Homework 7

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MATH 103A — Complex Analysis — Spring 2022

Problem 7.1 Let

$$f(z) = \frac{z^2+2}{(z^2+3)(z^2+2z+1)}$$

and let C_R denote the semicircle of radius R parameterized by $z(t) = Re^{it}$ with $t \in [0, \pi]$. Show that

$$\lim_{R\to\infty} \int_{C_R} f(z)\, dz = 0.$$

Solution. First note that for any $z \in C_R$ we have that $|z| = R$. Now let us see a chain of inequalities for all the polynomials that make up $f(z)$. For $\left|z^2+2\right|$ we can apply triangle inequality to obtain,

$$\left|z^2+2\right| \leqslant |z|^2 + 2 = R^2 + 2$$

now for $\left|z^2+3\right|$ we can apply reverse triangle inequality to obtain,

$$\left|z^2+3\right| \geqslant \left||z|^2 - |3|\right| = \left|R^2-3\right| = R^2 - 3$$

last but not least,

$$\begin{aligned} \left|z^2+2z+1\right| = \left|(z+1)^2\right| &= |z+1|^2 && \text{apply reverse triangle inequality} \\ &\geqslant ||z| + |1||^2 \\ &= |R-1|^2 \\ &= (R-1)^2. \end{aligned}$$

We apply all of these to obtain,

$$\left|\int_{C_R} \frac{z^2+2}{(z^2+3)(z^2+2z+1)}\right| \leqslant \frac{R^2+2}{(R^2-3)(R-1)^2} R\pi \to 0 \text{ as } R \to 0$$

then by the Squeeze Theorem we have,

$$\lim_{R\to\infty} \int_{C_R} f(z)\, dz = 0$$

as desired. $\square$

Problem 7.2 Let C be a positively oriented simply closed contour and let R be the region consisting of C and its interior.

(a) Show that the area A of the region R is given by the formula

$$A = \frac{1}{2i}\int_C \bar{z}\,dz.$$

Proof. We know the area of the region R to be $\int\int_R dxdy$ which is what want to show the formula gives. Now observe the following,

$$\begin{aligned}\frac{1}{2i}\int_C \bar{z}\,dz &= \frac{1}{2i}\int_C (x-iy)(dx+idy)\\ &= \frac{1}{2i}\int_C xdx + ixdy - iydx + ydy\\ &= \frac{1}{2i}\int_C \underbrace{(x-iy)}_{P(x,y)}dx + \underbrace{(y+ix)}_{Q(x,y)}dy\end{aligned}$$

we pause here to note that $P(x,y)$ and $Q(x,y)$ both have continuous partial derivatives, which lets us apply Greene's Theorem to get,

$$\begin{aligned}\frac{1}{2i}\int_C \bar{z}dz &= \frac{1}{2i}\int\int_R (Q_x - P_y)\,dA\\ &= \frac{1}{2i}\int\int_R (i - -i)\,dxdy\\ &= \frac{1}{2i}\int\int_R 2i\,dxdy\\ &= \frac{2i}{2i}\int\int_R dxdy\\ &= \int\int_R dxdy\end{aligned}$$

giving us the area of the region R as desired. □

(b) Compute the area A of the region enclosed by the *cardioid* C with parameterization

$$z(t) = \frac{1}{2} + e^{it} + \frac{1}{2}e^{2it}, \quad 0 \leqslant t \leqslant 2\pi.$$

Solution. Using the given parameterization we have,

$$dz = ie^{it} + ie^{2it}\,dt.$$

Now we use the formula derived above to compute the area,

$$\begin{aligned}
A = \frac{1}{2i}\int_C \bar{z}\,dz &= \frac{1}{2i}\int_C \left(\frac{1}{2} + e^{-it} + \frac{1}{2i}e^{-2it}\right)\left(ie^{it} + ie^{2it}\right)\,dt \\
&= \frac{1}{2i}\int_0^{2\pi} \frac{1}{2}ie^{it} + i + \frac{1}{2}i + \frac{1}{2}ie^{2it} + ie^{it} + \frac{1}{2}ie^{it}\,dt \\
&= \frac{1}{2i}\int_0^{2\pi} \frac{1}{2}ie^{2it} + 2ie^{it} + \frac{3}{2}i\,dt \\
&= \frac{1}{2i}\left[\frac{1}{4}e^{it} + 2e^{it} + \frac{3}{2}it\right]_0^{2\pi} \\
&= \frac{1}{2i}\left(\frac{1}{4} + 2 + 3\pi i - \frac{1}{4} - 2 + 0\right) \\
&= \frac{1}{2i}(3\pi i) \\
&= \frac{3\pi}{2}.
\end{aligned}$$

