Portfolio

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MATH 101 — Problem Solving — Fall 2021

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1 C^3 Solutions Submitted for Portfolio

1.1 I.C. Solutions

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1.2 O.C. Solutions

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1.3 Putnam Solutions

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PP41 Factor tactic, Get your hands dirty, Wishful thinking

PP42 Diophantine equation, Factor tactic, Primes and divisibility

1.4 I.C. Solutions being Resubmitted

Dec 3 ReSub. 44 Invariance Principle

50 Graph Theory

151 Geometry

1.5 O.C. Solutions being Resubmitted

Dec 3 ReSub. 30 Pigeon Hole Principle

1.6 New I.C. Submissions

124 Partitions and Bijections, Wishful thinking

2 Resubmissions

Problem 10/11 IC (44.) The numbers $1, 2, \ldots, 50$ are written on the blackboard. Then two numbers a and b are chosen and replaced by the single number $|a - b|$. After 49 operations a single number is left. Prove that it is odd.

Proof. If we have the numbers $1, 2, \ldots, 50$ that means half of them are even and half are odd. In other words we have 25 even numbers and 25 odd numbers. We know after 49 operations we will have a single number left. To determine if it is odd or even let's look at the 3 scenarios when taking the differences of even and odd numbers.

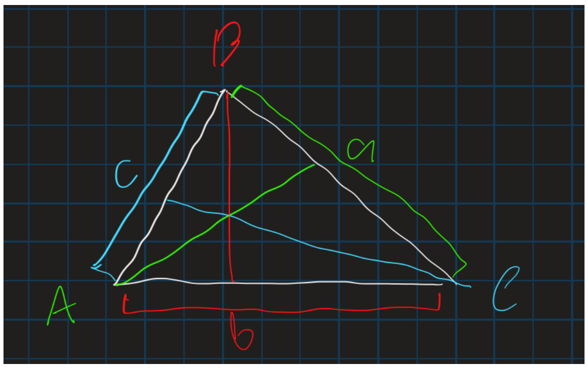
$$\begin{aligned}
\text{Two even numbers: } 2k - 2l &= 2(k - l) \\
\text{Odd and even numbers: } 2k + 1 - 2l &= 2(k - l) + 1 \\
\text{Two odd numbers: } 2k + 1 - (2l + 1) &= 2(k - l)
\end{aligned}$$

Therefore the number of odd numbers removed by using any operation will be either 0 or 2, meaning the parity of the number of odd numbers is invariant ☐

Problem 10/13 IC (50.) Every room in a house has an even number of doors. Prove that there are an even number of entrance doors to the house.

Solution. Let each room in the house be a vertex and outside be a vertex as well. All we have to show now is that this graph can't have exactly one vertex of odd degree. By the handshaking lemma there does not exist any such graph. Meaning the vertex corresponding to the outside is also even. ☐

Problem IC — 11/15 — 151. Is it possible for a triangle to have altitudes equal to 6, 10, and 20?



Proof. Consider the area of the triangle drawn above. It will be the following,

$$A = \frac{6}{2}a = \frac{10}{2}b = \frac{20}{2}c.$$

Which gives us the following equations,

$$a = \frac{1}{3}A$$
$$b = \frac{1}{5}A$$
$$c = \frac{1}{10}A$$

Recall though the sum of any 2 sides of a triangle has to be greater than the remaining third side. Which means we must have (triangle inequality)

$$a < b + c$$
$$\frac{10}{30}A < \frac{6}{30}A + \frac{3}{10}A = \frac{9}{10}A$$

that is if our altitudes were 6, 10 and 20, that would imply $\frac{10}{30}A < \frac{9}{10}A$ which is a contradiction. Therefore there can not be a triangle with those given altitudes. ☐

Problem 10/8 — OC — 30. Chose any $(n+1)$ element subset of $\{1, 2, \ldots, 2n\}$. Show that this subset contains two elements which are relatively prime.

