Recurrence and Ergodic Theorems

Kai Prince SFHEA

2025-07-29

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Theorem 0.1 (Bergelson (1985), Theorem 1.1)**.** *Let $(X, \mathscr{B}, \mu)$ be a probability space and suppose that $B_n \in \mathscr{B}$ such that $\mu(B_n) = b > 0$ for all $n \in \mathbb{N}$.*

Then there exists a positively dense index set $I \subset \mathbb{N}$ such that, for any finite subset $F \subseteq I$, we have

$$\mu\left(\bigcap_{i \in F} B_i\right) > 0.$$

Theorem 0.2 (cf. Lindenstrauss (2001), Theorem 1.2)**.** *Let Γ be a discrete amenable group acting on a measure space $(X, \mathscr{B}, \mu)$ by measure preserving transformation and let $\Phi = (\Phi_N)_{N \in \mathbb{N}}$ be a tempered Følner sequence.*

Then, for any $f \in L^1(\mu)$, there is a Γ-invariant $\bar{f} \in L^1(\mu)$ such that

$$\lim_{N \to \infty} \frac{1}{| \cdot | \Phi_N} \sum_{\gamma \in \Phi_N} f(\gamma . x) = \bar{f}(x)$$

for μ-almost every $x \in X$. In particular, if the Γ action is ergodic, then

$$\lim_{N \to \infty} \frac{1}{| \cdot | \Phi_N} \sum_{\gamma \in \Phi_N} f(\gamma . x) = \int f(x) \, d\mu(x)$$

for μ almost every x.

Corollary 0.1 (cf. Host (2019), Corollary 8)**.** *Let (X, Γ) be a topological dynamical system where Γ is an amenable group, μ an ergodic measure on X and Φ a tempered Følner sequence. Then μ-almost every $x \in X$ is generic for μ along Φ.*

Bergelson, V. (1985). 'Sets of recurrence of zm-actions and properties of sets of differences in zm', Journal of the London Mathematical Society, s2-31 (2), pp. 295–304. https://doi.org/10.1112/jlms/s2-31.2.295.

Lindenstrauss, E. (2001). 'Pointwise theorems for amenable groups', Inventiones mathematicae, 146 (2), pp. 259–295. https://doi.org/10.1007/s002220100162.

Host, B. (2019). '*A short proof of a conjecture of erdös proved by moreira, richter and robertson*', Available at: https://arxiv.org/abs/1904.09952.