Homework 2

Kevin Guillen

MATH 103A — Complex Analysis — Spring 2022

Problem 2.1

(a) Prove that

$$\arg zw = \arg z + \arg w = \{(\operatorname{Arg} z + \operatorname{Arg} w) + 2k\pi \mid k \in \mathbb{Z}\}$$

Proof. We know from class, specifically Proposition 3.1 (1), that $\arg zw = \arg z + \arg w$. So all we want to show now is that

$$\arg z + \arg w = \{(\operatorname{Arg} z + \operatorname{Arg} w) + 2k\pi \mid k \in \mathbb{Z}\}$$

We will do this by showing each set is contained in the other. First let us take an element a in $\arg z$ and an element b in $argw$. We compute their sum to be,

$$\begin{aligned} a + b &= \operatorname{Arg} z + 2\pi k \operatorname{Arg} w + 2\pi l \qquad & k, l \in \mathbb{Z} \\ &= \operatorname{Arg} z + \operatorname{Arg} w + 2\pi(k + l) \end{aligned}$$

we know $\mathbb{Z}$ is closed under addition so $k + l \in \mathbb{Z}$. Meaning then that $a + b$ is an element of $\{(\operatorname{Arg} z + \operatorname{Arg} w) + 2k\pi \mid k \in \mathbb{Z}\}$. Recall though a and b were abitrary though so we have,

$$\arg z + \arg w \subseteq \{(\operatorname{Arg} z + \operatorname{Arg} w) + 2k\pi \mid k \in \mathbb{Z}\}. \tag{1}$$

Now let c be an element of $\{(\operatorname{Arg} z + \operatorname{Arg} w) + 2k\pi \mid k \in \mathbb{Z}\}$, we know then it is of the form

$$c = \operatorname{Arg} z + \operatorname{Arg} w + 2\pi k \qquad k \in \mathbb{Z}$$

we always have though that $k = k + 0$ meaning the above can be expressed as,

$$c = \operatorname{Arg} z + \operatorname{Arg} w + 2\pi k = \operatorname{Arg} z + 2pi0 + \operatorname{Arg} w + 2\pi k$$

which is clear then that c is the sum of some element in $\arg z$ plus some element in $\arg w$. Meaning,

$$\{(\operatorname{Arg} z + \operatorname{Arg} w) + 2k\pi \mid k \in \mathbb{Z}\} \subseteq \arg z + \arg w. \tag{2}$$

Finally, (1) and (2) together give us the desired equality. □

(b) Show that if $\operatorname{Re} z > 0$ and $\operatorname{Re} w > 0$, then $\operatorname{Arg}(zw) = \operatorname{Arg} z + \operatorname{Arg} w$.

Solution. We know z and w are of the form $re^{i\theta}$ and $se^{i\varphi}$ respectively, and that their product is simply $rse^{i(\theta+\varphi)}$. Meaning $\operatorname{Arg} zw = \theta + \varphi$, but notice $\operatorname{Arg} z = \theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\operatorname{Arg} w = \varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Therefore,

$$\operatorname{Arg} zw = \operatorname{Arg} z + \operatorname{Arg} w$$

I know this is wrong, I'm just not sure how to incorporate the fact that φ and θ are in $(-\frac{\pi}{2}, \frac{\pi}{2})$ $\square$

Problem 2.2

(a) Let $z \in \mathbb{C}$. Using the principle of mathematical induction, show that the following formula holds for all integers $n \geqslant 1$

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}.$$

Proof. First, we show that the formula holds for $n = 1$ by working out the LHS and RHS of the formula. We see the LHS is,

$$1 + z$$

the RHS works out to be,

$$\frac{1 - z^2}{1 - z} = \frac{(1 - z)(1 + z)}{1 - z} = 1 + z.$$

Therefore the formula holds for $n = 1$.
We assume the formula holds for all $n < k$.
Using this assumption we now show the formula holds for $n = k$ through the following,

$$\begin{aligned}
\underbrace{1 + z + z^2 + \cdots + z^{k-1}}_{\text{assumption}} + z^k &= \underbrace{\frac{1 - z^k}{1 - z}}_{\text{formula}} + z^k \\
&= \frac{1 - z^k}{1 - z} + \frac{z^k(1 - z)}{1 - z} \\
&= \frac{1 - z^k + z^k - z^{k+1}}{1 - z} \\
&= \frac{1 - z^{k+1}}{1 - z}
\end{aligned}$$

We see through induction then that the formula holds for all integers $n \geqslant 1$ as desired.

