Følner sequence

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1 Introduction

Recall the definition of a group:

Definition 1.1 (Group)**.** A *group* is a non-empty set Γ together with a binary operation on Γ, denoted " $\cdot$ ", that combines any two elements γ and γ' of Γ to form an element of Γ, denoted $\gamma \cdot \gamma'$, such that the following three requirements, known as *group axioms*, are satisfied:

- *Associativity:* For all g, h, j in Γ, one has $(g \cdot h) \cdot j = g \cdot (h \cdot j)$.
- *Identity:* There exists an element e in Γ such that, for every g in Γ, one has $e \cdot g = g$ and $g \cdot e = g$. Such an element is unique and is called the *identity element*.
- *Unique Inverse:* For each g in Γ, there exists an element h in Γ such that $g \cdot h = e$ and $h \cdot g = e$, where e is the identity element. For each g, the element h is unique and is called the *inverse* of h and is denoted g^{-1}.

Definition 1.2. We define a *right-Følner sequence* in a group, Γ, as a sequence $\Phi = (\Phi_N)_{N \in \mathbb{N}}$ of finite subsets of Γ satisfying

$$\lim_{N \to \infty} \frac{|(\Phi_N \cdot \gamma^{-1}) \cdot \Phi_N|}{|\Phi_N|} = 1$$

for all $\gamma \in \Gamma$.

Definition 1.3. Similarly, we define a *left-Følner sequence* in a group, Γ, as a sequence $\Phi = (\Phi_N)_{N \in \mathbb{N}}$ of finite subsets of Γ satisfying

$$\lim_{N \to \infty} \frac{|(\gamma^{-1} \cdot \Phi_N) \cap \Phi_N|}{|\Phi_N|} = 1$$

for all $\gamma \in \Gamma$.

Definition 1.4. We call a sequence a *Følner sequence* if it is both a left and right Følner sequence.

A related definition is the following:

Definition 1.5. We call define *density of a subset* $A \subseteq \Gamma$ *with respect to a Følner sequence*, Φ, as

$$d_\Phi(A) = \lim_{N\to\infty} \frac{|\Phi_N \cap A|}{|\Phi_N|},$$

if it exists.

For $\mathbb{N}$, the *natural density*, d, is defined when the Følner sequence is constructed with $\Phi_N = [1, ..., N]$.

2 Results

Theorem 2.1 (The Test Theorem)**.** *This is a Theorem.*

Tip

This is a test tip.

3 More Information

You can learn more about controlling the appearance of HTML output here: https://quarto.org/docs/output-formats/html-basics.html