0\, d^3k\, \delta(k^2+m^2)\, \theta(k^0)$$

$$= \int_0^{\infty} dk^0\, d^3k\, \delta(\underbrace{\vec{k}^2 - (k^0)^2 + m^2}_{\omega^2 - (k^0)^2}) =$$

$$f(k^0) = \omega^2 - (k^0)^2$$

$$f(q) = 0 \Rightarrow q = \pm\omega$$

$$f'(q) = 2\omega$$

$$\delta(\omega^2 - (k^0)^2) = \frac{\delta(k^0-\omega)}{2\omega} + \cancel{\frac{\delta(k^0+\omega)}{2\omega}} \rightarrow 0 \because k^0 = -\omega < 0$$

$$\circleddash \int_0^\infty dk^0\, d^3k \; \frac{\delta(k^0 - \omega)}{2\omega}$$

$$= \int \frac{d^3k}{2\omega}$$

$$\therefore f(k) \propto \omega$$

Set

$$f(\omega) = (2\pi)^3\, 2\omega$$

and let

$$\widetilde{dk} = \frac{d^3k}{(2\pi)^3\, 2\omega}$$

$$\therefore \varphi(x) = \int \widetilde{dk} \left[a(\vec{k}) e^{ikx} + a^*(\vec{k}) e^{-ikx} \right]$$

$$\int d^3x\, e^{-ik'x} \varphi(x) = \int d^3x \frac{d^3k}{(2\pi)^3\, 2\omega} e^{-ik'x} \times \left[a(\vec{k}) e^{ikx} + a^*(\vec{k}) e^{-ikx} \right]$$

$$= \int \frac{d^3x\, d^3k}{(2\pi)^3\, 2\omega} a(\vec{k}) e^{i(k-k')x}$$

$$+ \int \frac{d^3x\, d^3k}{(2\pi)^3\, 2\omega} a^*(\vec{k}) e^{-i(k+k')x}$$

$$= \int \frac{d^3k}{2\omega}\, a(\vec{k})\, \delta^3(\vec{k}-\vec{k}')\, e^{-i(\omega-\omega')t}$$

$$+ \int \frac{d^3k}{2\omega}\, a^*(\vec{k})\, \delta^3(\vec{k}+\vec{k}')\, e^{i(\omega+\omega')t}$$

$$= \frac{1}{2\omega'}\, a(\vec{k}') + \frac{1}{2\omega'}\, a^*(-\vec{k}')\, e^{2i\omega' t}$$

$$\dot{\varphi}(x) = \int \widetilde{dk}\ i\omega\left[-a(\vec{k})e^{ikx} + a^*(\vec{k})e^{-ikx}\right]$$

$$\int d^3x\, e^{-ik'x}\, \dot{\varphi}(x) = \int d^3x\, \frac{d^3k}{(2\pi)^3 2\cancel{\omega}}\, e^{-ik'x}\, i\cancel{\omega}$$

$$\times \left[-a(\vec{k})e^{ikx} + a^*(\vec{k})e^{-ikx}\right]$$

$$= -i \int \frac{d^3x\, d^3k}{(2\pi)^3 2}\, a(\vec{k})\, e^{i(k-k')x}$$

$$+ i \int \frac{d^3x\, d^3k}{(2\pi)^3 2}\, a^*(\vec{k})\, e^{-i(k+k')x}$$

$$= -\frac{i}{2} \int d^3k\, a(\vec{k})\, \delta^3(\vec{k}-\vec{k}')\, e^{-i(\omega-\omega')t}$$

$$+ \frac{i}{2} \int d^3k\, a^*(\vec{k})\, \delta^3(\vec{k}+\vec{k}')\, e^{i(\omega+\omega')t}$$

$$= -\frac{i}{2}\, a(\vec{k}') + \frac{i}{2}\, e^{2i\omega' t}\, a^*(-\vec{k}')$$

$$\frac{1}{2\omega}\, a(\vec{k}) + \overset{\frac{1}{2\omega}e^{2i\omega t}a^*(-\vec{k})}{} = \int d^3x\, e^{-ikx}\, \varphi(x) \;\Big|\; i$$

$$-\frac{i}{2}\, a(\vec{k}) + \underset{\frac{i}{2}e^{2i\omega t}a^*(-\vec{k})}{} = \int d^3x\, e^{-ikx}\, \dot{\varphi}(x) \;\Big|\; -\frac{1}{\omega}$$

$$\frac{i}{2\omega}\, a(\vec{k}) + \frac{i}{2\omega}\, a(\vec{k}) = \int d^3x\, e^{-ikx}\left(i\varphi - \frac{1}{\omega}\dot{\varphi}\right) \;\Big|\; \frac{\omega}{i}$$

$$a(\vec{k}) = \int d^3x\, e^{-ikx}\left(\omega\varphi + i\dot{\varphi}\right)$$

$$= i\int d^3x\left(\dot{\varphi}\, e^{-ikx} - \varphi\, i\omega e^{-ikx}\right)$$

$$= i\int d^3x\left(\dot{\varphi}\, e^{-ikx} - \varphi\, (e^{-ikx})^{\cdot}\right)$$

$$= i\int d^3x\, e^{-ikx}\, \overleftrightarrow{\partial_0}\, \varphi(x)$$

$$f\, \overleftrightarrow{\partial_\mu}\, g := f\,\partial_\mu g - (\partial_\mu f)\, g$$

