Homework 3

Kevin Guillen

MATH-103A — Complex Analysis — Spring 2022

Problem 3.1 Let

$$M(z) = \frac{az+b}{cz+d}, \quad ad - bc \neq 0.$$

(a) Prove that $\lim_{z\to\infty} M(z) = \infty$ if $c = 0$.

Proof. If we $c = 0$ then we have that $M(x) = \dfrac{az+b}{d}$. We recall Theorem 5.11, which tell us that proving

$$\lim_{z\to 0} \frac{1}{M(\frac{1}{z})} = 0$$

is equivalent to what we are asked to prove. We first see that,

$$\frac{1}{M(\frac{1}{z})} = \frac{d}{a(\frac{1}{z}) + b}.$$

and recall from class that $\lim_{z\to 0} \dfrac{1}{z} = \infty$. So taking the desired limit we see,

$$\begin{aligned}\lim_{z\to 0} \frac{1}{M(\frac{1}{z})} &= \frac{d}{a\left(\lim_{z\to 0} \frac{1}{z}\right) + b} \\ &= \frac{1}{\infty} \\ &= 0\end{aligned}$$

which then implies that $\lim_{z\to\infty} M(z) = \infty$ as desired.

$\square$

(b) Prove that, if $c \neq 0$

$$\lim_{z\to\infty} M(z) = \frac{a}{c} \quad \text{and} \quad \lim_{z\to -d/c} M(z) = \infty.$$

Proof. We can apply Theorem 5.11 again, which gives us that $\lim_{z\to\infty} M(z) = \lim_{z\to 0} M(1/z)$. So working with the RHS of this equality we get,

$$\begin{aligned}
\lim_{z\to 0} M(1/z) = \lim_{z\to 0} \frac{a\left(\frac{1}{z}\right)+b}{c\left(\frac{1}{z}\right)+d} &= \lim_{z\to 0} \frac{\frac{a+bz}{z}}{\frac{c+dz}{z}} \\
&= \lim_{z\to 0} \frac{a+bz}{c+dz} \\
&= \frac{a}{c}.
\end{aligned}$$

Which means then that $\lim_{z\to\infty} M(z) = \frac{a}{c}$, as desired. □

(c) Compute $M'(z)$. For what z is $M'(z) = 0$? That is, describe the set $\{z \in \mathbb{C} \mid M'(z) = 0\}$.

Proof. We see that $M(z)$ is the quotient of two complex functions $f(z) = az + b$ and $g(z) = cz + d$. Meaning we can apply the Quotient rule from Theorem 6.12 to get,

$$M'(z) = \frac{f'(z)(cz+d) - (az+b)g'(z)}{(cz+d)^2}$$

We quickly verify what the derivative of this type of function,

$$\begin{aligned}
f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} &= \frac{az + ah + b - az - b}{h} \\
&= \frac{ah}{h} \\
&= a
\end{aligned}$$

also telling us that $g'(x) = c$. Putting this all back into the first equation we get,

$$\begin{aligned}
M'(z) &= \frac{a(cz+d) - (az+b)c}{(cz+d)^2} \\
&= \frac{acz + ad - azc - bc}{(cz+d)^2} \\
&= \frac{ad - bc}{(cz+d)^2}.
\end{aligned}$$

There is no z such that $M'(z) = 0$. This is because the only way for the derivative to be 0 would be for the numerator to be 0, but from what was given $ad - bc \neq 0$.

Meaning the set of all complex numbers such that $M'(z) = 0$ can be best described as the empty set, since that is what it is.

□

Problem 3.2 Example 5.7 in the Lecture Notes tells us that polynomials are continuous.

(a) Prove that the complex conjugation function $\sigma(z) := \overline{z}$ is continuous.

