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**117 - SS2 - HW3 - August 25th, 2021**

**[1]** Let $V$ and $W$ be finite-dimensional $\mathbb{F}$-vector spaces.

(a) Show that $\dim(\text{Hom}(V, W)) = \dim(V)\dim(W)$ by finding an explicit basis.

*Proof.* Since $V$ and $W$ are both finite, let the dimension of $V$ and the dimension of $W$ be denoted by $n$ and $m$ respectively. By definition that means the basis for $V$ and $W$ are the following.

$$\mathcal{B}_{\mathcal{V}} = \{v_1, v_2, \ldots, v_n\}$$

$$\mathcal{B}_{\mathcal{W}} = \{w_1, w_2, \ldots, w_m\}$$

Now let us define the linear maps $\pi_{ij} : V \to W$ for $1 \leq i \leq n$ and $1 \leq j \leq m$ by the following,

$$\pi_{ij}(v_p) = \begin{cases} w_j & p = i \\ 0 & p \neq i \end{cases}$$

These will serve as a basis for $\text{Hom}(V, W)$, and we will prove it with the following. Let $\alpha_{ij}$ be a scalar and assume we have,

$$\pi = \sum_{i,j}^{n,m} \alpha_{ij}\pi_{ij} = 0$$

This would mean for $\pi(v_i)$ and $i \in \{1, 2, \ldots, n\}$,

$$\pi(v_i) = \sum_{j}^{m} \alpha_{ij} w_j = 0$$

Recall though that the set of vector $w_j$ for $1 \leq j \leq m$ are linearly independent, and thus our maps $\pi_{ij}$ are also linearly independent.
Now take any function $\pi$ from $\text{Hom}(V, W)$. We can define it its values when inputting the basis of $V$ as $\pi(v_i) \in W$. Meaning when $i \in 1, 2, \ldots, n$ and $\alpha_{ij}$ as a scalar, we can express $\pi(v_i)$ as,

$$\pi(v_i) = \sum_{j}^{m} \alpha_{ij} w_j$$

Which means,

$$\pi = \sum_{i,j}^{n,m} \alpha_{ij}\pi_{ij}$$

because the linear functions agree on basis vectors. This means for $1 \leq i \leq n$ and $1 \leq j \leq m$,

$$\text{Hom}(V, W) = \text{span}(\{\pi_{ij}\})$$

This is the proof since we know there are $\dim(V)\dim(W)$ of these functions.

□

(b) Show that $\mathrm{Hom}(V, V) \cong V \otimes V^*$. Sorry if this isn't formal/rigorous enough I saved this for last since I get the reasoning, but I'm running out of time.

*Proof.* From class (prop 9.1) we showed that setting $U = V$ for $\mathrm{Hom}(U, V)$ that $V^* \otimes V \cong \mathrm{End}(V)$ and that really $\mathrm{End}(V)$ is just $\mathrm{Hom}(V, V)$.

We also know from class and a previous homework (or maybe I think a mastery problem) that $V^* \otimes V \cong V \otimes V^*$. Thus through transitivity we can compose these isomorphisms and we'll end up with $\mathrm{Hom}(V, V) \cong V \otimes V^*$. $\square$

**[2]** Let $T : \mathbb{F}^3 \to \mathbb{F}^3$ be the linear transformation with matrix:

$$\mathbf{A} = \begin{pmatrix} 1 & 4 & 3 \\ 3 & 4 & 1 \\ 4 & 4 & 4 \end{pmatrix}$$

Compute the standard matrix $[\Lambda^2 T]$ with respect to the standard basis $\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\}$ of $\Lambda^2(\mathbb{F}^3)$.

