Strange Loops

capturing knots with powerful notations

Katherine Ye | @hypotext

XING FU BORDERS

Photos okay—knots are hard to draw!

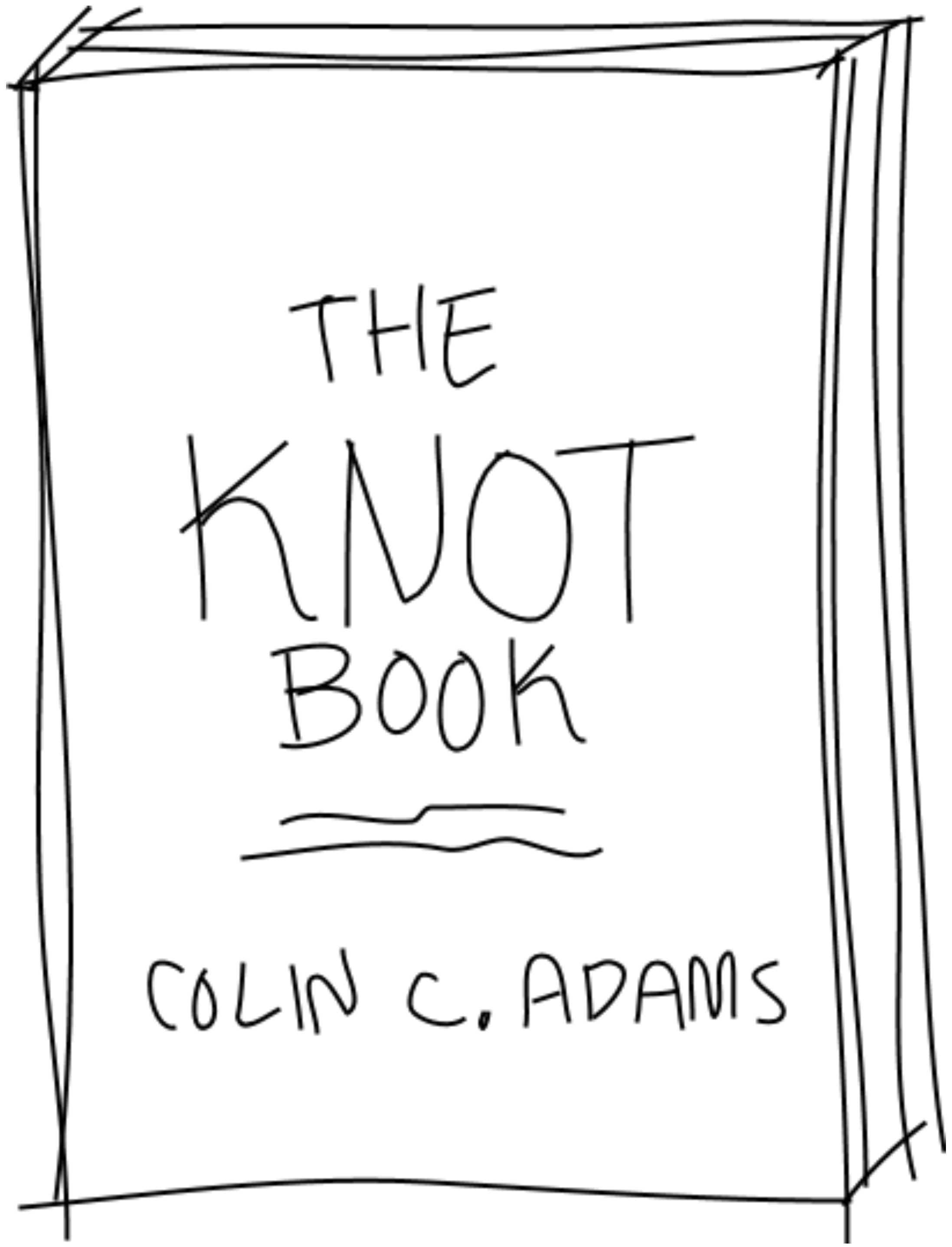
Little tells us that the enumeration of the 54 knots of [6] took him 6 years—from 1893 to 1899—the notation we shall soon describe made this just **one afternoon's work!**

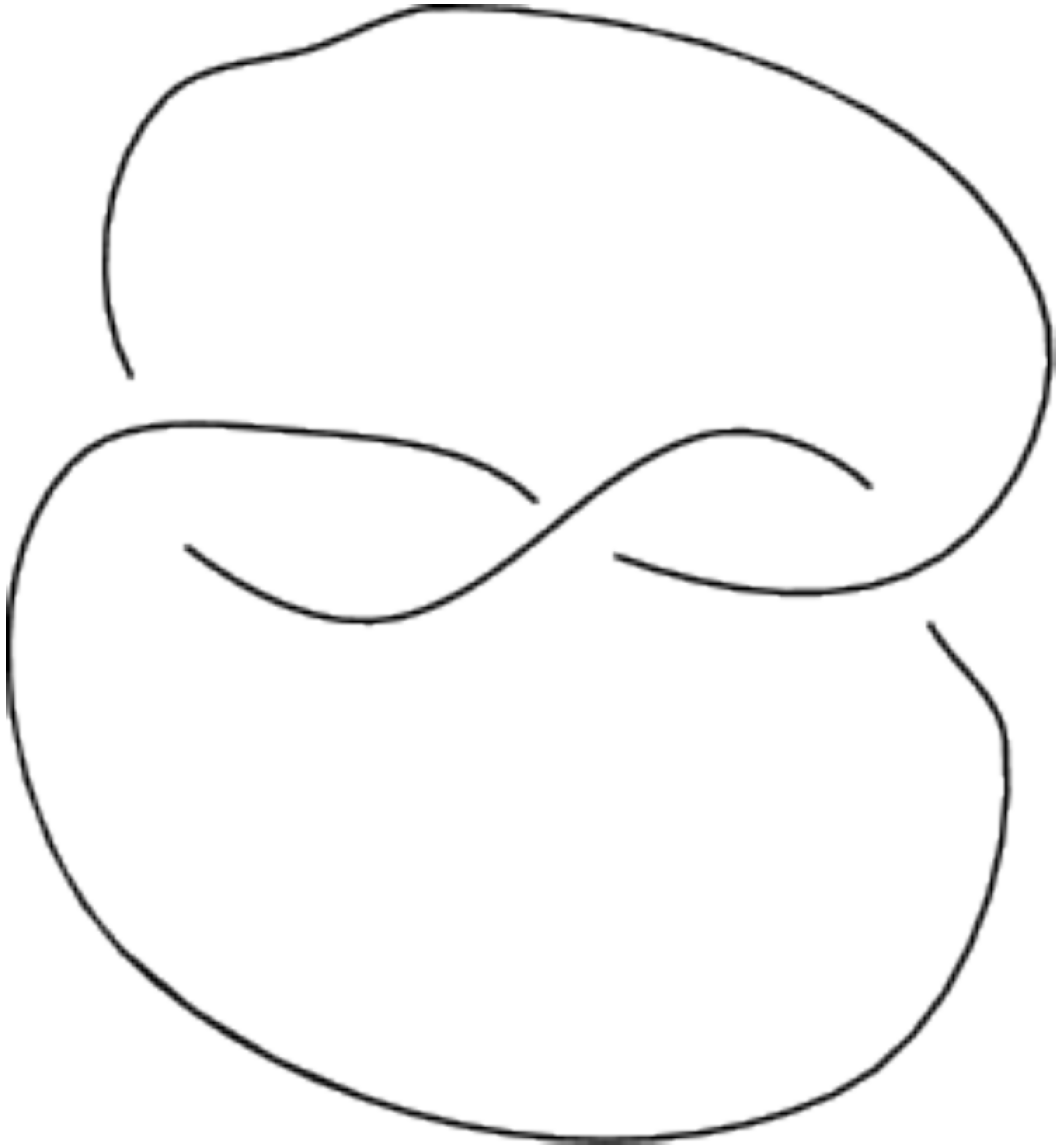
Outline

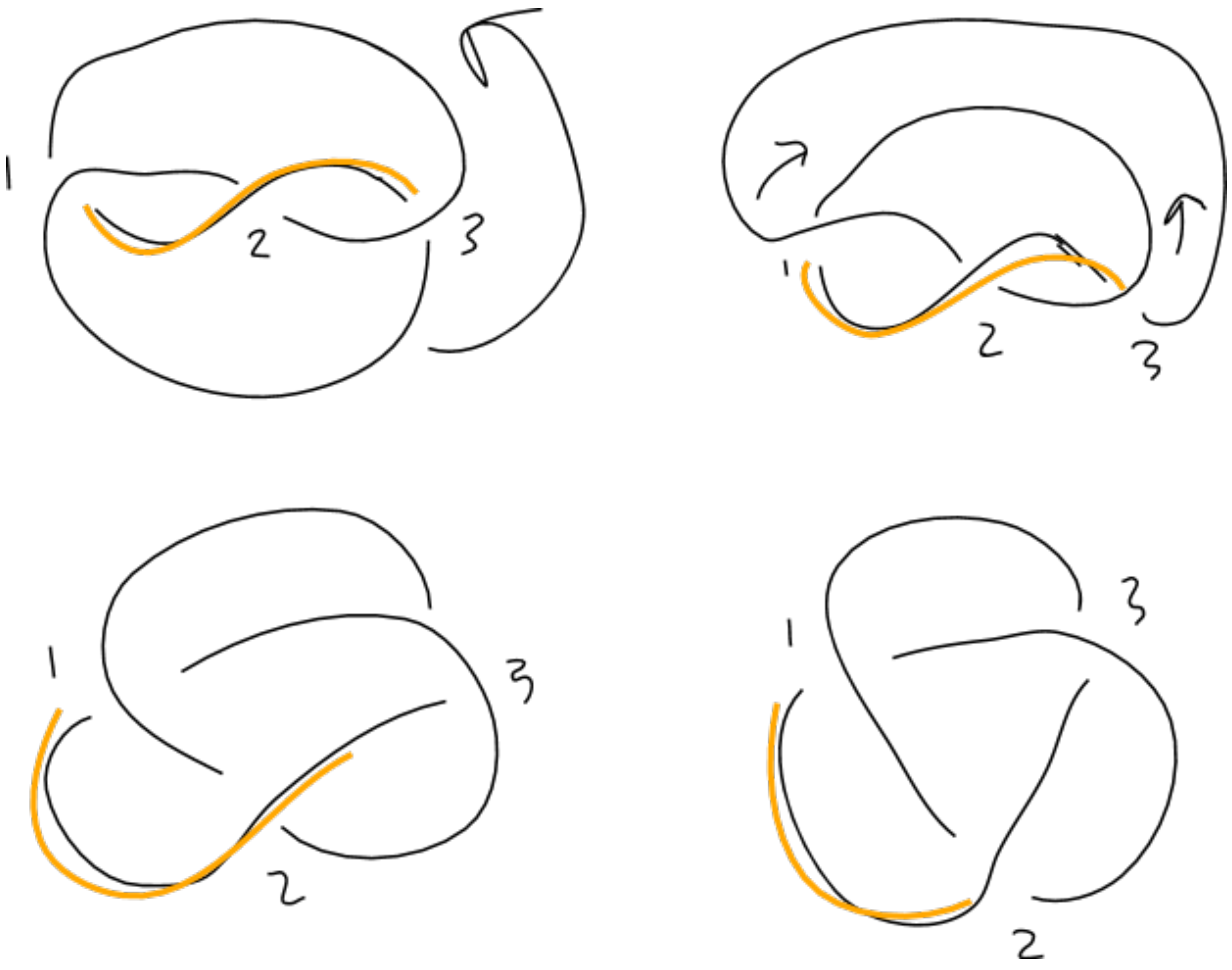
Crash course in knot theory

- Alexander-Briggs notation
- Dowker notation
- Intro to enumeration
- Conway notation *(Answer to the question!)*
- Modern tabulations
- Lessons

What's a knot?


THE
KNOT
BOOK
COLIN C. ADAMS




1
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3
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1
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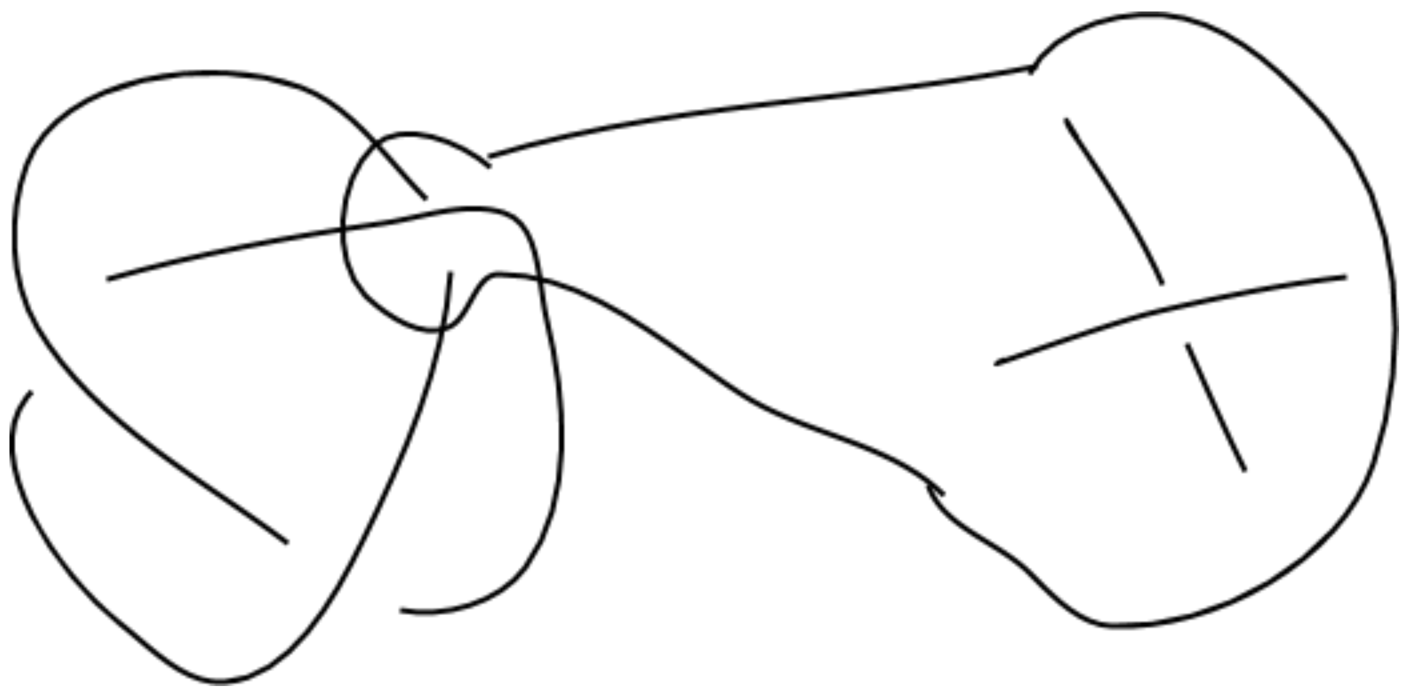
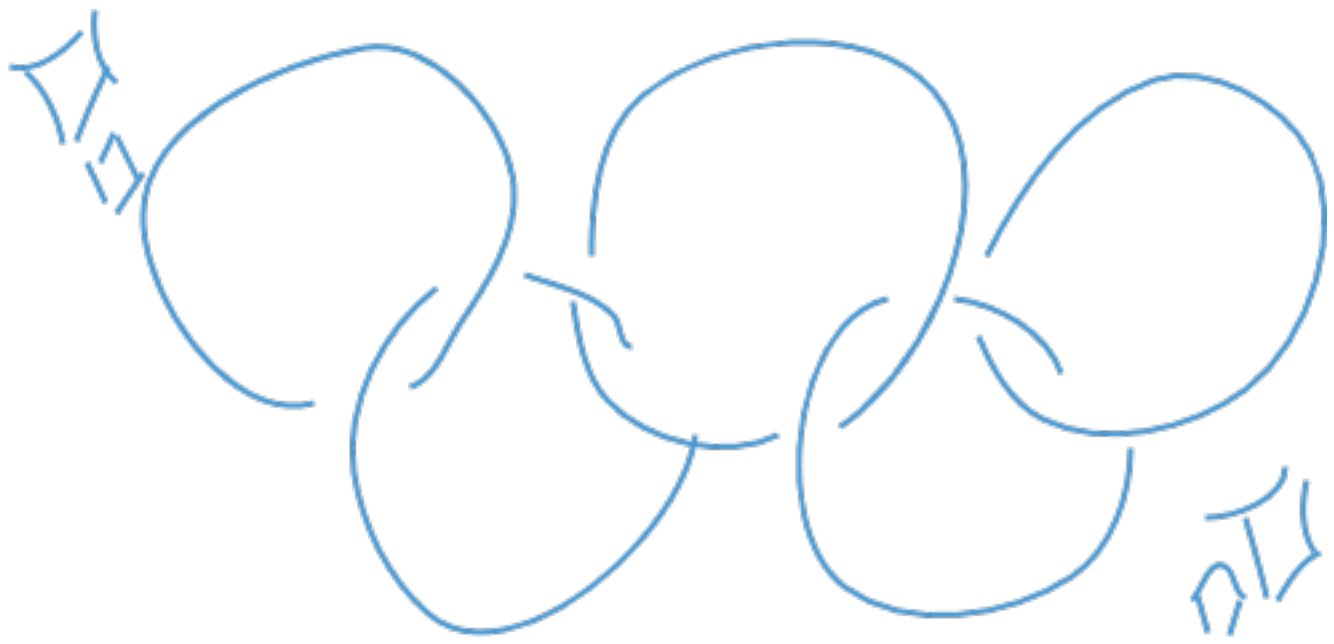
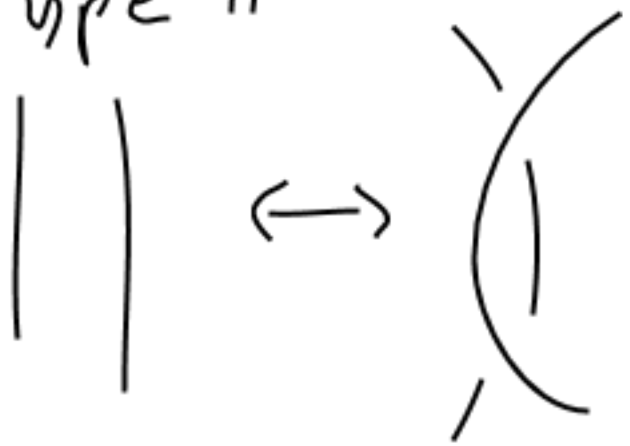




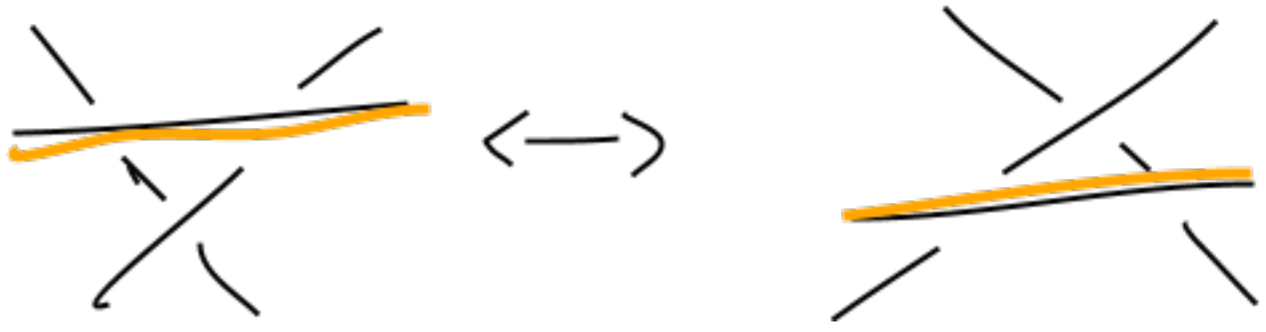
How do we tell knots apart?