$\square$

Problem 7.3 Let C be a closed contour and let $z_0 \in \mathbb{C}$ be a point not lying on C. The *winding number* of C about z_0 is defined by the integral

$$n(C, z_0) = \frac{1}{2\pi i} \int_C \frac{1}{z - z_0}\, dz.$$

(a) Compute $n(C_1, z_0)$ where C_1 is parameterized by

$$z(t) = z_0 + Re^{it}, \quad 0 \leqslant t \leqslant 2k\pi,\ k \in \mathbb{Z},\ R > 0.$$

Solution. From the parameterization of C_1 we have,

$$dz = Rie^{it}\, dt.$$

Now computing the winding number as defined,

$$\begin{aligned} n(C_1, z_0) &= \frac{1}{2\pi i} \int_0^{2k\pi} \frac{1}{z_0 + Re^{it} - z_0} (Rie^{it})\, dt \\ &= \frac{1}{2\pi i} \int_0^{2\pi k} \frac{Rie^{it}}{Re^{it}}\, dt \\ &= \frac{1}{2\pi i} \int_0^{2\pi k} i\, dt \\ &= \frac{1}{2\pi} \int_0^{2k\pi} dt \\ &= \frac{1}{2\pi} [t]_0^{2k\pi} \\ &= \frac{1}{2\pi} 2k\pi \\ &= k. \end{aligned}$$

We have then that the winding number of C_1 about z_0 is k. □

(b) Compute $n(C_2, z_0)$, where C_2 is any circle and z_0 is any point not lying on or interior to C_2.

Solution. Let R be the interior of C_2 joined with C_2 itself. Now consider the function,

$$\begin{aligned} f : R &\to \mathbb{C} \\ z &\mapsto \frac{1}{z - z_0} \end{aligned}$$

it is obvious from class and previous homework that f is holomorphic everywhere on R except when $z = z_0$, but z_0 is assumed to not lie in R, so that is not a worry. This then allows us to apply Cauchy-Goursat Theorem when computing the winding number to obtain,

$$n(C_2, z_0) = \frac{1}{2\pi i} \int_C \frac{1}{z - z_0}\, dz = \frac{1}{2\pi i} \int_C f(z)\, dz = 0$$

□

(c) Let C_3 be any closed contour and z_0 any point not lying on C_3, parameterized by $z : [a, b] \to \mathbb{C}$. For any such contour, we can always find real-valued (piece-wise) differentiable functions $r, \theta : [a, b] \to \mathbb{R}$ with $r(t) > 0$ such that $z(t) = z_0 + r(t)e^{i\theta(t)}$. Compute $n(C_3, z_0)$.

Solution. Given the parameterization of z we have that,

$$dz = r'(t)e^{i\theta(t)} - ir(t)e^{i\theta(t)}\theta'(t)\,dt$$

Now computing the winding number of C_3 about z_0,

$$\begin{aligned}
n(C_3, z_0) &= \frac{1}{2\pi i}\int_a^b \frac{1}{z_0 + r(t)e^{i\theta(t)} - z_0}(r'(t)e^{i\theta(t)} - ir(t)e^{i\theta(t)}\theta'(t))\,dt \\
&= \frac{1}{2\pi i}\int_a^b \frac{r'(t)e^{i\theta(t)} - ir(t)e^{i\theta(t)}\theta'(t)}{r(t)e^{i\theta(t)}}\,dt \\
&= \frac{1}{2\pi i}\int_a^b \frac{r'(t)}{r(t)} - i\theta'(t)\,dt \\
&= \frac{1}{2\pi i}\left(\int_a^b \frac{r'(t)}{r(t)}\,dt - i\int_a^b \theta'(t)\,dt\right) \\
&= \frac{1}{2\pi i}\left([\ln(r(t))]_a^b - i\,[\theta(t)]_a^b\right) \\
&= \frac{1}{2\pi i}(\ln(r(b)) - \ln(r(a)) - i(\theta(b) - \theta(a))) \\
&= \frac{1}{2\pi i}\left(\ln\left(\frac{r(b)}{r(a)}\right) - i(\theta(b) - \theta(a))\right) \\
&= \frac{1}{2\pi i}(\ln(1) - i(\theta(b) - \theta(a))) \\
&= \frac{\theta(a) - \theta(b)}{2\pi}.
\end{aligned}$$

