Homework 2

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Note: I used Symbolab to help do the matrix multiplication for P3 so, I apologize if I didn't write all the work required.

Problem P1 Let F be any field , $n \geqslant 0$ an integer, V and $n-$dimensional $F-$vector space. For any integer k such that $0 \leqslant k \leqslant n$, let G_k denote the set of $k - dimensional$ $F-$subspaces W of V
Prove that the action of $GL_F(V)$ on G_k given by

$$g.W = \{g(w) : w \in W\} \in G_k$$

for all $g \in GL_F(V)$, is transitive.

Proof. We defined in class $GL_F(V)$ as,

$$GL_F(V) = \{f : V \to V \mid F - \text{linear isomorphisms}\}$$

To show that the provided action is transitive we want to show that for any two subspaces $W, W' \in G_k$ there exists some $f \in GL_F(V)$ that satisfies:

$$f.W = f(W) = W'$$

Since both W and W' are in G_k we know they are of dimension k. Meaning that their bases can be expressed as $\{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_k\}$ for W and W' respectively. We know that linear transformations map subspaces to subspaces and there exists $f \in GL_F(V)$ such that,

$$f(\alpha_1 a_1 + \cdots + \alpha_k a_k) = \alpha_1 b_1 + \cdots + \alpha_k b_k$$

we know this is indeed in $GL_F(V)$ because we can see it is a linear transformation through the following. For any $w_1, w_2 \in W$, where $w_1 = (\alpha_1 a_1 + \cdots + \alpha_k a_k)$ and $w_2 = (\gamma_1 a_1 + \cdots + \gamma_k a_k)$ and α_i and γ_i are scalars, we have

$$\begin{aligned}
f(w_1 + w_2) &= f((\alpha_1 a_1 + \cdots + \alpha_k a_k) + (\gamma_1 a_1 + \cdots + \gamma_k a_k)) \\
&= f((\alpha_1 + \gamma_1)a_1 + \cdots + (\alpha_k + \gamma_k)a_k) \\
&= (\alpha_1 + \gamma_1)b_1 + \cdots + (\alpha_k + \gamma_k)b_k \\
&= (\alpha_1 b_1 + \cdots + \alpha_k b_k) + (\gamma_1 b_1 + \cdots + \gamma_k b_k) \\
&= f(\alpha_1 a_1 + \cdots + \alpha_k a_k) + f(\gamma_1 a_1 + \cdots + \gamma_k a_k) \\
&= f(w_1) + f(w_2)
\end{aligned}$$

and for $c \in F$,

$$\begin{aligned} cf(w_1) = cf(\alpha_1 a_1 + \cdots + \alpha_k a_k) = c(\alpha_1 a_1 + \cdots + \alpha_k a_k) &= c\alpha_1 b_1 + \cdots + c\alpha_1 b_1 \\ &= f(c(\alpha_1 a_1 + \cdots + \alpha_k a_k)) \\ &= f(cw_1) \end{aligned}$$

meaning we have $f(W) = W' = f.W$. then from our corollary we proved in class (Jan 18), we have that f is an isomorphism, since it maps all subspaces to subspaces, and thereby must be in $GL_F(V)$. Showing that the action is transitive. $\square$

Problem P2 Let $F = \mathbb{F}_{17}$ be the field with 17 elements. For any integer n, we will denote still by n its image in F.
Apply Gaussian elimination to find all the solutions to the linear system

$$\begin{aligned} 2x + 3y + 5z &= 10 \\ 4x + 5y + 8z &= 11 \\ 2x + 4y + 7z &= 2 \end{aligned}$$

Proof. We begin by writing our system of equations in matrix form,

$$\begin{bmatrix} 2 & 3 & 5 & 10 \\ 4 & 5 & 8 & 11 \\ 2 & 4 & 7 & 2 \end{bmatrix}$$

Next we will label under each matrix the operation we will be performing (Rn represents the nth row, where $x \cdot Rn$ is to mean multiply the nth row by x),

$$\underset{R2 = R2 - 2\cdot R3}{\begin{bmatrix} 2 & 3 & 5 & 10 \\ 4 & 5 & 8 & 11 \\ 2 & 4 & 7 & 2 \end{bmatrix}} \to \underset{R3 = R3 - R1}{\begin{bmatrix} 2 & 3 & 5 & 10 \\ 0 & 14 & 11 & 7 \\ 2 & 4 & 7 & 2 \end{bmatrix}} \to \underset{R3 = 14\cdot R3 - R2}{\begin{bmatrix} 2 & 3 & 5 & 10 \\ 0 & 14 & 11 & 7 \\ 0 & 1 & 2 & 9 \end{bmatrix}} \to \underset{R2 = 4\cdot R2}{\begin{bmatrix} 2 & 3 & 5 & 10 \\ 0 & 14 & 11 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

$$\to \underset{R1 = 5\cdot R1 - 3\cdot R2}{\begin{bmatrix} 2 & 3 & 5 & 10 \\ 0 & 5 & 10 & 11 \\ 0 & 0 & 0 & 0 \end{bmatrix}} \to \underset{R1 = 12\cdot R1}{\begin{bmatrix} 10 & 0 & 12 & 10 \\ 0 & 5 & 10 & 11 \\ 0 & 0 & 0 & 0 \end{bmatrix}} \to \underset{R2 = 7\cdot R2}{\begin{bmatrix} 1 & 0 & 8 & 0 \\ 0 & 5 & 10 & 11 \\ 0 & 0 & 0 & 0 \end{bmatrix}} \to \underset{R2 = 7\cdot R2}{\begin{bmatrix} 1 & 0 & 8 & 0 \\ 0 & 1 & 2 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