Proof. To show that σ is indeed continuous we can consider:

$$\begin{aligned}
\lim_{z \to z_0} |\sigma(z) - \sigma(z_0)| &= \lim_{z \to z_0} |\overline{z} - \overline{z_0}| & \\
&= \lim_{z \to z_0} |\overline{z - z_0}| & \text{By Proposition 2.4} \\
&= \lim_{z \to z_0} |z - z_0| & \text{Modulus is invariant to sign} \\
&= 0
\end{aligned}$$

meaning σ is continuous. □

(b) Prove that a polynomial in $\overline{z}$ is continuous. That is, prove that a polynomial given as

$$p(\overline{z}) = a_n \overline{z}^n + \cdots + a_1 \overline{z} + a_0, \quad a_i \in \mathbb{C},\ a_n \neq 0$$

is continuous.

Proof. Let $g(z) = a_n z^n + \cdots + a_1 z + a_0$ where a_i for $i = 1, \ldots, n$ are the same a_i from the given polynomial. Now let $\sigma(z) = \overline{z}$ be the complex conjugation function given from the previous part.

We know that $g(z)$ is clearly a polynomial and the lecture notes already tell us that polynomials are continuous. We just showed that $\sigma(z)$ is continuous. We see that,

$$g(\sigma(z)) = p(\overline{z})$$

meaning that $p(\overline{z})$ is just a composition of continuous functions. Therefore by Theorem 6.2 from the lecture notes we have that $p(\overline{z})$ is continuous as well. □

(c) Prove that the following functions are continuous by writing them as a sum or product of polynomials $p(z)$ and $q(\overline{z})$

(i) $R(z) := \operatorname{Re} z$

Proof. We recall from class that we can express $\operatorname{Re} z$ as $\frac{z + \overline{z}}{2}$, but this can be expressed as $\frac{z}{2} + \frac{\overline{z}}{2}$. Meaning we can let $p(z) = \frac{z}{2}$ and we see that,

$$\operatorname{Re} z = p(z) + p(\overline{z}) = \frac{z + \overline{z}}{2}$$

□

(ii) $I(z) \coloneqq \operatorname{Im} z$

Proof. Note that for $z = a + bi$ we have,

$$z - \overline{z} = a + bi - a + bi = 2bi$$

meaning we have a similar expression as before, $\operatorname{Im} z = \frac{z - \overline{z}}{2}$ which can be separated again as $q(z) = \frac{z}{2}$. Now we can express $\operatorname{Im} z$ as,

$$\operatorname{Im} z = q(z) - q(\overline{z}) = \frac{z - \overline{z}}{2}$$

□

(iii) $N(z) \coloneqq |z|^2$

Proof. Let $z = a + bi$ as before. We see that,

$$\begin{aligned} N(z) = a^2 + b^2 &= (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2 \\ &= (p(z) + p(\overline{z}))^2 + (q(z) - q(\overline{z}))^2 \end{aligned}$$

as desired. □

Problem 3.3 Show that the function $f : \mathbb{C} \to \mathbb{C}$ given by

$$f(z) = \begin{cases} \dfrac{\overline{z}^2}{z} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

is not differentiable at 0, possibly using Example 5.4 as an inspiration.

Proof. We know that if a function is differentiable at a point, then the limit given in Definition 6.8 is exists, and by existing it must be unique. Let us consider the derivative of f at 0,

$$\begin{aligned} f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h - 0} &= \lim_{h \to 0} \frac{f(h)}{h} \\ &= \lim_{h \to 0} \frac{\overline{h}^2/h}{h} \\ &= \lim_{h \to 0} \frac{\overline{h}^2}{h^2} \\ &= \lim_{h \to 0} \left(\frac{\overline{h}}{h}\right)^2 \end{aligned}$$

We know h is of the form $a + bi$ so we can consider approaching 0 alongside the real axis, that is $b = 0$. So we have,

$$\begin{aligned} f'(0) = \left(\frac{\overline{h}}{h}\right)^2 &= \left(\frac{a - bi}{a + bi}\right)^2 \\ &= \left(\frac{a}{a}\right)^2 && \text{since } b = 0 \\ &= 1^2 \\ &= 1 \end{aligned}$$