**Solution:** First let's get the computations for $T(e_i)$ out of the way,

$$T(e_1) = \begin{pmatrix} 1 & 4 & 3 \\ 3 & 4 & 1 \\ 4 & 4 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$$

$$T(e_2) = \begin{pmatrix} 1 & 4 & 3 \\ 3 & 4 & 1 \\ 4 & 4 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix}$$

$$T(e_3) = \begin{pmatrix} 1 & 4 & 3 \\ 3 & 4 & 1 \\ 4 & 4 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$$

Then we solving for the standard matrix with respect to the standard basis we get,

$$\begin{aligned}
T(e_1 \wedge e_2) &= T(e_1) \wedge T(e_2) \\
&= (1e_1 + 3e_2 + 4e_3){\wedge}(4e_1 + 4e_2 + 4e_3) \\
&= (4-12)e_1{\wedge}e_2 + (4-16)e_1{\wedge}e_3 + (12-16)e_2{\wedge}e_3 \\
&= -8(e_1{\wedge}e_2) - 12(e_1{\wedge}e_3) - 4(e_2{\wedge}e_3) \\
T(e_1{\wedge}e_3) &= T(e_1){\wedge}T(e_3) \\
&= (1e_1 + 3e_2 + 4e_3){\wedge}(3e_1 + 1e_2 + 4e_3) \\
&= (1-9)e_1{\wedge}e_2 + (4-12)e_1{\wedge}e_3 + (12-4)e_2{\wedge}e_3 \\
&= -8(e_1{\wedge}e_2) - 8(e_1{\wedge}e_3) + 8(e_2{\wedge}e_3) \\
T(e_2{\wedge}e_3) &= T(e_2){\wedge}T(e_3) \\
&= (4e_1 + 4e_2 + 4e_3){\wedge}(3e_1 + 1e_2 + 4e_3) \\
&= (4-12)e_1{\wedge}e_2 + (16-12)e_1{\wedge}e_3 + (16-4)e_2{\wedge}e_3 \\
&= -8(e_1{\wedge}e_2) + 4(e_1{\wedge}e_3) + 12(e_2{\wedge}e_3)
\end{aligned}$$

Now like in example 12.4 we can read off our coefficients to get the standard matrix and we get the following,

$$\begin{pmatrix} -8 & -8 & -8 \\ -12 & -8 & 4 \\ -4 & 8 & 12 \end{pmatrix}$$

**[3]** Let $V$ be a $\mathbb{F}$-vector space. Show that if $T, S \in \text{End}(V)$ such that $ST - TS$ commutes with $S$, then for every $k \in \mathbb{N}$:

$$S^kT - TS^k = kS^{k-1}(ST - TS)$$

*Proof.* Base case where $k = 1$

$$S^1T - TS^1 = 1S^0(S^1T - TS^1)$$

we see is true.

Now assume it holds for $k = n$

Now for $k = n + 1$

$$\begin{aligned}
S^{n+1}T - TS^{n+1}& \\
S^nST - TS^nS& \qquad \text{Recall though } ST - TS \text{ commutes with } S \\
(n+1)S^n(ST - TS)&
\end{aligned}$$

We know it holds for $k = n$ and thus by induction it holds for $k = n + 1$ $\square$

**[4]** Let $V$ be a $\mathbb{F}$-vector space. Show that if $T \in \text{End}(V)$ such that $T^2 - T + I = 0$, then $T$ is invertible.

*Proof.*

$$T^2 - T + I = 0$$

$$T^2 = T - I \qquad\qquad I = TT^{-1}$$

$$T^2 = T - TT^{-1}$$

$$T^2 = T(I - T^{-1})$$

$$T = (I - T^{-1})$$

Therefore $T$ is invertible. □

**[5]** Let $V$ be a $\mathbb{F}$-vector space. If $S, T \in \text{End}(V)$ such that $ST = 0$, does it follow that $TS = 0$?