$\square$

(b) If $\rho_1, \dots, \rho_n$ are the *distinct* n^{th} roots of unity, show that, using (a),

$$\sum_{i=1}^{n} \rho_i = 0.$$

Proof. We recall we can express the n^{th} roots of unity in terms of the principal root and roots of unity. We first recall that,

$$\beta_0 = \sqrt[n]{|\alpha|} e^{i \frac{\operatorname{Arg} \alpha}{n}}$$

so we can rewrite the given summation as,

$$\begin{aligned} \sum_{i=1}^{n} \rho_i &= \sum_{k=0}^{k-1} \beta_0 \zeta_n^k && \beta_0 \text{ is a constant} \\ &\beta_0 \sum_{k=0}^{k-1} \zeta_n^k. \end{aligned}$$

Giving us something we can finally apply (a) to and get,

$$\beta_0 \sum_{k=0}^{k-1} \zeta_n^k = \beta_0 \left(\frac{1 - (\zeta_n)^n}{1 - \zeta_n} \right).$$

Recall though that $\zeta_n = e^{\frac{2\pi i}{n}}$, so the above becomes,

$$\begin{aligned} \beta_0 \left(\frac{1 - (e^{\frac{2\pi i}{n}})^n}{1 - \zeta_n} \right) &= \beta_0 \left(\frac{1 - (e^{\frac{2\pi i n}{n}})}{1 - \zeta_n} \right) \\ &= \beta_0 \left(\frac{1 - (e^{2\pi i})}{1 - \zeta_n} \right) && \text{Euler's Identity: } e^{2\pi i} = 1 \\ &= \beta_0 \left(\frac{0}{1 - \zeta_n} \right) \\ &= 0. \end{aligned}$$

Therefore $\sum_{i=1}^{n} \rho_i = 0$, as desired. □

(c) We compute the following sum of real numbers

$$\cos \frac{\pi}{7} + \cos \frac{3\pi}{7} + \cos \frac{5\pi}{7} \tag{†}$$

(i) Let $w = e^{\frac{\pi i}{7}}$. What is $\operatorname{Re} w$ and w^7? Furthermore, rewrite (†) as

$$\operatorname{Re}(w^{a_1} + w^{a_2} + w^{a_3}), \quad \text{for some } 0 \leqslant a_i < 7.$$

Solution. We use Euler's formula to obtain that $\operatorname{Re} w = \cos(\dfrac{\pi}{7})$. Now we see w^7 is,

$$\begin{aligned} w^7 = (e^{\frac{\pi i}{7}})^7 &= e^{\frac{\pi i 7}{7}} \\ &= e^{\pi i} \\ &= -1. \end{aligned}$$

So it is clear that we can rewrite † as the desired equation using $a_1 = 1$, $a_2 = 3$, and $a_2 = 5$, to get,

$$\operatorname{Re}(w^1 + w^3 + w^5) = \cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7}$$

□

Letting $z = w$ we see we get,

(ii) Replacing z by $-z$ in (a), find a formula for

$$\frac{z^7+1}{z+1}.$$

Use this to deduce an identity involving w and its powers.

Proof. In this case we have $n = 6$ and $z = -z$, we apply (a) to see the formula for the given equation is just,

$$\begin{aligned} 1 + (-z) + (-z)^2 + (-z)^3 + (-z)^4 + (-z)^5 + (-z)^6 &= \frac{1-(-z)^7}{1-(-z)} \\ &= \frac{1-(-1)^7 z^7}{1+z} \\ &= \frac{1-(-1)z^7}{1+z} \\ 1 - z + z^2 - z^3 + z^4 - z^5 + z^6 &= \frac{z^7+1}{z+1} \end{aligned}$$

Let us consider then when we have $z = w$ we get,

$$\begin{aligned} 1 - w + w^2 - w^3 + w^4 - w^5 + w^6 &= \frac{w^7+1}{w+1} && \text{using (i)} \\ &= \frac{-1+1}{w+1} \\ &= 0 \end{aligned}$$

Giving us an identity for w,

$$\sum_{k=0}^{6} (-w)^k = 0$$

□

(iii) Using the identity you found in (iii), conclude that

$$w^{a_1} + w^{a_2} + w^{a_3} = \frac{1}{1-w}$$

where the a_i's are the numbers you found in (ii).

Solution. Let us expand the summation in our identity from the previous part to obtain,

$$\begin{aligned}
1 - w + w^2 - w^3 + w^4 - w^5 + w^6 &= 0 \\
(1 + w^2 + w^4 + w^6) - (w + w^3 + w^5) &= 0 \\
1 + w^2 + w^4 + w^6 &= w + w^3 + w^5 && \text{Let } w^2 = z \\
1 + z + z^2 + z^3 &= w + w^3 + w^5 && \text{Apply (a) to LHS} \\
\frac{1 - z^4}{1 - z} &= w + w^3 + w^5 && z = w^2 \\
\frac{1 - (w^2)^4}{1 - w^2} &= w + w^3 + w^5 \\
\frac{1 - w^8}{(1-w)(1+w)} &= w + w^3 + w^5 \\
\frac{1 - w^7 w}{(1-w)(1+w)} &= w + w^3 + w^5 && \text{apply (i) to } w^7 \\
\frac{1 + w}{(1-w)(1+w)} &= w + w^3 + w^5 \\
\frac{1}{(1-w)} &= w + w^3 + w^5.
\end{aligned}$$

Recall though that $a_1 = 1$, $a_2 = 3$, and $a_3 = 5$ giving us the desired result. □

(iv) Finally compute (†).

