Ch 8: Problems

8.1) Starting w/

$$\Delta(x-x') = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-x')}}{k^2+m^2-i\varepsilon}$$

verify

$$(-\Box^2+m^2)\,\Delta(x-x') = \delta^4(x-x')$$

Solution

$$(-\Box^2+m^2)\,\Delta(x-x') = \int \frac{d^4k}{(2\pi)^4} \frac{[-(-k^2)+m^2]\,e^{ik(x-x')}}{k^2+m^2-i\varepsilon}$$

$$\overset{\varepsilon\to 0}{=} \int \frac{d^4k}{(2\pi)^4}\, e^{ik(x-x')}$$

$$= \delta^4(x-x') \quad \checkmark$$

8.2) Starting w/

$$\Delta(x-x') = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-x')}}{k^2+m^2-i\varepsilon}$$

verify

$$\Delta(x-x') = i\theta(t-t')\int \widetilde{dk}\, e^{ik(x-x')}$$
$$+ i\theta(t'-t)\int \widetilde{dk}\, e^{-ik(x-x')}$$

Solution

$$\Delta(x) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ikx}}{k^2+m^2-i\varepsilon}$$

$$= \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \frac{e^{-ik^0t}\, e^{i\vec{k}\cdot\vec{x}}}{-(k^0)^2+\vec{k}^2+m^2-i\varepsilon}$$

$$= \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \underbrace{\int_{-\infty}^{\infty} \frac{dE}{2\pi} \frac{e^{-iEt}}{-E^2+\omega^2-i\varepsilon}}_{G(t) \text{ from the previous chapter}} \qquad \begin{pmatrix} E = k^0 \\ \omega = \sqrt{\vec{k}^2+m^2} \end{pmatrix}$$

$$= \frac{i}{2\omega} e^{-i\omega|t|}$$

$$= \frac{i}{2\omega}\left[e^{-i\omega t}\theta(t) + e^{i\omega t}\theta(-t)\right]$$

$$= \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \frac{i}{2\omega} e^{-i\omega t}\theta(t)$$
$$+ \underbrace{\int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \frac{i}{2\omega} e^{i\omega t}\theta(-t)}_{\vec{k}\to-\vec{k}}$$

$$= i\theta(t)\int \tilde{dk}\, e^{ikx} + i\theta(-t)\int \tilde{dk}\, e^{-ikx}$$

Then, let $x \to x - x'$ and $t \to t - t'$.

8.3) Starting w/ the result of the previous problem, verify

$$(-\Box + m^2)\,\Delta(x - x') = \delta^4(x - x')$$

Note that the time derivatives in the KG wave operator can act on either the field (which obeys the KG equation or the time-ordering step functions.

Solution

$$(-\Box + m^2)\,\Delta(x) = (-\Box + m^2)\left[i\theta(t)\int \tilde{dk}\, e^{ikx} + i\theta(-t)\int \tilde{dk}\, e^{-ikx}\right]$$

$$= i\theta(t)\int \tilde{dk}\,\underbrace{(-\Box+m^2)e^{ikx}}_{0} + i\theta(-t)\int\tilde{dk}\,\underbrace{(-\Box+m^2)e^{-ikx}}_{0}$$

$$+ i(-\Box\theta(t))\int\tilde{dk}\,e^{ikx} + 2i(-\partial_\mu\theta(t))\int\tilde{dk}\,\partial^\mu e^{ikx}$$

$$+ i(-\Box\theta(-t))\int\tilde{dk}\,e^{-ikx} + 2i\left(-\partial_\mu\theta(-t)\right)\int\tilde{dk}\,\partial^\mu e^{-ikx}$$

$$= i\ddot{\theta}(t)\int\tilde{dk}\,e^{ikx} + 2i\dot{\theta}(t)\int\tilde{dk}\,(-i\omega)e^{ikx}$$

