11/01 Submission

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MATH 101 — Problem Solving — Fall 2021

Problem IC - 10/22 - 81. Find a formula for the sum

$$\sum_{k=1}^{n} \frac{k}{(k+1)!}.$$

Proof. The formula for the given sum will be the following,

$$\frac{(n+1)!-1}{(n+1)!}.$$

We will prove this using induction.

Case $n = 1$:

$$\sum_{k=1}^{1} \frac{1}{(1+1)!} = \frac{1}{2} = \frac{(1+1)!-1}{(1+1)!}.$$

Induction step, assume it holds for $n \leqslant p$.

Case $n = p+1$:

$$\begin{aligned}
\sum_{k=1}^{p+1} \frac{k}{(k+1)!} &= \underbrace{\sum_{k=1}^{p} \frac{k}{(k+1)!}}_{\text{true for } n=p} + \frac{p+1}{(p+1+1)!} \\
&= \frac{(p+1)!-1}{(p+1)!} + \frac{p+1}{(p+2)!} \\
&= \left(\frac{p+2}{p+2}\right) \frac{(p+1)!-1}{(p+1)!} + \frac{p+1}{(p+2)!} \\
&= \frac{(p+2)!-p-2}{(p+2)!} + \frac{p+1}{(p+2)!} \\
&= \frac{(p+2)!-1}{(p+2)!} \\
&= \frac{(n+1)!-1}{(n+1)!}
\end{aligned}$$

as desired. □

Problem OC - 10/25 - 61. Show that $(n+1)^n \geqslant 2^n n!$

Proof. We will prove this using induction.

Case $n = 1$:

$$(1+1)^1 \geqslant 2^1 1! \to 2 \geqslant 2$$

Induction step, assume it holds for $n \geqslant k$.

Case $n = k+1$:

$$\begin{aligned} (k+1)^k &\geqslant 2^k k! && \text{multiply by } 2(k+1) \\ 2(k+1)^{k+1} &\geqslant (k+1)! 2^{k+1} \end{aligned}$$

The last thing we need is that we want to show $2(k+1)^{k+1} \leqslant (k+2)^{k+1}$ but this is of the form

$$\begin{aligned} 2n^n &\leqslant (n+1)^n \\ 2 &\leqslant (1+\frac{1}{n})^n \end{aligned}$$

We know by binomial expansion that this does indeed hold though since the first two terms will be $1 + \frac{n}{n}$. Therefore it holds for $n = k+1$. $\square$

Problem IC - 10/27 - 99. Prove for all real numbers x, y, z that $x^2 + y^2 + z^2 \geqslant xy + yz + zx$

Proof. We that for any real numbers x, y, z that the following holds,

$$(x-y)^2 + (y-z)^2 + (x-z)^2 \geqslant 0.$$

Expanding this inequality will yield the desired result.

$$\begin{aligned} (x-y)^2 + (y-z)^2 + (x-z)^2 &\geqslant 0 \\ x^2 + y^2 - 2xy + y^2 + z^2 - 2yz + x^2 + z^2 - 2xz &\geqslant 0 \\ 2x^2 + 2y^2 + 2z^2 - 2xy - 2yz - 2xz &\geqslant 0 \\ 2(x^2 + y^2 + z^2) &\geqslant 2(xy + yz + xz) \\ x^2 + y^2 + z^2 &\geqslant xy + yz + xz \end{aligned}$$

$\square$

Problem IC - 10/29 - 105. Verify algebraically the identity $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$ and then give a combinatorial proof.

Proof. Algebraically: Recall the formula for n choose k is simply $\dfrac{n!}{k!(n-k)!}$. Applying this to the LHS,

$$\begin{aligned}\binom{n}{r}\binom{r}{k} &= \frac{n!}{r!(n-r)!}\frac{r!}{k!(r-k)!} \\ &= \frac{n!r!}{r!k!(n-r)!(r-k)!} \\ &= \frac{n!}{k!(n-r)!(r-k)!}\end{aligned}$$

Applying to the RHS,

$$\begin{aligned}\binom{n}{k}\binom{n-k}{r-k} &= \frac{n!}{k!(n-k)!}\frac{(n-k)!}{(r-k)!(n-k-(r-k))!} \\ &= \frac{n!}{k!(n-k)!}\frac{(n-k)!}{(r-k)!(n-r)!} \\ &= \frac{n!(n-k)!}{k!(n-k)!(r-k)!(n-r)!} \\ &= \frac{n!}{k!(n-r)!(r-k)!}\end{aligned}$$

We see that the RHS does indeed equal the LHS, meaning the identity is true.
Not sure how to show combinatorially. □