# wavelet_entropy_example

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## 0.1 Wavelet entropy in Julia: an introduction

*Note: this is a draft. I guarantee nothing*

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[1]:
```
using Statistics, Entropies, Plots, Wavelets, LaTeXStrings
```

We'll start by generating a time series $f(t), t \in [1, 2, \ldots, N]$, which, for plotting convenience, we limit to $N = 300$ points.

[2]:
```
using Entropies, Plots
N, a = 300, 10
t = LinRange(0, 2*a* , N)
f = sin.(rand(1:15, N) ./ rand(1:10, N));
```

## 0.2 Maximal overlap discrete wavelet transform

A discrete wavelet transform is computed by sliding a particular type of wave filter across a time series signal $f(t)$, and seeing how well the filter matches the original signal at a particular point in time. It is computed using $W(\tau, s) = \dfrac{1}{\sqrt{s}} \sum_{t=0}^{N-1} f(t)\phi\left(\dfrac{t-\tau}{b}\right)$, where $t$ is the time index, $\tau$ is a time-shift parameter (changing $\tau$ will shift the wavelet relative to the original signal), and $s$ governs the width of the wavelet ($s$ large yields wide/expanded wavelets, which match lower frequencies better; $s$ small yields narrow/shrunken wavelets which match higher frequencies better). The value $W(\tau, s)$ indicates how well the signal, at a particular time $\tau$, matches the wavelet whose width is $s$.

If $\tau = k2^{-j}$ and $s = 2^{-j}$, we call the scales *dyadic scales*. Using dyadic scales requires relatively few computations to calculate wavelet coefficients $W(\tau, s)$. In constrast, if we wanted a continuous wavelet transform, computations would be more expensive, because we would need to compute the transform for a continous set of scaler and time-shifts (or, rather, with *small enough* steps, when using a computer).

To illustrate how wavelet entropy is computed, we'll first compute the detail coefficients for a maximal overlap discrete wavelet transform (MODWT), which assumes circular boundary conditions, so that there are $J$ coefficients for each time $t$. We will obtain a matrix $W_{t,s}$ with $N$ rows and $n_{scales}$ columns. Each row in $W$ contains the wavelet coefficients at time $t$ for scales $J = 1, 2, \ldots, n_{scales}$. There are `n_scales = Wavelets.WT.maxmodwttransformlevels(f) = floor(Int, log2(length(f))) = floor(Int, log2(N))` possible scales for our signal $f(t)$, and the number of scales will vary with signal length.

```
[3]: Wavelets.WT.maxmodwttransformlevels(f), floor(Int, log2(N))
```

```
[3]: (8, 8)
```

There are many possible choices of the wavelet function $\phi(\tau, s)$. Here, we'll pick the Haar wavelet, because it has the property of additive decomposition (Percival & Walden, 2000, p. 205), which means that we can reconstruct the original time series from the wavelet coefficients.

```
[4]: wl = Wavelets.WT.Haar()
     W = Entropies.get_modwt(f, wl)
```

```
[4]: 300×8 Matrix{Float64}:
  -0.805327    -0.416259    -0.0829205   …  -0.131756     0.0031218     0.482128
   0.964449    -0.402066     0.0295453      -0.142343     0.0153024     0.483365
  -0.352182     0.385695    -0.0777045      -0.157074     0.0275814     0.48007
   0.0919527    0.176019    -0.174326       -0.151768     0.0356763     0.477241
   0.12183     -0.0232232    0.165954       -0.133985     0.0282119     0.481788
   0.136161     0.235887     0.0929296   …  -0.0876612    0.0235376     0.488012
  -0.269068     0.0625423    0.200895       -0.09568      0.0191324     0.483977
   0.251183    -0.0753957    0.286199       -0.0404836    0.0189502     0.488918
  -0.187271     0.0230138    0.0624376      -0.0287558    0.0124681     0.489077
  -0.168847    -0.146102    -0.0305034      -0.0338128    0.00191506    0.489288
   0.330947    -0.0970086    0.00578065  …   0.0194427    0.000195283   0.489818
   0.0373443    0.265196    -0.0512026       0.0309431   -0.0114105     0.501971
  -0.409045    -0.00170483  -0.0863541       0.00964229  -0.0120787     0.499147

  -0.111252     0.138135     0.202075       -0.0542794    0.020785      0.490665
   0.0620574   -0.086148     0.375436       -0.0792524    0.0411169     0.495306
   0.0652335    0.039048     0.269899    …  -0.0782737    0.0422415     0.49516
  -0.322372    -0.0649236   -0.0187672      -0.0432454    0.0333716     0.495994
  -0.643059    -0.611285    -0.197527       -0.0734961    0.0138345     0.4874
   0.849551    -0.37947     -0.297733       -0.0706565    0.0187296     0.490627
   0.0506471    0.553344    -0.315088       -0.072656     0.0112537     0.495648
  -0.365294     0.292775    -0.265544    …  -0.0829076    0.0126834     0.489233
   0.0151187   -0.332411     0.0814964      -0.0968914    0.00593303    0.493076
   0.139524    -0.0977666    0.0541569      -0.104941     0.00637657    0.488547
   0.288664     0.291415     0.0899683      -0.108047     0.00475758    0.494788
  -0.157928     0.279462     0.188352       -0.0952414    0.00983415    0.493797
```