$$+ i\ddot{\theta}(-t)\int\tilde{dk}\,e^{-ikx} + 2i\dot{\theta}(-t)\int\tilde{dk}\,(i\omega)\,e^{-ikx} \;=$$

$$\dot{\theta}(t) = \delta(t)$$

$$\ddot{\theta}(t) = \dot{\delta}(t)$$

$$\dot{\theta}(-t) = -\delta(-t) = -\delta(t)$$

$$\ddot{\theta}(-t) = -\dot{\delta}(t)$$

$$= i\dot{\delta}(t)\int\tilde{dk}\,e^{ikx} + 2\delta(t)\int\tilde{dk}\,\omega\,e^{ikx}$$

$$-i\dot{\delta}(t)\int\tilde{dk}\,e^{-ikx} + 2\delta(t)\int\tilde{dk}\,\omega\,e^{-ikx} \;=$$

$$\delta(t)f(t) = \delta(t)f(0)$$

$$2\delta(t)e^{\pm i\omega t} = 2\delta(t) \Rightarrow 2\delta(t)\int\tilde{dk}\,\omega e^{ikx} = 2\delta(t)\int\tilde{dk}\,e^{i\vec{k}\cdot\vec{x}}\,\omega$$

$$2\delta(t)\int\tilde{dk}\,\omega e^{-ikx} = 2\delta(t)\int\tilde{dk}\,e^{-i\vec{k}\cdot\vec{x}}\,\omega$$

$$= 2\delta(t)\int\tilde{dk}\,e^{i\vec{k}\cdot\vec{x}}\,\omega$$

$$\ominus\; i\dot{\delta}(t)\int \widetilde{dk}\; e^{ikx} + \delta(t)\int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{x}}$$

$$-i\dot{\delta}(t)\int \widetilde{dk}\, e^{-ikx} + \delta(t)\int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{x}}$$

$$= i\dot{\delta}(t)\int \widetilde{dk}\; e^{ikx} - i\dot{\delta}(t)\int \widetilde{dk}\; e^{-ikx} + 2\underbrace{\delta(t)\delta^3(\vec{x})}_{\delta^4(x)} \ominus$$

$$\dot{\delta}(t)\int \widetilde{dk}\; e^{\pm ikx} = \frac{\partial}{\partial t}\left[\delta(t)\int \widetilde{dk}\; e^{\pm ikx}\right] - \delta(t)\int \widetilde{dk}\,(\mp i\omega)e^{\pm ikx}$$

$$= \frac{\partial}{\partial t}\left[\delta(t)\int \widetilde{dk}\; e^{\pm i\vec{k}\cdot\vec{x}}\right] \pm i\delta(t)\int \widetilde{dk}\; \omega e^{\pm i\vec{k}\cdot\vec{x}}$$

$$= \frac{\partial}{\partial t}\left[\delta(t)\int \widetilde{dk}\; e^{i\vec{k}\cdot\vec{x}}\right] \pm i\delta(t)\underbrace{\int \widetilde{dk}\; \omega e^{i\vec{k}\cdot\vec{x}}}_{\frac{1}{2}\delta^3(\vec{x})}$$

$$= \frac{\partial}{\partial t}\left[\delta(t)\int \widetilde{dk}\; e^{i\vec{k}\cdot\vec{x}}\right] \pm \frac{1}{2}i\delta^4(x)$$

$$\ominus\; \cancel{i\frac{\partial}{\partial t}\left[\delta(t)\int \widetilde{dk}\; e^{i\vec{k}\cdot\vec{x}}\right]} - \cancel{\frac{1}{2}\delta^4(x)}$$

$$\cancel{-i\frac{\partial}{\partial t}\left[\delta(t)\int \widetilde{dk}\; e^{i\vec{k}\cdot\vec{x}}\right]} - \cancel{\frac{1}{2}\delta^4(x)}$$

$$+\cancel{2}\delta^4(x)$$

$$= \delta^4(x)$$

8.4) Use

$$\varphi(x) = \int \tilde{dk} \left[a(\vec{k}) e^{ikx} + a^\dagger(\vec{k}) e^{-ikx} \right]$$

