Undergraduate Mathematics Formulae and Identities

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Contents

1 Preliminary Algebra

1.1 Coordinate Geometry

Equation of a line

$$y = mx + c$$

Equation of a circle

$$(x-h)^2 + (y-k)^2 = r^2$$

Equation of a parabola

$$y - k = a(x-h)^2 \qquad\qquad x - h = a(y-k)^2$$

Equation of an ellipse

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

Equation of a hyperbola

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

1.2 Logarithms and Exponentials

Properties of Exponentials

$$a^n a^m = a^{n+m} \qquad\qquad \frac{a^n}{a^m} = a^{n-m}$$

$$(ab)^n = a^n b^n \qquad\qquad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

$$(a^n)^m = a^{nm} \qquad\qquad \left(\frac{a}{b}\right)^{-n} = \frac{b}{a}^n$$

Properties of Logarithms

$$\log_a a = 1 \qquad\qquad \log_a 1 = 0$$

$$(\log_a b)(\log_b c) = \log_a c \qquad\qquad \log_a b = \frac{1}{\log_b a}$$

$$\log_a(xy) = \log_a x + \log_a y \qquad\qquad \log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$$

1.3 Binomial Expansion

$$^nC_k = \frac{n!}{k!(n-k)!} = \binom{n}{k}$$

Also,

$$\begin{aligned}
^nC_0 &= {}^nC_n = 1 \\
^nC_1 &= {}^nC_{n-1} = n \\
^nC_k &= {}^nC_{n-k}
\end{aligned}$$

For a binomial expression raised to the power n:

$$(x+y)^n = \sum_{k=0}^{k=n} {}^nC_k x^{n-k} y^k$$

Pascal's Triangle

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1 \\
1 \quad 9 \quad 36 \quad 84 \quad 126 \quad 126 \quad 84 \quad 36 \quad 9 \quad 1 \\
1 \quad 10 \quad 45 \quad 120 \quad 210 \quad 252 \quad 210 \quad 120 \quad 45 \quad 10 \quad 1
\end{array}$$

Commonly Factored Polynomials

$$\begin{array}{lcl}
x^2 - y^2 & = & (x-y)(x+y) \\
(x+y)^2 & = & x^2 + 2xy + y^2 \\
(x-y)^2 & = & x^2 - 2xy + y^2 \\
x^3 - y^3 & = & (x-y)(x^2 + xy + y^2) \\
x^3 + y^3 & = & (x+y)(x^2 - xy + y^2) \\
(x+y)^3 & = & x^3 + 3x^2y + 3xy^2 + y^3 \\
(x-y)^3 & = & x^3 - 3x^2y + 3xy^2 - y^3
\end{array}$$

2 Trigonometry

2.1 Common Values and the Unit Circle

θ°	0°	30°	45°	60°	90°
θ^c	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	N/D

Table 1: Commonly used trigonometric values. Refer the Unit Circle

2.2 Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1 \qquad \sec^2 x - \tan^2 x = 1$$

$$\csc^2 x - \cot^2 x = 1$$

2.3 Double Angle Formulae

$$\sin(2x) = 2\sin x\cos x \qquad \tan(2x) = \frac{2\tan x}{1-\tan^2 x}$$

$$\begin{aligned}\cos(2x) &= 2\cos^2 x - 1\\ &= \cos^2 x - \sin^2 x\\ &= 1 - 2\sin^2 x\end{aligned}$$

2.4 Sum and Difference Formulae

$$\sin(x \pm y) = \sin x\cos y \pm \cos x\sin y \qquad \cos(x \pm y) = \cos x\cos y \mp \sin x\sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x\tan y}$$

$$\begin{aligned}\arcsin x \pm \arcsin y &= \arcsin(x\sqrt{1-y^2} \pm y\sqrt{1-x^2})\\ \arccos x \pm \arccos y &= \arccos(xy \mp \sqrt{(1-x^2)(1-y^2)})\\ \arctan x \pm \arctan y &= \arctan(\frac{x \pm y}{1 \mp xy})\end{aligned}$$

2.5 Sum to Product Formulae

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) \qquad \sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$
$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) \qquad \cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

