Homework 4

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May I please have my proof for 6.1 graded, thank you.

Problem 4.1 (a) Show that Alt(4) is the derived subgroup of Sym(4).

(b) Find all composition series of Sym(4).

(c) Determine the higher derived subgroups of Sym(4).

Solution. (a) Well by definition given to us, the derived subgroup of any group is the smallest normal subgroup such that the factor group will be abelian. We know from class that Sym(4)/Alt(4) is commutative. This indicates to us that the derived subgroup will be contained in Alt(4). To show that the derived subgroup of Sym(4) is indeed Alt(4) look at the following.

$$(ab)(ad)(ab)(ad) = (abd)$$

We see that every 3-cycle is a commutator. We know there are 8 3-cycles in Sym(4), meaning the derived subgroup of it will be of size at least 9. Therefore its derived subgroup must be Alt(4). This is because the order of Sym(4) is 24, so the only possible subgroups will have to divide 24, and Alt(4) is of order 12, and the derived subgroup is contained in Alt(4)

(b) All composition series for Sym(4) are the following,

$$\mathrm{Sym}(4) \triangleright \mathrm{Alt}(4) \triangleright V \triangleright < a > \triangleright \{1\}$$

where a in $\{1, a\}$ is simply any non identity element in the Klein 4 group V. This is because any non identity in that group has order 2, and any subgroup of V is normal since V is abelian. We already know from class that the composition factors of this series is indeed simple.

(c) Let $G = \mathrm{Sym}(4)$. From part (a) we know that $G' = \mathrm{Alt}(()4)$. So, $G'' = \mathrm{Alt}(4)'$. To find the derived subgroup of Alt(4) we will use the fact that we know V (the Klein 4 group) is normal in Alt(4) because conjugation in Sym(4) does not change cycle structure. Because of this we know that $al4' \leqslant V$. We see though that every element in V is a commutator,

$$(14)(23) = (124)(134)(142)(143) = [(142), (143)]$$
$$(13)(24) = (123)(143)(132)(134) = [(132), (134)]$$
$$(12)(34) = (132)(142)(123)(124) = [(123), (124)]$$

Meaning any element in V is also an element in $\mathrm{Alt}(4)'$. Giving us $\mathrm{Alt}(4)' = V$. Therefore $G'' = V$.

Now to get $G''' = V'$. Recall though the derived subgroup of an abelian group is simply $\{1\}$. Therefore $G''' = \{1\}$.

Putting all this together we have the following,

$$\begin{aligned} G^0 &= \mathrm{Sym}(4) \\ G^1 &= \mathrm{Alt}(4) \\ G^2 &= V \text{ (Klein 4 group)} \\ G^3 &= \{1\} \end{aligned}$$

□

Problem 6.1 Show that for every cycle $(a_1, \ldots, a_k)$ in $\mathrm{Sym}(n)$ and every $\sigma \in \mathrm{Sym}(n)$ one has,

$$\sigma(a_1, \ldots, a_k)\sigma^{-1} = (\sigma(a_1), \ldots, \sigma(a_k))$$

Proof. This can be proved by showing that $\forall t \in \{1, \ldots, n\}$ the following holds,

$$(\sigma(a_1, \ldots, a_k)\sigma^{-1})(t) = (\sigma(a_1), \ldots, \sigma(a_k))(t).$$

This gives us 2 cases. The first is that t is equal to $\sigma(a_i)$ for some $i \in \{1, \ldots, k\}$. This means,

$$\begin{aligned} (\sigma \circ (a_1, \ldots, a_k)\sigma^{-1})(t) &= (\sigma \circ (a_1, \ldots, a_k))\sigma^{-1}(t) \\ &= (\sigma(a_1, \ldots, a_k))\sigma^{-1}(\sigma(a_i)) \\ &= \sigma(a_1, \ldots, a_k)(a_i) \qquad (*) \\ &= \sigma(a_{i+1}) \end{aligned}$$

(*) Recognizing the fact that if i were to be k, then $i+1$ would actually be 1 and not literally $i+1$.

Now the last case is that, for any $i \in \{1, \ldots, k\}$, $t \neq \sigma(a_i)$ which also implies $\sigma^{-1}(t) \neq a_i$. This means that the cycle $(a_1, \ldots, a_k)$ has no effect on $\sigma^{-1}(t)$. This gives us the following,

$$\begin{aligned} (\sigma \circ (a_1, \ldots, a_k)\sigma^{-1})(t) &= \sigma(a_1, \ldots, a_k)(\sigma^{-1}(t)) \\ &= \sigma(\sigma^{-1}(t)) \\ &= t \end{aligned}$$

Meaning for any value of t not in $(\sigma(a_1), \ldots, \sigma(a_k))$, $\sigma(a_1, \ldots, a_k)\sigma^{-1}$ leaves t fixed. All this together then means,

$$\sigma(a_1, \ldots, a_k)\sigma^{-1} = (\sigma(a_1), \ldots, \sigma(a_k))$$

as desired. □