Name: Kevin Guillen
Student ID: 1747199

117 - SS2 - HW3 - August 25th, 2021

[1] Let V and W be finite-dimensional $\mathbb{F}$-vector spaces.

(a) Show that $\dim(\text{Hom}(V, W)) = \dim(V)\dim(W)$ by finding an explicit basis.

Proof. Since V and W are both finite, let the dimension of V and the dimension of W be denoted by n and m respectively. By definition that means the basis for V and W are the following.

$$\mathcal{B}_{\mathcal{V}} = \{v_1, v_2, \dots, v_n\}$$

$$\mathcal{B}_{\mathcal{W}} = \{w_1, w_2, \dots, w_m\}$$

Now let us define the linear maps $\pi_{ij} : V \to W$ for $1 \leq i \leq n$ and $1 \leq j \leq m$ by the following,

$$\pi_{ij}(v_p) = \begin{cases} w_j & p = i \\ 0 & p \neq i \end{cases}$$

These will serve as a basis for $\text{Hom}(V, W)$, and we will prove it with the following. Let α_{ij} be a scalar and assume we have,

$$\pi = \sum_{i,j}^{n,m} \alpha_{ij}\pi_{ij} = 0$$

This would mean for $\pi(v_i)$ and $i \in \{1, 2, \dots, n\}$,

$$\pi(v_i) = \sum_{j}^{m} \alpha_{ij}w_j = 0$$

Recall though that the set of vector w_j for $1 \leq j \leq m$ are linearly independent, and thus our maps π_{ij} are also linearly independent.
Now take any function π from $\text{Hom}(V, W)$. We can define it its values when inputting the basis of V as $\pi(v_i) \in W$. Meaning when $i \in 1, 2, \dots, n$ and α_{ij} as a scalar, we can express $\pi(v_i)$ as,

$$\pi(v_i) = \sum_{j}^{m} \alpha_{ij}w_j$$

Which means,

$$\pi = \sum_{i,j}^{n,m} \alpha_{ij}\pi_{ij}$$

because the linear functions agree on basis vectors. This means for $1 \leq i \leq n$ and $1 \leq j \leq m$,

$$\text{Hom}(V, W) = \text{span}(\{\pi_{ij}\})$$

This is the proof since we know there are $\dim(V)\dim(W)$ of these functions.

$\square$

(b) Show that $\text{Hom}(V, V) \cong V \otimes V^*$.

[2] Let $T : \mathbb{F}^3 \to \mathbb{F}^3$ be the linear transformation with matrix:

$$\mathbf{A} = \begin{pmatrix} 1 & 4 & 3 \\ 3 & 4 & 1 \\ 4 & 4 & 4 \end{pmatrix}$$

Compute the standard matrix $[\Lambda^2 T]$ with respect to the standard basis $\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\}$ of $\Lambda^2(\mathbb{F}^3)$.

Solution: First let's get the computations for $T(e_i)$ out of the way,

$$T(e_1) = \begin{pmatrix} 1 & 4 & 3 \\ 3 & 4 & 1 \\ 4 & 4 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$$

$$T(e_2) = \begin{pmatrix} 1 & 4 & 3 \\ 3 & 4 & 1 \\ 4 & 4 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix}$$

$$T(e_3) = \begin{pmatrix} 1 & 4 & 3 \\ 3 & 4 & 1 \\ 4 & 4 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$$

Then we solving for the standard matrix with respect to the standard basis we get,

