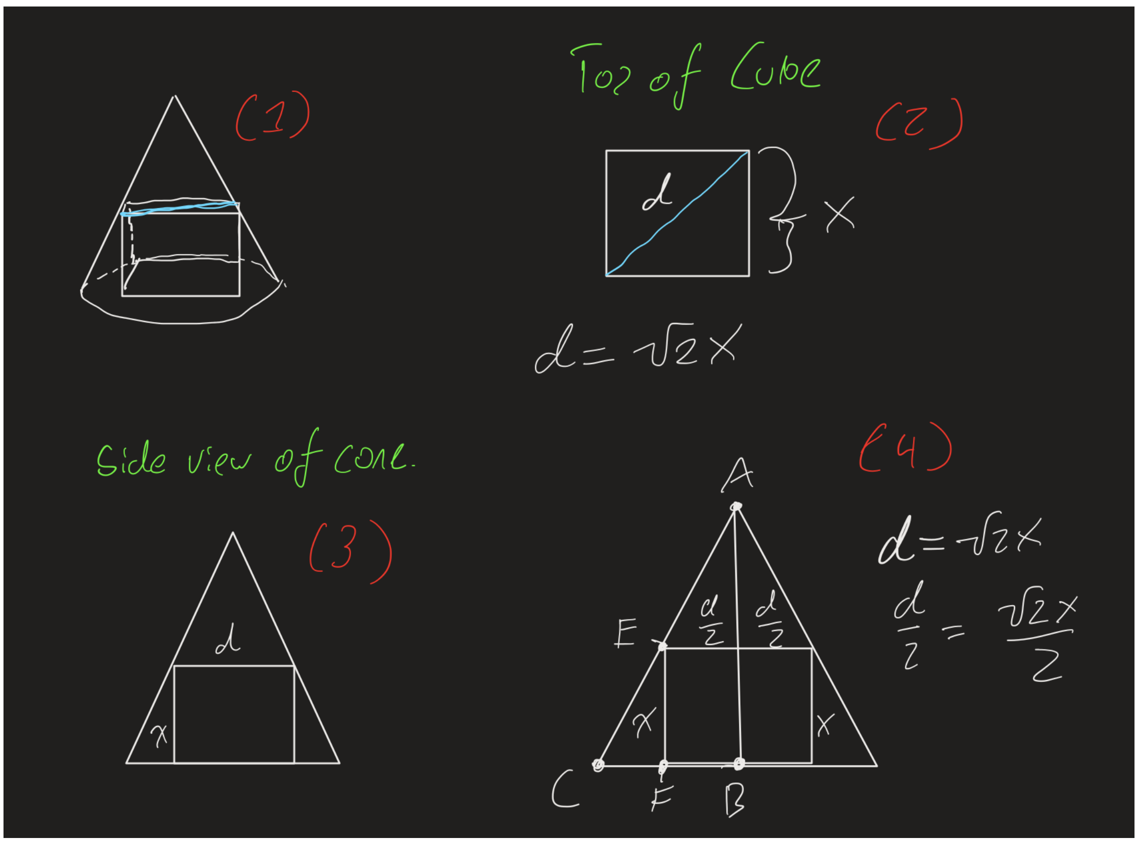
12/06 Submission

Kevin Guillen

MATH 101 — Math 200 — Fall 2021

Problem IC — 12-03 — PP 40 A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side length of the cube?



Proof. Consider the cube inscribed in a cone as in image (1), where the lengths of each side are denoted by x, and the diagonal of the top of the cube is denoted by d as seen in (2). We know that $d = \sqrt{2}x$. Now consider the side view of the cone such that our view is perpendicular to the top diagonal of the square as seen in (3). We can bisect this triangle along its apex to obtain two more triangles as seen in (4). Each of these new triangles will then have a rectangle of side lengths x and $\frac{d}{2}$. We see that the triangle formed by ABC is similar to CEF. This gives us the following equation,

$$\frac{1}{3} = \frac{1 - \frac{d}{2}}{x} = \frac{1 - \frac{\sqrt{2}x}{2}}{x}$$

now let's solve for x,

$$x = 3 - \frac{3\sqrt{2}x}{2} \tag{1}$$
$$x + \frac{3\sqrt{2}x}{2} = 3 \tag{2}$$
$$x(1 + \frac{3\sqrt{2}}{2}) = 3 \tag{3}$$
$$x = \frac{3}{1 + \frac{3\sqrt{2}}{2}} \approx 0.9661 \tag{4}$$

□

Problem IC — 12/03 — PP 41. Find the minimum value of $\dfrac{(x+\frac{1}{x})^6 - (x^6 + \frac{1}{x^6}) - 2)}{(x+\frac{1}{x})^3 + (x^3 + \frac{1}{x^3})}$ for $x > 0$

Proof. First let us manipulate the numerator so that we can substitute $x + \frac{1}{x}$ with y.

$$\begin{aligned}
(x + \frac{1}{x})^6 - (x^6 + \frac{1}{x^6}) - 2 &= (x + \frac{1}{x})^6 - (x^2 + \frac{1}{x^2})(x^4 + \frac{1}{x^4} - 1) - 2 &(5)\\
&= (x + \frac{1}{x})^6 - ((x + \frac{1}{x})^2 - 2)((x^2 + \frac{1}{x^2})^2 - 3) - 2 &(6)\\
&= (x + \frac{1}{x})^6 - ((x + \frac{1}{z})^2 - 2)(((x + \frac{1}{x})^2 - 2)^2 - 3) - 2 &(7)\\
&= y^6 - (y^2 - 2)((y^2 - 2)^2 - 3) - 2 &(8)\\
&= &(9)
\end{aligned}$$

□

Problem IC — 12-03 — PP 42 Let $k > 1$ be a positive integer. Show that the equation $x^2 - y^2 = k^3$ always has integral solutions in x and y.

Proof. Consider the following,

$$x^2 + y^2 = k^3$$
$$(x + y)(x - y) = (k)(k^2)$$

this gives us the system of equations,

$$(x + y) = k^2$$
$$(x - y) = k.$$

We know that $k^2 - k$ and $k^2 + k$ will yield an even integer for any $k \in \mathbb{Z}$ since squaring a number preserves parity. This lets us set $x = \frac{k^2+k}{2}$ and $y = \frac{k^2-k}{2}$. We see then,

$$(x + y)(x - y) = (\frac{k^2 + k}{2} + \frac{k^2 - k}{2})(\frac{k^2 + k}{2} - \frac{k^2 - k}{2}) \tag{10}$$

$$= \frac{2k^2}{2}\frac{2k}{2} \tag{11}$$

$$= k^2 k \tag{12}$$

$$= k^3 \tag{13}$$

meaning $x^2 - y^2 = k^3$ always has integral solutions in x and y. $\square$