Mathematical Formulae and Identities

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Contents

1 Preliminary Algebra

1.1 Coordinate Geometry

Equation of a line:

$$y = mx + c$$

Equation of a circle:

$$(x-h)^2 + (y-k)^2 = r^2$$

Equation of a parabola:

$$y - k = a(x-h)^2 \qquad\qquad x - h = a(y-k)^2$$

Equation of an ellipse:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

Equation of a hyperbola:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

1.2 Binomial Expansion

$${}^nC_k = \frac{n!}{k!(n-k)!} = \binom{n}{k}$$

Also,

$$\begin{aligned} {}^nC_0 &= {}^nC_n = 1 \\ {}^nC_1 &= {}^nC_{n-1} = n \\ {}^nC_k &= {}^nC_{n-k} \end{aligned}$$

For a binomial expression raised to the power n:

$$(x+y)^n = \sum_{k=0}^{k=n} {}^nC_k x^{n-k} y^k$$

Pascal's Triangle

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1 \\
1 \quad 9 \quad 36 \quad 84 \quad 126 \quad 126 \quad 84 \quad 36 \quad 9 \quad 1 \\
1 \quad 10 \quad 45 \quad 120 \quad 210 \quad 252 \quad 210 \quad 120 \quad 45 \quad 10 \quad 1
\end{array}$$

Commonly Factored Polynomials:

$$\begin{aligned}
x^2 - y^2 &= (x-y)(x+y) \\
(x+y)^2 &= x^2 + 2xy + y^2 \\
(x-y)^2 &= x^2 - 2xy + y^2 \\
x^3 - y^3 &= (x-y)(x^2 + xy + y^2) \\
x^3 + y^3 &= (x+y)(x^2 - xy + y^2) \\
(x+y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3 \\
(x-y)^3 &= x^3 - 3x^2y + 3xy^2 - y^3
\end{aligned}$$

2 Trigonometry

2.1 Common Values and the Unit Circle

θ°	0°	30°	45°	60°	90°
θ^c	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	N/D

Table 1: Commonly used trigonometric values. Refer the Unit Circle

2.2 Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1 \qquad \sec^2 x - \tan^2 x = 1$$

$$\csc^2 x - \cot^2 x = 1$$

2.3 Double Angle Formulae

$$\sin(2x) = 2\sin x \cos x \qquad \tan(2x) = \frac{2\tan x}{1 - \tan^2 x}$$

$$\begin{aligned}
\cos(2x) &= 2\cos^2 x - 1 \\
&= \cos^2 x - \sin^2 x \\
&= 1 - 2\sin^2 x
\end{aligned}$$

2.4 Sum and Difference Formulae

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y \qquad \cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

$$\arcsin x \pm \arcsin y = \arcsin\,(x\sqrt{1-y^2} \pm y\sqrt{1-x^2})$$
$$\arccos x \pm \arccos y = \arccos\,(xy \mp \sqrt{(1-x^2)(1-y^2)})$$
$$\arctan x \pm \arctan y = \arctan\,(\frac{x \pm y}{1 \mp xy})$$

2.5 Sum to Product Formulae

2.6 Product to Sum Formulae

2.7 Inverse Trigonometric Functions

$\sin^{-1}(-x) = -\sin^{-1} x$	$\cos^{-1}(-x) = \pi - \cos^{-1} x,$	$\lvert x\rvert \leq 1$
$\tan^{-1}(-x) = -\tan^{-1} x$	$\cot^{-1}(-x) = \pi - \cot^{-1} x,$	$x \in \mathbf{R}$
$\csc^{-1} x = \sin^{-1}\left(\frac{1}{x}\right)$	$\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right),$	$\lvert x\rvert \geq 1$
$\cot^{-1} x = \tan^{-1}\left(\frac{1}{x}\right),$		$x > 0$
$\cot^{-1} x = \pi + \tan^{-1}\left(\frac{1}{x}\right),$		$x < 0$
$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2},$		$\lvert x\rvert \leq 1$
$\csc^{-1} x + \sec^{-1} x = \frac{\pi}{2},$		$\lvert x\rvert \geq 1$

