Lorentz invariance

Lorentz transformation:

$$\bar{x}^\mu = \Lambda^\mu{}_\nu x^\nu$$

Λ includes ordinary spatial rotations:

$$\Lambda^0{}_0 = 1, \quad \Lambda^0{}_i = \Lambda^i{}_0 = 0, \quad \Lambda^i{}_j = R_{ij} : R^T R = 1$$

Set of all Lorentz transformations $\Rightarrow$ group

$$\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \, g_{\mu\nu} = g_{\rho\sigma}$$

$$\Lambda_{\nu\rho} \Lambda^\nu{}_\sigma = g_{\rho\sigma}$$

$$\Lambda_\nu{}^\rho \Lambda^\nu{}_\sigma = g^\rho{}_\sigma = \delta^\rho{}_\sigma$$

$$\therefore \Lambda_\nu{}^\rho = (\Lambda^{-1})^\rho{}_\nu$$

∞simal Lorentz transformations:

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$$

$$\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma g_{\mu\nu} = g_{\rho\sigma}$$

$$(\delta^\mu{}_\rho + \omega^\mu{}_\rho)(\delta^\nu{}_\sigma + \omega^\nu{}_\sigma) g_{\mu\nu} = g_{\rho\sigma}$$

$$(\delta^\mu{}_\rho \delta^\nu{}_\sigma + \delta^\mu{}_\rho \omega^\nu{}_\sigma + \omega^\mu{}_\rho \delta^\nu{}_\sigma + \omega^\mu{}_\rho \omega^\nu{}_\sigma) g_{\mu\nu} = g_{\rho\sigma}$$

$$g_{\rho\sigma} + \omega_{\rho\sigma} + \omega_{\sigma\rho} + O(\omega^2) = g_{\rho\sigma}$$

$$\therefore \omega_{\rho\sigma} = -\omega_{\sigma\rho}$$

$$\omega = \begin{pmatrix} 0 & \cdot & \cdot & \cdot \\ & 0 & \cdot & \cdot \\ & & 0 & \cdot \\ & & & 0 \end{pmatrix} : \exists \text{ only 6 independent ∞simal Lorentz transformations}$$

3 rotations and 3 boosts:

$$\omega_{ij} = -\varepsilon_{ijk} \hat{n}_k \delta\theta$$

$$\omega_{i0} = \hat{n}_i \delta\eta$$

$\delta\theta$: ∞simal rotation

$\delta\eta$: ∞simal rapidity

Only proper Lorentz transformations can be reached by compounding ∞simal ones:

$$(\Lambda^{-1})^\rho{}_\nu = \Lambda_\nu{}^\rho \Rightarrow \det(\Lambda)^{-1} = \det(\Lambda)$$

$$\Rightarrow \det(\Lambda)^2 = 1$$

$$\Rightarrow \det(\Lambda) = \pm 1$$

$\det(\Lambda) = 1$: proper

$\det(\Lambda) = -1$: improper

Proper Lorentz transformations $\Rightarrow$ subgroup

Another subgroup: Orthochronous : $\Lambda^0{}_0 \geq 1$

$$g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = g_{\rho\sigma}$$

$$g_{\mu\nu} \Lambda^\mu{}_0 \Lambda^\nu{}_0 = -1$$

$$g_{00} \Lambda^0{}_0 \Lambda^0{}_0 + g_{ij} \Lambda^i{}_0 \Lambda^j{}_0 = -1$$

$$-(\Lambda^0{}_0)^2 = -1 - \Lambda^i{}_0 \Lambda^i{}_0$$

$$(\Lambda^0{}_0)^2 = 1 + \Lambda^i{}_0 \Lambda^i{}_0$$

$$\Lambda^0{}_0 = \pm\sqrt{1 + \Lambda^i{}_0 \Lambda^i{}_0}$$

$$\therefore \Lambda^0{}_0 \geq 1 \quad \text{or} \quad \Lambda^0{}_0 \leq -1$$

Since $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$ and $\omega_{\mu\nu} = -\omega_{\nu\mu}$, it is orthochronous. Product of two orthochronous transformations is also orthochronous $\therefore$ Lorentz transformations that can be reached by compounding ∞simal ones are both proper and orthochronous and they form a subgroup.