$$[a(\vec{k}), a(\vec{k}')] = 0$$

$$[a^\dagger(\vec{k}), a^\dagger(\vec{k}')] = 0$$

$$[a(\vec{k}), a^\dagger(\vec{k}')] = (2\pi)^3 2\omega \delta^3(\vec{k} - \vec{k}')$$

$$a(\vec{k})|0\rangle = 0$$

$$\langle 0| a^\dagger(\vec{k}) = 0$$

to verify

$$\langle 0| T\{\varphi(x_1)\varphi(x_2)\}|0\rangle = \frac{1}{i} \Delta(x_2 - x_1)$$

Solution

See code_1 :

$$\langle 0| \varphi(x)\varphi(0)|0\rangle = \theta(t) \int \tilde{dk}\, e^{ikx}$$

$$\langle 0| \varphi(0)\varphi(x)|0\rangle = \theta(-t) \int \tilde{dk}\, e^{-ikx}$$

$$\therefore \langle 0| T\{\varphi(x)\varphi(0)\}|0\rangle = \theta(t) \int \tilde{dk}\, e^{ikx} + \theta(-t)\int \tilde{dk}\, e^{-ikx}$$

$$= \frac{1}{i} \Delta(x)$$

Now we can shift $x \to x_1$, $0 \to x_2$, and $\Delta(x) \to \Delta(x_1 - x_2)$.

8.5) The retarded and advanced Green functions for the Klein-Gordon wave operator satisfy $\Delta_{ret}(x-y)=0$ for $x^0 \gtrless y^0$ and $\Delta_{adv}(x-y)=0$ for $x^0 \leq y^0$. Find the pole prescription on the right-hand side of

$$\Delta(x) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ikx}}{k^2+m^2}$$

that yields these Green functions.

Solution

$$\Delta(x) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ikx}}{k^2+m^2}$$

$$= \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \frac{e^{-ik^0 t}}{-(k^0)^2+\vec{k}^2 +m^2}$$

$$= \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \int_{-\infty}^{\infty} \frac{dE}{2\pi} \frac{e^{-iEt}}{-E^2+\omega^2}$$

$$-E^2+\omega^2 = 0 \Rightarrow E = \pm\omega$$

$$e^{-iEt} = e^{-iR[\cos(\theta)+i\sin(\theta)]t}$$

$$= e^{R\sin(\theta)t + i(\cdots)}$$

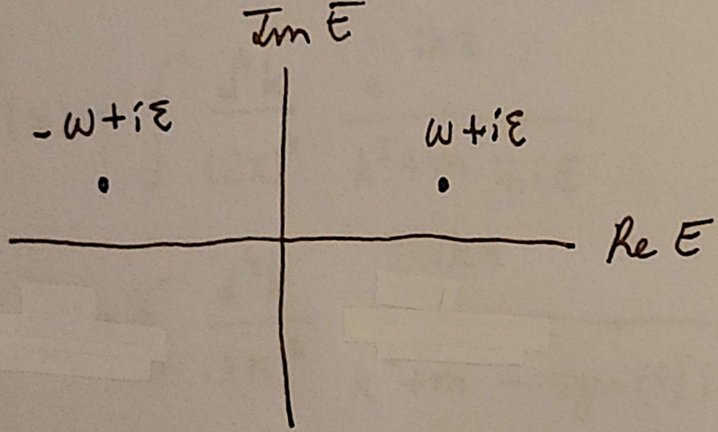
$t > 0 \Rightarrow$ close in the lower half plane

$t < 0 \Rightarrow$ close in the upper half plane

$\Delta_{ret}(t>0) = 0 \Rightarrow \Delta_{ret}(t<0) \neq 0 \Rightarrow$ close in the upper half plane

