P-adic Numbers

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March 10th 2022

Number Systems

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- $\mathbb{Z} = \{-2, -1, 0, 1, 2\}$ through $-$
- $\mathbb{Q} = \left\{\dots, -\frac{4}{7}, \frac{1}{2}, \dots\right\}$ through $/$

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- $\forall \varepsilon > 0$
- $\exists N$
- $\forall n > N$
- $|x_n - x| < \varepsilon$

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Settles on something that is not in $\mathbb{Q}$, it goes to e.
Which then gives us $\mathbb{R}$.

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An **absolute value** on a field $\mathbb{F}$ is a function $|\cdot|$ from $\mathbb{F}$ to $\mathbb{R}_{\geqslant 0}$ that satisfies the following properties for all $a, b \in \mathbb{F}$:

(1) Positive-definiteness: $|a| = 0 \iff a = 0$

(2) Multiplicativity: $|ab| = |a|\,|b|$

(3) Triangle Inequality: $|a + b| \leqslant |a| + |y|$

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$\mathbb{R}$ completes $\mathbb{Q}$.

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EX: $x = 90$ and $p = 3$ the $3-$adic valuation of x to be 2 since,

$$90 = 3^2 \cdot 10$$

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$x \in \mathbb{Q} - \mathbb{Z}$ then $v_p(x)$, we have $x = \dfrac{a}{b}$ and $a, b \in \mathbb{Z}$. So we define,

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Finally $v_p(0) = +\infty$

$P-$adic absolute value

A function $|\cdot|_p : \mathbb{Q} \to \mathbb{R}_{\geqslant 0}$ defined as,

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inducing the $p-$adic metric d_p

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Examples ($\mathbb{Z}$ and $\mathbb{Z}_p$)

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$p-$adic expansion of an integer is simply that number written in base p
$p-$adic expansion of a rational number is the formal power series, $r \in \mathbb{Q}$,

$$r = \sum_{i=k}^{\infty} a_i p^i$$

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Note that $10^{-4} = \dfrac{1}{10^4}$

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$$(\ldots 1 \cdot 3^1 + 1)3^3$$

Number Line?

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