MATH352 - FINAL PROJECT

Due Date: May 12th 2021

INSTRUCTIONS

Please, refer to the Syllabus for Instructions.

Report the code you use in the Appendix (if you are not using Jupyter, in that case, the code is in the body), report the results (pictures, tables, etc.) in the body of the report.

No page limit, but synthesis is appreciated.

THIS IS AN INDIVIDUAL WORK. ANY RESOURCE (INCLUDING ON-LINE) BEYOND THE MATERIAL ON CANVAS MUST BE CITED IN THE REPORT.

Problem 1. In this first part, we consider the Black and Scholes equation - Put Option (see Hands On session, week 7): V is the value of the Put option based on the asset with price x, E is the strike price, r the inflaction rate and σ the volatility:

$$\begin{cases} \dfrac{\partial V}{\partial \tau} + \dfrac{1}{2}\sigma^2 x^2 \dfrac{\partial^2 V}{\partial x^2} + rx\dfrac{\partial V}{\partial x} - rV = 0, & \tau \in (0,T], x \in (0, x_{\text{MAX}}) \\ V(x,T) = \max(E-x, 0), \\ V(0,\tau) = Ee^{-r(T-\tau)}, \quad V(x_{\text{MAX}}, \tau) = 0. \end{cases}$$

In the following, set $E=30$, $r=0.05$, $\sigma=0.1$, $x_{\text{MAX}}=100$, $T=1$. This is a final value problem. In the HandsOn session we converted it into a classical initial value problem with the change of variable $t=T-\tau$, so that the final time $\tau=T$ is mapped into the initial time $t=0$.

1. Write the problem in the **"divergence" initial-value** form (for $t=T-\tau$):

$$\begin{array}{ll} \dfrac{\partial V}{\partial t} - \dfrac{\partial}{\partial x}\left(\mu(x)\dfrac{\partial V}{\partial x}\right) + \beta(x)\dfrac{\partial V}{\partial x} + \rho V = 0, & t \in (0,T], x \in (0, x_{\text{MAX}}) \\ V(x,0) = \max(E-x,0), \\ V(0,t) = Ee^{-rt}, \quad V(x_{\text{MAX}}, t) = 0 \end{array},$$

   by finding appropriate coefficents $\mu(x) > 0$, $\beta(x)$ and ρ.

2. Write the weak formulation of the problem and its Finite Element approximation with the θ-method. Specify the expected convergence rates.

3. Solve the finite element approximation and compare the results with the finite difference solution found in HandsOn session Week 7. Try with different discretization parameters h and Δt. (Suggested initial values: $h=1$, $\Delta t=0.01$).

4. Report a plot of the solution $V(x,\tau=0)$. In particular, compute $V(32,\tau=0)$ knowing that the "exact" solution reads 0.1892. Comment on the accuracy of your solver as a function of h and Δt.

Problem 2. Let us consider now the 2D Black and Scholes equation Put Option, where the option with value V is based on two assets with price x and y. The equation reads for $\tau \in (0, T]$:

$$\frac{\partial V}{\partial \tau} + \frac{1}{2}\sigma_1^2 x^2 \frac{\partial^2 V}{\partial x^2} + \frac{1}{2}\sigma_2^2 y^2 \frac{\partial^2 V}{\partial y^2} + \rho\sigma_1\sigma_2 xy \frac{\partial^2 V}{\partial x \partial y} + rx\frac{\partial V}{\partial x} + ry\frac{\partial V}{\partial y} - rV = 0 \quad \text{with } (x, y) \in [0, x_{\mathrm{MAX}}] \times [0, y_{\mathrm{MAX}}]$$

with the final condition $V(x, y, T) = \max(0, E - (x + y))$ and the boundary conditions

$$\begin{cases} V(x, 0, \tau) = g(x, \tau) \\ V(x, y_{\mathrm{MAX}}, \tau) = 0 \\ V(0, y, \tau) = g(y, \tau) \\ V(x_{\mathrm{MAX}}, y, \tau) = 0 \end{cases}$$

where $g(x, \tau)$ (or $g(y, \tau)$) is the solution of the 1D Black and Scholes equation (when one of the asset is worthless, the problem reduces to the problem only for the other asset). Here, ρ is the correlation coefficient between the two assets. Data: $E = 40$, $r = 0.1$, $\sigma_1 = 0.1$, $\sigma_2 = 0.3$, $x_{\mathrm{MAX}} = y_{\mathrm{MAX}} = 100$, $T = 0.5$, $\rho = 0.5$.

PART 1: Finite Differences. Rewrite the problem as an initial value problem with the change of variable $t = T - \tau$. Write a code for the finite difference solution of the previous problem, by combining the 1D solver used in HandsOn7 for the boundary conditions g with a 2D solver. For the discretization of the mixed derivative, you can refer to the following scheme:

$$\frac{\partial^2 V}{\partial x \partial y}(x_i, y_j) \approx \frac{1}{2\Delta x}\left(\frac{\partial V}{\partial y}(x_{i+1}, y_j) - \frac{\partial V}{\partial y}(x_{i-1}, y_j)\right) \approx \frac{1}{4\Delta x \Delta y}(V_{i+1,j+1} - V_{i+1,j-1} - V_{i-1,j+1} + V_{i-1,j-1}).$$

Taking the value $V(18, 20, \tau = 0) = 2.0187$ as reference value, compare the accuracy of your solver for different values of the discretization parameters (suggested values to start with: $\Delta x = \Delta y = 1$ and $\Delta t = 0.01$).

