352 Quiz 9

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1. The upwind scheme is: in the FD discretization of an Advection-Diffusion problem, instead of approximating u' by the 2^{nd}-order formula used in a normal discretization, we approximate u' at a certain point with only the points staying on its upwind side. For a certain point, the upwind side is the side where the drift moves from for whatever kind of drift is concerned. For instance, in our case, we set the positive direction to be the right; if $\beta > 0$, for u_i, u_{i-1} is the upwind point whereas u_{i+1} is the downwind point. $\beta < 0$ is the other way around. Since the upwind side meets the drift earlier, we put our most concentration on it.

2. The upwind scheme with an accuracy of order $O(\Delta x)$ is

$$-\mu\frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} + \frac{\beta}{\Delta x}(u_i - u_{i-1}) = 0$$

which is equivalent to

$$-\mu(1 + \mathrm{P}_e)\frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} + \frac{\beta}{2\Delta x}(u_{i+1} - u_{i-1}) = 0$$

where $\mathrm{P}_e = \frac{|\beta|\Delta x}{2\mu} = \frac{\beta\Delta x}{2\mu}$ is the Peclet number. This aligns with our normal FD discretization for an Advection-Diffusion problem in one dimension except that now we have our μ modified. If we let

$$\mu_* = \mu(1 + \mathrm{P}_e)$$

we get

$$-\mu_*\frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} + \frac{\beta}{2\Delta x}(u_{i+1} - u_{i-1}) = 0.$$

In this case our new Peclet number is

$$\mathrm{P}_e^* = \frac{\beta\Delta x}{2\mu_*} = \frac{\beta\Delta x}{2\mu}\frac{1}{(1 + \mathrm{P}_e)} = \frac{\mathrm{P}_e}{1 + \mathrm{P}_e}.$$

Notice that $\mathrm{P}_e \geq 0$ since $\beta \geq 0$. Thus, $\mathrm{P}_e^* < 1$ regardless of what Δx is. Thus, the 1^{st}-order upwind scheme is unconditionally stable.

3. For $\beta < 0$, the equation becomes

$$-\mu\frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} + \frac{\beta}{\Delta x}(u_{i+1} - u_i) = 0.$$

Note that

$$\frac{\beta}{\Delta x}(u_{i+1} - u_i) = \frac{\beta}{2\Delta x}(u_{i+1} - u_{i-1}) - \frac{|\beta|}{2\Delta x}(u_{i+1} - 2u_i + u_i).$$

since $\beta < 0$ Making substitution, we get

$$-\mu(1 + \mathrm{P}_e)\frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} + \frac{\beta}{2\Delta x}(u_{i+1} - u_{i-1}) = 0$$

where $\mathrm{P}_e = \dfrac{|\beta|\Delta x}{2\mu}$ is the Peclet number. Now if we write the equation as

$$-\mu_* \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} + \frac{\beta}{2\Delta x}(u_{i+1} - u_{i-1}) = 0.$$

The new Peclet number becomes

$$\mathrm{P}_e^* = \frac{|\beta|\Delta x}{2\mu_*} = \frac{|\beta|\Delta x}{2\mu}\frac{1}{(1 + \mathrm{P}_e)} = \frac{\mathrm{P}_e}{1 + \mathrm{P}_e}.$$

Thus, the unconditional stability also holds.