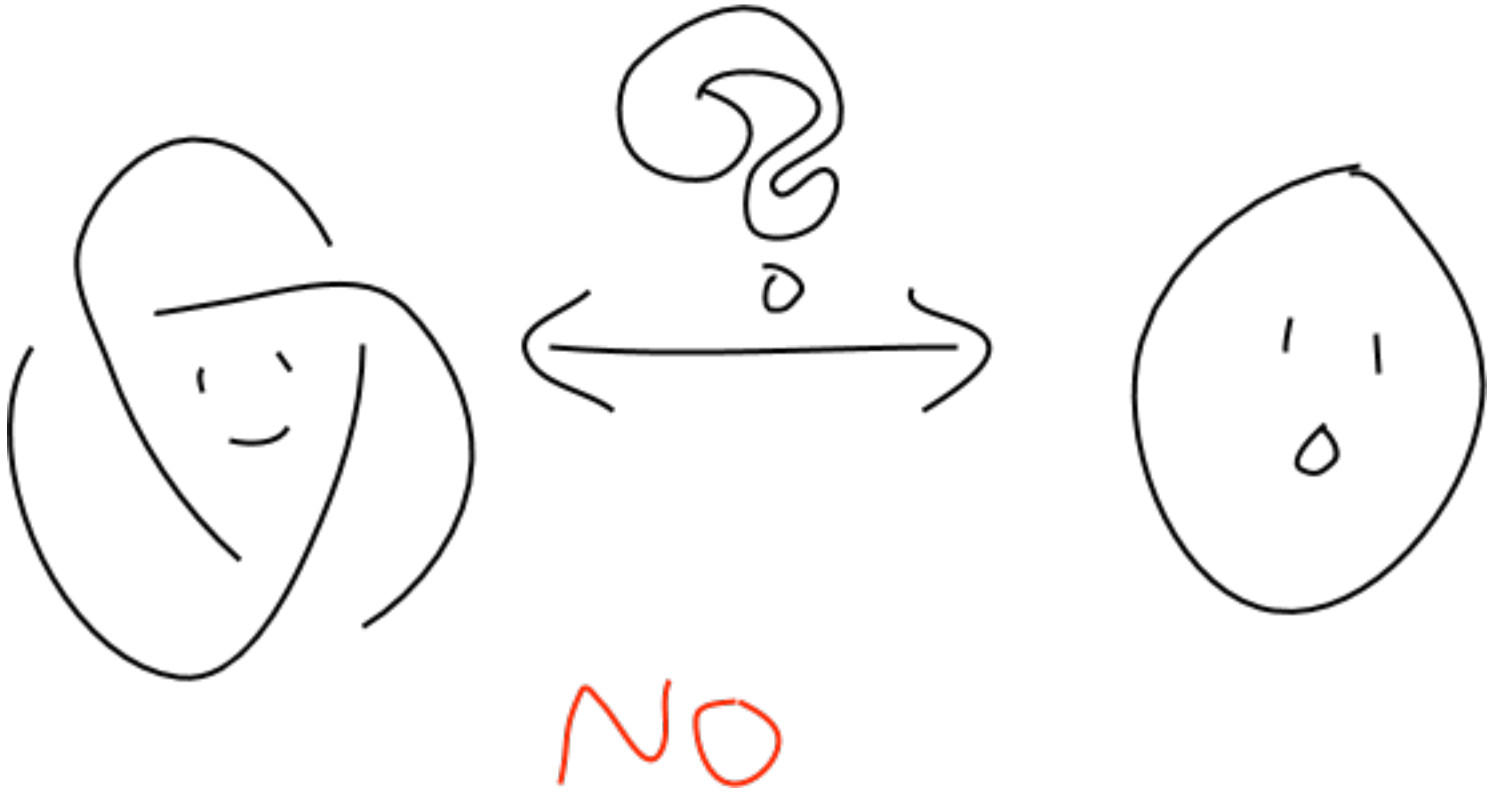
Type I

Type II



Type III




NO

Outline

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- **Alexander-Briggs notation**
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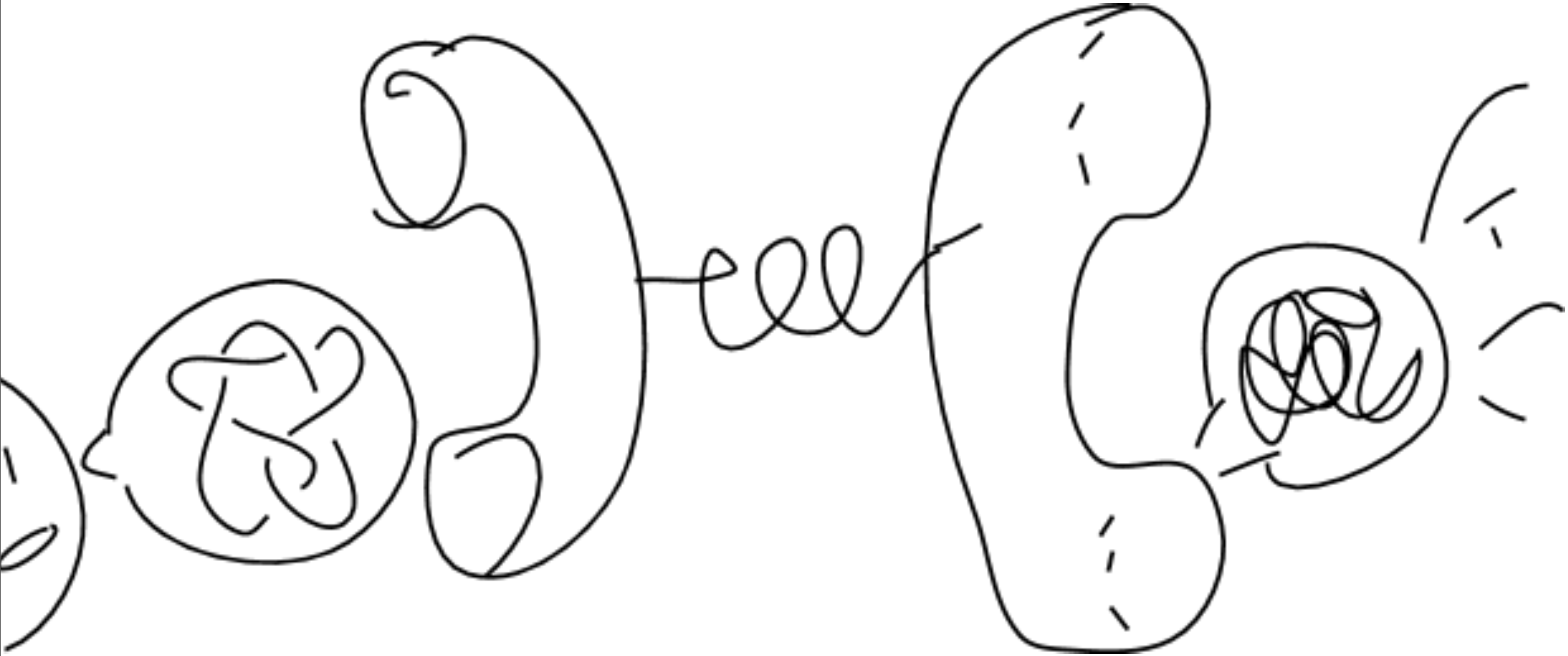
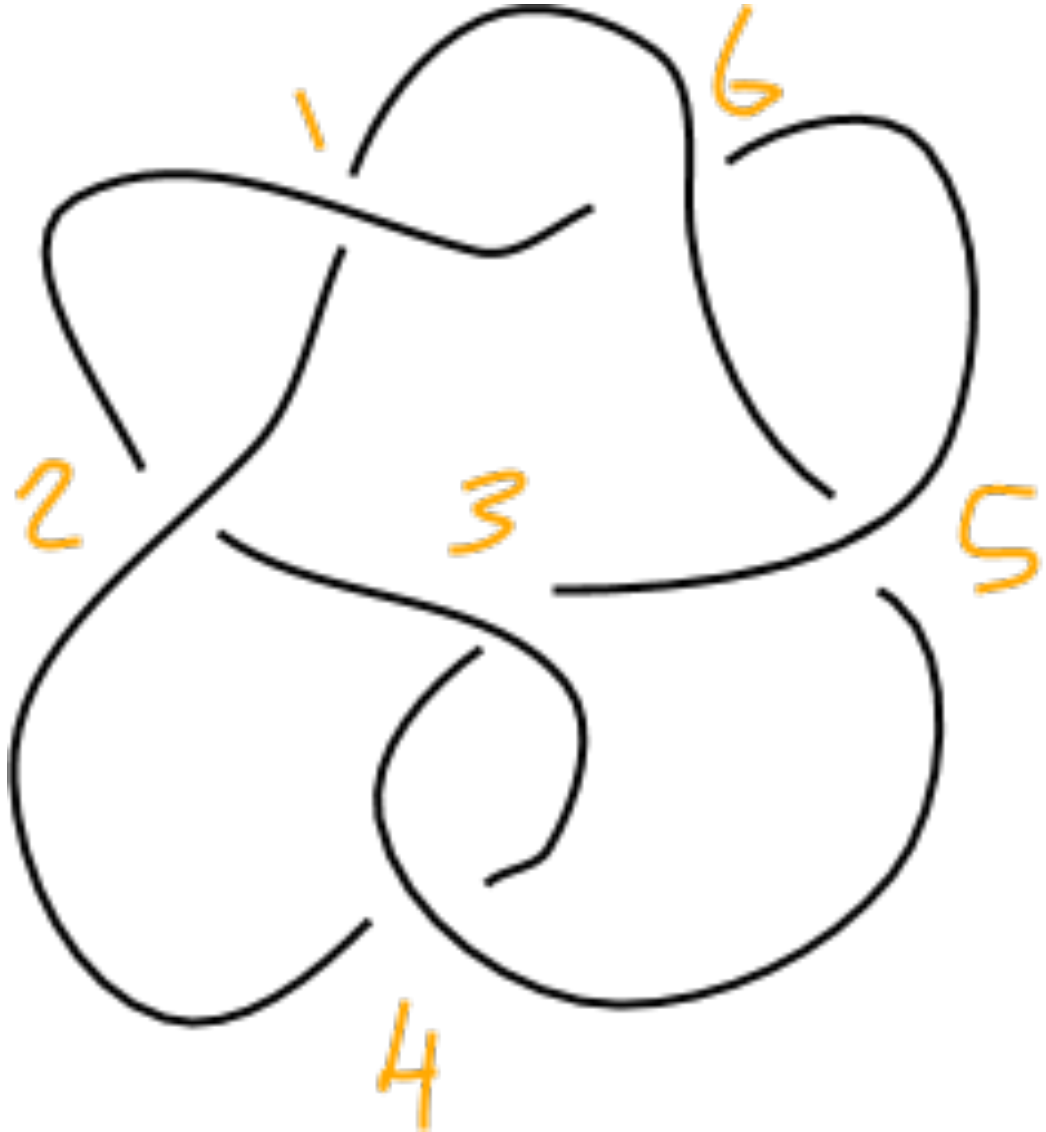
10_{163}

One thing I never figured out was: how do I tell a knot to a nerd over the phone? Keep in mind that the person you're talking to is just as much of a nerd as you are!

Conway

Stevedore

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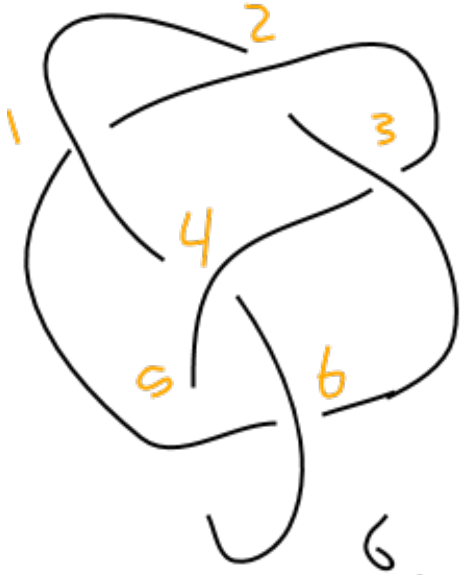
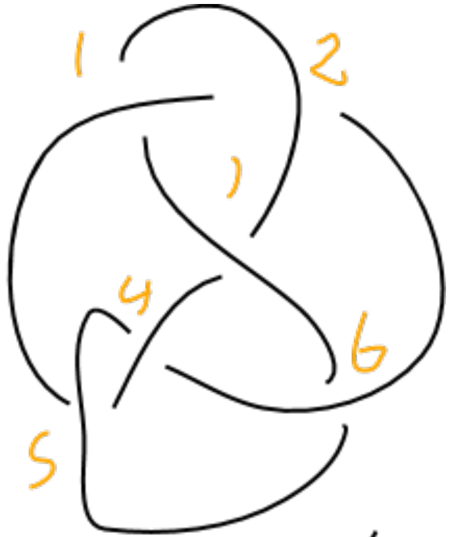


6_2



6_3

Alexander-Briggs compresses too much!

Outline

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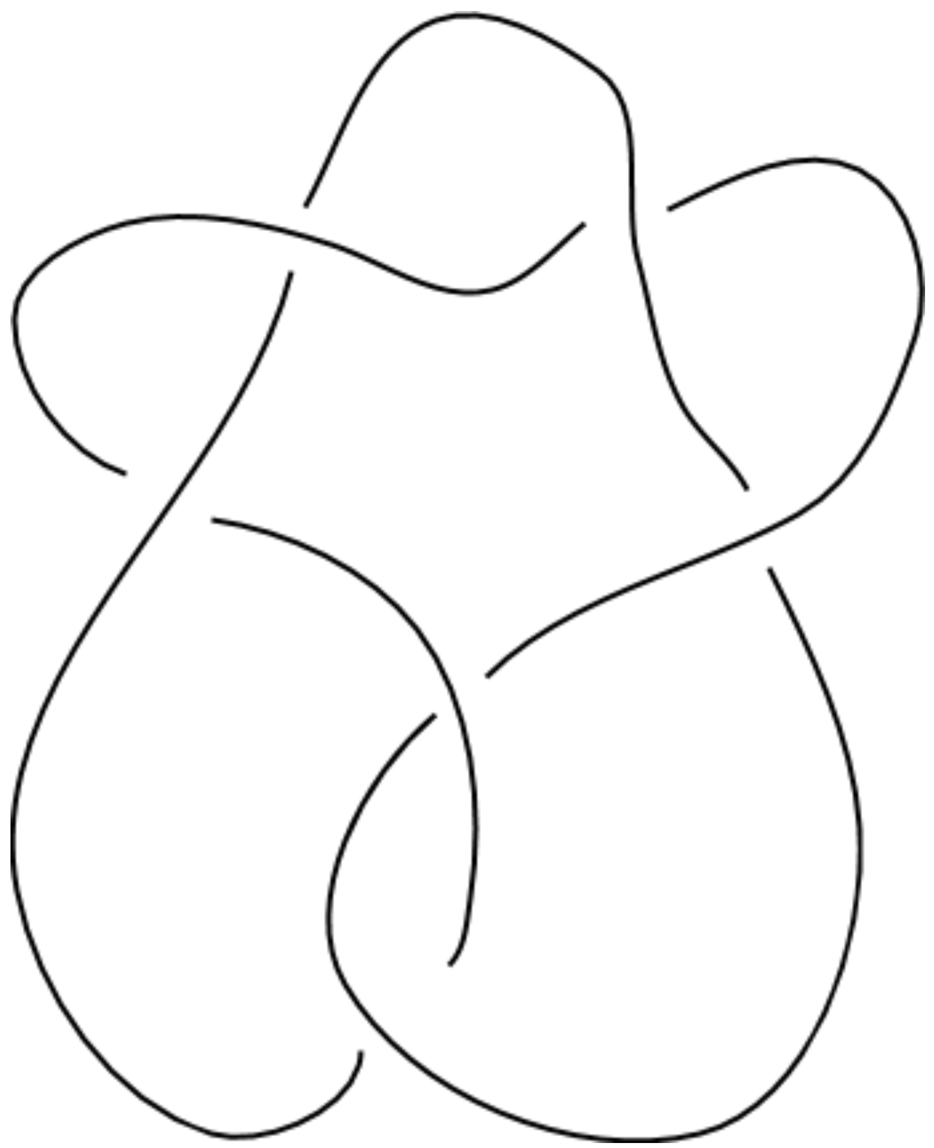

