Leap Frog Method

$$\frac{\partial^2 u}{\partial t^2} = \gamma^2 \frac{\partial^2 u}{\partial x^2} \quad (+f). \qquad \begin{cases} u(x,0) = u_0(x) \\ \frac{\partial u}{\partial t}(x,0) = v_0(x). \end{cases} \qquad \begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases}$$

$$\left.\frac{\partial^2 u}{\partial t^2}\right|_{t^n} = \frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2} + O(\Delta t^2)$$

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{x_i} = \frac{x_{i+1} - 2x_i + x_{i-1}}{\Delta x^2} + O(\Delta x^2)$$

Combining these two, we obtain

$$(1) \qquad u_i^{n+1} - 2u_i^n + u_i^{n-1} = c^2 (u_{i+1}^n - 2u_i^n + u_{i-1}^n), \quad c^2 = \frac{\gamma^2 \Delta t^2}{\Delta x^2}$$

Also consider the centered discretization for time.

$$\left.\frac{\partial u}{\partial t}\right|_{t^n} = \frac{u^{n+1} - u^{n-1}}{2\Delta t} + O(\Delta t^2)$$

When $n=0$, $\left.\frac{\partial u}{\partial t}\right|_{t^0} = \frac{u^1 - u^{-1}}{2\Delta t} = v_0(x)$

This yields

$$u^1 - u^{-1} = 2\Delta t \, v_0(x) \qquad (*)$$

Therefore, when $n=0$, we have

$$u_i^1 - 2u_i^0 + u_i^{-1} = c^2 (u_{i+1}^0 - 2u_i^0 + u_{i-1}^0)$$

Substituting u_i^{-1} by $(*)$, we obtain

$$u_i^1 + u_i^1 - 2\Delta t\, V_{0_i}(x) = 2u_i^0 + c^2(u_{i+1}^0 - 2u_i^0 + u_{i-1}^0)$$

$$\Rightarrow \quad u_i^1 = u_i^0 + \frac{c^2}{2}(u_{i+1}^0 - 2u_i^0 + u_{i-1}^0) + \Delta t\, V_{0_i}(x),$$

which is the desired initial condition.

Further, if we write the leap frog method (1) in a matrix form, we get

$$\boxed{\underline{u}^{n+1} = (2I + c^2 A)\,\underline{u}^n - \underline{u}^{n-1}}$$

where $A = \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 \cdots & 0 \\ 0 & 1 & -2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & 1 & -2 \end{bmatrix}$.