

Undergraduate Mathematics Formulae and Identities

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1 Preliminary Algebra

1.1 Coordinate Geometry

Equation of a line

$$y = mx + c$$

Equation of a circle

$$(x - h)^2 + (y - k)^2 = r^2$$

Equation of a parabola

$$y - k = a(x - h)^2 \qquad x - h = a(y - k)^2$$

Equation of an ellipse

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$$

Equation of a hyperbola

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$$

1.2 Logarithms and Exponentials

Properties of Exponentials

$$a^n a^m = a^{n+m}$$

$$(ab)^n = a^n b^n$$

$$(a^n)^m = a^{nm}$$

$$\frac{a^n}{a^m} = a^{n-m}$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

$$\left(\frac{a}{b}\right)^{-n} = \frac{b^n}{a}$$

Properties of Logarithms

$$\log_a a = 1$$

$$(\log_a b)(\log_b c) = \log_a c$$

$$\log_a(xy) = \log_a x + \log_a y$$

$$\log_a 1 = 0$$

$$\log_a b = \frac{1}{\log_b a}$$

$$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$$

1.3 Binomial Expansion

$${}^nC_k = \frac{n!}{k!(n-k)!} = \binom{n}{k}$$

Also,

$$\begin{aligned} {}^nC_0 &= {}^nC_n = 1 \\ {}^nC_1 &= {}^nC_{n-1} = n \\ {}^nC_k &= {}^nC_{n-k} \end{aligned}$$

For a binomial expression raised to the power n :

$$(x+y)^n = \sum_{k=0}^{k=n} {}^nC_k x^{n-k} y^k$$

Pascal's Triangle

$$\begin{array}{cccccccccccccccc} & & & & & & & & 1 & & & & & & & & & \\ & & & & & & & 1 & & 1 & & & & & & & & \\ & & & & & & 1 & & 2 & & 1 & & & & & & & \\ & & & & & 1 & & 3 & & 3 & & 1 & & & & & & \\ & & & & 1 & & 4 & & 6 & & 4 & & 1 & & & & & \\ & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & & & \\ & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & & & \\ 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 & & & \\ & 1 & & 8 & & 28 & & 56 & & 70 & & 56 & & 28 & & 8 & & 1 \\ & & 1 & & 9 & & 36 & & 84 & & 126 & & 126 & & 84 & & 36 & & 9 & & 1 \\ 1 & & & 10 & & 45 & & 120 & & 210 & & 252 & & 210 & & 120 & & 45 & & 10 & & 1 \end{array}$$

Commonly Factored Polynomials

$$\begin{aligned} x^2 - y^2 &= (x-y)(x+y) \\ (x+y)^2 &= x^2 + 2xy + y^2 \\ (x-y)^2 &= x^2 - 2xy + y^2 \\ x^3 - y^3 &= (x-y)(x^2 + xy + y^2) \\ x^3 + y^3 &= (x+y)(x^2 - xy + y^2) \\ (x+y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3 \\ (x-y)^3 &= x^3 - 3x^2y + 3xy^2 - y^3 \end{aligned}$$

2 Trigonometry

2.1 Common Values and the Unit Circle

θ°	0°	30°	45°	60°	90°
θ^c	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin \theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan \theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	N/D

Table 1: Commonly used trigonometric values. Refer the Unit Circle

2.2 Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$

$$\sec^2 x - \tan^2 x = 1$$

$$\csc^2 x - \cot^2 x = 1$$

2.3 Double Angle Formulae

$$\sin(2x) = 2 \sin x \cos x$$

$$\tan(2x) = \frac{2 \tan x}{1 - \tan^2 x}$$

$$\begin{aligned} \cos(2x) &= 2 \cos^2 x - 1 \\ &= \cos^2 x - \sin^2 x \\ &= 1 - 2 \sin^2 x \end{aligned}$$

