Mathematical Formulae and Identities

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Contents

1 Preliminary Algebra

1.1 Coordinate Geometry

Equation of a line:

$$y = mx + c$$

Equation of a circle:

$$(x-h)^2 + (y-k)^2 = r^2$$

Equation of a parabola:

$$y - k = a(x-h)^2 \qquad\qquad x - h = a(y-k)^2$$

Equation of an ellipse:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

Equation of a hyperbola:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

1.2 Binomial Expansion

$$^nC_k = \frac{n!}{k!(n-k)!} = \binom{n}{k}$$

Also,

$$\begin{aligned} ^nC_0 &= {}^nC_n = 1 \\ ^nC_1 &= {}^nC_{n-1} = n \\ ^nC_k &= {}^nC_{n-k} \end{aligned}$$

For a binomial expression raised to the power n:

$$(x+y)^n = \sum_{k=0}^{k=n} {}^nC_k x^{n-k} y^k$$

2 Trigonometry

2.1 Common Values & the Unit Circle

2.2 Pythagorean Identities

$$\begin{aligned} \sin^2 x + \cos^2 x &= 1 \qquad\qquad \sec^2 x - \tan^2 x = 1 \\ \csc^2 x - \cot^2 x &= 1 \end{aligned}$$

θ°	0°	30°	45°	60°	90°
θ^c	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	N/D

Table 1: Commonly used trigonometric values. Refer the Unit Circle

2.3 Double Angle Formulae

$$\sin(2x) = 2\sin x\cos x \qquad \tan(2x) = \frac{2\tan x}{1-\tan^2 x}$$

$$\begin{aligned}\cos(2x) &= 2\cos^2 x - 1\\ &= \cos^2 x - \sin^2 x\\ &= 1 - 2\sin^2 x\end{aligned}$$

2.4 Sum and Difference Formulae

$$\sin(x \pm y) = \sin x\cos y \pm \cos x\sin y \qquad \cos(x \pm y) = \cos x\cos y \mp \sin x\sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x\tan y}$$

$$\begin{aligned}\arcsin x \pm \arcsin y &= \arcsin(x\sqrt{1-y^2} \pm y\sqrt{1-x^2})\\ \arccos x \pm \arccos y &= \arccos(xy \mp \sqrt{(1-x^2)(1-y^2)})\\ \arctan x \pm \arctan y &= \arctan(\frac{x \pm y}{1 \mp xy})\end{aligned}$$

2.5 Sum to Product Formulae

2.6 Product to Sum Formulae

2.7 Inverse Trigonometric Functions

$$
\begin{array}{lll}
\sin^{-1}(-x) = -\sin^{-1} x & \cos^{-1}(-x) = \pi - \cos^{-1} x, & |x| \leq 1 \\
\tan^{-1}(-x) = -\tan^{-1} x & \cot^{-1}(-x) = \pi - \cot^{-1} x, & x \in \mathbf{R} \\
\csc^{-1} x = \sin^{-1}\left(\dfrac{1}{x}\right) & \sec^{-1} x = \cos^{-1}\left(\dfrac{1}{x}\right), & |x| \geq 1 \\
\cot^{-1} x = \tan^{-1}\left(\dfrac{1}{x}\right), & & x > 0 \\
\cot^{-1} x = \pi + \tan^{-1}\left(\dfrac{1}{x}\right), & & x < 0 \\
\sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2}, & & |x| \leq 1 \\
\csc^{-1} x + \sec^{-1} x = \dfrac{\pi}{2}, & & |x| \geq 1
\end{array}
$$

2.8 Complex Numbers

Euler's Formula:

$$re^{i\theta} = r(\cos\theta + i\sin\theta)$$

De Moivre's Formula:

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

Exponential Definition of Trigonometric Functions:

$$\cos(ix) = \frac{(e^x + e^{-x})}{2} \qquad \sin(ix) = i\frac{(e^x - e^{-x})}{2} \qquad \tan(ix) = i\frac{(e^x - e^{-x})}{(e^x + e^{-x})}$$

Exponential Definition of Hyperbolic Functions:

$$\cosh(x) = \frac{(e^x + e^{-x})}{2} \qquad \sinh(x) = \frac{(e^x - e^{-x})}{2} \qquad \tanh(x) = \frac{(e^x - e^{-x})}{(e^x + e^{-x})}$$

Relationship between hyperbolic and trigonometric functions:

$$
\begin{array}{rr}
\cosh x = \cos ix & \cos x = \cosh ix \\
i\sinh x = \sin ix & i\sin x = \sinh ix
\end{array}
$$

