Homework 1

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Problem P1 Let M be a left $R-$module.

(a) Let $N_1 \subseteq N_2 \subseteq N_3 \subseteq \dots$ be an ascending chan of $R-$submodules in M. Prove that the union $\bigcup_{j=1}^{\infty} N_j$ is an $R-$submodule of M.

(b) Let $R = \mathcal{C}(\mathbb{R})$ denote the ring of (real-valued) continuous functions on $\mathbb{R}$, with pointwise addition and multiplication (as in class). defined

$$\mathcal{C}_c(\mathbb{R}) = \{f \in \mathcal{C}(\mathbb{R}) : \exists N = N(f) \in \mathbb{N} \text{ such that } f(x) = 0 \text{ for all } |x| > N\}$$

Prove that $\mathcal{C}_C(\mathbb{R})$ is an $R-$submodule of R. Is it a subring?

(a) *Proof.* First we will define N to be the following,

$$N = \bigcup_{j=1}^{\infty} N_j.$$

Now we must show that N is a subgroup of M under addition and that it is closed under scalar operation for it to be a submodule. So we will show that it is non-empty, closed under addition, and closed under scalars. Inverses is handled through the proof of scalars since $-1_R \in R$.

We know N is non-empty since it is a union of non-empty sets (because N_j is given to be a submodule). So let $x, y \in N$ we know then there exists some $a, b \in \mathbb{N}$ such that $x \in N_a$ and $y \in N_b$. We can then let $k = \max\{a, b\}$. Which means $N_a \subseteq N_k$ and $N_b \subseteq N_k$ and therefore $x, y \in N_k$.

Now because we know N_k to be a submodule of M (since k is either equal to a or b), we have the following

$$x + y \in N_k$$

and because $N_k \subseteq N$ we have,

$$x + y \in N$$

as desired.

Now let $r \in R$ and $x \in N$. Like before this means that there exists some $a \in \mathbb{N}$ such that $x \in N_a$. Where N_a is a submodule of M. So we know the following

$$rx \in N_a$$

and because $N_a \subseteq N$ we have,

$$rx \in N$$

as desired.

All this together then means that $\bigcup_{j=1}^{\infty} N_j$ is indeed a submodule of M. $\square$

(b) *Proof.* In order to show that $\mathcal{C}_C(\mathbb{R})$ is an $R-$submodule we will show that it is non-empty, closed under addition, and closed under scalars.

First consider the zero function, which we will denote as o, that maps everything to $0_{\mathbb{R}}$. Since we know

$$o(x) = 0, \ \forall x \in \mathbb{R}$$

then we know it must be in $\mathcal{C}_C(\mathbb{R})$ since for $N = 1$ we have

$$o(x) = 0, \ \forall |x| > 1.$$

Now let $f, g \in \mathcal{C}_C(\mathbb{R})$. Then we know there exists $N_f, N_g \in \mathbb{N}$ such that the following hold,

$$f(x) = 0, \ \forall |x| > N_f \tag{1}$$
$$g(x) = 0, \ \forall |x| > N_g \tag{2}$$

Now we want to show that $(f + g) \in \mathcal{C}_C(\mathbb{R})$. We know addition is defined pointwise so,

$$(f + g)(x) = f(x) + g(x)$$

And we know the addition of continuous function is again continuous. Now let $N_{f+g} = \max\{N_f, N_g\}$, we know then that the following holds,

$$f(x) + g(x) = 0, \ \forall |x| > N_{f+g}. \tag{3}$$

This is because,

$$\begin{aligned} f(x) &= 0, \ \forall |x| > N_{f+g} \geqslant N_f && \text{by (1)} \\ g(x) &= 0, \ \forall |x| > N_{f+g} \geqslant N_g && \text{by (2)} \end{aligned}$$

and $0 + 0 = 0$. Since $N_{f+g} \in \mathbb{N}$ we have then that $(f + g) \in \mathcal{C}_C(\mathbb{R})$ as desired.

Now we will show that $\mathcal{C}_C(\mathbb{R})$ is closed under scalars. Let $r \in R = \mathcal{C}(\mathbb{R})$ and $f \in \mathcal{C}_C(\mathbb{R})$. We know then there exists $N_f \in \mathbb{N}$ such that,

$$f(x) = 0, \ \forall |x| > N_f.$$

We know then that the following holds,

$$(rf)(x) = r(x)f(x) = 0, \ \forall |x| > N_f.$$

This is because $r(x)$ will evaluate to some real number and we know $f(x) = 0$ for all $|x| > N_f$. Any real number times 0 will again be 0, and the product of continuous functions is again continuous, meaning $(rf) \in \mathcal{C}_C(\mathbb{R})$ as desired.