2.4 Sum and Difference Formulae

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y \qquad \cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

$$\arcsin x \pm \arcsin y = \arcsin(x\sqrt{1-y^2} \pm y\sqrt{1-x^2})$$

$$\arccos x \pm \arccos y = \arccos(xy \mp \sqrt{(1-x^2)(1-y^2)})$$

$$\arctan x \pm \arctan y = \arctan\left(\frac{x \pm y}{1 \mp xy}\right)$$

2.5 Sum to Product Formulae

$$\begin{aligned}\sin x + \sin y &= 2 \sin \left(\frac{x+y}{2} \right) \cos \left(\frac{x-y}{2} \right) & \sin x - \sin y &= 2 \cos \left(\frac{x+y}{2} \right) \sin \left(\frac{x-y}{2} \right) \\ \cos x + \cos y &= 2 \cos \left(\frac{x+y}{2} \right) \cos \left(\frac{x-y}{2} \right) & \cos x - \cos y &= -2 \sin \left(\frac{x+y}{2} \right) \sin \left(\frac{x-y}{2} \right)\end{aligned}$$

2.6 Product to Sum Formulae

$$\sin(x+y) \sin(x-y) = \sin^2 x - \sin^2 y \qquad \cos(x+y) \cos(x-y) = \cos^2 x - \sin^2 y$$

2.7 Laws of Sines and Cosines

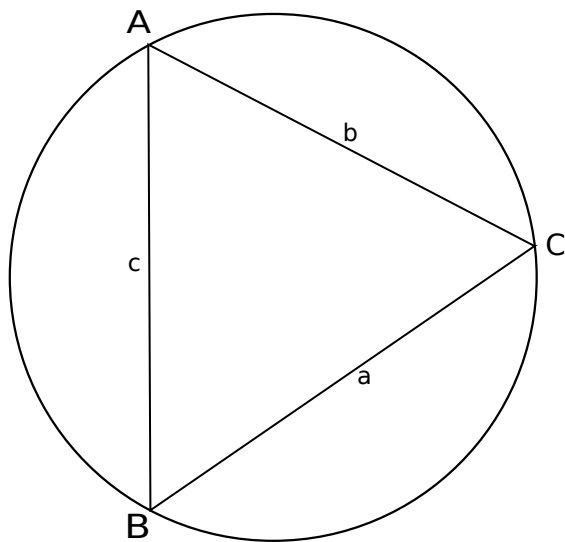


Figure 2: $\triangle ABC$ in a circumscribed circle of circumradius r

Law of Sines

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \frac{1}{2r}$$

Law of Cosines

$$\begin{aligned}a^2 &= b^2 + c^2 - 2bc \cos A & A &= \cos^{-1} \left(\frac{b^2 + c^2 - a^2}{2bc} \right) \\b^2 &= a^2 + c^2 - 2ac \cos B & B &= \cos^{-1} \left(\frac{a^2 + c^2 - b^2}{2ac} \right) \\c^2 &= a^2 + b^2 - 2ab \cos C & C &= \cos^{-1} \left(\frac{a^2 + b^2 - c^2}{2ab} \right)\end{aligned}$$

2.8 Inverse Trigonometric Functions

$$\begin{aligned}\sin^{-1}(-x) &= -\sin^{-1} x & \cos^{-1}(-x) &= \pi - \cos^{-1} x, & |x| &\leq 1 \\ \tan^{-1}(-x) &= -\tan^{-1} x & \cot^{-1}(-x) &= \pi - \cot^{-1} x, & x &\in \mathbf{R} \\ \csc^{-1} x &= \sin^{-1} \left(\frac{1}{x} \right) & \sec^{-1} x &= \cos^{-1} \left(\frac{1}{x} \right), & |x| &\geq 1 \\ \cot^{-1} x &= \tan^{-1} \left(\frac{1}{x} \right), & & & x &> 0 \\ \cot^{-1} x &= \pi + \tan^{-1} \left(\frac{1}{x} \right), & & & x &< 0 \\ \sin^{-1} x + \cos^{-1} x &= \frac{\pi}{2}, & & & |x| &\leq 1 \\ \csc^{-1} x + \sec^{-1} x &= \frac{\pi}{2}, & & & |x| &\geq 1\end{aligned}$$

2.9 Trigonometry in the Complex Plane

Euler's Formula

$$re^{i\theta} = r(\cos \theta + i \sin \theta)$$

De Moivre's Formula

$$(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta)$$

Exponential Definition of Trigonometric Functions

$$\cos(ix) = \frac{(e^x + e^{-x})}{2} \quad \sin(ix) = i \frac{(e^x - e^{-x})}{2} \quad \tan(ix) = i \frac{(e^x - e^{-x})}{(e^x + e^{-x})}$$

