Homework 2

Kevin Guillen

MATH 200 — Algebra I — Fall 2021

I'd like my proof for problem 3.2 (It's the last one) to be graded please, thank you.

Problem 2.6. Show that for any non-empty subset X of a group G, the normalizer of X, $N_G(X)$ and the centralizer of X, $C_G(X)$ is again a subgroup of G. Show also that $C_G(X)$ is contained in $N_G(X)$.

Proof. **Normalizer** We know the normalizer of a subset X is defined as the following,

$$N_G(X) = \left\{ g \in G \mid gXg^{-1} = X \right\}.$$

So consider $x, y \in N_G(x)$. Let $z = xy$, we want to show that $z \in N_G(X)$. In other words we want to show $zXz^{-1} = X$, based on the above. We can see through the following that this is indeed true,

$$\begin{aligned} zXz^{-1} &= (xy)X(xy)^{-1} && (xy)^{-1} = y^{-1}x^{-1} \\ &= xyXy^{-1}x^{-1} && y \in N_G(x) \\ &= xXx^{-1} && x \in N_G(x) \\ &= X. \end{aligned}$$

Meaning $N_G(X)$ is closed under group operation.

Let $y \in N_G(X)$, based on the definition of the normalizer though,

$$\begin{aligned} yXy^{-1} &= X && \text{taking } y \text{ on the right} \\ yX &= Xy && \text{taking } y^{-1} \text{ on the left} \\ X &= y^{-1}Xy && y = (y^{-1})^{-1} \\ X &= y^{-1}X(y^{-1})^{-1} \end{aligned}$$

that y^{-1} is indeed in $N_G(X)$. Thus by the subgroup criterion, $N_G(X)$ is indeed a subgroup.

Centralizer: We know the definition of the centralizer of a subset X is the following,

$$C_G(X) = \left\{ g \in G \mid gxg^{-1} = x, \forall x \in X \right\}.$$

So consider $a, b \in C_G(X)$. Let $z = ab$, we want to show that $z \in C_G(X)$. In other words we want to show $zxz^{-1} = x$ for all $x \in X$. We see through the following that this does indeed hold.

$$\begin{aligned} zxz^{-1} &= (ab)x(ab)^{-1} && (ab)^{-1} = b^{-1}a^{-1} \\ &= (ab)x(b^{-1}a^{-1}) && \text{We know associativity holds in } G \\ &= a(bx^{-1})a^{-1} && b \in C_G(X) \\ &= axa^{-1} && a \in C_G(X) \\ &= x \end{aligned}$$

Meaning $C_G(X)$ is closed under group operation.

Let $y \in C_G(X)$. By definition that means for all $x \in X$, $yxy^{-1} = x$, but consider the following,

$$\begin{aligned}
yxy^{-1} &= x && \text{taking } y^{-1} \text{ on the left}\\
xy^{-1} &= y^{-1}x && \text{taking } y \text{ on the right}\\
x &= y^{-1}xy && y = (y^{-1})^{-1}\\
x &= y^{-1}x(y^{-1})^{-1}.
\end{aligned}$$

This means that for any $y \in C_G(X)$, that y^{-1} is also in $C_G(X)$. Thus $C_G(X)$ is a subgroup.

Now we want to show that the centralizer is contained in the normalizer. Expanding on the definition of the normalizer $gXg^{-1} = X \to gX = Xg$. This means there exists some $s, t \in X$ such that $gs = tg$. What we see though is that this is simply a weaker property when compared to the centralizer definition. Expanding on the definition of the centralizer, for all $x \in X$ we have $gxg^{-1} = x \to gx = xg$. Meaning any $g \in C_G(X)$ has the property that $gs = tg$ where $t = s = x$, which means it is also in $N_G(X)$, thus $C_G(X) \subset N_G(X)$ □

Problem 2.7. Let $f : G \to H$ be a group homomorphism.

(a) If $U \leqslant G$ then $f(U) \leqslant H$.

(b) If $V \leqslant H$ then $f^{-1}(V) = \{g \in G \mid f(g) \in V\}$ is a subgroup of G.

(c) Show that f is injective if and only if $\ker(f) = \{1\}$

(a) *Proof.* Let $x, y \in f(U)$, and let $z = xy$, we want to show $z \in f(U)$. Since $x, y \in f(U)$, that means there exists $x', y' \in U$ such that $f(x') = x$ and $f(y') = y$. Giving us,

$$\begin{aligned}
z &= xy\\
&= f(x')f(y') && \text{f is a homomorphism so,}\\
&= f(x'y')
\end{aligned}$$

Because U is a subgroup then $x'y' \in U$, meaning $z = f(x'y') \in f(U)$, thus $f(U)$ is closed under group operation.

