Mathematical Formulae and Identities

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Contents

1 Preliminary Algebra

1.1 Coordinate Geometry

Equation of a line:

$$y = mx + c$$

Equation of a circle:

$$(x-h)^2 + (y-k)^2 = r^2$$

Equation of a parabola:

$$y - k = a(x-h)^2 \qquad\qquad x - h = a(y-k)^2$$

Equation of an ellipse:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

Equation of a hyperbola:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

1.2 Logarithms and Exponentials

Properties of Exponentials

$$a^n a^m = a^{n+m} \qquad\qquad \frac{a^n}{a^m} = a^{n-m}$$

$$(ab)^n = a^n b^n \qquad\qquad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

$$(a^n)^m = a^{nm} \qquad\qquad \left(\frac{a}{b}\right)^{-n} = \frac{b}{a}^n$$

Properties of Logarithms

$$\log_a a = 1 \qquad\qquad \log_a 1 = 0$$

$$(\log_a b)(\log_b c) = \log_a c \qquad\qquad \log_a b = \frac{1}{\log_b a}$$

$$\log_a(xy) = \log_a x + \log_a y \qquad\qquad \log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

1.3 Binomial Expansion

$$^{n}C_k = \frac{n!}{k!(n-k)!} = \binom{n}{k}$$

Also,

$$\begin{aligned} ^{n}C_0 &= {}^{n}C_n = 1 \\ ^{n}C_1 &= {}^{n}C_{n-1} = n \\ ^{n}C_k &= {}^{n}C_{n-k} \end{aligned}$$

For a binomial expression raised to the power n:

$$(x+y)^n = \sum_{k=0}^{k=n} {}^{n}C_k x^{n-k} y^k$$

Pascal's Triangle

$$\begin{array}{ccccccccccccccccccccc}
 & & & & & & & & & & 1 & & & & & & & & & & \\
 & & & & & & & & & 1 & & 1 & & & & & & & & & \\
 & & & & & & & & 1 & & 2 & & 1 & & & & & & & & \\
 & & & & & & & 1 & & 3 & & 3 & & 1 & & & & & & & \\
 & & & & & & 1 & & 4 & & 6 & & 4 & & 1 & & & & & & \\
 & & & & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & & & \\
 & & & & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & & & \\
 & & & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 & & & \\
 & & 1 & & 8 & & 28 & & 56 & & 70 & & 56 & & 28 & & 8 & & 1 & & \\
 & 1 & & 9 & & 36 & & 84 & & 126 & & 126 & & 84 & & 36 & & 9 & & 1 & \\
1 & & 10 & & 45 & & 120 & & 210 & & 252 & & 210 & & 120 & & 45 & & 10 & & 1
\end{array}$$

Commonly Factored Polynomials:

$$\begin{aligned}
x^2 - y^2 &= (x-y)(x+y) \\
(x+y)^2 &= x^2 + 2xy + y^2 \\
(x-y)^2 &= x^2 - 2xy + y^2 \\
x^3 - y^3 &= (x-y)(x^2 + xy + y^2) \\
x^3 + y^3 &= (x+y)(x^2 - xy + y^2) \\
(x+y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3 \\
(x-y)^3 &= x^3 - 3x^2y + 3xy^2 - y^3
\end{aligned}$$

2 Trigonometry

2.1 Common Values and the Unit Circle

θ°	0°	30°	45°	60°	90°
θ^c	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	N/D

Table 1: Commonly used trigonometric values. Refer the Unit Circle

2.2 Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1 \qquad \sec^2 x - \tan^2 x = 1$$

$$\csc^2 x - \cot^2 x = 1$$

2.3 Double Angle Formulae

$$\sin(2x) = 2\sin x\cos x \qquad \tan(2x) = \frac{2\tan x}{1-\tan^2 x}$$

$$\begin{aligned} \cos(2x) &= 2\cos^2 x - 1 \\ &= \cos^2 x - \sin^2 x \\ &= 1 - 2\sin^2 x \end{aligned}$$

