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Show that if $U$ and $W$ are finite-dimensional vector subspaces of a $\mathbb{F}$-vector space $V$, then:

$$\dim(U) + \dim(W) = \dim(U + W) + \dim(U \cap W)$$

This is the analogue of the *Inclusion-Exclusion Principle* for sets adapted to vector spaces. In a certain sense the dimension for vector spaces plays the same role cardinality has with respect to sets.

*Proof.* $U$ and $W$ are finite dimensional, so we have $\dim(U \cap W) = n$. Meaning our basis can be expressed as the set of vectors

$$\{v_1, v_2, \dots, v_n\}$$

This set the basis for $U \cap W$. Meaning this set is linearly independent in $U$ and in $W$. Which means this set of vectors is a subset to the basis for $U$ and $W$. Giving us the basis for $U$ as,

$$\{v_1, v_2, \dots, v_n, u_1, \dots u_i\}$$

And the basis for $W$ as,

$$\{v_1, v_2, \dots, v_n, w_1, \dots, w_j\}$$

This implies $\dim(U) = n + i$ and $\dim(W) = n + j$.

Now our goal is to show the union of $\mathcal{B}_U$ and $\mathcal{B}_W$ serves as a basis for $U + W$.

For any $v \in V$ we know this vector is simply $v = u + w$ for $u \in U$ and $w \in W$. We also know $u$ and $w$ can be expressed as a linear combination of the vectors in it's basis for coeffectients in $\mathbb{F}$. Therefore we have,

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots \alpha_n v_n + \beta_1 u_1 + \dots \beta_i u_i + \gamma_1 v_1 + \dots \gamma_n v_n + \delta w_1 + \dots + \delta_j w_j$$

$$v = (\alpha_1 + \gamma_1)v_1 + \dots + (\alpha_n + \gamma_n)v_n + \beta_1 u_1 + \dots \beta_i u_i + \delta_1 w_1 + \dots \delta_j w_j$$

Therefore the union of $\mathcal{B}_U$ and $\mathcal{B}_W$ spans the whole vector space of $U + W$

Now we want to show these vectors are linearly independent,

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n + \beta_1 u_1 + \dots \beta_i u_i + \delta_1 w_1 + \dots + \delta_j w_j = 0$$

$$\delta_1 w_1 + \dots + \delta_j w_j = -(\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n + \beta_1 u_1 + \dots \beta_i u_i)$$

Which means $\delta_1 w_1 + \dots + \delta_j w_j$ is a vector in the span of $\mathcal{B}_U$, therefore $\delta_1 w_1 + \dots + \delta_j w_j \in U$. Remeber though that $\{w_1, \dots, w_j\}$ is the basis for $W$, and thus $\delta_1 w_1 + \dots + \delta_j w_j$ is in $W$ as well, since it is in both $W$ and $U$ it must also be in their intersection. That means our set of vectors $\{v_1, v_2, \dots, v_n\}$ can be used to express $\delta_1 w_1 + \dots + \delta_j w_j$,

$$\delta_1 w_1 + \dots + \delta_j w_j = \beta_1 v_1 + \dots \beta_n v_n$$

$$\beta_1 v_1 + \dots \beta_n v_n - (\delta_1 w_1 + \dots + \delta_j w_j) = 0$$

Recall though the set of vectors $\{v_1, v_2, \ldots, v_n, w_1, \ldots, w_j\}$ is linearly independent, so the only way to satisfy this is if all $\delta_i$ and $\beta_i$ are equal to 0. The same reasoning applies to

$$\beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_n v_n + \delta_1 w_1 + \cdots + \delta_j w_j$$

in that all coeffectients will have to be 0 to satisfy the equation. Making the above vectors linearly independent. Therefore,

$$\{v_1, v_2, \ldots, v_n, u_1, \ldots u_i, w_1, \ldots, w_j\}$$

are linearly independent. Meaning it satisfies all the criteria to be a basis for $U + W$.

We see though that $\dim(U+W) = n+i+j$. Recall though that $\dim(U) = n+i$ and $\dim(W) = n + j$ and $\dim(U \cap W) = n$.

$$\dim(U) + \dim(W) = n + i + n + j = 2n + i + j$$

$$\dim(U + W) + \dim(U \cap W) = n + i + j + n = 2n + i + j$$

Therefore $\dim(U) + \dim(W) = \dim(U + W) + \dim(U \cap W)$ as desired. $\square$