2.6 Product to Sum Formulae

$$\sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y \qquad \cos(x+y)\cos(x-y) = \cos^2 x - \sin^2 y$$

2.7 Laws of Sines and Cosines

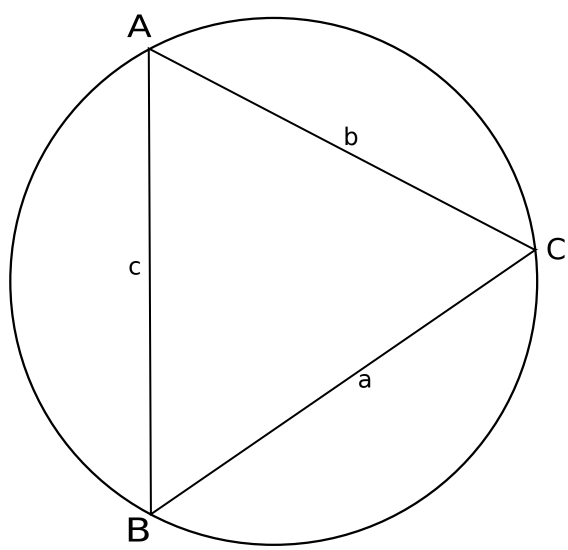

Figure 2: ΔABC in a circumscribed circle of circumradius r

Law of Sines

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \frac{1}{2r}$$

Law of Cosines

$$a^2 = b^2 + c^2 - 2bc\cos A \qquad A = \cos^{-1}\left(\frac{b^2+c^2-a^2}{2bc}\right)$$

$$b^2 = a^2 + c^2 - 2ac\cos B \qquad A = \cos^{-1}\left(\frac{a^2+c^2-b^2}{2ac}\right)$$

$$c^2 = a^2 + b^2 - 2ab\cos C \qquad C = \cos^{-1}\left(\frac{a^2+b^2-c^2}{2ab}\right)$$

2.8 Inverse Trigonometric Functions

$$\sin^{-1}(-x) = -\sin^{-1}x \qquad \cos^{-1}(-x) = \pi - \cos^{-1}x, \qquad |x| \le 1$$
$$\tan^{-1}(-x) = -\tan^{-1}x \qquad \cot^{-1}(-x) = \pi - \cot^{-1}x, \qquad x \in \mathbf{R}$$
$$\csc^{-1}x = \sin^{-1}\left(\frac{1}{x}\right) \qquad \sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right), \qquad |x| \ge 1$$
$$\cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right), \qquad x > 0$$
$$\cot^{-1}x = \pi + \tan^{-1}\left(\frac{1}{x}\right), \qquad x < 0$$
$$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}, \qquad |x| \le 1$$
$$\csc^{-1}x + \sec^{-1}x = \frac{\pi}{2}, \qquad |x| \ge 1$$

2.9 Trigonometry in the Complex Plane

Euler's Formula

$$re^{i\theta} = r(\cos\theta + i\sin\theta)$$

De Moivre's Formula

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

Exponential Definition of Trigonometric Functions

$$\cos(ix) = \frac{(e^x + e^{-x})}{2} \qquad \sin(ix) = i\frac{(e^x - e^{-x})}{2} \qquad \tan(ix) = i\frac{(e^x - e^{-x})}{(e^x + e^{-x})}$$

Exponential Definition of Hyperbolic Functions

$$\cosh(x) = \frac{(e^x + e^{-x})}{2} \qquad \sinh(x) = \frac{(e^x - e^{-x})}{2} \qquad \tanh(x) = \frac{(e^x - e^{-x})}{(e^x + e^{-x})}$$

Relationship between hyperbolic and trigonometric functions

$$\cosh x = \cos ix \qquad \cos x = \cosh ix$$
$$i \sinh x = \sin ix \qquad i \sin x = \sinh ix$$

2.10 Hyperbolic Identities

$$\cosh^2 x - \sinh^2 x = 1 \qquad \operatorname{sech}^2 x + \tanh^2 x = 1 \qquad \operatorname{csch}^2 x + \coth = 1$$
$$\sinh(2x) = 2 \sinh x \cosh x \qquad \cosh(2x) = \cosh^2 x + \sinh^2 x \qquad \sinh x + \cosh x = e^x$$

