Følner sequence

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2025-08-06

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1 Følner Sequences

Definition 1.1. We define a *right-Følner sequence* in G as a sequence $\Phi = (\Phi_N)_{N\in\mathbb{N}}$ of finite subsets of G satisfying

$$\lim_{N\to\infty} \frac{|\cdot|(\Phi_N \cdot g^{-1}) \cdot \Phi_N}{|\cdot|\Phi_N} = 1,$$

for all $g \in G$.

Definition 1.2. Similarly, we define a *left-Følner sequence* in G as a sequence $\Phi = (\Phi_N)_{N\in\mathbb{N}}$ of finite subsets of G satisfying

$$\lim_{N\to\infty} \frac{|\cdot|(g^{-1} \cdot \Phi_N) \cap \Phi_N}{|\cdot|\Phi_N} = 1,$$

for all $g \in G$.

Definition 1.3. We call a sequence in G a *Følner sequence* if it is both a left and right Følner sequence.

1.1 Alternative definitions for Monoids

Definition 1.4. Let M be a countably-infinite left-cancellative monoid with discrete topology. We define a *left-Følner sequence* in M as a sequence of finite subsets $\Phi = (\Phi_N)_{N\in\mathbb{N}}$ satisfying

$$\lim_{N\to\infty} \frac{|\cdot|(m \cdot \Phi_N) \cdot \Phi_N}{|\cdot|\Phi_N} = 1$$

for all $g \in M$.

Definition 1.5. Similarly, for a countably-infinite right-cancellative monoid with discrete topology M, we define a *right-Følner sequence* in M as a sequence of finite subsets $\Phi = (\Phi_N)_{N \in \mathbb{N}}$ satisfying

$$\lim_{N \to \infty} \frac{|\cdot|(\Phi_N \cdot m) \cdot \Phi_N}{|\cdot|\Phi_N} = 1$$

for all $g \in M$.

1.2 Equivalent definitions using Set Differences

Equivalent definitions can be constructed by using set differences instead of intersections.

For example, the equivalent definition of a left-Følner sequence, Φ, in M requires

$$\lim_{N \to \infty} \frac{|\cdot|(\Phi_N \cdot m) \triangle \Phi_N}{|\cdot|\Phi_N} = 0,$$

to be satisfied for all $m \in M$.

This alternative definition will be useful when looking at proving some of the properties of density.

2 Tempered Følner Sequences

Definition 2.1 (Lindenstrauss (2001), Definition 1.1)**.** A sequence of sets $\Phi = (\Phi_N)_{N \in \mathbb{N}}$ will be said to be *tempered* if, for some $b > 0$ and all $n \in \mathbb{N}$,

$$|\cdot| \bigcup_{1 \leq k < N} \Phi_k^{-1} \Phi_N \leq b |\cdot| \Phi_N. \tag{1}$$

is referred to as the *Shulman condition.*

Proposition 2.1 (Lindenstrauss (2001), Proposition 1.4)**.**

1. *Every Følner sequence $\Phi = (\Phi_N)_{N \in \mathbb{N}}$ has a tempered subsequence.*
2. *Every amenable group has a tempered Følner sequence.*

Lindenstrauss, E. (2001). 'Pointwise theorems for amenable groups', Inventiones mathematicae, 146 (2), pp. 259–295. https://doi.org/10.1007/s002220100162.