$\Rightarrow$ no poles in the lower half plane

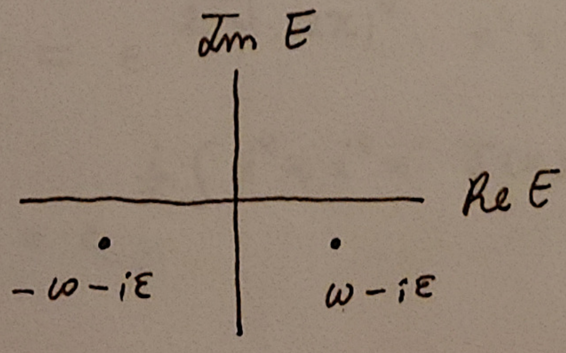
Retarded:



$\Delta_{adv}(t<0) = 0 \Rightarrow \Delta_{adv}(t>0) \neq 0 \Rightarrow$ close in the lower half plane

$\Rightarrow$ no poles in the upper half plane

Advanced:



Retarded: $(E-(\omega+i\varepsilon))(-E+(-\omega+i\varepsilon))$

(−) $= -E^2+\omega^2+i\varepsilon\cdot 2E+O(\varepsilon^2)$

Advanced: $(E-(\omega-i\varepsilon))(-E+(-\omega-i\varepsilon))$

(+) $= -E^2+\omega^2-i\varepsilon\cdot 2E+O(\varepsilon^2)$

Feynman: $(E-(\omega-i\varepsilon))(-E+(-\omega+i\varepsilon))$

$$= -E^2+\omega^2-i\varepsilon\cdot 2\omega+O(\varepsilon^2)$$

$$\Delta_{\pm}(x) = \int \frac{d^4k}{(2\pi)^4}\frac{e^{ikx}}{k^2+m^2\mp i\varepsilon}$$

$$= \int \frac{d^4k}{(2\pi)^4}\frac{e^{ikx}}{k^2+m^2-\operatorname{sgn}(t)\,i\varepsilon}$$

8.6) Let $Z_0(J) = e^{iW_0(J)}$ and evaluate the real and imaginary parts of $W_0(J)$.

Solution

$$Z_0(J) = e^{\frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\frac{\tilde{J}(k)\tilde{J}(-k)}{k^2+m^2-i\varepsilon}}$$

$$= e^{\frac{i}{2}\int d^4x\, d^4x'\, J(x)\Delta(x-x')J(x')}$$

$$\therefore W_0(J) = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} \frac{\tilde{J}(k)\tilde{J}(-k)}{k^2+m^2-i\varepsilon}$$

$$= \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} \tilde{J}(k)\tilde{J}(-k)\left[P.V\frac{1}{k^2+m^2} + i\pi\delta(k^2+m^2)\right]$$

Action contains $J\varphi$; φ is real, and so must be J

$\therefore \tilde{J}(k)\tilde{J}(-k) = |\tilde{J}(k)|^2$ as in the previous chapter.

$$\therefore \mathrm{Re}\, W_0(J) = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} \frac{|\tilde{J}(k)|^2}{k^2+m^2}$$

$$\mathrm{Im}\, W_0(J) = \frac{1}{2}\pi\int \frac{d^4k}{(2\pi)^4} |\tilde{J}(k)|^2\, \delta(k^2+m^2)$$

It seems that there is nothing much to do w/ the real part.

$$\delta(k^2+m^2) = \delta(-E^2+\omega^2),\quad E = k^0,\ \omega = \sqrt{\vec{k}^2+m^2}$$

$$= \delta(E^2-\omega^2)$$

$$= \frac{\delta(E-\omega)+\delta(E+\omega)}{2\omega}$$

I will stop here. I couldn't see what's more that can be done.