2.8 Trigonometry in the Complex Plane

Euler's Formula:

$$re^{i\theta} = r(\cos\theta + i\sin\theta)$$

De Moivre's Formula:

$$(\cos\theta + i\sin\theta)^n = \cos\,(n\theta) + i\sin\,(n\theta)$$

Exponential Definition of Trigonometric Functions:

$$\cos\,(ix) = \frac{(e^x + e^{-x})}{2} \qquad \sin\,(ix) = i\frac{(e^x - e^{-x})}{2} \qquad \tan\,(ix) = i\frac{(e^x - e^{-x})}{(e^x + e^{-x})}$$

Exponential Definition of Hyperbolic Functions:

$$\cosh\,(x) = \frac{(e^x + e^{-x})}{2} \qquad \sinh\,(x) = \frac{(e^x - e^{-x})}{2} \qquad \tanh\,(x) = \frac{(e^x - e^{-x})}{(e^x + e^{-x})}$$

Relationship between hyperbolic and trigonometric functions:

$$\cosh x = \cos ix \qquad \cos x = \cosh ix$$
$$i\sinh x = \sin ix \qquad i\sin x = \sinh ix$$

2.9 Hyperbolic Identities

$$\cosh^2 x - \sinh^2 x = 1 \qquad \operatorname{sech}^2 x + \tanh^2 x = 1 \qquad \operatorname{csch}^2 x + \coth = 1$$
$$\sinh(2x) = 2\sinh x \cosh x \qquad \cosh(2x) = \cosh^2 x + \sinh^2 x \qquad \sinh x + \cosh x = e^x$$

2.10 Inverse Hyperbolic Functions

$$\cosh^{-1} x = \ln(\sqrt{1+x^2} + x) \qquad \sinh^{-1} x = \ln(\sqrt{1+x^2} + x)$$
$$\begin{aligned} \tanh^{-1} x &= \ln\sqrt{\frac{1+x}{1-x}} \\ &= \frac{1}{2}\ln\frac{1+x}{1-x} \end{aligned}$$

3 Vector Algebra

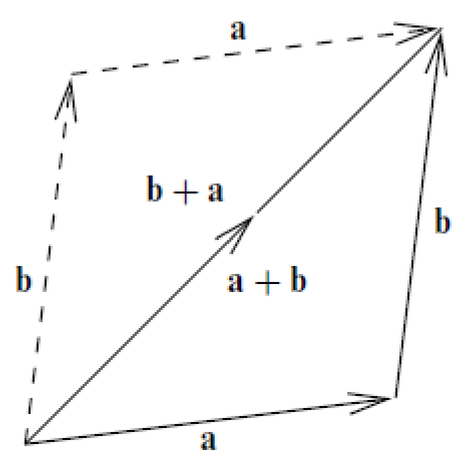

Figure 2: Addition of two vectors using Parallelogram Law

3.1 Scalar Product

$$\begin{aligned} a \cdot b &= |a||b|\cos\theta \\ &= \langle a|b\rangle \end{aligned}$$

If $a = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k}$ and $b = x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k}$ then,

$$a \cdot b = (x_1x_2)\hat{\imath} + (y_1y_2)\hat{\jmath} + (z_1z_2)\hat{k}$$

If $a \cdot b = 0$ then, $a \perp b$

3.2 Vector Product

$$
\begin{aligned}
a \times b &= |a||b| \sin\theta \\
&= |b\rangle\langle a|
\end{aligned}
$$

If $a = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k}$ and $b = x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k}$ then,

$$
a \times b = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}
$$

3.3 Equations of lines, planes and spheres

Equation of a line: In Fig.3, the vector $\mathbf{r}$ can be written as $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$

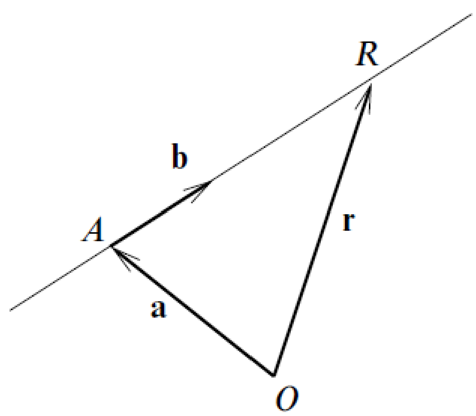

Figure 3: The equation of a line. The vector b is in the direction AR and λb is the vector from A to R.