Solution. Recall $w = e^{\frac{\pi * i}{7}}$, but from class we can use Euler's formula to express it as $w = \cos\frac{\pi}{7} + i\sin\frac{\pi}{7}$. Plugging this into what we showed in the last part,

$$\frac{1}{1 - \cos\frac{\pi}{7} - i\sin\frac{\pi}{7}}. \tag{3}$$

To help compute it though we have,

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2}.$$

Applying this to (3) we get,

$$\begin{aligned}
\frac{1}{1 - \cos\frac{\pi}{7} - i\sin\frac{\pi}{7}} &= \frac{1 - \cos\frac{\pi}{7} + i\sin\frac{\pi}{7}}{\left(\sin^2\frac{\pi}{7}\right) + \left(1 - \cos\frac{\pi}{7}\right)^2} \\
&= \frac{1 - \cos\frac{\pi}{7} + i\sin\frac{\pi}{7}}{2 - 2\cos\frac{\pi}{7}} \\
&= \frac{1 - \cos\frac{\pi}{7}}{2(1 - \cos\frac{\pi}{7})} + i\frac{\sin\frac{\pi}{7}}{2(1 - \cos\frac{\pi}{7})} \\
&= \frac{1}{2} + i\frac{\sin\frac{\pi}{7}}{2(1 - \cos\frac{\pi}{7})}.
\end{aligned}$$

Recall though † was simply the real component of this, so $\dagger = \frac{1}{\cdot}2$

$\square$

Problem 2.3

(a) Recall that a set is open if every point of the set is an interior point. Prove that a set $U \subseteq \mathbb{C}$ is open if and only if it does not contain any of its boundary points; that is, $\partial U \cap U = \varnothing$. Then deduce that the complement of a closed set is open.

Proof. ($\Rightarrow$) Assuming that U is an open set, that means by definition every point in U is an interior point. If there existed a point p in U that was a boundary point, we know from class that means for all $\varepsilon > 0$ the $\varepsilon-$neighborhood of p contains points in U and points not in U. Such a p in U would contradict the fact that U is open, since every point in an open set is an interior point, that means there is supposed to exist an ε for p such that all points in the ε-neighborhood of p are contained in U. Therefore if U is open, it does not contain any of its boundary points.
($\Leftarrow$) Assuming that U does not contain any of its boundary points that means then U^c contains it's boundary points. Which is to say that $\partial U \subseteq U^c$ which from class we know that means U^c is closed, and by Definition 4.1 its complement is open, but $(U^c)^c = U$, therefore U is open, if it does not contain any of its boundary points.
We have shown both directions meaning we have a set U is open if and only if it does not contain any of its boundary points. □

(b) Prove that an open disk $D_\varepsilon(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < \varepsilon\}$ is a domain; that is, a non-empty open and connected subset of $\mathbb{C}$.

Proof. (non-empty) We show this is non-empty by simply considering z_0 which is in $\mathbb{C}$ and $|z_0 - z_0| = |0| = 0 < \varepsilon$ since by definition $\varepsilon > 0$.
(connected) To show that the open disk is connected we only need 1 line segment. Given any two points p and q in $D_\varepsilon(z_0)$ we know we have the line,

$$f(x) = p + x(q - p)$$

for $x \in [0, 1]$. What we have to show now though is that ALL points in this line are indeed in the open disk, which is just showing

$$|f(x) - z_0| < \varepsilon.$$

So let us expand the LHS of the inequality,

$$\begin{aligned}
|f(x) - z_0| &= |p + x(q - p) - z_0| \\
&= |p + xq - xp - z_0| \\
&= |(1 - x)p + xq - z_0| \\
&= |(1 - x)p - z_0 + xq + xz_0 - xz_0| \\
&= |(1 - x)(p - z_0) + x(q - z_0)| \qquad \text{triangle identity}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant (1-x)\,|p-z_0| + x\,|q-z_0| && \text{p and q are points in the disk}\\
&\leqslant (1-x)\varepsilon + x\varepsilon\\
&= \varepsilon\\
|f(x)-z_0| &\leqslant \varepsilon
\end{aligned}
$$

we see that all the points of $f(x)$ do indeed lie in the $D_\varepsilon(z_0)$.
All this together shows that $D_\varepsilon(z_0)$ is a domain

$\square$

Problem 2.4 Let $f : G \to \mathbb{C}$ be a complex function, and suppose z_0 is an accumulation point of G. Show that

$$\lim_{z \to z_0} f(z) = w_0 \quad \text{if and only if} \quad \lim_{z \to z_0} |f(z) - w_0| = 0.$$

Thereby deduce that

$$\lim_{z \to z_0} \overline{f(z)} = \overline{w}_0 \quad \text{if and only if} \quad \lim_{z \to z_0} f(z) = w_0.$$

Problem 2.5 Compute the following limits and prove your claim by using only the ε-δ definition.

(a) $\lim_{z \to i} \overline{z}$

(b) $\lim_{z \to 1+i} z^2$

Collaborators:

References:

- [Book(s): Title, Author]
- [Online: Link]
- [Notes: Link]

Fin.