8.7) Repeat the analysis of this section for the complex scalar field that was introduced in problem 3.5 and further studied in problem 5.1. Write your source term in the form $J^\dagger \varphi + J\varphi^\dagger$, and find an explicit formula, analogous to

$$Z_0(J) = e^{\frac{i}{2}\int \frac{d^4k}{(2\pi)^4} \frac{\tilde{J}(k)\tilde{J}(-k)}{k^2+m^2-i\varepsilon}}$$

$$= e^{\frac{i}{2}\int d^4x\, d^4x'\, J(x)\Delta(x-x')J(x')}$$

for $Z_0(J^\dagger, J)$. Write down the appropriate generalization of

$$\langle 0|T\{\varphi(x_1)\ldots\}|0\rangle = \left[\frac{1}{i}\frac{\delta}{\delta J(x_1)}\right]\cdots Z_0(J)\Big|_{J=0}$$

and use it to compute $\langle 0|T\{\varphi(x_1)\varphi(x_2)\}|0\rangle$, $\langle 0|T\{\varphi^\dagger(x_1)\varphi(x_2)\}|0\rangle$, and $\langle 0|T\{\varphi^\dagger(x_1)\varphi^\dagger(x_2)\}|0\rangle$. Then verify your results by using the method of problem 8.4. Finally, give the appropriate generalization of

$$\langle 0|T\{\varphi(x_1)\dots\varphi(x_{2n})\}|0\rangle$$

$$= \frac{1}{i^n}\sum_{\text{pairings}} \Delta(x_{i_1}-x_{i_2})\dots\Delta(x_{i_{2n-1}}-x_{i_{2n}})$$

Solution

$$\mathcal{L}_0 = -\partial^\mu\varphi^\dagger\partial_\mu\varphi^\dagger - m^2\varphi^\dagger\varphi$$

$$Z_0(J\ ,J^\dagger) := \langle 0|0\rangle_{J,J^\dagger} = \int D\varphi\, D\varphi^\dagger\, e^{i\int d^4x(\mathcal{L}_0+J^\dagger\varphi+J\varphi^\dagger)}$$

$$\varphi(x) = \int\frac{d^4k}{(2\pi)^4}\, e^{ikx}\qquad\tilde{\varphi}(k)$$

$$\varphi^\dagger(x) = \int\frac{d^4k}{(2\pi)^4}\, e^{-ikx}\qquad\tilde{\varphi}^\dagger(k)$$

$$S_0 = \int d^4x\left(\mathcal{L}_0 + J^\dagger\varphi + J\varphi^\dagger\right)$$

$$= \int d^4x\left[-\partial^\mu\varphi^\dagger\partial_\mu\varphi - m^2\varphi^\dagger\varphi + J^\dagger\varphi + J\varphi^\dagger\right]$$

$$= \int d^4x\left\{-\int\frac{d^4k}{(2\pi)^4}(-ik^\mu)e^{-ikx}\tilde{\varphi}^\dagger(k)\int\frac{d^4k'}{(2\pi)^4}\,ik'_\mu e^{ik'x}\tilde{\varphi}(k')\right.$$

$$- m^2 \int \frac{d^4k}{(2\pi)^4}\, e^{-ikx}\, \tilde{\varphi}^\dagger(k) \int \frac{d^4k'}{(2\pi)^4}\, e^{ik'x}\, \tilde{\varphi}(k')$$

$$+ \int \frac{d^4k}{(2\pi)^4}\, e^{-ikx}\, \tilde{J}^\dagger(k) \int \frac{d^4k'}{(2\pi)^4}\, e^{ik'x}\, \tilde{\varphi}(k')$$

$$+ \int \frac{d^4k}{(2\pi)^4}\, e^{ikx}\, \tilde{J}(k) \int \frac{d^4k'}{(2\pi)^4}\, e^{-ik'x}\, \tilde{\varphi}^\dagger(k') \Bigg\}$$

$$= - \int \frac{d^4x\, d^4k\, d^4k'}{(2\pi)^4 (2\pi)^4}\, k\cdot k'\, e^{-i(k-k')x}\, \tilde{\varphi}^\dagger(k)\tilde{\varphi}(k')$$

$$- \int \frac{d^4x\, d^4k\, d^4k'}{(2\pi)^4 (2\pi)^4}\, m^2\, e^{-i(k-k')x}\, \tilde{\varphi}^\dagger(k)\tilde{\varphi}(k')$$

