Ch 1: Problems

1.1) Show that the Dirac matrices must be even-dimensional. Hint: Show that the eigenvalues of β are all ± 1 and that $\mathrm{tr}\,\beta = 0$. To show that $\mathrm{tr}\,\beta = 0$, consider e.g. $\mathrm{tr}\,\alpha_1^2 \beta = 0$. Similarly, show that $\mathrm{tr}\,\alpha_i = 0$.

Solution

$$\alpha_i \alpha_j + \alpha_j \alpha_i = 2\delta_{ij} \Rightarrow \alpha_i^2 = 1 \quad \forall i$$

$$\alpha_i \beta + \beta \alpha_i = 0$$

$$\beta \alpha_i \beta + \beta^2 \alpha_i = 0, \qquad \beta^2 = 1$$

$$\underbrace{\mathrm{tr}\,\beta \alpha_i \beta}_{\mathrm{tr}\,\alpha_i} + \mathrm{tr}\,\alpha_i = 0 \Rightarrow \mathrm{tr}\,\alpha_i = 0 \quad \forall i$$

$$M^2 = 1 = (u^\dagger M_* u)^2 = u^\dagger M_*^2 u$$

$$\therefore uu^\dagger = 1 = M_*^2 = \mathrm{diag}(\lambda_1^2, \lambda_2^2, \dots) \Rightarrow \lambda_i^2 = 1 \quad \forall i$$

$$\therefore \alpha_{i*} = \mathrm{diag}(\underbrace{1, \dots, 1}_{N_+}, \underbrace{-1, \dots, -1}_{N_-})$$

$$\mathrm{tr}\,\alpha_i = 0 \Rightarrow N_+ = N_-$$

$$\therefore D(\alpha) = 2N_+ \text{ : even-dimensional}$$

$$\underbrace{\alpha_i \alpha_i}_{1} \beta + \alpha_i \beta \alpha_i = 0$$

$$\mathrm{tr}\,\beta + \underbrace{\mathrm{tr}\,\alpha_i \beta \alpha_i}_{\mathrm{tr}\,\beta} = 0 \Rightarrow \mathrm{tr}\,\beta = 0$$

$$\left.\begin{aligned} \mathrm{tr}\,\beta &= 0 \\ \beta^2 &= 1 \end{aligned}\right\} D(\beta) = 2N_+ \text{ : even-dimensional similarly to } \alpha_i$$

Since we have already eliminated the case of 2×2, the minimum dimension should be 4×4.

1.2) W/ the hamiltonian

$$H = \int d^3x \; a^\dagger(\vec{x}) \left[-\frac{\hbar^2}{2m} \vec{\nabla}^2 + \mathcal{U}(\vec{x}) \right] a(\vec{x})$$

$$+ \frac{1}{2} \int d^3x \, d^3y \; V(\vec{x} - \vec{y}) \, a^\dagger(\vec{x}) \, a^\dagger(\vec{y}) \, a(\vec{y}) \, a(\vec{x})$$

show that the state defined by

$$|\psi(t)\rangle = \int d^3x_1 \dots d^3x_n \ \psi(\vec{x}_1, \dots, \vec{x}_n, t) a^\dagger(\vec{x}_1) \dots a^\dagger(\vec{x}_n)|0\rangle$$

obey the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = H |\psi(t)\rangle$$

iff the wavefunction obeys

$$i\hbar \frac{\partial}{\partial t} \psi = \left\{ \sum_{j=1}^{n} \left[-\frac{\hbar^2}{2m} \vec{\nabla}_j^2 + \mathcal{U}(\vec{x}_j) \right] + \sum_{j=1}^{n} \sum_{k=1}^{j-1} V(\vec{x}_j - \vec{x}_k) \right\} \psi$$

Your demonstration should apply both to the case of bosons, where the particle creation and annihilation operators obey the commutation relations

$$[a(\vec{x}), a(\vec{x}')] = 0$$

$$[a^\dagger(\vec{x}), a^\dagger(\vec{x}')] = 0$$

$$[a(\vec{x}), a^\dagger(\vec{x}')] = \delta^3(\vec{x} - \vec{x}')$$

and to the fermions, where the particle creation and annihilation operators obey the anticommutation relations

$$\{a(\vec{x}),\ a(\vec{x}\,')\} = 0$$

$$\{a^\dagger(\vec{x}),\ a^\dagger(\vec{x}\,')\} = 0$$

$$\{a(\vec{x}),\ a^\dagger(\vec{x}\,')\} = \delta^3(\vec{x}-\vec{x}\,')$$

<u>Solution</u> See code_1 and code_3

1.3) Show explicitly that $[N, H] = 0$, where H is given in the previous problem and

$$N = \int d^3x\ a^\dagger(\vec{x})\, a(\vec{x})$$

<u>Solution</u> See code_2.