2.4 Sum and Difference Formulae

$$\sin(x \pm y) = \sin x\cos y \pm \cos x\sin y \qquad \cos(x \pm y) = \cos x\cos y \mp \sin x\sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x\tan y}$$

$$\begin{aligned} \arcsin x \pm \arcsin y &= \arcsin(x\sqrt{1-y^2} \pm y\sqrt{1-x^2}) \\ \arccos x \pm \arccos y &= \arccos(xy \mp \sqrt{(1-x^2)(1-y^2)}) \\ \arctan x \pm \arctan y &= \arctan(\frac{x \pm y}{1 \mp xy}) \end{aligned}$$

2.5 Sum to Product Formulae

2.6 Product to Sum Formulae

2.7 Inverse Trigonometric Functions

$$
\begin{array}{lll}
\sin^{-1}(-x) = -\sin^{-1} x & \cos^{-1}(-x) = \pi - \cos^{-1} x, & |x| \leq 1 \\
\tan^{-1}(-x) = -\tan^{-1} x & \cot^{-1}(-x) = \pi - \cot^{-1} x, & x \in \mathbf{R} \\
\csc^{-1} x = \sin^{-1}\left(\frac{1}{x}\right) & \sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right), & |x| \geq 1 \\
\cot^{-1} x = \tan^{-1}\left(\frac{1}{x}\right), & & x > 0 \\
\cot^{-1} x = \pi + \tan^{-1}\left(\frac{1}{x}\right), & & x < 0 \\
\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}, & & |x| \leq 1 \\
\csc^{-1} x + \sec^{-1} x = \frac{\pi}{2}, & & |x| \geq 1
\end{array}
$$

2.8 Trigonometry in the Complex Plane

Euler's Formula:

$$re^{i\theta} = r(\cos\theta + i\sin\theta)$$

De Moivre's Formula:

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

Exponential Definition of Trigonometric Functions:

$$\cos(ix) = \frac{(e^x + e^{-x})}{2} \qquad \sin(ix) = i\frac{(e^x - e^{-x})}{2} \qquad \tan(ix) = i\frac{(e^x - e^{-x})}{(e^x + e^{-x})}$$

Exponential Definition of Hyperbolic Functions:

$$\cosh(x) = \frac{(e^x + e^{-x})}{2} \qquad \sinh(x) = \frac{(e^x - e^{-x})}{2} \qquad \tanh(x) = \frac{(e^x - e^{-x})}{(e^x + e^{-x})}$$

Relationship between hyperbolic and trigonometric functions:

$$
\begin{array}{rr}
\cosh x = \cos ix & \cos x = \cosh ix \\
i\sinh x = \sin ix & i\sin x = \sinh ix
\end{array}
$$

2.9 Hyperbolic Identities

$$
\begin{array}{lll}
\cosh^2 x - \sinh^2 x = 1 & \operatorname{sech}^2 x + \tanh^2 x = 1 & \operatorname{csch}^2 x + \coth = 1 \\
\sinh(2x) = 2\sinh x\cosh x & \cosh(2x) = \cosh^2 x + \sinh^2 x & \sinh x + \cosh x = e^x
\end{array}
$$

2.10 Inverse Hyperbolic Functions

$$\cosh^{-1} x = \ln(\sqrt{1+x^2} + x) \qquad \sinh^{-1} x = \ln(\sqrt{1+x^2} + x)$$

$$\begin{aligned} \tanh^{-1} x &= \ln \sqrt{\frac{1+x}{1-x}} \\ &= \frac{1}{2} \ln \frac{1+x}{1-x} \end{aligned}$$

