Homework 2

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Problem 13.3.4 The construction of the regular 7-gon amounts to contractibility of $\cos(2\pi/7)$. We shall see later that $\alpha = 2\cos(2\pi/7)$ satisfies the equation $x^3 + x^2 - 2x - 1 = 0$. Use this to prove that the regular 7-gon is not constructible by straightedge and compass.

Proof. To begin we will show that the polynomial given,

$$p(x) = x^3 + x^2 - 2x - 1$$

is irreducible over $\mathbb{Q}$. To do so, we use the Rational Root Theorem, where if $p(x)$ has a root in $\mathbb{Q}$ it will be of the form $\frac{p}{q}$ where q divides the leading coefficient and p divides the constant term. So the only potential roots are ± 1 we see though that,

$$\begin{aligned} p(1) &= 1 + 1 - 2 - 1 = -1 \\ p(-1) &= -1 + 1 + 2 - 1 = 1 \end{aligned}$$

so it has not roots in $\mathbb{Q}$, meaning it is irreducible over $\mathbb{Q}$. This means then that α is of degree 3 and $[\mathbb{Q}(\alpha) : \mathbb{Q}] \neq 2^k$ for some $k \in \mathbb{N}$, but by Proposition 23 in D&F α would have to be a power of 2 in order for it to be constructed, meaning then that the regular 7-gon cannot be constructed by straightedge and compass. $\square$

Problem 13.3.5 Use the fact that $\alpha = 2\cos(2\pi/5)$ satisfies the equation $x^2 + x - 1 = 0$ to conclude that the regular 5-gon is constructible by straightedge and compass.

Proof. We do like before and try to determine if,

$$p(x) = x^2 + x - 1$$

has roots in $\mathbb{Q}$. Similarly if it did, it would have to be ± 1 by the Rational Root Test. We see,

$$\begin{aligned} p(1) &= 1 + 1 - 1 = 1 \\ p(-1) &= 1 - 1 - 1 = -1 \end{aligned}$$

meaning it has no roots in $\mathbb{Q}$, and so it is irreducible. Giving us that the degree of α is 2, which is clearly a power of 2 so it is constructible. We know we we are able to bisect and angle so we are also able to construct $\cos(2\pi/5)$ from $2\cos(2\pi/5)$. Now we just need to show that $\sin(2\pi/5)$ is constructible, but recalling our trig identities this is equivalent to showing,

$$\sin(2\pi/5) = \sqrt{1 - \cos^2(2\pi/5)}$$

the RHS is constructible. Recall though we are able to multiply constructions which is the squaring, we can subtract and we can take roots. Therefore $\sin(2\pi/5)$ is also constructible. Which means then that the regular 5-gon is constructible by straightedge and compass. $\square$

Problem 13.4.1 Determine the splitting field and its degree over $\mathbb{Q}$ for $x^4 - 2$.

Proof. Let $p(x) = x^4 - 2$. We see that the real roots of this polynomial are $\pm\sqrt[4]{2}$ and the complex roots are $\pm i\sqrt[4]{2}$. The relation between these two pairs of roots is that the complex root is just the real root multiplied by i, we also know from class that $\mathbb{Q}(-\alpha) = \mathbb{Q}(\alpha)$, meaning the splitting field for $p(x)$ is simply $\mathbb{Q}(i, \sqrt[4]{2})$ To calculate the degree of the splitting field over $\mathbb{Q}$, we use Theorem 14 from D&F and obtain,

$$\left[\mathbb{Q}(i, \sqrt[4]{2}) : \mathbb{Q}\right] = \left[\mathbb{Q}(i, \sqrt[4]{2}) : \mathbb{Q}(\sqrt[4]{2})\right] \left[\mathbb{Q}(\sqrt[4]{2}) : \mathbb{Q}\right]$$

We know that $i \notin \mathbb{Q}(\sqrt[4]{2})$ so $x^2 + 1$ which has root i is irreducible. Meaning

$$\left[\mathbb{Q}(i, \sqrt[4]{2}) : \mathbb{Q}(\sqrt[4]{2})\right] = 2$$

next $\sqrt[4]{2}$ is the root of the irreducible polynomial $x^4 + 2$ so,

$$\left[\mathbb{Q}(\sqrt[4]{2} : \mathbb{Q})\right] = 4$$

multiplying the two we have the degree of $\mathbb{Q}(i, \sqrt[4]{2})$ over $\mathbb{Q}$ to be 8.

$\square$

Problem 13.4.2 Determine the splitting field and its degree over $\mathbb{Q}$ for $x^4 + 2$.

Proof. Let $q(x) = x^4 + 2$ and its splitting field to be E_q, and let $p(x) = x^4 - 2$ as before and its splitting field be $E_p = \mathbb{Q}(i, \sqrt[4]{2})$. Our claim now is that $E_q = E_p$. To do so we first want to show that

$$\gamma = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$

is in both E_q and E_p.

