Ch 4: Spin-statistics theorem

$$H_0 = \int \widetilde{dk}\ \omega\, a^\dagger(\vec{k}) a(\vec{k})$$

$$\omega = \sqrt{\vec{k}^2 + m^2}$$

$$[a(\vec{k}), a(\vec{k}')]_\mp = 0$$

$$[a^\dagger(\vec{k}), a^\dagger(\vec{k}')]_\mp = 0$$

$$[a(\vec{k}), a^\dagger(\vec{k}')]_\mp = (2\pi)^3 2\omega\, \delta^3(\vec{k}-\vec{k}')$$

Consider adding terms to the hamiltonian that will result in local, Lorentz-inv. interactions. Let

$$\varphi^+(\vec{x}, 0) := \int \widetilde{dk}\ e^{i\vec{k}\cdot\vec{x}}\, a(\vec{k})$$

$$\varphi^-(\vec{x}, 0) := \int \widetilde{dk}\ e^{-i\vec{k}\cdot\vec{x}}\, a^\dagger(\vec{k})$$

Time-evolve w/ H_0:

$$\varphi^+(\vec{x}, t) = e^{iH_0 t} \varphi^+(\vec{x}, 0)\, e^{-iH_0 t} = \int \widetilde{dk}\ e^{ikx}\, a(\vec{k})$$

$$\varphi^-(\vec{x}, t) = e^{iH_0 t} \varphi^-(\vec{x}, 0)\, e^{-iH_0 t} = \int \widetilde{dk}\ e^{-ikx}\, a^\dagger(\vec{k})$$

$$\varphi(x) = \varphi^+(x) + \varphi^-(x)$$: usual hermitian free field

Under proper orthochronous Lorentz transf.:

$$\mathcal{U}(\Lambda)^{-1} \varphi(x) \mathcal{U}(\Lambda) = \varphi(\Lambda^{-1} x)$$

We have shown that

$$\mathcal{U}(\Lambda)^{-1} a(\vec{k}) \mathcal{U}(\Lambda) = a(\Lambda^{-1} \vec{k})$$

$$\mathcal{U}(\Lambda)^{-1} a^\dagger(\vec{k}) \mathcal{U}(\Lambda) = a^\dagger(\Lambda^{-1} \vec{k})$$

Thus,

$$\mathcal{U}(\Lambda)^{-1} \varphi^\pm(x) \mathcal{U}(\Lambda) = \varphi^\pm(\Lambda^{-1} x)$$

$\therefore$ φ^+ and φ^- are Lorentz scalars. We will then have local, Lorentz-invariant interactions if we take the interaction lagrangian $\mathcal{L}_1$ to be a hermitian function of φ^+ and φ^-.

Transition amplitude $T_{i\to f}$ from $|i\rangle$ at $t=-\infty$ to $|f\rangle$ at $t=+\infty$:

$$T_{i\to f} = \langle f | T \exp\left[-i\int_{-\infty}^{\infty} dt\, H_I(t)\right] |i\rangle$$

$H_I(t) := e^{iH_0 t} H_1 e^{-iH_0 t}$: perturbing hamiltonian in the interaction pic.

H_1 : interaction ham. in Schr. pic.

T : time-ordering symbol

Key point : For $T_{i\to f}$ to be Lorentz-inv, time-ordering must be frame-indep.
Time-ordering of two spacetime points x and x' is frame-indep if their separation is timelike,

$$(x-x')^2 < 0$$

Spacelike-separated x and x' can have different temporal ordering in different frames. $\therefore$ we require

$$[\mathcal{H}_I(x), \mathcal{H}_I(x')] = 0 \text{ whenever } (x-x')^2 > 0$$

Obviously, $[\varphi^+(x), \varphi^+(x')]_\mp = 0 = [\varphi^-(x), \varphi^-(x')]_\mp$.

