Homework 6

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MATH 200 — Algebra I — Fall 2021

Can I please have my 9.1 graded, thank you.

Problem 3.3 Let G be a group. Show that $Z(G)$ and G' are characteristic subgroups of G.

Proof. (a) Showing the center of G is characteristic subgroup of G. First let $\varphi \in \mathrm{Aut}(G)$. We need to show $\varphi(Z(G)) = Z(G)$. Let $z \in Z(G)$, and $x \in G$. Then we know there exists some $y \in G$ such that $x = \varphi(y)$. This gives us the following,

$$\begin{aligned}\varphi(z)x &= \varphi(z)\varphi(y)\\ &= \varphi(zy)\\ &= \varphi(yz)\\ &= \varphi(y)\varphi(z) = x\varphi(z)\end{aligned}$$

We can see then that $\varphi(z)$ commutes with all elements of G. Therefore $\varphi(Z(G)) \subset Z(G)$. Applying the same reasoning to φ^{-1} we get that $\varphi^{-1}(Z(G)) =\subset Z(G)$ which gives us that $Z(G) \subset \varphi(Z(G))$ and thus $Z(G) = \varphi(Z(G))$ as desired.

(b) Show the commutator subgroup of G is characteristic. Consider any $\varphi \in \mathrm{Aut}(G)$. We want to show that $\varphi(G') = G'$. Let x be a commutator then $x = a^{-1}b^{-1}ab$. We have then the following,

$$\begin{aligned}\varphi(x) &= \varphi(a^{-1}b^{-1}ab)\\ &= \varphi(a^{-1})\varphi(b^{-1})\varphi(a)\varphi(b)\\ &= \varphi(a)^{-1}\varphi(b)^{-1}\varphi(a)\varphi(b)\end{aligned}$$

that $\varphi(x)$ is indeed a commutator as well. Now let $g \in G'$ then we know for some commutators x_i for $1 \leqslant i \leqslant n$ that $g = x_1 \ldots x_n$. Now with the following,

$$\begin{aligned}\varphi(g) &= \varphi(x_1 \ldots x_n)\\ &= \varphi(x_1)\ldots\varphi(x_n)\end{aligned}$$

we see that $\varphi(g)$ must be in G' because every $\varphi(x_i)$ is in G', and G' is a subgroup. Therefore $\varphi(G') = G'$ meaning G' is characteristic.

$\square$

Problem 9.1 (a) Determine the unit group of the ring $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$, the ring of Gaussian integers.
(b) Show that the ring $\mathbb{Z}[\sqrt{2}] = \left\{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\right\}$ has infintely many units and find all units of finite order.

(a) *Proof.* If $x \in \mathbb{Z}[i]$ is a unit that means there exists $y \in \mathbb{Z}[i]$ such that $xy = 1$. We know that the Guassian integers are a subring of $\mathbb{C}$. Meaning we have consider the norm squared of these values. Where the norm of is defined as the following,

$$N(a + bi) = \sqrt{(a + bi)(a - bi)} = \sqrt{a^2 + b^2}$$

When two values are equal we know their norm will be equal. Therfore we have, $N(xy)^2 = N(1)^2 = 1$. Recall though both x and y are of the form $a + bi$ and $c + di$ respectively. This gives us the following,

$$\begin{aligned} N(xy)^2 &= 1 && \text{Norm is multiplicative} \\ N(x)^2 N(y)^2 &= 1 \\ N(a + bi)^2 N(c + di)^2 &= 1 \\ (a^2 + b^2)(c^2 + d^2) &= 1 \end{aligned}$$

Thus $(a^2 + b^2)$ and $(c^2 + d^2)$ must both be non negative integers and must be equal to 1. In other words if $a^2 + b^2 = 1$ then a^2 and b^2 must be less than or equal to 1. This gvies us the following solutions as $\pm 1 + 0i$ and $0 \pm 1i$. Thus the only units in the Guassian integers are

$$\{1, i, -1, -i\}$$

□

(b) *Proof.* If $x \in \mathbb{Z}[\sqrt{2}]$ is a unit that means there exists a $y \in \mathbb{Z}[\sqrt{2}]$ such that $xy = 1$. From 111b we know the norm of this ring is simply,

$$\begin{aligned} N : \mathbb{Z}[\sqrt{2}] &\to \mathbb{N} \\ a + b\sqrt{2} &\mapsto a^2 - 2b^2 \end{aligned}$$

Next we will use the fact that $N(x) = 1$ if and only if x is unit. The reasoning for this is simple in that if x is a unit there there exists a y such that $xy = 1$. Taking the norm then implies that $N(xy) = N(x)N(y) = 1$ which is only possible if $N(x) = 1$ and $N(y)$. Then if $N(x) = 1$ that means $x \cdot \bar{x} = 1$ (where $\bar{x}$ is the conjugate of x). Using this to our advantage we can see that $3 + 2\sqrt{2}$ must be a unit because $9 - 2 \cdot 4 = 1$. Let us consider the power of $(3 + 2\sqrt{2})$ now though. We see through the following,

$$\begin{aligned} N((3 + 2\sqrt{2})^n) &=? \\ N(\underbrace{(3 + 2\sqrt{2}) \cdot \dots \cdot (3 + 2\sqrt{(2)})}_{n \text{ times}}) &= && \text{Norm is multiplicative} \\ \underbrace{N(3 + 2\sqrt{2}) \cdot \dots \cdot N(3 + 2\sqrt{2})}_{n \text{ times}} &= \end{aligned}$$

$$1 \cdot \cdots \cdot 1 = 1$$

Thus we have all powers of $(3+2\sqrt{2})$ are units. From here we must take a look at the order of $(3+2\sqrt{2})$. It is obvious it does not have a finite order since $1 < (3+2\sqrt{2})$ and when taking powers of $(3+2\sqrt{2})$ we are simply multiplying and adding positive numbers which means it will only be increasing and can never decrease to be 1. As a result the order of $(3+2\sqrt{2})$ is infinite, and we showed every power of it is a unit, meaning there is an infinite number of units in $\mathbb{Z}[\sqrt{2}]$. Therefore the only units of finite order must be -1 and 1. $\square$