Homework 6

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MATH 103A — Complex Analysis — Spring 2022

Problem 6.1 Let $z : [a, b] \to \mathbb{C}$ be a parameterization of a smooth arc C and suppose $f(z)$ is holomorphic at a point $z_0 = z(t_0)$ on $\mathbb{C}$. Show that

$$(f \circ z)'(t_0) = f'(z(t_0))\, z'(t_0)$$

Proof. Since z is a parameterization of a smooth arc C it can be expressed as $z = x + iy$, meaning then that,

$$z_0 = z(t_0) = x(t_0) + iy(t_0)$$

and we know f can be expressed as,

$$f(z) = f(x, y) = u(x, y) + iv(x, y).$$

So if we consider $f(z(t))$ we have,

$$f(z(t)) = f(x(t), y(t)) = u(x(t), y(t)) + iv(x(t), y(t)).$$

So if we want to take the derivative of $f(z(t_0))$ with respect to t_0 we have,

$$\begin{aligned} f(z(t_0))' &= u_x x_t(t_0) + u_y y_t(t_0) + i(v_x x_t(t_0) + v_y y_t(t_0)) \\ &= u_x(z(t_0))x_t(t_0) + u_x(z(t_0))y_t(t_0) + i(v_x(z(t_0))x_t(t_0) + v_y(z(t_0))y_t(t_0)) \end{aligned}$$

Since $f(z)$ is holomorphic at a point $z_0 = z(t_0)$ we have that,

$$f'(z(t_0)) = u_x(z(t_0)) + iv_x(z(t_0))$$

and we have $z'(t_0) = x_t(t_0) + iy_t(t_0)$ so their product will be,

$$\begin{aligned} f'(z(t_0))z'(t_0) &= (u_x(z(t_0)) + iv_x(z(t_0)))\,(x_t(t_0) + iy_t(t_0)) \\ &= u_x(z(t_0))x_t(t_0) + iv_x(z(t_0))x_t(t_0) + iu_x(z(t_0))y_t(t_0) - v_x(z(t_0))y_t(t_0) \\ &= u_x(z(t_0))x_t(t_0) - v_x(z(t_0))y_t(t_0) + i(v_x(z(t_0))x_t(t_0) + u_x(z(t_0))y_t(t_0)) \\ &\quad \text{apply Cauchy Riemann equations} \\ &= u_x(z(t_0))x_t(t_0) + u_x(z(t_0))y_t(t_0) + i(v_x(z(t_0))x_t(t_0) + v_y(z(t_0))y_t(t_0)) \end{aligned}$$

giving us the desired equality. $\square$

Problem 6.2 Let $\alpha, \beta \in \mathbb{R}$. Evaluate the following integral of real-valued functions

$$\int_0^\pi e^{\alpha x} \cos \beta x \; dx \quad \text{and} \quad \int_0^\pi e^{\alpha x} \sin \beta x \; dx$$

simultaneously by computing a *single* integral of a complex-valued function.

Proof. We have $f(x) = e^{\alpha x} \cos(\beta x) + i e^{\alpha x} \sin(\beta x)$, which we can rewrite as,

$$\begin{aligned} f(x) &= e^{\alpha x}(\cos(\beta x) + i \sin(\beta x)) \\ &= e^{\alpha x} e^{i \beta x} \\ &= e^{\alpha x + i \beta x} \\ &= e^{(\alpha + i \beta) x}. \end{aligned}$$

Now if we integrate $f(x)$ from 0 to π we have,

$$\begin{aligned} \int_0^\pi f(x) \; dx &= \int_0^\pi e^{(\alpha + i\beta)x} \; dx \\ &= \left[\frac{e^{(\alpha+i\beta)x}}{\alpha + i\beta} \right]_0^\pi \\ &= \frac{\left[e^{(\alpha+i\beta)x}\right]_0^\pi}{\alpha + i\beta} \\ &= \frac{e^{\alpha\pi + i\beta\pi} - 1}{\alpha + i\beta} \\ &= \frac{e^{\alpha\pi}(\cos(\beta\pi) + i \sin(\beta\pi)) - 1}{\alpha + i\beta} \\ &= \frac{\alpha - i\beta}{\alpha^2 + \beta^2} \left(e^{\alpha\pi} \cos(\beta\pi) + i e^{\alpha\pi} \sin(\beta\pi) - 1 \right) \\ &= \frac{\alpha \left(e^{\alpha\pi} \cos(\beta\pi) + i e^{\alpha\pi} \sin(\beta\pi) - 1 \right) - i\beta \left(e^{\alpha\pi} \cos(\beta\pi) + i e^{\alpha\pi} \sin(\beta\pi) - 1 \right)}{\alpha^2 + \beta^2} \\ &= \frac{\alpha e^{\alpha\pi} \cos(\beta\pi) + i\alpha e^{\alpha\pi} \sin(\beta\pi) - \alpha + -\beta e^{\alpha\pi} i \cos(\beta\pi) + \beta e^{\alpha\pi} \sin(\beta\pi) + i\beta}{\alpha^2 + \beta^2} \\ &= \frac{e^{\alpha\pi} \left(\alpha \cos(\beta\pi) + \beta \sin(\beta\pi) \right) - \alpha}{\alpha^2 + \beta^2} + i \frac{e^{\alpha\pi} \left(\alpha \sin(\beta\pi) - \beta \cos(\beta\pi) \right) + \beta}{\alpha^2 + \beta^2}. \end{aligned}$$

