Ch 8: Path integral for free field theory

$$\mathcal{H}_0 = \frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{1}{2}m^2\varphi^2$$

$q(t) \rightarrow \varphi(\vec{x},t)$ (classical field)

$Q(t) \rightarrow \varphi(\vec{x},t)$ (operator field)

$f(t) \rightarrow J(\vec{x},t)$ (classical source)

$\mathcal{H}_0 \rightarrow (1-i\varepsilon)\mathcal{H}_0$ as before

and this is equivalent to $m^2 \rightarrow (1-i\varepsilon)m^2$.

From now on, when we write m^2, it will $m^2 - i\varepsilon$.

Path integral (= functional integral) of our free theory:

$$Z_0(J) = \langle 0|0\rangle_J = \int D\varphi\, e^{i\int d^4x\,(\mathcal{L}_0 + J\varphi)}$$

$$\mathcal{L}_0 = -\frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2$$

$D\varphi \propto \prod_x d\varphi(x)$: functional measure

4d Fourier transform:

$$\tilde{\varphi}(k) = \int d^4x \, e^{-ikx} \varphi(x)$$

$$\varphi(x) = \int \frac{d^4k}{(2\pi)^4} e^{ikx} \tilde{\varphi}(k)$$

$$S_0 = \int d^4x \left(\mathcal{L}_0 + J\varphi \right)$$

$$= \int d^4x \left(-\frac{1}{2}(\partial\varphi)^2 - \frac{1}{2} m^2 \varphi^2 + J\varphi \right]$$

$$= \int d^4x \Big\{ -\frac{1}{2} \Big[\int \frac{d^4k}{(2\pi)^4} ik_\mu e^{ikx} \tilde{\varphi}(k) \Big] \Big[\int \frac{d^4k'}{(2\pi)^4} ik'^\mu e^{ik'x} \tilde{\varphi}(k') \Big]$$

$$-\frac{1}{2} m^2 \Big[\int \frac{d^4k}{(2\pi)^4} e^{ikx} \tilde{\varphi}(k) \Big] \Big[\int \frac{d^4k'}{(2\pi)^4} e^{ik'x} \tilde{\varphi}(k') \Big]$$

$$+ \frac{1}{2} \Big[\int \frac{d^4k}{(2\pi)^4} e^{ikx} \tilde{J}(k) \Big] \Big[\int \frac{d^4k'}{(2\pi)^4} e^{ik'x} \tilde{\varphi}(k') \Big]$$

$$+ \frac{1}{2} \Big[\int \frac{d^4k'}{(2\pi)^4} e^{ik'x} \tilde{J}(k') \Big] \Big[\int \frac{d^4k}{(2\pi)^4} e^{ikx} \tilde{\varphi}(k) \Big] \Big\}$$

$$= +\frac{1}{2} \int \frac{d^4x \, d^4k \, d^4k'}{(2\pi)^4 (2\pi)^4} k \cdot k' e^{i(k+k')x} \tilde{\varphi}(k) \tilde{\varphi}(k')$$

$$-\frac{1}{2} \int \frac{d^4x \, d^4k \, d^4k'}{(2\pi)^4 (2\pi)^4} m^2 e^{i(k+k')x} \tilde{\varphi}(k) \tilde{\varphi}(k')$$

$$+\frac{1}{2}\int \frac{d^4x\, d^4k\, d^4k'}{(2\pi)^4(2\pi)^4}\, e^{i(k+k')x}\, \tilde{J}(k)\tilde{\varphi}(k')$$

$$+\frac{1}{2}\int \frac{d^4x\, d^4k\, d^4k'}{(2\pi)^4(2\pi)^4}\, e^{i(k+k')x}\, \tilde{J}(k')\tilde{\varphi}(k)$$

$$= \frac{1}{2}\int \frac{d^4k\, d^4k'}{(2\pi)^4}\, \delta^4(k+k')\Big\{(k\cdot k' - m^2)\tilde{\varphi}(k)\tilde{\varphi}(k') + \tilde{J}(k)\tilde{\varphi}(k') + \tilde{J}(k')\tilde{\varphi}(k)\Big\}$$

$$= \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\Big[-(k^2+m^2)\tilde{\varphi}(k)\tilde{\varphi}(-k) + \tilde{J}(k)\tilde{\varphi}(-k) + \tilde{J}(-k)\varphi(k)\Big]$$

