Homework 5

Kevin Guillen

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May I please have my proof for 5.3 graded, thank you.

Problem 5.2 Let G be a p-group for a prime p and let N be a non-trivial normal subgroup of G. Show that $N \cap Z(G) > 1$.

Proof. We can consider G acting on the set N via conjugation.

$$\alpha(g, n) = gng^{-1}$$

Because N is given to be a non-trivial normal subgroup we know that $\alpha(g, h) \in N$. Now consider the set of fixed points under this group action,

$$\begin{aligned} N^G &= \{n \in N \mid \alpha(g, n) = n, \forall g \in G\} \\ &= \left\{n \in N \mid gng^{-1} = n, \forall g \in G\right\} \\ &= \{n \in N \mid gn = ng, \forall g \in G\} \\ &= N \cap Z(G) \end{aligned}$$

From the orbit equation we have,

$$\left|N^G\right| = |N| - \sum_{x \in \mathcal{R} \setminus X^G} [G : G_x]$$

Because N was given to be non-trivial, by Lagrange, it must be of order p^a where $a \in \{1, \ldots, k\}$ (k being number such that $G = p^k$). We know each G_x is a proper subgroup meaning it has an order of p^z for $z \in \{1, \ldots, k-1\}$. This means that p divides N^G, meaning $N \cap Z(G) > 1$ as desired. □

Problem 5.3
(a) Let G be a group such that $G/Z(G)$ is cyclic. Show that G is abelian.
(b) Show that if a group G has order p^2, for some p, then G is abelian.

(a) *Proof.* Given that $G/Z(G)$ is cyclic that means there exists an element $g \in G$ such that,

$$G/Z(G) = \langle gZ(G) \rangle.$$

Now for any element $a \in G$, we know there exists some $n \in \mathbb{Z}$ such that $aZ(G) = (gZ(G))^n$. Which implies the following,

$$aZ(G) = (gZ(G))^n$$

$$
\begin{aligned}
&= \underbrace{gZ(G) \cdot gZ(G) \cdot \cdots \cdot gZ(G)}_{n} \\
aZ(G) &= g^n Z(G) && aH = bH \longleftrightarrow b^{-1}a \in H \\
\to (g^n)^{-1}a &\in Z(G) \\
g^{-n}a &\in Z(G).
\end{aligned}
$$

This means for any element $a \in G$, there exists some $n \in \mathbb{Z}$, and some $z \in Z(G)$ such that,

$$
\begin{aligned}
g^{-n}a &= z \\
a &= g^n z
\end{aligned}
$$

So consider any two elements $a, b \in G$, they are of the form $a = g^n z_1$ and $b = g^m z_2$. So we see from the following,

$$
\begin{aligned}
ab &= g^n z_1 g^m z_2 && z_1, z_2 \in Z(G) \\
&= g^n g^m z_1 z_2 \\
&= g^{n+m} z_1 z_2 \\
&= g^{m+n} z_1 z_2 \\
&= g^m g^n z_1 z_2 \\
&= g^m z_2 g^n z_1 \\
&= ba
\end{aligned}
$$

that G is abelian. □

(b) *Proof.* Given that G is a p-group, that means its center is non-trivial. Because the center of any group is always a subgroup, by Lagrange the order of $Z(G)$ must be either p^2 or p. If it is the first case we are done since that would imply $Z(G) = G$ making G abelian. If it is of order p, we know $Z(G)$ is normal, meaning we can take the quotient group $Z/Z(G)$, and it will have to be of order p, meaning it is cyclic and according to part (a) it must be abelian. □

Problem 5.5 (Frattini Argument) Let G be a finite group, p a prime, $H \trianglelefteq G$ and $P \in \mathrm{Syl}_p(H)$. Show that $G = HN_G(P)$. (Hint: Let $g \in G$ and consider P and gPg^{-1}. Show that both are Sylow p-subgroups of H)

Proof. Let $g \in G$. We know from the given that P is a p-sylow subgroup of H, meaning $P \subseteq H$. Also since H is normal if we perform conjugation on P with g, we have that $gPg^{-1} \subseteq H$. We also know that a subgroup under conjugation will be another subgroup of the same order, in other words gPg^{-1} is another p-sylow subgroup. Recall though by Sylow's theorem 5.12(c) any two p-sylow subgroups are conjugate. Meaning there exists some $h \in H$ such that $hPh^{-1} = gPg^{-1}$. Consider the following though,

$$
\begin{aligned}
hPh^{-1} &= gPg^{-1} \\
P &= h^{-1}gPg^{-1}h
\end{aligned}
$$

this means that $h^{-1}g \in N_G(P)$. Using this fact, we can express any element $g \in G$ as $g = h(h^{-1}g)$, where $h \in H$ and $h^{-1}g \in N_G(P)$. Giving us the desired equality, $G = HN_G(P)$. □

Problem 5.6 Show that every group of order 1000 is solvable.

Proof. Let G represent a group of order 1000. Note that $1000 = 5^3 2^3$. We know from class that the $\left|\mathrm{Syl}_5(G)\right| \equiv 1 \bmod 5$ and $\left|\mathrm{Syl}_5(G)\right| \mid 8$. Meaning $\left|\mathrm{Syl}_5(G)\right| = 1$. Let H represent this unique Sylow 5-subgroup. Because it is unique it is normal. Meaning we can take the quotient group G/H which will have order 2^3 and is therefore solvable (Theorem 5.10). For the same reasoning H is also solvable because it is of order 5^3, this means G is solvable as desired. □

Problem 6.4 Show that every group of order 72 is solvable.

Proof. Let G represent a group of order 72. Note that 72's prime decomposition is $3^2 \cdot 2^3$. We know that $|\mathrm{Syl}_3(G)| \equiv 1 \bmod 3$ and that $|\mathrm{Syl}_3(G)| \mid 8$, meaning $|\mathrm{Syl}_3(G)| = 1$ or 4. In the first case that means there exists a unique Sylow 3-subgroup and it is normal, we will refer to this subgroup as H. Because H is of order 9, it is abelian and therefore solvable, meaning G/H will also be solvable since its order is 2^3 (Theorem 5.10). So in all we have that H is solvable and that G/H is solvable, therefore G is solvable.

If $\left|\mathrm{Syl}_3(G) = 4\right|$, let $H \in \mathrm{Syl}_3(G)$. H will not be normal in this case. We know the subgroups in $\mathrm{Syl}_3(G)$ are conjugates of H. So consider G acting on H via conjugation. Recall that $[G : N_G(H)] = \mathrm{Syl}_3(G) = 4$. We can let G act on $N_G(H)$ via right multiplication on the right cosets of $N_G(T)$. This will give us the homomorphism,

$$f : G \to \mathrm{Sym}(4)$$

with $\ker(f) \leqslant N_G(H)$ and we also have though $|\mathrm{Sym}(4)| < |G|$, meaning $\ker(f) \neq 1$. All together we have, $\ker(f) \leqslant N_G(H) < G$, meaning $\ker(f) \neq G$. Therefore $\ker(f)$ is non-trivial and because the kernel of a homomorphism is a normal subgroup we have that G is not simple and therefore solvable. □