From this we can quickly see that $\mathcal{C}_C(\mathbb{R})$ is not a subring of $\mathcal{C}(\mathbb{R})$. This is because $\mathcal{C}(\mathbb{R})$ contains a multiplicative identity which is the constant function that maps everything to 1.

This function is not in the set $\mathcal{C}_C(\mathbb{R})$ and therefore $\mathcal{C}_C(\mathbb{R})$ cannot be a subring of $\mathcal{C}(\mathbb{R})$ since it can't share the same multiplicative identity.

$\square$

Problem P2 Let M be a left R—module. The *annihlator* of M in R is defined as:

$$Ann_R(M) = \{r \in R : rm = 0 \text{ for all } m \in M\}$$

(a) Prove that $Ann_R(M)$ is a bilateral ideal of R.

(b) If M_1 and M_2 are two left R—modules, prove that

$$Ann_R(M_1 \times M_2) = Ann_R(M_1) \cap Ann_R(M_2)$$

(c) Compute $Ann_R(M)$ when $R = \mathbb{Z}$ and $M = (\mathbb{Z}/112\mathbb{Z})^\times$ is the multiplicative abelian group of units in $\mathbb{Z}/112\mathbb{Z}$

(a) *Proof.* Let $Ann_R(M)$ be denoted as I. We know this I is non-empty since $0 \in R$ and $0m = 0$ for all $m \in M$. So now let $a, b \in I$, we will show that $a + b \in I$. Let $m \in M$. Consider the following,

$$\begin{aligned} (a+b)m &= am + bm && \text{by definition} \\ &= 0 + 0 && \text{since } a, b \in I \end{aligned}$$

and since m was arbitrary we have then that $(a + b)m = 0$ for all $m \in M$, meaning $(a + b) \in I$.
Let $r \in R$ and $a \in I$. We want to show $ra \in I$, so let $m \in M$. We see through the following,

$$\begin{aligned} (ra)m &= r(am) && a \in I \\ &= r0 \\ &= 0 \end{aligned}$$

ra is in I. Now we want to show that $ar \in I$,

$$\begin{aligned} (ar)m &= a(rm) && \text{M is closed under scalars so } rm \in M, \text{ and } a \in I \\ &= 0 \end{aligned}$$

Now with all this together we have that $Ann_R(M)$ is a bilateral ideal of R $\square$

(b) *Proof.* Let $r \in Ann_R(M_1 \times M_2)$. That means then for all $(m_1, m_2) \in M_1 \times M_2$,

$$r(m_1, m_2) = (rm_1, rm_2) = (0, 0)$$

since scalar multiplication is done component wise when working with the cross product of R-modules, $rm_1 = 0$ and $rm_2 = 0$ for all $m_1 \in M_1$ and for all $m_2 \in M_2$ therefore, $r \in Ann_R(M_1) \cap Ann_R(M_2)$.
Because r was arbitrary, $Ann_R(M_1 \times M_2) \subseteq Ann_R(M_1) \cap Ann_R(M_2)$

Now consider $r \in Ann_R(M_1) \cap Ann_R(M_2)$. Let $(m_1, m_2) \in M_1 \times M_2$, we have the following,

$$\begin{aligned} r(m_1, m_2) &= (rm_1, rm_2) && r \in Ann_R(M_1) \text{ and } r \in Ann_R(M_2) \\ &= (0, 0) \end{aligned}$$

which means $r \in Ann_R(M_1 \times M_2)$ and since r was arbitrary we have $Ann_R(M_1) \cap Ann_R(M_2) \subseteq Ann_R(M_1 \times M_2)$.

Together we then have that $Ann_R(M_1 \times M_2) = Ann_R(M_1) \cap Ann_R(M_2)$ as desired. □

(c) Let $z \in \mathbb{Z}$ and $\overline{m} \in (\mathbb{Z}/112\mathbb{Z})^\times$. For z to be an element of the annihilator of M in $\mathbb{Z}$ we must have

$$z\overline{m} = 0$$

for all $\overline{m} \in (\mathbb{Z}/112\mathbb{Z})^\times$. This means then that $112 \mid z$, because $\overline{m} \neq 0$, but

$$112 \mid z \iff 7 \mid z \wedge 2 \mid z$$

giving us the following congruencies,

$$\begin{aligned} z &\equiv 0 \bmod 7 \\ z &\equiv 0 \bmod 2 \end{aligned}$$

and by CRT the solution is $z \equiv 0 \bmod 14$ which is to say $z \in 14\mathbb{Z}$. Therefore the annihilator of $(\mathbb{Z}/112\mathbb{Z})^\times$ in $\mathbb{Z}$ is $14\mathbb{Z}$.