Now let's consider the limit when h is approaching zero along the diagonal where $a = b$. We have,

$$\begin{aligned} f'(0) = \left(\frac{\overline{h}}{h}\right)^2 &= \left(\frac{a - ai}{a + ai}\right)^2 \\ &= \frac{a^2 - a^2 - 2a^2 i}{a^2 - a^2 + 2a^2 i} \\ &= \frac{-2a^2 i}{2a^2 i} \\ &= -1 \end{aligned}$$

which does not equal the limit when approaching along the real line. Therefore f is not differentiable at 0. $\square$

Problem 3.4 Let G be a domain and $f : G \to \mathbb{C}$ a function that is differentiable at every point in G. Consider the domain

$$G^* = \{z \in \mathbb{C} \mid \overline{z} \in G\}$$

and the function

$$f^* : G^* \to \mathbb{C},\ z \mapsto \overline{f(\overline{z})}$$

Show that f^* is differentiable at every point in G^*.

Proof. We know if f^* is differentiable at a point $z_0 \in G^*$ then we'd have the following limit exist,

$$f^*(z_0) = \lim_{z \to z_0} \frac{f^*(z) - f^*(z_0)}{z - z_0}$$

we know though that this limit will be equal to,

$$\begin{aligned}
\lim_{z \to z_0} \frac{f^*(z) - f^*(z_0)}{z - z_0} &= \lim_{\overline{z} \to \overline{z_0}} \frac{f^*(\overline{z}) - f^*(\overline{z_0})}{\overline{z} - \overline{z_0}} &\quad (1)\\
&= \lim_{\overline{z} \to \overline{z_0}} \frac{\overline{f(z)} - \overline{f(z_0)}}{\overline{z} - \overline{z_0}} &\quad (2)\\
&= \lim_{\overline{z} \to \overline{z_0}} \frac{\overline{f(z) - f(z_0)}}{\overline{z - z_0}} &\quad (3)\\
&= \lim_{\overline{z} \to \overline{z_0}} \overline{\left(\frac{f(z) - f(z_0)}{z - z_0}\right)} &\quad (4)
\end{aligned}$$

Note though that elements are in the domain G^* if their conjugate is in the domain G, meaning $\overline{z}$ and $\overline{z_0}$ are in G. We see once we input these into f^* we are really taking the derivative of f at $\overline{z}$ in G, and then taking the conjugation of that. We know can take this derivative because f is defined to be differentiable at every point in G. In other words we know this derivative exists:

$$\lim_{\overline{z} \to \overline{z_0}} \frac{f(z) - f(z_0)}{z - z_0} \qquad (5)$$

We recall one more property about limits though and that is,

$$\lim_{z \to z_0} f(x) = L \implies \lim_{z \to z_0} \overline{f(x)} = \overline{L}$$

because we know f to be differentiable at $\overline{z_0}$ we have that (5) is equal to some value L and therefore (4) is equal to $\overline{L}$, but then

$$\lim_{z \to z_0} \frac{f^*(z) - f^*(z_0)}{z - z_0} = \overline{L}$$

which means that f^* is differentiable at every point in G^*.

□

Problem 3.5 For each function, determine all points at which the derivative exists. When the derivative exists, find its value. Use Example 6.10 from the Lecture Notes as an inspiration.

(a) $f(z) = z + i\overline{z}$

Proof. We know the derivative is of the form,

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

so expanding the RHS we get,

$$\begin{aligned}\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} &= \lim_{h \to 0} \frac{z + h + i\overline{z} + i\overline{h} - z - i\overline{z}}{h} \\ &= \lim_{h \to 0} \frac{h + i\overline{h}}{h}\end{aligned}$$

We know that h is of the form $a + bi$, so if we consider approaching 0 alongside the real axis we'd have $h = \overline{h}$, which turns the limit to,

$$\begin{aligned}\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} &= \lim_{h \to 0} \frac{h + ih}{h} \\ &= 1 + i.\end{aligned}$$

Now if we consider when approaching 0 alongside the imaginary axis, that is $h = -\overline{h}$ we have,

$$\begin{aligned}\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} &= \lim_{h \to 0} \frac{h - ih}{h} \\ &= 1 - i\end{aligned}$$

Because a limit is unique that would be then that $1 + i = 1 - i$, but this isn't true, therefore the limit does not exist, meaning there are no points where the function of the derivative exists.