Exponential Definition of Hyperbolic Functions

$$\cosh(x) = \frac{(e^x + e^{-x})}{2} \quad \sinh(x) = \frac{(e^x - e^{-x})}{2} \quad \tanh(x) = \frac{(e^x - e^{-x})}{(e^x + e^{-x})}$$

Relationship between hyperbolic and trigonometric functions

$$\begin{array}{ll}\cosh x = \cos ix & \cos x = \cosh ix \\ i \sinh x = \sin ix & i \sin x = \sinh ix\end{array}$$

2.10 Hyperbolic Identities

$$\begin{array}{lll}\cosh^2 x - \sinh^2 x = 1 & \operatorname{sech}^2 x + \tanh^2 x = 1 & \operatorname{csch}^2 x + \coth^2 x = 1 \\ \sinh(2x) = 2 \sinh x \cosh x & \cosh(2x) = \cosh^2 x + \sinh^2 x & \sinh x + \cosh x = e^x\end{array}$$

2.11 Inverse Hyperbolic Functions

$$\begin{array}{ll}\cosh^{-1} x = \ln(\sqrt{1+x^2} + x) & \sinh^{-1} x = \ln(\sqrt{1+x^2} + x) \\ \tanh^{-1} x = \ln \sqrt{\frac{1+x}{1-x}} & \\ = \frac{1}{2} \ln \frac{1+x}{1-x} & \end{array}$$

3 Vector Algebra

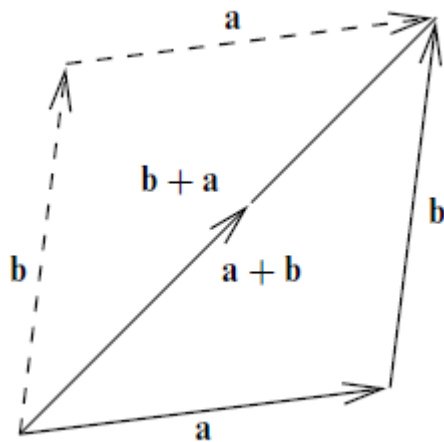


Figure 3: Addition of two vectors using Parallelogram Law

3.1 Scalar Product

Definition 1 (Dot Product) For two vectors a and b , the scalar product, or the dot product, is defined as:

$$\begin{aligned}a \cdot b &= |a||b| \cos \theta \\ &= \langle a|b \rangle\end{aligned}$$

Where θ is the angle between the two vectors.

If $a = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $b = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$ then,

$$a \cdot b = (x_1x_2)\hat{i} + (y_1y_2)\hat{j} + (z_1z_2)\hat{k}$$

If $a \perp b$, then $a \cdot b = 0$ and, if $a \parallel b$, then $a \cdot b = |a||b|$

3.2 Vector Product

Definition 2 (Cross Product) For two vectors a and b , the vector product, or the cross product, is defined as:

$$\begin{aligned} a \times b &= |a||b| \sin \theta \\ &= |b|\langle a| \end{aligned}$$

Where θ is the angle between the two vectors.

If $a = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $b = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$ then,

$$a \times b = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

The resultant is a vector $a \times b$ that is mutually perpendicular to both, a and b .

3.3 Triple Products

Scalar Triple Product

$$[a \ b \ c] = (a \times b) \cdot c = a \cdot (b \times c) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

3.4 Equations of lines, planes and spheres

Equation of a line In Fig.4, the vector \mathbf{r} can be written as $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$

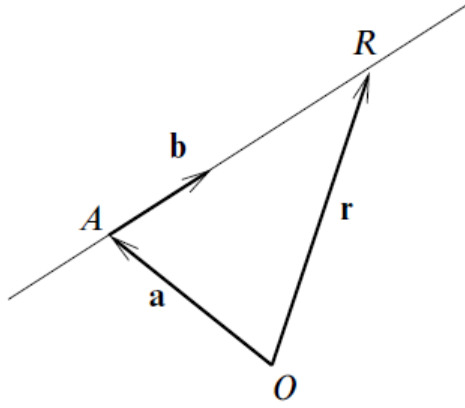


Figure 4: The equation of a line. The vector \mathbf{b} is in the direction AR and $\lambda\mathbf{b}$ is the vector from A to R .