Because of the property of additive decomposition, we can write $\mathbf{x} = \sum_{i=1}^{n_{scales}} \mathbf{W_i}$ (here $\mathbf{x}$ and $\mathbf{W_i}$ are column vectors, and $\mathbf{W_i}$ contain the coefficients at scale $i$). A single time series value at time $k$ can thus be reconstructed $\sum_{i=1}^{n_{scales}} W_{k,i}$. Let's verify that our original signal and wavelet-reconstructed signal are equal (within some tolerance, due to imprecisions introduced during calculations):

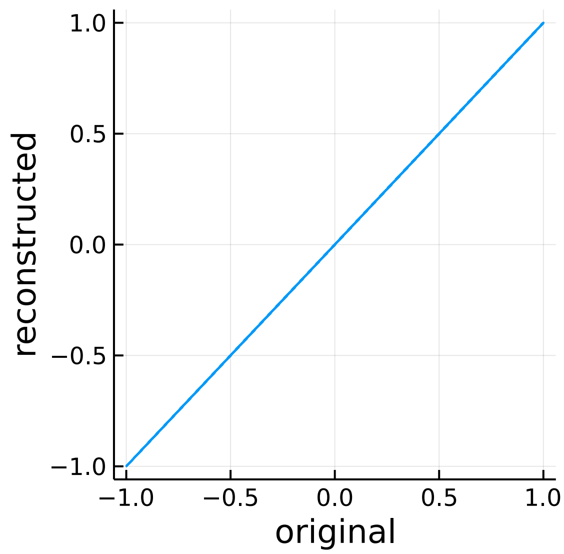
```
[5]: reconstructed = [sum(W[t, :]) for t = 1:N]

     all(f .- reconstructed .< 1e-10)
```

[5]: true

[6]:
```julia
plot(f, reconstructed, size = (300, 300), label = "", xlabel = "original",␣
↪ylabel = "reconstructed")
```

[6]:




### 0.3 Relation between original time series and coefficients

The detail coefficients at successive levels are the parts of the signals left by passing the original signal through successive low-pass filters, such that the coefficients at level $j$ contains the part of the time series whose frequencies did not pass further down the cascade of low-pass filters. The coefficients at dyadic scale $j$ corresponds to a frequency range $(\frac{1}{2^{j+1}}, \frac{1}{2^j}]$. Let's visualize this:

