352 Quiz 11

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First, for the θ-method, assuming that $Re(\lambda(A)) < 0$, the followings hold:

- Accuracy: all θ-methods are consistent and the order p satisfies

  $p = 1$, if $\theta \neq \frac{1}{2}$
  $p = 2$, if $\theta = \frac{1}{2}$

- Stability: the θ-method is

  unconditionally stable if $\theta \geq \frac{1}{2}$
  conditionally stable if $\theta < \frac{1}{2}$

Let us see the Space-Time Accuracy of the θ-method then.
The full discretization is

$$(I - \theta \Delta t A)u^{k+1} = (I + (1-\theta)\Delta t A)u^k + (1-\theta)(g^k + f^k) + \theta(g^{k+1} + f^{k+1})$$

where

$$A_{(n-1)\text{x}(n-1)} = \frac{1}{\Delta x^2}\begin{bmatrix} -2 & 1 & 0 & 0 & ... & 0 \\ 1 & -2 & 1 & 0 & ... & 0 \\ 0 & 1 & -2 & 1 & ... & 0 \\ ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & ... & ... & 1 & -2 \end{bmatrix}$$

$$u = u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ \vdots \\ \vdots \\ u_{n-1}(t) \end{bmatrix}, \quad g = g(t) = \frac{1}{\Delta x^2}\begin{bmatrix} g_1(t) \\ 0 \\ 0 \\ \vdots \\ \vdots \\ g_2(t) \end{bmatrix}, \quad f = f(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \\ \vdots \\ \vdots \\ f_{n-1}(t) \end{bmatrix}$$

Now, assume that the order of space discretization is q. The accuracy of the full discretization is of order

$$O(\Delta x^q + \Delta t^{p(\theta)})$$

where

$p(\theta) = 1$, if $\theta \neq \frac{1}{2}$

$p(\theta) = 2$, if $\theta = \frac{1}{2}$