$$\begin{aligned}
T(e_1 \wedge e_2) &= T(e_1) \wedge T(e_2) \\
&= (1e_1 + 3e_2 + 4e_3) \wedge (4e_1 + 4e_2 + 4e_3) \\
&= (4 - 12) e_1 \wedge e_2 + (4 - 16) e_1 \wedge e_3 + (12 - 16) e_2 \wedge e_3 \\
&= -8(e_1 \wedge e_2) - 12(e_1 \wedge e_3) - 4(e_2 \wedge e_3) \\
T(e_1 \wedge e_3) &= T(e_1) \wedge T(e_3) \\
&= (1e_1 + 3e_2 + 4e_3) \wedge (3e_1 + 1e_2 + 4e_3) \\
&= (1 - 9) e_1 \wedge e_2 + (4 - 12) e_1 \wedge e_3 + (12 - 4) e_2 \wedge e_3 \\
&= -8(e_1 \wedge e_2) - 8(e_1 \wedge e_3) + 8(e_2 \wedge e_3) \\
T(e_2 \wedge e_3) &= T(e_2) \wedge T(e_3) \\
&= (4e_1 + 4e_2 + 4e_3) \wedge (3e_1 + 1e_2 + 4e_3) \\
&= (4 - 12) e_1 \wedge e_2 + (16 - 12) e_1 \wedge e_3 + (16 - 4) e_2 \wedge e_3 \\
&= -8(e_1 \wedge e_2) + 4(e_1 \wedge e_3) + 12(e_2 \wedge e_3)
\end{aligned}$$

Now like in example 12.4 we can read off our coefficients to get the standard matrix and we get the following,

$$\begin{pmatrix} -8 & -8 & -8 \\ -12 & -8 & 4 \\ -4 & 8 & 12 \end{pmatrix}$$

[3] Let V be a $\mathbb{F}$-vector space. Show that if $T, S \in \text{End}(V)$ such that $ST - TS$ commutes with S, then for every $k \in \mathbb{N}$:

$$S^kT - TS^k = kS^{k-1}(ST - TS)$$

[4] Let V be a $\mathbb{F}$-vector space. Show that if $T \in \text{End}(V)$ such that $T^2 - T + I = 0$, then T is invertible.

Proof.

$$\begin{aligned} T^2 - T + I &= 0 \\ T^2 &= T - I && I = TT^{-1} \\ T^2 &= T - TT^{-1} \\ T^2 &= T(I - T^{-1}) \\ T &= (I - T^{-1}) \end{aligned}$$

Therefore T is invertible. □

[5] Let V be a $\mathbb{F}$-vector space. If $S, T \in \text{End}(V)$ such that $ST = 0$, does it follow that $TS = 0$?

Proof. Consider the vector space $\mathbb{R}^2$ over $\mathbb{R}$. We have in $\text{End}(V)$ the following,

$$S = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

$$T = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

We see though that,

$$ST = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

but,

$$TS = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix} \neq ST$$

So, no. If we have two linear transformation S and T such that $ST = 0$ it does not follow that $TS = 0$ □

[6] Let $\mathbb{P}_n[x]$ denote the $\mathbb{F}$-vector space of all polynomials with degree less than or equal to n whose coefficients come from $\mathbb{F}$. Suppose that $L \in \mathrm{End}(V)$ such that $Lp(x) = p(x+1)$ for every $p(x) \in \mathbb{P}_n[x]$. Prove that if D is the differentiation operator defined through the power rule, then:

$$I + \frac{D}{1!} + \frac{D^2}{2!} + \cdots + \frac{D^{n-1}}{(n-1)!} = L$$

[7] Let V be a $\mathbb{F}$-vector space with subspaces U and W. Prove that if $T \in \mathrm{End}(V)$ such that U and W are invariant under T, then the subspace spanned by U and W is invariant under T.

Proof. Let the vector space Z represent the subspace spanned by $U + W$.

$$Z = \mathrm{span}(\{U + W\})$$

Meaning any vector $z \in Z$ is of the form $z = u + w$ where u and w are vectors of U and W respectively. This gives us,

$$T(z) = T(u + w) = T(u) + T(w) \subseteq U + W$$

$\square$

[8] Let V be a $\mathbb{F}$-vector space with $E, F : V \to V$ projections.

(a) Prove that $\mathrm{im}(E) = \mathrm{im}(F)$ if and only if $EF = F$ and $FE = E$.