□

(b) $g(z) = z \operatorname{Im} z$

Proof. We know the derivative is of the form,

$$g'(z) = \lim_{h \to 0} \frac{g(z+h) - g(z)}{h}$$

so expanding the RHS we get,

$$\begin{aligned}\lim_{h \to 0} \frac{g(z+h) - g(z)}{h} &= \lim_{h \to 0} \frac{(z+h) \operatorname{Im}(z+h) - z \operatorname{Im} z}{h} \\ &= \lim_{h \to 0} \frac{z \operatorname{Im}(z+h) + h \operatorname{Im}(z+h) - z \operatorname{Im} z}{h}\end{aligned}$$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{\color{red}{z \operatorname{Im} z} + z \operatorname{Im} h + h \operatorname{Im} z + h \operatorname{Im} h - \color{red}{z \operatorname{Im} z}}{h} \\
&= \lim_{h\to 0} \frac{z \operatorname{Im} h + h \operatorname{Im} z + h \operatorname{Im} h}{h} && \text{apply 3.2(c)(iii)} \\
&= \lim_{h\to 0} \left(\frac{zh - z\overline{h}}{2} + \frac{hz - h\overline{z}}{2} + \frac{h^2 - h\overline{h}}{2} \right) h^{-1} \\
&= \lim_{h\to 0} \frac{zh - z\overline{h} + hz - h\overline{z} + h^2 - h\overline{h}}{2h}
\end{aligned}
$$

Now let us consider when h approaches 0 along the real axis, that is $h = \overline{h}$,

$$
\begin{aligned}
\lim_{h\to 0} \frac{g(z+h) - g(z)}{h} &= \lim_{h\to 0} \frac{zh - zh + hz - h\overline{z} + h^2 - hh}{2h} \\
&= \lim_{h\to 0} \frac{hz - h\overline{z}}{2h} \\
&= \lim_{h\to 0} \frac{z - \overline{z}}{2}
\end{aligned}
$$

Now let's see when h approaches 0 along the imaginary axis, that is $h = -\overline{h}$,

$$
\begin{aligned}
\lim_{h\to 0} \frac{g(z+h) - g(z)}{h} &= \lim_{h\to 0} \frac{zh + zh + hz - h\overline{z} + h^2 + hh}{2h} \\
&= \lim_{h\to 0} \frac{3zh + 2h^2 - h\overline{z}}{2h} \\
&= \lim_{h\to 0} \frac{3z + 2h - \overline{z}}{2} && h \to 0 \\
&= \lim_{h\to 0} \frac{3z - \overline{z}}{2}.
\end{aligned}
$$

Because limits are unique if $g'(z)$ existed we would have,

$$
\begin{aligned}
\frac{3z - \overline{z}}{2} &= \frac{z - \overline{z}}{2} \\
3z - \overline{z} &= z - \overline{z} \\
2z &= 0 \\
z &= 0
\end{aligned}
$$

meaning $g'(z)$ can only exist if $z = 0$, now we just need to check if it actually exists. We see that it does through the following,

$$
g'(0) = \lim_{h\to 0} \frac{h^2 - h\overline{h}}{2h} = \lim_{h\to 0} \frac{h - \overline{h}}{2} = 0.
$$

□

Problem 3.6 By definition, a function $f : G \to \mathbb{C}$ is differentiable at $z_0 \in G$ if the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. Unpacking the limit definition, we see that f is differentiable at z_0 if and only if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\text{if } 0 < |z - z_0| < \delta, \quad \text{then } \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon.$$

By appealing only to the definition, we show that $\sigma : \mathbb{C} \to \mathbb{C}$ defined by $\sigma(z) = \bar{z}$ is not differentiable anywhere by completing the following steps.

(i) Let $z_0 \in \mathbb{C}$ and assume that $\sigma'(z_0)$ exists. Choose $\delta > 0$ according to the definition using $\varepsilon = 1/2$ and write down the resulting statement.

