11/22 Resubmission

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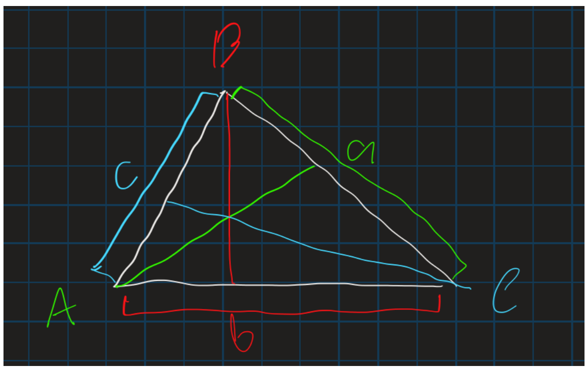
MATH 101 — Problem Solving — Fall 2021

Problem IC — 11/15 — 151. Is it possible for a triangle to have altitudes equal to 6, 10, and 20?



Proof. Consider the area of the triangle drawn above. It will be the following,

$$A = \frac{6}{2}a = \frac{10}{2}b = \frac{20}{2}c.$$

Which gives us the following equations,

$$a = \frac{1}{3}A$$
$$b = \frac{1}{5}A$$
$$c = \frac{1}{10}A$$

Recall though by the triangle inequality we have that $a < b + c$, we see through the following though,

$$a < b + c$$
$$\frac{10}{30}A < \frac{6}{30}A + \frac{3}{10}A = \frac{9}{10}A$$

that if our altitudes were 6, 10 and 20, that would imply $\frac{10}{30}A < \frac{9}{10}A$ which is a contradiction. Therefore there can not be a triangle with those given altitudes. ☐

Problem OC — 11/01 — 68 Imagine an $n \times n$ chessboard. How many ways is it possible to choose four squares, no three in the same row or columns which are the vertices of a rectangle?

Proof. If we have an $n \times n$ chessboard that means we have n columns and n rows. We can treat each row and column as sides of a rectangles. If we choose 2 unique vertical lines and 2 unique horizontal lines their intersections will generate a rectangle. There are n choose 2 ways to pick our horizontal lines, and n choose 2 ways to pick our vertical lines This means for an $n \times n$ chessboard we have,

$$\binom{n}{2}\binom{n}{2}$$

rectangles

□

Problem OC — 11/10 — 86. Prove that there are no positive integers x, y such that $x + y$, $2x + y$, and $x + 2y$ are squares.

Proof. Assuming this were to be true, then for integers a, b, and c we have,

$$x + y = a^2$$
$$2x + y = b^2$$
$$x + 2y = c^2$$

Using the 2nd equation to solve for y we get, $y = b^2 - 2x$. Plugging this into the 1st equation we get, $x = b^2 - a^2$. Using the 1st equation to solve for x we get, $x = a^2 - y$. Pluggin this into the 2nd equation we get $y = 2a^2 - b^2$.

Now plugging in these value into the 3rd equation, we get,

$$a^2 - y + 4a^2 - 2b^2 = c^2$$
$$3a^2 = b^2 + c^2$$

Thus there is only integer solutions if this diophantine equation holds. We know any integers squared mod 4 will have values 0 or 1. Therfore $b^2 + c^2 \in \{0, 1, 2\}$, and $3a^2 \in \{0, 3\}$. This together means we must have $3a^2 \equiv 0 \bmod 4$ and $(b^2 + c^2) \equiv 0 \bmod 4$. Meaning a is even and is of the form $a = 2k$, also means that b and c are even and of the form $b = 2l$ and $c = 2n$. Plugging this back into the equation we get,

$$12k^2 = 4l^2 + 4n^2.$$

This means there has to be a solution for k, l, and n. But this can be since $k + l + n < a + b + c$. Therefore there is no integers x and y such that $x + y$, $2x + y$, and $x + 2y$ are squares. □