This then gives us,

$$\begin{aligned} x + 8z &= 0 \\ y + 2z &= 9 \end{aligned}$$

solving for our leading variables in terms of z

$$\begin{aligned} x &= 9z \\ y &= 9 + 15z. \end{aligned}$$

Which means the set of all solutions for the linear system is the following,

$$\{(9z, 9 + 15z, z) \mid z \in \mathbb{F}_{17}\}$$

$\square$

Problem P3 Consider the positively oriented orthonormal vectors in $V = \mathbb{R}^3$:

$$v_1 = \frac{1}{\sqrt{2}}(1,-1,0),\ v_2 = \frac{1}{\sqrt{3}}(1,1,1),\ \text{and } v_3 = v_1 \times v_2$$

(the vector, or cross, product)
Let T be the rotation of $V = \mathbb{R}^3$ about the axis v_3 by $90°$

(1) Compute the matrix $[T]_{\mathcal{B}'} = [T]_{\mathcal{B}'}^{\mathcal{B}'}$ with respect to the basis

$$\mathcal{B}' = (v_1, v_2, v_3)$$

(2) Compute the matrix of T with respect to the standard basis $\mathcal{B} = (e_1, e_2, e_3)$

Proof. First we must compute v_3 which evaluates to be,

$$v_3 = \left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \sqrt{\frac{2}{3}}\right) = \frac{1}{\sqrt{6}}(-1,-1,2)$$

Now because we are rotating only 90 degrees about v_3 with positively oriented orthonormal vectors, we know v_3 should remain unchanged after our rotation. While $T(v_1) = v_2$ and $T(v_2) = -v_1$, putting this together we have,

$$\begin{aligned} T(v_1) &= 1v_2 \\ T(v_2) &= -1v_1 \\ T(v_3) &= 1v_3 \end{aligned}$$

which means we have $[T]_{\mathcal{B}'}^{\mathcal{B}'} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Now let $P : V \to V$ be $P(v) = v$. Now we need to calculate $[P]_{\mathcal{B}'}^{\mathcal{B}}$,

$$\begin{aligned} P(v_1) &= \frac{1}{\sqrt{(2)}}(1,-1,0) = \frac{1}{\sqrt{2}}e_1 - \frac{1}{\sqrt{2}}e_2 \\ P(v_2) &= \frac{1}{\sqrt{3}}(1,1,1) = \frac{1}{\sqrt{3}}e_1 + \frac{1}{\sqrt{3}}e_2 + \frac{1}{\sqrt{3}}e_3 \\ P(v_3) &= \frac{1}{\sqrt{6}}(-1,-1,2) = -\frac{1}{\sqrt{6}}e_1 - \frac{1}{\sqrt{6}}e_2 + \frac{2}{\sqrt{6}}e_3 \end{aligned}$$

all together gives us,

$$[P]_{\mathcal{B}'}^{\mathcal{B}} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{bmatrix}$$

which then means,

$$\frac{1}{[P]_{\mathcal{B}'}^{\mathcal{B}}} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{2} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix}.$$

Recall though that,

$$[\mathsf{T}]_{\mathcal{B}'}^{\mathcal{B}'} = \frac{1}{[\mathsf{P}]_{\mathcal{B}'}^{\mathcal{B}}}[\mathsf{T}]_{\mathcal{B}}^{\mathcal{B}}[\mathsf{P}]_{\mathcal{B}'}^{\mathcal{B}},$$

so solving for $[\mathsf{T}]_{\mathcal{B}}^{\mathcal{B}}$ we get,

$$[\mathsf{T}]_{\mathcal{B}}^{\mathcal{B}} = [\mathsf{P}]_{\mathcal{B}'}^{\mathcal{B}}[\mathsf{T}]_{\mathcal{B}'}^{\mathcal{B}'}\frac{1}{[\mathsf{P}]_{\mathcal{B}'}^{\mathcal{B}}}$$

which is what we want to compute, so now we can plug in what we know to get,

$$\begin{aligned}
[\mathsf{T}]_{\mathcal{B}}^{\mathcal{B}} &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{2} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}}. \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{6} & \frac{-2\sqrt{6}+1}{6} & \frac{-\sqrt{6}-3}{3\sqrt{6}} \\ \frac{2\sqrt{(6)}+1}{6} & \frac{1}{6} & \frac{3-\sqrt{6}}{3\sqrt{6}} \\ \frac{3-\sqrt{6}}{3\sqrt{6}} & \frac{-\sqrt{6}-3}{3\sqrt{6}} & \frac{2}{3} \end{bmatrix}.
\end{aligned}$$

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