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MAT-331 Abstract Algebra

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A Mathematical Analysis of the 15 Puzzle

The 15-puzzle is a common game consisting of 15 sliding pieces and one empty space. It typically consists of a 4×4 square with each piece labeled with numbers 1-15 respectively. However, sometimes (such as my first experience with the toy) it consists of an image broken into 16 squares and then one square removed from a corner. The object of the game is to take the puzzle from a jumbled order and return it back to its original order by sliding the pieces. It looks like this:

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & 8 \\ \hline 9 & 10 & 11 & 12 \\ \hline 13 & 14 & 15 & \\ \hline \end{array} \tag{1}$$

We will see, by modeling our arrangements with permutations, that not every arrangements is solvable, there's exactly 653837184000 unsolvable arrangements, and the permutation of the puzzle can be viewed as the group S_{16} for general arrangements.

To understand how we can model this puzzle with permutations we must first look at an example and evaluate how the pieces move, and write their movement with cycle decomposition notation. If we look at the example given by Keith Conrad in his paper "The 15-Puzzle" that describes the configuration that moves the pieces from these positions,

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & 8 \\ \hline 9 & 10 & 11 & 12 \\ \hline 13 & 14 & 15 & 16 \\ \hline \end{array} \Longrightarrow \begin{array}{|c|c|c|c|}\hline 8 & 12 & 2 & 5 \\ \hline 11 & 1 & 6 & 16 \\ \hline 7 & 14 & 10 & 15 \\ \hline 9 & 4 & 3 & 13 \\ \hline \end{array} \tag{2}$$

(where 16 is the missing piece) we can take the positions of each piece originally and compare them to the positions after the scrambling. This makes it easier to follow the movements. For the example of Equation 2, we can write these movements in this way:

$$\begin{array}{cccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 6 & 3 & 15 & 14 & 4 & 7 & 9 & 1 & 13 & 11 & 5 & 2 & 16 & 10 & 12 & 8 \end{array} \tag{3}$$

Here we can see that what was originally in position 1 is now in position 6, what was in position 6 is now in position 7, 7 in 9, and so on. We can now follow each of these numbers until it forms a cycle when 8 goes to position 1. Written in disjoint cycle decomposition, Equation 2 and 3 becomes (1, 6, 7, 9, 13, 16, 8)(2, 3, 15, 12)(4, 14, 10, 11, 5).

Now that we know how to model each movement from the identity (Equation 1) we can show that not every arrangement is solvable. To show this, I will provide another example made by Keith Conrad in the following puzzle:

$$\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & 8 \\ \hline 9 & 10 & 11 & 12 \\ \hline 13 & 14 & 15 & \\ \hline \end{array} \Longrightarrow \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & 8 \\ \hline 9 & 10 & 11 & 12 \\ \hline 13 & 15 & 14 & \\ \hline \end{array} \tag{4}$$

We see this is not possible if we model the movement from the left to the right as the transposition (14, 15). Because the empty space is in the same location after the rearrangement, the empty space had to move an even number of spaces. This is the case because if we move the empty space up one slot, we have to move it back down one slot to get it to the original placement. The same is true for the right and left movements. Thus, there must be an even number of transpositions taken in order to make the permutation (14, 15). However, (14, 15) itself is an odd permutation and therefore there is a contradiction.

In fact, it can be seen true that if the empty space (16) is found to be on any of the green spaces in Figure 3, it will take an even permutation in order to move the puzzle back to the identity solution. However, if the empty space is found on a white square, it will take an odd permutation. This is true because it will take an even number of up/down or left/right movements to get the empty piece to space 16.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

(5)

I will provide one example below that was made by Hazel Grant in his 2002 essay entitled "The Fifteen Puzzle". Taking into account that the empty space is on a green square, we know that an even permutation is required in order to solve the scrambling. Following a similar tactic in Equation 3, we will find that this puzzle, in fact, requires an odd permutation.

13	9	2	3
14	16	4	15
10	11	1	7
12	5	6	8

$\Longrightarrow$

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

(6)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
13	9	2	3	14	16	4	15	10	11	1	7	12	5	6	8

We can see that the permutation of the following solution is (16, 8, 15, 6)(14, 5)(13, 12, 7, 4, 3, 2, 9, 10, 11, 1). Because this permutation consists of three odd permutations, the entire permutation itself is odd since the product of three odd permutations is also odd. However, we stated previously that an even permutation was required if the empty space was placed on a green space. Therefore, Equation 6 is impossible to solve. It is for this very reason that Sam Loyd (a well known recreational mathematician from the 1890's) knew that those who claimed they could solve equation 4 were lying and "offered a $1000 prize (worth over $25000 today)" for anyone who could show him a solution. It was rumored that Loyd invented the 15 puzzle himself, however, according to Keith Conrad, "that is false".

We can also see that half of the possible configurations of the 15 puzzle cannot be solved. "Every movement of pieces in the 15-puzzle starting from the standard configuration that brings the empty space back to its original position must be an even permutation of the other 15 pieces". This is a proof done by Keith Conrad in his essay on the 15-puzzle. Since the number of permutations of 15 objects is 15!, there are 1307674368000 different arrangements where the empty space remains stationary. However, the number of even (or odd) permutations of 15 objects is 15!/2. Thus, there is 653837184000 unsolvable positions in which we can place the 16 pieces of the puzzle initially to where we can not get back to the identity configuration. That is pretty crazy!