2.9 Hyperbolic Identities

$$
\begin{array}{lll}
\cosh^2 x - \sinh^2 x = 1 & \operatorname{sech}^2 x + \tanh^2 x = 1 & \operatorname{csch}^2 x + \coth = 1 \\
\sinh(2x) = 2\sinh x\cosh x & \cosh(2x) = \cosh^2 x + \sinh^2 x & \sinh x + \cosh x = e^x
\end{array}
$$

2.10 Inverse Hyperbolic Functions

$$\cosh^{-1} x = \ln(\sqrt{1+x^2} + x) \qquad \sinh^{-1} x = \ln(\sqrt{1+x^2} + x)$$

$$\begin{aligned} \tanh^{-1} x &= \ln\sqrt{\frac{1+x}{1-x}} \\ &= \frac{1}{2}\ln\frac{1+x}{1-x} \end{aligned}$$

3 Vector Algebra

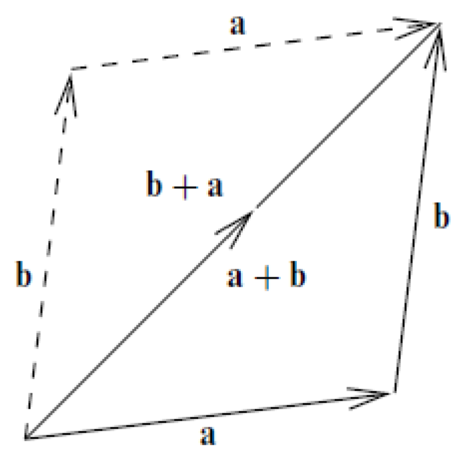

Figure 2: Addition of two vectors using Parallelogram Law

3.1 Scalar Product

$$\begin{aligned} a \cdot b &= |a||b|\cos\theta \\ &= \langle a|b\rangle \end{aligned}$$

If $a = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k}$ and $b = x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k}$ then,

$$a \cdot b = (x_1x_2)\hat{\imath} + (y_1y_2)\hat{\jmath} + (z_1z_2)\hat{k}$$

If $a \cdot b = 0$ then, $a \perp b$

3.2 Vector Product

$$\begin{aligned} a \times b &= |a||b|\sin\theta \\ &= |b\rangle\langle a| \end{aligned}$$

If $a = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k}$ and $b = x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k}$ then,

$$a \times b = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

3.3 Equations of lines, planes and spheres

Equation of a line: In Fig.3, the vector $\mathbf{r}$ can be written as $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$

4 Limits

5 Differential Calculus

5.1 Derivative Rules and Properties

$$\frac{\mathrm{d}}{\mathrm{d}x}$$

5.2 Common Derivatives

$$\frac{\mathrm{d}}{\mathrm{d}x}a^x = (\ln a)a^x \qquad \frac{\mathrm{d}}{\mathrm{d}x}\log_a x = \frac{1}{x\ln a}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x \qquad \frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x \qquad \frac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2 x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\csc x = -\csc x\cot x \qquad \frac{\mathrm{d}}{\mathrm{d}x}\sec x = \sec x\tan x \qquad \frac{\mathrm{d}}{\mathrm{d}x}\cot x = -\csc^2 x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1} x = \frac{1}{1+x^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\sec^{-1} x = \frac{1}{x\sqrt{x^2-1}} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\csc^{-1} x = -\frac{1}{x\sqrt{x^2-1}} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\cot^{-1} x = -\frac{1}{1+x^2}$$

6 Integral Calculus

6.1 Fundamental Theorem of Calculus

Theorem 1 (First Fundamental Theorem of Calculus) *If f is continuous on $[a, b]$, then the function defined by*

$$S(x) = \int_a^x f(t)\,dt$$

is continuous on $[a, b]$ and differentiable on (a, b), and $S'(x) = f(x)$.

Written in Leibniz notation,

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,dt = f(x)$$

Theorem 2 (Second Fundamental Theorem of Calculus) *If f is a continuous function on $[a, b]$, then*

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is the anti-derivative of f, i.e. $F' = f$.