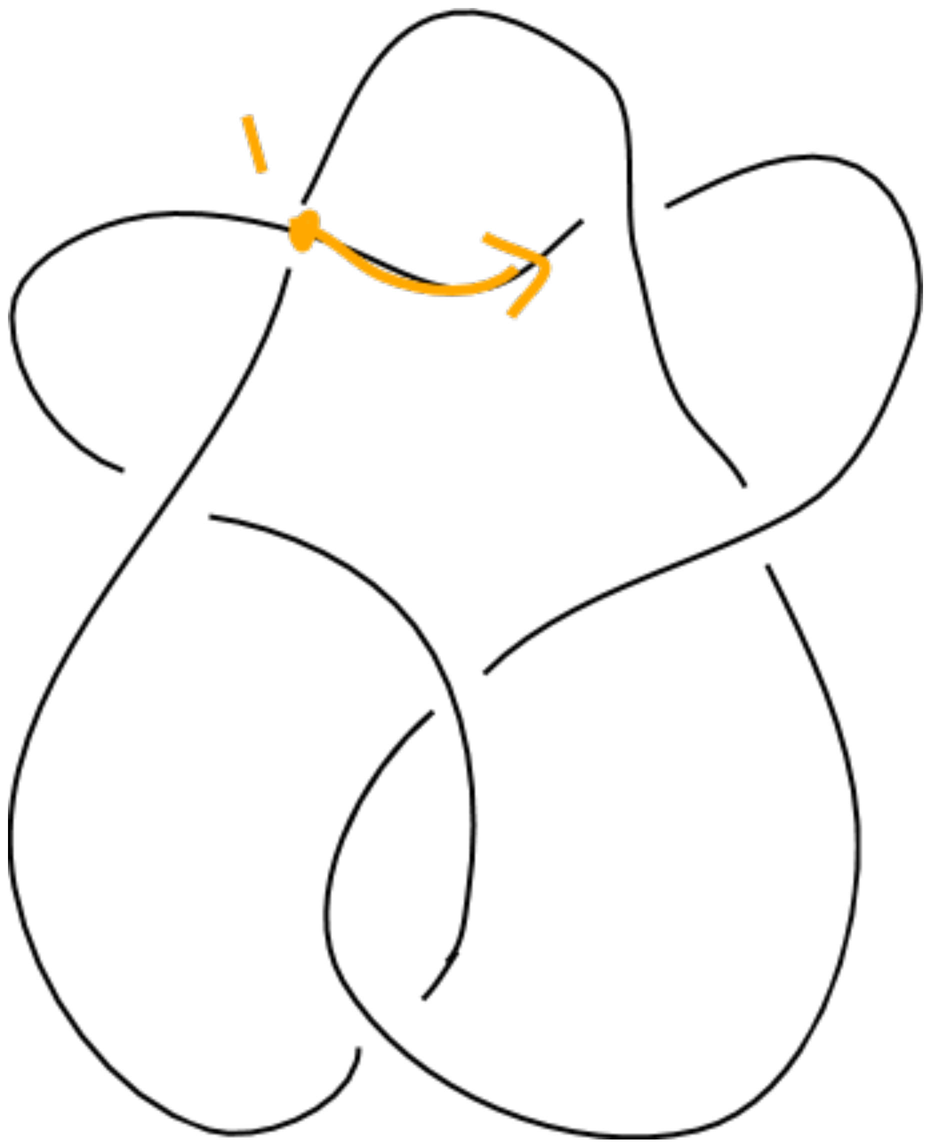

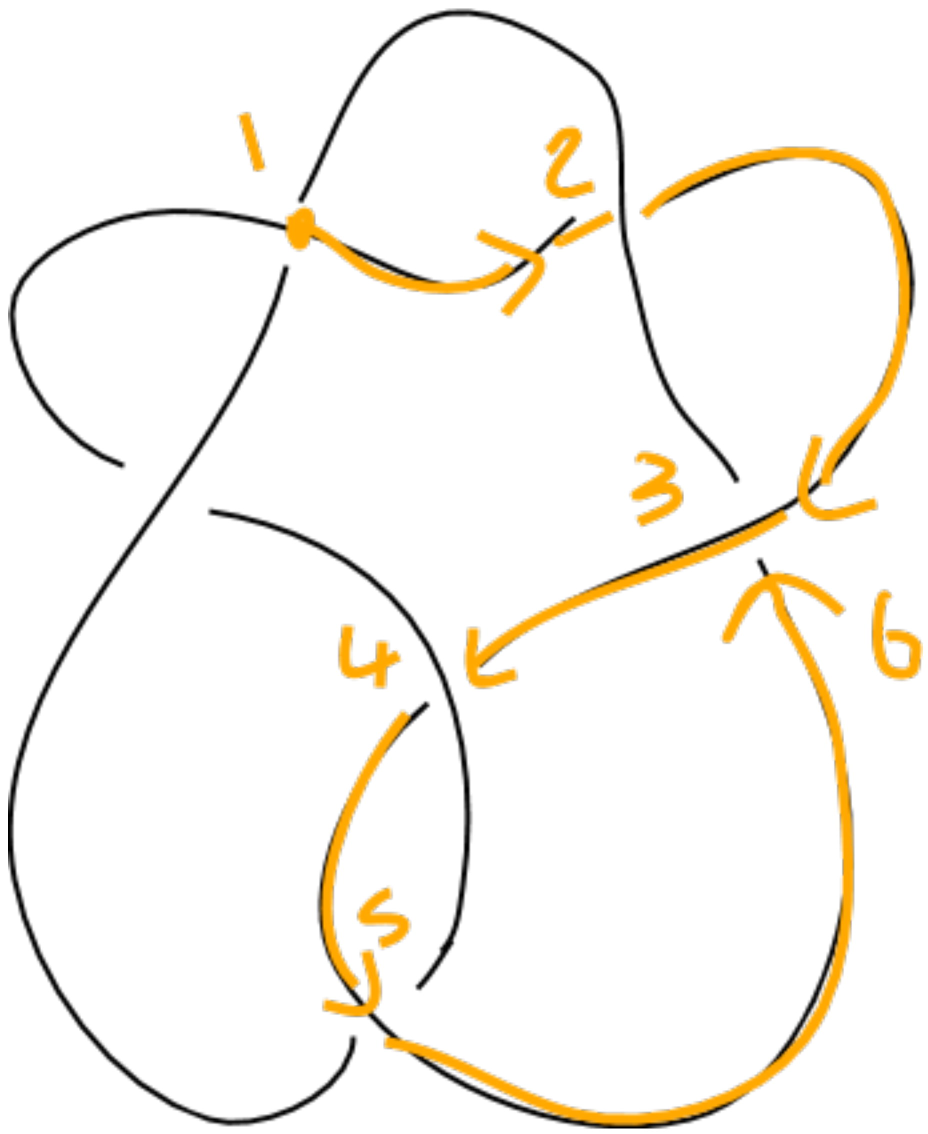
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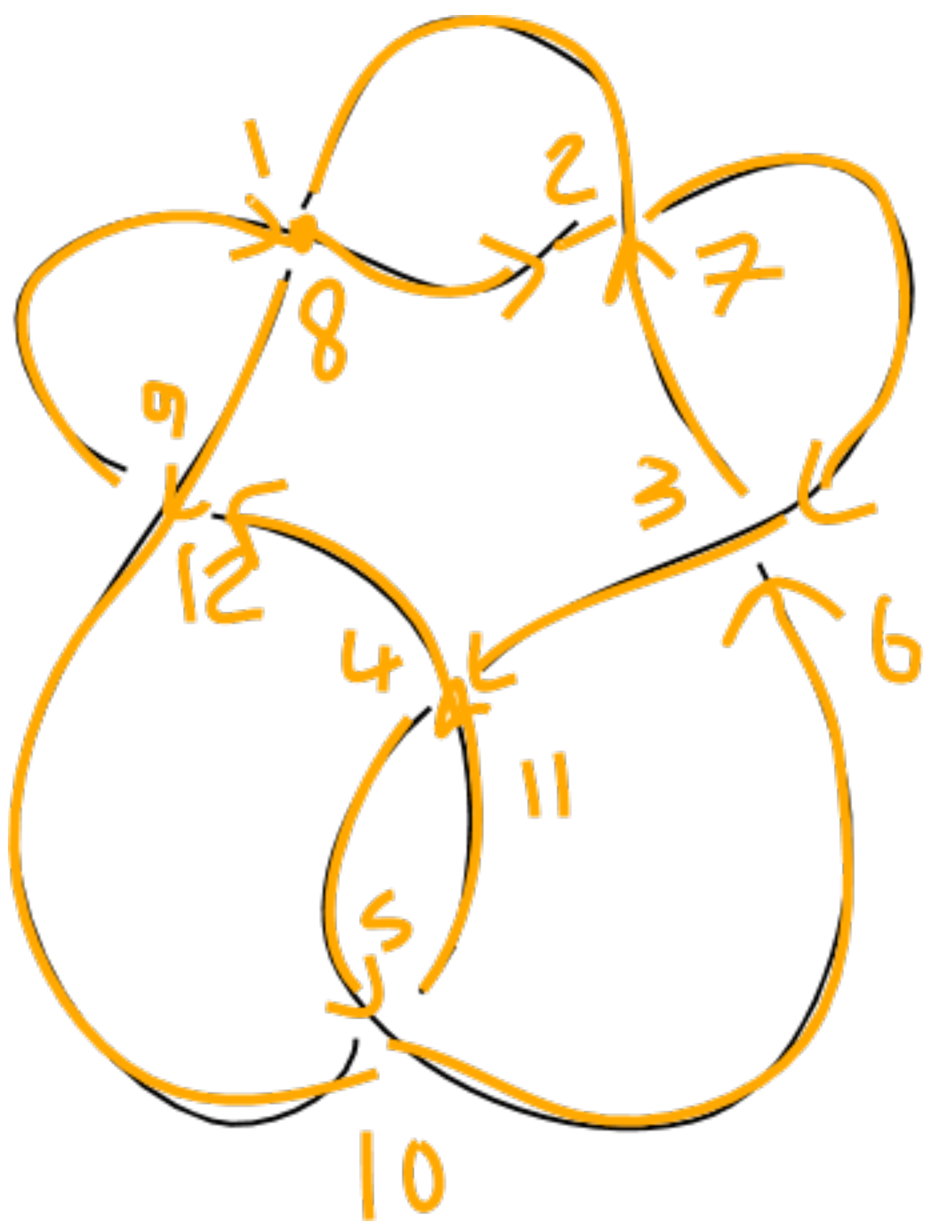
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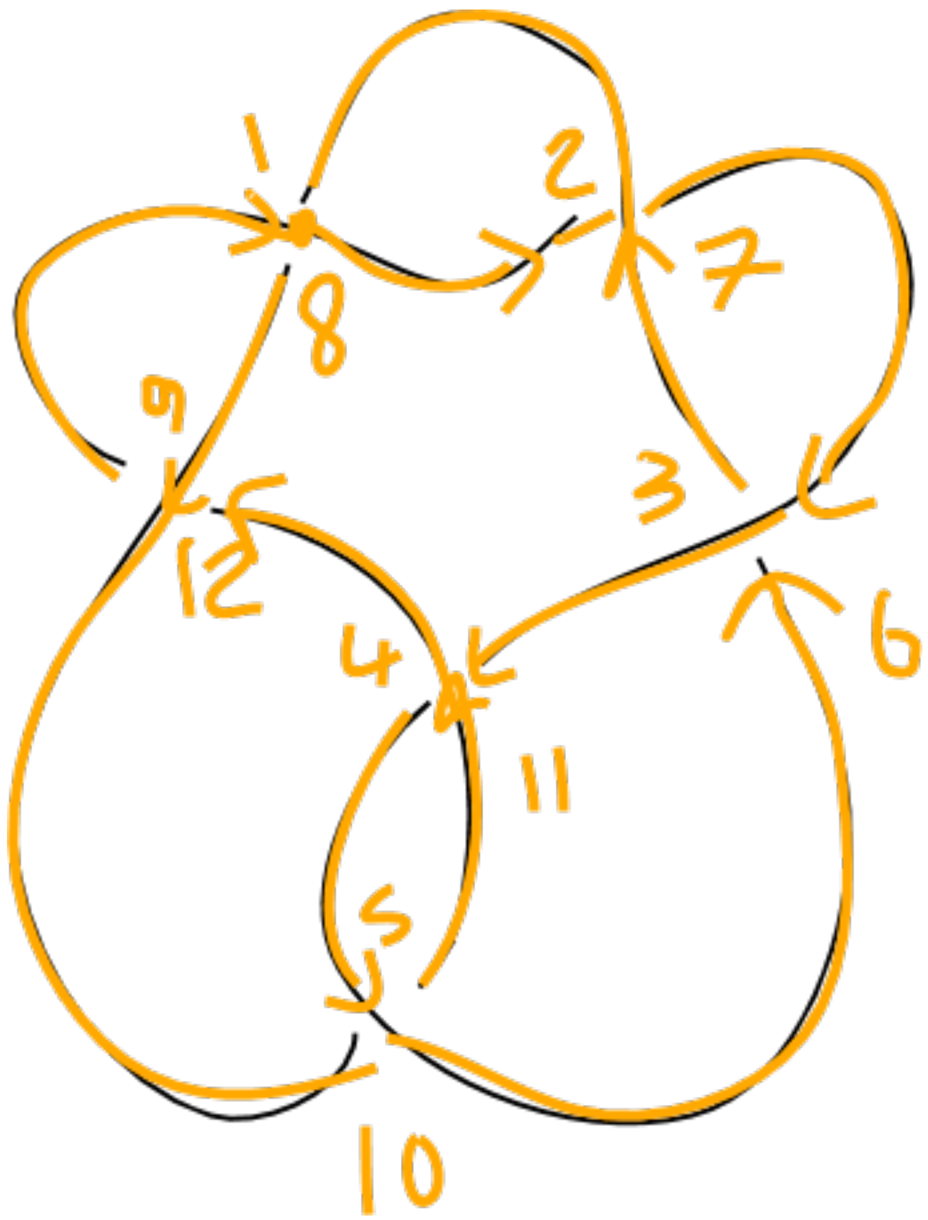
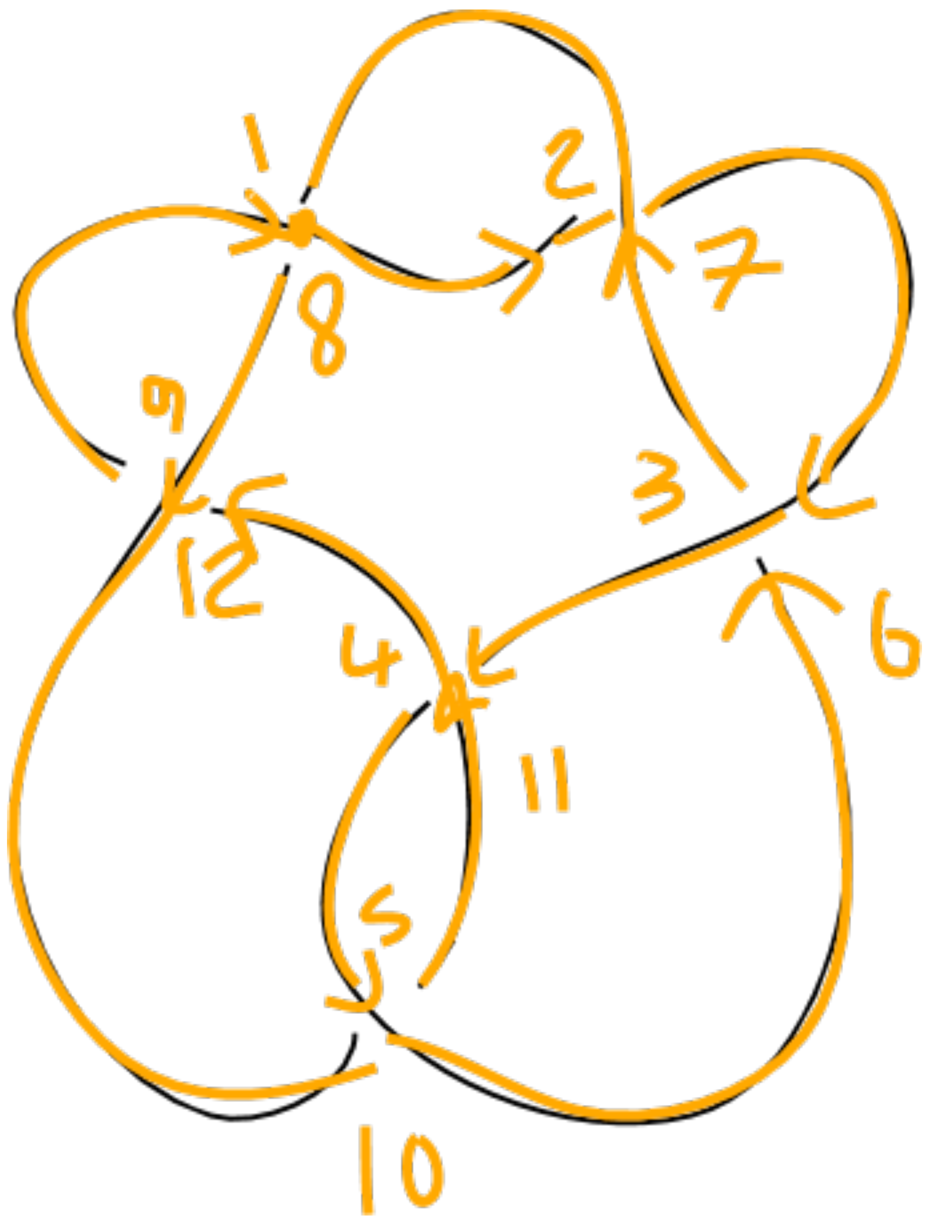
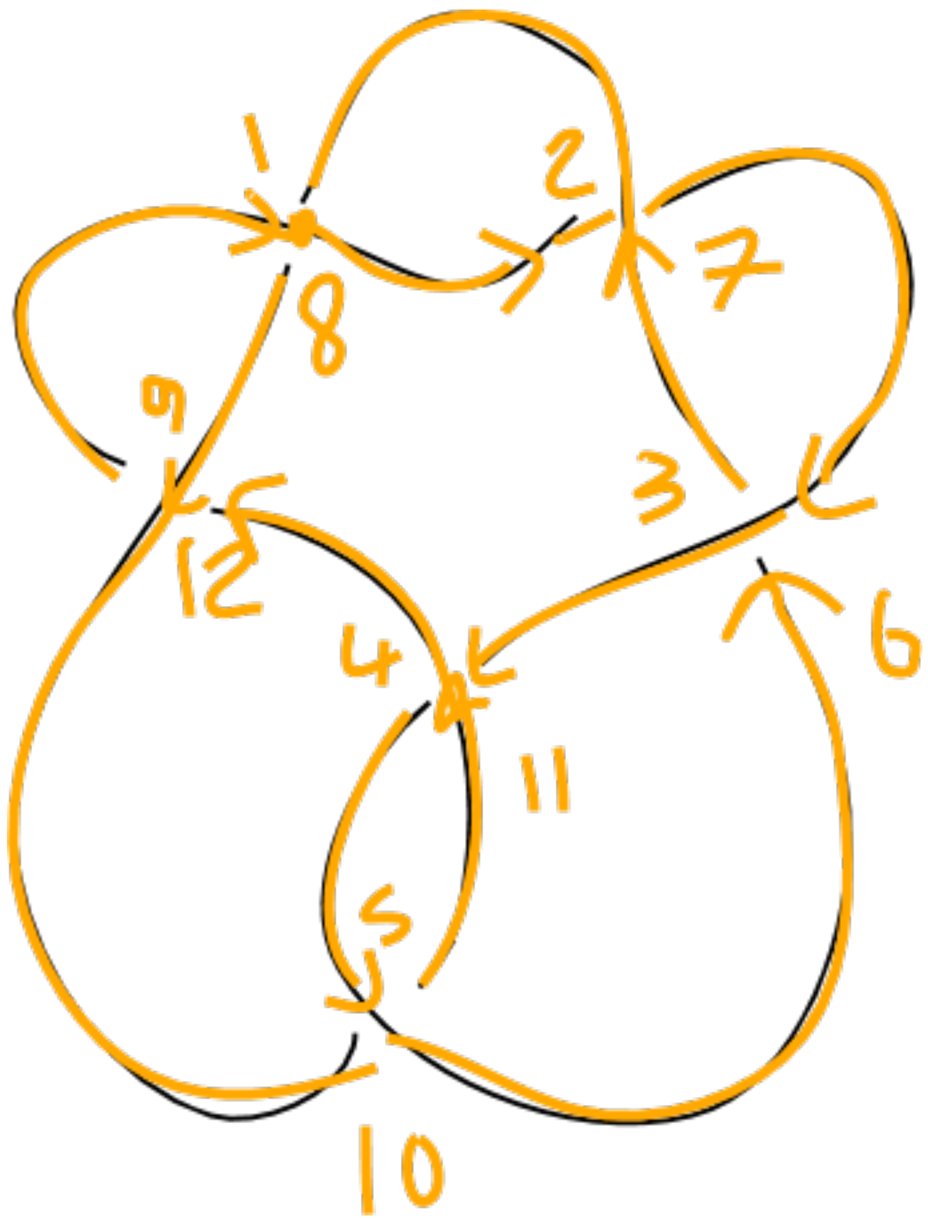