BONUS: This is not necessary. This bonus is worth 10 points. Recalling the Taylor expansion in 2D

$$f(x + \Delta x, y + \Delta y) = f(x, y) + \frac{\partial f}{\partial x}(x, y)\Delta x + \frac{\partial f}{\partial y}(x, y)\Delta y + \frac{1}{2!}\left[\frac{\partial^2 f}{\partial x^2}(x, y)\Delta x^2 + 2\frac{\partial^2 f}{\partial x \partial y}(x, y)\Delta x \Delta y + \frac{\partial^2 f}{\partial y^2}(x, y)\Delta y^2\right] + \sum_{n=3}^{\infty} \frac{1}{n!}\sum_{k=0}^{n} \binom{n}{k} \frac{\partial^n f}{\partial x^k \partial y^{n-k}} \Delta x^k \Delta y^{n-k},$$

for $h = \Delta x = \Delta y$, find the convergence rate p of the previous formula for the mixed second derivative (i.e. the error as $\mathcal{O}(h^p)$).

PART 2: Finite Elements.

1) Verify that the problem admits the following "divergence" initial value form, for $t = T - \tau$:

$$\frac{\partial V}{\partial t} - \nabla \cdot (\mathbb{T}\nabla V) + \boldsymbol{\beta} \cdot \nabla V + rV = 0$$

where

$$\mathbb{T} \equiv \frac{1}{2}\begin{pmatrix} \sigma_1^2 x^2 & \rho\sigma_1\sigma_2 xy \\ \rho\sigma_1\sigma_2 xy & \sigma_2^2 y^2 \end{pmatrix}$$

is a matrix (or tensor) and

$$\boldsymbol{\beta} \equiv \begin{pmatrix} \left(\sigma_1^2 + \frac{1}{2}\rho\sigma_1\sigma_2 - r\right)x \\ \left(\sigma_2^2 + \frac{1}{2}\rho\sigma_1\sigma_2 - r\right)y \end{pmatrix}$$

is a vector.

Note that here $\mathbb{T}\nabla V$ denotes the matrix-vector product

$$\frac{1}{2}\begin{pmatrix} \sigma_1^2 x^2 & \rho\sigma_1\sigma_2 x y \\ \rho\sigma_1\sigma_2 x y & \sigma_2^2 y^2 \end{pmatrix}\begin{pmatrix} \frac{\partial V}{\partial x} \\ \frac{\partial V}{\partial y} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \sigma_1^2 x^2 \frac{\partial V}{\partial x} + \rho\sigma_1\sigma_2 x y \frac{\partial V}{\partial y} \\ \rho\sigma_1\sigma_2 x y \frac{\partial V}{\partial x} + \sigma_2^2 y^2 \frac{\partial V}{\partial y} \end{pmatrix}$$

and that $\nabla \cdot (\mathbb{T}\nabla V)$ stands for the divergence of the $\mathbb{T}\nabla V$ vector:

$$\nabla \cdot (\mathbb{T}\nabla V) \equiv \frac{\partial}{\partial x}\left(\frac{1}{2}\left(\sigma_1^2 x^2 \frac{\partial V}{\partial x} + \rho\sigma_1\sigma_2 x y \frac{\partial V}{\partial y}\right)\right) + \frac{\partial}{\partial y}\left(\frac{1}{2}\left(\rho\sigma_1\sigma_2 x y \frac{\partial V}{\partial x} + \sigma_2^2 y^2 \frac{\partial V}{\partial y}\right)\right).$$

2) Write the weak formulation of the problem in this divergence form. To this aim, keep in mind that for a test function $\varphi(x, y)$ vanishing on the boundary

$$-\int_\Omega \nabla \cdot (\mathbb{T}\nabla V)\varphi d\omega = \int_\Omega \mathbb{T}\nabla V \cdot \nabla\varphi d\omega$$

where

$$\mathbb{T}\nabla V \cdot \nabla\varphi \equiv \frac{1}{2}\begin{pmatrix} \sigma_1^2 x^2 \frac{\partial V}{\partial x} + \rho\sigma_1\sigma_2 x y \frac{\partial V}{\partial y} \\ \rho\sigma_1\sigma_2 x y \frac{\partial V}{\partial x} + \sigma_2^2 y^2 \frac{\partial V}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial \varphi}{\partial x} \\ \frac{\partial \varphi}{\partial y} \end{pmatrix} = \frac{1}{2}\left(\left(\sigma_1^2 x^2 \frac{\partial V}{\partial x} + \rho\sigma_1\sigma_2 x y \frac{\partial V}{\partial y}\right)\frac{\partial \varphi}{\partial x} + \left(\rho\sigma_1\sigma_2 x y \frac{\partial V}{\partial x} + \sigma_2^2 y^2 \frac{\partial V}{\partial y}\right)\frac{\partial \varphi}{\partial y}\right).$$

3) Write the finite element+$\theta-$method discretization of the problem. Specify the expected accuracy of the discretization, assuming that the solution is infinitely regular.

BONUS - This part is not necessary. It is worth 40 bonus points.

Write a Fenics code for the problem. To do so, consider that the function `inner` can perform the needed products. Alternatively, the derivative $\frac{\partial V}{\partial x}$ can be written as `V.dx(0)` while $\frac{\partial V}{\partial y}$ is `V.dx(1)` (and similar for φ). Compare the solution with the finite difference one.