Homework 3

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MATH 201 — Algebra II — Winter 2022

Problem 1 Let $M = \mathbb{Z}^2$ and let N be the $\mathbb{Z}$-submodule generated by the 2 elements

$$\begin{bmatrix} 120 \\ 240 \end{bmatrix} \text{ and } \begin{bmatrix} 360 \\ -300 \end{bmatrix}$$

In (a) and (b) we use the notation and terminology adopted in class on the 3rd of February. The terminology in (c) and (d) will be defined on the 8th of February.

(a) Find an element $\nu \in \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ such that $\nu(N)$ is maximal in $\Sigma_{M,N}$: prove that it is indeed maximal.

(b) Find an element $y_1 \in M$ such that $M = \mathbb{Z}y_1 \bigoplus \ker(\nu)$ and $N = \mathbb{Z}a_\nu y_1 \bigoplus (\ker(\nu) \cap N)$.

(c) Find the invariant factors of the quotient $\mathbb{Z}$-module M/N.

(d) Find the elementary divisors of the quotient $\mathbb{Z}$-module M/N

(a) *Proof.* Any $\nu_A \in \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ is of the form,

$$\nu_A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = ax + by = \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

where $A = \begin{bmatrix} a & b \end{bmatrix}$ Now let's consider the following,

$$\nu_A\left(\begin{bmatrix} 120 \\ 240 \end{bmatrix}\right) = 120a + 240b = 60(2a + 4b)$$
$$\nu_A\left(\begin{bmatrix} 360 \\ -300 \end{bmatrix}\right) = 360a - 300b = 60(6a - 5b)$$

Let $A = \begin{bmatrix} 1 & 1 \end{bmatrix}$. We see then that $\nu_A(N) = (60)$ and is indeed maximal. This is because there is no element $y \in N$ and $\nu_A \in \mathrm{Hom}(M, \mathbb{Z})$ that can map y to a factor of 60 that is not 60. This is because 60 is the GCD of all 4 numbers in the given matrices. Meaning $a_1 = a_\nu = 60$ $\square$

(b) *Proof.* First we want a $y \in N$ that under ν_A maps to the generator of $\nu_A(N)$. In other words $\nu_A(y) = 60$. So we can simply let $y = \begin{bmatrix} 360 \\ -300 \end{bmatrix}$ we see this holds since,

$$\nu_A\left(\begin{bmatrix} 360 \\ -300 \end{bmatrix}\right) = 360 - 300 = 60.$$

Now we can let our y_1 be the divisor of y such that $a_\nu y_1 = y$,

$$60y_1 = \begin{bmatrix} 360 \\ -300 \end{bmatrix} \to y_1 = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$$

$$\ker(\nu_A) = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{Z}^2 \mid x + y = 0 \right\} = \mathbb{Z} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

We also have $\ker(\nu_A) \cap N$ to be,

$$\ker(\nu_A) \cap N = \left\{ a \begin{bmatrix} 120 \\ 240 \end{bmatrix} + b \begin{bmatrix} 360 \\ -300 \end{bmatrix} \mid a, b \in \mathbb{Z} \right\}$$

Writing a in terms of b we get $a = 5.5b$ so,

$$\ker(\nu_A) \cap N = \left\{ (5.5b) \begin{bmatrix} 120 \\ 240 \end{bmatrix} + b \begin{bmatrix} 360 \\ -300 \end{bmatrix} \mid b \in \mathbb{Z} \right\}$$
$$\ker(\nu_A) \cap N = \left\{ \begin{bmatrix} 1020b \\ 1020b \end{bmatrix} \mid b \in \mathbb{Z} \right\}$$

Now let $a_2 = gcd(1020, 1020) = 1020$ and $y_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ we have then that,

$$M = \mathbb{Z} \begin{bmatrix} 360 \\ -300 \end{bmatrix} \bigoplus \mathbb{Z} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$N = \mathbb{Z}60 \begin{bmatrix} 360 \\ -300 \end{bmatrix} \bigoplus \mathbb{Z}1020 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\square$

(c)

(d)