1	2	3	4	5	6	7	8	9	10	11	12
8	7	6	11	10	3	2	1	12	5	4	9



1	2	3	4	5	6	7	8	9	10	11	12
8	7	6	11	10	3	2	1	12	5	4	9

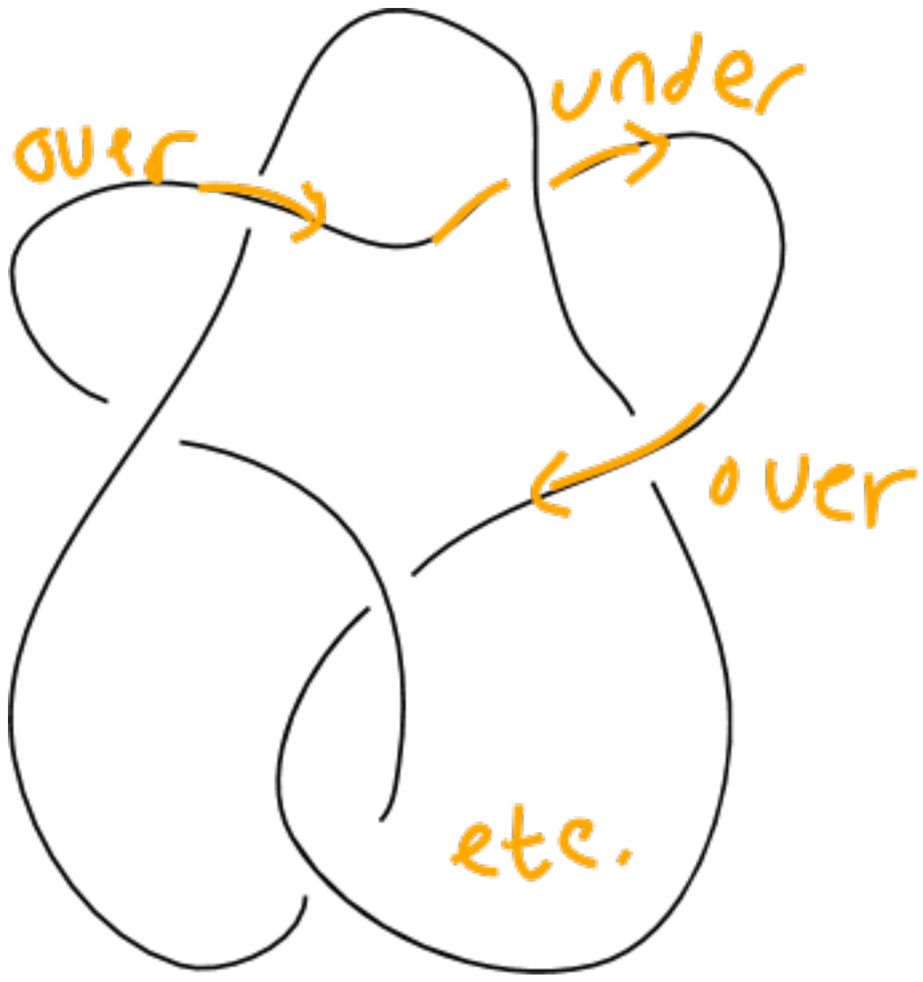
1	3	5	7	9	11
8	6	10	2	12	4



1	2	3	4	5	6	7	8	9	10	11	12
8	7	6	11	10	3	2	1	12	5	4	9

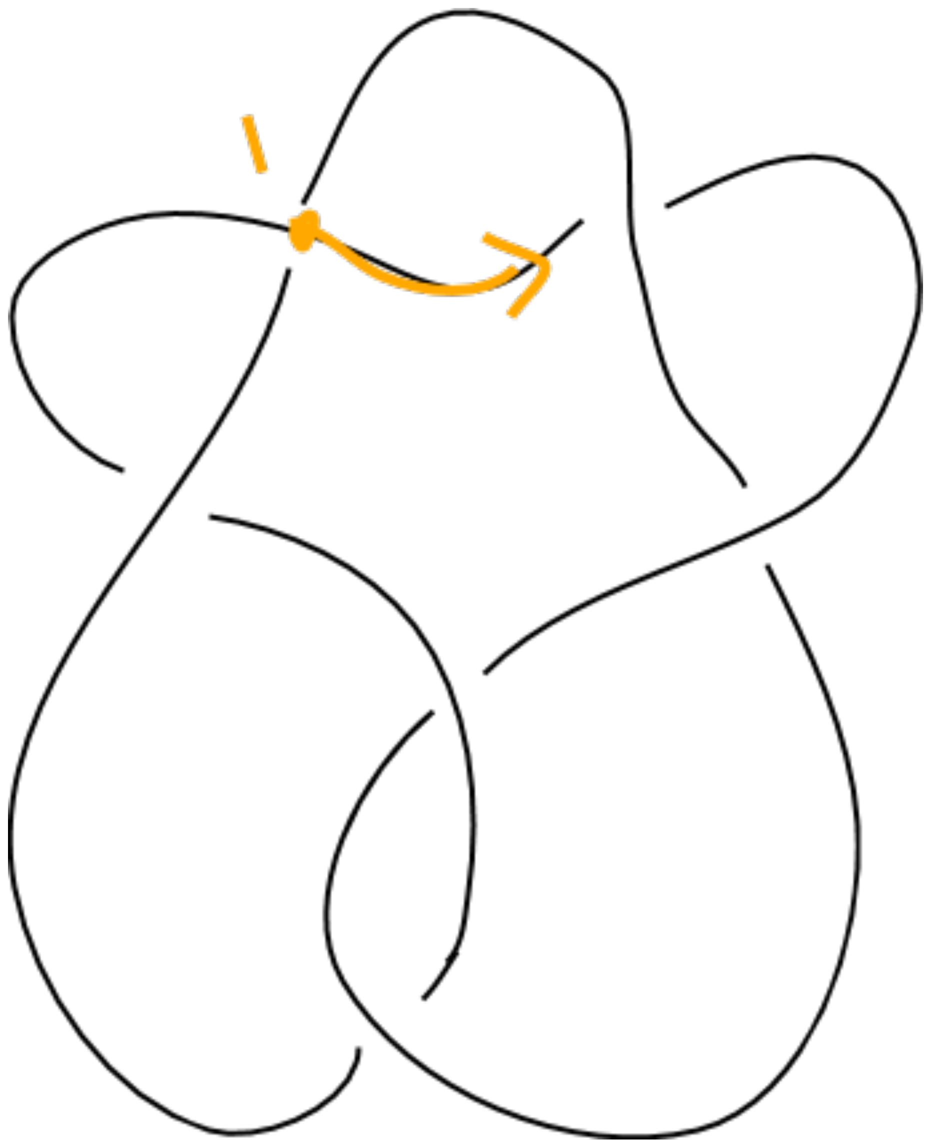
1	3	5	7	9	11
8	6	10	2	12	4

8 6 10 2 12 4

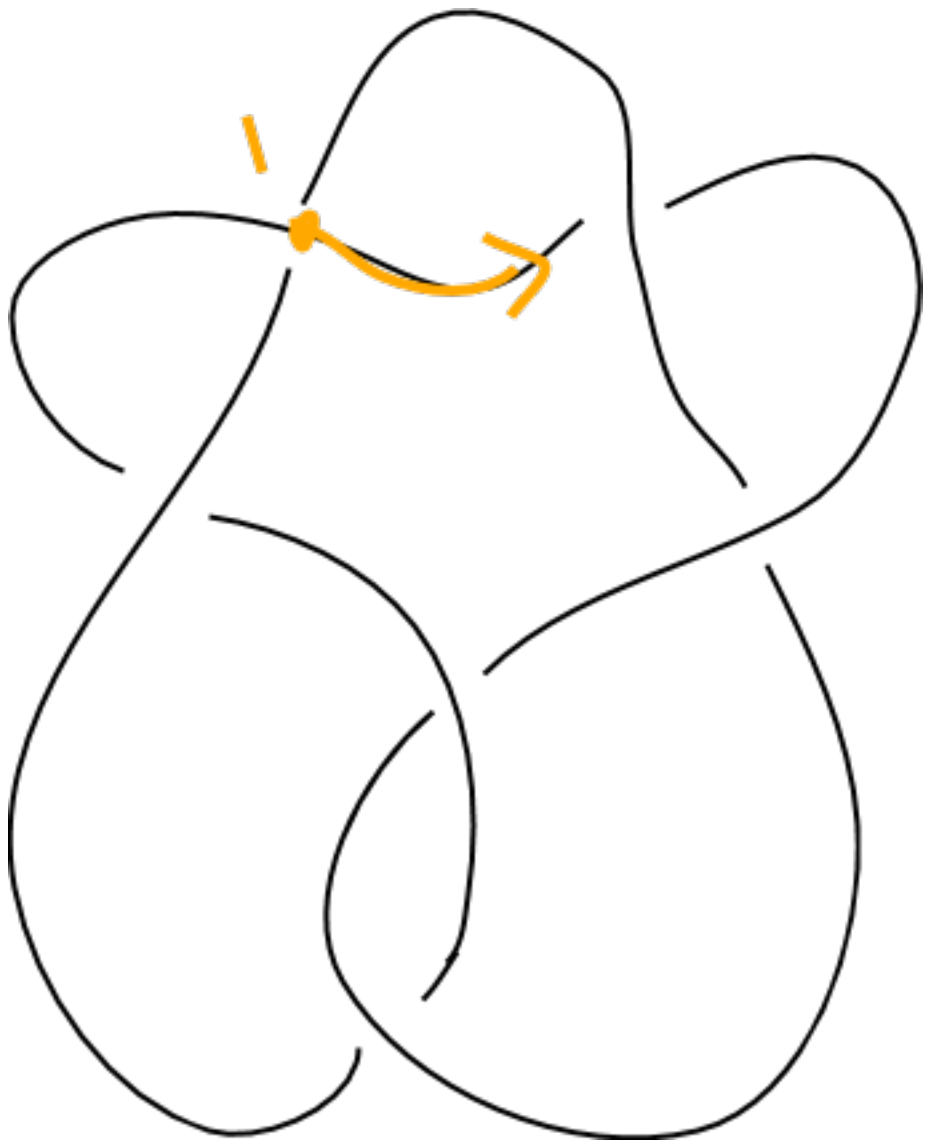


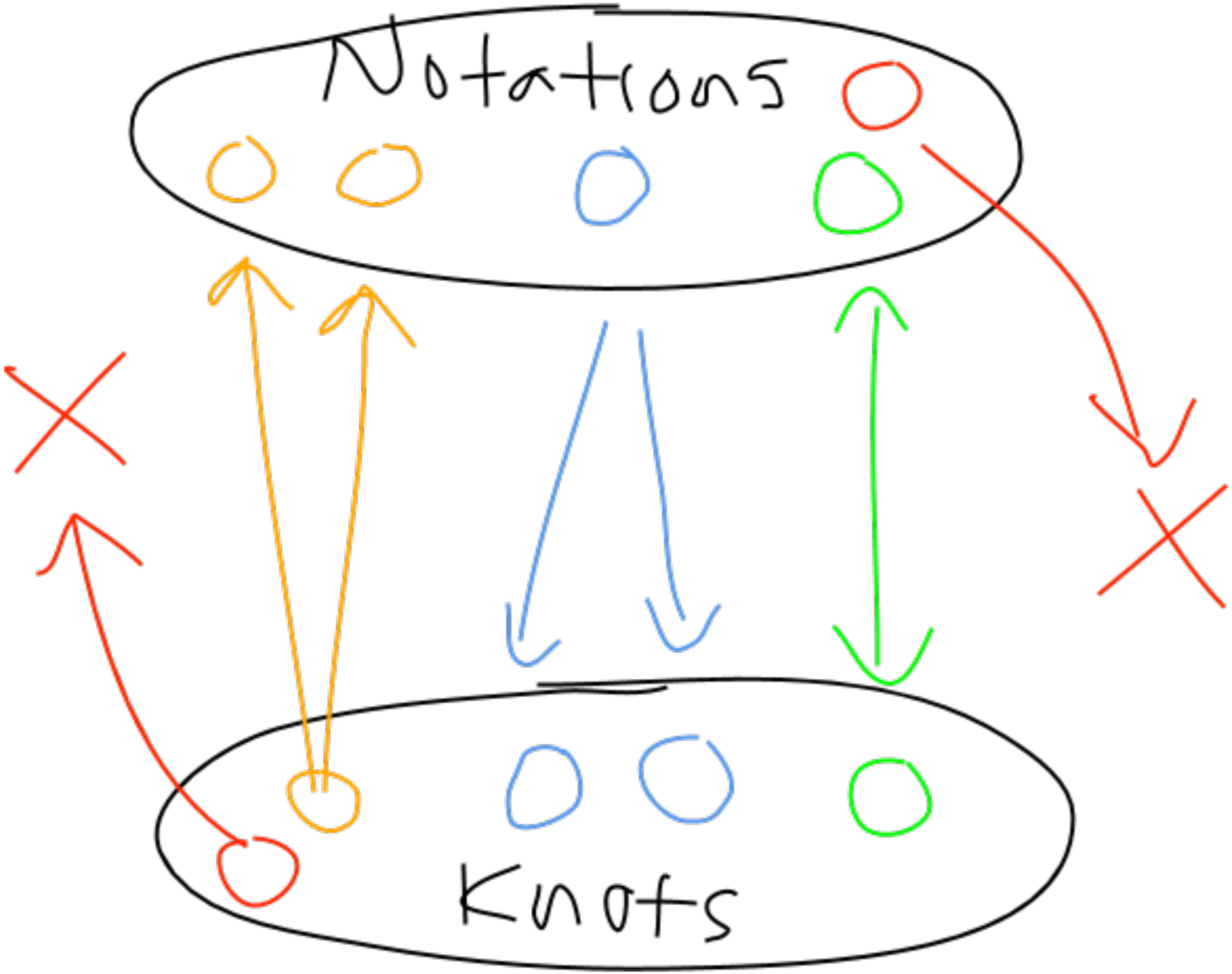
over
under
over
etc.


