# Homework 8

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MATH 103A — Complex Analysis — Spring 2022

**Problem 8.1**

(a) Let $C$ denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Compute

$$\int_C \frac{\cos z}{z(z^2+8)}$$

***Solution.*** Let,

$$f(z) = \frac{\cos z}{z^2+8}$$

we have then that,

$$\int_C \frac{\cos z}{z(z^2+8)}\,dz = \int_C \frac{f(z)}{z-0}\,dz.$$

Let $z_0 = 0$. Since $f(z)$ is holomorphic at all points on and interior to $C$ and $z_0$ is in the interior of $C$, we have by Cauchy's Integral Formula that,

$$\begin{aligned}
f(z_0) &= \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz \\
\frac{1}{8} &= \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz \\
\frac{\pi i}{4} &= \int_C \frac{f(z)}{z-z_0}\,dz
\end{aligned}$$

therefore,

$$\int_C \frac{\cos z}{z(z^2+8)} = \frac{\pi i}{4}.$$

□

(b) Let $C$ denote the circle centered at $i$ of radius 2, positively oriented. Compute

$$\int_C \frac{1}{(z^2+4)^2}$$

***Solution.*** Let us note that,

$$(z^2+4) = (z+2i)(z-2i).$$

Now let $f(z) = \dfrac{1}{z+2i}$, we can rewrite the given integral as,

$$\begin{aligned}\int_C \frac{1}{(z^2+4)^2} &= \int_C \frac{1}{(z-2i)^2(2+2i)^2} \\ &= \int_C \frac{f(z)}{(z-2i)^2}\,dz.\end{aligned}$$

Let $z_0 = 2i$. As before, we know that $f(z)$ is holomorphic at all points on and interior to $C$ since the place it is not holomorphic is when $z = -2i$ which is not in or part of the contour, and since $z_0$ is inside the contour, we can apply the generalization of Cauchy's Integral Formula to obtain,

$$f'(z_0) = \frac{1!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2} \tag{1}$$

We obtain the derivative of $f$ to be,

$$f'(z) = -\frac{2}{(z+2i)^3}$$

which lets us calculate the LHS of (1) to be,

$$f'(2i) = -\frac{2}{(4i)^3} = \frac{2}{64i} = -\frac{i}{32}.$$

Solving the integral now in (1) we get,

$$\begin{aligned}-\frac{i}{32} &= \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2} \\ \frac{\pi}{16} &= \int_C \frac{f(z)}{(z-z_0)^2}\end{aligned}$$

therefore,

$$\int_C \frac{1}{(z^2+4)^2} = \frac{\pi}{16}.$$

$\square$

**Problem 8.2** Let C be the circle of radius 3, positively oriented, centered at the origin. Show that if

$$g(w) = \int_C \frac{2z^2 - z - 2}{z - w}\, dz, \quad |w| \neq 3,$$

then $g(2) = 8\pi i$. What is the value of $g(w)$ when $|w| > 3$?

***Solution.*** Let $f(z) = 2z^2 - z - 2$, we rewrite g now as,

$$g(w) = \int_C \frac{2z^2 - z - 2}{z - w}\, dz = \int_C \frac{f(z)}{z - w}\, dz$$

so evaluating at $w = 2$ we know 2 is in the interior of C, and $f(z)$ is holomorphic on C, so we can applying Cauchy's Integral formula to obtain,

$$\begin{aligned} f(2) &= \frac{1}{2\pi i} \int_C \frac{2z^2 - z - 2}{z - 2}\, dz \\ 4 &= \frac{1}{2\pi i} \int_C \frac{2z^2 - z - 2}{z - 2}\, dz \\ 8\pi i &= \int_C \frac{2z^2 - z - 2}{z - 2}\, dz \end{aligned}$$

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We have that $g(w)$ is holomorphic over C for $|w| > 3$, so by Cauchy-Goursat Theorem $g(w) = 0$ when $|w| > 3$. □

**Problem 8.3** Let $C$ be the unit circle parametrised as $z(t) = e^{it}$, $-\pi \leqslant t \leqslant \pi$. First show that for any $a \in \mathbb{R}$,

$$\int_C \frac{e^{az}}{z}\, dz = 2\pi i$$

Then write this integral in terms of $t$, using the definition of a contour integral, to derive the integration formula

$$\int_0^{\pi} e^{a\cos t}\cos(a\sin t)\, dt = \pi.$$

***Solution.*** Let $f(z) = e^{az}$, we can rewrite the above integral as,

$$\int_C \frac{e^{az}}{z}\, dz = \int_C \frac{f(z)}{z-0}\, dz.$$

Let $z_0 = 0$, it is clear that $z_0$ is in the interior of $C$, and $f(z)$ is holomorphic at all points on and interior to $C$. So, we can apply Cauchy's Integral Formula to obtain,

$$\begin{aligned}
f(z_0) &= \frac{1}{2\pi i}\int_C \frac{f(z)}{z-0}\, dz \\
1 &= \frac{1}{2\pi i}\int_C \frac{f(z)}{z-0}\, dz \\
2\pi i &= \int_C \frac{f(z)}{z-0}\, dz \\
2\pi i &= \int_C \frac{e^{az}}{z}\, dz, \quad \forall a \in \mathbb{R}
\end{aligned}$$

as desired.