2.11 Inverse Hyperbolic Functions

$$\cosh^{-1} x = \ln(\sqrt{1+x^2} + x) \qquad \sinh^{-1} x = \ln(\sqrt{1+x^2} + x)$$
$$\begin{aligned} \tanh^{-1} x &= \ln \sqrt{\frac{1+x}{1-x}} \\ &= \frac{1}{2} \ln \frac{1+x}{1-x} \end{aligned}$$

3 Vector Algebra

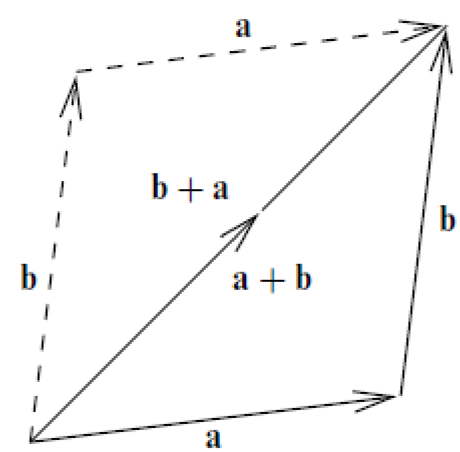

Figure 3: Addition of two vectors using Parallelogram Law

3.1 Scalar Product

Definition 1 (Dot Product) *For two vectors a and b, the scalar product, or the dot product, is defined as:*

$$\begin{aligned} a \cdot b &= |a||b| \cos\theta \\ &= \langle a|b\rangle \end{aligned}$$

Where θ is the angle between the two vectors.

If $a = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k}$ and $b = x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k}$ then,

$$a \cdot b = (x_1 x_2)\hat{\imath} + (y_1 y_2)\hat{\jmath} + (z_1 z_2)\hat{k}$$

If $a \perp b$, then $a \cdot b = 0$ and, if $a \parallel b$, then $a \cdot b = |a||b|$

3.2 Vector Product

Definition 2 (Cross Product) *For two vectors a and b, the vector product, or the cross product, is defined as:*

$$\begin{aligned} a \times b &= |a||b| \sin\theta \\ &= |b\rangle\langle a| \end{aligned}$$

Where θ is the angle between the two vectors.

If $a = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k}$ and $b = x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k}$ then,

$$a \times b = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

The resultant is a vector $a \times b$ that is mutually perpendicular to both, a and b.

3.3 Triple Products

Scalar Triple Product

$$[a\ b\ c] = (a \times b) \cdot c = a \cdot (b \times c) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

3.4 Equations of lines, planes and spheres

Equation of a line In Fig.4, the vector $\mathbf{r}$ can be written as $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$

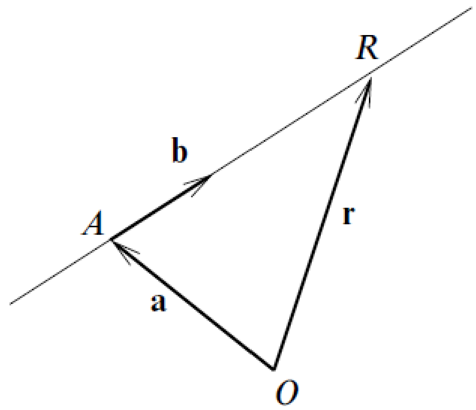

Figure 4: The equation of a line. The vector b is in the direction AR and λb is the vector from A to R.

General Equation of a line:

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

Parametric Equation of a line:

$$x = x_1 + at \qquad y = y_1 + bt \qquad z = z_1 + ct$$

Where t is a parameter

Equation of a Plane

4 Limits

4.1 Precise Definition of a Limit

Definition 3 (Limit of a function) *Let $f(x)$ be a function defined on an open interval around x_0. We say that the limit of $f(x)$ as x approaches x_0 is L, i.e. $\lim_{x\to x_0} f(x) = L$, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all x*

$$0 < |x - x_0| < \delta \;\Rightarrow\; |f(x) - L| < \varepsilon.$$

4.2 Common Limits

$$\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^{mx} = e^{mk} \qquad \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = e \qquad \lim_{x\to\infty}\left(1-\frac{1}{x}\right)^{x} = \frac{1}{e}$$

$$\lim_{x\to\infty}\left(\frac{x}{x+k}\right)^{x} = \frac{1}{e^k} \qquad \lim_{x\to 0}(1+x)^{\frac{1}{x}} = e \qquad \lim_{x\to 0}\left(\frac{a^x-1}{x}\right) = \ln a$$