*Proof.* Consider the vector space $\mathbb{R}^2$ over $\mathbb{R}$. We have in $\text{End}(V)$ the following,

$$S = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

$$T = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

We see though that,

$$ST = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

but,

$$TS = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix} \neq ST$$

So, no. If we have two linear transformation $S$ and $T$ such that $ST = 0$ it does not follow that $TS = 0$ □

**[6]** Let $\mathbb{P}_n[x]$ denote the $\mathbb{F}$-vector space of all polynomials with degree less than or equal to $n$ whose coefficients come from $\mathbb{F}$. Suppose that $L \in \text{End}(V)$ such that $Lp(x) = p(x+1)$ for every $p(x) \in \mathbb{P}_n[x]$. Prove that if $D$ is the differentiation operator defined through the power rule, then:

$$I + \frac{D}{1!} + \frac{D^2}{2!} + \cdots + \frac{D^{n-1}}{(n-1)!} = L$$

*Proof.* We are given that $Lp(x) = p(x+1)$. We know $p(x) \in \mathbb{P}_n[x]$ is of the form

$$p(x) = a_n x^n + \cdots + a_1 x + a_o$$

This means $Dp(x)$, where $D$ is the differentiation operator, is,

$$\frac{d}{dx}p(x) = na_n x^{n-1} + \cdots + 2a_2 x + a_1 + 0$$

We see for different powers of $D$,

$$\frac{d^2}{dx^2}p(x) = n(n-1)a_n x^{n-2} + \cdots + 6a_3 x + 2a_2 + 0$$

$$\frac{d^3}{dx^3}(x) = n(n-1)(n-2)x^{n-3} + \cdots + 24a_4 x + 6a_3 + 0$$

$$\vdots$$

$$\frac{d^{n-1}}{dx^{n-1}}p(x) = n \cdot (n-1) \cdot (n-2) \ldots 2 \cdot a_n x + 0 = n! a_n x$$

Now consider $\left(I + \frac{D}{1!} + \frac{D^2}{2!} + \cdots + \frac{D^{n-1}}{(n-1)!}\right) p(x)$

$$= p(x) + \frac{D}{1!}p(x) + \frac{D^2}{2!}p(x) + \cdots + \frac{D^{n-1}}{(n-1)!}p(x) + \frac{D^n}{n!}p(x)$$

$$= (a_n x^n + \cdots + a_1 x + a_0) + (na_n x^{n-1} + \cdots + 2a_2 x + a_1)$$

$$+ (\frac{n(n-1)}{2}a_n x^{n-2} + \cdots + 3a_3 x + a_2) + (\frac{n(n-1)(n-2)}{6}x^{n-3} + \cdots + 4a_4 x + a_3 + 0)$$

$$+ \cdots + na_n x + a_n$$

$$= a_n(x^n + nx^{n+1} + nx^{n-1} + \frac{n(n-1)}{2}x^{n-2} + \cdots + nx + 1) + \cdots + a_3(x^3 + 3x^2 + 3x + 1)$$

$$+ a_2(x^2 + 2x + 1) + a_1(x+1) + a_0$$

$$= a_n(x+1)^n + \cdots + a_3(x+1)^3 + a_2(x+1)^2 + a_1(x+1) + a_0$$

$$= p(x+1)$$

$$= Lp(x)$$

Now if $p(x)$ were to be $\deg(p(x)) < n$ say for this case specifically $n-1$ through the same method as shown we'd see $D^{n-1}p(x) = (n-1)!a_{n-1}$

And we see

$$
\begin{aligned}
\left(I+\frac{D}{1!}+\frac{D^2}{2!}+\cdots+\frac{D^{n-1}}{(n-1)!}\right)p(x) &= a_{n-1}(x^{n-1}+(n-1)x^{n-2}+\cdots+1)\\
&\quad + a_2(x^2+2x+1)+a_1(x+1)+a_0\\
&= a_{n-1}(x+1)^{n-1}+\cdots+a_2(x+1)^2+a_1(x+1)+a_0\\
&= p(x+1)\\
&= Lp(x)
\end{aligned}
$$

therefore we have as desired that $\left(I+\frac{D}{1!}+\frac{D^2}{2!}+\cdots+\frac{D^{n-1}}{(n-1)!}\right)p(x)=L$ □

**[7]** Let $V$ be a $\mathbb{F}$-vector space with subspaces $U$ and $W$. Prove that if $T \in \mathrm{End}(V)$ such that $U$ and $W$ are invariant under $T$, then the subspace spanned by $U$ and $W$ is invariant under $T$.

