Homework 4

Kevin Guillen

MATH 103A — Complex Analysis — Spring 2022

Problem 4.1 Prove that the function

$$f(z) = e^{-\theta}\cos(\ln r) + ie^{-\theta}\sin(\ln r)$$

is differentiable when $r > 0$ and $0 < \theta < 2\pi$, and find $f'(z)$ in terms of $f(z)$.

Proof. We can use Theorem 7.4 to show that this function is differentiable under the given conditions for r and θ. We first see that,

$$f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$$

where $u(r,\theta) = e^{-\theta}\cos(\ln r)$ and $v(r,\theta) = e^{-\theta}\sin(\ln r)$. Meaning we must show that both these functions' partial derivatives exist with respect to r and θ, are continuous, and the CR equations are satisfied. We see the partial derivatives are,

$$u_r = -\frac{e^{-\theta}\sin(\ln r)}{r} \qquad v_r = \frac{e^{-\theta}\cos(\ln r)}{r}$$
$$u_\theta = -e^{-\theta}\cos(\ln r) \qquad v_\theta = -e^{-\theta}\sin(\ln r)$$

which we know are continuous when $r > 0$ and $\theta \in (0, 2\pi)$. We also have that $ru_r = v_\theta$ and $u_\theta = -rv_r$, meaning then that this function is differentiable when $r > 0$ and $\theta \in (0, 2\pi)$.

We know by by Discussion 7.7 that $f'(re^{i\theta}) = e^{-i\theta}(u_r(r,\theta) + iv_r(r,\theta))$, applying this to what we have we get,

$$\begin{aligned}
f'(z) = f'(re^{i\theta}) &= e^{-i\theta}\left(-\frac{e^{-\theta}\sin(\ln r)}{r} + i\frac{e^{-\theta}\cos(\ln r)}{r}\right) \\
&= \frac{e^{-i\theta}}{r}(-e^{-\theta}\sin(\ln r) + ie^{-\theta}\cos(\ln r)) \\
&= \frac{e^{-i\theta}}{r}(i^2e^{-\theta}\sin(\ln r) + ie^{-\theta}\cos(\ln r)) \\
&= i\frac{e^{-i\theta}}{r}(ie^{-\theta}\sin(\ln r) + e^{-\theta}\cos(\ln r)) \\
&= i\frac{e^{-i\theta}}{r}f(re^{i\theta}) \\
&= i\frac{e^{-i\theta}}{r}f(z)
\end{aligned}$$

as desired. $\square$

Problem 4.2 Let $f = u + iv$ be a complex-valued function defined on an open set $G \subseteq \mathbb{C}$. Suppose that the first-order partial derivatives of $\operatorname{Re} f = u$ and $\operatorname{Im} f = v$ exist and are continuous on G.

(a) Recall that if $z = x + iy$, then

$$x = \frac{z + \bar{z}}{2} \quad \text{and} \quad y = \frac{z - \bar{z}}{2i}$$

Treat $f = f(x, y)$ as a function in two real-variables, and *formally* apply the chain rule in Calculus to obtain the expressions

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \quad \text{and} \quad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

Solution. By chain rule we have $\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z}$. Before manipulating this, we evaluate the following terms as,

$$\frac{\partial x}{\partial z} = \frac{1}{2} \qquad\qquad \frac{\partial y}{\partial z} = \frac{1}{2i}$$

now plugging in and doing some algebra to our original equation given to us by the chain rule we get,

$$\begin{aligned}
\frac{\partial f}{\partial z} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} \\
&= \frac{\partial f}{\partial x}\frac{1}{2} + \frac{\partial f}{\partial y}\frac{1}{2i} \\
&= \frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{1}{i}\right) \\
&= \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)
\end{aligned}$$

as desired.

Again by chain rule we have $\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}}$. Like before we note that,

$$\frac{\partial x}{\partial \bar{z}} = \frac{1}{2} \qquad\qquad \frac{\partial y}{\partial \bar{z}} = -\frac{1}{2i}.$$

Finally plugging this into the equation given by the chain rule and doing some algebra we get,

$$\begin{aligned}
\frac{\partial f}{\partial \bar{z}} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}} \\
&= \frac{\partial f}{\partial x}\frac{1}{2} - \frac{\partial f}{\partial y}\frac{1}{2i} \\
&= \frac{1}{2}\left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\frac{1}{i}\right) \\
&= \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)
\end{aligned}$$

as desired.

