

# Homework 5

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May I please have my proof for 5.3 graded, thank you.

**Problem 5.2** Let  $G$  be a  $p$ -group for a prime  $p$  and let  $N$  be a non-trivial normal subgroup of  $G$ . Show that  $N \cap Z(G) > 1$ .

*Proof.* We can consider  $G$  acting on the set  $N$  via conjugation.

$$\alpha(g, n) = gng^{-1}$$

Because  $N$  is given to be a non-trivial normal subgroup we know that  $\alpha(g, h) \in N$ . Now consider the set of fixed points under this group action,

$$\begin{aligned} N^G &= \{n \in N \mid \alpha(g, n) = n, \forall g \in G\} \\ &= \{n \in N \mid gng^{-1} = n, \forall g \in G\} \\ &= \{n \in N \mid gn = ng, \forall g \in G\} \\ &= N \cap Z(G) \end{aligned}$$

From the orbit equation we have,

$$|N^G| = |N| - \sum_{x \in \mathcal{R} \setminus X^G} [G : G_x]$$

Because  $N$  was given to be non-trivial, by Lagrange, it must be of order  $p^a$  where  $a \in \{1, \dots, k\}$  ( $k$  being number such that  $G = p^k$ ). We know each  $G_x$  is a proper subgroup meaning it has an order of  $p^z$  for  $z \in \{1, \dots, k-1\}$ . This means that  $p$  divides  $N^G$ , meaning  $N \cap Z(G) > 1$  as desired.  $\square$

**Problem 5.3**

- (a) Let  $G$  be a group such that  $G/Z(G)$  is cyclic. Show that  $G$  is abelian.
- (b) Show that if a group  $G$  has order  $p^2$ , for some  $p$ , then  $G$  is abelian.

(a) *Proof.* Given that  $G/Z(G)$  is cyclic that means there exists an element  $g \in G$  such that,

$$G/Z(G) = \langle gZ(G) \rangle.$$

Now for any element  $a \in G$ , we know there exists some  $n \in \mathbb{Z}$  such that  $aZ(G) = (gZ(G))^n$ . Which implies the following,

$$aZ(G) = (gZ(G))^n$$

$$\begin{aligned}
&= \underbrace{gZ(G) \cdot gZ(G) \cdots gZ(G)}_n \\
&\alpha Z(G) = g^n Z(G) \qquad \qquad \qquad \alpha H = bH \iff b^{-1}\alpha \in H \\
&\rightarrow (g^n)^{-1}\alpha \in Z(G) \\
&g^{-n}\alpha \in Z(G).
\end{aligned}$$

This means for any element  $\alpha \in G$ , there exists some  $n \in \mathbb{Z}$ , and some  $z \in Z(G)$  such that,

$$\begin{aligned}
g^{-n}\alpha &= z \\
\alpha &= g^n z
\end{aligned}$$

So consider any two elements  $\alpha, \beta \in G$ , they are of the form  $\alpha = g^n z_1$  and  $\beta = g^m z_2$ . So we see from the following,

$$\begin{aligned}
\alpha\beta &= g^n z_1 g^m z_2 & z_1, z_2 \in Z(G) \\
&= g^n g^m z_1 z_2 \\
&= g^{n+m} z_1 z_2 \\
&= g^{m+n} z_1 z_2 \\
&= g^m g^n z_1 z_2 \\
&= g^m z_2 g^n z_1 \\
&= \beta\alpha
\end{aligned}$$

that  $G$  is abelian. □

- (b) *Proof.* Given that  $G$  is a  $p$ -group, that means its center is non-trivial. Because the center of any group is always a subgroup, by Lagrange the order of  $Z(G)$  must be either  $p^2$  or  $p$ . If it is the first case we are done since that would imply  $Z(G) = G$  making  $G$  abelian. If it is of order  $p$ , we know  $Z(G)$  is normal, meaning we can take the quotient group  $G/Z(G)$ , and it will have to be of order  $p$ , meaning it is cyclic and according to part (a) it must be abelian. □

