

# Actions

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**Definition 0.1** (Bekka and Mayer (2000) Section 2). An *action* of a group,  $G$ , on a measurable space  $(X, \mathcal{B})$  is a measurable mapping

$$G \times X \rightarrow X, (g, x) \mapsto g.x$$

with the following properties:

1. Associativity: For all  $g, g' \in G, x \in X$ , then  $g.(g'.x) = (g \cdot g').x$
2. Identity: There exists an identity element  $e \in G$  such that  $e.x = x$  for all  $x \in X$ .
3. Quasi-Invariance: For any  $B \in \mathcal{B}$  and for all  $g \in G$ , we have  $\mu(g.B) = 0$  if and only if  $\mu(B) = 0$ .

The action of  $G$  is also ergodic if it satisfies the additional property:

4. If  $B \in \mathcal{B}$  and  $\mu(B) = \mu(g.B)$  for any  $g \in G$ , then  $\mu(B) = 0$  or  $\mu(X \setminus B) = 0$ .

**Definition 0.2.** A *topological dynamical system under the action of  $G$* , denoted  $(X, G)$ , is a compact metric space  $X$  that has continuous surjective maps,  $(g, x) \mapsto g.x$ , for all  $g \in G$ .

**Definition 0.3.** Let  $x \in X$ ,  $\Phi = (\Phi_N)_{N \in \mathbb{N}}$  be a Følner sequence in  $\Gamma$  and  $\mu$  a probability measure on  $X$ . Where  $\delta_x$  is the Dirac mass at  $x$ , if

$$\frac{1}{|\Phi_N|} \sum_{g \in \Phi_N} \delta_{g.x} \xrightarrow{\text{weakly}^*} \mu \text{ as } N \rightarrow \infty,$$

then we say  $x$  is *generic for  $\mu$  with respect to  $\Phi$*  and we denote this with  $x \in \text{gen}(\mu, \Phi)$ .<sup>1</sup>

Bekka, M. B. and Mayer, M. (2000). *Ergodic theory and topological dynamics of group actions on homogeneous spaces*. Cambridge University Press. <https://doi.org/10.1017/cbo9780511758898>.

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<sup>1</sup>consider tempered separately as the FCP construction only depends on sequential compactness