General Equation of a line:

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

Parametric Equation of a line:

$$x = x_1 + at \qquad y = y_1 + bt \qquad z = z_1 + ct$$

Where t is a parameter

4 Limits

4.1 Precise Definition of a Limit

Definition 3 (Limit of a function) Let $f(x)$ be a function defined on an open interval around x_0 . We say that the limit of $f(x)$ as x approaches x_0 is L , i.e. $\lim_{x \rightarrow x_0} f(x) = L$, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

4.2 Common Limits

$$\begin{array}{lll} \lim_{x \rightarrow \infty} \left(1 + \frac{k}{x}\right)^{mx} = e^{mk} & \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e & \lim_{x \rightarrow \infty} \left(1 - \frac{1}{x}\right)^x = \frac{1}{e} \\ \lim_{x \rightarrow \infty} \left(\frac{x}{x+k}\right)^x = \frac{1}{e^k} & \lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e & \lim_{x \rightarrow 0} \left(\frac{a^x - 1}{x}\right) = \ln a \\ \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 & \lim_{x \rightarrow 0} \frac{\sin ax}{bx} = \frac{a}{b} & \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0 \end{array}$$

4.3 L'Hôpital's Rule

Theorem 1 (L'Hôpital's Rule) Suppose f and g are differentiable functions such that

1. $g'(x) \neq 0$ on an open interval I containing a ;
2. $\lim_{x \rightarrow a} f(x) = 0$ and $\lim_{x \rightarrow a} g(x) = 0$, or $\lim_{x \rightarrow a} f(x) = \pm\infty$ and $\lim_{x \rightarrow a} g(x) = \pm\infty$;
3. $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$ exists.

Then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}.$$

5 Sequences and Series

5.1 Common Summation Formulae

$$\begin{aligned} \sum_{i=1}^n i &= \frac{n(n+1)}{2} & \sum_{i=1}^n i^2 &= \frac{n(n+1)(2n+1)}{6} \\ \sum_{i=1}^n i^3 &= \frac{n^2(n+1)^2}{4} & \sum_{i=1}^n r^i &= \frac{1-r^{n+1}}{1-r} \end{aligned}$$

5.2 Geometric Series

The Geometric Series

$$S = \sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 \dots$$

Is convergent if $|r| < 1$ and,

$$S = \frac{a}{1-r}$$

5.3 Power-Series

The power series

$$\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges for $p > 1$

5.4 Series Convergence Tests

5.5 Common Power Series

$$\begin{aligned}e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \\ \sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \\ \cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\end{aligned}$$

6 Differential Calculus

6.1 Formal Definition of a derivative

Definition 4 (Limit Definition of a derivative) *The derivative of a continuous function $f(x)$ on an interval I is defined as the limit of the difference quotient*

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Where Δx is a small change in x

6.2 Derivative Rules and Properties

The differential operator is a linear operator, i.e.

$$\begin{aligned}(f(x) \pm g(x))' &= f'(x) \pm g'(x) \\ (c \cdot f(x))' &= c \cdot f'(x)\end{aligned}$$

Product Rule

$$(f(x) \cdot g(x))' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

Quotient Rule:

$$\left(\frac{f(x)}{g(x)} \right)' = \frac{f'(x) \cdot g(x) - g'(x) \cdot f(x)}{(g(x))^2}$$