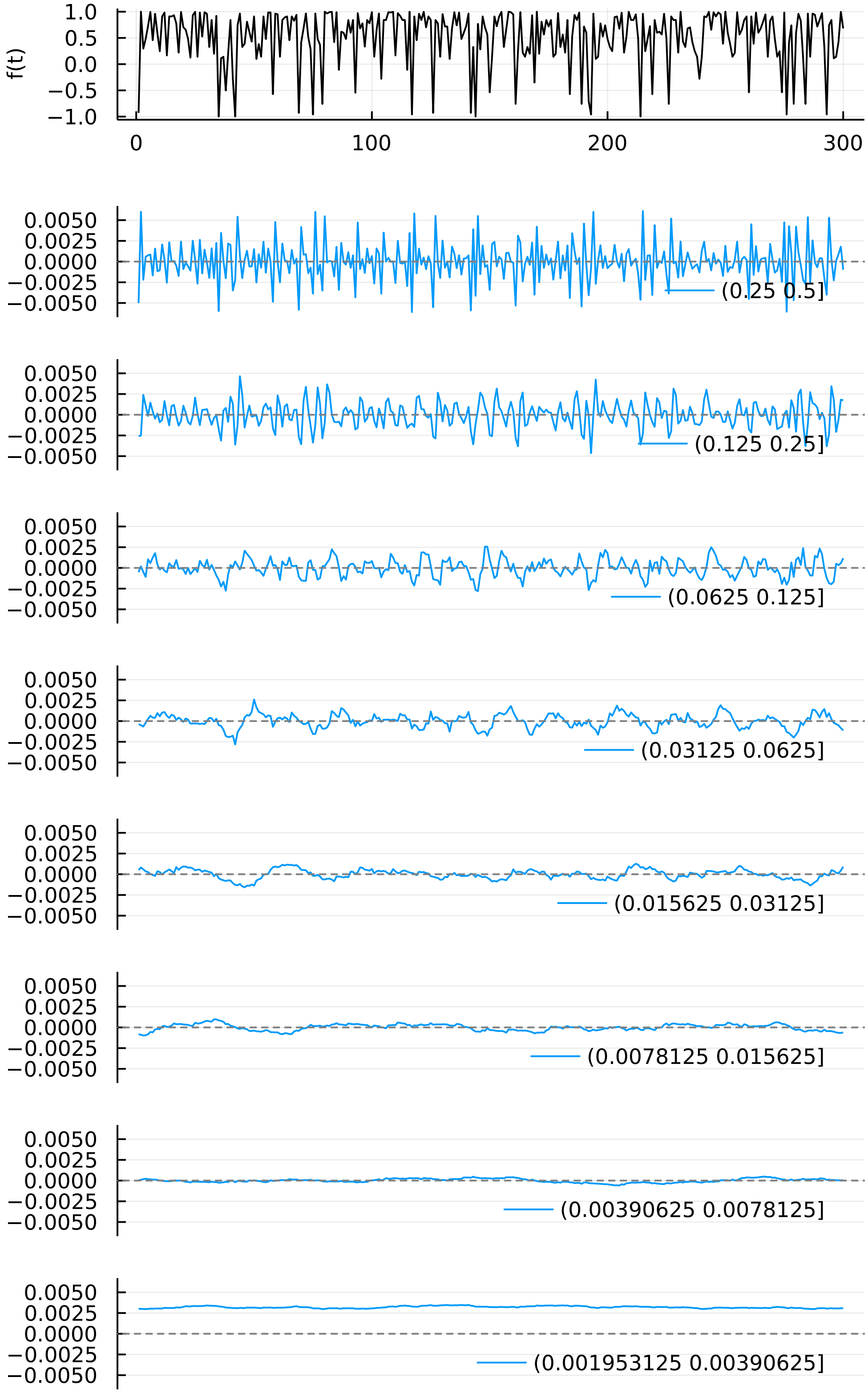
[7]:
```julia
# Normalize coefficients to total energy to have the same axis scales, see below
Etot = sum(W .^ 2)
emax = maximum(W ./ Etot)*1.1

# Plot the coefficients for each dyadic scale j, corresponding to frequency␣
↪range (1/2^(j+1), 1/2^j].
ps = Plots.Plot[]
for j = 1:size(W, 2)
    ylbl = "s_$j"
    lbl = string("(", 1/2^(j+1), " ", 1/2^j, "]")
    push!(ps, plot(W[:, j] ./ Etot,
                    fg_legend = :transparent, bg_legend = :transparent, legend␣
↪= :bottomright,
                    xticks = false, xaxis = false,
```

```
                    ylabel = latexstring(ylbl), label = lbl, ylim = (-emax,␣
↪emax)))
    hline!([0], ls = :dash, label = "", lq = 0.8, c = :grey)
end

# Merge original plot and coefficient plots.
p_ts = plot(f, ylabel = "f(t)", xlabel = "t", c = :black, label = "")
p_wl = plot(p_ts, ps..., link = :x, layout = grid(length(ps)+1, 1), guidefont =␣
↪font(8), size = (500, 800), xlabel = "")
```

[7]:


f(t)
1.0
0.5
0.0
−0.5
−1.0
0
100
200
300
0.0050
0.0025
0.0000
−0.0025
−0.0050
(0.25 0.5]
(0.125 0.25]
(0.0625 0.125]
(0.03125 0.0625]
(0.015625 0.03125]
(0.0078125 0.015625]
(0.00390625 0.0078125]
(0.001953125 0.00390625]

For our example time series, we can see that most of the variability is located at intermediate-high frequencies.