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 \qquad \lim_{x\to 0}\frac{\sin ax}{bx} = \frac{a}{b} \qquad \lim_{x\to 0}\frac{1-\cos x}{x} = 0$$

4.3 L'Hôpital's Rule

Theorem 1 (L'Hôpital's Rule) *Suppose f and g are differentiable functions such that*

1. $g'(x) \neq 0$ *on an open interval* I *containing* a;
2. $\lim_{x\to a} f(x) = 0$ *and* $\lim_{x\to a} g(x) = 0$, *or* $\lim_{x\to a} f(x) = \pm\infty$ *and* $\lim_{x\to a} g(x) = \pm\infty$;
3. $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ *exists.*

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

5 Sequences and Series

5.1 Common Summation Formulae

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} \qquad \sum_{i=1}^{n} r^i = \frac{1-r^{n+1}}{1-r}$$

5.2 Geometric Series

The Geometric Series

$$S = \sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 \dots$$

Is convergent if $|r| < 1$ and,

$$S = \frac{a}{1-r}$$

5.3 Power-Series

The power series

$$\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges for $p > 1$

5.4 Series Convergence Tests

5.5 Common Power Series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$
$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$
$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

6 Differential Calculus

6.1 Formal Definition of a derivative

Definition 4 (Limit Definition of a derivative) *The derivative of a continuous function $f(x)$ on an interval I is defined as the limit of the difference quotient*

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Where Δx is a small change in x

6.2 Derivative Rules and Properties

The differntial operator is a linear operator, i.e.

$$(f(x) \pm g(x))' = f'(x) \pm g'(x)$$
$$(c \cdot f(x)' = c \cdot f'(x)$$

Product Rule

$$(f(x) \cdot g(x))' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

Quotient Rule:

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x) \cdot g(x) - g'(x) \cdot f(x)}{(g(x))^2}$$

Chain Rule

$$(f(g(x)))' = f'(g(x)) \cdot g'(x)$$

Theorem 2 (Leibnitz' Theorem) *For a function $f(x) = u(x)v(x)$, the n^{th} derivative, $f^{(n)}(x)$ is given by*

$$f^{(n)} = \sum_{r=0}^{n} {}^nC_r u^{(r)} v^{(n-r)}$$

6.3 Common Derivatives

$$\frac{\mathrm{d}}{\mathrm{d}x}a^x = (\ln a)a^x \qquad \frac{\mathrm{d}}{\mathrm{d}x}\log_a x = \frac{1}{x\ln a}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x \qquad \frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x \qquad \frac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2 x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\csc x = -\csc x\cot x \qquad \frac{\mathrm{d}}{\mathrm{d}x}\sec x = \sec x\tan x \qquad \frac{\mathrm{d}}{\mathrm{d}x}\cot x = -\csc^2 x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1} x = \frac{1}{1+x^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\sec^{-1} x = \frac{1}{x\sqrt{x^2-1}} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\csc^{-1} x = -\frac{1}{x\sqrt{x^2-1}} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\cot^{-1} x = -\frac{1}{1+x^2}$$

6.4 Mean Value Theorem

Theorem 3 (Mean Value Theorem) *If a function $f(x)$ is continuous and differentiable in the range (a, c), then there exists atleast one value b, $a < b < c$, such that*

$$f'(b) = \frac{f(c) - f(a)}{c - a}$$

6.5 Applications of Derivatives

6.5.1 Tangent to a Curve

Tangent to a curve $f(x)$ at a point a:

$$\left.\frac{\mathrm{d}}{\mathrm{d}x}f(x)\right|_{x=a}$$

For a straight line passing through points $(\mathrm{x}_1, \mathrm{y}_1)$ and $(\mathrm{x}_2, \mathrm{y}_2)$, the slope, m, is constant and is calculated by:

$$m = \tan\theta = \frac{y_2 - y_1}{x_2 - x_1}$$

Where θ is the angle of the line with the x-axis

6.5.2 Analysis of a Curve

Critical Points $x = a$ is a critical point of $f(x)$ if $f'(a) = 0$ or $f'(a)$ doesn't exist.