Proof. Let S denote the set $\{1, 2, \ldots, 2n\}$. To prove this we will use the pigeon hole principle and the fact that two neighboring numbers are relatively prime. Our "pigeonholes" in this case will be a list of n numbers from S in ascending order where every number is not adjacent to any other number. The max amount of numbers that can be chosen from $\{1, 2, \ldots, 2n\}$ such that no numbers are relatively prime would be n, because consecutive numbers in our list will have a difference of at least 2. So if we choose $n+1$ numbers and match them with our pigeonholes we won't be able to give each number its own hole because there is only n numbers in S that can be not adjacent to any other number, meaning 1 number has to be adjacent to some other number making them relatively prime. □

3 New Submissions

Problem IC — 11/05 — 124. (Putnam) Define a selfish set to be a set which has its own cardinality as an element. FInd, with a proof, the number of subsets of $\{1, 2, \ldots, n\}$ which are minimal selfish sets, that is, selfish sets none of whose proper subsets is selfish.

Proof. Consider an arbitrary set A. If A were to be a minimal selfish set, then by definition every element of A would need to be greater than or equal to the cardinality of A. This observation will be needed.

Let A_n be defined as follows,

$$A_n := \{1, 2, \ldots, n\}.$$

Also let $S(A_n)$ denote the number of subsets where are minimal selfish sets in A_n.

Let $B \subseteq A$ and be a minimal selfish set. There are 2 cases here, the first is that n is in B and the second is that n is not in B.

If n is not contained in B then we know know B will also have to be a minimal selfish set of A_{n-1} which there are $S(A_{n-1})$ of.

If n is indeed contained in B then we know that the element 1 cannot be in B because then B would not be a minimal selfish set. This is because $\{1\}$ is a minimal selfish set. So this subset must also be a subset of $\{2, \ldots, n\}$. If we remove the element n from this set then we know it must be a subset of $\{2, \ldots, n-1\}$. Next if we were to subtract 1 from every element this new set, let's refer to it as C, has got to be a subset of $\{1, \ldots, n-2\}$. Meaning if B was a minimal selfish set of A_n that contained n, then our derived set C, must be a minimal selfish set of A_{n-2}. Which we know there are $S(A_{n-2})$ of.

Putting this together we get,

$$S(A_n) = S(A_{n-1} + S(A_{n-2})).$$

We see for $A_1 = \{1\}$. That it only has 1 subset that is a minimal selfish set, that it $\{1\}$. Next we see for $A_2 = \{1, 2\}$ that it only contains 1 subset that is a minimal selfish set at that is $\{1\}$ again. So we have $A_1 = A_2 = 1$. Notice though that these are the same starting values as the fibonacci sequence, and that we define the value of $S(A_n)$ for $n > 2$ as the sum of the previous 2 terms. Thus $S(A_n) = \text{Fib}(n)$, where $\text{Fib}(n)$ is the nth term in the fibonacci sequence. □

4 Strategies

Generalization

Specialization IC 122, OC 68, PP14

Relax conditions OC 18, 68

Get your hands dirty IC: 14, 46 OC: 71, 72, 77, 90 Putnam: 19, 20, 41

Wishful thinking IC: 148 OC: 69 Putnam: 15, 20, 41

Find a penultimate step IC: 143, 153, 46 Putnam: 20, 37

Formulate intermediate goals IC: 122, 110

5 Tactics and techniques

External principle

Find and exploit symmetries Putnam: 15

Invariance principle IC: 44

Pigeon hole principle OC: 30

Counting in two different ways IC 110

6 Tools and mathematical content

Graph Theory IC: 50 OC: 68

Complex numbers

Generating functions IC: 109

Factor tactic IC: 99, 135 OC: 90 Putnam: 19, 29,37, 41, 412

Arithmetic and geometric sequences and series IC: 81, OC sept 27 Putnam: 29

Polynomials Putnam: 14

Inequalities IC:99 Putnam: 29

Pascal's triangle and the binomial theorem OC: 1

Partitions and bijections OC: 69

Principle of inclusions-exclusion

Recurrence relations IC: 19, OC: 83

Primes and divisibility IC: 143, 135, 136 OC: 83 Putnam: 42

Congruence IC: 143, 135, 136 Putnam: 19, 29

Diophantine equations Putnam: 42

Geometry IC 148, 151, 153 Putnam: 40

Overall I feel satisfied with the diversity of my problems and the diversity of problems presented to us in class. Maybe there is tendency towards certain types of problems for me but even in the areas where I didn't have many submissions I feel good with those because those are generally what I struggled with more. It felt like intuitively I could see it and understand why, but since some of these areas (graph theory) are completely new to me it was difficult to express in a rigorous manner. If the subject wasn't new then it might have been very old to me (Geometry) so even though on paper I should be familiar with terms and ideas from that area of math it was still a struggle.