5. We can view $\mathbb{F}^n$ as coloumn vectors of length n i.e.

$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

First we'll show it is an abelian group under addition $(\mathbb{F}^n, +)$

The identity e is simply $\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{F}^n$

for any $v \in \mathbb{F}^n$ we see

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} 0+v_1 \\ \vdots \\ 0+v_n \end{pmatrix} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} v_1+0 \\ \vdots \\ v_n+0 \end{pmatrix} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

The inverse for any $v \in \mathbb{F}^n$ is simply $-v \in \mathbb{F}^n$

$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} -v_1 \\ \vdots \\ -v_n \end{pmatrix} = \begin{pmatrix} v_1 - v_1 \\ \vdots \\ v_n - v_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$ since each component is done in $\mathbb{F}$

$$\begin{pmatrix} -v_1 \\ \vdots \\ -v_n \end{pmatrix} + \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} -v_1 + v_1 \\ \vdots \\ -v_n + v_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

Associativity: $v, w, x \in \mathbb{F}^n$

$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} + \left(\begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} + \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right) = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} (w_1 + x_1) \\ \vdots \\ (w_n + x_n) \end{pmatrix}$$

$$= \begin{pmatrix} (v_1 + (w_1 + x_1)) \\ \vdots \\ (v_n + (w_n + x_n)) \end{pmatrix}$$

We know $\mathbb{F}$ is associative therefore

$$= \begin{pmatrix} (v_1 + w_1) + x_1 \\ \vdots \\ (v_n + w_n) + x_n \end{pmatrix}$$

$$= \begin{pmatrix} (v_1 + w_1) \\ \vdots \\ (v_n + w_n) \end{pmatrix} + \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$= \left(\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \right) + \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \checkmark$$

Commutative: $v, w \in \mathbb{F}^n$

$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{pmatrix} = \begin{pmatrix} w_1 + v_1 \\ \vdots \\ w_n + v_n \end{pmatrix} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} + \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \checkmark$$

Therefore $(\mathbb{F}^n, +)$ is an abelian gp.

Next $\forall \alpha, \beta \in \mathbb{F}$, $v \in \mathbb{F}^n$

$$\alpha\left(\beta\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}\right) = \alpha\begin{pmatrix} \beta v_1 \\ \vdots \\ \beta v_n \end{pmatrix} = \begin{pmatrix} (\alpha\beta)v_1 \\ \vdots \\ (\alpha\beta)v_n \end{pmatrix} = (\alpha\beta)\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \checkmark$$

We see for 1

$$1 \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} 1 \cdot v_1 \\ \vdots \\ 1 \cdot v_n \end{pmatrix} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} v_1 \cdot 1 \\ \vdots \\ v_n \cdot 1 \end{pmatrix} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \cdot 1 \checkmark$$

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for $\alpha \in \mathbb{F}$, $v, w \in \mathbb{F}^n$

$$\alpha\left(\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}\right) = \begin{pmatrix} \alpha(v_1 + w_1) \\ \vdots \\ \alpha(v_1 + w_1) \end{pmatrix} = \begin{pmatrix} \alpha v_1 + \alpha w_1 \\ \vdots \\ \alpha v_1 + \alpha w_1 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha v_1 \\ \vdots \\ \alpha v_n \end{pmatrix} + \begin{pmatrix} \alpha w_1 \\ \vdots \\ \alpha w_n \end{pmatrix}$$

$$= \alpha\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} + \alpha\begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \checkmark$$

$\forall \alpha, \beta \in \mathbb{F}$, $v \in \mathbb{F}^n$

$$(\alpha+\beta)\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} (\alpha+\beta)v_1 \\ \vdots \\ (\alpha+\beta)v_n \end{pmatrix} = \begin{pmatrix} \alpha v_1 + \beta v_1 \\ \vdots \\ \alpha v_n + \beta v_n \end{pmatrix} = \begin{pmatrix} \alpha v_1 \\ \vdots \\ \alpha v_n \end{pmatrix} + \begin{pmatrix} \beta v_1 \\ \vdots \\ \beta v_n \end{pmatrix}$$

$$= \alpha\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} + \beta\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \checkmark$$

Therefor $\mathbb{F}^n$ is indeed a ~~field~~ vector space.