Homework 9

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Problem 11.5.1 Prove that if M is a cyclic R—module then $\mathcal{T}(M) = \mathcal{S}(M)$, i.e., the tensor algebra $\mathcal{T}(M)$ is commutative.

Proof. Because M is cyclic, we have without loss of generality that $M = R/I$ for some ideal of R. Then $\mathcal{S}(M) = \mathcal{T}(M)/J$ where J is the 2-sided ideal generated by elements of the form

$$m_1 \otimes m_2 - m_2 \otimes m_1$$

Writing $m_i = r_i(1 + I)$, we have $m_1 \otimes m_2 - m_2 \otimes m_1 = 0$ therefore $\mathcal{S}(M) = \mathcal{T}(M)$. □

Problem 11.5.4 Prove that $m \wedge n_1 \wedge n_2 \wedge \cdots \wedge n_k = (-1)^k(n_1 \wedge n_2 \wedge \cdots \wedge n_k \wedge m)$. In particular, $x \wedge (y \wedge z) = (y \wedge z) \wedge x$ for all $x, y, z \in M$.

Proof. We have R to be commutative with a 1, now let M be an (R, R) bimodule such that for all $r \in R$ and $m \in M$ we have,

$$rm = mr.$$

Now for all $x, y, z \in M$ we have,

$$x \wedge (y \wedge z) = (y \wedge z) \wedge x$$

Now we will do induction on n. It is clear for $n = 0, 1$ the result holds, now we assume it holds for all $n \leqslant k$.

Now we prove for $n = k + 1$,

$$\begin{aligned} m \wedge n_1 \wedge n_2 \wedge \cdots \wedge n_k \wedge n_{k+1} &= (m \wedge n_1 \wedge n_2 \wedge \cdots \wedge n_k) \wedge n_{k+1} \\ &= (-1)^k(n_1 \wedge n_2 \wedge \cdots \wedge n_k \wedge m) \wedge n_{k+1} \\ &= (-1)^k(n_1 \wedge n_2 \wedge \cdots \wedge n_{k+1} \wedge m) \\ &= (-1)^k(n_1 \wedge n_2 \wedge \cdots \wedge n_{k+1} \wedge m). \end{aligned}$$

Giving us the desired equality,

$$m \wedge n_1 \wedge n_2 \wedge \cdots \wedge n_k = (-1)^k(n_1 \wedge n_2 \wedge \cdots \wedge n_k \wedge m).$$

□

Problem 11.5.6 If A is an R—algebra in which $a^2 = 0$ for all $a \in A$ and $\varphi : M \to A$ is an R—module homomorphism, prove there is a unique R—algebra homomorphism $\Phi : \bigwedge(M) \to A$ such that $\Phi\mid_M = \varphi$

Proof. As before we have R to be commutative with a 1 and M an (R, R) bimodule, such that for all $r \in R$ and $m \in M$ such that,

$$rm = mr.$$

Through the properties of a tensor algebra there exists a unique R—module homomorphism $\overline{\Phi} : \overline{I}(M) \to A$, $\overline{\Phi}_M = \varphi$

$A(M)$ is ideal of $\mathcal{T}(M)$ is generated by simple tensor of the form $m \otimes m$.

$$\begin{aligned}\overline{\Phi}(m \otimes m) &= \overline{\Phi}(m^2)\\ &= 0 \in A\\ A(M) &\subseteq \ker \overline{\Phi}\end{aligned}$$

then by the 1st isomorphism theorem for R—algebras,

$$\begin{aligned}&\Phi : \bigwedge(M) \to A, \text{ defined by } \Phi(\overline{t}) = \overline{\Phi}(t), \Phi\mid_M = \varphi\\ &\psi : \bigwedge(M) \to A\end{aligned}$$

then,

$$\begin{aligned}\psi(\wedge m_i) &= \prod \psi(m_i)\\ &= \prod \varphi(m_i)\\ &= \prod \Phi(m_i)\\ &= \Phi(\wedge m_i)\end{aligned}$$

so Φ is unique R—algebra homomorphism. □

Problem 11.5.8 Let R be an integral domain and let F be its field of fractions

(a) Considering F as an R—module, prove that $\bigwedge^2 F = 0$

(b) Let I be any R—submodule of F (for example, any ideal in R). Prove that $\bigwedge^i I$ is a torsion R-module for $i \geqslant 2$(i.e., for every $x \in \bigwedge^i I$ there is some nonzero $r \in R$ with $rx = 0$)