3 Vector Algebra

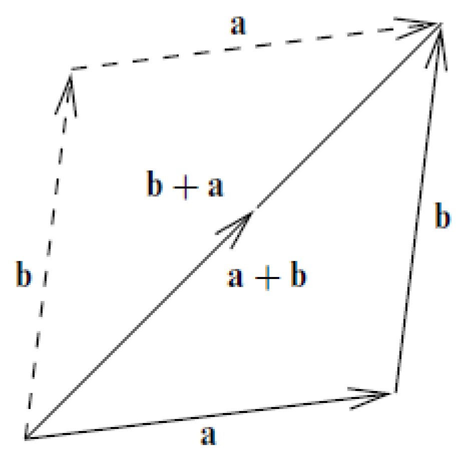

Figure 2: Addition of two vectors using Parallelogram Law

3.1 Scalar Product

$$\begin{aligned} a \cdot b &= |a||b| \cos\theta \\ &= \langle a|b\rangle \end{aligned}$$

If $a = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k}$ and $b = x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k}$ then,

$$a \cdot b = (x_1x_2)\hat{\imath} + (y_1y_2)\hat{\jmath} + (z_1z_2)\hat{k}$$

If $a \cdot b = 0$ then, $a \perp b$

3.2 Vector Product

$$\begin{aligned} a \times b &= |a||b| \sin\theta \\ &= |b\rangle\langle a| \end{aligned}$$

If $a = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k}$ and $b = x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k}$ then,

$$a \times b = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

3.3 Equations of lines, planes and spheres

Equation of a line: In Fig.3, the vector $\mathbf{r}$ can be written as $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$

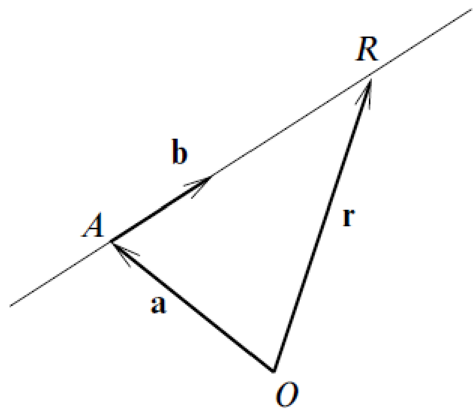

Figure 3: The equation of a line. The vector b is in the direction AR and λb is the vector from A to R.

4 Limits

4.1 Precise Definition of a Limit

Definition 1 (Limit of a function) *Let $f(x)$ be a function defined on an open interval around x_0. We say that the limit of $f(x)$ as x approaches x_0 is L, i.e. $\lim_{x\to x_0} f(x) = L$, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all x*

$$0 < |x - x_0| < \delta \;\Rightarrow\; |f(x) - L| < \varepsilon.$$

4.2 Common Limits

$$\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^{mx} = e^{mk} \qquad \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = e \qquad \lim_{x\to\infty}\left(1-\frac{1}{x}\right)^{x} = \frac{1}{e}$$

$$\lim_{x\to\infty}\left(\frac{x}{x+k}\right)^{x} = \frac{1}{e^k} \qquad \lim_{x\to 0}(1+x)^{\frac{1}{x}} = e \qquad \lim_{x\to 0}\left(\frac{a^x-1}{x}\right) = \ln a$$

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 \qquad \lim_{x\to 0}\frac{\sin ax}{bx} = \frac{a}{b} \qquad \lim_{x\to 0}\frac{1-\cos x}{x} = 0$$

4.3 L'Hôpital's Rule

Theorem 1 (L'Hôpital's Rule) *Suppose f and g are differentiable functions such that*

1. $g'(x) \neq 0$ *on an open interval* I *containing* a;
2. $\lim_{x\to a} f(x) = 0$ *and* $\lim_{x\to a} g(x) = 0$, *or* $\lim_{x\to a} f(x) = \pm\infty$ *and* $\lim_{x\to a} g(x) = \pm\infty$;
3. $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ *exists.*

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

5 Sequences and Series

5.1 Common Summation Formulae

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} \qquad \sum_{i=1}^{n} r^i = \frac{1-r^{n+1}}{1-r}$$

5.2 Geometric Series

The Geometric Series

$$S = \sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 \dots$$

Is convergent if $|r| < 1$ and,

$$S = \frac{a}{1-r}$$

5.3 Power-Series

$$\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges for $p > 1$

5.4 Series Convergence Tests

5.5 Taylor Series

Also refer Definition 2.