Let us begin with E_q first. Showing $i, \sqrt{2} \in E_q$ is suffice for showing $\gamma \in E_q$. Consider the polynomials $x^4 + 2$ and $x^4 - 1$, let a be a root of the first one and b be a root of the second one. We see that,

$$a^4 = -2$$
$$b^4 = 1$$

then that means $(ab)^4 = a^4b^4 = -2$, so (ab) is also a root of $x^4 + 2$. We know though that the roots of $x^4 - 1$ are ± 1 and $\pm i$ so roots of $x^4 + 2$ are $\pm a$ and $\pm ia$. Going back to E_q, this is the field generated by the roots $\pm a$ and $\pm ia$ over $\mathbb{Q}$, so we have,

$$(ia)a^{-1} = i \in E_q$$

Now all that is left is showing $\sqrt{2} \in E_q$. Let a be a root of $x^4 + 2$, and then let $c = a^2$, we have then that c is a root for the equation $x^2 + 2$ since,

$$c^2 = (a^2)^2 = a^4 = -2$$

we know though the roots of $x^2 + 2$ are explicitly $\pm i\sqrt{2}$, in either case though we know $i \in E_q$ so we have,

$$c \cdot i^{-1} = \pm\sqrt{2} \in E_q$$

and therefore $\gamma \in E_q$

Showing i and $\sqrt{2}$ in E_p is much easier since we already know $\sqrt[4]{2} \in E_p$ so we have $(\sqrt[4]{2})^2 = \sqrt{2} \in E_p$, and we already know i is in E_p from the previous problem so we have $\gamma \in E_p$.

Finally let α be a root of $q(x)$ and let β be a root of $p(x)$ we have then that $\alpha^4 = -2$ and $\beta^4 = 2$. Recall that we know γ is in both of these polynomials splitting field and that,

$$\gamma^2 = i$$
$$\gamma^4 = -1$$

Notice though that $(\gamma\beta)^4 = \gamma^4\beta^4 = -2$ and so $\gamma\beta$ is a root of $q(x)$. Applying what we observed earlier the roots of $q(x)$ are $\pm\gamma\beta$ and $\pm i\gamma\beta$. We know though because i, γ, and β are in E_q then these roots are also in E_q, recall though these roots are what generate E_p, so $E_p \subseteq E_q$.

Now we notice that $(\gamma\alpha)^4 = \gamma^4\alpha^4 = 2$, meaning $\gamma\alpha$ is a root of $p(x)$, but recalling from the previous problem, the roots are then $\pm\gamma\alpha$ and $\pm i\gamma\alpha$, but because α, γ, and i are in E_p, these roots are in E_p. These are the same roots that generate E_q though, so we have $E_q \subseteq E_p$, showing containment both ways and therefore $E_q = E_p$.

Meaning that $\mathbb{Q}(i, \sqrt[4]{2})$ is the splitting field of $q(x)$ and from the previous problem its degree is 8. $\square$

Problem 13.4.3 Determine the splitting field and its degree over $\mathbb{Q}$ for $x^4 + x^2 + 1$.

Proof. Let $p(x) = x^4 + x^2 + 1$ be the given polynomial. We see that we can factor this polynomial as,

$$p(x) = x^4 + x^2 + 1 = (x^2 - x + 1)(x^2 + x + 1)$$

So solving for the root of $p(x)$ is simply solving for the roots of the two factors on the right, using the quadratic formula we obtain,

$$\pm\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

now let $z = \frac{1}{2} - i\frac{\sqrt{3}}{2}$, then the roots of $p(x)$ are simply $z, -z, \overline{z}, \overline{-z}$. Recalling the fact from problem 1 that $\mathbb{Q}(\alpha) = \mathbb{Q}(-\alpha)$ we have the splitting field for $p(x)$ to be $\mathbb{Q}(z, \overline{z})$. Notice though that,

$$z + \overline{z} = \frac{1}{2} - i\frac{\sqrt{3}}{2} + \frac{1}{2} + i\frac{\sqrt{3}}{2} = 1$$

meaning the additive inverse of z is simply $\overline{z}$, the splitting field is simply $\mathbb{Q}(z)$. Now for the degree, we know z was the root of a factor of $p(x)$ specifically $x^2 - x + 1$, which is irreducible over $\mathbb{Q}$ since z is complex, therefore we have,

$$[\mathbb{Q}(z) : \mathbb{Q}] = 2$$

□

Problem 13.4.4 Determine the splitting field and its degree over $\mathbb{Q}$ for $x^6 - 4$.

Proof. Let $p(x) = x^6 - 4$ be the given polynomial. We see $p(x)$ can be factored to be,

$$p(x) = x^6 - 4 = (x^3 - 2)(x^3 + 2)$$

So like before the roots of $p(x)$ are simply the roots of the polynomials on the RHS. These polynomials are discussed as an example in chapter 13 section 4 of D&F, so using the primitive 3rd root of unity ζ_3, we know the roots of $(x^3 - 2)$ to be $\sqrt[3]{2}$, $\zeta_3\sqrt[3]{2}$, and $(\zeta_3)^2\sqrt[3]{2}$. Similarly the roots for $x^3 + 2$ are $-\sqrt[3]{2}$, $-\zeta_3\sqrt[3]{2}$, and $-(\zeta_3)^2\sqrt[3]{2}$. Which means the splitting field is $\mathbb{Q}(\zeta_3, \sqrt[3]{2})$.
Now we can calculate the degree as before,

$$\left[\mathbb{Q}(\zeta_3, \sqrt[3]{2}) : \mathbb{Q}\right] = \left[\mathbb{Q}(\zeta_3, \sqrt[3]{2}) : \mathbb{Q}(\sqrt[3]{2})\right]\left[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}\right].$$

We already know the degree of $\left[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}\right]$ to be 3, since $\sqrt[3]{2}$ is a root of the polynomial $x^3 - 2$ which is irreducible over $\mathbb{Q}$. Finally ζ_3 is a root of the polynomial $x^2 + x + 1$ which is irreducible over $\mathbb{Q}(\sqrt[3]{2})$, so $\left[\mathbb{Q}(\zeta_3, \sqrt[3]{2}) : \mathbb{Q}(\sqrt[3]{2})\right]$ is of degree 2. All together we then have,

$$\left[\mathbb{Q}(\zeta_3, \sqrt[3]{2}) : \mathbb{Q}\right] = 6$$

$\square$