*Proof.* Let the vector space $Z$ represent the subspace spanned by $U+W$.

$$Z = \mathrm{span}(\{U+W\})$$

Meaning any vector $z \in Z$ is of the form $z = \alpha u + \beta w$ where $u$ and $w$ are vectors of $U$ and $W$ respectively with $\alpha$ and $\beta$ begin scalars. This gives us,

$$T(z) = T(\alpha u + \beta w) = \alpha T(u) + \beta T(w)$$

Recall though since $U$ and $W$ are invariant under $T$ we have that $T(u) \in U$ and $T(w) \in W$ and $Z$ is the span of $U+W$, therefore,

$$\alpha T(u) + \beta T(w) \in Z$$

Meaning $Z$ is invariant under $T$. $Z$ was defined to be the span of $U+W$. So we have that if $U$ and $W$ are invariant under $T$ then the subspace spanned by $U$ and $W$ is also invariant under $T$. □

**[8]** Let $V$ be a $\mathbb{F}$-vector space with $E, F : V \to V$ projections.

(a) Prove that $\mathrm{im}(E) = \mathrm{im}(F)$ if and only if $EF = F$ and $FE = E$.

*Proof.* For the forward direction we will assume $\mathrm{im}(E) = \mathrm{im}(F)$. This means for any vector $x \in V$ there exists some vector $y \in V$ such that $E(x) = F(y)$. Now consider,

$$
\begin{aligned}
EF(y) &= EE(x)\\
&= E(E(x)) && \text{Recall though all projections are idempotent}\\
&= E(x)\\
&= F(y)\\
EF &= F
\end{aligned}
$$

Now consider,

$$\begin{aligned}
FE(x) &= FF(y) \\
&= F(F(y)) && \text{Recall though all projections are idempotent} \\
&= F(y) \\
&= E(x) \\
FE &= E
\end{aligned}$$

as desired. So now we have if $\text{im}(E) = \text{im}(F)$ implies $EF = F$ and $FE = E$ Now for the reverse direction, we will assume $EF = F$ and $FE = E$. This means for any vector $x \in V$ we have, $EF(x) = F(x)$ and $FE(x) = E(x)$.
We know that for vector $x$, there exists some vector $y \in \text{im}(E)$ such that $E(x) = y$. Now consider the following,

$$\begin{aligned}
E(x) &= y && \text{Recall our assumption } FE = E \\
FE(x) &= y \\
F(E(x)) &= y
\end{aligned}$$

Recall though $y$ was in the image of $E$ and now we can see that it is also in the image of $F$. Therefore $\text{im}(E) \subseteq \text{im}(F)$.
We also know though that for a vector $x$, there exists some vector $y \in \text{im}(F)$ such that $F(x) = y$. Now consider the following,

$$\begin{aligned}
F(x) &= y && \text{Recall our assupmtion } EF = F \\
EF(x) &= y \\
E(F(x)) &= y
\end{aligned}$$

$y$ started in the image of $F$, but we can see that it is also in the image of $E$. This gives us that $\text{im}(F) \subseteq \text{im}(E)$
Putting this all together we have,

$$\text{im}(E) \subseteq \text{im}(F) \text{ and } \text{im}(F) \subseteq \text{im}(E) \rightarrow \text{im}(F) = \text{im}(E)$$

as desired. □

(b) Prove that $\ker(E) = \ker(F)$ if and only $EF = E$ and $FE = F$

*Proof.* First we will go in the forward direction and assume $\ker(E) = \ker(F)$. This means whenever a vector $x \in V$ satisfies $E(x) = 0$ then it must also satisfy $F(x) = 0$.
We want to show that $EF = E$ and $FE = F$. First let's work with $EF = E$. If this equality were to hold that would mean for any vector $x$ in $V$ we would have,

$$(E - EF(x) = 0)$$

So let's assume that it doesn't hold, that would mean there exists some vector $y$ in $V$ such that,

$$(E - EF)(y) \neq 0$$

We see though such a $y$ would imply this about the kernel of $E$,

$$(E - EF)(y) \neq 0$$
$$(E(y) - EF(y)) \neq 0$$
$$E(y - F(y)) \neq 0$$

It's that since $E(y - F(y)) \neq 0$ that means it is NOT in the kernel of $E$. Recall though our assumption was that $\ker(E) = \ker(F)$, that means this is also not in the kernel of $F$. But,

$$F(y - F(y)) \neq 0$$
$$F(y) - F(F(y)) \neq 0 \qquad \text{Recall though every projection is idempotent}$$
$$F(y) - F(y) \neq 0$$
$$F(y) \neq F(y)$$