Proof. For the forward direction we will assume $\mathrm{im}(E) = \mathrm{im}(F)$. This means for any vector $x \in V$ there exists some vector $y \in V$ such that $E(x) = F(y)$. Now consider,

$$\begin{aligned} EF(y) &= EE(x) \\ &= E(E(x)) && \text{Recall though all projections are idempotent} \\ &= E(x) \\ &= F(y) \\ EF &= F \end{aligned}$$

Now consider,

$$\begin{aligned} FE(x) &= FF(y) \\ &= F(F(y)) && \text{Recall though all projections are idempotent} \\ &= F(y) \\ &= E(x) \\ FE &= E \end{aligned}$$

as desired. So now we have if $\mathrm{im}(E) = \mathrm{im}(F)$ implies $EF = F$ and $FE = E$

Now for the reverse direction, we will assume $EF = F$ and $FE = E$. This means for any vector $x \in V$ we have, $EF(x) = F(x)$ and $FE(x) = E(x)$.
We know that for vector x, there exists some vector $y \in \text{im}(E)$ such that $E(x) = y$. Now consider the following,

$$\begin{aligned} E(x) &= y && \text{Recall our assumption } FE = E \\ FE(x) &= y \\ F(E(x)) &= y \end{aligned}$$

Recall though y was in the image of E and now we can see that it is also in the image of F. Therefore $\text{im}(E) \subseteq \text{im}(F)$.
We also know though that for a vector x, there exists some vector $y \in \text{im}(F)$ such that $F(x) = y$. Now consider the following,

$$\begin{aligned} F(x) &= y && \text{Recall our assupmtion } EF = F \\ EF(x) &= y \\ E(F(x)) &= y \end{aligned}$$

y started in the image of F, but we can see that it is also in the image of E. This gives us that $\text{im}(F) \subseteq \text{im}(E)$
Putting this all together we have,

$$\text{im}(E) \subseteq \text{im}(F) \text{ and } \text{im}(F) \subseteq \text{im}(E) \to \text{im}(F) = \text{im}(E)$$

as desired. □

(b) Prove that $\ker(E) = \ker(F)$ if and only $EF = E$ and $FE = F$

Proof. First we will go in the forward direction and assume $\ker(E) = \ker(F)$. This means whenever a vector $x \in V$ satisfies $E(x) = 0$ then it must also satisfy $F(x) = 0$.
We want to show that $EF = E$ and $FE = F$. First let's work with $EF = E$. If this equality were to hold that would mean for any vector x in V we would have,

$$(E - EF(x) = 0)$$

So let's assume that it doesn't hold, that would mean there exists some vector y in V such that,

$$(E - EF)(y) \neq 0$$

We see though such a y would imply this about the kernel of E,

$$\begin{aligned} (E - EF)(y) &\neq 0 \\ (E(y) - EF(y)) &\neq 0 \\ E(y - F(y)) &\neq 0 \end{aligned}$$

It's that since $E(y - F(y)) \neq 0$ that means it is NOT in the kernel of E. Recall though our assumption was that $\ker(E) = \ker(F)$, that means this is also not in the kernel of F. But,

$$
\begin{aligned}
F(y - F(y)) &\neq 0 \\
F(y) - F(F(y)) &\neq 0 \qquad\qquad \text{Recall though every projection is idempotent} \\
F(y) - F(y) &\neq 0 \\
F(y) &\neq F(y)
\end{aligned}
$$

Which is a contradiction. Therefore if the kernel of the projection E and F are the same then $EF = E$.

We can take a similar look at $FE = F$ and see if this weren't true there would exist some vector y in V such that,

$$(F - FE)(y) \neq 0$$

Assuming this vector did indeed exist,

$$
\begin{aligned}
(F - FE)(y) &\neq 0 \\
F(y) - FE(y) &\neq 0 \\
F(y - E(y)) &\neq 0
\end{aligned}
$$

it would mean $(y - E(y))$ is not in the kernel of E. We see though,

$$
\begin{aligned}
E(y - E(y)) &\neq 0 \\
E(y) - E(E(y)) &\neq 0 \qquad\qquad \text{Projections are idempotent} \\
E(y) - E(y) &\neq 0 \\
E(y) &\neq E(y)
\end{aligned}
$$

which is again a contradiction. Therefore if the kernel of E and F are equal then $EF = E$ and $FE = F$

Putting all this together: $\ker(F) = \ker(E)$ if and only if $EF = E$ and $FE = E$

$\square$

[9] (a) Prove that if E is a projection on a finite-dimensional $\mathbb{F}$-vector space, then there exists a basis $\mathcal{B}$ such that the matrix representative $[E]_{\mathcal{B}}$ has the following special form: $e_{ij} = 0$ if $i \neq j$ and $e_{ii} = 0$ or 1 for all i and j.

