

# Følner sequence

Kai Prince SFHEA

2025-08-13

## 1 Introduction

Recall the definition of a group:

**Definition 1.1** (Group). A *group* is a non-empty set  $\Gamma$  together with a binary operation on  $\Gamma$ , denoted “ $\cdot$ ”, that combines any two elements  $\gamma$  and  $\gamma'$  of  $\Gamma$  to form an element of  $\Gamma$ , denoted  $\gamma \cdot \gamma'$ , such that the following three requirements, known as *group axioms*, are satisfied:

- *Associativity*: For all  $g, h, j$  in  $\Gamma$ , one has  $(g \cdot h) \cdot j = g \cdot (h \cdot j)$ .
- *Identity*: There exists an element  $e$  in  $\Gamma$  such that, for every  $g$  in  $\Gamma$ , one has  $e \cdot g = g$  and  $g \cdot e = g$ . Such an element is unique and is called the *identity element*.
- *Unique Inverse*: For each  $g$  in  $\Gamma$ , there exists an element  $h$  in  $\Gamma$  such that  $g \cdot h = e$  and  $h \cdot g = e$ , where  $e$  is the identity element. For each  $g$ , the element  $h$  is unique and is called the *inverse* of  $h$  and is denoted  $g^{-1}$ .

**Definition 1.2.** We define a *right-Følner sequence* in a group,  $\Gamma$ , as a sequence  $\Phi = (\Phi_N)_{N \in \mathbb{N}}$  of finite subsets of  $\Gamma$  satisfying

$$\lim_{N \rightarrow \infty} \frac{|(\Phi_N \cdot \gamma^{-1}) \cap \Phi_N|}{|\Phi_N|} = 1$$

for all  $\gamma \in \Gamma$ .

**Definition 1.3.** Similarly, we define a *left-Følner sequence* in a group,  $\Gamma$ , as a sequence  $\Phi = (\Phi_N)_{N \in \mathbb{N}}$  of finite subsets of  $\Gamma$  satisfying

$$\lim_{N \rightarrow \infty} \frac{|(\gamma^{-1} \cdot \Phi_N) \cap \Phi_N|}{|\Phi_N|} = 1$$

for all  $\gamma \in \Gamma$ .

**Definition 1.4.** We call a sequence a *Følner sequence* if it is both a left and right Følner sequence.

A related definition is the following:

**Definition 1.5.** We call define *density of a subset*  $A \subseteq \Gamma$  with respect to a Følner sequence,  $\Phi$ , as

$$d_{\Phi}(A) = \lim_{N \rightarrow \infty} \frac{|\Phi_N \cap A|}{|\Phi_N|},$$

if it exists.

For  $\mathbb{N}$ , the *natural density*,  $d$ , is defined when the Følner sequence is constructed with  $\Phi_N = [1, \dots, N]$ .

## 2 Results

**Theorem 2.1** (The Test Theorem). *This is a Theorem.*



Tip

This is a test tip.

## 3 More Information

You can learn more about controlling the appearance of HTML output here:  
<https://quarto.org/docs/output-formats/html-basics.html>