(c) Give an example of an integral domain R and an R—module I in F with $\bigwedge^i I \neq 0$ for every $i \geqslant 0$ (cf. the example following corollary 37)

(a) *Proof.* Let F be an $\mathrm{R}-$module, we have,

$$\begin{aligned}
\frac{a}{b} \otimes \frac{c}{d} &\in \mathcal{T}^2(\mathrm{F}) \\
\frac{a}{b} \otimes \frac{c}{d} &= \frac{ad}{bd} \otimes \frac{cb}{bd} \\
&= abcd\left(\frac{1}{bd} \otimes \frac{1}{bd}\right) \\
\frac{a}{b} \wedge \frac{c}{d} &= 0 \in \wedge^2(\mathrm{F})
\end{aligned}$$

□

(b) *Proof.* Let I be any $\mathrm{R}-$submodule of F, we have,

$$\frac{a_1}{b_1} \wedge \frac{a_2}{b_2} \wedge \cdots \wedge \frac{a_k}{b_k} \in \wedge^2(\mathrm{I})$$

then $a_i \neq 0$ and $b - i \neq 0$, $a_1 a_2 b_1 b_2 \neq 0 \in \mathrm{R}$,

$$a_1 a_2 b_1 b_2 \left(\frac{a_1}{b_1} \wedge \frac{a_2}{b_2} \wedge \cdots \wedge \frac{a_k}{b_k}\right) = \frac{a_1 a_2}{1} \wedge \frac{a_1 a_2}{1} \wedge \cdots \wedge \frac{a_k}{b_k}$$

every element of $\wedge^{\mathrm{K}}(\mathrm{I})$ is torsion as desired. □

(c) *Proof.* Let us consider $\mathrm{R} = \mathrm{Z}[x_1, x_2, \ldots, x_n]$ and $\mathrm{I} = (x-1, x_2, \ldots, x_n)$

Now we consider some j and let,

$$\alpha_j x_j - \beta_j x_j = \sum_{i \neq j} (\beta_i - \alpha_i) x_i$$

Since R is a domain we have x_j divides the right hand side,

$$\sum_{i \neq j} (\beta_i - \alpha_i) x_i = x_j h_j$$

Here, $h_i \in \mathrm{I}$ such that $\alpha_i - \beta_i = h_i$

Now we consider $\prod(\mathrm{I})$ as column vectors that means $\sum \alpha_k x_k$ as $[\alpha_1, \alpha_2, \ldots, \alpha_n]^{\mathrm{T}}$. Tow column vectors A and B represent the same element I, there exists a third column vector H, such that $\mathrm{A} = \mathrm{B} + \mathrm{H}$.

Now consider the elements of R^k as square matrix. The determinant of such matrix A, as an element of R reduced mod I. If the matrix A and B represent the same elements of I^i then matrix H is such $\mathrm{A} = \mathrm{B} + \mathrm{H}$. Now consider determinants of both sides mod I, which we compute using the combinatorial formula,

$$\det(\mathrm{B} + \mathrm{H}) = \sum_{\sigma \in \sigma_n} \varepsilon(\sigma) \prod (\beta_{\sigma(i),j} + h_{\sigma(i),j})$$

$h_{i,j}$ is divisible by some x_i and hence goes to the quotient R/I and so,

$$\det(\mathrm{A}) = \det(\mathrm{B} + \mathrm{H}) \equiv \det(\mathrm{B}) \mod \mathrm{I}$$

Thus the map $\det : \mathrm{I}^i \to \mathrm{R}/\mathrm{I}$ is well defined alternating bilinear map. This map is nontrivial since

$$\det(x_1 \otimes \cdots \otimes x_n) = 1$$

therefor for all i, $\wedge^i(\mathrm{I}) \neq 0$ □

Problem 11.5.9 Let $R = \mathbb{Z}[G]$ be the group ring of the group $G = \{1, \sigma\}$ of order 2. Let $M = \mathbb{Z}e_1 + \mathbb{Z}e_2$ be the free $\mathbb{Z}$-module of rank 2 with basis e_1 and e_2. Define $\sigma(e_1) = e_1 + 2e_2$ and $\sigma(e_2) = -e_2$. Prove that this makes M into an R—module and that the R—module $\wedge^2 M$ is a group of order 2 with $e_1 \wedge e_2$ as generator.