Chain Rule

$$(f(g(x)))' = f'(g(x)) \cdot g'(x)$$

Theorem 2 (Leibnitz' Theorem) *For a function $f(x) = u(x)v(x)$, the n^{th} derivative, $f^{(n)}(x)$ is given by*

$$f^{(n)} = \sum_{r=0}^n {}^nC_r u^{(r)} v^{(n-r)}$$

6.3 Common Derivatives

$\frac{d}{dx} a^x = (\ln a) a^x$	$\frac{d}{dx} \log_a x = \frac{1}{x \ln a}$	
$\frac{d}{dx} \sin x = \cos x$	$\frac{d}{dx} \cos x = -\sin x$	$\frac{d}{dx} \tan x = \sec^2 x$
$\frac{d}{dx} \csc x = -\csc x \cot x$	$\frac{d}{dx} \sec x = \sec x \tan x$	$\frac{d}{dx} \cot x = -\csc^2 x$
$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$
$\frac{d}{dx} \sec^{-1} x = \frac{1}{x\sqrt{x^2-1}}$	$\frac{d}{dx} \csc^{-1} x = -\frac{1}{x\sqrt{x^2-1}}$	$\frac{d}{dx} \cot^{-1} x = -\frac{1}{1+x^2}$

6.4 Mean Value Theorem

Theorem 3 (Mean Value Theorem) *If a function $f(x)$ is continuous and differentiable in the range (a, c) , then there exists atleast one value b , $a < b < c$, such that*

$$f'(b) = \frac{f(c) - f(a)}{c - a}$$

6.5 Applications of Derivatives

6.5.1 Tangent to a Curve

Tangent to a curve $f(x)$ at a point a :

$$\left. \frac{d}{dx} f(x) \right|_{x=a}$$

For a straight line passing through points (x_1, y_1) and (x_2, y_2) , the slope, m , is constant and is calculated by:

$$m = \tan \theta = \frac{y_2 - y_1}{x_2 - x_1}$$

Where θ is the angle of the line with the x-axis

6.5.2 Analysis of a Curve

Critical Points $x = a$ is a critical point of $f(x)$ if $f'(a) = 0$ or $f'(a)$ doesn't exist.

Slope

1. $f(x)$ is increasing on an interval I if $f'(x) > 0$, i.e. it has a positive slope on that interval.
2. $f(x)$ is decreasing on an interval I if $f'(x) < 0$, i.e. it has a negative slope on that interval.
3. $f(x)$ is constant on an interval I if $f'(x) = 0$.

Concavity

1. $f(x)$ is concave up on an interval I if $f''(x) > 0$.
2. $f(x)$ is concave down on an interval I if $f''(x) < 0$.

Inflection Points $x = a$ is an inflection point of $f(x)$ if the concavity changes at $f(a)$.

Extrema $f(a)$ is a stationary point on an interval I if $f'(a) = 0$.

1. If $f''(a) > 0$, then $f(a)$ is a local minimum.
2. If $f''(a) < 0$, then $f(a)$ is a local maximum.
3. If $f''(a) = 0$, then the second derivative test fails.

6.5.3 Taylor Polynomials

Definition 5 (Taylor Polynomial) Let $f(x)$ be a real-valued function that is infinitely differentiable at $x = x_0$. The Taylor series expansion for the function $f(x)$ centered around the point $x = x_0$ is given by

$$\sum_{n=0}^{\infty} f^{(n)}(x_0) \frac{(x - x_0)^n}{n!}$$

Where $f^{(n)}(x_0)$ is the n^{th} derivative of $f(x)$ at $x = x_0$.

7 Integral Calculus

7.1 Fundamental Theorem of Calculus

Theorem 4 (First Fundamental Theorem of Calculus) If f is continuous on $[a, b]$, then the function defined by

$$S(x) = \int_a^x f(t) dt$$

is continuous on $[a, b]$ and differentiable on (a, b) , and $S'(x) = f(x)$.

Written in Leibniz notation,

$$\frac{d}{dx} \int_a^x f(t) dt = f(x)$$

Theorem 5 (Second Fundamental Theorem of Calculus) If f is a continuous function on $[a, b]$, then

$$\int_a^b f(x) dx = F(b) - F(a)$$

where F is the anti-derivative of f , i.e. $F' = f$.

