[3] Let V be a $\mathbb{F}$-vector space. Show that if $T, S \in \text{End}(V)$ such that $ST - TS$ commutes with S, then for every $k \in \mathbb{N}$:

$$S^kT - TS^k = kS^{k-1}(ST - TS)$$

Proof. Base case where $k = 1$

$$S^1T - TS^1 = 1S^0(S^1T - TS^1)$$

we see is true.

Now assume it holds for $k = n$

Now for $k = n + 1$

$$\begin{aligned} & S^{n+1}T - TS^{n+1} \\ & S^nST - TS^nS && \text{Recall though } ST - TS \text{ commutes with } S \\ & (n+1)S^n(ST - TS) \end{aligned}$$

We know it holds for $k = n$ and thus by induction it holds for $k = n + 1$ □