Another interesting element of the 15 puzzle is that by the figure below, we can view the set of permutations of the 15 puzzle as the group S_{15}. We can see this if we view the puzzle is a sort of snake path as Tom Howe did in his essay "Two Approaches to Analyzing the Permutations of the 15 Puzzle":

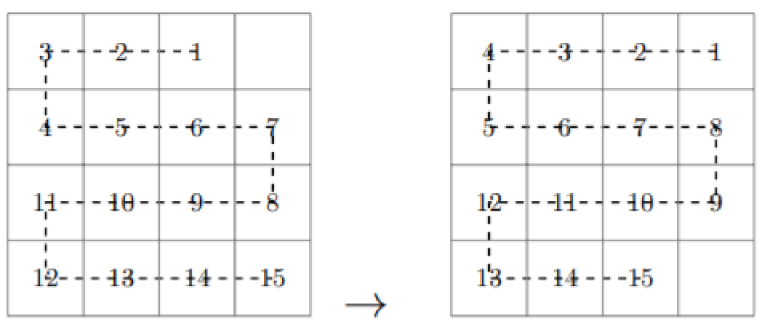
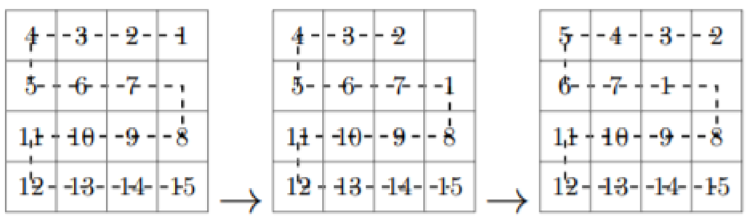


Figure 1: Typical 1-15 arrangement of the 15-puzzle. Notice that both puzzles feature the same permutation of the fifteen numbered squares despite the fact that the blank square has a different location in each configuration.

Notice that by moving the blank block along the snaking path, we can move the blank to any cell without changing the order of the remaining blocks. However, if we move the empty space up or down, we will change the permutation of the 15 remaining elements. We will see from the image that there are exactly 9 ways that the blank space can move up or down without following the snaked path. Howe lists several examples on how to find these 9, however, I will give a detailed explanation on how to find the first one here and simply list the rest for simplicity sake. In doing this, we will show there are exactly 9 permutations describing all legal moves and the set of all solvable arrangements of the 15 puzzle forms the group A_{15}.

To start, we will begin from the left side of Figure 1 and move the empty cell along the snaked line until it is in position 7. At this point we have not changed the permutation. Now we will swap the empty space with the number in position 1. This will change our permutation and place the empty space in the space labeled 1. We can then move the empty space along the snaked path from 1 to 7 again. This will not change our permutation.



The permutation that took place was (1, 7, 6, 5, 4, 3, 2). Lets call this permutation τ_1. We can follow this same procedure for all remaining 8 ways we can move up and down (not along the snaked path) and we will find the following resulting permutations:

- $\circ\ \tau_1 = (1, 7, 6, 5, 4, 3, 2)$ $\circ\ \tau_2 = (3, 5, 4)$ $\circ\ \tau_3 = (2, 6, 5, 4, 3)$
- $\circ\ \tau_4 = (7, 9, 8)$ $\circ\ \tau_5 = (6, 10, 9, 8, 7)$ $\circ\ \tau_6 = (5, 11, 10, 9, 8, 7, 6)$
- $\circ\ \tau_7 = (11, 13, 12)$ $\circ\ \tau_8 = (10, 14, 13, 12, 11)$ $\circ\ \tau_9 = (9, 15, 14, 13, 12, 11, 10)$

because these permutation arise from moving the empty space not along the snaked path, the group generated from these nine permutations generate all permutation of the puzzle. Because A_n is generated by the 3-cycles (1, 2, i) for $3 \leq i \leq n$, these nine permutations form the set A_{15}:

- $(1,2,3) = \tau_1^2\tau_2^2\tau_1^{-2}$
- $(2,3,4) = \tau_1\tau_2^2\tau_1^{-1}$
- $(3,4,5) = \tau_2^2$
- $(4,5,6) = \tau_3^{-1}\tau_2^2\tau_3$
- $(5,6,7) = \tau_6^2\tau_4^2\tau_6^{-2}$
- $(6,7,8) = \tau_6\tau_2^2\tau_6^{-1}$
- $(7,8,9) = \tau_4^2$
- $(8,9,10) = \tau_5^{-1}\tau_4^2\tau_5$
- $(9,10,11) = \tau_9^2\tau_7^2\tau_9^{-2}$
- $(10,11,12) = \tau_9\tau_7^2\tau_9^{-1}$
- $(11,12,13) = \tau_7^2$
- $(12,13,14) = \tau_8^{-1}\tau_7^2\tau_8$
- $(13,14,15) = \tau_9^{-1}(\tau_8^{-1}\tau_7^2\tau_8)\tau_9$

From this, we can see that the solvable configurations of the 15 puzzle is the group $A_{15} \subseteq S_{15}$.

The 15 Puzzle has a lot of interesting characteristics that can be modeled with permutations and the group A_{15}. As a wise man once said, “permutations are obviously super useful in counting things. Anything where rearrangements are being done, permutations are at play.” My hope, is you have a new found appreciation for them, or perhaps you now really want to play with a 15-puzzle. My only recommendation if you do, is do not remove the pieces.

1 Sources

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[4] Hazel Grant. *The Fifteen Puzzle*.The University of British Columbia, 2002. Print.

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