352 Quiz 10

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Classical

As it has already been seen, a second-order discretization of the space derivative is

$$\frac{\partial^2 u}{\partial x^2}(x_i) \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}$$

where $u_i = u_i(t) = u(x_i, t)$. We let $u_i^n = u(x_i, t^k)$ where $\{t^k\}$ is the set of discretization points for time and $f_i(t) = f(x_i, t)$. Also notice that $u_0 = u(0, t) = g_1(t)$, $u_n = u(1, t) = g_2(t)$, and $u^0 = u(x, 0) = u_0(x)$.

The system becomes

$$\begin{aligned}
\frac{\partial u}{\partial t}(x_1) &= \frac{du_1}{dt} = \frac{1}{\Delta x^2}(u_2 - 2u_1 + u_0) + f_1(t) \\
\frac{\partial u}{\partial t}(x_2) &= \frac{du_2}{dt} = \frac{1}{\Delta x^2}(u_3 - 2u_2 + u_1) + f_2(t) \\
&\vdots \\
&\vdots \\
\frac{\partial u}{\partial t}(x_{n-1}) &= \frac{du_{n-1}}{dt} = \frac{1}{\Delta x^2}(u_n - 2u_{n-1} + u_{n-1}) + f_{n-1}(t)
\end{aligned}$$

We rewrite this system as a matrix product

$$\frac{du}{dt} = Au + g + f$$

where

$$A_{(n-1)\mathrm{x}(n-1)} = \frac{1}{\Delta x^2}\begin{bmatrix} -2 & 1 & 0 & 0 & ... & 0 \\ 1 & -2 & 1 & 0 & ... & 0 \\ 0 & 1 & -2 & 1 & ... & 0 \\ ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & ... & ... & 1 & -2 \end{bmatrix}$$

$$u = u(t) = \begin{bmatrix} u_1(t) \\ \mathrm{u}_2(t) \\ u_3(t) \\ \vdots \\ \vdots \\ \mathrm{u}_{n-1}(t) \end{bmatrix}, \quad g = g(t) = \frac{1}{\Delta x^2}\begin{bmatrix} g_1(t) \\ 0 \\ 0 \\ \vdots \\ \vdots \\ g_2(t) \end{bmatrix}, \quad f = f(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \\ \vdots \\ \vdots \\ f_{n-1}(t) \end{bmatrix}$$

Now note that $u^k = u(t^k)$, $g^k = g(t^k)$, and $f^k = f(t^k)$. With the Explicit Euler Method approximating the time derivative, we get the discretization:

$$\frac{du^{k+1}}{dt} = \frac{du}{dt}(t^{k+1}) \approx \frac{1}{\Delta t}(u^{k+1} - u^k) = Au^{k+1} + g^{k+1} + f^{k+1}$$

which is equivalent to

$$(I - \Delta tA)u^{k+1} = u^k + \Delta t(g^{k+1} + f^{k+1}).$$

θ-Method for Time Discretization

EE:

$$\frac{du}{dt}(t^k) \approx \frac{1}{\Delta t}(u^{k+1} - u^k) = Au^k + g^k + f^k$$

IE:

$$\frac{du}{dt}(t^{k+1}) \approx \frac{1}{\Delta t}(u^{k+1} - u^k) = Au^{k+1} + g^{k+1} + f^{k+1}$$

$$(1-\theta) * \text{EE} + \theta * \text{IE} \implies$$

$$\frac{1}{\Delta t}(u^{k+1} - u^k) = (1-\theta)Au^k + \theta Au^{k+1} + (1-\theta)(g^k + f^k) + \theta(g^{k+1} + f^{k+1}) \implies$$

$$(I - \theta\Delta tA)u^{k+1} = (I + (1-\theta)\Delta tA)u^k + (1-\theta)(g^k + f^k) + \theta(g^{k+1} + f^{k+1})$$