Which is a contradiction. Therefore if the kernel of the projection $E$ and $F$ are the same then $EF = E$.

We can take a similar look at $FE = F$ and see if this weren't true there would exist some vector $y$ in $V$ such that,

$$(F - FE)(y) \neq 0$$

Assuming this vector did indeed exist,

$$(F - FE)(y) \neq 0$$
$$F(y) - FE(y) \neq 0$$
$$F(y - E(y)) \neq 0$$

it would mean $(y - E(y))$ is not in the kernel of $E$. We see though,

$$E(y - E(y)) \neq 0$$
$$E(y) - E(E(y)) \neq 0 \qquad \text{Projections are idempotent}$$
$$E(y) - E(y) \neq 0$$
$$E(y) \neq E(y)$$

which is again a contradiction. Therefore if the kernel of $E$ and $F$ are equal then $EF = E$ and $FE = F$

Putting all this together: $\ker(F) = \ker(E)$ if and only if $EF = E$ and $FE = E$

□

**[9]** (a) Prove that if $E$ is a projection on a finite-dimensional $\mathbb{F}$-vector space, then there exists a basis $\mathcal{B}$ such that the matrix representative $[E]_{\mathcal{B}}$ has the following special form: $e_{ij} = 0$ if $i \neq j$ and $e_{ii} = 0$ or 1 for all $i$ and $j$.

*Proof.* First we will show why this exists. One important property about projections is that for any projection $E$ it must be idempotent, in other words it must satisfy the following $E^2 = E$

If we recall how multiplication is defined between square matrices between matrix $A$ and $B$ to obtain $C$ we get the following,

$$C = \begin{pmatrix} c_{11} & c_{22} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{pmatrix}$$

where $c_{ij} = a_{i1}b_{ij} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$

Recall though in this situation $A = B = E$. So if all our non diagonal terms are non zero the only time we could possibly have a non zero value would be when we are multiplying elements along the diagonal. In other words for $i = j$ $c_{ij} = a_{ij}b_{ji} = a_{ii}a_{ii}$. If $e_{ii} = 0$ then $c_{ii} = 0$ if $e_{ii} = 1$ then $c_{ii} = 1$. Meaning regardless if $e_{ii} = 0$ or 1. A projection such that the matrix representative has this special form will satisfy the requirement of being a projection.

Now if there did exists some other matrix representative not of this form we'd see it wouldn't satisfy $E^2 = E$. Since first consider if the diagonal could be something other than 0 or 1. We'd see it the requirement would fail due to $x^2 \neq x$ for any $x \neq 0, 1$. Then the same reasoning applies to if non diagonal entries $\square$

(b) An *involution* is a linear transformation $U$ on a $\mathbb{F}$-vector space $V$ such that $U^2 = I$. Show that if $\mathrm{char}(\mathbb{F}) \neq 2$, then the equation $U = 2E - I$ establishes a one-to-one correspondence between all projections $E$ and all involutions $U$.

