Homework 4

Kevin Guillen

MATH 201 — Algebra II — Winter 2022

Problem 1 Consider the complex matrix

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & -2 & 1 \end{pmatrix}$$

(1) Compute the characteristic polynomial of A: Show your work.

(2) Determine the Jordan form of A: Show your work.

(1) ***Solution.*** To find the characteristic polynomial we must calculate $\det(A - \lambda I_3)$. This works out to be,

$$\begin{aligned} \det\left(\begin{pmatrix} 1-\lambda & 2 & 0 \\ 2 & 1-\lambda & 2 \\ 0 & -2 & 1-\lambda \end{pmatrix}\right) &= (1-\lambda)((1-\lambda)^2+4) - 2(2-2\lambda) + 0 \\ &= (1-\lambda)(\lambda^2 - 2\lambda + 5) - 4 + 4\lambda \\ &= -\lambda^3 + 2\lambda - 5\lambda + \lambda^2 + 5 \\ &= -\lambda^3 + 3\lambda^2 - 3\lambda + 1. \end{aligned}$$

So we have $-\lambda^3 + 3\lambda^2 - 3\lambda + 1$ to be the characteristic polynomial of A. □

(2) ***Solution.*** First we will find the roots of the characteristic polynomial to determine the eigenvalues for A,

$$-\lambda^3 + 3\lambda^2 - 3\lambda + 1 = -(\lambda - 1)^3$$

it is clear that 1 is the eigenvalue of A with multiplicity 3.

We have 3 scenarios for the potential Jordan from of A that is, one 3×3 block, one 2×2 block with one 1×1 block, or three 1×1 blocks. Let us consider the dimension of the eigenspace for $\lambda = 1$,

$$(A - 1I_3) = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & -2 & 0 \end{pmatrix}$$

so we solve the following,

$$\begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & -2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

where α is a scalar. Therefore the eigenspace for $\lambda = 1$ is spanned by 1 vector meaning it has dimension 1. So we have that the number of Jordan blocks for $\lambda = 1$ to be 1. So we must have one 3×3 jordan block. Meaning the Jordan form of A has to be,

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

□

Problem 2 Consider the complex matrix

$$B = \begin{pmatrix} 4 & 5-5i \\ 5+5i & -1 \end{pmatrix}$$

Find a basis (v_1, v_2) of $\mathbb{C}^2$ such that

(a) Each of v_1, v_2 is an eigenvector of B.

(b) $< v_1 \mid v_1 >_{std} = 1 =< v_2 \mid v_2 >_{std}$ and $< v_1 \mid v_2 >_{std} = 0$ at the same time.

Solution. Let us first find the eigenvalues of the given matrix B,

$$\begin{aligned} \det\left(\begin{pmatrix} 4-\lambda & 5-5i \\ 5+5i & -1-\lambda \end{pmatrix}\right) &= (4-\lambda)(-1-\lambda) - (5-5i)(5+5i) \\ &= -4 + \lambda - 4\lambda + \lambda^2 + 25 - 25i + 25i + 25 \\ &= \lambda^2 - 3\lambda - 54 \end{aligned}$$

solving for λ,

$$\lambda = \frac{3 \pm \sqrt{9 - 4(-54)}}{2} = \frac{3 \pm \sqrt{225}}{2} = \frac{3 \pm 15}{2}$$

so $\lambda_1 = 9$ and $\lambda_2 = -6$

Now solving for the eigenvector for λ_1 first we get,

$$\begin{pmatrix} -5 & 5-5i \\ 5+5i & -10 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1-i \\ 1 \end{pmatrix}$$

so $w_1 = \begin{pmatrix} 1-i \\ 1 \end{pmatrix}$ next we need to get the eigenvector for λ_2,

$$\begin{pmatrix} 10 & 5-5i \\ 5+5i & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1+i \\ 2 \end{pmatrix}$$

and we have $w_2 = \begin{pmatrix} -1+i \\ 2 \end{pmatrix}$. Now we must scale these eigenvectors with some α and βin $\mathbb{C}$, so that $< \alpha w_1 \mid \alpha w_1 >= 1 =< \beta w_2 \mid \beta w_2 >$ and $< \alpha w_1 \mid \beta w_2 >= 0$. Once that is met we can simply set $v_1 = \alpha w_1$ and $v_2 = \beta w_2$

To solve for α to satisfy $< \alpha w_1 \mid \alpha w_1 >= 1$ we simply solve for α in the following,

$$\begin{aligned} (\alpha + \alpha i)(\alpha - \alpha i) + \alpha^2 &= 1 \\ 2\alpha^2 + \alpha^2 &= 1 \\ 3\alpha^2 &= 1 \\ \alpha &= \sqrt{\frac{1}{3}} \end{aligned}$$

Now verifying $< \alpha w_1 \mid \alpha w_1 >= 1$,

$$< \alpha w_1 \mid \alpha w_1 >= \begin{pmatrix} \sqrt{\frac{1}{3}}(1+i) & \sqrt{\frac{1}{3}} \end{pmatrix} \begin{pmatrix} \sqrt{\frac{1}{3}}(1-i) \\ \sqrt{\frac{1}{3}} \end{pmatrix} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$$

Now solving for β to satisfy $< \beta w_2 \mid \beta w_2 >= 1$ we do like before and solve for β in the following,

$$\begin{aligned} (-\beta - \beta i)(-\beta + \beta i) + 4\beta &= 1 \\ 2\beta^2 + 4\beta^2 &= 1 \\ 6\beta^2 &= 1 \\ \beta &= \sqrt{\frac{1}{6}} \end{aligned}$$

Now verifying $< \beta w_2 \mid \beta w_2 >= 1$,

$$< \beta w_2 \mid \beta w_2 >= \begin{pmatrix} \sqrt{\frac{1}{6}(-1-i)} & 2\sqrt{\frac{1}{6}} \end{pmatrix} \begin{pmatrix} \sqrt{\frac{1}{6}}(-1+i) \\ 2\sqrt{\frac{1}{6}} \end{pmatrix} = \frac{1}{6} + \frac{1}{6} + \frac{4}{6} = 1$$

Finally verifying $< \alpha w_1 \mid \beta w_2 >= 0$,

$$< \alpha w_1 \mid \beta w_2 >= \begin{pmatrix} \sqrt{\frac{1}{3}}(1+i) & \sqrt{\frac{1}{3}} \end{pmatrix} \begin{pmatrix} \sqrt{\frac{1}{6}}(-1+i) \\ 2\sqrt{\frac{1}{6}} \end{pmatrix} = -\frac{\sqrt{2}}{3} + \frac{\sqrt{2}}{3} = 0$$

as desired. Therefore we have,

$$(v_1, v_2) = \left(\sqrt{\frac{1}{3}} \begin{pmatrix} (1-i) \\ 1 \end{pmatrix}), \sqrt{\frac{1}{6}} \begin{pmatrix} -1+i \\ 2 \end{pmatrix} \right)$$

$\square$