Slope

1. $f(x)$ is increasing on an interval I if $f'(x) > 0$, i.e. it has a positive slope on that interval.

2. $f(x)$ is decreasing on an interval I if $f'(x) < 0$, i.e. it has a negative slope on that interval.

3. $f(x)$ is constant on an interval I if $f'(x) = 0$.

Concavity

1. $f(x)$ is concave up on an interval I if $f''(x) > 0$.
2. $f(x)$ is concave down on an interval I if $f''(x) < 0$.

Inflection Points $x = a$ is an inflection point of $f(x)$ if the concavity changes at $f(a)$.

Extrema $f(a)$ is a stationary point on an interval I if $f'(a) = 0$.

1. If $f''(a) > 0$, then $f(a)$ is a local minimum.
2. If $f''(a) < 0$, then $f(a)$ is a local maximum.
3. If $f''(a) = 0$, then the second derivative test fails.

6.5.3 Taylor Polynomials

Definition 5 (Taylor Polynomial) *Let $f(x)$ be a real-valued function that is infinitely differentiable at $x = x_0$. The Taylor series expansion for the function $f(x)$ centered around the point $x = x_0$ is given by*

$$\sum_{n=0}^{\infty} f^{(n)}(x_0)\frac{(x-x_0)^n}{n!}$$

Where $f^{(n)}(x_0)$ is the n^{th} derivative of $f(x)$ at $x = x_0$.

6.6 Partial Derivatives

7 Integral Calculus

7.1 Fundamental Theorem of Calculus

Theorem 4 (First Fundamental Theorem of Calculus) *If f is continuous on $[a, b]$, then the function defined by*

$$S(x) = \int_a^x f(t)\,dt$$

is continuous on $[a, b]$ and differentiable on (a, b), and $S'(x) = f(x)$.

Written in Leibniz notation,

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,dt = f(x)$$

Theorem 5 (Second Fundamental Theorem of Calculus) *If f is a continuous function on $[a, b]$, then*

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is the anti-derivative of f, i.e. $F' = f$.

7.2 Common Antiderivatives and Integrals

Antiderivatives

$$\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + c \qquad \int \frac{1}{ax+b}\,\mathrm{d}x = \frac{1}{a}\ln|ax+b| + c$$

$$\int \cos a\,\mathrm{d}x = \frac{1}{a}\sin ax + c \qquad \int \sin ax\,\mathrm{d}x = -\frac{1}{a}\cos ax + c \qquad \int \sec^2 x\,\mathrm{d}x =$$

8 Laplace and Fourier Transforms

8.1 Laplace Transform

Definition 6 (Laplace Transform) *The Laplace transform of a function $f(t)$, defined for all real numbers $t \geq 0$, is the function $F(s)$, which is a unilateral transform defined by*

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

where s is a complex number frequency parameter
$s = \sigma + i\omega$, with real numbers σ and ω.

8.2 Common Laplace Transforms

$$\mathcal{L}(1) = \frac{1}{s} \qquad \mathcal{L}(e^{at}) = \frac{1}{s-a} \qquad \mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}(\sin(at)) = \frac{a}{s^2+a^2} \qquad \mathcal{L}(\cos(at)) = \frac{s}{s^2+a^2} \qquad \mathcal{L}(t\sin(at)) = \frac{2as}{(s^2+a^2)^2}$$

$$\mathcal{L}(t\cos(at)) = \frac{s^2-a^2}{(s^2+a^2)^2} \qquad \mathcal{L}(\sinh(at)) = \frac{a}{s^2-a^2} \qquad \mathcal{L}(\cosh(at)) = \frac{s}{s^2-a^2}$$

$$\mathcal{L}(\delta(t-c)) = e^{-cs} \qquad \mathcal{L}(\sqrt{t}) = \frac{\sqrt{\pi}}{2s^{3/2}} \qquad \mathcal{L}(f'(t)) = sF(s) - f(0)$$

$$\mathcal{L}(f^{(n)}(t)) = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0)\cdots - sf^{n-2}(0) - f^{n-1}(0)$$

