Homework 8

Due: Friday, May 28th

Mandatory:

(1) If $n, m \in \mathbb{N}$, let us say that $n \leq m$ if and only if n divides m. Show that the relation $\leq$ is a partial order on $\mathbb{N}$. Is the set $\{2, 3, 4, 5\}$ a chain?

(2) Let R be a commutative ring with $1 \neq 0$. The *Jacobson radical* of R, denoted $J(R)$, is defined as the intersection of all maximal ideals of R. In symbols:

$$J(R) := \bigcap_{M=\text{maximal ideal of } R} M.$$

Prove that $J(R)$ is a proper ideal of R, and determine $J(\mathbb{Z})$.

(3) Recall that the *Gaussian integers* $\mathbb{Z}[i]$ is the following subring of $\mathbb{C}$:

$$\mathbb{Z}[i] := \{a + bi : a, b \in \mathbb{Z}\}.$$

Use the fact that $(1 + i)(1 - i) = 2$ to show that the ideal of $\mathbb{Z}[i]$ generated by 2 is not a prime ideal.

(4) Let A and B be commutative rings both with 1 and let $f : A \to B$ be a unital ring homomorphism. If P is a prime ideal of B, prove that

$$f^{-1}(P) = \{a \in A : f(a) \in P\}$$

is a prime ideal of A.

Optional:

(5) Give an example to show that the result of problem (4) is not true if "prime ideal" is replaced with "maximal ideal."

(6) Let $p \in \mathbb{N}$ be a prime. Consider the following subset of the rationals:

$$\mathbb{Z}_{(p)} := \{\frac{a}{b} \in \mathbb{Q} : \gcd(a, b) = 1 \text{ and } p \nmid b\}.$$

In words, $\mathbb{Z}_{(p)}$ is the set of rational numbers $x \in \mathbb{Q}$ such that when x is written in lowest terms, say $x = \frac{a}{b}$, then p does not divide b.

(a) Show that $\mathbb{Z}_{(p)}$ is a subring of $\mathbb{Q}$. (The ring $\mathbb{Z}_{(p)}$ is called the *localization of* $\mathbb{Z}$ *at* (p).)

(b) Use the usual bar notation for the elements of $\mathbb{Z}/p\mathbb{Z}$; that is, write $\overline{k} = k + p\mathbb{Z}$ for each $k \in \mathbb{Z}$. Define a map

$$\begin{aligned} \phi : \mathbb{Z}_{(p)} &\to \mathbb{Z}/p\mathbb{Z} \\ x &\mapsto \overline{a} \cdot (\overline{b})^{-1} \text{ if } x = \frac{a}{b}. \end{aligned}$$

Prove that ϕ is a well-defined, unital, and surjective ring homomorphism. Deduce that

$$p\mathbb{Z}_{(p)} = \{\frac{a}{b} \in \mathbb{Q} : \gcd(a, b) = 1,\ p \nmid b, \text{ and } p \mid a\}$$

is a maximal ideal of $\mathbb{Z}_{(p)}$.

(c) Show that $p\mathbb{Z}_{(p)}$ is the *unique* maximal ideal of $\mathbb{Z}_{(p)}$.