Note that $a(\vec{k})$ is time-indep.

$$\mathcal{L} = -\frac{1}{2}\underbrace{\partial^\mu\varphi\,\partial_\mu\varphi}_{\substack{=\partial^0\varphi\,\partial_0\varphi + \partial^i\varphi\,\partial_i\varphi \\ = -\dot{\varphi}^2 + (\vec{\nabla}\varphi)^2}} - \frac{1}{2}m^2\varphi^2 + \Omega_0$$

$$= \frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}(\vec{\nabla}\varphi)^2 - \frac{1}{2}m^2\varphi^2 + \Omega_0$$

Conjugate momentum:

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}(x)} = \dot{\varphi}(x)$$

Hamiltonian density:

$$\mathcal{H}(x) = \pi(x)\dot{\varphi}(x) - \mathcal{L}$$

$$= \pi^2(x) - \frac{1}{2}\pi^2(x) + \frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{1}{2}m^2\varphi^2 - \Omega_0$$

$$= \frac{1}{2}\pi^2(x) + \frac{1}{2}(\vec{\nabla}\varphi(x))^2 + \frac{1}{2}m^2\varphi^2(x) - \Omega_0$$

Hamiltonian: See code-1.

$$H = \int d^3x\, \mathcal{H}(x)$$

$$= -\Omega_0 V + \frac{1}{2}\int \widetilde{dk}\, \omega\left[a^*(\vec{k})a(\vec{k}) + a(\vec{k})a^*(\vec{k})\right]$$

So far, it's been all classical.

Canonical quantization:

$$[\varphi(\vec{x},t),\ \varphi(\vec{x}',t)] = 0$$

$$[\pi(\vec{x},t),\ \pi(\vec{x}',t)] = 0$$

$$[\varphi(\vec{x},t),\ \pi(\vec{x}',t)] = i\,\delta^3(\vec{x}-\vec{x}')$$

See code-2:

$$[a(\vec{k}),\ a(\vec{k}')] = 0$$

$$[a^\dagger(\vec{k}),\ a^\dagger(\vec{k}')] = 0$$

$$[a(\vec{k}),\ a^\dagger(\vec{k}')] = (2\pi)^3\, 2\omega\, \delta^3(\vec{k}-\vec{k}')$$

Back to hamiltonian:

$$H = -\Omega_0 V + \frac{1}{2}\int \widetilde{dk}\ \omega\,[a^\dagger(\vec{k})a(\vec{k}) + \underbrace{a(\vec{k})a^\dagger(\vec{k})}_{(2\pi)^3 2\omega\,\delta^3(0) + a^\dagger(\vec{k})a(\vec{k})}]$$

$$= -\Omega_0 V + \int \widetilde{dk}\ \omega\ a^\dagger(\vec{k})a(\vec{k})$$

$$+ \frac{1}{2}\int \frac{d^3k}{\cancel{(2\pi)^3 2\omega}}\ \underbrace{\cancel{(2\pi)^3}\ \cancel{2\omega}\ \delta^3(0)}_{V}\ \omega$$

$$= -\Omega_0 V + \underbrace{V \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\,\omega}_{=:\mathcal{E}_0} + \int \widetilde{dk}\,\omega\, a^\dagger(\vec{k})\,a(\vec{k})$$

$=: \mathcal{E}_0$: total zero-point energy of all osillators

$$= (\mathcal{E}_0 - \Omega_0)V + \int \widetilde{dk}\,\omega\, a^\dagger(k)\,a(\vec{k})$$

$\mathcal{E}_0$ is ∞, so we can integrate it up to some UV cutoff, $\Lambda \gg m$:

$$\mathcal{E}_0 = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\,\omega$$

$$\rightarrow \frac{1}{2}\int_0^\Lambda \frac{q^2\,dq}{(2\pi)^3}\int_0^{4\pi} d\Omega\,\sqrt{q^2+m^2}\,, \quad q := |\vec{k}|$$

$$\approx \frac{1}{2}\,\cancel{4}\pi\,\frac{q^4}{\cancel{4}}\Big|_0^\Lambda \frac{1}{(2\pi)^3}$$

$$= \frac{1}{2}\,\frac{\Lambda^4}{8\pi^2}$$

$$= \frac{\Lambda^4}{16\pi^2}$$

This is physically justified if the formalism of quantum field theory breaks down at some large energy scale.

For now, since Ω_0 is arbitrary, set

$$\Omega_0 = \mathcal{E}_0$$

W/ this choice, the ground state has zero energy.