(b) An *involution* is a linear transformation U on a $\mathbb{F}$-vector space V such that $U^2 = I$. Show that if $\text{char}(\mathbb{F}) \neq 2$, then the equation $U = 2E - I$ establishes a one-to-one correspondence between all projections E and all involutions U.

(c) Prove that the only eigenvalues of a projection are 0 and 1. Furthermore, prove that the only eigenvalues of an involution are -1 and 1. (This does not require the vector space to be finite-dimensional.)

[10] Find all the (complex) eigenvalues and eigenvectors of the following matrices over $\mathbb{C}$:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 1 & 1 \\ 0 & i \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{E} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Solution:

A) Solving for the eigenvalues and eigenvectors of A, $\det(A - \lambda I) = \lambda^2$. Solving for $\lambda^2 = 0$ we get $\lambda = 0$.

Now to get the corresponding eigenvector we get,

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Thus the eigenvalue for A is 0 and its corresponding eigenvector is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

B) Solving for the eigenvalues and eigenvectors of B, $\det(B - \lambda I) = \lambda^2 - \lambda - \lambda i + i$. Solving for λ we get, $\lambda = i, 1$.

Now lets obtain the corresponding eigenvector for $\lambda_1 = i$,

$$\begin{pmatrix} 1 - i & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Now for $\lambda_2 = 1$,

$$\begin{pmatrix} 0 & 0 \\ 0 & i - 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Thus we have for the eigenvalues $\lambda_1 = i$ and $\lambda_2 = 1$ the corresponding eigenvectors are $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ respectively.

C) For C we have $\det(C - \lambda I) = i - \lambda - \lambda i + \lambda^2$. Solving for λ we get the following eigenvalues: 1, i.

Let's obtain the corresponding eigenvector for $\lambda_1 = i$,

$$\begin{pmatrix} 1 - i & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We get for the first equation $(1-i)x + y = 0$. We see if we set $x = (-1-i)$ we get, $-1 + i - i - 1 + y = -2 + y = 0$. Therefore $y = 2$. Thus the corresponding eigenvector is $\begin{pmatrix} -1 - i \\ 2 \end{pmatrix}$ Now to obtain the corresponding eigenvector for $\lambda_2 = 1$,

$$\begin{pmatrix} 0 & 1 \\ 0 & i - 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Thus our eigenvector for eigenvalues $\lambda_1 = i$ and $\lambda_2 = 1$ are $\begin{pmatrix} -1 - i \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ respectively.

D) For D we have, $\det(D - \lambda I) = -\lambda^3 + 3\lambda^2 = -\lambda^2(\lambda - 3)$. Solving for λ we get $\lambda = 0, 3$.

Now solving for the corresponding eigenvector for $\lambda_1 = 3$, (Some steps I'm skipping over since it would be a lot of matrices to type out)

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \to \cdots \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This gives us that $x_1 = x_2$ by the first row, and $x_2 = x_3$ by the second row. Thus our corresponding eigenvector is, $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Now for the corresponding eigenvector for $\lambda_2 = 0$,

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We see the corresponding eigenvectors to be $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Thus for the eigenvalue $\lambda_1 = 3$ the corresponding eigenvector is $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and for $\lambda_2 = 0$ the corresponding eigenvectors are $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

D) For D we have $\det(D - \lambda I) = -\lambda^3 + 3\lambda^2 - 3\lambda + 1 = -(\lambda - 1)^3$. Solving for λ we get $\lambda = 1$.

Now solving for the corresponding eigenvector we get,

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Thus we see the corresponding eigenvector for $\lambda = 1$ is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$