2022 Summer Phys 135-3 Discussion Notes

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Contents

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(1) *From Sec. 16-3 in the textbook. The transverse velocity is zero at the top and bottom of the wave, which means that it is the highest in the middle and thus the kinetic energy is highest in the middle. The elastic potential energy is also the highest in the middle and lowest at the top and bottom. This means that the total energy in the middle is higher than at the top. This doesn't seem to make sense because energy has to be conserved.*

The answer is, energy is always conserved. The question is, how do we see that? Let's review the issue in two approaches: one that is intuitive, without any math, and the other that involves math but explicitly proves that *money* is conserved.

We have to look at this problem by contrasting it with the spring-potato system. Suppose I have a potato that is connected to an ideal spring on horizontal, frictionless rails so that the potato undergoes this periodic back-and-forth motion through out the time. Suppose initially I stretch the potato beyond its equilibrium point and release it from rest. Now, let's call the axis along which the rails are aligned the y axis. We know from mechanics that the position has a sinusoidal dependence on time, i.e.

$$y(t) = y_m \cos(\omega t) \tag{1}$$

(and confirm that $y(0)$ is the distance between the initial stretched position and the equilibrium position and that $\frac{dy}{dt}\big|_{t=0}$ satisfies the condition that the potato is at rest initially). When I plot this position, I see something as in Fig. 1.

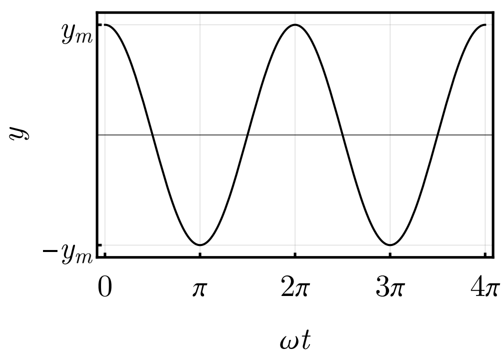

Figure 1:

This looks like Fig. 16-10 in the textbook (after all, both x and t are under the same cosine function). At the top and bottom points, the potato comes to rest momentarily and there appears the maximum restoring force that pulls the potato back toward the equilibrium point. At the equilibrium point, the potato has its maximum speed but no potential energy since the spring is not stretched anymore. Then, we see in Fig. 1 that there is a constant exchange between the kinetic and potential energies: If one of them increases, then the other ones must decrease.

However, this is not exactly the situation for an oscillating string. It is true that we have the maximum kinetic energy at the points where the transverse displacement vanishes (i.e. the middle points). But now these sections experience the highest elastic potential energy, as well. Accordingly, the top and bottom sections have the minimum potential energy. Here, we might think there is a problem with the conservation of energy.

It makes perfect sense when you think about it as follows: In Fig. 16-10 in the textbook, for example, we are looking at a snapshot of this string. But this guy is oscillating. How does the oscillation take place? Some external agent provides a driving force to the string and then there appears a tension throughout the string. At the middle sections, the elastic potential energy is highest because the tension pulls apart the line segments at these points. This tension will be used to bring down the adjacent top and bottom points to the middle part in the next point and time, while raising or lowering this middle section (You may say that the nodes do not move in a standing wave; in that case, just recall that a standing wave is just a linear summation of two waves going in opposite directions). Thus, this high energy concentrated at the middle sections are spent to bring these parts up and down in the next position and time instant. Your money is conserved here.

If you have been reading too carefully, you will see where we are heading with this: There must be some *energy flow* (very much like a *charge flow*, which we simply call *current*) which is spread throughout the string and which carries this *energy density* (very much like a charge density) from one point to the next along the string. How does this work?

I will do some math now, which may look a bit involved, but if you know how to write $\int_a^b dx\, f'g$ as $\left[fg\right]_a^b - \int dx\, fg'$ and how to approximate $(1+x)^n$ as $1+nx$ for small x, then you should be perfectly able to follow this discussion.

Suppose I have a string which is initially unstretched, so it just lies along the x axis. Suppose the transverse motion will be along the y axis, as in the textbook, when I start moving the string up and down to give some energy to it. Now, let's focus on a line segment on this string. Initially, we have Δx. When I swing the string up and down, there will be some tension moving across the string to create the oscillations. Suppose our line element gets stretched a bit and its length becomes $\Delta\ell$, as shown in Fig. 2.

