11-29 Submission

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MATH 101 — Problem Solving — Fall 2021

Problem IC — 11-22 — PP2 Show that every positive integer is a sum of one or more numbers of the form 2^r3^s, where r and s are nonnegative integers and no summand divides another. (For example 23 = 9 + 8 + 6.)

Proof. We see for 0 the set of its sums of the form 2^r3^s, $S(0) = \varnothing$. For $S(1) = \{1\}$.

Assume this hold for all $k < n$.

Now in the case that $k = n$, if n is even,

$$S(n) = \left\{2a \mid a \in S(\frac{n}{2})\right\}$$

which is comprised of elements of the form 2^r3^s since $\frac{n}{2} < n$.

In the case that n is odd, we can take the greatest $\alpha \in \mathbb{N}$ such that $3^\alpha < n$ and we get,

$$S(n) = \{3^\alpha\} \cup S(\frac{n - 3^\alpha}{2})$$

which also holds due this being true for $k < n$. □

Problem IC — 11-22 — PP14 For which real numbers c is there a straight line that intersects the curve

$$x^4 + 9x^3 + cx^2 + 9x + 4$$

in four distinct points.

Proof. We need this function to have two inflection points for then we know there is a line that will intersect it at 4 distinct points. To do this we have to look at its 2nd derivative which is,

$$f''(x) = 12x^2 + 54x + 2c.$$

Now we need to solve for when the discriminant of the 2nd derivative is greater than zero, that way we have two distinct real roots.

$$\begin{aligned} b^2 - 4ac &> 0 \\ 54^2 - 4 \cdot 12 \cdot 2c &> 0 \\ 2916 - 96c &> 0 \end{aligned}$$

This is only true for $c \in (-\infty, 30.375)$

□

Problem IC — 11/29 — PP15 A square of side $2a$, always lying in the first quadrant, moves so that two consecutive vertices are always on the $x-$ and $y-$axes. Fin the locus of the center of the square.

Proof. The locus of the center of the square is clearly on the line $y = x$. Now we just need to find the minimum part of the line. This is when both vertices are on the same axis, say the y-axis. The center in which case is (a, a). Now for the maximum. Continuing with the position mentioned, if the vertex at the origin begins to move to the right, and the vertex strictly on the on the y axis, the center reaches its maximum when the restricted vertices are placed at $(0, a)$ and $(a, 0)$. This means are center will be $\sqrt{a^2 + a^2} = \sqrt{2a^2} = \sqrt{2}a$ far from the origin. Thus the locus of the center of the square is on $y = x$ for $x \in [a, \sqrt{2}a]$ □

Problem OC — 11/24 — PP21 A class with 2N students score $1, 2, \ldots, 10$. Each of these score occurred at least once, the average was 7.4. Show that the group can be divided into two groups such that the average is 7.4.

Proof. First let us consider the sum of all the scores which will simply be $S = x_1 + x_2 + \cdots + x_{2N} = (7.4)2N$. Which we can express as,

$$S = \frac{5}{5}(7.4)2N = \frac{74N}{5}.$$

This gives us that 5 divides N and that the total sum is even. Now let $x_1, x_2, \ldots x_{2N}$ be the sequence of scores in ascending order. Next let us define $y_k = x_{2k} - x_{2k-1}$. We know the y_k will either be 1 or 0, this is because for any two consecutive scores we have that they will be equal or 1 apart and since every score occurs at least once. Let S' be the sum of $y_1 + y_2 + \ldots y_N$. Because S is even, S' must also be even. This means there is some $m < N$, such that,

$$y_1 + y_2 + \cdots + y_m = \frac{S'}{2}.$$

Now if we consider the scores of $x_2, x_4, \ldots x_{2m}, x_{2m+1}, \ldots, x_{2N-1}$ and their sum denoted by T,

$$\begin{aligned}
T &= x_2 + x_4 + \cdots + x_{2m} + x_{2m+1} + \cdots + x_{2N-1} && (y_k = x_{2k} - x_{2k-1}) \\
T &= (y_1 + x_1) + \ldots (y_m + x_{2m-1}) + x_{2m+1} + \cdots + x_{2N-1} \\
T &= (y_1 + y_2 + \cdots + y_m) + (x_1 + x_3 + \ldots x_{2m+1} + \cdots + x_{2N-1}) && \left(y_1 + \cdots + y_m = \frac{S'}{2}\right) \\
T &= \frac{1}{2}(S' + 2x_1 + 2x_3 + \cdots + x_{2N-1}) && (y_1 + y_2 + \cdots + y_N = S') \\
T &= \frac{1}{2}(((y_1 + x_1) + \cdots + (y_N + x_{2N-1})) + (x_1 + x_3 + \cdots + x_{2N-1})) && (y_k = x_{2k} - x_{2k-1}) \\
T &= \frac{1}{2}(x_2 + \cdots + x_{2N} + x_1 + x_3 \ldots x_{2N-1}) \\
T &= \frac{1}{2}(x_1 + x_2 + \cdots + x_{2N}) \\
T &= \frac{1}{2}S
\end{aligned}$$

But $\frac{1}{2}S$ is simply $7.4 \cdot N$. Thus the average of this collection of student scores is 7.4 and as a consequence the average of the student's scores not in this group must also be 7.4. Meaning we have broken the students into two groups where the average is still the same. $\square$