Two discrete transformations that take us out of this subgroup: parity and time reversal.

$$\mathcal{P}^\mu{}_\nu = (\mathcal{P}^{-1})^\mu{}_\nu = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} : \text{orthochronous, improper}$$

$$\mathcal{T}^\mu{}_\nu = (\mathcal{T}^{-1})^\mu{}_\nu = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} : \text{non-orthochronous, improper}$$

Lorentz invariance $\equiv$ under proper orthochronous subgroup

In quantum theory, symmetries are represented by unitary (antiunitary) operators $\therefore$ we associate a unitary $\mathcal{U}(\Lambda)$ to each proper orthochronous Λ.

$$\mathcal{U}(\Lambda'\Lambda) = \mathcal{U}(\Lambda')\mathcal{U}(\Lambda)$$

Recall the general structure from QM:

$$\mathcal{U} = 1 + iG\varepsilon$$

$$\mathcal{U}^\dagger \mathcal{U} = 1$$

$G^\dagger = G$: generator of symmetry

Here,

$$\mathcal{U} = 1 + \frac{i}{2\hbar} \underbrace{\omega_{\mu\nu}}_{\text{antisymmetric}} \overbrace{M^{\mu\nu}}^{\text{generator of Lorentz group}}$$

$M^{\mu\nu}$: antisymmetric

$$\mathcal{U}(\Lambda)^{-1} \mathcal{U}(\Lambda') \mathcal{U}(\Lambda) = \mathcal{U}(\Lambda^{-1}\Lambda'\Lambda)$$

$$\Lambda' = 1 + \omega$$

$$\mathcal{U}(\Lambda)^{-1} \left(1 + \frac{i}{2\hbar} \omega_{\mu\nu} M^{\mu\nu}\right) \mathcal{U}(\Lambda) = \mathcal{U}(\Lambda^{-1}\Lambda'\Lambda)$$

$$\begin{aligned} \mathcal{U}(\Lambda^{-1}\Lambda'\Lambda) &= \mathcal{U}(\Lambda^{-1}(1+\omega)\Lambda) \\ &= \mathcal{U}(1 + \Lambda^{-1}\omega\Lambda) \\ &= 1 + \frac{i}{2\hbar} (\Lambda^{-1}\omega\Lambda)_{\mu\nu} M^{\mu\nu} \\ &= 1 + \frac{i}{2\hbar} (\Lambda^{-1})_\mu{}^\rho \omega_{\rho\sigma} \Lambda^\sigma{}_\nu M^{\mu\nu} \\ &= 1 + \frac{i}{2\hbar} \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu \omega_{\rho\sigma} M^{\mu\nu} \end{aligned}$$

$$1 + \frac{i}{2\hbar} \omega_{\mu\nu} \mathcal{U}(\Lambda)^{-1} M^{\mu\nu} \mathcal{U}(\Lambda) = 1 + \frac{i}{2\hbar} \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu \omega_{\rho\sigma} M^{\mu\nu}$$

$$\omega_{\mu\nu} \underbrace{\mathcal{U}(\Lambda)^{-1} M^{\mu\nu} \mathcal{U}(\Lambda)}_{\text{already antisym. in } \mu,\nu} = \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu \omega_{\rho\sigma} M^{\mu\nu}$$

$$= \omega_{\mu\nu} \underbrace{\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma M^{\rho\sigma}}_{\frac{1}{2}\left(\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma M^{\rho\sigma} - \underbrace{\Lambda^\nu{}_\rho \Lambda^\mu{}_\sigma \underbrace{M^{\rho\sigma}}_{-M^{\sigma\rho}}}_{+\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma M^{\rho\sigma}}\right)}$$

$$= \omega_{\mu\nu} \underbrace{\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma M^{\rho\sigma}}_{\text{already antisym. in } \mu,\nu}$$

$$\therefore \mathcal{U}(\Lambda)^{-1} M^{\mu\nu} \mathcal{U}(\Lambda) = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma M^{\rho\sigma}$$

$\therefore$ each index gets its own Lorentz transformation.