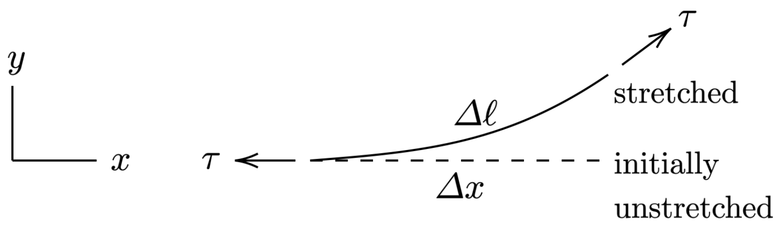

Figure 2:

The line segment has been drawn with some curvature but since we are looking at a very small segment, we can approximate it as a straight line. Moreover, we assume that the rise along the transverse direction is small, i.e. $\Delta y \ll \Delta x$. The kinetic energy of this segment is given by

$$\Delta K = \frac{1}{2}\Delta m\, u^2 \tag{2}$$

where Δm is the mass of this line segment and $u = \partial y/\partial t$ is the transverse velocity of this wave motion. Now, we can get the potential energy of this segment from the work done on it by the tension force. Initially, the length of the segment is Δx and after stretching with the tension force τ from both ends, the length becomes $\Delta\ell = \sqrt{\Delta x^2 + \Delta y^2}$ (from geometry). We have said that Δy is much less than Δx, so it's not exactly zero. Then, the change in the length of the segment is $\Delta\ell - \Delta x$ and the work done on it is

$$\Delta U = \tau(\Delta\ell - \Delta x) \tag{3}$$

Thus, the total energy of this segment is

$$\begin{aligned}
\Delta E &= \Delta K + \Delta U \\
&= \frac{1}{2}\Delta m\, u^2 + \tau(\Delta\ell - \Delta x) \\
&= \frac{1}{2}(\mu\Delta\ell)\left(\frac{\partial y}{\partial t}\right)^2 + \tau(\Delta\ell - \Delta x) \\
&= \frac{1}{2}\mu\sqrt{\Delta x^2 + \Delta y^2}\left(\frac{\partial y}{\partial t}\right)^2 + \tau\left(\sqrt{\Delta x^2 + \Delta y^2} - \Delta x\right) \\
&= \frac{1}{2}\mu\sqrt{\Delta x^2 + \Delta y^2}\left(\frac{\partial y}{\partial t}\right)^2 + \tau\left[\Delta x\left(1 + \frac{\Delta y^2}{\Delta x^2}\right)^{1/2} - \Delta x\right] \\
&\approx \frac{1}{2}\mu\,\Delta x\left(\frac{\partial y}{\partial t}\right)^2 + \tau\left\{\Delta x\left[1 + \frac{1}{2}\left(\frac{\Delta y}{\Delta x}\right)^2\right] - \Delta x\right\} \\
&\approx \frac{1}{2}\mu\,\Delta x\left(\frac{\partial y}{\partial t}\right)^2 + \frac{1}{2}\tau\,\Delta x\left(\frac{\partial y}{\partial x}\right)^2 \\
&\approx \left[\frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 + \frac{1}{2}\tau\left(\frac{\partial y}{\partial x}\right)^2\right]\Delta x
\end{aligned} \tag{4}$$

where you know that $y = y(x,t) = y_m \sin(kx - \omega t)$. Now finally, the total energy of the string is just the integral of this expression over its length:

$$E = \int_a^b dE = \int_a^b dx\left[\frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 + \frac{1}{2}\tau\left(\frac{\partial y}{\partial x}\right)^2\right] \tag{5}$$

We call the first term the *kinetic energy density* and the second term the *potential energy density* (because these are the quantities that give the corresponding energies when integrated over the length of the string).