4 Limits

5 Differential Calculus

5.1 Derivative Rules and Properties

$$
\frac{\mathrm{d}}{\mathrm{d}x}
$$

5.2 Common Derivatives

$$\frac{\mathrm{d}}{\mathrm{d}x} a^x = (\ln a) a^x \qquad \frac{\mathrm{d}}{\mathrm{d}x} \log_a x = \frac{1}{x \ln a}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \sin x = \cos x \qquad \frac{\mathrm{d}}{\mathrm{d}x} \cos x = -\sin x \qquad \frac{\mathrm{d}}{\mathrm{d}x} \tan x = \sec^2 x$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \csc x = -\csc x \cot x \qquad \frac{\mathrm{d}}{\mathrm{d}x} \sec x = \sec x \tan x \qquad \frac{\mathrm{d}}{\mathrm{d}x} \cot x = -\csc^2 x$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{\mathrm{d}}{\mathrm{d}x} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{\mathrm{d}}{\mathrm{d}x} \tan^{-1} x = \frac{1}{1+x^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \sec^{-1} x = \frac{1}{x\sqrt{x^2-1}} \qquad \frac{\mathrm{d}}{\mathrm{d}x} \csc^{-1} x = -\frac{1}{x\sqrt{x^2-1}} \qquad \frac{\mathrm{d}}{\mathrm{d}x} \cot^{-1} x = -\frac{1}{1+x^2}$$

5.3 Applications of Derivatives

5.3.1 Tangent to a Curve

Tangent to a curve $f(x)$ at a point a:

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x)\Big|_{x=a}$$

For a straight line passing through points $(\mathrm{x}_1, \mathrm{y}_1)$ and $(\mathrm{x}_2, \mathrm{y}_2)$, the slope, m, is constant and is calculated by:

$$m = \tan\theta = \frac{y_2 - y_1}{x_2 - x_1}$$

Where θ is the angle of the line with the x-axis

5.4 Taylor Polynomials Expansions

Definition 1 (Taylor Polynomial) *Let $f(x)$ be a real-valued function that is infinitely differentiable at $x = x_0$. The Taylor series expansion for the function $f(x)$ centered around the point $x = x_0$ is given by*

$$\sum_{n=0}^{\infty} f^{(n)}(x_0) \frac{(x-x_0)^n}{n!}$$

Where $f^{(n)}(x_0)$ is the n^{th} derivative of $f(x)$ at $x = x_0$.

6 Integral Calculus

6.1 Fundamental Theorem of Calculus

Theorem 1 (First Fundamental Theorem of Calculus) *If f is continuous on $[a, b]$, then the function defined by*

$$S(x) = \int_a^x f(t)\, dt$$

is continuous on $[a, b]$ and differentiable on (a, b), and $S'(x) = f(x)$.

Written in Leibniz notation,

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(t)\, dt = f(x)$$

Theorem 2 (Second Fundamental Theorem of Calculus) *If f is a continuous function on $[a, b]$, then*

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

where F is the anti-derivative of f, i.e. $F' = f$.

