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117 - SS2 - HW3 - August 25th, 2021

[1] Let V and W be finite-dimensional $\mathbb{F}$-vector spaces.

(a) Show that $\dim(\text{Hom}(V, W)) = \dim(V)\dim(W)$ by finding an explicit basis.

Proof. Since V and W are both finite, let the dimension of V and the dimension of W be denoted by n and m respectively. By definintion that means the basis for V and W are the following.

$$\mathcal{B}_{\mathcal{V}} = \{v_1, v_2, \ldots, v_n\}$$

$$\mathcal{B}_{\mathcal{W}} = \{w_1, w_2, \ldots, w_m\}$$

Now let us define the lienar maps $\pi_{ij} : V \to W$ for $1 \leq i \leq n$ and $1 \leq j \leq m$ by the following,

$$\pi_{ij}(v_p) = \begin{cases} w_j & p = i \\ 0 & p \neq i \end{cases}$$

These will serve as a basis for $\text{Hom}(V, W)$, and we will prove it with the following. Let α_{ij} be a scalar and assume we have,

$$\pi = \sum_{i,j}^{n,m} \alpha_{ij}\pi_{ij} = 0$$

This would mean for $\pi(v_i)$ and $i \in \{1, 2, \ldots, n\}$,

$$\pi(v_i) = \sum_{j}^{m} \alpha_{ij} w_j = 0$$

Recalle though that the set of vector w_j for $1 \leq j \leq m$ are linearly independent, and thus our maps π_{ij} are also linearly independent.
Now take any function π from $\text{Hom}(V, W)$. We can define it its values when inputting the basis of V as $\pi(v_i) \in W$. Meaning when $i \in 1, 2, \ldots, n$ and α_{ij} as a scalar, we can express $\pi(v_i)$ as,

$$\pi(v_i) = \sum_{j}^{m} \alpha_{ij} w_j$$

Which means,

$$\pi = \sum_{i,j}^{n,m} \alpha_{ij}\pi_{ij}$$

because the linear functions agree on basis vectors. This means for $1 \leq i \leq n$ and $1 \leq j \leq m$,

$$\text{Hom}(V, W) = \text{span}(\{\pi_{ij}\})$$

This is the proof since we know there are $\dim(V)\dim(W)$ of these functions.

□

(b) Show that $\mathrm{Hom}(V, V) \cong V \otimes V^*$.

[2] Let $T : \mathbb{F}^3 \to \mathbb{F}^3$ be the linear transformation with matrix:

$$\mathbf{A} = \begin{pmatrix} 1 & 4 & 3 \\ 3 & 4 & 1 \\ 4 & 4 & 4 \end{pmatrix}$$

Compute the standard matrix $[\Lambda^2 T]$ with respect to the standard basis $\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\}$ of $\Lambda^2(\mathbb{F}^3)$.

[3] Let V be a $\mathbb{F}$-vector space. Show that if $T, S \in \mathrm{End}(V)$ such that $ST - TS$ commutes with S, then for every $k \in \mathbb{N}$:

$$S^k T - T S^k = k S^{k-1}(ST - TS)$$

[4] Let V be a $\mathbb{F}$-vector space. Show that if $T \in \mathrm{End}(V)$ such that $T^2 - T + I = 0$, then T is invertible.

Proof.

$$\begin{aligned} T^2 - T + I &= 0 \\ T^2 &= T - I & \qquad I = TT^{-1} \\ T^2 &= T - TT^{-1} \\ T^2 &= T(I - T^{-1}) \\ T &= (I - T^{-1}) \end{aligned}$$

Therefore T is invertible. □

[5] Let V be a $\mathbb{F}$-vector space. If $S, T \in \mathrm{End}(V)$ such that $ST = 0$, does it follow that $TS = 0$?

Proof. Consider the vector space $\mathbb{R}^2$ over $\mathbb{R}$. We have in $\mathrm{End}(V)$ the following,

$$S = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

$$T = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

We see though that,

$$ST = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

but,

$$TS = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix} \neq ST$$

So, no. If we have two linear transformation S and T such that $ST = 0$ it does not follow that $TS = 0$ $\square$

[6] Let $\mathbb{P}_n[x]$ denote the $\mathbb{F}$-vector space of all polynomials with degree less than or equal to n whose coefficients come from $\mathbb{F}$. Suppose that $L \in \mathrm{End}(V)$ such that $Lp(x) = p(x+1)$ for every $p(x) \in \mathbb{P}_n[x]$. Prove that if D is the differentiation operator defined through the power rule, then:

$$I + \frac{D}{1!} + \frac{D^2}{2!} + \cdots + \frac{D^{n-1}}{(n-1)!} = L$$

[7] Let V be a $\mathbb{F}$-vector space with subspaces U and W. Prove that if $T \in \mathrm{End}(V)$ such that U and W are invariant under T, then the subspace spanned by U and W is invariant under T.

Proof. Let the vector space Z represent the subspace spanned by $U + W$.

$$Z = \mathrm{span}(\{U + W\})$$

Meaning any vector $z \in Z$ is of the form $z = u + w$ where u and w are vectors of U and W respectively. This gives us,

$$T(z) = T(u + w) = T(u) + T(w) \subseteq U + W$$

$\square$

[8] Let V be a $\mathbb{F}$-vector space with $E, F : V \to V$ projections.

(a) Prove that $\mathrm{im}(E) = \mathrm{im}(F)$ if and only if $EF = F$ and $FE = E$.

(b) Prove that $\ker(E) = \ker(F)$ if and only $EF = E$ and $FE = F$.

[9] (a) Prove that if E is a projection on a finite-dimensional $\mathbb{F}$-vector space, then there exists a basis $\mathcal{B}$ such that the matrix representative $[E]_{\mathcal{B}}$ has the following special form: $e_{ij} = 0$ if $i \neq j$ and $e_{ii} = 0$ or 1 for all i and j.

(b) An *involution* is a linear transformation U on a $\mathbb{F}$-vector space V such that $U^2 = I$. Show that if $\mathrm{char}(\mathbb{F}) \neq 2$, then the equation $U = 2E - I$ establishes a one-to-one correspondence between all projections E and all involutions U.

(c) Prove that the only eigenvalues of a projection are 0 and 1. Furthermore, prove that the only eigenvalues of an involution are -1 and 1. (This does not require the vector space to be finite-dimensional.)

[10] Find all the (complex) eigenvalues and eigenvectors of the following matrices over $\mathbb{C}$:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 1 & 1 \\ 0 & i \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{E} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$