Given $x \in f(U)$, we want to show $x^{-1} \in f(U)$. By $x \in f(U)$ that means there exists $x' \in U$ such that $x = f(x')$. Since U is a subgroup there exists $x'^{-1} \in U$, meaning $f(x'^{-1}) \in f(U)$. Recall though f is a homomorphism that means it respects inverses, thus $f(x'^{-1}) = f(x')^{-1}$, which will be x^{-1}. We verify through the following,

$$\begin{aligned}
xx^{-1} &= f(x')f(x')^{-1}\\
&= f(x'x'^{-1})\\
&= f(1)\\
&= 1.
\end{aligned}$$

Thus we have that if $U \leqslant G$ then $f(U) \leqslant H$. □

(b) *Proof.* Let $x, y \in f^{-1}(V)$, that means there exists $x', y' \in V$ such that $f(x) = x'$ and $f(y) = y'$. Recall though V is a subgroup so $x'y' \in V$, but $x'y' = f(x)f(y)$ and f is a homomorphism so $f(x)f(y) = f(xy) \in V$ which means $xy \in f^{-1}(V)$.

Let $x \in f^{-1}(V)$ then $f(x) \in V$, and because V is a subgroup we have $f(x)^{-1} \in V$. Recall though f is a homomorphism, and so $f(x)^{-1} = f(x^{-1}) \in V$ and thus $x^{-1} \in f^{-1}(V)$. □

(c) *Proof.* $\Rightarrow$ Given that f is injective and a group homomorphism, that means it respects the identity element, meaning $f(1_G) = 1_H$. That also means whenever $f(x) = f(y) \to x = y$. Take an element $x \in \ker(f)$, by definition that means $f(x) = 1_H$, recall though $f(1_G) = 1_H$. So we have $f(x) = f(1_G)$, but by definition that means $x = 1_G$. Therefore if f is injective, the kernel of f is $\{1\}$

$\Leftarrow$ Given that f is a group homomorphism and that $\ker(f) = \{1\}$. We want to show that f is injective. Consider $x, y \in G$ such that $f(x) = f(y)$. Now consider the following,

$$\begin{aligned} f(xy^{-1}) &= f(x)f(y^{-1}) \\ &= f(x)f(y)^{-1} && f(x) = f(y) \\ &= f(x)f(x)^{-1} \\ &= 1_H. \end{aligned}$$

Recall though $\ker(f) = \{1_G\}$, and we see $f(xy^{-1}) = 1_H$ that means $x = y$, and thus f is injective. □

Problem 2.9. Let G and A be groups and assume that A is abelian. Show that the set $\mathrm{Hom}(G, A)$ of group homomorphisms from G to A is again an abelian group under the multiplication defined by

$$(f_1 \cdot f_2)(g) := f_1(g)f_2(g) \qquad \text{for } f_1, f_2 \in \mathrm{Hom}(G, A) \text{ and } g \in G$$

Proof. From this point forward let $H = \mathrm{Hom}(G, A)$

Closure. Let $f_1, f_2 \in H$, let $f_3 = f_1 \cdot f_2$ we want to show that $f_3 \in H$. Now let $a, b \in G$, we have the following,

$$\begin{aligned} f_3(ab) = (f_1f_2)(ab) &= f_1(ab)f_2(ab) && \text{Recall though } f_1, f_2 \text{ are group homomorphisms} \\ &= f_1(a)f_1(b)f_2(a)f_2(b) && \text{All these elements are in } A\text{, and } A \text{ is abelian} \\ &= f_1(a)f_2(a)f_1(b)f_2(b) \\ &= (f_1f_2)(a)(f_1f_2)(b) \\ &= f_3(a)f_3(b). \end{aligned}$$

We see then that f_3 is a group homomorphism meaning it is also in H. Thus H is closed under the multiplication.

Identity. Simply let the the be the identity element be, $f_e : G \to A$, $g \mapsto 1_A$. It is obvious that this is a homomorphism, and is in H. We see through the following that it does indeed serve the role of the identity element. Let $f_1 \in H, g \in G$

$$\begin{aligned} (f_ef_1f_e)(g) &= f_e(g)f_1(g)f_e(g) \\ &= 1_Af_1(g)1_A && \text{Recall } f_1(g) \text{ is an element of } A \\ &= f_1(g) \end{aligned}$$

Inverse. Let $f_1 \in H$. We see the inverse is simply $f_1^{-1} \in H$, we verify through the following where $g \in G$,

$$\begin{aligned} (f_1f_1^{-1})(g) &= f_1(g)f_1^{-1}(g) \\ &= f_1(g)f_1(g^{-1}) \\ &= f_1(gg^{-1}) \\ &= f_1(1_G) \\ &= 1_A \\ &= f_e(g) \end{aligned}$$

