Math201 PSet 1

[DF] = Dummit and Foote.

$*$ = optional with extra credit.

P1 Let R be a ring and M be a left R-module.

(a) Let $N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots$ be an ascending chain of R-submodules in M. Prove that the union $\cup_{j=1}^{\infty} N_j$ is an R-submodule of M.

(b) Let $R = \mathscr{C}(\mathbb{R})$ denote the ring of (real-valued) continuous functions on $\mathbb{R}$, with pointwise addition and multiplication (as in class). Define

$$\mathscr{C}_c(\mathbb{R}) = \{f \in \mathscr{C}(\mathbb{R}) : \exists N = N(f) \in \mathbb{N} \text{ such that } f(x) = 0 \text{ for all } |x| > N\}$$

Prove that $\mathscr{C}_c(\mathbb{R})$ is an R-submodule of R. Is it a subring?

P2 Let M be a left R-module. The *annihilator* of M in R is defined as:

$$Ann_R(M) = \{r \in R : rm = 0 \text{ for all } m \in M\}.$$

(a) Prove that $Ann_R(M)$ is a bilateral ideal of R.

(b) If M_1 and M_2 are two left R-modules, prove that

$$Ann_R(M_1 \times M_2) = Ann_R(M_1) \cap Ann_R(M_2).$$

(c) Compute $Ann_R(M)$ when $R = \mathbb{Z}$ and $M = (\mathbb{Z}/112\mathbb{Z})^{\times}$ is the multiplicative abelian group of units in $\mathbb{Z}/112\mathbb{Z}$. (*Hint.* Use the Chinese remainder theorem and part (b).)