Density

Kai Prince SFHEA

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Definition 0.1. For $A \subseteq G$ and Følner sequence $\Phi = (\Phi_N)_{n=1}^{\infty}$, we write

$$_{\Phi}(A) = \limsup_{N\to\infty} \frac{|\cdot|A \cap \Phi_N}{|\cdot|\Phi_N}$$
$$_{\Phi}(A) = \liminf_{N\to\infty} \frac{|\cdot|A \cap \Phi_N}{|\cdot|\Phi_N}$$

to be the *upper and lower densities of A with respect to Φ*, respectively. If these agree, then we can write

$$d_{\Phi}(A) = \lim_{N\to\infty} \frac{|\cdot|A \cap \Phi_N}{|\cdot|\Phi_N}$$

to be the *density of A with respect to Φ*. We also define the *upper Banach density of A* by

$$(A) = \sup\{d_{\Phi}(A) : \text{for Følner sequences } \Phi \text{ where } d_{\Phi}(A) \text{ exists}\}.$$

Proposition 0.1 (Monotonicity)**.** *Let $A, B \subseteq M$ be subsets. For $A \subseteq B$, we have*

$$_{\Phi}(A) \leq_{\Phi} (B),$$

and

$$_{\Phi}(A) \leq_{\Phi} (B).$$

Proposition 0.2 (Translation Invariance)**.** *Let $B \subseteq M$ be a subset. For all $m \in M$, we have that*

$$_{\Phi}(B) =_{m\cdot\Phi} (B) =_{\Phi} (m \cdot B)$$

and

$$_{\Phi}(B) =_{m\cdot\Phi} (B) =_{\Phi} (m \cdot B).$$

Lemma 0.1 (Upper/Lower Density Pairwise Additive)**.** *Let $A, B \subseteq M$ be subsets such that $d_\Phi(A)$ and $d_\Phi(B)$ both exist, then we have*

$$
\begin{aligned}
{}_\Phi(A \cup B) +_\Phi (A \cap B) &= d_\Phi(A) + d_\Phi(B), \\
{}_\Phi(A \cup B) +_\Phi (A \cap B) &= d_\Phi(A) + d_\Phi(B).
\end{aligned}
$$

Further, we have $d_\Phi(A \cap B)$ exists if and only if $d_\Phi(A \cup B)$ exists.