8 6 10 2 12 4
?



N crossings
Over- or understrand
Forward or backward direction
= ~4N notations




Notations
Knots


8 6 10 2 12 4
?

Reverse-engineering Dowker

8 6 10 2 12 4

$\downarrow$

1 3 5 7 9 11

8 6 10 2 12 4

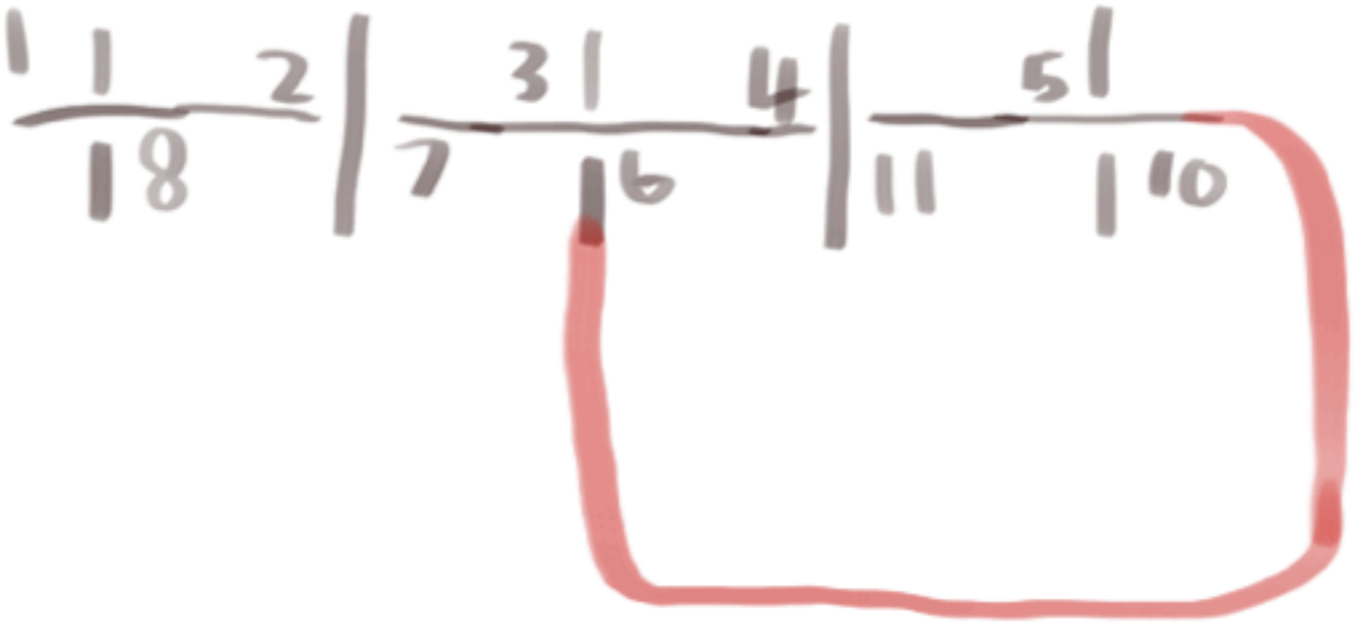
1	3	5	7	9	11
8	6	10	2	12	4

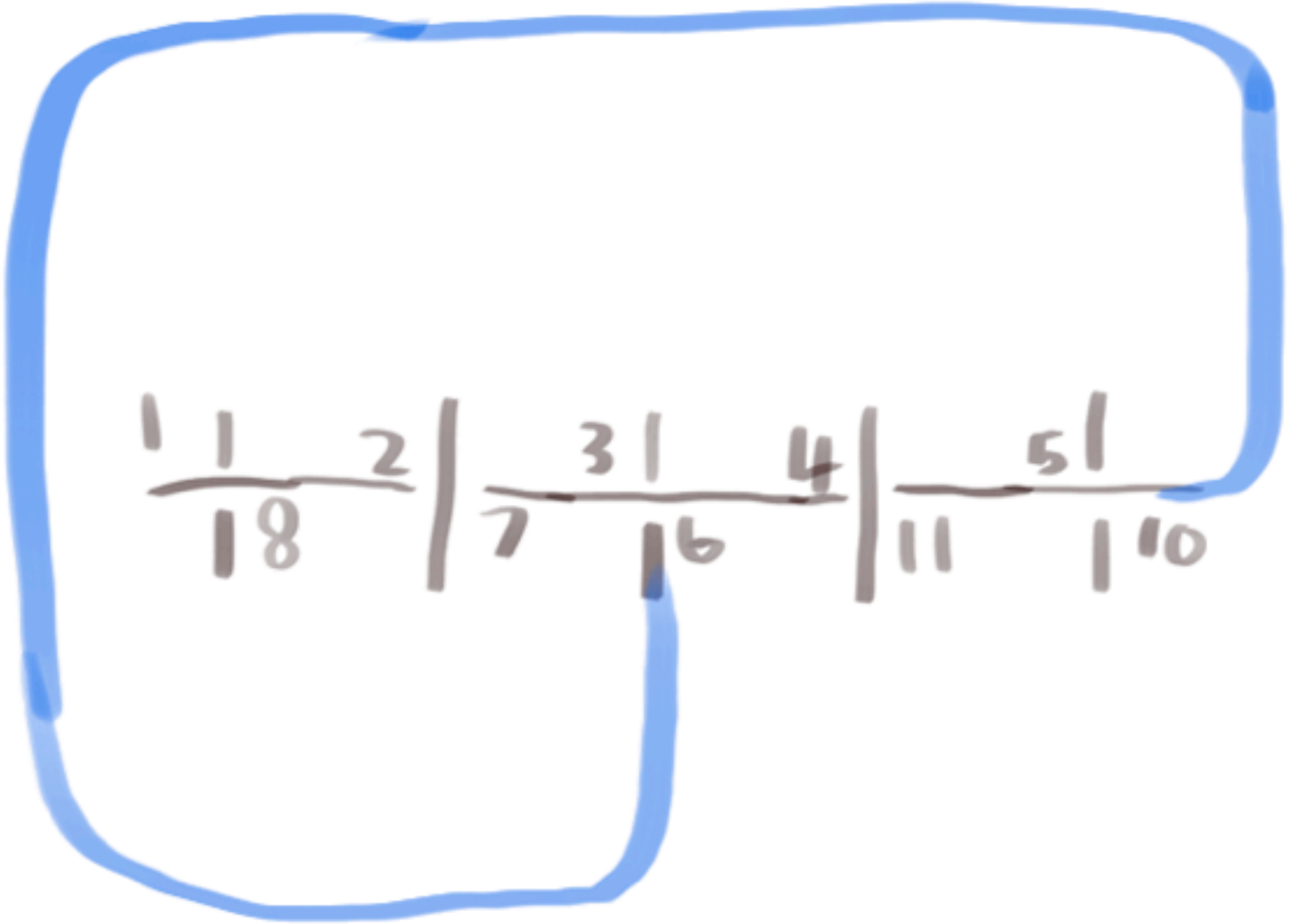
~~1~~ ~~3~~ ~~5~~ 7 9 11

8 6 10 ~~2~~ 12 ~~4~~

~~1~~ ~~3~~ ~~5~~ 7 9 11

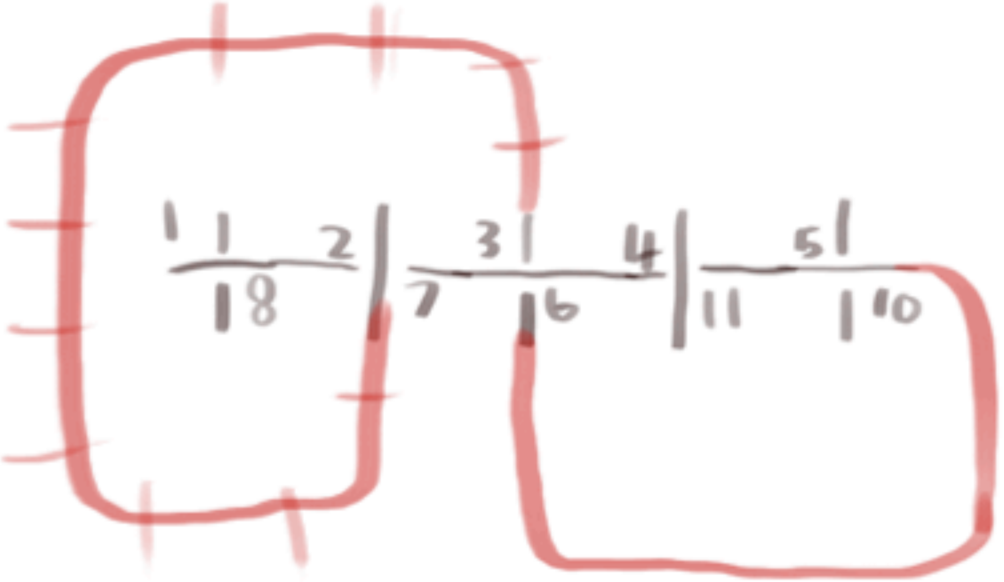
8 ~~6~~ 10 ~~2~~ 12 ~~4~~

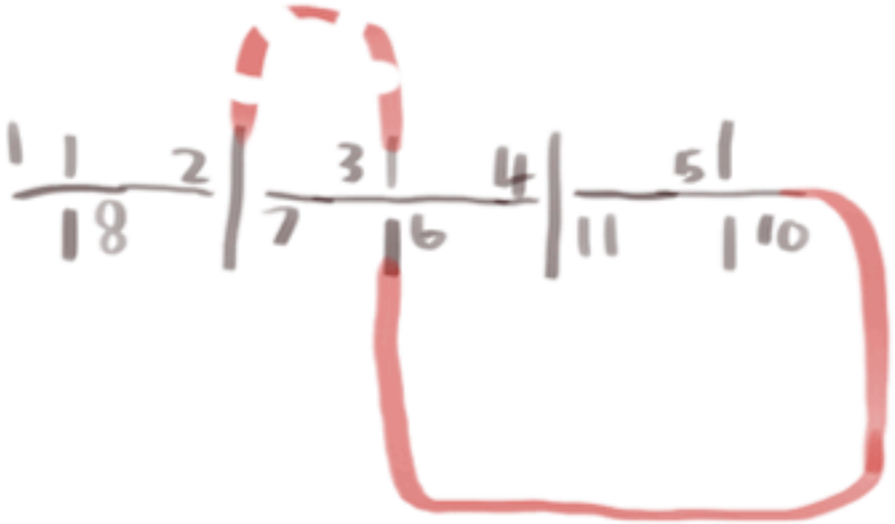


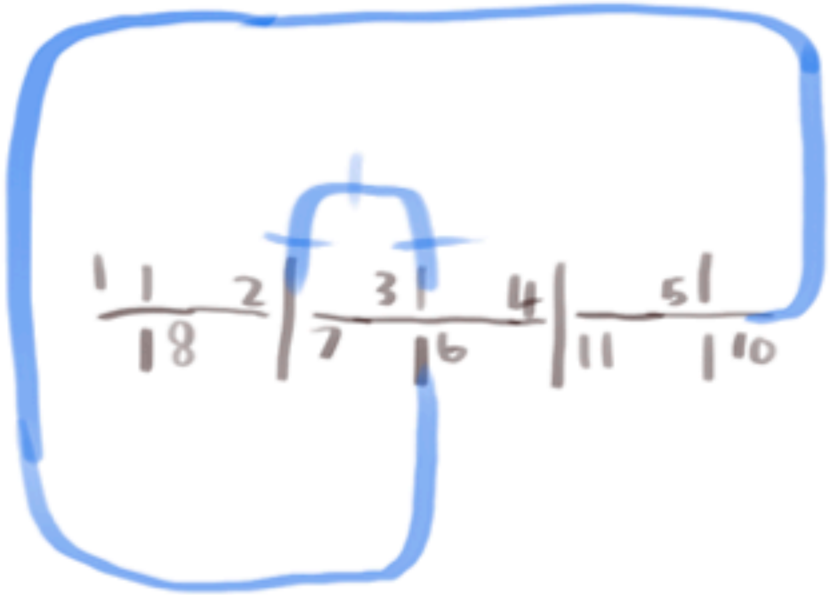


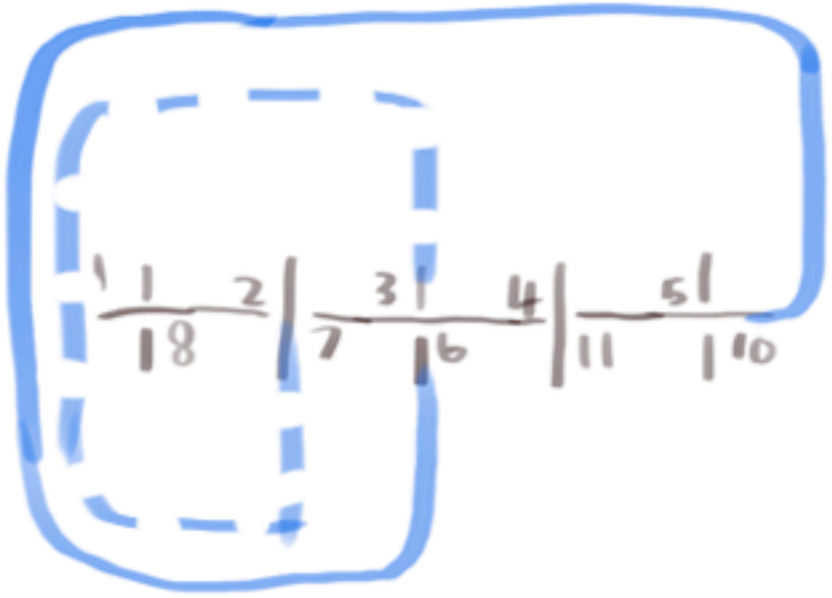