6.2 Common Antiderivatives

$$\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + c \qquad \int \frac{1}{ax+b}\,\mathrm{d}x = \frac{1}{a}\ln|ax+b| + c$$

$$\int \cos a\,\mathrm{d}x = \frac{1}{a}\sin ax + c \qquad \int \sin ax\,\mathrm{d}x = -\frac{1}{a}\cos ax + c$$

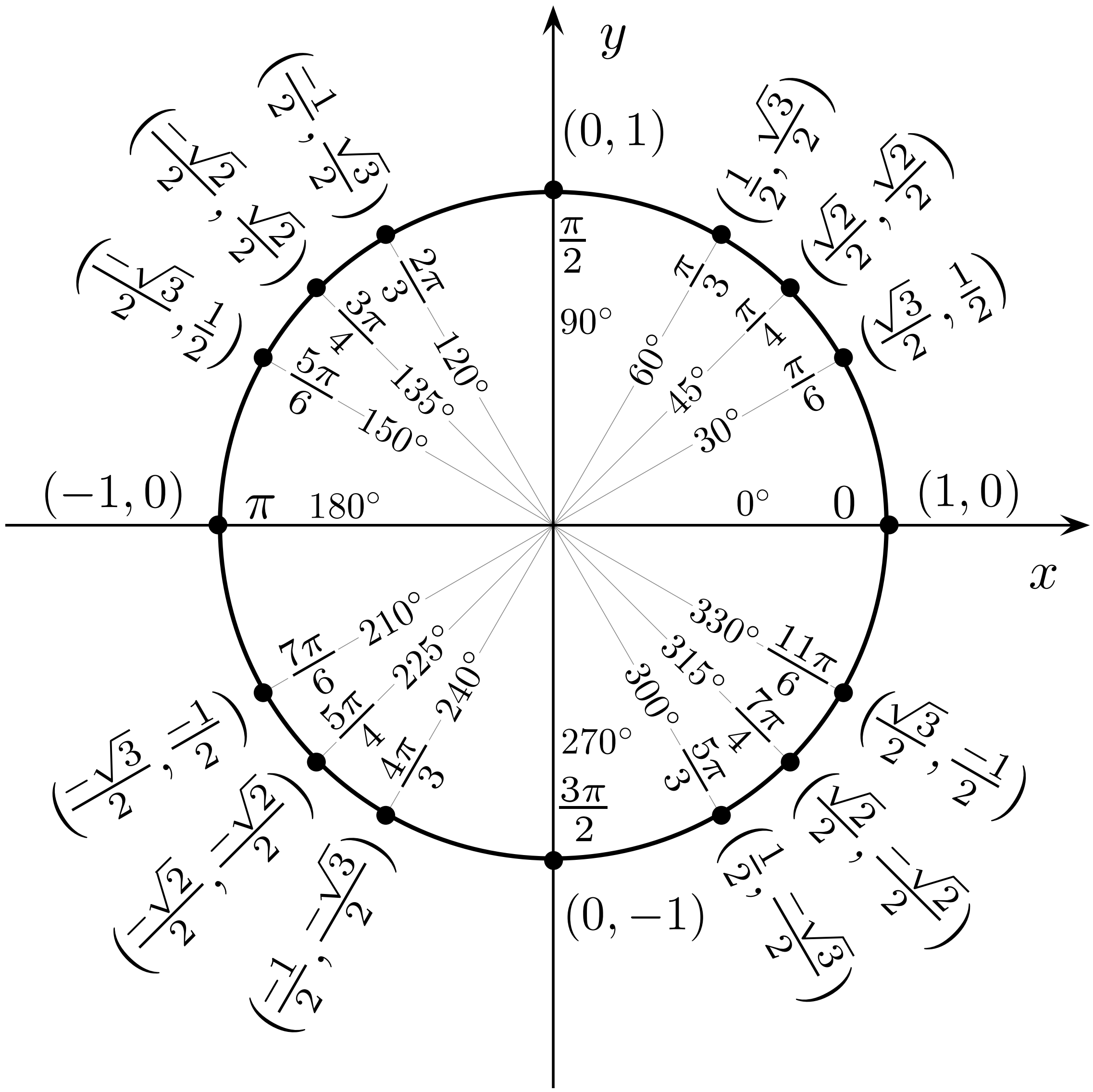

Figure 1: Unit Circle

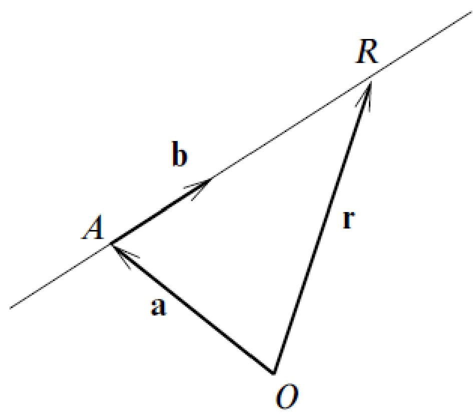

Figure 3: The equation of a line. The vector b is in the direction AR and λb is the vector from A to R.