In the next section of the textbook, i.e. in Sec. 16-4, it discusses the wave equation:

$$\mu\frac{\partial^2 y}{\partial t^2} = \tau\frac{\partial^2 y}{\partial x^2} \tag{6}$$

Now we are going to use this to show that the total energy is constant. We know that a quantity is constant in physics if its time derivative vanishes:

$$\frac{dE}{dt} = 0 \tag{7}$$

Then, we want to show that

$$\frac{d}{dt}\int_a^b dx \left(\frac{1}{2}\mu\dot{y}^2 + \frac{1}{2}\tau y'^2\right) = \int_a^b dx \left[\frac{\partial}{\partial t}\left(\frac{1}{2}\mu\dot{y}^2\right) + \frac{\partial}{\partial t}\left(\frac{1}{2}\tau y'^2\right)\right] \tag{8}$$

where I've introduced some shorthand notation. I use overdot for time derivative and prime for position derivative. Since these are partial derivatives, we won't have any technical problems due to the order of said derivatives. With this notation, the wave equation becomes

$$\mu\ddot{y} = \tau y'' \tag{9}$$

Multiply both sides by the transverse speed, $u = \dot{y}$:

$$\mu\dot{y}\ddot{y} = \tau\dot{y}y'' \tag{10}$$

Recall the chain rule of derivative, i.e. I can write $\frac{\partial}{\partial t} fg = \dot{f}g + f\dot{g}$, so we see that $\dot{y}\ddot{y} = \frac{1}{2}\dot{y}\ddot{y} + \frac{1}{2}\ddot{y}\dot{y} = \frac{1}{2}\frac{\partial(\dot{y}^2)}{\partial t}$, so the left-hand side of Eq. (10) becomes

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\mu\dot{y}^2\right) = \tau\dot{y}y'' \tag{11}$$

Recognize the left-hand side as the time derivative of the kinetic energy density. Now, integrate both sides over x:

$$\begin{aligned}
\int_a^b dx \frac{\partial}{\partial t}\left(\frac{1}{2}\mu\dot{y}^2\right) &= \int_a^b dx\, \tau\dot{y}y'' \\
&= \left[\tau\dot{y}y'\right]_a^b - \int_a^b dx\, \tau\dot{y}'y' \\
&= \left[\tau\dot{y}y'\right]_a^b - \int_a^b dx\, \tau\left(\frac{1}{2}\dot{y}'y' + \frac{1}{2}y'\dot{y}'\right) \\
&= \left[\tau\dot{y}y'\right]_a^b - \int_a^b dx\, \tau\frac{1}{2}\frac{\partial(y'^2)}{\partial t}
\end{aligned} \tag{12}$$

or

$$\int_a^b dx \frac{\partial}{\partial t}\left(\frac{1}{2}\mu\dot{y}^2\right) + \int_a^b dx \frac{\partial}{\partial t}\left(\frac{1}{2}\tau\dot{y}^2\right) = \left[\tau\dot{y}y'\right]_a^b \tag{13}$$

and finally

$$\frac{d}{dt}\int_a^b dx \left(\frac{1}{2}\mu\dot{y}^2 + \frac{1}{2}\tau y'^2\right) = \left[\tau\dot{y}y'\right]_a^b \tag{14}$$

The left-hand side is simply dE/dt. What the heck is the right-hand side? We call this term the *surface term* because it is evaluated at the boundaries (It's called a surface term because we like to imagine things in three dimensions in general). What do we have at the boundaries? There are a couple of options.

Suppose we have an infinitely long string and the *action* happens at a finite portion of it that we can observe (which is the case for *localized* particles in atoms or in traps in quantum mechanics). In that case, the string doesn't move at the ends, i.e. the transverse speed will be zero at both end points a and b, i.e. $\dot{y}(a) = \dot{y}(b) = 0$, so the surface term drops and we see that $dE/dt = 0$.

Suppose we have an infinitely long string but it's oscillating everywhere, not just at a finite portion of it (which is the case in the particle-collision experiments). Then, we model our universe as a *big box* of side L, having periodic ends (i.e. whatever happens at $x = 0$ is happening at $x = L$). This periodicity makes the surface term vanish again.

Suppose we have a finite string but we clamp it at both ends. Then, we'll have a situation like the first case above, so the surface term will drop again.

Suppose we have a finite string but with open/free ends. Free ends don't move by themselves, so the transverse velocity is zero at the ends, so the surface term drops again.