~~1~~ ~~3~~ ~~5~~ ~~7~~ 9 11

8 ~~6~~ 10 ~~2~~ 12 ~~4~~



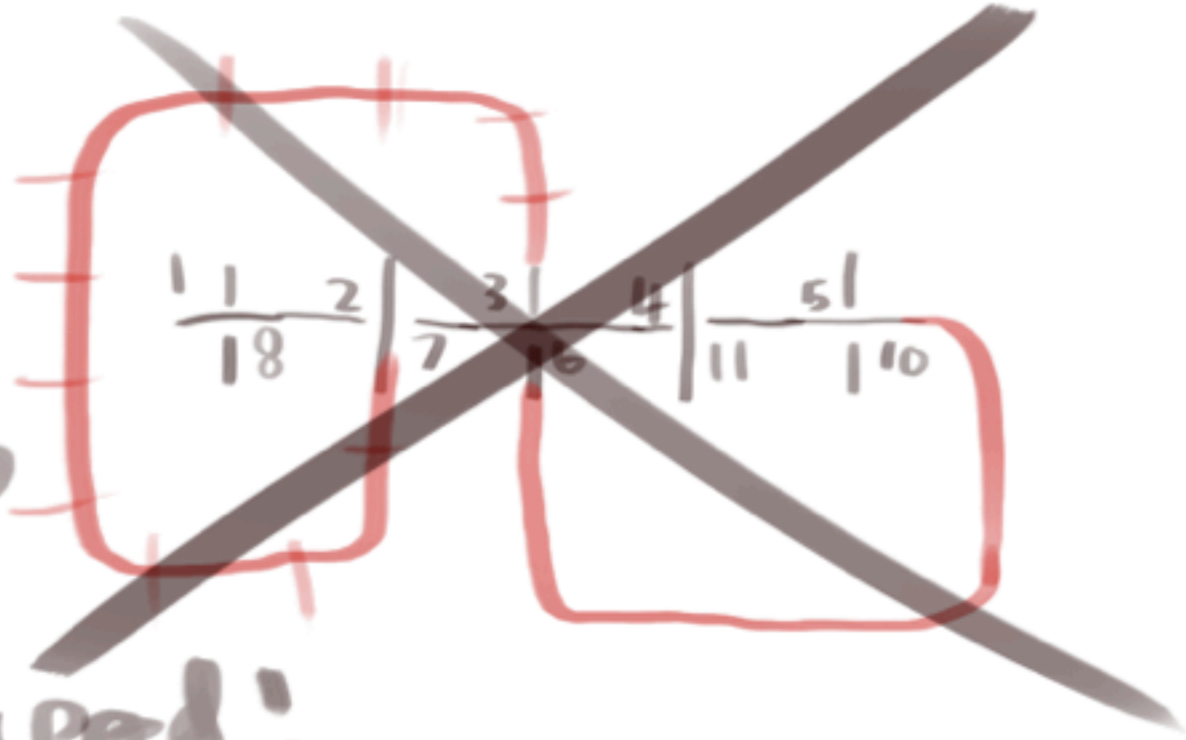




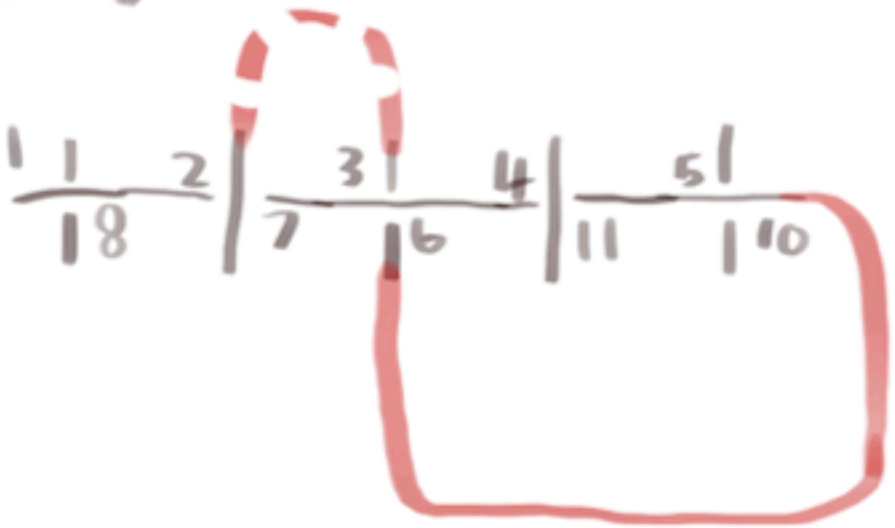


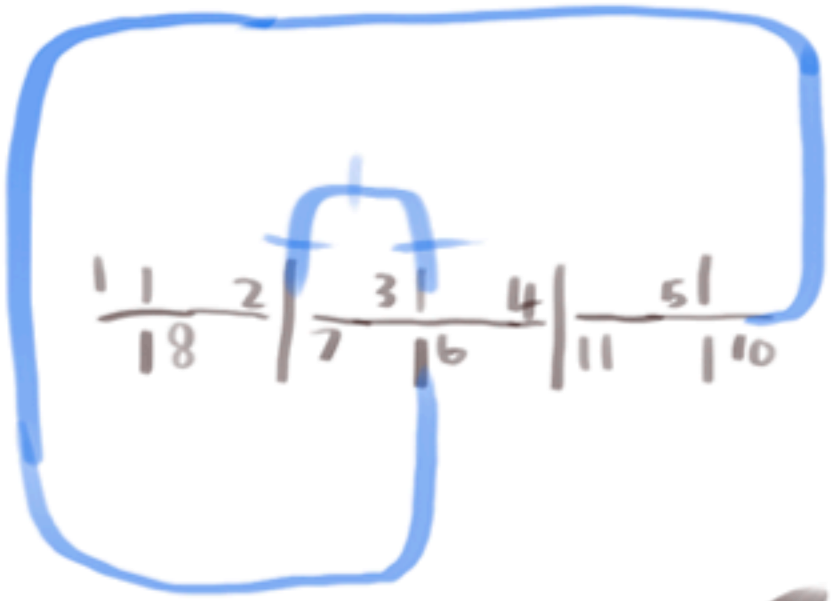
~~1~~ ~~3~~ ~~5~~ ~~7~~ 9 11

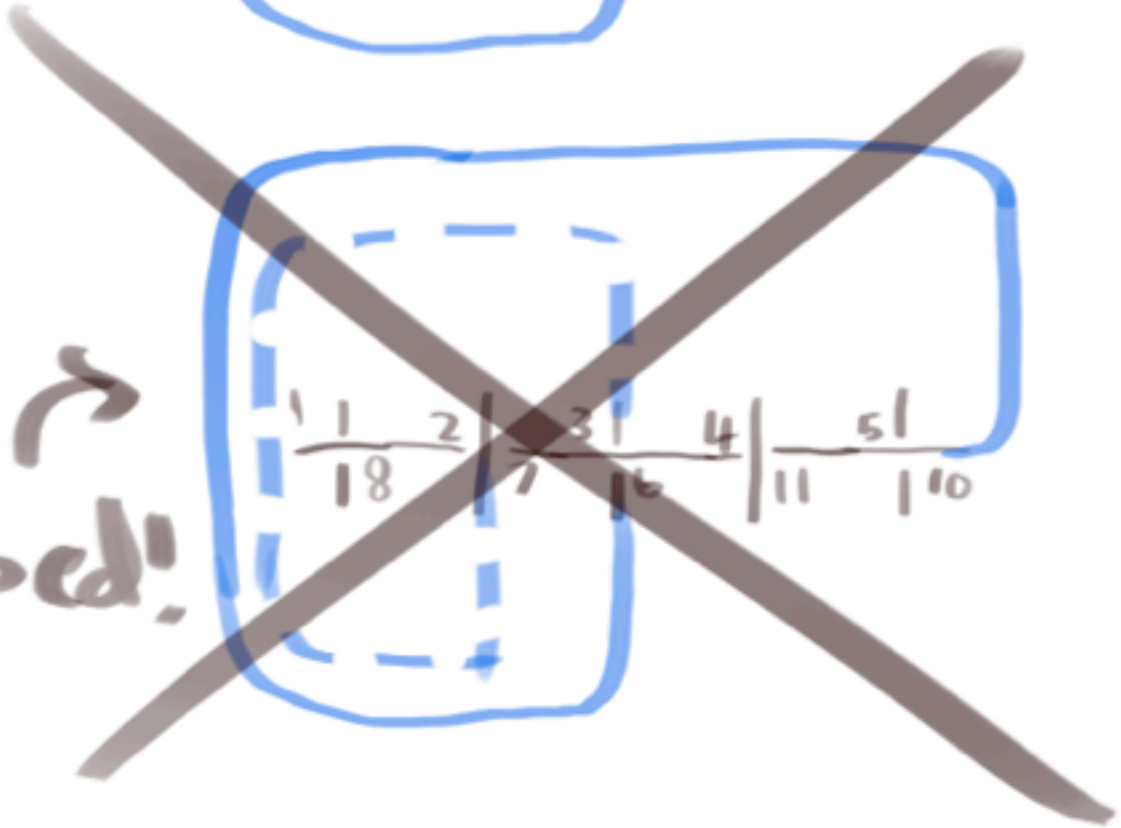
8 ~~6~~ 10 ~~2~~ 12 ~~4~~



Trapped!



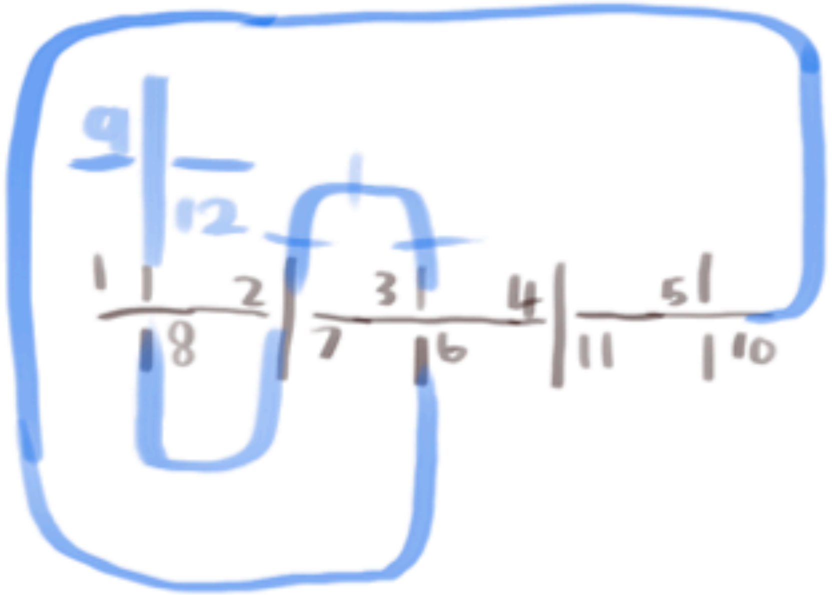


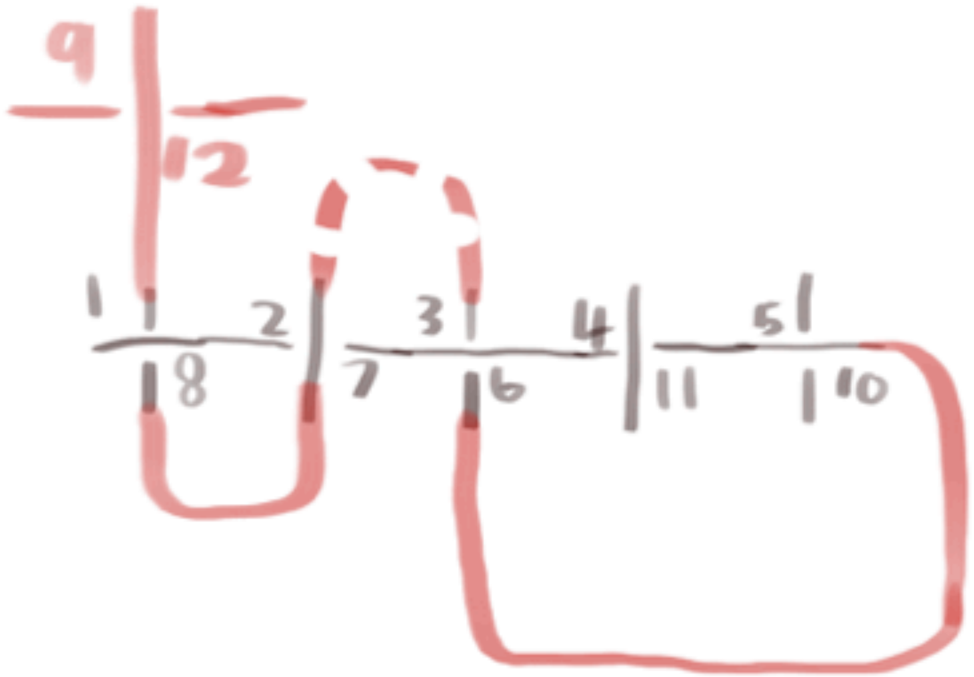


Trapped!

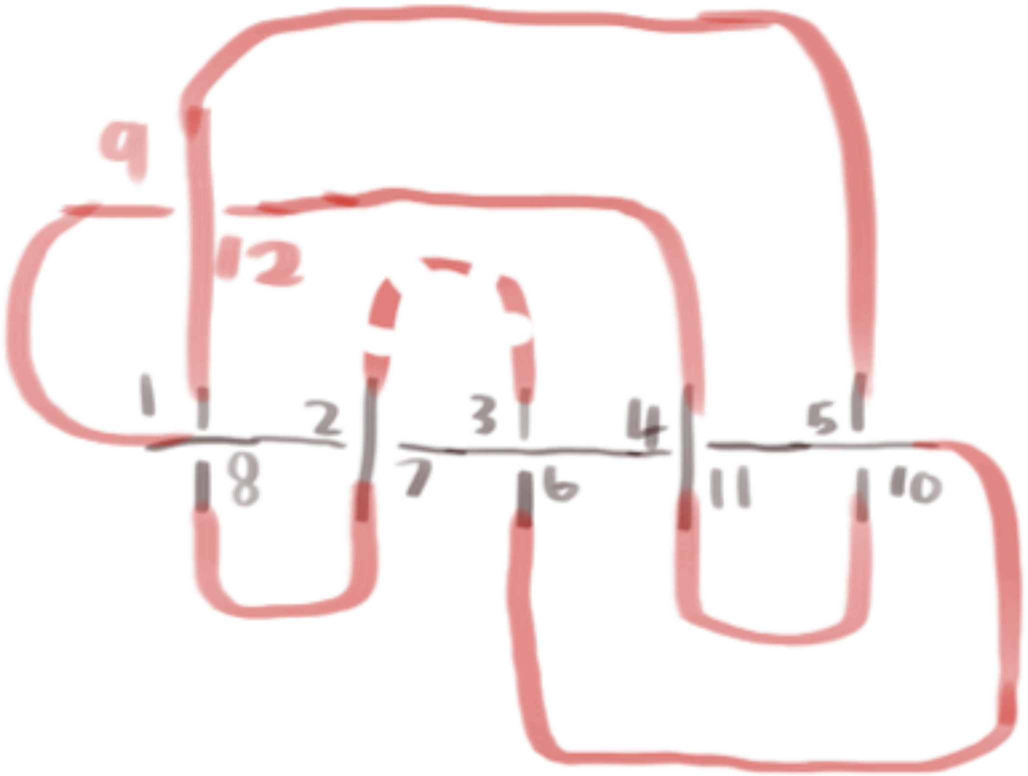
~~1~~	~~3~~	~~5~~	~~7~~	$\boxed{\cancel{9}}$	11
~~8~~	~~6~~	10	~~2~~	$\boxed{\cancel{12}}$	~~4~~

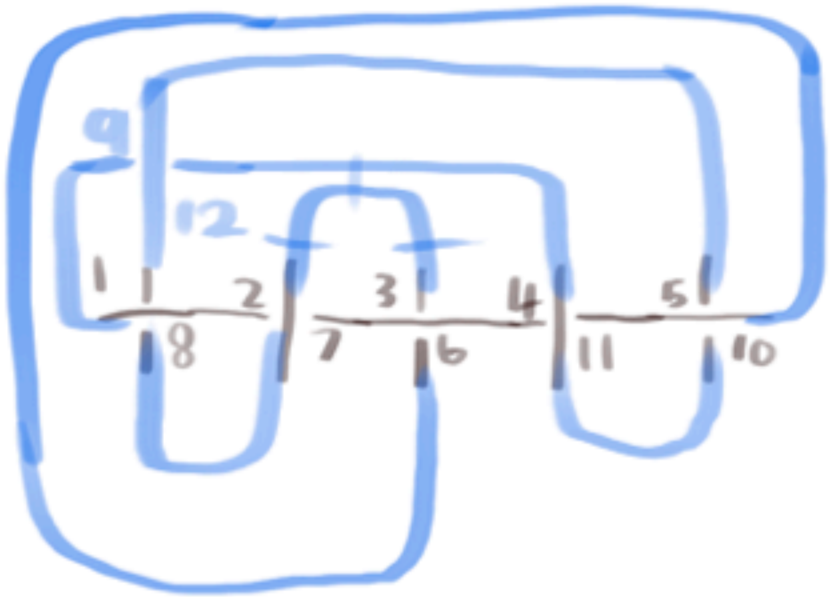
New crossing!



