Name:

Topology - Summer Session 1 - HW2 - 7/3/2021

(3.3) For each topology on $\{a, b, c, d\}$ from Question 3.2, list the open sets in the subspace topology for the subset $\{a, b, c\}$.

Proof. Recalling from 3.2 the only list of open sets that formed a topology were the following,

$$V_1 = \emptyset, \{a\}, \{a, b\}, \{a, b, c, d\}$$
$$V_2 = \emptyset, \{a, c\}, \{a, b, c\}, \{a, c, d\}, \{a, b, c, d\}.$$

So taking the intersection of $\{a, b, c\}$ with every open set in each topology will give us each the list of open sets in the subspace for the subset $\{a, b, c\}$

$$\forall U \in V_1,\ U \cap \{a, b, c\} = \emptyset, \{a\}, \{a, b\}, \{a, b, c\}$$

$$\forall U \in V_2,\ U \cap \{a, b, c\} = \emptyset, \{a, c\}, \{a, b, c\}.$$

□

(3.8) Let $f : \mathbb{R} \to \mathbb{Z}$ be the floor function which rounds a real number x down to the nearest integer:

$$f(x) = n \text{ provided that } n \in \mathbb{Z} \text{ and } n \leq x < n + 1$$

Determine whether f is continuous.

Proof. We know in $\mathbb{Z}$ that for any $n \in \mathbb{Z}$, $\{n\}$ will be an open set. Taking the inverse image of this open set we see we get,

$$f^{-1}(n) = [n, n + 1)$$

which is a closed set in $\mathbb{R}$. Thus, by definition the floor function is not continuous. □

(3.11) Let T be a set and B a collection of subsets of T. Show that if every element of T belongs to at least one subset in B and B is closed under finite intersections, then the collection of all unions of sets in B forms a topology on T

Proof. Assuming that if $\forall x \in T$, $\exists U \in B$ such that, $x \in U$, and that B is closed under finite intersections, we want to show the collection of all unions of sets in B forms a topology on T. Let's refer to this collection as $\mathscr{B}$.

The first thing we need to show is that the empty set is in B'. Since $\mathscr{B}$ is defined as all unions of sets in B, let's take the union of sets in B indexed by I, where $I = \emptyset$ to get

$$\bigcup_{i \in I} U_i = \emptyset, \text{ where } U_i \in B.$$

Then we have that $\emptyset \in \mathscr{B}$, satisfying the first requirement.

Now to show that T is in $\mathscr{B}$. This is similar to as before in that B is defined in that for any element in T there is at least one subset in B that contains that element. Since B is also defined to be subsets of T taking any union of these subsets can never equal anything more than T. Now $\mathscr{B}$ is defined to be any union of set in B, thus taking the union of all sets in B will result in T.

Thus, we have it that $T \in \mathscr{B}$

Now to show that the union of elements of any subcollection is closed. Taking a union of subcollections in $\mathscr{B}$ to be U. In other words U is obtained by,

$$U = \bigcup_{i \in I} U_i$$

where I is some index set, and $U_i \in \mathscr{B}$. We know $\mathscr{B}$ to be defined as the collection of any unions of set in B, there each U_i can be expressed as,

$$U_i = \bigcup_{j \in J} B_j$$

where J is some index set, and $B_j \in B$. Expanding this out we get,

$$U = \bigcup_{i \in I} \bigcup_{j \in J} B_{j,i}$$

we see that this is simply a union of sets in B, and $\mathscr{B}$ is defined to be the collection of any union of sets in B, thus $U \in \mathscr{B}$.

Now we need to show $\mathscr{B}$ is closed under finite intersection. Take U_1, $U_2 \in \mathscr{B}$. They are defined as,

$$U_1 = \bigcup_{i \in I} B_i$$
$$U_2 = \bigcup_{j \in J} B_j$$

for I and J as some index and $B_i,\ B_j \in B$. Taking the intersection of these two we get,

$$U_1 \cap U_2 = \bigcup_{i \in I} B_i \cap \bigcup_{j \in J} B_j = \bigcup_{i \in I} \bigcup_{j \in J} (B_i \cap B_j)$$

Since B_i and B_j are elements of B, and B was defined to be closed under intersection, this then just becomes a union of set in B, which by definition is then in $\mathscr{B}$. It is obvious that this will hold for finite intersection through induction.

Thus, $\mathscr{B}$ does indeed form a topology for T

$\square$