A Short Proof of a Generalised Conjecture of Erdős for Amenable Groups

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2025-08-06

Abstract

We follow steps provided by the paper published by (Host, 2019), 'A Short Proof of a Conjecture of Erdős Proved by Moreira, Richter and Robertson', as well as using the results provided by (Kra et al., 2022), to generalise the proof of Erdős's conjecture for amenable groups. The main result that we aim to prove is: 'every positively dense subset of an amenable group contains the group sumset of k infinite sets for every natural number k'.

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1 Introduction

Theorem 1.1 (cf. (Host, 2019), Theorem 1)**.** *Let* $(\Gamma, \cdot)$ *be an amenable group. If* $A \subseteq \Gamma$ *has positive density, then there exists infinite subsets* B *and* C *of* Γ *such that* $B \cdot C \subset A$*.*

Theorem 1.2 (cf. (Host, 2019), Proposition 2)**.** *There exists a set of positive density not containing any sumset of positive density and an infinite set.*

Theorem 1.3 (cf. (Kra et al., 2022), Theorem 1.1)**.** *Let* $(\Gamma, \cdot)$ *be an amenable group. If* $A \subseteq \Gamma$ *has positive density then, for every* $k \in \mathbb{N}$*, there are infinite subsets* $B_1, ..., B_k \subset \Gamma$ *such that* $B_1 \cdot \cdots \cdot B_k \subset A$*.*

2 Preliminaries

We will use $\mathbb{N} = \{1, 2, 3, ...\}$ and e to denote the identity of the group Γ.

2.1 Amenable Groups and Actions

Throughout this paper, unless otherwise specified, we will let $(\Gamma, \cdot)$ be a second countable discrete group. This also means that the Haar measure of Γ is the counting measure.

Definition 2.1. [Embed #def-tempered from folner.qmd]

For simplicity of this paper, we will use the alternative and equivalent definition that a group, Γ, is *amenable* (and second countable) if and only if it has a Følner sequence.

Definition 2.2. [Embed #prp-temperedFolner from folner.qmd]

Definition 2.3. [Embed #def-density from density.qmd]

2.2 Topological Dynamics of Group Actions

Definition 2.4. [Embed #def-action from actions.qmd]

Definition 2.5. [Embed #def-topologicalDynamicalSystem from actions.qmd]

Definition 2.6. [Embed #def-generic from actions.qmd]

2.3 Recurrence Results & Ergodic Theorems

Theorem 2.1. *[Embed #thm-recurrence from recurrence-and-Ergodic-theorems.qmd]*

Theorem 2.2. *[Embed #thm-GeneralisedPET from recurrence-and-Ergodic-theorems.qmd]*

2.4 Erdős Cubes & Cubic Measures

2.5 Factor Maps

2.6 Key Dynamical Results

Theorem 2.3 (cf. Host (2019), Theorem 3)**.** *Let (X, Γ) be a topological dynamical system where Γ is an amenable group, $x_0 \in X$, E be a clopen subset of X and*

$$A = \{\gamma \in \Gamma : \gamma . x_0 \in E\}.$$

Let $X \times X$ be acted upon by $\Gamma \times \Gamma$. Let $x_1 \in X$ and ν be a measure on $X \times X$ such that (x_0, x_1) is generic along some Følner sequence $\Phi = (\Phi_N)_{N \in \mathbb{N}}$.

Assume there exists $\varepsilon > 0$ and a sequence $(s_i)_{i \in \mathbb{N}}$ in Γ such that

$$s_i . x_0 \to x_1 \text{ as } i \to \infty \text{ and } \nu(s_i^{-1} . E \times E) \geq \varepsilon \text{ for all } i.$$

Then there exist infinite subsets $B, C \subseteq \Gamma$ such that $B \cdot C \subset A$.

Theorem 2.4 (cf. Kra et al. (2022), Theorem 3.5)**.** *Let (X, μ, Γ) be an ergodic system under the action of Γ and let $E \subset X$ be an open set with $\mu(E) > 0$.*

If $a \in X$ is generic for μ with respect to some Følner sequence Φ then, for every $k \in \mathbb{N}$, there exists 2-dimensional Erd"{o}s cube $\underline{x} = (x_{00}, x_{01}, x_{10}, x_{11}) \in X^{[[2]]}$ with $x_{00} = a$ and $x_{11} \in E$.

Theorem 2.5 (cf. Host (2019), Theorem 4.4)**.** *Let (X, μ, Γ) be an ergodic system under the action of Γ and let $E \subset X$ be an open set with $\mu(E) > 0$.*

If $a \in X$ is generic for μ with respect to some Følner sequence Φ then, for every $k \in \mathbb{N}$, there exists k-dimensional Erd"{o}s cubes $\underline{x} \in X^{[[k]]}$ with $x_{\vec{0}} = a$ and $x_{\vec{1}} \in E$.

3 Proof of Theorem Theorem 2.4

3.1 Furstenberg's Correspondence Princple

Theorem 3.1. *[Embed #thm-FCP from furstenberg-correspondence.qmd]*

3.2 Kronecker Factor

Proposition 3.1. *[Embed #prp-KroneckerGeneric from factor-maps.qmd]*

3.3 Choosing a point x_1

3.4 The joining ν

3.5 Proof Conclusion

4 Proof of Corollary (ref)

5 Proof of Theorem (ref)

Discussion

Host, B. (2019). '*A short proof of a conjecture of erdös proved by moreira, richter and robertson*', Available at: https://arxiv.org/abs/1904.09952.

Kra, B., et al. (2022). '*Infinite sumsets in sets with positive density*', Available at: https://arxiv.org/abs/2206.01786.