$$+ \int \frac{d^4x\, d^4k\, d^4k'}{(2\pi)^4 (2\pi)^4}\, e^{-i(k-k')x}\, \tilde{J}^\dagger(k)\, \tilde{\varphi}(k')$$

$$+ \int \frac{d^4x\, d^4k\, d^4k'}{(2\pi)^4 (2\pi)^4}\, e^{-i(k-k)x}\, \tilde{J}(k)\, \tilde{\varphi}^\dagger(k')$$

$$= - \int \frac{d^4k}{(2\pi)^4}\, k^2\, \tilde{\varphi}^\dagger(k)\, \tilde{\varphi}(k) - m^2 \int \frac{d^4k}{(2\pi)^4}\, \tilde{\varphi}^\dagger(k)\, \tilde{\varphi}(k)$$

$$+ \int \frac{d^4k}{(2\pi)^4} \left[\tilde{J}^\dagger(k)\tilde{\varphi}(k) + \tilde{J}(k)\tilde{\varphi}^\dagger(k) \right]$$

$$= \int \frac{d^4k}{(2\pi)^4} \left\{ - \tilde{\varphi}^\dagger(k)(k^2 + m^2)\tilde{\varphi}(k) + \tilde{J}^\dagger(k)\tilde{\varphi}(k) + \tilde{J}(k)\tilde{\varphi}^\dagger(k) \right\}$$

See code-2:

$$\tilde{\varphi}(k) = \tilde{\chi}(k) + \frac{\tilde{J}(k)}{k^2+m^2}$$

$$\tilde{\varphi}^\dagger(k) = \tilde{\chi}^\dagger(k) + \frac{\tilde{J}^\dagger(k)}{k^2+m^2}$$

$$D\varphi = D\chi, \quad D\varphi^\dagger = D\chi^\dagger$$

$$\therefore S_0 = \int \frac{d^4k}{(2\pi)^4} \Big\{ \underbrace{-\tilde{\chi}^\dagger(k)(k^2+m^2)\tilde{\chi}(k)}_{\text{this part is } \langle 0|0\rangle_{J,J^\dagger=0}} + \frac{\tilde{J}^\dagger(k)\tilde{J}(k)}{k^2+m^2} \Big\}$$

so it gives 1.

$$Z_0(J,J^\dagger) = \langle 0|0\rangle_{J,J^\dagger} = e^{i\int \frac{d^4k}{(2\pi)^4} \frac{\tilde{J}^\dagger(k)\tilde{J}(k)}{k^2+m^2-i\varepsilon}}$$

$$= e^{i\int d^4x\, d^4x'\, J^\dagger(x)\Delta(x-x')J(x')}$$

where $\Delta(x)$ is the same as in the real scalar theory.

$$\langle 0|T\{\varphi(x_1)\dots\varphi^\dagger(x_2)\dots\}|0\rangle$$

$$= \left[\frac{1}{i}\frac{\delta}{\delta J(x_1)}\right]\cdots\left[\frac{1}{i}\frac{\delta}{\delta J^\dagger(x_2)}\right]\cdots Z_0(J,J^\dagger)\Big|_{J,J^\dagger=0}$$

$$\langle 0|T\{\varphi(x_1)\varphi(x_2)\}|0\rangle = \left[\frac{1}{i}\frac{\delta}{\delta J(x_1)}\right]\left[\frac{1}{i}\frac{\delta}{\delta J(x_2)}\right] Z_0(J,J^\dagger)\Big|_{J,J^\dagger=0}$$

$$= \frac{1}{i^2}\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)} e^{i\int d^4x\, d^4x'\, J^\dagger(x)\Delta(x-x')J(x')}\Big|_{J,J^\dagger=0}$$

$$= \frac{1}{i^2}\frac{\delta}{\delta J(x_1)}\Big\{ i\int d^4x\, d^4x'\, J^\dagger(x)\Delta(x-x')\delta^4(x'-x_2) \times e^{i\int d^4x\, d^4x'\, J^\dagger(x)\Delta(x-x')J(x')}\Big\}_{J,J^\dagger=0}$$

$$= \frac{1}{i}\frac{\delta}{\delta J(x_1)}\Big\{\int d^4x\, J^\dagger(x)\Delta(x-x_2)\, e^{i(\dots)}\Big\}_{J,J^\dagger=0}$$