Proof. We have the mapping $\varphi M \to M$ by $\varphi(e_1) = e_1 + 2e_2$ and $\varphi(e_2) = -e_2$. By using this mapping we make M into an R—module and compute the exterior power $\wedge^2(M)$ over R.

We have φ to be an endomorphism of order 2, and $\sigma^2 = 1$. Now we define the following,

$$R \times M \to M$$
$$(a \cdot 1 + b\sigma) \cdot m = am + b\varphi(m)$$

Now note that for any group G, ring R, and S—module M, If,

$$\alpha : R \to \mathrm{End}_S(M) \qquad\qquad \beta : G \to \mathrm{Aut}_S(M)$$

such that

$$\alpha \subseteq C_{\mathrm{End}_S(M)}(\mathrm{im}\,\beta)$$

then the induced map given by,

$$\gamma : R[G] \to \mathrm{End}_S(M)$$

given by $\gamma(\sum r_i g_i) = \sum \alpha(r_i) \circ \beta(g_i)$ is a well defined ring homomorphism. So we have M to be a $Z[G]$ module. As R is commutative, so M is an (R, R)-bimodule such that $rm = mr$. Giving us

$$\begin{aligned}
-(e_1 \wedge e_2) &= e_1 \wedge (-e_2) \\
&= e_1 \wedge \sigma \cdot e_2 \\
&= \sigma \cdot e_1 \wedge e_2 \\
&= e_1 + 2e_2 \wedge e_2 \\
&= e_1 \wedge e_2 \\
2(e_1 \wedge e_2) &= 0 \\
\sigma(e_1 \wedge e_2) &= 0
\end{aligned}$$

here, $\sigma^2(M)$ is generated by $e_1 \wedge e_2$ therefore $R(e_1 \wedge e_2) = \{0, e_1 \wedge e_2\}$

Now we consider the mapping,

$$\det : M^2 \to \mathbb{Z}/2\mathbb{Z}$$
$$(ae_1 + be_2, ce_1 + de_2) \mapsto ad - bc \mod 2$$

Therefore det is an alternating $\mathbb{Z}$—billinear form. To show that is is R—bilinear though we show that $\det(v, \sigma w) = \det(v\sigma, w)$,

$$\det(ae_1 + be_2, \sigma(ce_1 + de_2)) = \det(ae_1 + be_2, ce_1 + (2c - d)e_2)$$

$$
\begin{aligned}
&= a(2c-d)-bc \\
&= ad - c(2a-b) \\
&= \det(ae_1 + (2a-b)e_2, ce_1 + de_2)
\end{aligned}
$$

and $\det(e_1, e_2) = 1 \neq 0$ so $e_1 \wedge e_2$ is nonzero in $\bigwedge^2(M)$ therefore $\bigwedge^2(M) \cong \mathbb{Z}/2\mathbb{Z}$

$\square$

Problem 11.5.10 Prove that $z - (1/k!)\mathrm{Alt}(z) = (1/k!)\sum_{\sigma \in S_k}(z - \varepsilon(\sigma)\sigma z)$ for any $k-$tensor z and use this to prove that the kernel of the R-module homomorphism $(1/k!)\mathrm{Alt}$ in proposition 40 is $\mathcal{A}^k(M)$.

Proof. Let $z \in T^k(M)$ then,

$$
\begin{aligned}
z - \frac{1}{k!}\mathrm{Alt}(z) &= z - \frac{1}{k!}\sum_{\sigma \in S_k} \varepsilon(\sigma)\sigma z \\
&= \frac{1}{k!}\left(zk! - \sum_{\sigma \varepsilon S_k} \varepsilon(\sigma)\sigma z\right) \\
&= \frac{1}{k!}\left(\sum_{\sigma \in S_k} z - \sum_{\sigma \in z_k} \varepsilon(\sigma)\sigma z\right) \\
&= \frac{1}{k!}\left(\sum_{\sigma \in S_k} (z - \varepsilon(\sigma)\sigma z)\right)
\end{aligned}
$$

thus $z - \frac{1}{k!}\mathrm{Alt}(z) = \frac{1}{k!}\left(\sum_{\sigma \in S_k}(z - \varepsilon(\sigma)\sigma z)\right)$ as needed.