$$\mathcal{U}(\Lambda)^{-1} p^\mu \mathcal{U}(\Lambda) = \Lambda^\mu{}_\nu p^\nu$$

$$\mathcal{U}(\Lambda)^{-1} T^{\mu\nu\rho\ldots} \mathcal{U}(\Lambda) = \Lambda^\mu{}_{\mu'} \Lambda^\nu{}_{\nu'} \Lambda^\rho{}_{\rho'} \cdots T^{\mu'\nu'\rho'\ldots}$$

$$\mathcal{U}(\Lambda)^{-1} M^{\mu\nu} \mathcal{U}(\Lambda) = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma M^{\rho\sigma}$$

$$\left(1 - \frac{i}{2\hbar}\omega_{\rho\sigma} M^{\rho\sigma}\right) M^{\mu\nu} \left(1 + \frac{i}{2\hbar}\omega_{\rho\sigma} M^{\rho\sigma}\right) = (\delta^\mu{}_\rho + \omega^\mu{}_\rho)(\delta^\nu{}_\sigma + \omega^\nu{}_\sigma) M^{\rho\sigma}$$

$$\left(M^{\mu\nu} - \frac{i}{2\hbar}\omega_{\rho\sigma} M^{\rho\sigma} M^{\mu\nu}\right)\left(1 + \frac{i}{2\hbar}\omega_{\rho\sigma} M^{\rho\sigma}\right) = (\delta^\mu{}_\rho + \omega^\mu{}_\rho)(M^{\rho\nu} + \omega^\nu{}_\sigma M^{\rho\sigma})$$

$$M^{\mu\nu} + \frac{i}{2\hbar}\omega_{\rho\sigma}[M^{\mu\nu}, M^{\rho\sigma}] + O(\omega^2) = M^{\mu\nu} + \omega^\nu{}_\sigma M^{\mu\sigma} + \omega^\mu{}_\rho M^{\rho\nu} + O(\omega^2)$$

$$\frac{i}{2\hbar}\omega_{\rho\sigma} \underbrace{[M^{\mu\nu}, M^{\rho\sigma}]}_{\text{already antisym. in } \rho,\sigma} = \omega^\nu{}_\sigma M^{\mu\sigma} + \omega^\mu{}_\rho M^{\rho\nu}$$

$$= \omega_{\rho\sigma} g^{\rho\nu} M^{\mu\sigma} + \underbrace{\omega^\mu{}_\sigma}_{\omega_{\rho\sigma} g^{\rho\mu}} M^{\sigma\nu}$$

$$= \omega_{\rho\sigma} \underbrace{(g^{\rho\nu} M^{\mu\sigma} + g^{\rho\mu} M^{\sigma\nu})}_{\text{antisymmetrize wrt } \rho,\sigma}$$

$$= \omega_{\rho\sigma} \frac{1}{2}(g^{\rho\nu} M^{\mu\sigma} + g^{\rho\mu} M^{\sigma\nu} - g^{\sigma\nu} M^{\mu\rho} - g^{\sigma\mu} M^{\rho\nu})$$

$$\begin{aligned} [M^{\mu\nu}, M^{\rho\sigma}] &= -i\hbar(g^{\rho\nu} M^{\mu\sigma} + g^{\rho\mu} M^{\sigma\nu} - g^{\sigma\nu} M^{\mu\rho} - g^{\sigma\mu} M^{\rho\nu}) \\ &= -i\hbar(g^{\nu\rho} M^{\mu\sigma} - g^{\mu\rho} M^{\nu\sigma} - g^{\nu\sigma} M^{\mu\rho} + g^{\mu\sigma} M^{\nu\rho}) \\ &= i\hbar(g^{\mu\rho} M^{\nu\sigma} - g^{\nu\rho} M^{\mu\sigma} - g^{\mu\sigma} M^{\nu\rho} - g^{\nu\sigma} M^{\mu\rho}) \\ &= i\hbar(g^{\mu\rho} M^{\nu\sigma} - (\mu \leftrightarrow \nu)) - (\rho \leftrightarrow \sigma) \end{aligned}$$

Useful identity:

$$\varepsilon_{ijk}\varepsilon_{lmn} = \delta_{il}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{jn}\delta_{kl} + \delta_{in}\delta_{jl}\delta_{km} - \delta_{im}\delta_{jl}\delta_{kn} - \delta_{il}\delta_{jn}\delta_{km} - \delta_{in}\delta_{jm}\delta_{kl}$$

let

$J_i = \frac{1}{2}\varepsilon_{ijk} M^{jk} \leftrightarrow M^{jk} = \varepsilon_{ijk} J_i$: angular momentum

$K_i = M^{i0}$: boost

See code-1:

$[J_i, J_j] = i\hbar\varepsilon_{ijk} J_k$: usual angular momentum commutator

$[J_i, K_j] = i\hbar\varepsilon_{ijk} K_k$: boost is a vector

$[K_i, K_j] = -i\hbar\varepsilon_{ijk} J_k$: two successive boosts give rotation.

