# Homework 8

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MATH 103A — Complex Analysis — Spring 2022

**Problem 8.1**

(a) Let C denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Compute

$$\int_C \frac{\cos z}{z(z^2+8)}$$

***Solution.*** Let,

$$f(z) = \frac{\cos z}{z^2+8}$$

we have then that,

$$\int_C \frac{\cos z}{z(z^2+8)}\,dz = \int_C \frac{f(z)}{z-0}\,dz.$$

Let $z_0 = 0$. Since $f(z)$ is holomorphic on C and $z_0$ is in the interior of C, we have by Cauchy's Integral Formula that,

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz \\ \frac{1}{8} &= \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz \\ \frac{\pi i}{4} &= \int_C \frac{f(z)}{z-z_0}\,dz \end{aligned}$$

therefore,

$$\int_C \frac{\cos z}{z(z^2+8)} = \frac{\pi i}{4}.$$

□

(b) Let C denote the circle centered at $i$ of radius 2, positively oriented. Compute

$$\int_C \frac{1}{(z^2+4)^2}$$

***Solution.*** Let us note that,

$$(z^2+4) = (z+2i)(z-2i).$$

Now let $f(z) = \dfrac{1}{z+2i}$, we can rewrite the given integral as,

$$\begin{aligned}
\int_C \frac{1}{(z^2+4)^2} &= \int_C \frac{1}{(z-2i)^2(2+2i)^2} \\
&= \int_C \frac{f(z)}{(z-2i)^2}\,dz.
\end{aligned}$$

Let $z_0 = 2i$. As before, we know that $f(z)$ is holomorphic on the given contour since the place it is not holomorphic is when $z = -2i$ which is not in or part of the contour, and since $z_0$ is inside the contour, we can apply the generalization of Cauchy's Integral Formula to obtain,

$$f'(z_0) = \frac{1!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2} \tag{1}$$

We obtain the derivative of $f$ to be,

$$f'(z) = -\frac{2}{(z+2i)^3}$$

which lets us calculate the LHS of (1) to be,

$$f'(2i) = -\frac{2}{(4i)^3} = \frac{2}{64i} = -\frac{i}{32}.$$

Solving the integral now in (1) we get,

$$\begin{aligned}
-\frac{i}{32} &= \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2} \\
\frac{\pi}{16} &= \int_C \frac{f(z)}{(z-z_0)^2}
\end{aligned}$$

therefore,

$$\int_C \frac{1}{(z^2+4)^2} = \frac{\pi}{16}.$$

$\square$

**Problem 8.2** Let C be the circle of radius 3, positively oriented, centered at the origin. Show that if

$$g(w) = \int_C \frac{2z^2 - z - 2}{z - w}\, dz, \quad |w| \neq 3,$$

then $g(2) = 8\pi i$. What is the value of $g(w)$ when $|w| > 3$?

***Solution.*** Let $f(z) = 2z^2 - z - 2$, we rewrite g now as,

$$g(w) = \int_C \frac{2z^2 - z - 2}{z - w}\, dz = \int_C \frac{f(z)}{z - w}\, dz$$

so evaluating at $w = 2$ we know 2 is in the interior of C, and $f(z)$ is holomorphic on C, so we can applying Cauchy's Integral formula to obtain,

$$\begin{aligned} f(2) &= \frac{1}{2\pi i} \int_C \frac{2z^2 - z - 2}{z - 2}\, dz \\ 4 &= \frac{1}{2\pi i} \int_C \frac{2z^2 - z - 2}{z - 2}\, dz \\ 8\pi i &= \int_C \frac{2z^2 - z - 2}{z - 2}\, dz \end{aligned}$$

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We have that $g(w)$ is holomorphic over C for $|w| > 3$, so by Cauchy-Goursat Theorem $g(w) = 0$ when $|w| > 3$. □

**Problem 8.3** Let $C$ be the unit circle parametrised as $z(t) = e^{it}$, $-\pi \leqslant t \leqslant \pi$. First show that for any $a \in \mathbb{R}$,

$$\int_C \frac{e^{az}}{z}\, dz = 2\pi i$$

Then write this integral in terms of $t$, using the definition of a contour integral, to derive the integration formula

$$\int_0^{\pi} e^{a \cos t} \cos(a \sin t)\, dt = \pi.$$

***Solution.*** □

**Problem 8.4** Let $f$ be an entire function such that there exists an $M > 0$ such that $\mathrm{Re}(f(z)) \geqslant M$ for all $z \in \mathbb{C}$. Prove that $f$ is constant.

***Solution.*** $\square$

**Problem 8.5** Let $f$ be an entire function such that $|f(z)| \leqslant A|z|$ for all $z$, where $A$ is a fixed positive number. Show that $f(z) = \alpha z$, where $\alpha$ is a complex constant.

***Solution.*** $\square$

**Collaborators:**

**References:**

- [Book(s): Title, Author]
- [Online: Link]
- [Notes: Link]

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