In all these cases, we see that the surface term drops so that

$$\frac{dE}{dt} = 0 \tag{15}$$

so our money is constant. But what happens when the right-hand side does not vanish? This may be the case for a finite string with one end oscillating all the time to keep the string in harmonic motion. Then, the transverse momentum at one end may not match the other, depending on how you hold or fix the string, so we see that the right-hand side can become just some number:

$$\frac{dE}{dt} = \text{constant, } C \tag{16}$$

In this case, we see that the energy is linearly increasing in time. But that's totally expected. There is someone providing constant energy to the string. Then, we have to imagine this string plus this external agent as a system and analyze them together. But again, in that case, there is some energy generated at some point and this energy is imparted to the string to make it oscillate. So, the energy is conserved again.

We have this quantity under the integral as the energy density:

$$\mathcal{E} = \frac{1}{2}\mu\dot{y}^2 + \frac{1}{2}\tau y'^2 \tag{17}$$

(It's called a density because its x integration gives the total energy.) When we trace our finger over the string in Fig. 16-10, we are actually looking at this object. You have every right to believe that the energy is not conserved in different parts of the body of the string. In fact, it goes for this specific case like cosine squared, so whenever y becomes zero, the energy density becomes maximum and whenever y becomes a top or bottom point, the energy density becomes zero. See Fig. 3 but don't consider the scale too seriously.

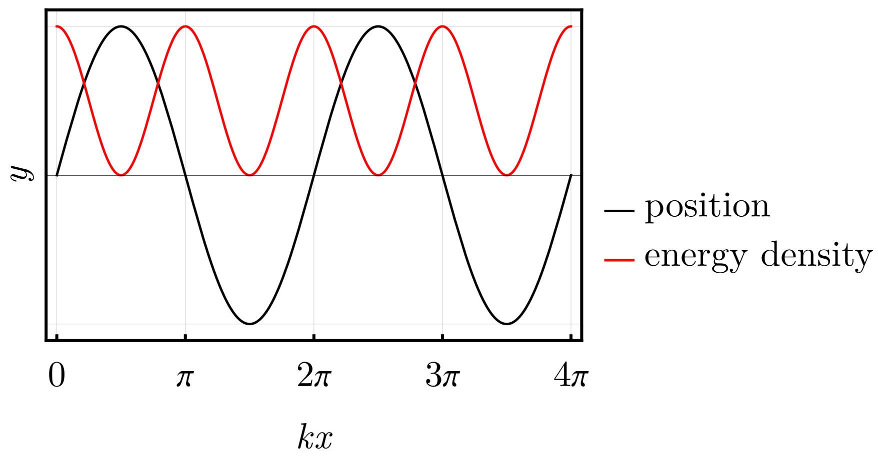

Figure 3:

(2) *MW Assignments Question 4*

Let us draw a free-body diagram of the rope at two different points, say A and B as in Fig. 4.

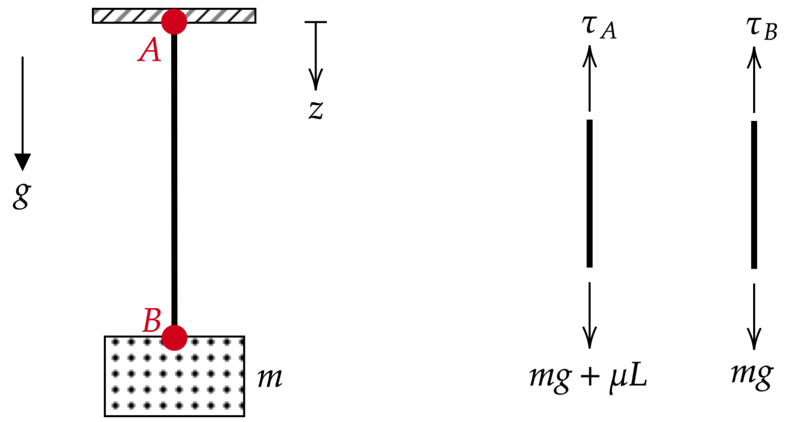

Figure 4:

Let's look at point A. The tension at that point will feel the weight of the entire rope and also the mass m. At point B, the tension will try to counter only the weight of the mass m. Thus, we see that the tension has a profile $\tau = mg + \mu(L - z)$. Since the tension gets bigger and bigger as we move up the rope, we see naively from the relation $v = \sqrt{\tau/\mu}$ that the speed should also increase.