Using the parametrization of $C$ we have that,

$$dz = ie^{it}dt.$$

So, rewriting the integral in terms of $t$ we have,

$$\int_{-\pi}^{\pi} \frac{e^{ae^{it}}}{e^{it}} ie^{it}\, dt = \int_{-\pi}^{\pi} ie^{ae^{it}}\, dt$$

Now we use Euler's Formula and some manipulation to obtain,

$$\begin{aligned}
2\pi i = \int_{-\pi}^{\pi} ie^{ae^{it}}\, dt &= \int_{-\pi}^{\pi} ie^{a(\cos t + i\sin t)}\, dt \\
&= i\int_{-\pi}^{\pi} e^{a\cos t}e^{ia\sin t}\, dt \\
&= i\int_{-\pi}^{\pi} e^{a\cos t}(\cos(a\sin t) + i\sin(a\sin t))\, dt \\
&= i\int_{-\pi}^{\pi} e^{a\cos t}\cos(a\sin t) + ie^{a\cos t}\sin(a\sin t)\, dt \\
&= i\int_{-\pi}^{\pi} e^{a\cos t}\cos(a\sin t)\, dt + i\int_{-\pi}^{\pi} ie^{a\cos t}\sin(a\sin t)\, dt \\
&= -\int_{-\pi}^{\pi} e^{a\cos t}\sin(a\sin t)\, dt + i\int_{-\pi}^{\pi} e^{a\cos t}\cos(a\sin t)\, dt.
\end{aligned}$$

Comparing the real and imaginary parts we have that,

$$\int_{-\pi}^{\pi} e^{a\cos t}\cos(a\sin t)\,dt = 2\pi.$$

Now we know cos is an even function and that sin is odd, but in this case sin is being inputted into a cos, so the function we are integrating right above must be even as well. Which gives us,

$$\begin{aligned}\int_{-\pi}^{\pi} e^{a\cos t}\cos(a\sin t)\,dt &= 2\int_{0}^{\pi} e^{a\cos t}\cos(a\sin t)\,dt\\ 2\pi &= 2\int_{0}^{\pi} e^{a\cos t}\cos(a\sin t)\,dt\\ \pi &= \int_{0}^{\pi} e^{a\cos t}\cos(a\sin t)\,dt\end{aligned}$$

as desired.

$\square$

**Problem 8.4** Let $f$ be an entire function such that there exists an $M > 0$ such that $\mathrm{Re}(f(z)) \geqslant M$ for all $z \in \mathbb{C}$. Prove that $f$ is constant.

***Solution.*** Given that $\mathrm{Re}(f(z)) \geqslant M$ that implies that $-\mathrm{Re}(f(z)) \leqslant -M$. We also know that $f$ is entire so we have that $e^{-f(z)}$ is also entire. If we consider the modulus of this we get,

$$\left|e^{-f(z)}\right| = e^{-\mathrm{Re}(f(z))} \leqslant e^{-M}$$

so we can let $g(z) = e^{-f(z)}$, we see by above that is is bounded, so by Liouville's Theorem we have that is is constant, and by earlier homework we know $g'$ must evaluate to 0, which gives us,

$$g'(z) = -e^{-f(z)}f'(z) = 0$$

which means that $f'(z)$ is 0, but by Theorem 8.4 we have that $f(z)$ must be constant, as desired. $\square$

**Problem 8.5** Let $f$ be an entire function such that $|f(z)| \leqslant A\,|z|$ for all $z$, where $A$ is a fixed positive number. Show that $f(z) = \alpha z$, where $\alpha$ is a complex constant.

***Solution.*** Let us apply Cauchy's Inequalities for $n = 2$ we get,

$$\left|f''(z_0)\right| \leqslant \frac{2!}{R^2} \max_{z \in C_R(z_0)} |f(z)|\,. \tag{2}$$

We know for any circle $C_R(z_0)$, that is a circle of radius $R$ with center $z_0$, that,

$$|z - z_0| = R$$

for all $z$ on the circle $C_R(z_0)$, which then gives us,

$$-R \leqslant |z| - |z_0| \leqslant R \implies |z| \leqslant |z_0| + R$$

We are given that $|f(z)| \leqslant A\,|z|$, using what we derived above we get,

$$|f(z)| \leqslant A\,|z| \leqslant A(|z_0| + R).$$

Now going back to (2) we have,

$$\left|f''(z_0)\right| \leqslant \frac{2!}{R^2} A(|z_0| + R) \to 0 \text{ as } R \to \infty$$

therefore $f''(z_0) = 0$. Now using a fact from homework 4 and the fact that $z_0$ was an arbitrary point on $\mathbb{C}$, we have $f(z) = \alpha z + \beta$. Now we consider when $z = 0$ we get,

$$|f(0)| \leqslant A\,|0| \implies -0 \leqslant f(0) \leqslant 0 \implies f(0) = 0$$

giving us,

$$\alpha \cdot 0 + \beta = 0 \implies \beta = 0.$$

Therefore $f(z) = \alpha z$ as desired. $\square$

**Collaborators:**

**References:**

- [Book(s): Title, Author]
- [Online: Link]
- [Notes: Link]

Fin.