$\square$

(b) Define $\frac{\partial f}{\partial x} \coloneqq \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$, and similarly for $\frac{\partial f}{\partial y}$. Prove that f is holomorphic on G if and only if $\frac{\partial f}{\partial \bar{z}} = 0$.

Proof. ($\Leftarrow$) First we will move in the reverse direction and assume that $\frac{\partial f}{\partial \bar{z}} = 0$. Recall though that we obtained the expression $\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$. Meaning we have,

$$\begin{aligned} 0 &= \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \\ &= \frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} \\ &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) \\ 0 &= \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \end{aligned}$$

which implies the following two equalities,

$$\begin{aligned} 0 &= \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} & \qquad 0 &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial y} &= \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \\ v_y &= u_x & u_y &= -v_x \end{aligned}$$

meaning we not only have that the first order partial derivatives exist, but also that the Cauchy-Riemann equations are satisfied, giving us that f is holomorphic as desired.

($\Rightarrow$) With the work we did in the reverse direction, we can easily prove the forward direction by following what we did in reverse order. We assume that f is holomorphic meaning the Cauchy-Riemann equations are satisfied giving us that,

$$\begin{aligned} v_y &= u_x & \qquad u_y &= -v_x \\ 0 &= u_x - v_y & 0 &= v_x + u_y \\ 0 &= \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} & 0 &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \end{aligned}$$

Recall we defined the following $\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$ and similarly for $\frac{\partial f}{\partial y}$, plugging these into the expression we obtained for $\frac{\partial f}{\partial \bar{z}}$ from the previous part we have,

$$\begin{aligned} \frac{\partial f}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \\ &= \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\right) \\ &= \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}(0 + i0) \\
&= 0.
\end{aligned}
$$

Meaning if f is holomorphic then $\dfrac{\partial f}{\partial \bar{z}} = 0$. Thereby proving the desired statement.

$\square$

(c) (i) If f is holomorphic on G, prove that $f'(z) = \dfrac{\partial f}{\partial z}$.

Proof. We assume f to be holomorphic meaning we have that the CR-equations are satisfied, and that its derivative is equal to $u_x + iv_x$. Recall our expression for $\dfrac{\partial f}{\partial z}$, and note

$$
\begin{aligned}
\frac{\partial f}{\partial z} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \\
&= \frac{1}{2}\left(u_x + iv_x - i(u_y + iv_y)\right) \\
&= \frac{1}{2}\left(u_x + v_y + i(v_x - u_y)\right) && \text{apply CR-equations} \\
&= \frac{1}{2}\left(u_x + u_x + i(v_x + v_x)\right) \\
&= \frac{1}{2}\left(2u_x + i(2v_x)\right) \\
&= u_x + iv_x.
\end{aligned}
$$

Therefore if f is holomorphic on G, we have that $f'(z) = \dfrac{\partial f}{\partial z}$

$\square$

(ii) The *Jacobian* of the function $(x, y) \mapsto (u(x, y), v(x, y))$ is the determinant of the matrix

$$
\begin{pmatrix}
\dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\[2ex]
\dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y}
\end{pmatrix}
$$

If f is holomorphic on G, prove that the Jacobian equals $|f'(z)|^2 \geqslant 0$.

Proof. First let us compute the Jacobian, which is just taking the determinant of the matrix above turns out to be

$$
\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = u_x v_y - u_y v_x.
$$

Now recall that if f is holomorphic we have the derivative to be $f' = u_x + iv_x$ and the CR-equations to be satisfied, so let us compute $|f'|^2$,

$$
\left|f'\right|^2 = |u_x + iv_x|^2 = u_x u_x + v_x v_x \qquad \text{apply CR-equations}
$$

$$= u_x v_y - u_y v_x$$

which is equal to the computation for the Jacobian. Giving us that the Jacobian is equal to $|f'(z)|^2$, as desired. $\square$

Problem 4.3 Suppose f is entire, with real and imaginary parts u and v satisfying

$$u(x, y)\, v(x, y) = 3$$

for all $z = x + iy$. Show that f is constant.