$\square$

Problem 7.4 Let $a, b \in \mathbb{C}$ and let C_R be the circle of radius R centered at the origin, traversed once in the positive orientation. If $|a| < R < |b|$, show that

$$\int_{C_R} \frac{1}{(z-a)(z-b)}\, dz = \frac{2\pi i}{a-b}.$$

Solution. First we can break down the integral using partial fractions.

$$\begin{aligned}
\frac{1}{(z-a)(z-b)} &= \frac{A}{(z-a)} + \frac{B}{(z-b)} \\
1 &= A(z-b) + B(z-a) && \text{let } z = b \\
1 &= A(b-b) + B(b-a) \\
\frac{1}{b-a} &= B
\end{aligned}$$

jumping back to the equation on the second line,

$$\begin{aligned}
1 &= A(z-b) + B(z-a) && \text{let } z = a \\
1 &= A(a-b) + B(a-a) \\
\frac{1}{a-b} &= A.
\end{aligned}$$

Now we compute the given integral using everything we found,

$$\int_{C_R} \frac{1}{(z-a)(z-b)}\, dz = \frac{1}{a-b} \underbrace{\int_{C_R} \frac{1}{z-a}\, dz}_{(1)} + \frac{1}{b-a} \underbrace{\int_{C_R} \frac{1}{z-b}\, dz}_{(2)}.$$

Recall from the previous problem, that the winding number of a contour, C_R, about a point a is,

$$n(C_R, a) = \frac{1}{2\pi i} \int_{C_R} \frac{1}{z-a}\, dz$$

it is clear though that the winding number of C_R about a should be 1 since C_R is just a circle being traversed once with positive orientation and $|a| < R$. This means that the integral (1) should evaluate to $2\pi i$. Applying the same reasoning to C_R about b, the winding number should work out to be 0, because we are given that $R < |b|$, meaning the contour goes around B 0 times. Which means the integral (2) has to evaluate to 0. Plugging this back in we get,

$$\begin{aligned}
\int_{C_R} \frac{1}{(z-a)(z-b)}\, dz &= \frac{1}{a-b} 2\pi i + \frac{1}{b-a} 0 \\
&= \frac{2\pi i}{a-b}
\end{aligned}$$

as desired. □

Problem 7.5 Let

$$f(z) = \frac{1}{z^2+1}.$$

Determine whether f has an antiderivative on the given domain G. You must prove your claims.

(a) $G = \mathbb{C} \setminus \{i, -i\}$.

Solution. Using the fact that $z^2+1 = (z+i)(z-i)$, let us use partial fraction decomposition as before to expand f,

$$\begin{aligned} \frac{1}{(z+i)(z-i)} &= \frac{A}{z+i} + \frac{B}{z-i} \\ 1 &= A(z-i) + B(z+i) && \text{choose } z = i \\ 1 &= A(i-i) + B(i+i) \\ \frac{1}{2i} &= B \end{aligned}$$

going back to the equation on the second line,

$$\begin{aligned} 1 &= A(z-i) + B(z+i) && \text{choose } z = -i \\ 1 &= A(-i-i) + B(-i+i) \\ -\frac{1}{2i} &= A. \end{aligned}$$

Now let us integrate f over the contour C, where C is a circle with radius 1 centered around i traversed once with positive orientation. So we have,

$$\int_C f(z)\,dz = -\frac{1}{2i}\int_C \frac{1}{z+i}\,dz + \frac{1}{2i}\int_C \frac{1}{z-i}\,dz.$$

We apply similar reasoning found in the previous problem, where these are the integrals used in calculating the winding number of our contour about i and about $-i$. We know though that $-i$ is outside our contour so the first integral must evaluate to 0. We know that i is the center of our contour, so the winding number should be 1, meaning the second integral evaluates to $2\pi i$. Plugging this back in we get,

$$\begin{aligned} \int_C f(z)\,dz &= \frac{1}{2i}0 + \frac{1}{2i}2\pi i \\ &= \pi. \end{aligned}$$

Which is non-zero meaning the antiderivative does not exist on the given domain G.

□

(b) $G = \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$.

Solution. Due to time, I'm not sure how to rigorously prove this. So I'll just explain my intuition. I believe the antiderivative exists over this domain. This is because the only parts where the function can fail will be $\pm - i$ are not only non-existent in this domain, but it is impossible to create a closed contour containing them like we did previously. This is because there is no way to have a contour "wrap" around $\pm i$ because they are on the line $z = x + iy$ for $x = 0$, but this domain has $\operatorname{Re} z > 0$ which is to say $x > 0$. This means then that we can't have a integral over a closed contour that evaluates to something non-zero. $\square$

Collaborators:

References:

- [Book(s): Title, Author]
- [Online: Link]
- [Notes: Link]

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