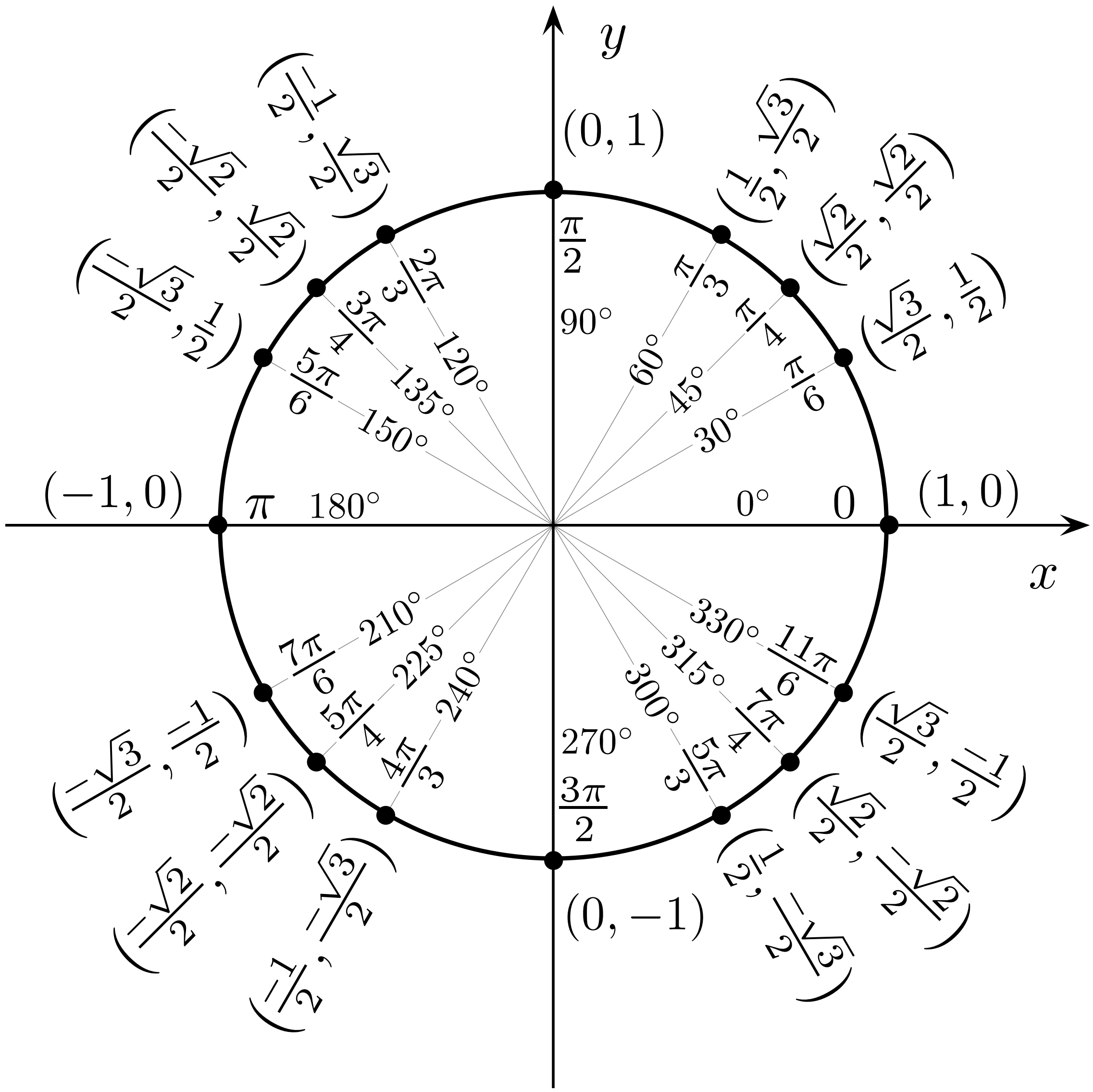

Figure 1: Unit Circle