Proof. Let us consider taking the partial derivatives of the product of u and v, meaning we'll take the derivatives with respect to x and to y using chain rule,

$$\begin{aligned} uv_x + u_x v &= 0 \ (\star) \\ uv_y + u_y v &= 0. \end{aligned}$$

Using that the fact that f was entire, we know it satisfies the Cauchy-Riemann equations which let us turn the last equation to

$$uu_x - v_x v = 0 \ (\spadesuit).$$

But because we know $\star$ and $\spadesuit$ are 0, we know the following,

$$\begin{aligned} (\spadesuit)u + (\star)v &= 0 \\ u^2 u_x - v_x vu + v_x vu + u_x v^2 &= 0 \\ u_x(u^2 + v^2) &= 0 \end{aligned}$$

giving us that u_x is equal to 0. For similar reason we know the following is true too,

$$\begin{aligned} (\star)u - (\spadesuit)v &= 0 \\ u^2 v_x + u_x vu - uvu_x + v_x v^2 &= 0 \\ v_x(u^2 + v^2) &= 0 \end{aligned}$$

giving us that v_x is equal to 0.

Now because f is entire we know its derivative exists and can be expressed as $f' = u_x + iv_x$, plugging in what we know though we have that $f' = 0$, meaning that f must be constant, as desired. $\square$

Problem 4.4 Prove that, if $G \subseteq \mathbb{C}$ is a domain and $f : G \to \mathbb{C}$ is a complex-valued function with $f''(z)$ defined and equal to 0 for all $z \in G$, then $f(z) = az + b$ for some $a, b \in \mathbb{C}$.

Proof. We apply Theorem 8.4 and see that because $f''(z) = 0$ we have that $f'(z)$ must be constant, which we will express as $f'(z) = a$, where a is some constant. Now consider the function $g(z) = f(z) - az$. If we take the derivative of g we know it is,

$$\begin{aligned} g'(z) &= f'(z) - a \\ g'(z) &= a - a \\ g'(z) &= 0. \end{aligned}$$

Applying Theorem 8.4 again, because $g'(z) = 0$ we have that $g(z) = b$ where b is some constant. Now if we solve for $f(z)$ in $g(z)$ we get,

$$\begin{aligned} g(z) &= f(z) + az \\ az + g(z) &= f(z) \\ az + b &= f(z). \end{aligned}$$

We see then that $f(z) = az + b$ for some $a, b \in \mathbb{C}$, as desired. □

Problem 4.5 Find all solutions to the equation $e^{2z} - 2ie^z = 1$.

Proof. First let us rewrite our equation as,

$$\begin{aligned} e^{2z} - 2ie^z &= 1 \\ e^{2z} - 2ie^z - 1 &= 0 \\ (e^z)^2 - 2ie^z - 1 &= 0 \end{aligned}$$

now let $u(z) = e^z$, which let's us rewrite our equation again,

$$u^2 - 2iu - 1 = 0.$$

Now recall from Homework 1 we derived a formula to obtain the solutions of this equations as $z = \dfrac{-b \pm \Delta^{1/2}}{2a}$ where $\Delta = b^2 - 4ac$. Solving for these we see that

$$\Delta = (-2i)^2 - 4(-1) = -4 + 4 = 0$$

giving us,

$$z = \frac{2i \pm 0}{2} = i.$$

Recall that $u(z) = e^z$ meaning we have that $e^z = i$ as the solutions. Using the definitions of the complex exponential function we have that,

$$\begin{aligned} e^z &= i \\ e^x e^{iy} &= e^{i\pi/2} \end{aligned}$$

giving us that $e^x = 1$ which means that $x = 0$. We get that y is,

$$y = \arg i = \frac{\pi}{2} + 2k\pi$$

giving us the solutions to be $z = i\left(\dfrac{\pi}{2} + 2k\pi\right)$ where $k \in \mathbb{Z}$.

□

Collaborators: Peers at section on Wednesday.