6.2 Common Antiderivatives

$$\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + c \qquad \int \frac{1}{ax+b}\,\mathrm{d}x = \frac{1}{a}\ln|ax+b| + c$$

$$\int \cos a\,\mathrm{d}x = \frac{1}{a}\sin ax + c \qquad \int \sin ax\,\mathrm{d}x = -\frac{1}{a}\cos ax + c$$

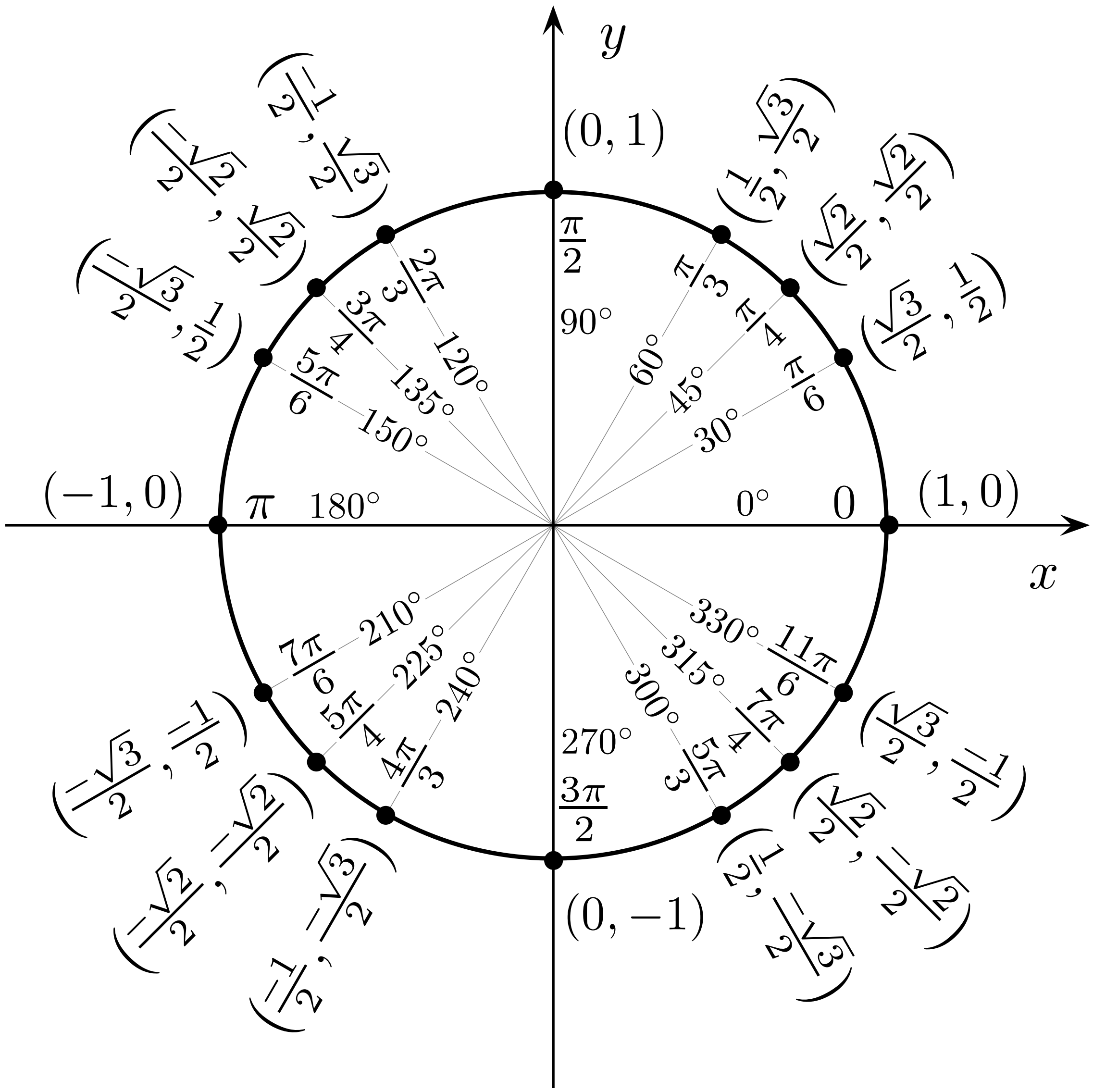

Figure 1: Unit Circle