Actions

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2025-08-13

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Definition 0.1 (Bekka and Mayer (2000) Section 2)**.** An *action* of a group, Γ, on a measurable space $(X, \mathscr{B})$ is a measurable mapping

$$\Gamma \times X \to X,\ (\gamma, x) \mapsto \gamma.x$$

with the following properties:

1. Associativity: For all $\gamma, \gamma' \in \Gamma, x \in X$, then $\gamma.(\gamma'.x) = (\gamma \cdot \gamma').x$
2. Identity: There exists an identity element $e \in \Gamma$ such that $e.x = x$ for all $x \in X$.
3. Quasi-Invariance: For any $B \in \mathscr{B}$ and for all $\gamma \in \Gamma$, we have $\mu(\gamma.B) = 0$ if and only if $\mu(B) = 0$.

The action of Γ is also ergodic if it satisfies the additional property:

4. If $B \in \mathscr{B}$ and $\mu(B) = \mu(\gamma.B)$ for any $\gamma \in \Gamma$, then $\mu(B) = 0$ or $\mu(X \setminus B) = 0$.

Remark 0.1. We don't require invertability in order to use actions and could instead use a monoid, M, defining the pre-image of M on $(X, \mathscr{B})$ as a measurable mapping

$$M \times X \to \mathscr{B},\ (m, x) \mapsto m^{-1}.x$$

such that

$$m^{-1}.x = \{x' \in X : m.x' = x\}.$$

Definition 0.2. A *topological dynamical system under the action of* Γ, denoted (X, Γ), is a compact metric space X that has continuous surjective maps, $(\gamma, x) \mapsto \gamma.x$, for all $\gamma \in \Gamma$.

Definition 0.3. Let $x \in X$, $\Phi = (\Phi_N)_{N \in \mathbb{N}}$ be a Følner sequence in Γ and μ a probability measure on X. Where δ_x is the Dirac mass at x, if

$$\frac{1}{|\Phi_N|} \sum_{\gamma \in \Phi_N} \delta_{\gamma.x} \underset{\text{weakly}*}{\longrightarrow} \mu \text{ as } N \to \infty,$$

then we say x *is generic for* μ *with respect to* Φ and we denote this with $x \in \mathrm{gen}(\mu, \Phi)$.[1]

We are interested in how the action of a group Γ transforms functions on $(X, \mathscr{B}, \mu)$ so we must identify the associated definitions within functional analysis.

We define the map $U_\gamma : \mathrm{L}^2(X) \to \mathrm{L}^2(X)$ where $f \mapsto f \circ U_\gamma$ as the *Koopman operator* induced by $\gamma \in \Gamma$. We find that the Koopman operator induced by Γ is a unitary operator and the group homomorphism $U : \Gamma \to \mathscr{U}(\mathrm{L}^2(X))$ is the *unitary representation* of Γ on $\mathrm{L}^2(X)$, where $\mathscr{U}(\mathrm{L}^2(X))$ is the set of all unitary operators on $\mathrm{L}^2(X)$.

Bekka, M. B. and Mayer, M. (2000). *Ergodic theory and topological dynamics of group actions on homogeneous spaces*. Cambridge University Press. https://doi.org/10.1017/cbo9780511758898.

[1]consider tempered separately as the FCP construction only depends on sequential compactness