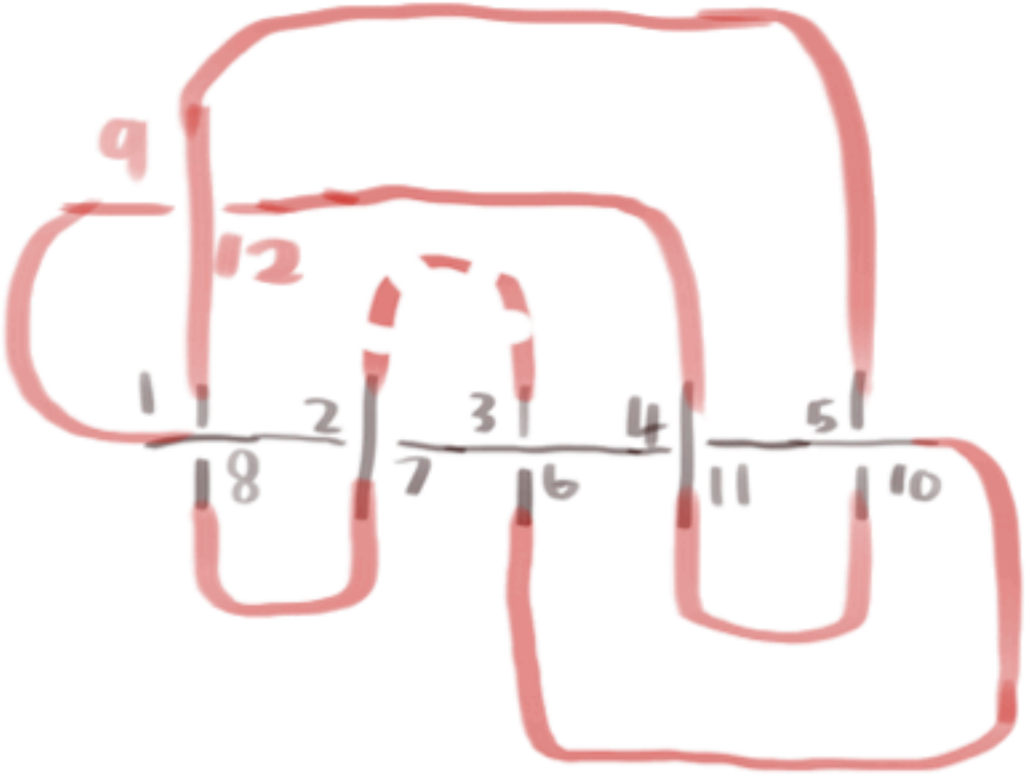


1	3	5	7	9	11
8	6	10	2	12	4

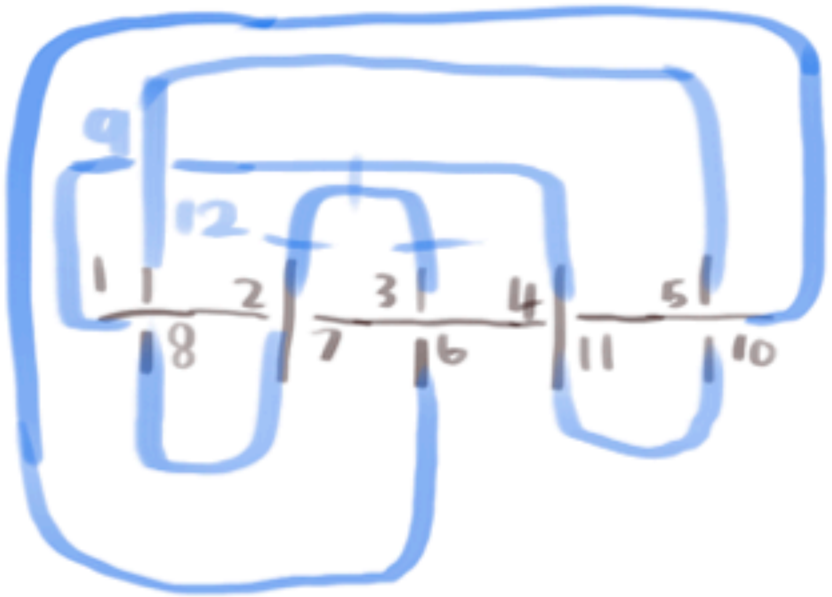




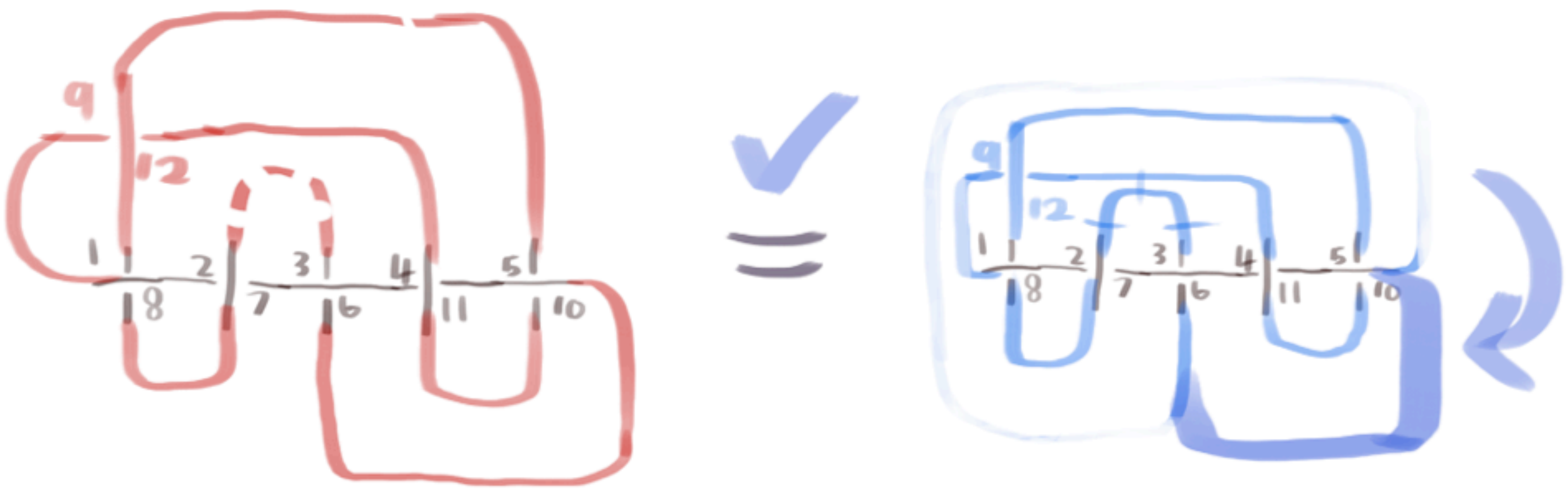
~~1~~	~~3~~	~~5~~	~~7~~	~~9~~	~~11~~
~~8~~	~~6~~	~~10~~	~~2~~	~~12~~	~~4~~

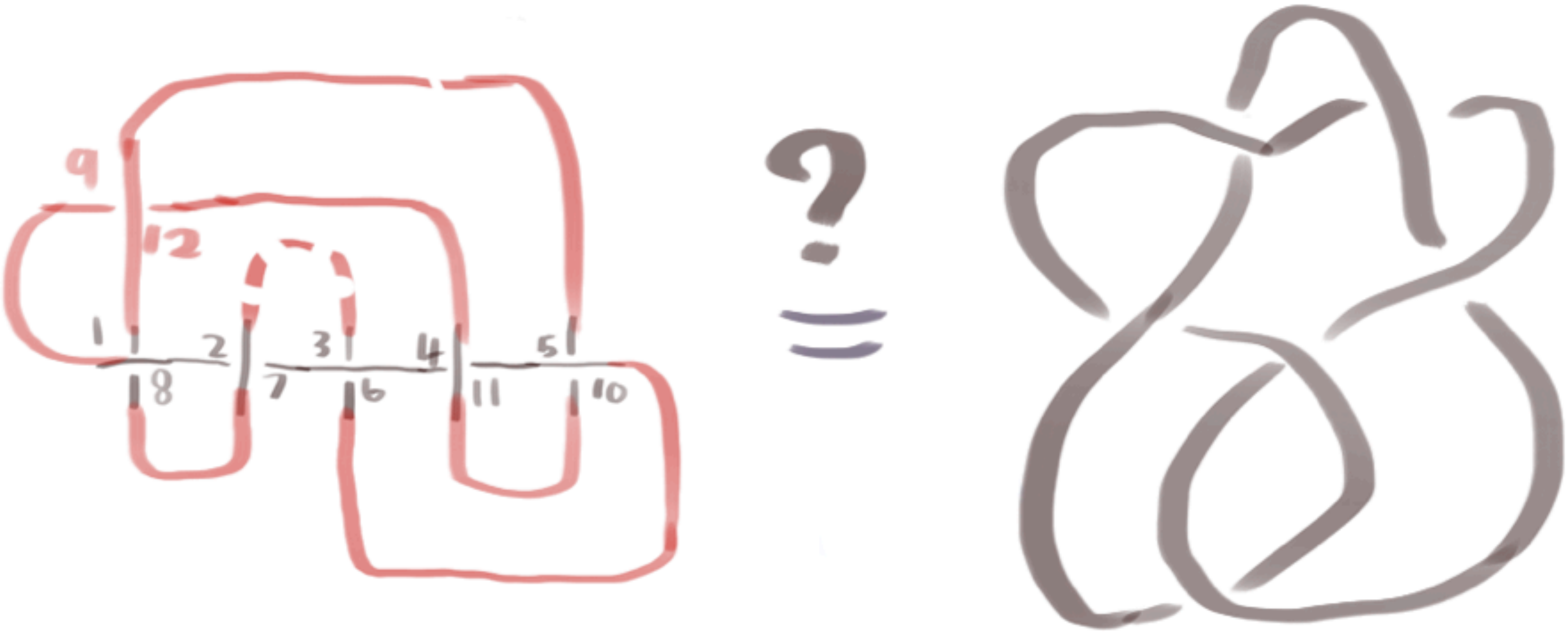


$\overset{?}{=}$



1	3	5	7	9	11
8	6	10	2	12	4




9
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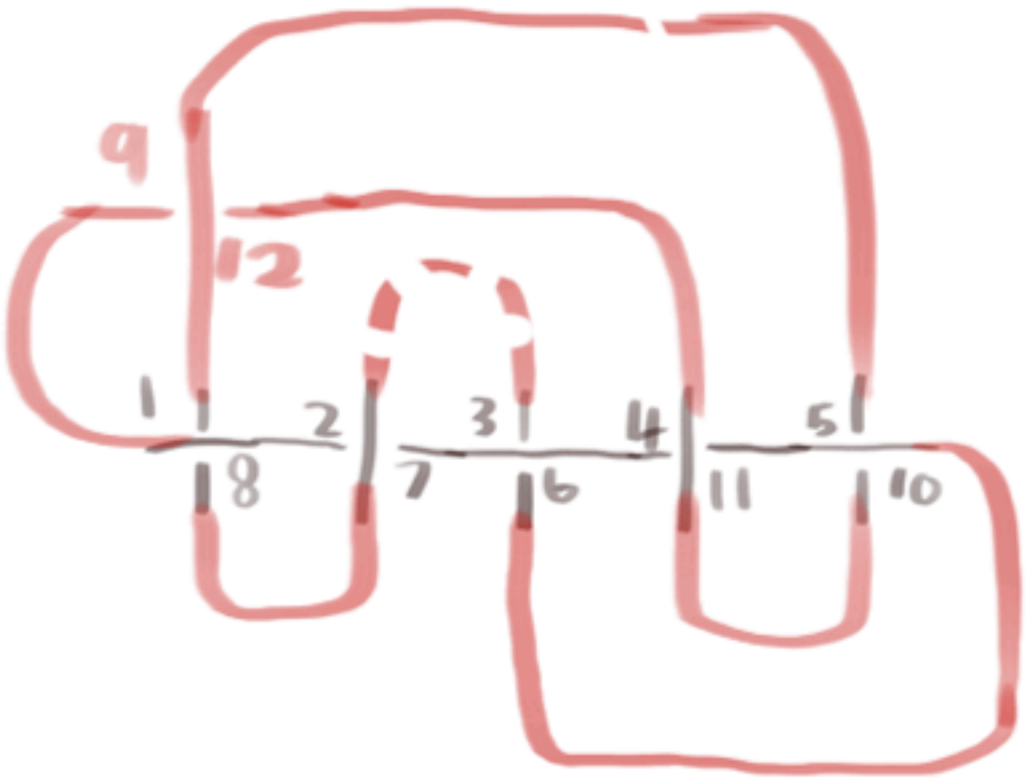
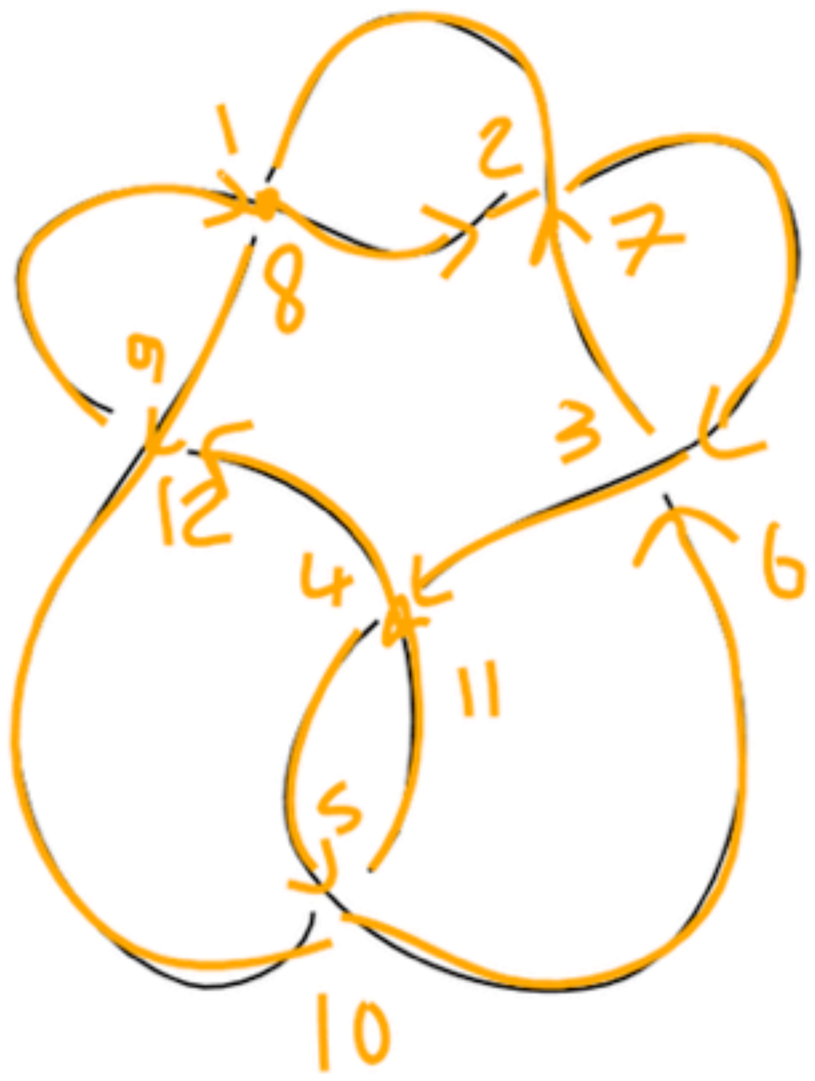
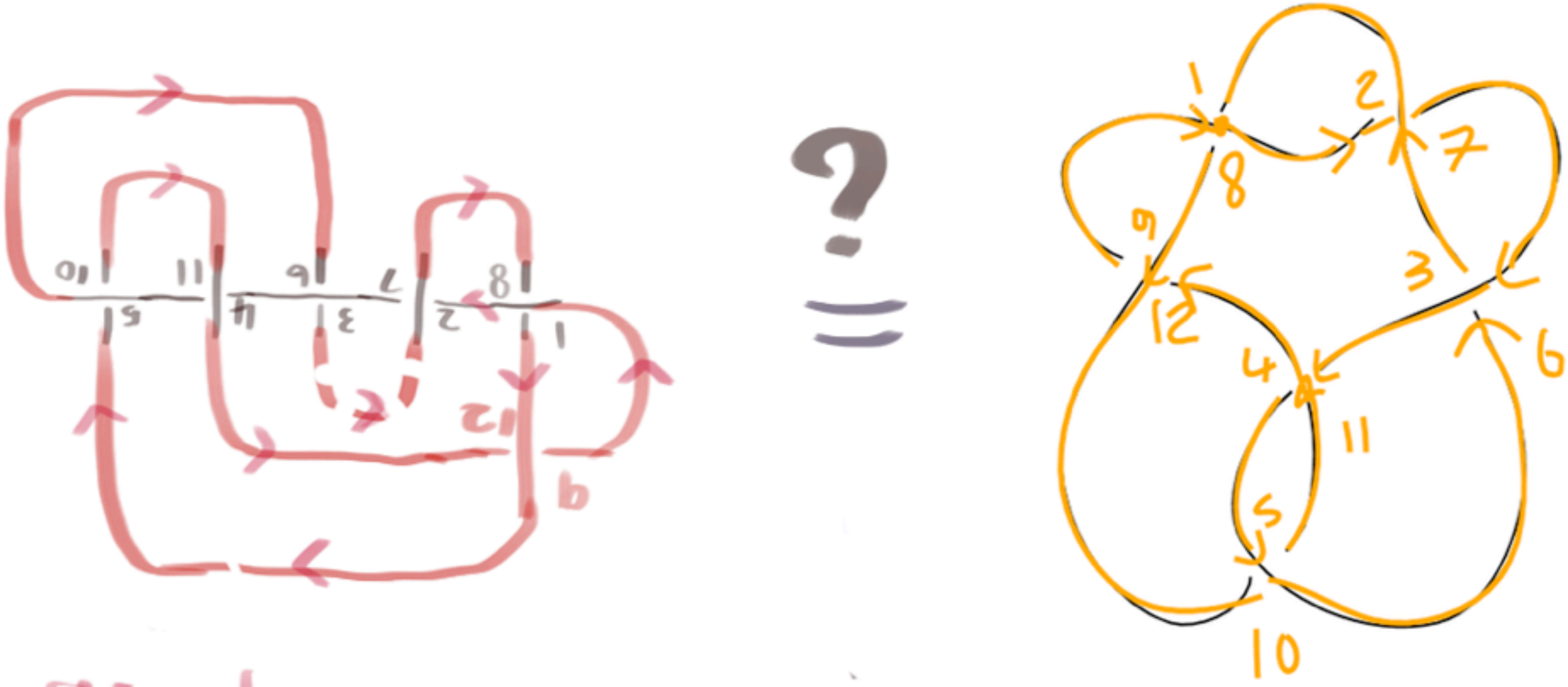


$\overset{?}{=}$

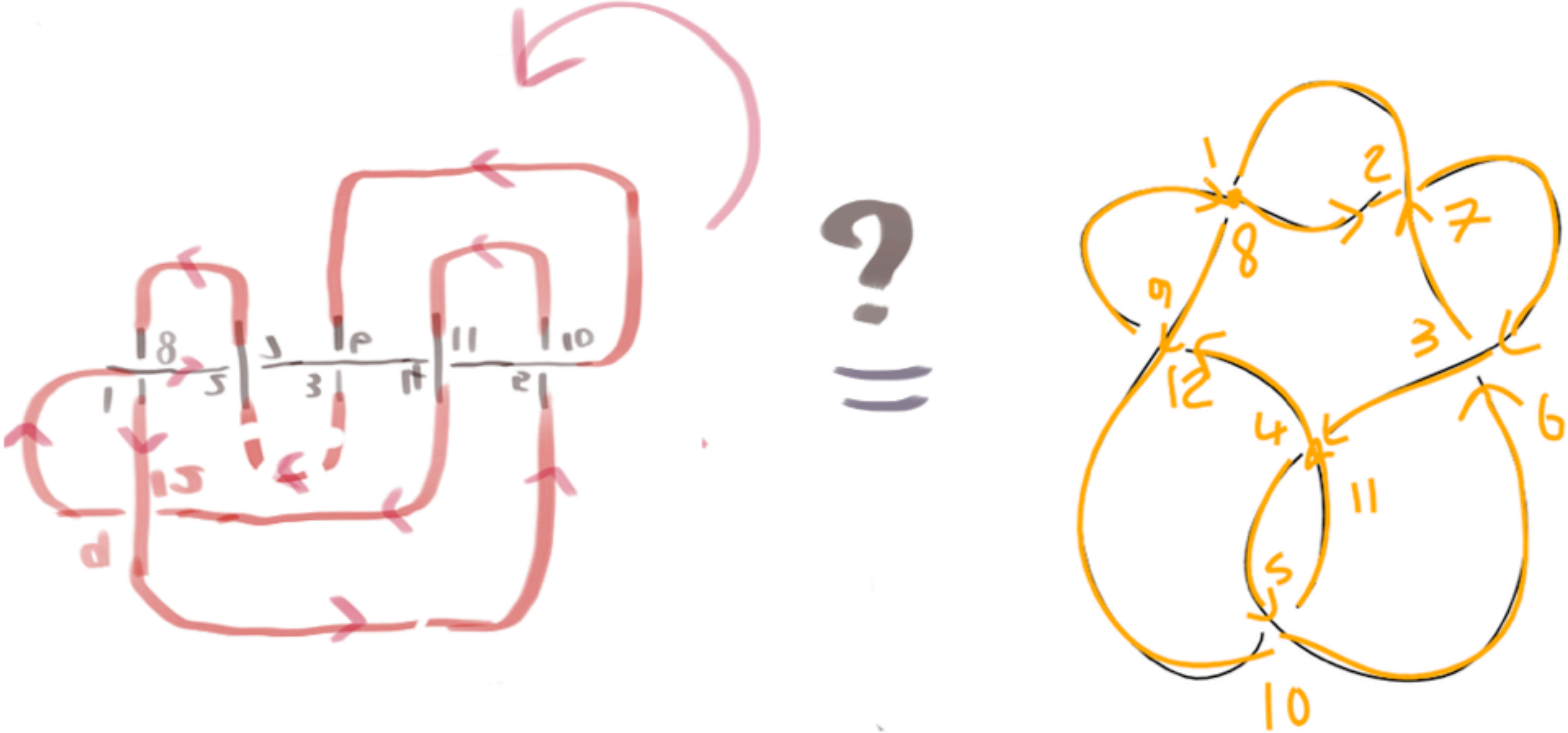


(Cheating!)