Now recall that $\int_0^\pi f(x) \; dx = \int_0^\pi e^{\alpha x} \cos(\beta x) \; dx + i \int_0^\pi e^{\alpha x} \sin(\beta x) \; dx$. Therefore we have

$$\begin{aligned} \int_0^\pi e^{\alpha x} \cos \beta x \; dx &= \frac{e^{\alpha\pi} \left(\alpha \cos(\beta\pi) + \beta \sin(\beta\pi) \right) - \alpha}{\alpha^2 + \beta^2} \\ \int_0^\pi e^{\alpha x} \sin \beta x \; dx &= \frac{e^{\alpha\pi} \left(\alpha \sin(\beta\pi) - \beta \cos(\beta\pi) \right) + \beta}{\alpha^2 + \beta^2} \end{aligned}$$

as desired. $\square$

Problem 6.3 Let $z_1, z_2 \in \mathbb{C}$. Compute the integral

$$\int_C dz = \int_C 1\, dz$$

where C is any contour joining z_1 to z_2.

Solution. Let $\sigma : [0,1] \to \mathbb{C}$ be the parameterization of C. Where $\sigma(0) = z_1$ and $\sigma(1) = z_2$ since C is any contour joining z_1 and z_2. Now let $f(z)$ be the constant function that maps every complex number to 1. Note then that $f(\sigma(t)) = 1$ for $t \in [0,1]$. We have then that,

$$\begin{aligned}
\int_C 1\, dz &= \int_C f(z)\, dz && \text{apply Def 12.3} \\
&= \int_0^1 f(\sigma(t))\sigma'(t)\, dt \\
&= \int_0^1 \sigma'(t) && \text{apply F.T.C} \\
&= \sigma(1) - \sigma(0) \\
&= z_2 - z_1.
\end{aligned}$$

□

Problem 6.4 Let C denote the unit circle with positive orientation. Compute the integral

$$\frac{1}{2\pi i}\int_C \frac{(1+z)^n}{z^{k+1}}\,dz = \star$$

for any integers $0 \leqslant k \leqslant n$.

Solution. The parameterization of C is $z(t) = e^{it}$ for $0 \leqslant t \leqslant 2\pi$ which then gives us that $dz = ie^{it}$, so we have,

$$\begin{aligned}
\star &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{(1+e^{it})^n}{(e^{it})^{k+1}} ie^{it}\,dt && \text{using Binomial theorem}\\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}}{e^{it(k+1)}}\sum_{r=0}^{n}\binom{n}{r}e^{itr}\,dt\\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{-ikt}\sum_{r=0}^{n}\binom{n}{r}e^{itr}\,dt\\
&= \frac{1}{2\pi}\int_0^{2\pi} \sum_{r=0}^{n}\binom{n}{r}e^{it(r-k)}\,dt\\
&= \frac{1}{2\pi}\sum_{r=0}^{n}\binom{n}{r}\int_0^{2\pi} e^{it(r-k)}\,dt
\end{aligned}$$

We note here that integral above works out to be $\dfrac{-i(e^{i2\pi(r-k)}-1)}{r-k}$ for $r \neq k$, but using Eulers identity in the numerator the expression works out to be 0. The only nonzero answer is when $r = k$ since we get,

$$\begin{aligned}
\int_0^{2\pi} e^0\,dt &= \int_0^{2\pi} 1\,dt\\
&= 2\pi
\end{aligned}$$

meaning the sum works out to be $2\pi\binom{n}{r}$, where $r = k$ turning our integral to,

$$\star = \frac{1}{2\pi}2\pi\binom{n}{r} = \binom{n}{k}$$

□

Problem 6.5 Integrate the function $f(z) = \overline{z}$ over the following contours:

(a) C_1: the line segment joining 0 to $1 + i$;

Solution. We can parameterization the line segment as $z(t) = t + it$ for $0 \leqslant t \leqslant 1$. Giving us that $dz = (1 + i)tdt$. All together now we integrating the given function over this contour we get,

$$\begin{aligned}\int_{C_1} \overline{z}\ dz &= \int_0^1 (1-i)(1+i)x\ dx \\ &= 2\left[\frac{x^2}{2}\right]_0^1 \\ &= 2 \cdot \frac{1}{2} \\ &= 1.\end{aligned}$$

□

(b) C_2: the line segment joining 0 to 1, following by the line segment joining 1 to $1 + i$.

Solution. Let A_1 and A_2 denote the first and second line segment respectively. We have the parameterization of A_1 as $z(t) = t$ for $0 \leqslant t \leqslant 1$ which gives us that $dz = dt$. The parameterization of A_2 as $z(s) = 1 + is$ for $0 \leqslant s \leqslant 1$ which gives us $dz = ids$. So we have $C_1 = A_1 + A_2$, now integrating $f(z)$ over the given contour we get,

$$\begin{aligned}\int_{C_1} f(z)\ dz &= \int_{A_1} f(z)\ dz + \int_{A_2} f(z)\ dz \\ &= \int_0^1 t\ dt + \int_0^1 (1-is)\cdot i\ ds \\ &= \left[\frac{t^2}{2}\right]_0^1 + \int_0^1 s + i\ ds \\ &= \frac{1}{2} + \left[\frac{s^2}{2} + is\right]_0^1 \\ &= \frac{1}{2} + \frac{1}{2} + i \\ &= 1 + i.\end{aligned}$$

□

Collaborators:

References:

- [Book(s): Title, Author]
- [Online: Link]
- [Notes: Link]

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