$\downarrow$

$\exists$ no derivative to kill this term, so it gives 0 at the end.

$$= 0$$

Simile, $\langle 0|T\{\varphi^\dagger(x_1)\varphi^\dagger(x_2)\}|0\rangle = 0$. The number of

φ and $\varphi^\dagger$ should match.

$\langle 0 | T\{\varphi^\dagger(x_1)\varphi(x_2)\} | 0 \rangle$

$$= \frac{1}{i}\frac{\delta}{\delta J^\dagger(x_1)}\left\{ \int d^4x\, J^\dagger(x)\Delta(x-x_2)\, e^{i\int d^4x\, d^4x'\, J^\dagger(x)\Delta(x-x')J(x')} \right\}_{J,J^\dagger=0}$$

$$= \frac{1}{i}\left\{ \Delta(x_1-x_2)\, e^{i(\cdots)} + \int d^4x\, J^\dagger(x)\Delta(x-x_2)(\cdots) \right\}_{J,J^\dagger=0}$$

$$= \frac{1}{i}\Delta(x_1-x_2)$$

$$\varphi(x) = \int \widetilde{dk}\left[a(\vec{k})e^{ikx} + b^\dagger(\vec{k})e^{-ikx} \right]$$

$$\varphi^\dagger(x) = \int \widetilde{dk}\left[a^\dagger(\vec{k})e^{-ikx} + b(\vec{k})e^{ikx} \right]$$

See code-3:

$$\langle 0 | T\{\varphi(x_1)\varphi(x_2)\} | 0 \rangle = 0$$

$$\langle 0 | T\{\varphi^\dagger(x_1)\varphi^\dagger(x_2)\} | 0 \rangle = 0$$

$$\langle 0 | T\{\underset{(x)}{\varphi^\dagger(x_1)}\,\underset{(0)}{\varphi(x_2)}\} | 0 \rangle = \theta(t)\int \widetilde{dk}\, e^{ikx} + \theta(-t)\int \widetilde{dk}\, e^{-ikx}$$

$$= \frac{1}{i}\Delta(x) = \frac{1}{i}\Delta(x_1-x_2)$$

$$\langle 0|T\{\varphi^\dagger(x_1)\dots\varphi^\dagger(x_n)\varphi(y_1)\dots\varphi(y_n)\}|0\rangle$$

$$= \frac{1}{i}\sum_{\text{pairings}} \Delta(x_{i_1}-y_{j_1})\dots\Delta(x_{i_n}-y_{j_n})$$

8.8) A harmonic oscillator (in units w/ $m=\hbar=1$) has a ground-state wavefunction $\langle q|0\rangle \propto e^{-\omega q^2/2}$. Now consider a real scalar field $\varphi(x)$ and define a field eigenstate $|A\rangle$ that obeys

$$\varphi(\vec{x},0)|A\rangle = A(\vec{x})|A\rangle$$

where the function $A(\vec{x})$ is everywhere real. For a free-field theory speified by the hamiltonian

$$\mathcal{H}_0 = \frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{1}{2}m^2\varphi^2$$

Show that the ground-state wavefunctional is

$$\langle A|0\rangle \propto e^{-\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\omega(\vec{k})\tilde{A}(\vec{k})\tilde{A}(-\vec{k})}$$

where $\tilde{A}(\vec{k}) := \int d^3x\, e^{-i\vec{k}\cdot\vec{x}} A(\vec{x})$ and $\omega(\vec{k}) := \sqrt{\vec{k}^2+m^2}$.

