Nov 15 Submission

Kevin Guillen

MATH 101 — Problem Solving — Fall 2021

Problem IC — 11/10 — 135. Show if $a^2 + b^2 = c^2$ then $3|ab$

Proof. First let us figure out what the remainder of any square is modulo 3. By Fermat's theorem we have $n^3 \equiv n \bmod 3$. This means $3|(n^3 - n) \to 3|n(n^2 - 1)$. Because 3 is a prime that means it must divide one of these factors. In the case that 3 divides n, then it must also divide n^2. In the case that it divides $(n^2 - 1)$ that means $n^2 \equiv 1 \bmod 3$. Meaning the only possible remainders are 0 and 1.

If $3 \nmid ab$ that would mean neither a or b are divisble by 3. Implying they are of the form $a^2 \equiv 1 \bmod 3$ and $b^2 \equiv 1 \bmod 3$. Therefore $c^2 \equiv 2 \bmod 3$, but that is impossible since the only possible remainders for a square mod 3 are 0 and 1. Therefore if the equation holds then $3|ab$. □

Problem IC — 11/10 — 136. If $x^3 + y^3 = z^3$ show that at least 1 of x, y, z is divisible by 7.

Proof. First let us determine what the cubes of any integer modulo 7 would be. By Fermat's theorem we have that

$$n^7 \equiv n \bmod 7$$

Which means $7|(n^7 - n) \to 7|n(n^{3-1})(n^3 + 1)$. Because 7 is a prime it must divide one of these factors. In the case that 7 divides n then it must divide n^3, implying $n^3 \equiv 0 \bmod 7$. In the case that 7 divides $(n^3 - 1)$ then that means $n^3 \equiv 1 \bmod 7$. Finally if 7 divides $(n^3 + 1)$ that means $n^3 \equiv -1 \bmod 7$.

Now in the case that neither x^3 or y^3 are divisible by 7. That means they have a remainder of ± 1 when dividing by 7. Without loss of generality say x^3 has remainder -1 and y^3 has remainder 1. Then their sum has to have reaminder 0 meaning z^3 will be divisible by 7. In the case they both have remainder 1 that would result in a contradiction because $z^3 \equiv 2 \bmod 7$ is not possible. Therefoer at least one of these integers is disvisble by 7 if the given equation holds. □

Problem IC — 11/10 — 139. For what values of n can $\{1, 2, \dots, n\}$ be partitioned into three subsets with equal sums?

Proof. If we are able to partition the set into 3 subsets that all have the same sum that would be the sum of all the terms is divisble by 3. This gives us the following requirement,

$$3 | \sum_{k=1}^{n} k$$

We can obtain a formula for the summation through the following,

$$1 + 2 + \cdots + (n-1) + n = n + (n-1) + \cdots + 2 + 1$$

adding both sides to each other we get

$$\underbrace{(n+1) + (n+1) + \cdots + (n+1)}_{n} = n(n+1)$$

now we have to divide by 2 to undo our addition and we get the following,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

This means in order to get 3 paritions that have equal sum, 3 must divide $\frac{n(n+1)}{2}$. Therfore n must satisfy either $n \equiv 0 \bmod 3$ or $n \equiv 2 \bmod 3$. We see though in the case that $n = 3$, such a partition is not possible. Therefore there is also a lower bound for n. We see this lower bound is simply $n = 5$. We see this throught the following,

$$\{1, 4\}, \{2, 3\}, \{5\}. \tag{1}$$

Therfore $n >= 5$ and either $n \equiv 0 \bmod 3$ or $n \equiv 2 \bmod 3$ □

Problem IC — 11/12 — 143. Find all positive integers n such that $2^4 + 2^7 + 2^n$ is a perfect square.

Proof. This is same as finding n such that $n^2 + 144 = k^2$. Consider the following though,

$$\begin{aligned}
2^n + 144 &= k^2 \\
2^n &= k^2 - 144 \\
2^n &== (k+12)(k-12)
\end{aligned}$$

Therefore we have that $(k+12)$ and $(k-12)$ must both be powers of 2 and that they must differ by 24. We can see that 8 and 32 differ by 24 and are both powers of 2. This gives us,

$$8 \cdot 32 = 2^3 2^5 = 2^8.$$

Thus the only integer n that can satisfy this is $n = 8$. This is because the distance between powers of two is always increasing there will never be another pair of powers of 2 such that their difference is 24. □

Problem OC — 11/10 — 88. Prove that there does not exist a natural number n such that $n(n+1)$ is a perfect square.

Proof. Assume $n(n+1)$ is indeed a perfect square. That means it can expressed as, $n(n+1) = k^2$ for some $k \in \mathbb{Z}$. Consider the following though,

$$\begin{aligned} n(n+1) &= k^2 \\ n^2 + n &= k^2 \\ n^2 + k^2 &= -n \\ (n+k)(n-k) &= -n \end{aligned}$$

But either $(n+k)$ or $(n-k)$ is greater than $|n|$, so this is a contradiction. Therfore $n(n+1)$ cannot be a perfect square. □

Problem OC — 11/12 — 90. Prove there is a unique integer n such that $2^8 + 2^{11} + 2^n$ is a perfect square.

Proof. This is similair to our IC class problem 143. First we see we are looking to satisfy the following,

$$\begin{aligned} 2^8 + 2^{11} + 2^n &= k^2 \\ 2^n + 2304 &= k^2 \\ 2^n &= k^2 - 2304 \\ 2^n &= (k-48)(k+48) \end{aligned}$$

Thus there has to be two powers of 2 such that their difference is 96. Consider 128, we see $128 - 96 = 32$, and botb 128 and 32 are powers of 2. This gives us the following,

$$32 \cdot 128 = 2^5 2^7 = 2^{12}.$$

Therefore $n = 12$ meaning there does exist indeed exist an n such that the sum given is a perfect square. □