Name: Kevin Guillen
Student ID: 1747199

117 - SS2 - MP3 - August 13th, 2021

- [5] Let the $\mathbb{R}$-vector space of all smooth functions on $\mathbb{R}^3$ be denoted by $C^\infty(\mathbb{R}^3)$ where a smooth function on $\mathbb{R}^3$ is a function $f : \mathbb{R}^3 \to \mathbb{R}$ that has continuous partial derivatives of every order. Define a differential 1-form on $\mathbb{R}^3$ to be a symbol of the form:

$$\omega = f_1 dx + f_2 dy + f_3 dz$$

for $f_1, f_2, f_3 \in C^\infty(\mathbb{R}^3)$ and let the collection of all differential 1-forms on $\mathbb{R}^3$ be denoted by $\Omega^1(\mathbb{R}^3)$

  - [a] Show that $\Omega^1(\mathbb{R}^3)$ can be regarded as a $\mathbb{R}$-vector space.

Proof. First we will begin my showing the set $(\Omega^1, +)$ forms an abelian group.
Associative: We see for any vectors $f, g, h \in \Omega^1$ we have the following,

$$\begin{aligned} f + (g+h) = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} + \left[\begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} + \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} \right] &= \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} + \begin{pmatrix} g_1 + h_1 \\ g_2 + h_2 \\ g_3 + h_3 \end{pmatrix} \\ &= \begin{pmatrix} f_1 + (g_1 + h_1) \\ f_2 + (g_2 + h_2) \\ f_3 + (g_3 + h_3) \end{pmatrix} \\ &= \begin{pmatrix} (f_1 + g_1) + h_1 \\ (f_2 + g_2) + h_3 \\ (f_3 + g_3) + h_3 \end{pmatrix} \\ &= \begin{pmatrix} f_1 + g_1 \\ f_2 + g_2 \\ f_3 + g_3 \end{pmatrix} + \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} \\ &= \left[\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} + \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} \right] + \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} \\ &= (f+g) + h \end{aligned}$$

Identity: Let the identity of Ω^1 be the following,

$$0 = \begin{pmatrix} f_0 \\ f_0 \\ f_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Which is simply the vector composed of of zero functions. We can see it satisfies the requirements by the following, for any $f \in \Omega^1$

$$\begin{aligned}
0 + f &= \begin{pmatrix} f_0 \\ f_0 \\ f_0 \end{pmatrix} + \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} f_0 + f_1 \\ f_0 + f_2 \\ f_0 + f_3 \end{pmatrix} \\
&= \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = f
\end{aligned}$$

$$\begin{aligned}
f + 0 &= \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} + \begin{pmatrix} f_0 \\ f_0 \\ f_0 \end{pmatrix} = \begin{pmatrix} f_1 + f_0 \\ f_2 + f_0 \\ f_3 + f_0 \end{pmatrix} \\
&= \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = f
\end{aligned}$$

Inverse: We see for any vector f in Ω^1 the inverse of f is defined as the following,

$$f^{-1} = \begin{pmatrix} -f_1 \\ -f_2 \\ -f_3 \end{pmatrix}$$

We can see that this does indeed serve as an inverse since,

$$\begin{aligned}
f + f^{-1} &= \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} + \begin{pmatrix} -f_1 \\ -f_2 \\ -f_3 \end{pmatrix} = \begin{pmatrix} f_1 - f_1 \\ f_2 - f_2 \\ f_3 - f_3 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0
\end{aligned}$$

$$\begin{aligned}
f^{-1} + f &= \begin{pmatrix} -f_1 \\ -f_2 \\ -f_3 \end{pmatrix} + \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} -f_1 + f_1 \\ -f_2 + f_2 \\ -f_3 + f_3 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0
\end{aligned}$$

Commutative: Let $f, g \in \Omega^1$. We can see based on the following these elements are commutative.

$$f + g = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} + \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = \begin{pmatrix} f_1 + g_1 \\ f_2 + g_2 \\ f_3 + g_3 \end{pmatrix} \tag{1}$$

$$= \begin{pmatrix} g_1 + f_1 \\ g_2 + f_2 \\ g_3 + f_3 \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} + \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = g + f \tag{2}$$

So we have it that $(\Omega^1, +)$ is indeed an abelian group. Now we will show that it behaves well with scalars.
Let $\alpha, \beta \in \mathbb{R}$ and $f \in \Omega^1$

$$\begin{aligned} \alpha \left[\beta \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \right] = \alpha \begin{pmatrix} \beta f_1 \\ \beta f_2 \\ \beta f_3 \end{pmatrix} &= \begin{pmatrix} \alpha\beta f_1 \\ \alpha\beta f_2 \\ \alpha\beta f_3 \end{pmatrix} \\ &= \begin{pmatrix} (\alpha\beta) f_1 \\ (\alpha\beta) f_3 \\ (\alpha\beta) f_3 \end{pmatrix} \\ &= (\alpha\beta) \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \end{aligned}$$

Distribution holds, for $\alpha, \beta \in \mathbb{R}$ and $f, g \in \Omega^1$.

$$\alpha \left[\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} + \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} \right] = \alpha \begin{pmatrix} f_1 + g_1 \\ f_2 + g_2 \\ f_3 + g_3 \end{pmatrix}$$

$\square$