Associativity. Let $f_1, f_2, f_2 \in H$. We see through the following that associativity holds. Let $g \in G$

$$\begin{aligned}
((f_1 f_2) f_3)(g) &= ((f_1 f_2)(g) f_3(g) \\
&= (f_1(g) f_2(g)) f_3(g) && \text{Recall these are element in } A\text{, and } A \text{ is a group} \\
&= f_1(g)(f_2(g) f_2(g)) \\
&= f_1(g)(f_2 f_3)(g) \\
&= (f_1(f_2 f_3))(g)
\end{aligned}$$

as we can see associativity does indeed hold.

Commutativity. Let $f_1, f_2 \in H$. We see through the following commutativity holds. Let $g \in G$,

$$\begin{aligned}
(f_1 f_2)(g) &= f_1(g) f_2(g) && \text{These are elements of } A \text{ and } A \text{ is abelian} \\
&= f_2(g) f_1(g) \\
&= (f_2 f_1)(g)
\end{aligned}$$

as we can see commutativity does indeed hold.

With all this together that means $\mathrm{Hom}(G, A)$ is indeed an abelian group, as desired. □

Problem 3.1. Let M and N be normal subgroups of a group G. Show that also $M \cap N$ and MN are normal subgroups of G.

Proof. To begin we know from 2.11 example (c) that the intersection of any collection of subgroups of a group is again a subgroup. Meaning $M \cap N$ is a subgroup of G. All that is left to show now is that it is a normal subgroup of G. We see immediately though that for all $k \in M \cap N$ and for all $g \in G$ that $gkg^{-1} \in M$, and $gkg^{-1} \in N$. This is because any element that is in the intersection of M and N must be in both those subgroups, and those subgroups were said to be normal. This means for all $k \in M \cap N$ and for all $g \in G$ that $gkg^{-1} \in M \cap N$. Therefore proving that $M \cap N$ is indeed a normal subgroup of G. □

Problem 3.2. Let G be a group and let X be a subset of G. Show that $C_G(X) \trianglelefteq N_G(X)$

Proof. In this same homework we worked out from problem 2.6 that $C_G(X)$ is indeed contained in $N_G(X)$. So to solve this problem we just need to show that it is indeed a subgroup and then that it is normal. Let's begin with showing that it is a subgroup.

Closure. Let $a, b \in C_G(X)$, we want to show $(ab) \in C_G(X)$. For all $x \in X$ we see through the following,

$$\begin{aligned}
(ab)x(ab)^{-1} &= (ab)x(b^{-1}a^{-1}) && \text{we know associativity holds} \\
&= a(bxb^{-1})a^{-1} && b \in C_G(X) \\
&= axa^{-1} && a \in C_G(X) \\
&= x
\end{aligned}$$

that $C_G(X)$ is indeed closed under group operation.

Inverse. Now for any $a \in C_G(X)$ we will show that a^{-1} is also in $C_G(X)$. By definition of a being in $C_G(X)$ we have for all $x \in X$,

$$\begin{aligned}
axa^{-1} &= x \\
a^{-1}axa^{-1} &= a^{-1}x \\
xa^{-1}a &= a^{-1}xa \\
x &= a^{-1}xa.
\end{aligned}$$

Meaning a^{-1} is also in $C_G(X)$. Therefore $C_G(X) \leqslant N_G(X)$.

Now to show that is normal. We will use Theorem 3.1 (iii) to prove that this subgroup is indeed normal. Consider the map $f : N_G(X) \to \mathrm{Aut}(X)$ where $n \mapsto (x \mapsto nxn^{-1})$. We know $(x \mapsto axa^{-1})$ is indeed an automorphism based on Example 2.6 (c). So first we want to show that f is a homomorphism.

$$\begin{aligned}
f(a)f(b) &= (x \mapsto axa^{-1})(x \mapsto bxb^{-1}) \\
&= (x \mapsto (abxb^{-1}a^{-1})) && (ab)^{-1} = b^{-1}a^{-1} \\
&= (x \mapsto (ab)x(ab)^{-1}) \\
&= f(ab)
\end{aligned}$$

Now with that out of the way to apply the theorem stated earlier we need to show that $\ker(f) = C_G(X)$. We will do this by considering an element $a \in \ker(f)$, and for all $x \in X$, we see through the following,

$$\begin{aligned}
x &= axa^{-1} \\
xa &= ax
\end{aligned}$$

that $a \in C_G(X)$ and thus $\ker(f) = C_G(X)$. Since we see by being in the kernel of f an element must satisfy the definition of being in the centralizer of the subset X.

$\square$