Solution

$$H = \frac{1}{2} p^2 + \frac{1}{2} \omega^2 q^2$$

$$H|0\rangle = \frac{\omega}{2}|0\rangle$$

$$\left(\frac{1}{2} p^2 + \frac{1}{2} \omega^2 q^2\right)|0\rangle = \frac{\omega}{2}|0\rangle$$

$$\frac{1}{2}\langle q|p^2|0\rangle + \frac{1}{2}\omega^2 q^2 \langle q|0\rangle = \frac{\omega}{2}\langle q|0\rangle$$

$$-\frac{\partial^2}{\partial q^2}\langle q|0\rangle + \omega^2 q^2 \langle q|0\rangle = \omega \langle q|0\rangle$$

$\langle q|0\rangle = N e^{-\omega q^2/2}$: checked using Mathematica

$$H = \int d^3x \left[\frac{1}{2}\pi^2 + (\vec{\nabla}\varphi)^2 + \frac{1}{2} m^2 \varphi^2\right]_{t=0}$$

$$= \int d^3x \left\{ \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \tilde{\pi}(\vec{k}) \int \frac{d^3k'}{(2\pi)^3} e^{i\vec{k}'\cdot\vec{x}} \tilde{\pi}(\vec{k}') \right.$$

$$+ \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} i\vec{k} e^{i\vec{k}\cdot\vec{x}} \tilde{\varphi}(\vec{k}) \cdot \int \frac{d^3k'}{(2\pi)^3} i\vec{k}' e^{i\vec{k}'\cdot\vec{x}} \tilde{\varphi}(\vec{k}')$$

$$\left. + \frac{1}{2} m^2 \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \tilde{\varphi}(\vec{k}) \int \frac{d^3k'}{(2\pi)^3} e^{i\vec{k}'\cdot\vec{x}} \tilde{\varphi}(\vec{k}') \right\}$$

$$= \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left[\tilde{\pi}(\vec{k})\tilde{\pi}(-\vec{k}) + (\vec{k}^2 + m^2)\tilde{\varphi}(\vec{k})\tilde{\varphi}(-\vec{k})\right]$$

$$= \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left[\tilde{\pi}(\vec{k})\tilde{\pi}(-\vec{k}) + \omega^2 \tilde{\varphi}(\vec{k})\tilde{\varphi}(-\vec{k})\right]$$

$$= \frac{1}{2}\sum_{\vec{k}} \tilde{\pi}_{\vec{k}}\tilde{\pi}_{-\vec{k}} + \frac{1}{2}\sum_{\vec{k}} \omega_{\vec{k}}^2 \tilde{\varphi}_{\vec{k}}\tilde{\varphi}_{-\vec{k}}$$

So it is a sum of a bunch of harmonic oscillators. For a single mode, we then naively expect

$$\langle A|0\rangle = N e^{-\frac{1}{2}\omega_{\vec{k}} \tilde{A}_{\vec{k}}\tilde{A}_{-\vec{k}}} \quad \text{(single mode)}$$

and hence, for the grand total of all the oscillators, we multiply them*:

$$\langle A|0\rangle = \prod_{\vec{k}} N e^{-\frac{1}{2}\omega_{\vec{k}} \tilde{A}_{\vec{k}}\tilde{A}_{-\vec{k}}}$$

$$= \mathcal{N} e^{-\frac{1}{2}\sum_{\vec{k}}\omega_{\vec{k}} \tilde{A}_{\vec{k}}\tilde{A}_{-\vec{k}}}$$

$$= \mathcal{N} e^{-\frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\, \omega\, \tilde{A}(\vec{k})\tilde{A}(-\vec{k})}$$

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* because, if we have n particles, we do not write $\sum_{i=1}^{n} a^\dagger(\vec{k}_i)|0\rangle$ but rather $\left(\prod_{i=1}^{n} a^\dagger(\vec{k}_i)\right)|0\rangle$.