Name: Kevin Guillen
Student ID: 1747199

117 - SS2 - MP3 - August 13th, 2021

[1] Let V and W be $\mathbb{F}$-vector spaces (of any dimension) and $f : V \to W$ a linear transformation. Show that the induced map $\overline{f} : V/\ker(f) \to W$ in the following diagram is injective:

$$\begin{tikzcd} V \arrow[r, "\pi"] \arrow[dr, "f"'] & V/\ker(f) \arrow[d, dashed, "\overline{f}"] \\ & W \end{tikzcd}$$

where $\pi(v) = v + \ker(f)$ and $f = \overline{f} \circ \pi$.

Proof. Take $v_1 + \ker(f)$ and $v_2 + \ker(f)$ in $V/\ker(f)$ such that $v_1 + \ker(f) \neq v_2 + \ker(f)$. The induced map $\overline{f}(v)$ is simply $\overline{f}(v + \ker(f)) = f(v)$. So assuming

$$\overline{f}(v_1 + \ker(f)) = \overline{f}(v_2 + \ker(f))$$

we get the following,

$$\begin{aligned} f(v_1 + \ker(f)) &= f(v_2 + \ker(f)) \\ f(v_1 + \ker(f)) - f(v_2 + \ker(f)) &= 0 \\ f((v_1 - v_2) + \ker(f)) &= 0 \\ &\to v_1 = v_2 \end{aligned}$$

which is a contradiction since we said they were not equal. Thus the induced map is indeed injective. □

[2] Let V be a $\mathbb{F}$-vector space of dimension n. Suppose that $m < n$ and that $y_1, \dots, y_m \in V^*$.

(a) Prove that there exists a non-zero vector $x \in V$ such that $[x, y_j] = 0$ for $1 \leq j \leq m$. What does this result say about the solutions of linear equations?

Proof. We can prove this using the rank-nullity theorem from linear algebra. To apply it we will first define the map

$$\begin{aligned} \phi : V &\to \mathbb{F}^m \\ x &\mapsto (y_1(x), \dots, y_m(x)) \end{aligned}$$

We are given that the dimension of V is n, we know from class that the dimension of $\mathbb{F}^m$ is simply m. We also know that $m < n$, recall the rank-nullity theorem

$$\dim(V) = \dim(\mathbb{F}^m) + \dim(\ker(\phi))$$

thus by the rank nullity theorem we know that the kernal of ϕ is non trivial. Therefore there exists a non-zero vector $x \in V$ such that $[x, y_j] = 0$ for $1 \leq j \leq m$ □

(b) Under what conditions on the scalars $\alpha_1, \ldots, \alpha_m \in \mathbb{F}$ is it true that there exists a vector $x \in V$ such that $[x, y_j] = \alpha_j$ for $1 \leq j \leq m$? What does this result say about the solutions of linear equations?

[3] Provide an example of a $\mathbb{F}$-vector space V with three $\mathbb{F}$-vector subspaces U, W_1, and W_2 such that $U \oplus W_1 = U \oplus W_2$, but $W_1 \neq W_2$. Note that this means that there is no cancellation law for direct sums. What is the geometric picture corresponding to this situation?

Solution:

Let $V = \mathbb{R}^2$. Now let the vector subspaces $U, W_1,\ W_2$ be $\text{span}((1,0)), \text{span}((0,1))$, and $\text{span}((1,1))$ respectively. We can see that,

$$U \oplus W_1 = U \oplus W_2$$

$$\text{span}((1,0)) \oplus \text{span}((0,1)) = \text{span}((1,0)) \oplus \text{span}((1,1)$$

While $W_1 \neq W_2$ because $\text{span}((1,1) \neq \text{span}((0,1))$

[4] Given a finite-dimensional $\mathbb{F}$-vector space V, form the direct sum $W = V \oplus V^*$, and prove that the correspondence $(x, y) \to (y, x)$ is an isomorphism between W and W^*.

Proof. First we can see that $W^* = (V \oplus V^*)^* = V^* \oplus V^{**}$. Recall though from class that since V is of finite dimension then there exists a natural isomorphism between V and V^{**}. The natural ismorphism is simply $\phi(x_0)(y) = [x_0, y]$ □

[5] Let U and V be $\mathbb{F}$-vector spaces. A bilinear form $\omega : U \oplus V \to \mathbb{F}$ is *degenerate* if, as a function of one of its two arguments, it vanishes identically for some non-zero value of its other argument; otherwise it is *non-degenerate.*

(a) Give an example of a degenerate bilinear form (not identically zero) on the $\mathbb{C}$-vector space $\mathbb{C}^2 \oplus \mathbb{C}^2$.