7.2 Common Antiderivatives and Integrals

Antiderivatives

$$\begin{aligned}\int \frac{1}{x} dx &= \ln |x| + c & \int \frac{1}{ax+b} dx &= \frac{1}{a} \ln |ax+b| + c \\ \int \cos ax dx &= \frac{1}{a} \sin ax + c & \int \sin ax dx &= -\frac{1}{a} \cos ax + c & \int \sec^2 x dx &= \end{aligned}$$

8 Laplace and Fourier Transforms

9 Vector Calculus

9.1 Gradient

9.2 Curl

9.3 Divergence

10 Calculus of Variations

10.1 Euler - Lagrange Equation

11 Ordinary Differential Equations

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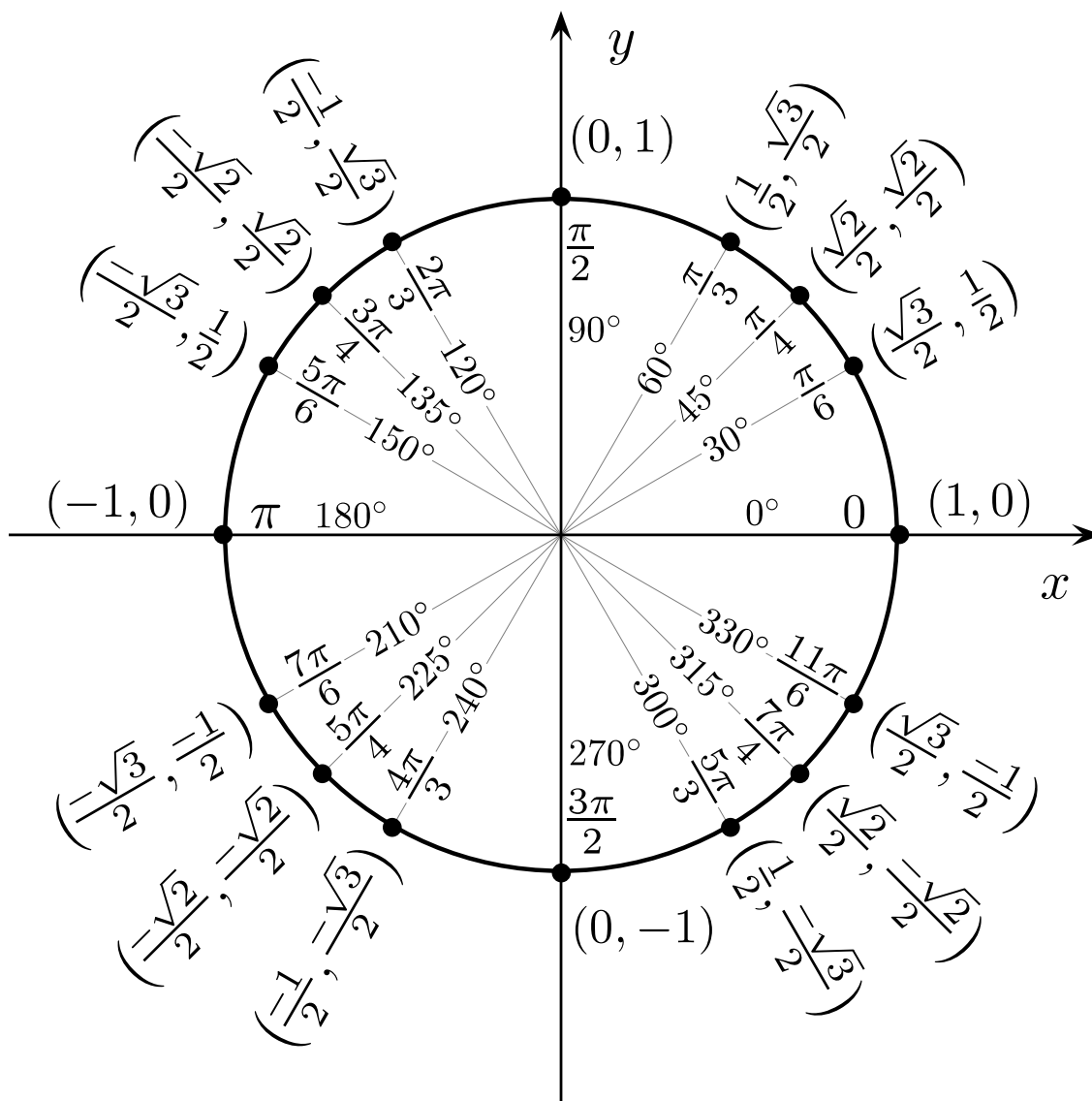


Figure 1: Unit Circle