Disclaimer: I apologize in advanced if this is very wrong. I was scratching my head trying to think of a way to explicitly choose a δ for $\varepsilon = \frac{1}{2}$, but I'm not sure how given that we are assuming $\sigma'(z_0)$ to exist, but not knowing what it actually is. So I decided that maybe we are being asked to just rewrite the definition using $\sigma(z)$ and the other given items, and that I'm misunderstanding the problem.

Solution. We assume that $\sigma'(z_0)$ exists, and so by definition we have then for every $\varepsilon > 0$ theres exists a $\delta > 0$ such that

$$\text{if } 0 < |z - z_0| < \delta, \text{ then } \left| \frac{\sigma(z) - \sigma(z_0)}{z - z_0} - \sigma'(z_0) \right| < \varepsilon.$$

Let us choose $\varepsilon = 1/2$, then by definition there exists a $\delta > 0$ such that

$$0 < |z - z_0| < \delta \implies \left| \frac{\sigma(z) - \sigma(z_0)}{z - z_0} - \sigma'(z_0) \right| < \frac{1}{2} \tag{6}$$

. □

(ii) Consider $z = z_0 + \frac{\delta}{2}$ and conclude from (a) that $|1 - \sigma'(z_0)| < \varepsilon$.

Solution. Let us consider $z = z_0 + \frac{\delta}{2}$ we see that,

$$0 < |z - z_0| = \left| z_0 + \frac{\delta}{2} - z_0 \right| = \frac{\delta}{2} < \delta$$

since this δ is from our choice of $\varepsilon = 1/2$, by (6) we have that ,

$$\left| \frac{\overline{z_0} + \frac{\delta}{2} - \overline{z_0}}{z_0 + \frac{\delta}{2} - z_0} - \sigma'(z_0) \right| < \frac{1}{2}$$

$$\left|\frac{\frac{\delta}{2}}{\frac{\delta}{2}} - \sigma'(z_0)\right| < \frac{1}{2}$$
$$\left|1 - \sigma'(z_0)\right| < \frac{1}{2}$$

as desired.

□

(iii) Consider $z = z_0 + i\frac{\delta}{2}$ and conclude from (a) that $|1 + \sigma'(z_0)| < \varepsilon$.

Solution. In a similar fashion we see,

$$0 < |z - z_0| = \left|z_0 + i\frac{\delta}{2} - z_0\right| = \left|i\frac{\delta}{2}\right| = \frac{\delta}{2} < \delta$$

so by (6) we have that,

$$\left|\frac{\overline{z_0} - i\frac{\delta}{2} - \overline{z_0}}{z_0 + i\frac{\delta}{2} - z_0} - \sigma'(z_0)\right| < \frac{1}{2}$$
$$\left|-\frac{i\frac{\delta}{2}}{i\frac{\delta}{2}} - \sigma'(z_0)\right| < \frac{1}{2}$$
$$\left|-1 - \sigma'(z_0)\right| < \frac{1}{2}$$
$$\left|1 + \sigma'(z_0)\right| < \frac{1}{2}.$$

as desired. □

(iv) Using the triangle inequality together with (ii) and (iii), obtain a contradiction.

Solution. Using what we know of (ii) and (iii) we have that,

$$\left|1 + \sigma'(z_0)\right| + \left|1 - \sigma'(z_0)\right| < \frac{1}{2} + \frac{1}{2}$$

we can apply the triangle inequality on the LHS and obtain,

$$\left|1 + \sigma'(z_0) + 1 - \sigma'(z_0)\right| \leqslant \left|1 + \sigma'(z_0)\right| + \left|1 - \sigma'(z_0)\right| < \frac{1}{2} + \frac{1}{2}$$
$$|2| < 1$$
$$2 < 1$$

which is a contradiction. Therefore $\sigma(z)$ is not differentiable anywhere. □

Collaborators: $\frac{\bar{\mathrm{d}}}{\bar{\mathrm{x}}}$

References:

- [Book(s): Title, Author]
- [Online: Link]
- [Notes: Link]

Fin.