### 0.4 Relative signal energy and wavelet entropy

The *energy* of the signal at a particular scale level (frequency range) $j$ is $E_j = \sum_{t=1}^{N} (W_{tj})^2$. The total energy of the signal is $E_{tot} = \sum_{t=1}^{N} \sum_{j=1}^{n_{scales}} (W_{t,j})^2$. Based on the quantities $E_j$ and $E_{tot}$, we can define the *relative energy* at each frequency range $\left(\frac{1}{2^{j+1}}, \frac{1}{2^j}\right]$ as $p_j = \frac{E_j}{E_{tot}}$. The relative energy at level $j$ thus gives an indication of strongly that frequency range is present in the original signal. Normalizing each $p_j$ to the total energy, this naturally gives rise of the notion of a *probability distribution over the frequency bands corresponding to each dyadic levels.*

[8]:
```
s = [Entropies.energy_at_scale(W, j) ./ Etot for j = 1:size(W, 2)]
```

[8]:
```
8-element Vector{Float64}:
 0.2738045207764152
 0.11935778797910458
 0.053979061628271575
 0.032981485968457264
 0.015640601580414436
 0.0073874016856513465
 0.002526898148804093
 0.49432224223288157
```

Computing entropy from `s` is of course trivial:

[9]:
```
Entropies.genentropy(Probabilities( s), base = 2)
```

[9]:
```
1.9377788046525701
```

Entropy is maximized for a uniform probability distribution, and is always lower for non-flat distributions.

[10]:
```
genentropy(Probabilities([0.2, 0.2, 0.2, 0.2, 0.2]), base = 2),␣
↪genentropy(Probabilities([0.1, 0.2, 0.3, 0.2, 0.2]), base = 2)
```

[10]:
```
(2.321928094887362, 2.2464393446710154)
```

Similarly, the wavelet entropy is maximized when energy is spread out across all frequency bands. The following three signals illustrate this nicely.

[11]:
```
x = sin.(t);
y = sin.(t .+  cos.(t/0.5));
z = sin.(rand(1:15, N) ./ rand(1:10, N)) .+ rand(N)*0.1

est = TimeScaleMODWT()
h_x = Entropies.genentropy(x, est, base = 2)
h_y = Entropies.genentropy(y, est, base = 2)
```

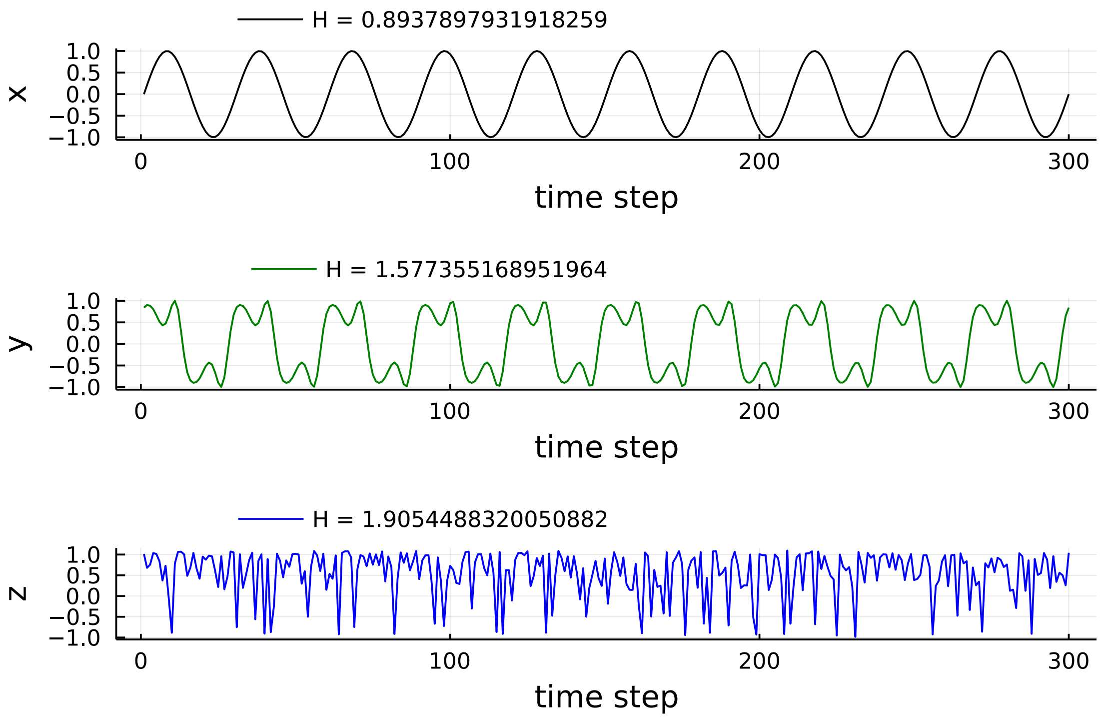
```julia
h_z = Entropies.genentropy(z, est, base = 2)

px = plot(x, ylabel = "x", c = :black, label = "H = $(h_x)")
py = plot(y, ylabel = "y", c = :green, label = "H = $(h_y)")
pz = plot(z, ylabel = "z", c = :blue, label = "H = $(h_z)")

# Get coefficients, then compute relative energies corresponding to
# different frequency bands.
wl = Wavelets.WT.Haar()
 x = Entropies.relative_wavelet_energies(Entropies.get_modwt(x, wl))
 y = Entropies.relative_wavelet_energies(Entropies.get_modwt(y, wl))
 z = Entropies.relative_wavelet_energies(Entropies.get_modwt(z, wl))
# Or Entropies.time_scale_density(ts, wl) directly

plot(px, py, pz, layout = grid(3, 1), xlabel = "time step",
    legend = :outertop, fg_legend = :transparent, bg_legend = :transparent)
```

