Names: Mingji Chen
Kevin Guillen

Topology - Summer Session 1 - HW4 - July 23, 2021

(6.1) Write down a homotopy equivalence between $(0,1)$ and $[0,1]$.

Proof. First let's define

$$f : [0,1] \to (0,1)$$
$$x \mapsto \frac{x+1}{3}$$

Then we will define

$$g : (0,1) \hookrightarrow [0,1]$$
$$x \mapsto x.$$

It's obvious that both of these are continuous functions, and we see $g \circ f \cong Id_{[0,1]}$ by,

$$H : [0,1] \times [0,1] \to [0,1]$$
$$(x,t) \mapsto x \cdot t + (1-t) \cdot \left(\frac{x+1}{3}\right)$$

We can see $H(x,0) = g \circ f$ and $H(x,1) = x, \ \forall x \in [0,1]$.

We can also see $f \circ g \cong Id_{(0,1)}$ by,

$$H : (0,1) \times [0,1] \to (0,1)$$
$$(x,t) \mapsto x \cdot t + (1-t) \cdot \left(\frac{x+1}{3}\right)$$

We can see $H(x,0) = f \circ g$ and $H(x,1) = x, \ \forall x \in (0,1)$.

Thus f and g is a homotopy equivalence between $(0,1)$ and $[0,1]$ $\square$

(6.2) List all homotopy classes of maps $(0,1) \to (0,1)$

Proof. Since $(0,1)$ is contractible, then there exists a homotopy H between the Id and a constant point $\{0\}$. Now consider $f : (0,1) \to (0,1)$. Then $H \circ f : (0,1) \times I \to (0,1)$ is a homotopy between f and the constant map. Since f is homotopic to the constant map and the constant map is homotopic to the Id, $f \simeq Id$, therefor there is only on homotopy class. $\square$

(6.3) Prove that a discrete space consisting of m points in homotopy equivalent to a discrete space consisting of n points if, and only if, $m = n$.

Proof. If $m \neq n$, denote these points as $X = \{x_1, x_2, ..., x_m\}$ and $Y = \{y_1, y_2, ..., y_n\}$. Assume then that $m > n$ then at least two points of X maps to one point of Y. Without loss of generality we have,

$$\begin{aligned} f : X &\to Y & g : Y &\to X \\ x_1 &\mapsto y_1 & y_1 &\mapsto x_1 \\ x_m &\mapsto y_1 & y_1 &\mapsto x_m \\ x_i &\mapsto y_i & y_j &\mapsto x_i \end{aligned}$$

Then we have,

$$\begin{aligned} H : X \times I &\to X \\ (x, 0) &\mapsto g \circ f(x) \\ (x, 1) &\mapsto x \end{aligned}$$

then we have $g \circ f(x_1) = g \circ f(x_1)\ g(y_1)$.

For a fixed $x \in X$ the path $r : I \to X$, $t \mapsto H(x, t)$ is continuous. Since I is connected and X is discrete then r must be constant.

We have then $x_1 = r(0) = r(1) = x_m$, each point in Y has only one preimage and vice versa, therefore $m = n$.

On the other hand if $m = n$ then $X = \{x_1, x_2, ...x_n\}$ and $Y = \{y_1, y_2, ...y_n\}$. Let $f : X \to Y$, $x_i \mapsto y_i$ and $g : Y \to X$, $y_i \mapsto x_i$. Then it is obvious that $f \circ g \simeq Id_Y$ and $g \circ f \simeq Id_X$

□

(6.5) Show that the map $f : S^1 \to S^1$ given by $f(x, y) = (-x, -y)$ is homotopic to the identity map.

Proof. For $f : S^1 \to S^1$, $(x, y) \mapsto (-x, -y)$. We can consider the homotopy H as follows,

$$\begin{aligned} H : S^1 \times [0, 1] &\to S^1 \\ ((cos(\theta), sin(\theta)), t) &\mapsto (cos(\theta + (1 - t)\pi), sin(\theta + (1 - t)\pi)) \end{aligned}$$

We can see that $H((cos(\theta), sin(\theta)), 0) = (cos(\theta + \pi), sin(\theta + \pi))$, this is rotating the point around the circle 180 degrees, which means it is sending it to it's antipodal point, therefore it is equal to f. Now for $H((cos(\theta), sin(\theta)), 1) = (cos(\theta + 0), sin(\theta + 0))$ which is just the identity map. Therefore $f \simeq Id_{S^1}$ □