Homework 2

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MATH 200 — Algebra I — Fall 2021

Problem 2.6. Show that for any non-empty subset X of a group G, the normalizer of X, $N_G(X)$ and the centralizer of X, $C_G(X)$ is again a subgroup of G. Show also that $C_G(X)$ is contained in $N_G(X)$.

Proof. **Normalizer** We know the normalizer of a subset X is defined as the following,

$$N_G(X) = \left\{ g \in G \mid gXg^{-1} = X \right\}.$$

So consider $x, y \in N_G(x)$. Let $z = xy$, we want to show that $z \in N_G(X)$. In other words we want to show $zXz^{-1} = X$, based on the above. We can see through the following that this is indeed true,

$$\begin{aligned} zXz^{-1} &= (xy)X(xy)^{-1} && (xy)^{-1} = y^{-1}x^{-1} \\ &= xyXy^{-1}x^{-1} && y \in N_G(x) \\ &= xXx^{-1} && x \in N_G(x) \\ &= X. \end{aligned}$$

Meaning $N_G(X)$ is closed under group operation.

Let $y \in N_G(X)$, based on the definition of the normalizer though,

$$\begin{aligned} yXy^{-1} &= X && \text{taking } y \text{ on the right} \\ yX &= Xy && \text{taking } y^{-1} \text{ on the left} \\ X &= y^{-1}Xy && y = (y^{-1})^{-1} \\ X &= y^{-1}X(y^{-1})^{-1} \end{aligned}$$

that y^{-1} is indeed in $N_G(X)$. Thus by the subgroup criterion, $N_G(X)$ is indeed a subgroup.

Centralizer: We know the definition of the centralizer of a subset X is the following,

$$C_G(X) = \left\{ g \in G \mid gxg^{-1} = x, \forall x \in X \right\}.$$

So consider $a, b \in C_G(X)$. Let $z = ab$, we want to show that $z \in C_G(X)$. In other words we want to show $zxz^{-1} = x$ for all $x \in X$. We see through the following that this does indeed hold.

$$\begin{aligned} zxz^{-1} &= (ab)x(ab)^{-1} && (ab)^{-1} = b^{-1}a^{-1} \\ &= (ab)x(b^{-1}a^{-1}) && \text{We know associativity holds in } G \\ &= a(bx^{-1})a^{-1} && b \in C_G(X) \\ &= axa^{-1} && a \in C_G(X) \\ &= x \end{aligned}$$

Meaning $C_G(X)$ is closed under group operation.

Let $y \in C_G(X)$. By definition that means for all $x \in X$, $yxy^{-1} = x$, but consider the following,

$$
\begin{aligned}
yxy^{-1} &= x && \text{taking } y^{-1} \text{ on the left} \\
xy^{-1} &= y^{-1}x && \text{taking } y \text{ on the right} \\
x &= y^{-1}xy && y = (y^{-1})^{-1} \\
x &= y^{-1}x(y^{-1})^{-1}.
\end{aligned}
$$

This means that for any $y \in C_G(X)$, that y^{-1} is also in $C_G(X)$. Thus $C_G(X)$ is a subgroup.

Now we want to show that the centralizer is contained in the normalizer. Expanding on the definition of the normalizer $gXg^{-1} = X \to gX = Xg$. This means there exists some $s, t \in X$ such that $gs = tg$. What we see though is that this is simply a weaker property when compared to the centralizer definition. Expanding on the definition of the centralizer, for all $x \in X$ we have $gxg^{-1} = x \to gx = xg$. Meaning any $g \in C_G(X)$ has the property that $gs = tg$ where $t = s = x$, which means it is also in $N_G(X)$, thus $C_G(X) \subset N_G(X)$ ☐

Problem 2.7. Let $f : G \to H$ be a group homomorphism.

(a) If $U \leqslant G$ then $f(U) \leqslant H$.

(b) If $V \leqslant H$ then $f^{-1}(V) = \{g \in G \mid f(g) \in V\}$ is a subgroup of G.

(c) Show that f is injective if and only if $\ker(f) = \{1\}$

Proof. (a) Let $x, y \in f(U)$, and let $z = xy$, we want to show $z \in f(U)$. Since $x, y \in f(U)$, that means there exists $x', y' \in U$ such that $f(x') = x$ and $f(y') = y$. Giving us,

$$
\begin{aligned}
z &= xy \\
&= f(x')f(y') && \text{f is a homomorphism so,} \\
&= f(x'y')
\end{aligned}
$$

Because U is a subgroup then $x'y' \in U$, meaning $z = f(x'y') \in f(U)$, thus $f(U)$ is closed under group operation.

Given $x \in f(U)$, we want to show $x^{-1} \in f(U)$. By $x \in f(U)$ that means there exists $x' \in U$ such that $x = f(x')$. Since U is a subgroup there exists $x'^{-1} \in U$, meaning $f(x'^{-1}) \in f(U)$. Recall though f is a homomorphism that means it respects inverses, thus $f(x'^{-1}) = f(x')^{-1}$, which will be x^{-1}. We verify through the following,

$$
\begin{aligned}
xx^{-1} &= f(x')f(x')^{-1} \\
&= f(x'x'^{-1}) \\
&= f(1) \\
&= 1
\end{aligned}
$$

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