Let $\tilde{\chi}(k) = \tilde{\varphi}(k) - \dfrac{\tilde{J}(k)}{k^2+m^2}$ so $\mathcal{D}\varphi = \mathcal{D}\chi$:

$$S_0 = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\Big[\frac{\tilde{J}(k)\tilde{J}(-k)}{k^2+m^2} - \tilde{\chi}(k)(k^2+m^2)\tilde{\chi}(-k)\Big]$$

Just as for the harmonic oscillator, the integral over χ simply yields a factor of $Z_0(0) = \langle 0|0\rangle_{J=0}$ $= 1.$

$$\therefore Z_0(J) = e^{\frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\frac{\tilde{J}(k)\tilde{J}(-k)}{k^2+m^2-i\varepsilon}}$$

Going back to the position space w/ a little algebra, we get

$$Z_0(J) = e^{\frac{i}{2}\int d^4x\, d^4x'\, J(x)\,\Delta(x-x')\,J(x')}$$

where

$$\Delta(x) := \int \frac{d^4k}{(2\pi)^4} \frac{e^{ikx}}{k^2+m^2-i\varepsilon}$$: Feynman propagator

is the Green function for the KG equation:

$$(-\Box + m^2)\,\Delta(x-x') = \delta^4(x-x')$$

$$(-\Box + m^2)\,\Delta(x-x') = \int \frac{d^4k}{(2\pi)^4} \left.\frac{(+k^2+m^2)\,e^{ik(x-x')}}{k^2+m^2-i\varepsilon}\right|_{\varepsilon\to 0}$$

$$= \int \frac{d^4k}{(2\pi)^4}\, e^{ik(x-x')}$$

$$= \delta^4(x-x')$$

Evaluation of the k^0 integral via residue theorem:

$$\Delta(x) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ikx}}{k^2+m^2-i\varepsilon}$$

$$= \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \frac{e^{-ik^0t}\, e^{i\vec{k}\cdot\vec{x}}}{-(k^0)^2+\vec{k}^2+m^2-i\varepsilon}$$

$$= \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \underbrace{\int_{-\infty}^{\infty} \frac{dE}{2\pi} \frac{e^{-iEt}}{-E^2+\omega^2-i\varepsilon}}_{G(t) \text{ from the previous chapter}}$$

$$(E = k^0,\ \omega = \sqrt{\vec{k}^2+m^2})$$

$$= \frac{i}{2\omega} e^{-i\omega|t|}$$

$$= \frac{i}{2\omega}\left[e^{-i\omega t}\theta(t) + e^{i\omega t}\theta(-t)\right]$$

$$= \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \frac{i}{2\omega} e^{-i\omega t} \theta(t)$$

$$+ \underbrace{\int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}}}_{\vec{k}\to-\vec{k}} \frac{i}{2\omega} e^{i\omega t} \theta(-t)$$

$$= i\,\theta(t) \int \widetilde{dk}\, e^{ikx} + i\,\theta(-t)\int \widetilde{dk}\, e^{-ikx}$$

We can do the successive integrals in terms of Bessel functions now.

$$\langle 0 | T\{\varphi(x_1)\dots\} | 0\rangle = \left[\frac{1}{i}\frac{\delta}{\delta J(x_1)}\right]\cdots \quad Z_0(J)\Big|_{J=0}$$

We can do a very similar calulation like in the previous chapter to get

$$\langle 0 | T\{\varphi(x_1)\varphi(x_2)\} | 0\rangle = \frac{1}{i}\Delta(x_2 - x_1)$$

$$\langle 0 | T\{\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\} | 0\rangle = \frac{1}{i^2}(\Delta_{12}\Delta_{34} + \Delta_{13}\Delta_{24} + \Delta_{14}\Delta_{23})$$

$$(\Delta_{ij} := \Delta(x_i - x_j))$$

$$\langle 0 | T\{\varphi(x_1)\dots\varphi(x_{2n})\} | 0\rangle = \frac{1}{i^n}\sum_{\text{pairings}} \Delta_{i_1 i_2}\cdots\Delta_{i_{2n-1} i_{2n}}$$

This is known as the Wick theorem.