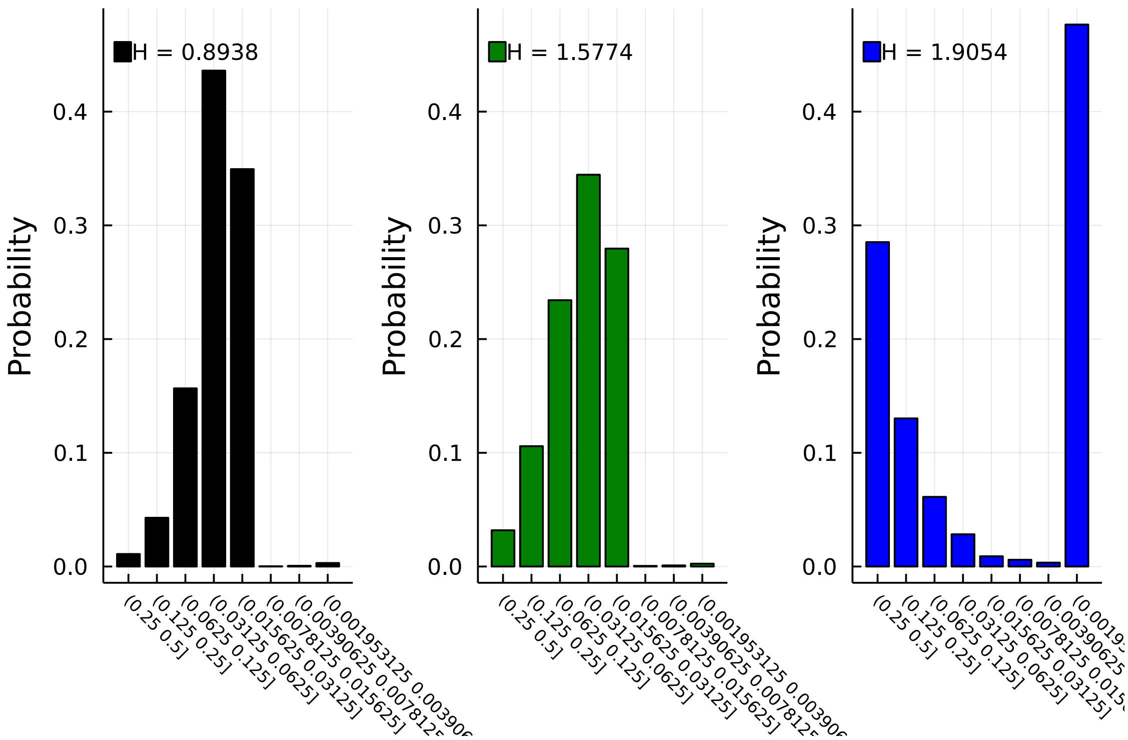
[11]:




[12]:
```julia
frequency_bands = [string("(", 1/2^(j+1), " ", 1/2^j, "]") for j = 1:length( x)]
distx = bar( x, c = :black, ylabel = "Probability", xticks = (1:length( x),␣
 ↪frequency_bands), xrotation =-45, xtickfont = font(6), label = "H =␣
 ↪$(round(h_x, digits = 4))")
```

```julia
disty = bar( y, c = :green, ylabel = "Probability", xticks = (1:length( x),
    frequency_bands), xrotation =-45, xtickfont = font(6), label = "H =
    $(round(h_y, digits = 4))")
distz = bar( z, c = :blue, ylabel = "Probability", xticks = (1:length( x),
    frequency_bands), xrotation =-45, xtickfont = font(6), label = "H =
    $(round(h_z, digits = 4))")
plot(distx, disty, distz, layout = grid(1, 3), link = :y, legend = :topleft,
    fg_legend = :transparent, bg_legend = :transparent)
```

[12]:




The time series with the widest range of frequencies (z) has the highest wavelet entropy, while the smoothest time series (x) which has most of its energy at few frequency bands has the lowest wavelet entropy.

[ ]:

```

```

[ ]:

```

```