However,

$$[\varphi^+(x), \varphi^-(x')]_\mp = \int \widetilde{dk}\, \widetilde{dk'}\, e^{ikx} e^{-ik'x'} [a(\vec{k}), a^\dagger(\vec{k}')]_\mp$$

$$= \int \frac{d^3k}{(2\pi)^3 \cancel{2\omega}} \frac{d^3k'}{\cancel{(2\pi)^3} 2\omega'} e^{ikx - ik'x'} \cancel{(2\pi)^3} \cancel{2\omega} \delta^3(\vec{k}-\vec{k}')$$

$$= \int \frac{d^3k\, d^3k'}{(2\pi)^3 2\omega'} e^{i\vec{k}\cdot\vec{x} - i\vec{k}'\cdot\vec{x}'} e^{-i\omega t + i\omega' t'} \delta^3(\vec{k}-\vec{k}')$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} e^{-i\omega(t-t')} \quad (=)$$

Go to a frame where $t - t' = 0$ so that $(x-x')^2 = -(t-t')^2 + (\vec{x}-\vec{x}')^2 = r^2 > 0$.

$$(=) \int \frac{d^3k}{(2\pi)^3 2\omega} e^{i\vec{k}\cdot\vec{r}}$$

$$= \frac{1}{16\pi^3} \int_0^\infty dq\, q^2 \underbrace{\underbrace{\int_{-1}^{1} d\psi}\; \underbrace{\int_0^{2\pi} d\varphi}_{2\pi}\; \underbrace{e^{iqr\psi}}} \frac{1}{\sqrt{q^2+m^2}}$$

$$= \frac{1}{iqr}\left(e^{iqr} - e^{-iqr}\right)$$

$$= \frac{1}{iqr}\, 2i \sin(qr)$$

$$= \frac{2}{qr} \sin(qr)$$

$$= \frac{1}{\cancel{16}_{4}\,\pi^{\cancel{3}2}} \int_0^\infty dq\, q^{\cancel{2}} \frac{\cancel{2}}{\cancel{q}r} \sin(qr)\, \cancel{2\pi} \frac{1}{\sqrt{q^2+m^2}}$$

$$= \frac{1}{4\pi^2 r} \int_0^\infty dq \frac{q \sin(qr)}{\sqrt{q^2+m^2}}, \quad p = \frac{q}{m}$$

$$= \frac{1}{4\pi^2 r} \int_0^\infty dp\, m \frac{\cancel{m}p \sin(p\, mr)}{\sqrt{\cancel{m^2}p^2 + \cancel{m^2}}}$$

$$= \frac{m}{4\pi^2 r} \int_0^\infty dt \frac{t \sin(mrt)}{\sqrt{t^2+1}}, \quad t = \sinh(u)$$

$$dt = \cosh(u)\, du$$

$$\int_0^\infty \to \int_0^\infty$$

$$= \frac{m}{4\pi^2 r} \int_0^\infty \cancel{\cosh(u)}\, du \, \frac{\sinh(u) \sin(mr \sinh(u))}{\cancel{\sqrt{\sinh(u)^2 + 1}}}$$

$$= \frac{m}{4\pi^2 r} \int_0^\infty du \sinh(u) \sin(mr \sinh(u))$$

See functions.wolfram.com/Bessel-TypeFunctions/BesselK/07/01/01/0005 :

$$K_\nu(x) = \operatorname{cosec}\left(\frac{\pi \nu}{2}\right) \int_0^\infty dt \sin(x \sinh(t)) \sinh(\nu t)$$

$$\therefore \int_0^\infty du \sinh(u) \sin(mr \sinh(u)) = \frac{K_1(mr)}{\operatorname{cosec}\left(\frac{\pi}{2}\right)}$$

$$= K_1(mr)$$

$$\therefore [\varphi^+(x), \varphi^-(x')]_\mp = \frac{m}{4\pi^2 r} K_1(mr)$$

$$=: C(r)$$

Note that $C(r) > 0 \quad \forall r > 0$. For small m, $K_1(mr) = \frac{1}{mr} + O(mr)^0$, so even for $m=0$, we have $mr\, K_1(mr) = 1$ and hence

$$[\varphi^+(x), \varphi^-(x')]_\mp = \frac{1}{4\pi^2 r^2}$$

which is never 0 $\therefore$ $\mathcal{H}_I(x)$, involving both φ^+ and φ^-, will not satisfy

$$[\mathcal{H}_I(x), \mathcal{H}_I(x')] = 0 \quad \text{for} \quad (x-x')^2 > 0$$

generically.

