Actions

Kai Prince SFHEA

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Definition 0.1 (Bekka and Mayer (2000) Section 2)**.** An *action* of a group, G, on a measurable space $(X, \mathscr{B})$ is a measurable mapping

$$G \times X \to X, \ (g, x) \mapsto g.x$$

with the following properties:

1. Associativity: For all $g, g' \in G, x \in X$, then $g.(g'.x) = (g \cdot g').x$
2. Identity: There exists an identity element $e \in G$ such that $e.x = x$ for all $x \in X$.
3. Quasi-Invariance: For any $B \in \mathscr{B}$ and for all $g \in G$, we have $\mu(g.B) = 0$ if and only if $\mu(B) = 0$.

The action of G is also ergodic if it satisfies the additional property:

4. If $B \in \mathscr{B}$ and $\mu(B) = \mu(g.B)$ for any $g \in G$, then $\mu(B) = 0$ or $\mu(X \setminus B) = 0$.

Remark 0.1. We don't require invertability in order to use actions and could instead use a monoid, M, defining the pre-image of M on $(X, \mathscr{B})$ as a measurable mapping

$$M \times X \to \mathscr{B}, \ (m, x) \mapsto m^{-1}.x$$

such that

$$m^{-1}.x = \{x' \in X : m.x' = x\}.$$

Definition 0.2. A *topological dynamical system under the action of* G, denoted (X, G), is a compact metric space X that has continuous surjective maps, $(g, x) \mapsto g.x$, for all $g \in G$.

Definition 0.3. Let $x \in X$, $\Phi = (\Phi_N)_{N \in \mathbb{N}}$ be a Følner sequence in Γ and μ a probability measure on X. Where δ_x is the Dirac mass at x, if

$$\frac{1}{|\Phi_N|} \sum_{g \in \Phi_N} \delta_{g.x} \underset{\text{weakly}*}{\longrightarrow} \mu \text{ as } N \to \infty,$$

then we say x *is generic for* μ *with respect to* Φ and we denote this with $x \in \mathrm{gen}(\mu, \Phi)$.[1]

We are interested in how the action of a group G transforms functions on $(X, \mathscr{B}, \mu)$ so we must identify the associated definitions within functional analysis.

We define the map $U_g : \mathrm{L}^2(X) \to \mathrm{L}^2(X)$ where $f \mapsto f \circ U_g$ as the *Koopman operator* induced by $g \in G$. We find that the Koopman operator induced by G is a unitary operator and the group homomorphism $U : G \to \mathscr{U}(\mathrm{L}^2(X))$ is the *unitary representation* of G on $\mathrm{L}^2(X)$, where $\mathscr{U}(\mathrm{L}^2(X))$ is the set of all unitary operators on $\mathrm{L}^2(X)$.

Bekka, M. B. and Mayer, M. (2000). *Ergodic theory and topological dynamics of group actions on homogeneous spaces*. Cambridge University Press. https://doi.org/10.1017/cbo9780511758898.

[1] consider tempered separately as the FCP construction only depends on sequential compactness