Week 3 Problems

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MATH 101 — Problem Solving — Fall 2021

Problem 10/8 OC (30.) Chose any $(n+1)$ element subset of $\{1, 2, \ldots, 2n\}$. Show that this subset contains two elements which are relatively prime.

Proof. Let S denote the set $\{1, 2, \ldots, 2n\}$. To prove this we will use the pigeon hole principle and the fact that two neighboring numbers are relatively prime. Our "pigeonholes" in this case will be a list of n+1 numbers in ascending order. The max amount of numbers that can be chosen from $\{1, 2, \ldots, 2n\}$ such that no numbers are relatively prime would be n, because consecutive numbers in our list will have a difference of at least 2. Since we are choosing $n+1$ numbers though 1 number in our list will have to have a difference of 1 with the number next to it in the list. As a result these two numbers will be relatively prime. □

Problem 10/11 IC (44.) The numbers $1, 2, \ldots, 50$ are written on the blackboard. Then two numbers a and b are chosen and replaced by the single number $|a - b|$. After 49 operations a single number is left. Prove that it is odd.

Proof. If we have the numbers $1, 2, \ldots, 50$ that means half of them are even and half are odd. In other words we have 25 even numbers and 25 odd numbers. We know after 49 operations we will have a single number left. To determine if it is odd or even let's look at the 3 scenarios when taking the differences of even and odd numbers.

$$\begin{aligned} \text{Two even numbers: } 2k - 2l &= 2(k - l) \\ \text{Odd and even numbers: } 2k + 1 - 2l &= 2(k - l) + 1 \\ \text{Two odd numbers: } 2k + 1 - (2l + 1) &= 2(k - l) \end{aligned}$$

Since in the set of number $\{1, 2, 3, \ldots, 50\}$. Half of these numbers are odd, more specifically there are an odd number of odd numbers. This means our last number will have to be odd because the only way to remove odd numbers when summing is to add two of them together, meaning we would have needed an even amount. □

Problem 10/11 IC (46.) Seven quarters are initially all heads up. On a single move you can choose any four and turn them over (change heads to tails and tails to heads). Is it possible to obtain all tails up after a sequence of such moves?

Solution. This is impossible, we will show this by showing the number of heads will always be odd and the number of tails will always be even. This is important since the state of the coins before the "winning" move would have to be where we have 4 heads and 3 tails, since we'd simply flip all 4 heads to tails. Consider this though, after the first move we will have 3H and 4T.

We have (2k + 1) heads and (2n) tails. If we flip 1 heads and 3 tails this changes the coin state by adding 2 heads and removing 2 tails, (2(k+1) + 1) heads and (2(n-1)) tails.

If we flip 2 heads and 2 tails this does nothing.

If we flip 3 heads to 1 tails this changes the coin state by adding 2 tails and removing 2 heads, (2(k-1) + 1) heads and (2(n + 1)).

If we can flip 4 tails to heads that would give us (2(k + 2) + 1) heads which is still odd.

If we can flip 4 heads to tails that would give us (2(k-2) + 1) heads which is also still odd.

Since we can't obtain a state of having an even number of heads, we can never have a sequence that results in all tails up. □

Problem 10/13 IC (50.) Every room in a house has an even number of doors. Prove that there are an even number of entrance doors to the house.

Solution. Let each room in the house be a vertex and outside be a vertex as well. All we have to show now is that this graph can't have exactly one vertex of odd degree. By the handshaking lemma there does not exist any such graph. So we have an even number of vertices which have odd degree. So connecting each adjacent door by an edge each entrance door has odd degree, therefore by handshaking lemma we have an even number of entrance doors. □