To resolve the problem, try using a particular linear combo of $\varphi^+(x)$ and $\varphi^-(x)$:

$$\varphi_\lambda(x) := \varphi^+(x) + \lambda\varphi^-(x)$$

$$\varphi_\lambda^\dagger(x) := \varphi^-(x) + \lambda^*\varphi^+(x)$$

where $\lambda \in \mathbb{C}$. Then,

$$[\varphi_\lambda(x), \varphi_\lambda^\dagger(x')]_\mp = [\varphi^+(x) + \lambda\varphi^-(x), \varphi^-(x') + \lambda^*\varphi^+(x')]_\mp$$

$$= [\varphi^+(x), \varphi^-(x')]_\mp + \lambda^*[\varphi^+(x), \varphi^+(x')]_\mp$$

$$+ \lambda[\varphi^-(x), \varphi^-(x')]_\mp + |\lambda|^2[\varphi^-(x), \varphi^+(x')]_\mp$$

$$= C(r) + |\lambda|^2 \left(\mp C(r) \right)$$

$$= (1 \mp |\lambda|^2)\, C(r)$$

$$[\varphi_\lambda(x), \varphi_\lambda(x')]_{\mp} = [\varphi^+(x) + \lambda\varphi^-(x), \varphi^+(x') + \lambda\varphi^-(x')]_{\mp}$$

$$= 0 \qquad = 0$$

$$= \lambda\,[\varphi^+(x), \varphi^-(x')]_{\mp} + \lambda\,[\varphi^-(x), \varphi^+(x')]_{\mp}$$

$$= \lambda\,[\, C(r) \mp C(r)\,]$$

$$= \lambda\,(1 \mp 1)\, C(r)$$

If we want $\varphi_\lambda(x)$ to either commute or anticommute w/ both $\varphi_\lambda(x')$ and $\varphi_\lambda^\dagger(x')$ at spacelike separations, then we must choose $|\lambda| = 1$ and commutators. Only then, we can find a suitable $H_I(x)$ by making it a hermitian function of $\varphi_\lambda(x)$.

But this has simply returned us to the theory of a real scalar $\because$ for

$\lambda = e^{i\alpha}$, $e^{-i\alpha/2}\varphi_\lambda(x)$ is hermitian.

In fact, if we make the replacements $a(\vec{k}) \to e^{i\alpha/2} a(\vec{k})$ and $a^\dagger(\vec{k}) \to e^{-i\alpha/2} a^\dagger(\vec{k})$, then the commutation relations of a and $a^\dagger$ are unchanged, and $e^{-i\alpha/2}\varphi_\lambda(x) = \varphi(x)$ $= \varphi^+(x) + \varphi^-(x)$. $\therefore$ our attempt to start w/ a and $a^\dagger$ as fundamental objects has simply led us back to the real, commuting, scalar field $\varphi(x)$ as the fundamental object.

Consider again $\varphi(x)$ as fundamental, w/ a Lagrangian given by some function of Lorentz scalars $\varphi(x)$ and $\partial^\mu\varphi\,\partial_\mu\varphi$. Then, quantization will result in $[\varphi(x), \varphi(x')]_\mp = 0$ for $t = t'$. If we choose anticommutators, then $\varphi^2 = 0 = (\partial_\mu\varphi)^2$, resulting in a trivial

$\mathcal{L}$ that is at most linear in φ and independent of $\dot{\varphi}$. This does not lead to the correct physics.

The situation generalizes to fields of higher spin, in any number of space-time dimensions. One choice of quantization always leads to a trivial $\mathcal{L}$, so this choice is not allowed. The allowed choice is always commutators for fields of integer spin and anticommutators for fields of half-integer spin.