Density

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Definition 0.1. For $A \subseteq \Gamma$ and Følner sequence $\Phi = (\Phi_N)_{n=1}^{\infty}$, we write

$$\overline{d}_\Phi(A) = \limsup_{N\to\infty} \frac{\lambda A \cap \Phi_N}{\lambda \Phi_N}$$
$$\underline{d}_\Phi(A) = \liminf_{N\to\infty} \frac{\lambda A \cap \Phi_N}{\lambda \Phi_N}$$

to be the *upper and lower densities of A with respect to* Φ, respectively. If these agree, then we can write

$$d_\Phi(A) = \lim_{N\to\infty} \frac{\lambda A \cap \Phi_N}{\lambda \Phi_N}$$

to be the *density of A with respect to* Φ. We also define the *upper Banach density of A* by

$$\overline{d}(A) = \sup\{d_\Phi(A) : \text{for Følner sequences } \Phi \text{ where } d_\Phi(A) \text{ exists}\}.$$

Proposition 0.1 (Monotonicity)**.** *Let $A, B \subseteq M$ be subsets. For $A \subseteq B$, we have*

$$\overline{d}_\Phi(A) \leq \overline{d}_\Phi(B),$$

and

$$\underline{d}_\Phi(A) \leq \underline{d}_\Phi(B).$$

Proposition 0.2 (Translation Invariance)**.** *Let $B \subseteq M$ be a subset. For all $m \in M$, we have that*

$$\overline{d}_\Phi(B) = \overline{d}_{m\cdot\Phi}(B) = \overline{d}_\Phi(m \cdot B)$$

and

$$\underline{d}_\Phi(B) = \underline{d}_{m\cdot\Phi}(B) = \underline{d}_\Phi(m \cdot B).$$

Lemma 0.1 (Upper/Lower Density Pairwise Additive)**.** *Let $A, B \subseteq M$ be subsets such that $d_\Phi(A)$ and $d_\Phi(B)$ both exist, then we have*

$$\overline{d}_\Phi(A \cup B) + \underline{d}_\Phi(A \cap B) = d_\Phi(A) + d_\Phi(B),$$
$$\underline{d}_\Phi(A \cup B) + \overline{d}_\Phi(A \cap B) = d_\Phi(A) + d_\Phi(B).$$

Further, we have $d_\Phi(A \cap B)$ exists if and only if $d_\Phi(A \cup B)$ exists.