Now we for the latter statement. Let $z \in A^k(M)$ and suppose that i and $i+j$ components of z are equal, we have,

$$\sigma z = \sigma(1i+1)z$$

moreover,

$$\varepsilon(\sigma)\sigma z + \varepsilon(\sigma(ij+1))\sigma(ij+1)z = 0$$

Now we consider the following equation,

$$\mathrm{Alt}(z) = \sum_{\sigma \in S_k} \varepsilon(\sigma)\sigma z$$

the RHS of this can be broke up into a summation over the cosets of $< (ij+1) >$ each of which is zero, giving us

$$\frac{1}{k!}\mathrm{Alt}(z) = 0$$

therefore

$$A^K(M) \subseteq \ker \frac{1}{k!}\mathrm{Alt}$$

Now let $z \in \ker \frac{1}{k!}\mathrm{Alt}$ then we have,

$$\frac{1}{k!}\sum_{\sigma \in S_k}(z - \varepsilon(\sigma)\sigma z) = z$$

for each σ, $z - \varepsilon(\sigma)\sigma z \in A^k(M)$. Therefore $z \in A^k(M)$ and thus $\ker \frac{1}{k!}\mathrm{Alt} = A^k(M)$ as desired. □

Problem 11.5.11 Prove that the image of Alt_k is the unique largest subspace of $\mathcal{T}^k(V)$ on which each permutation σ in the symmetric group S_k acts as multiplication by the scalar $\varepsilon(\sigma)$.

Proof. We have V to be an F—vector space. Now S_k acts on the tensor power $T^k(V)$ by permuting the components. Let $k!$ be a unit in the ring R and M an R—module. The map $(1/k!)\mathrm{Alt}$ induces and R—module isomorphism between the k^{th} exterior power of M and the R—sub module of alternating k—tensors:

$$\frac{1}{k!}\mathrm{Alt} : \wedge^k M \cong \{\text{alternating} k - \text{tensors}\}$$

and that Alt_k is defined on $T^k(V)$ by the following,

$$\mathrm{Alt}_K(z) = \sum_{\sigma \in S_k}\varepsilon(\sigma)\sigma z$$

Now let $z \in T^k(V)$ such that for all $\sigma \in S_k$ we have,

$$\sigma z = \varepsilon(\sigma)z$$

then,

$$\begin{aligned}\mathrm{Alt}_k(z) &= \frac{1}{k!}\sum_{\sigma \in S_k}\varepsilon(\sigma)\sigma z \\ &= \frac{1}{k!}\sum_{\sigma \in S_k}\varepsilon(\sigma)\varepsilon(\sigma)z \\ &= \frac{1}{k!}\sum_{\sigma \in S_k} z \\ &= z\end{aligned}$$

Therefore $z \in \operatorname{im}\mathrm{Alt}_k$. Specifically any subspace of $T^k(V)$ upon which every permutation $\sigma \in S_k$ acts as scalar multiplication by $\varepsilon(\sigma)$ is in $\operatorname{im}\mathrm{Alt}_k$

So it can be seen that $\sigma \in S_k$ acts on $\operatorname{im}\mathrm{Alt}_k$ as multiplication by $\varepsilon(\sigma)$ as $\operatorname{im}\mathrm{Alt}_k \cong_F \wedge^k(V)$. Therefore $\operatorname{im}\mathrm{Alt}_k$ is the unique largest subspace of $T^k(V)$ on which each permutation σ in the symmetric group S_k acts as multiplication by the scalar $\varepsilon(\sigma)$. □

Problem 11.5.13 Let F be any field in which $-1 \neq 1$ and let V be a vector space over F. Prove that $V \oplus_F V = \mathcal{S}^2(V) \oplus \bigwedge^2(V)$ i.e., that every 2-tensor may be written uniquely as a sum of a symmetric and an alternating tensor.

Proof. We note that $\dim \mathcal{S}^2(V) = \dfrac{n(n+1)}{2}$ and that $\dim \bigwedge^2(V) = \dfrac{n(n+1)}{2}$. Therefore, $\mathcal{S}^2(V) \oplus \bigwedge^2(V) = n^2 = \dim V \otimes_F V$. We get the desired result by prove both these spaces intersect trivially, so assume that $v \in S^2(V) \cap \bigwedge^2(V)$, then we have,

$$\begin{cases} \sigma v = v \\ \sigma v = sgn(\sigma) v \end{cases} \iff v = sgn(\sigma) v \iff v(1 - sgn(\sigma)) = 0$$

we have though that $sgn(\sigma) = 1$ since we are in the symmetric group S_2. The above equation implies that $v(1-1) - 0$, but we assumed $-1 \neq 1$ so that can't be, therefore v is forced to be 0, giving us the desired result. $\square$