**Problem 5.5** (Fratini Argument) Let  $G$  be a finite group,  $p$  a prime,  $H \trianglelefteq G$  and  $P \in \text{Syl}_p(H)$ . Show that  $G = HN_G(P)$ . (Hint: Let  $g \in G$  and consider  $P$  and  $gPg^{-1}$ . Show that both are Sylow  $p$ -subgroups of  $H$ )

*Proof.* Let  $g \in G$ . We know from the given that  $P$  is a  $p$ -sylow subgroup of  $H$ , meaning  $P \subseteq H$ . Also since  $H$  is normal if we perform conjugation on  $P$  with  $g$ , we have that  $gPg^{-1} \subseteq H$ . We also know that a subgroup under conjugation will be another subgroup of the same order, in other words  $gPg^{-1}$  is another  $p$ -sylow subgroup. Recall though by Sylow's theorem 5.12(c) any two  $p$ -sylow subgroups are conjugate. Meaning there exists some  $h \in H$  such that  $hPh^{-1} = gPg^{-1}$ . Consider the following though,

$$\begin{aligned}
hPh^{-1} &= gPg^{-1} \\
P &= h^{-1}gPg^{-1}h
\end{aligned}$$

this means that  $h^{-1}g \in N_G(P)$ . Using this fact, we can express any element  $g \in G$  as  $g = h(h^{-1}g)$ , where  $h \in H$  and  $h^{-1}g \in N_G(P)$ . Giving us the desired equality,  $G = HN_G(P)$ . □

**Problem 5.6** Show that every group of order 1000 is solvable.

*Proof.* Let  $G$  represent a group of order 1000. Note that  $1000 = 5^3 2^3$ . We know from class that the  $|Syl_5(G)| \equiv 1 \pmod{5}$  and  $|Syl_5(G)| \mid 8$ . Meaning  $|Syl_5(G)| = 1$ . Let  $H$  represent this unique Sylow 5-subgroup. Because it is unique it is normal. Meaning we can take the quotient group  $G/H$  which will have order  $2^3$  and is therefore solvable (Theorem 5.10). For the same reasoning  $H$  is also solvable because it is of order  $5^3$ , this means  $G$  is solvable as desired.  $\square$

**Problem 6.4** Show that every group of order 72 is solvable.

*Proof.* Let  $G$  represent a group of order 72. Note that 72's prime decomposition is  $3^2 \cdot 2^3$ . We know that  $|Syl_3(G)| \equiv 1 \pmod{3}$  and that  $|Syl_3(G)| \mid 8$ , meaning  $|Syl_3(G)| = 1$  or 4. In the first case that means there exists a unique Sylow 3-subgroup and it is normal, we will refer to this subgroup as  $H$ . Because  $H$  is of order 9, it is abelian and therefore solvable, meaning  $G/H$  will also be solvable since its order is  $2^3$  (Theorem 5.10). So in all we have that  $H$  is solvable and that  $G/H$  is solvable, therefore  $G$  is solvable.

If  $|Syl_3(G)| = 4$ , let  $H \in Syl_3(G)$ .  $H$  will not be normal in this case. We know the subgroups in  $Syl_3(G)$  are conjugates of  $H$ . So consider  $G$  acting on  $H$  via conjugation. Recall that  $[G : N_G(H)] = |Syl_3(G)| = 4$ . We can let  $G$  act on  $N_G(H)$  via right multiplication on the right cosets of  $N_G(H)$ . This will give us the homomorphism,

$$f : G \rightarrow \text{Sym}(4)$$

with  $\ker(f) \leq N_G(H)$  and we also have though  $|\text{Sym}(4)| < |G|$ , meaning  $\ker(f) \neq 1$ . All together we have,  $\ker(f) \leq N_G(H) < G$ , meaning  $\ker(f) \neq G$ . Therefore  $\ker(f)$  is non-trivial and because the kernel of a homomorphism is a normal subgroup we have that  $G$  is not simple and therefore solvable.  $\square$