*Proof.* Assuming we are not working in a field of characteristic 2. First let us begin with some involution $U$ we can obtain its respective projection $E$ through the following,

$$E = \frac{U + I}{2}$$

We know this satisfies the property of begin a projection through the following.

$$\begin{aligned} E^2 = \frac{U + I}{2}^2 &= \frac{U + I}{2}\frac{U + I}{2} \\ &= \frac{U^2 + U + U + U^2}{4} && U \text{ is an involution so,} \\ &= \frac{2U + 2I}{4} \\ &= \frac{2(U + I)}{2(2)} \\ &= \frac{U + I}{2} = E \end{aligned}$$

Then we see for any projection we can obtain its respective involution through,

$$U = 2E - I$$

We can see this indeed satisfies being an involution through the following.

$$\begin{aligned} U^2 &= (2E - I)^2 \\ &= 4E^2 - 2E - 2E + I^2 \quad \text{Recall though } E \text{ is a projection so,} \quad = 4E - 4E + I \\ &= 0 + I \\ &= I \end{aligned}$$

We have thus showed the 1-1 correspondence due to being able to obtain any projections respective involution and vice versa. □

(c) Prove that the only eigenvalues of a projection are 0 and 1. Furthermore, prove that the only eigenvalues of an involution are $-1$ and 1. (This does not require the vector space to be finite-dimensional.)

**[10]** Find all the (complex) eigenvalues and eigenvectors of the following matrices over $\mathbb{C}$:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 1 & 1 \\ 0 & i \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{E} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

**Solution:**

**A)** Solving for the eigenvalues and eigenvectors of $A$, $\det(A - \lambda I) = \lambda^2$. Solving for $\lambda^2 = 0$ we get $\lambda = 0$.

Now to get the corresponding eigenvector we get,

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Thus the eigenvalue for $A$ is 0 and its corresponding eigenvector is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

**B)** Solving for the eigenvalues and eigenvectors of $B$, $\det(B - \lambda I) = \lambda^2 - \lambda - \lambda i + i$. Solving for $\lambda$ we get, $\lambda = i, 1$.

Now lets obtain the corresponding eigenvector for $\lambda_1 = i$,

$$\begin{pmatrix} 1 - i & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Now for $\lambda_2 = 1$,

$$\begin{pmatrix} 0 & 0 \\ 0 & i - 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Thus we have for the eigenvalues $\lambda_1 = i$ and $\lambda_2 = 1$ the corresponding eigenvectors are $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ respectively.

**C)** For $C$ we have $\det(C - \lambda I) = i - \lambda - \lambda i + \lambda^2$. Solving for $\lambda$ we get the following eigenvalues: 1, $i$.

Let's obtain the corresponding eigenvector for $\lambda_1 = i$,

$$\begin{pmatrix} 1-i & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We get for the first equation $(1-i)x + y = 0$. We see if we set $x = (-1-i)$ we get, $-1+i-i-1+y = -2 + y = 0$. Therefore $y = 2$. Thus the corresponding eigenvector is $\begin{pmatrix} -1-i \\ 2 \end{pmatrix}$ Now to obtain the corresponding eigenvector for $\lambda_2 = 1$,

$$\begin{pmatrix} 0 & 1 \\ 0 & i-1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Thus our eigenvector for eigenvalues $\lambda_1 = i$ and $\lambda_2 = 1$ are $\begin{pmatrix} -1-i \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ respectively.

**D)** For $D$ we have, $\det(D - \lambda I) = -\lambda^3 + 3\lambda^2 = -\lambda^2(\lambda - 3)$. Solving for $\lambda$ we get $\lambda = 0, 3$.

Now solving for the corresponding eigenvector for $\lambda_1 = 3$, (Some steps I'm skipping over since it would be a lot of matrices to type out)

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \to \cdots \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This gives us that $x_1 = x_2$ by the first row, and $x_2 = x_3$ by the second row. Thus our corresponding eigenvector is, $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Now for the corresponding eigenvector for $\lambda_2 = 0$,

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We see the corresponding eigenvectors to be $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Thus for the eigenvalue $\lambda_1 = 3$ the corresponding eigenvector is $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and for $\lambda_2 = 0$ the corresponding eigenvectors are $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

**D)** For $D$ we have $\det(D - \lambda I) = -\lambda^3 + 3\lambda^2 - 3\lambda + 1 = -(\lambda - 1)^3$. Solving for $\lambda$ we get $\lambda = 1$. Now solving for the corresponding eigenvector we get,

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Thus we see the corresponding eigenvector for $\lambda = 1$ is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$