6 Differential Calculus

6.1 Derivative Rules and Properties

The differntial operator is a linear operator

$$(f(x) \pm g(x))' = f'(x) \pm g'(x) \qquad\qquad (c \cdot f(x)' = c \cdot f'(x)$$

Product Rule:

$$(f(x) \cdot g(x))' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

Quotient Rule:

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x) \cdot g(x) - g'(x) \cdot f(x)}{(g(x))^2}$$

Chain Rule:

$$(f(g(x)))' = f'(g(x)) \cdot g'(x)$$

6.2 Common Derivatives

$$\frac{\mathrm{d}}{\mathrm{d}x} a^x = (\ln a) a^x \qquad \frac{\mathrm{d}}{\mathrm{d}x} \log_a x = \frac{1}{x \ln a}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \sin x = \cos x \qquad \frac{\mathrm{d}}{\mathrm{d}x} \cos x = -\sin x \qquad \frac{\mathrm{d}}{\mathrm{d}x} \tan x = \sec^2 x$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \csc x = -\csc x \cot x \qquad \frac{\mathrm{d}}{\mathrm{d}x} \sec x = \sec x \tan x \qquad \frac{\mathrm{d}}{\mathrm{d}x} \cot x = -\csc^2 x$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{\mathrm{d}}{\mathrm{d}x} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{\mathrm{d}}{\mathrm{d}x} \tan^{-1} x = \frac{1}{1+x^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \sec^{-1} x = \frac{1}{x\sqrt{x^2-1}} \qquad \frac{\mathrm{d}}{\mathrm{d}x} \csc^{-1} x = -\frac{1}{x\sqrt{x^2-1}} \qquad \frac{\mathrm{d}}{\mathrm{d}x} \cot^{-1} x = -\frac{1}{1+x^2}$$

6.3 Applications of Derivatives

6.3.1 Tangent to a Curve

Tangent to a curve $f(x)$ at a point a:

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) \Big|_{x=a}$$

For a straight line passing through points $(\mathrm{x}_1, \mathrm{y}_1)$ and $(\mathrm{x}_2, \mathrm{y}_2)$, the slope, m, is constant and is calculated by:

$$m = \tan\theta = \frac{y_2 - y_1}{x_2 - x_1}$$

Where θ is the angle of the line with the x-axis

6.3.2 Taylor Polynomials

Definition 2 (Taylor Polynomial) *Let $f(x)$ be a real-valued function that is infinitely differentiable at $x = x_0$. The Taylor series expansion for the function $f(x)$ centered around the point $x = x_0$ is given by*

$$\sum_{n=0}^{\infty} f^{(n)}(x_0)\frac{(x-x_0)^n}{n!}$$

Where $f^{(n)}(x_0)$ is the n^{th} derivative of $f(x)$ at $x = x_0$.

7 Integral Calculus

7.1 Fundamental Theorem of Calculus

Theorem 2 (First Fundamental Theorem of Calculus) *If f is continuous on $[a, b]$, then the function defined by*

$$S(x) = \int_a^x f(t)\,dt$$

is continuous on $[a, b]$ and differentiable on (a, b), and $S'(x) = f(x)$.

Written in Leibniz notation,

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,dt = f(x)$$

Theorem 3 (Second Fundamental Theorem of Calculus) *If f is a continuous function on $[a, b]$, then*

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is the anti-derivative of f, i.e. $F' = f$.