(b) Give an example of a non-degenerate bilinear form on the $\mathbb{C}$-vector space $\mathbb{C}^2 \oplus \mathbb{C}^2$.

[6] Does there exist a $\mathbb{F}$-vector space V and a bilinear form $\omega : V \oplus V \to \mathbb{F}$ such that ω is not identically zero, but $\omega(x, x) = 0$ for every $x \in V$?

[7] Let $\{e_1, e_2\}$ and $\{e'_1, e'_2, e'_3\}$ be the standard bases for the $\mathbb{R}$-vector spaces $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively, where
$e_i = (\delta_{1i}, \delta_{2i})$ and $e'_i = (\delta_{1i}, \delta_{2i}, \delta_{3i})$ for δ_{pq} representing the Kronecker delta. Given that $x = (1, 1) \in \mathbb{R}^2$ and $y = (1, 1, 1) \in \mathbb{R}^3$, find the coordinates of $x \otimes y \in \mathbb{R}^2 \otimes \mathbb{R}^3$ with respect to the standard product basis $\{e_i \otimes e'_j \mid 1 \leq i \leq 2, 1 \leq j \leq 3\}$.

[8] Let $\mathcal{S}_k$ represent the permutation group on k elements.

(a) Prove that if $\sigma, \tau \in \mathcal{S}_k$, then there exists a unique $\pi \in \mathcal{S}_k$ such that $\sigma\pi = \tau$.

(b) Prove that if $\sigma, \tau, \pi \in \mathcal{S}_k$ such that $\pi\sigma = \pi\tau$, then $\sigma = \tau$.

[9] Let $\mathcal{S}_k$ represent the permutation group on k elements. Prove that every permutation in $\mathcal{S}_k$ is the product of transpositions of the form $(j, j+1)$, where $1 \leq j < k$. Is this factorization unique?

[10] Let V be a finite-dimensional $\mathbb{F}$-vector space.

(a) A bilinear form $b_1 : V \times V \to \mathbb{F}$ is called *symmetric* if $b_1(v, w) = b_1(w, v)$. Similarly, a bilinear form $b_2 : V \times V \to \mathbb{F}$ is called *skew-symmetric* if $b_2(v, w) = -b_2(w, v)$. Prove that any bilinear form $\omega : V \times V \to \mathbb{F}$ can be written as a sum of symmetric and skew-symmetric bilinear forms. You may assume that $\mathrm{char}(\mathbb{F}) \neq 2$.

(b) What if $\mathrm{char}(\mathbb{F}) = 2$ in part (a)? Does the decomposition of ω into symmetric and skew-symmetric bilinear forms no longer work?

(c) For a field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2$ it is known that skew-symmetric and alternating bilinear forms are the same. If instead we consider $\mathrm{char}(\mathbb{F}) = 2$ then symmetric and skew-symmetric bilinear forms are the same, and since alternating bilinear forms are skew-symmetric no matter the characteristic of a field it follows that alternating bilinear forms are symmetric. Is it true that all symmetric bilinear forms on a field of characteristic 2 are alternating?

(d) A 2-tensor $x_1 \otimes y_1 \in V \otimes V$ is called *symmetric* if $x_1 \otimes y_1 = y_1 \otimes x_1$. Similarly, a 2-tensor $x_2 \otimes y_2 \in V \otimes V$ is called *skew-symmetric* if $x_2 \otimes y_2 = -y_2 \otimes x_2$. Prove that $V \otimes V = \mathrm{Sym}^2(V) \oplus \mathrm{Skew}^2(V)$, where $\mathrm{Sym}^2(V)$ and $\mathrm{Skew}^2(V)$ represent the symmetric and skew-symmetric 2-tensors on V, respectively. You may assume that $\mathrm{char}(\mathbb{F}) \neq 2$.

(e) What if $\mathrm{char}(\mathbb{F}) = 2$ in part (c)? Does the decomposition of $V \otimes V$ no longer hold true?