

# A Short Proof of a Generalised Conjecture of Erdős for Amenable Groups

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## Abstract

We follow steps provided by the paper published by [\(Host, 2019\)](#), ‘A Short Proof of a Conjecture of Erdős Proved by Moreira, Richter and Robertson’, as well as using the results provided by [\(Kra et al., 2022\)](#), to generalise the proof of Erdős’s conjecture for amenable groups. The main result that we aim to prove is: ‘every positively dense subset of an amenable group contains the group sumset of  $k$  infinite sets for every natural number  $k$ ’.

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# 1 Introduction

**Theorem 1.1** (cf. (Host, 2019), Theorem 1). *Let  $(\Gamma, \cdot)$  be an amenable group. If  $A \subseteq \Gamma$  has positive density, then there exists infinite subsets  $B$  and  $C$  of  $\Gamma$  such that  $B \cdot C \subset A$ .*

**Theorem 1.2** (cf. (Host, 2019), Proposition 2). *There exists a set of positive density not containing any sumset of positive density and an infinite set.*

**Theorem 1.3** (cf. (Kra et al., 2022), Theorem 1.1). *Let  $(\Gamma, \cdot)$  be an amenable group. If  $A \subseteq \Gamma$  has positive density then, for every  $k \in \mathbb{N}$ , there are infinite subsets  $B_1, \dots, B_k \subset \Gamma$  such that  $B_1 \cdot \dots \cdot B_k \subset A$ .*

# 2 Preliminaries

We will use  $\mathbb{N} = \{1, 2, 3, \dots\}$  and  $e$  to denote the identity of the group  $\Gamma$ .

## 2.1 Amenable Groups and Actions

Throughout this paper, unless otherwise specified, we will let  $(\Gamma, \cdot)$  be a second countable discrete group. This also means that the Haar measure of  $\Gamma$  is the counting measure.

**Definition 2.1.** [Embed #def-tempered from folner.qmd]

For simplicity of this paper, we will use the alternative and equivalent definition that a group,  $\Gamma$ , is *amenable* (and second countable) if and only if it has a Følner sequence.

**Definition 2.2.** [Embed #prp-temperedFolner from folner.qmd]

**Definition 2.3.** [Embed #def-density from density.qmd]

## 2.2 Topological Dynamics of Group Actions

**Definition 2.4.** [Embed #def-action from actions.qmd]

**Definition 2.5.** [Embed #def-topologicalDynamicalSystem from actions.qmd]

**Definition 2.6.** [Embed #def-generic from actions.qmd]

## 2.3 Recurrence Results & Ergodic Theorems

**Theorem 2.1.** [Embed #thm-recurrence from recurrence-and-Ergodic-theorems.qmd]

**Theorem 2.2.** [Embed #thm-GeneralisedPET from recurrence-and-Ergodic-theorems.qmd]

## 2.4 Erdős Cubes & Cubic Measures

## 2.5 Factor Maps

## 2.6 Key Dynamical Results

**Theorem 2.3** (cf. Host (2019), Theorem 3). *Let  $(X, \Gamma)$  be a topological dynamical system where  $\Gamma$  is an amenable group,  $x_0 \in X$ ,  $E$  be a clopen subset of  $X$  and*

$$A = \{\gamma \in \Gamma : \gamma.x_0 \in E\}.$$

*Let  $X \times X$  be acted upon by  $\Gamma \times \Gamma$ . Let  $x_1 \in X$  and  $\nu$  be a measure on  $X \times X$  such that  $(x_0, x_1)$  is generic along some Følner sequence  $\Phi = (\Phi_N)_{N \in \mathbb{N}}$ .*

*Assume there exists  $\varepsilon > 0$  and a sequence  $(s_i)_{i \in \mathbb{N}}$  in  $\Gamma$  such that*

$$s_i.x_0 \rightarrow x_1 \text{ as } i \rightarrow \infty \text{ and } \nu(s_i^{-1}.E \times E) \geq \varepsilon \text{ for all } i.$$

*Then there exist infinite subsets  $B, C \subseteq \Gamma$  such that  $B \cdot C \subset A$ .*

**Theorem 2.4** (cf. Kra et al. (2022), Theorem 3.5). *Let  $(X, \mu, \Gamma)$  be an ergodic system under the action of  $\Gamma$  and let  $E \subset X$  be an open set with  $\mu(E) > 0$ .*

*If  $a \in X$  is generic for  $\mu$  with respect to some Følner sequence  $\Phi$  then, for every  $k \in \mathbb{N}$ , there exists 2-dimensional Erdős cube  $\underline{x} = (x_{00}, x_{01}, x_{10}, x_{11}) \in X^{[2]}$  with  $x_{00} = a$  and  $x_{11} \in E$ .*

**Theorem 2.5** (cf. Host (2019), Theorem 4.4). *Let  $(X, \mu, \Gamma)$  be an ergodic system under the action of  $\Gamma$  and let  $E \subset X$  be an open set with  $\mu(E) > 0$ .*

*If  $a \in X$  is generic for  $\mu$  with respect to some Følner sequence  $\Phi$  then, for every  $k \in \mathbb{N}$ , there exists  $k$ -dimensional Erdős cubes  $\underline{x} \in X^{[k]}$  with  $x_{\vec{0}} = a$  and  $x_{\vec{1}} \in E$ .*

## 3 Proof of Theorem Theorem 2.4

### 3.1 Furstenberg's Correspondence Principle

**Theorem 3.1.** *[Embed #thm-FCP from furstenberg-correspondence.qmd]*

### 3.2 Kronecker Factor

**Proposition 3.1.** *[Embed #prp-KroneckerGeneric from factor-maps.qmd]*

### 3.3 Choosing a point $x_1$

### 3.4 The joining $\nu$

### 3.5 Proof Conclusion

## 4 Proof of Corollary (ref)

## 5 Proof of Theorem (ref)

## Discussion

Host, B. (2019). '*A short proof of a conjecture of erdős proved by moreira, richter and robertson*', Available at: <https://arxiv.org/abs/1904.09952>.

Kra, B., et al. (2022). '*Infinite sumsets in sets with positive density*', Available at: <https://arxiv.org/abs/2206.01786>.