7.2 Common Antiderivatives

$$\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + c \qquad \int \frac{1}{ax+b}\,\mathrm{d}x = \frac{1}{a}\ln|ax+b| + c$$

$$\int \cos a\,\mathrm{d}x = \frac{1}{a}\sin ax + c \qquad \int \sin ax\,\mathrm{d}x = -\frac{1}{a}\cos ax + c$$

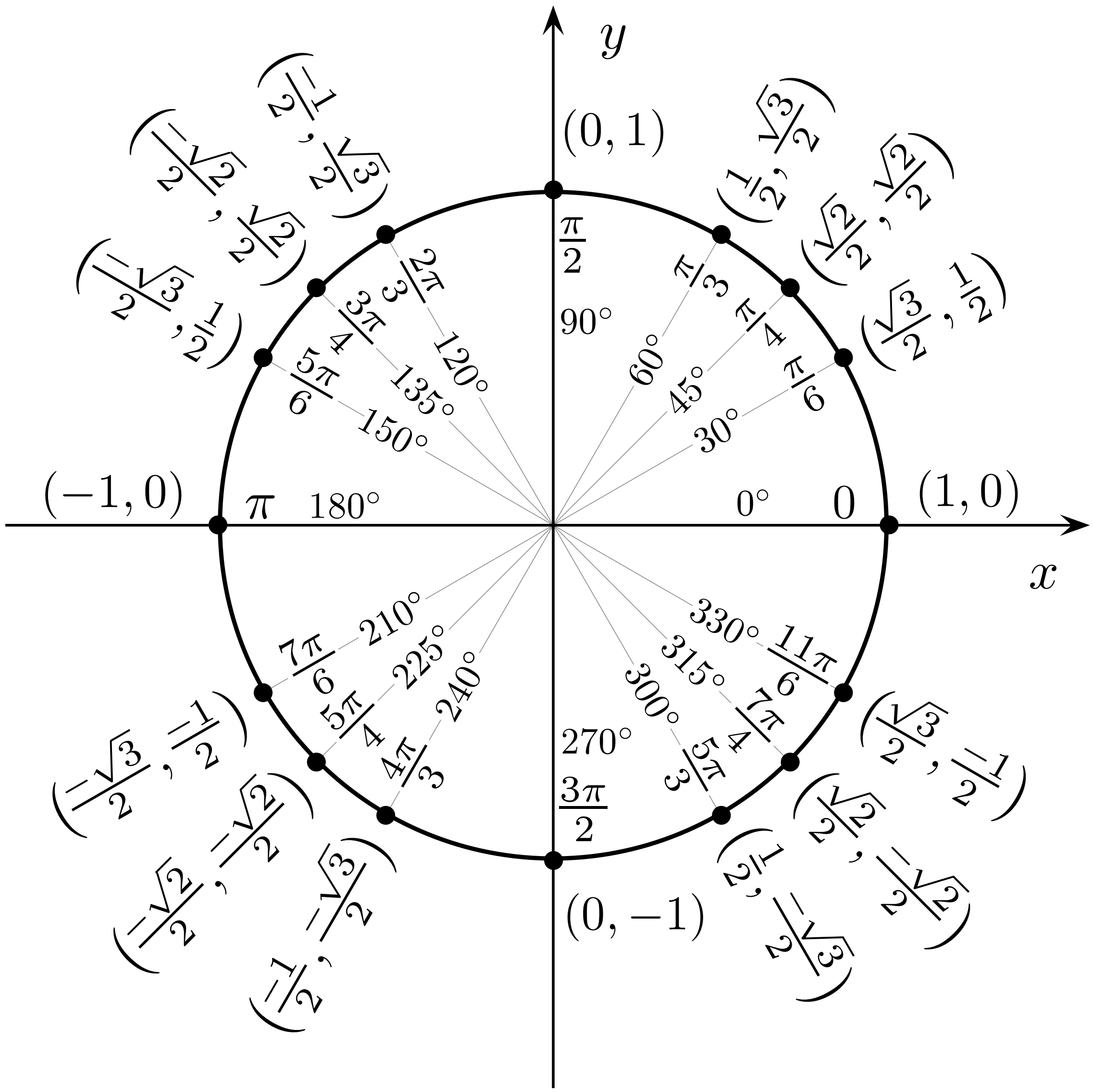

Figure 1: Unit Circle