$$\mathcal{U}(\Lambda)^{-1} p^\mu \mathcal{U}(\Lambda) = \Lambda^\mu{}_\nu p^\nu$$

$$\left(1 - \frac{i}{2\hbar}\omega_{\rho\sigma} M^{\rho\sigma}\right) p^\mu \left(1 + \frac{i}{2\hbar}\omega_{\rho\sigma} M^{\rho\sigma}\right) = (\delta^\mu{}_\nu + \omega^\mu{}_\nu) p^\nu$$

$$\left(p^\mu - \frac{i}{2\hbar}\omega_{\rho\sigma} M^{\rho\sigma} p^\mu\right)\left(1 + \frac{i}{2\hbar}\omega_{\rho\sigma} M^{\rho\sigma}\right) = p^\mu + \omega^\mu{}_\nu p^\nu$$

$$p^\mu + \frac{i}{2\hbar}\omega_{\rho\sigma}[p^\mu, M^{\rho\sigma}] + O(\omega^2) = p^\mu + \omega^\mu{}_\nu p^\nu$$

$$\frac{i}{2\hbar}\omega_{\rho\sigma} \underbrace{[p^\mu, M^{\rho\sigma}]}_{\text{already antisym. in } \rho,\sigma} = \omega_{\rho\sigma} g^{\mu\rho} \delta^\sigma{}_\nu p^\nu$$

$$= \omega_{\rho\sigma} \underbrace{g^{\mu\rho} p^\sigma}_{\text{antisymmetrize wrt } \rho,\sigma}$$

$$= \omega_{\rho\sigma} \frac{1}{2}(g^{\mu\rho} p^\sigma - g^{\mu\sigma} p^\rho)$$

$$\begin{aligned} [p^\mu, M^{\rho\sigma}] &= -i\hbar(g^{\mu\rho} g^\sigma - g^{\mu\sigma} p^\rho) \\ &= i\hbar(g^{\mu\sigma} p^\rho - (\rho \leftrightarrow \sigma)) \end{aligned}$$

See code-2:

$$[J_i, H] = 0$$

$$[J_i, p_j] = i\hbar\varepsilon_{ijk} p_k$$

$$[K_i, H] = i\hbar p_i$$

$$[K_i, p_j] = i\hbar\delta_{ij} H$$

Obviously,

$$[p_i, p_j] = 0$$

$$[p_i, H] = 0$$

All these commutators of $\vec{J}, \vec{K}, \vec{p}$, and H form the Lie algebra of the Poincaré group.

What happens to a quantum scalar field, $\varphi(x)$, under Lorentz?

$$e^{+iHt/\hbar} \varphi(\vec{x}, 0) e^{-iHt/\hbar} = \varphi(\vec{x}, t)$$

Generalize:

$$e^{-ipx/\hbar} \varphi(0) e^{ipx/\hbar} = \varphi(x)$$

$T(a) := e^{ipa/\hbar}$: Spacetime translation

$$T(x)\varphi(0)T(x)^{-1} = \varphi(x) = \varphi(0 + x)$$

$$T(a)\varphi(x)T(a)^{-1} = \varphi(x + a) \quad \Big| \quad a \to -a$$

$$T(-a)\varphi(x)T(-a)^{-1} = \varphi(x - a)$$

$$T(a)^{-1}\varphi(x)T(a) = \varphi(x - a)$$

∞simal translation:

$$T(a) = 1 - \frac{i}{\hbar} pa$$

Expectation:

$$\mathcal{U}(\Lambda)^{-1}\varphi(x)\mathcal{U}(\Lambda) = \varphi(\underbrace{\Lambda^{-1}x}_{\bar{x}})$$

Derivatives:

$$\mathcal{U}(\Lambda)^{-1}\partial^\mu\varphi(x)\mathcal{U}(\Lambda) = \Lambda^\mu{}_\nu \bar{\partial}^\nu \varphi(\bar{x})$$

$\bar{\partial}$: derivative wrt $\bar{x}$

$$\mathcal{U}(\Lambda)^{-1}\partial^2\varphi(x)\mathcal{U}(\Lambda) = \bar{\partial}^2\varphi(\bar{x})$$

$\therefore$ KG is Lorentz-inv. : $\left(-\partial^2 + \frac{m^2c^2}{\hbar^2}\right)\varphi(x) = 0$