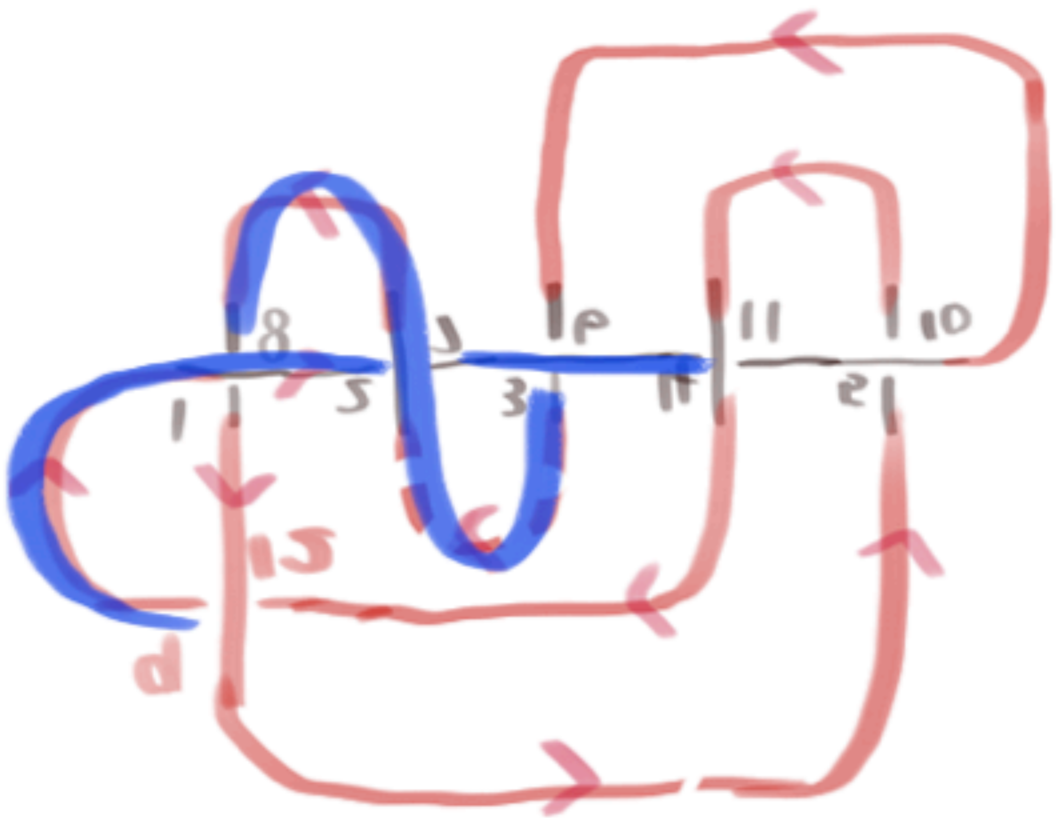


flip horiz.

(to correct beginning mistake)



rotate 180° ccw



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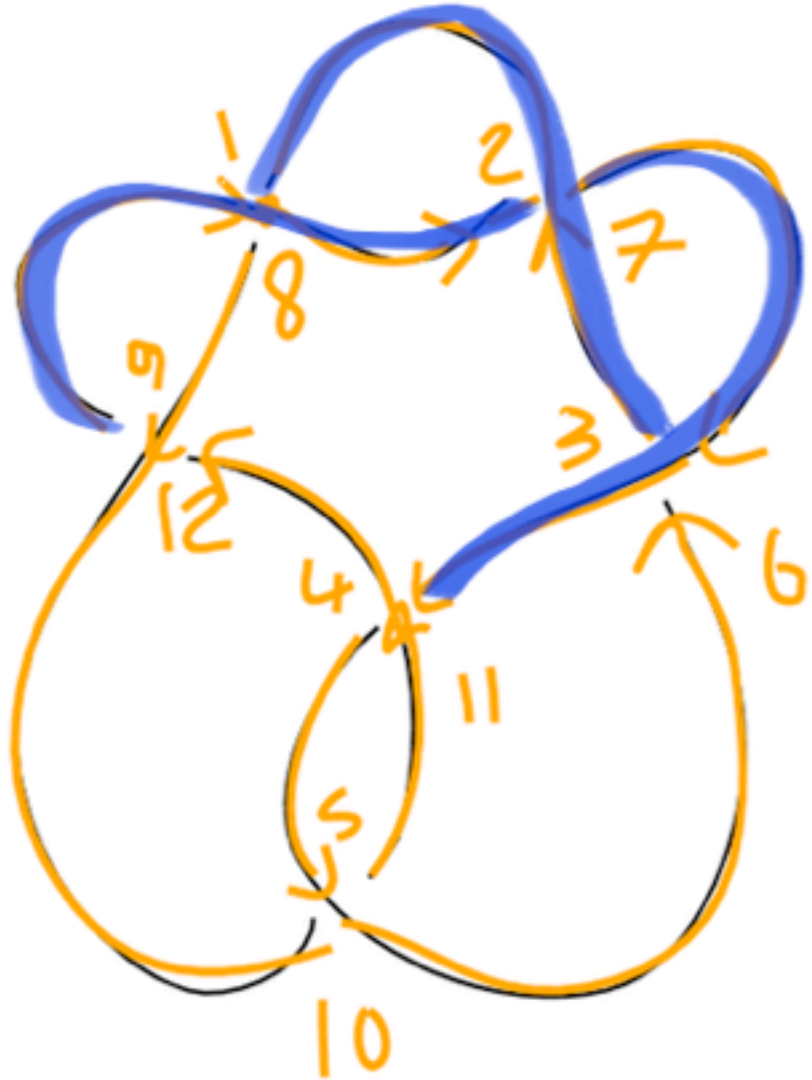

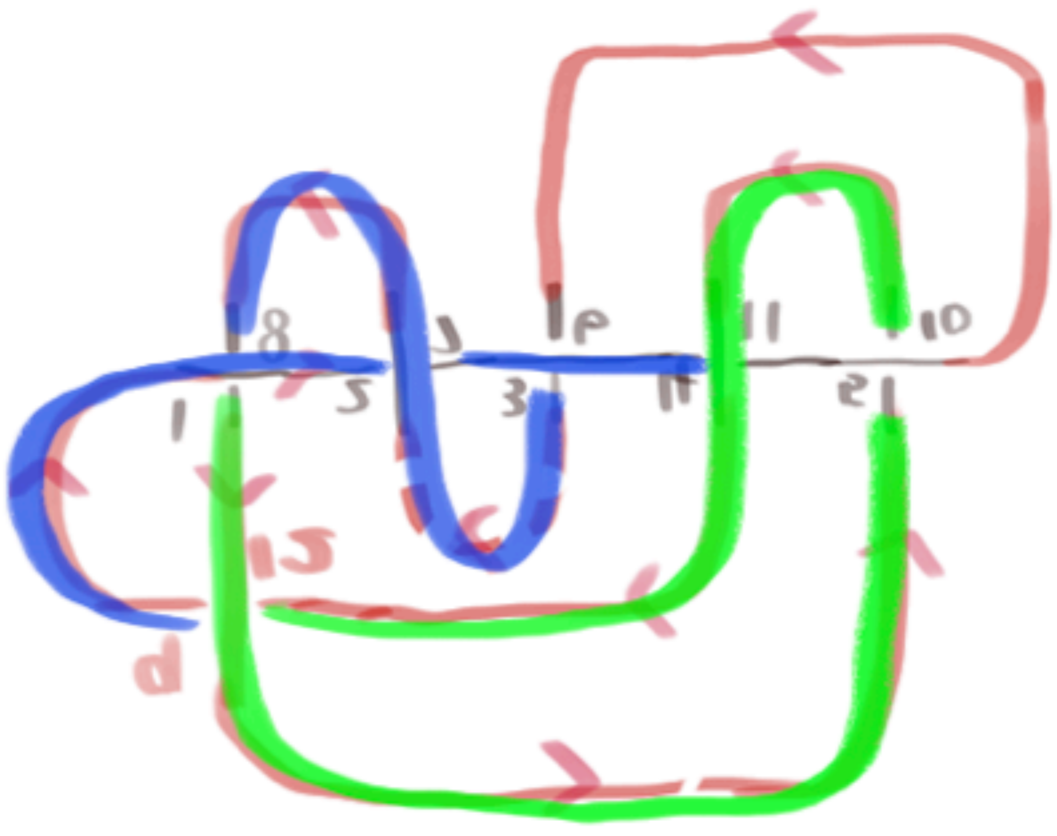

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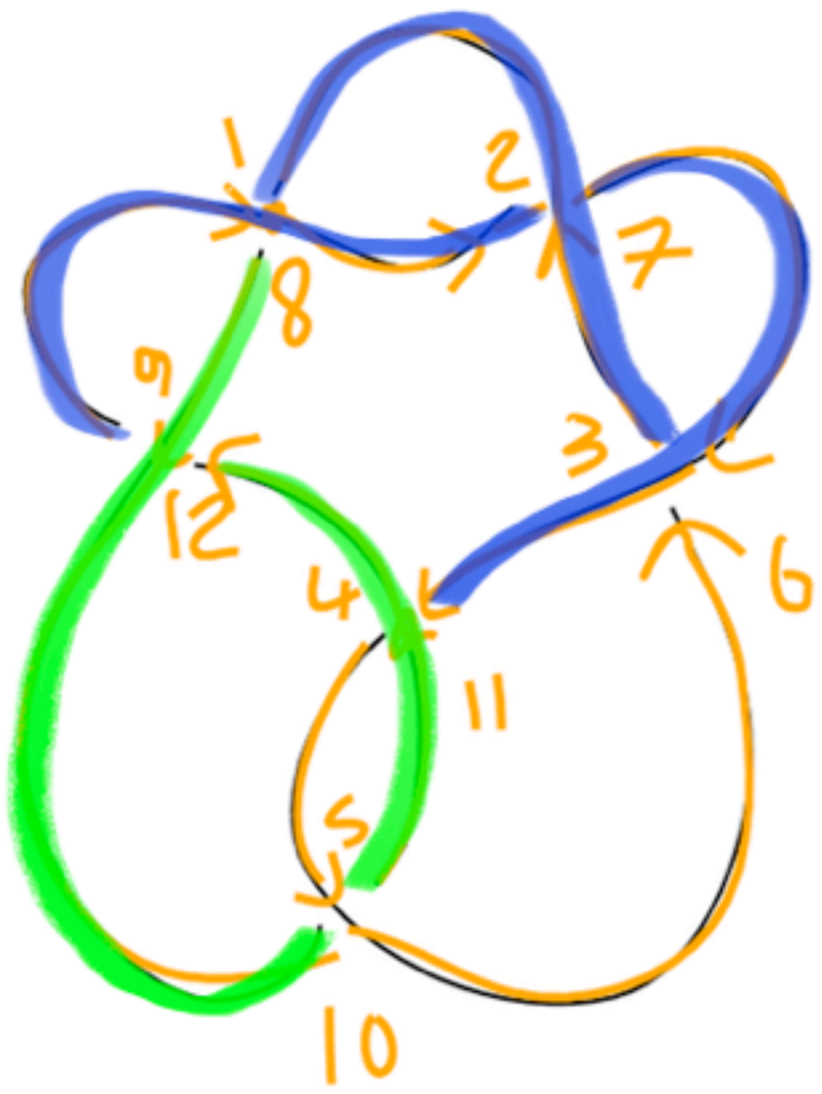
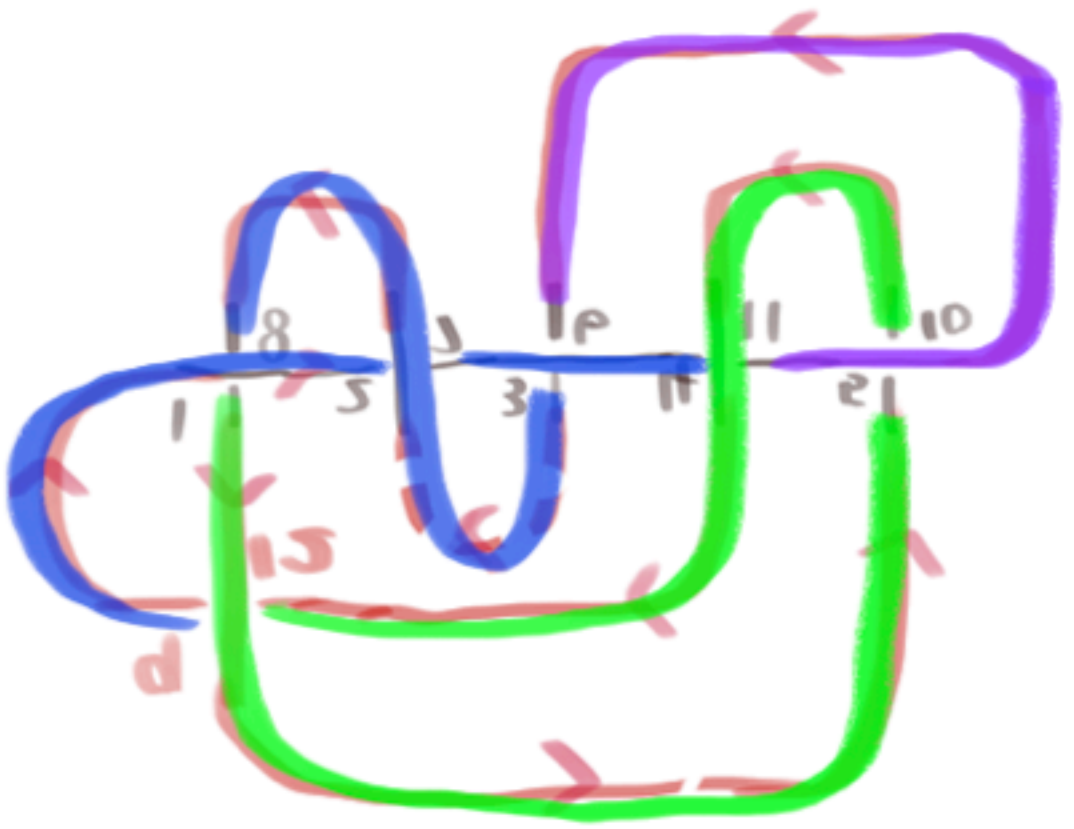
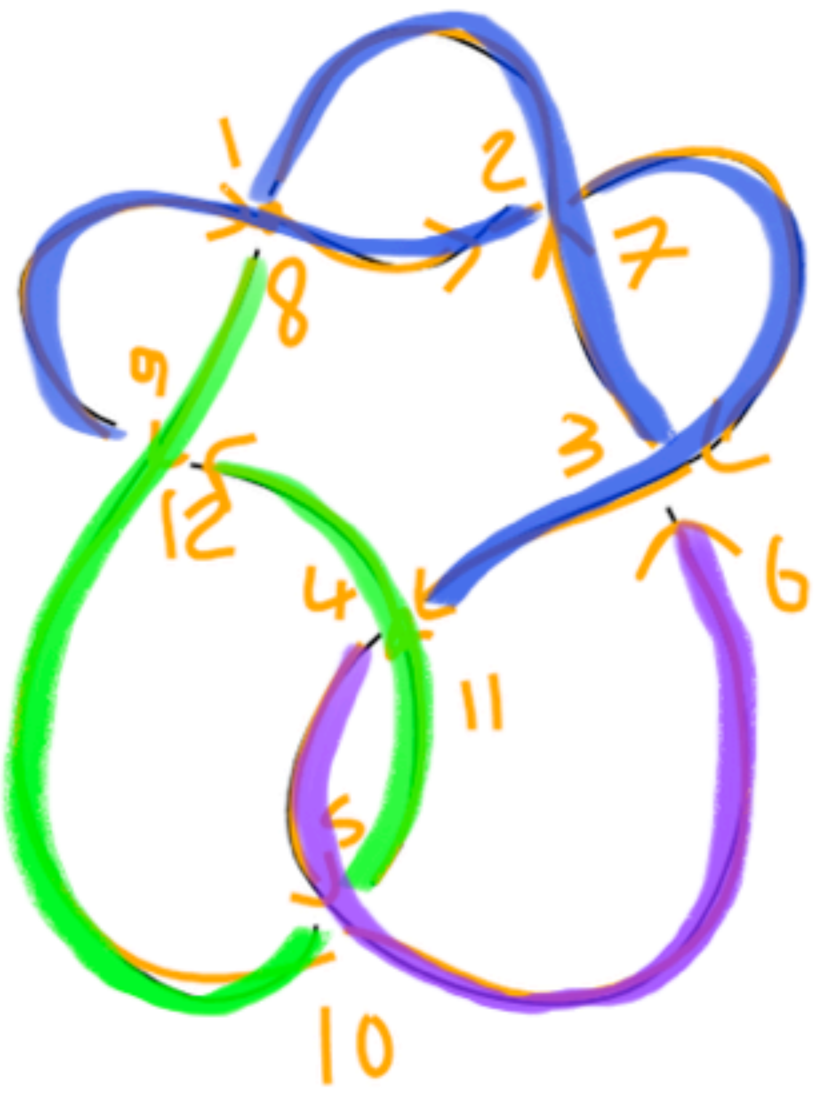
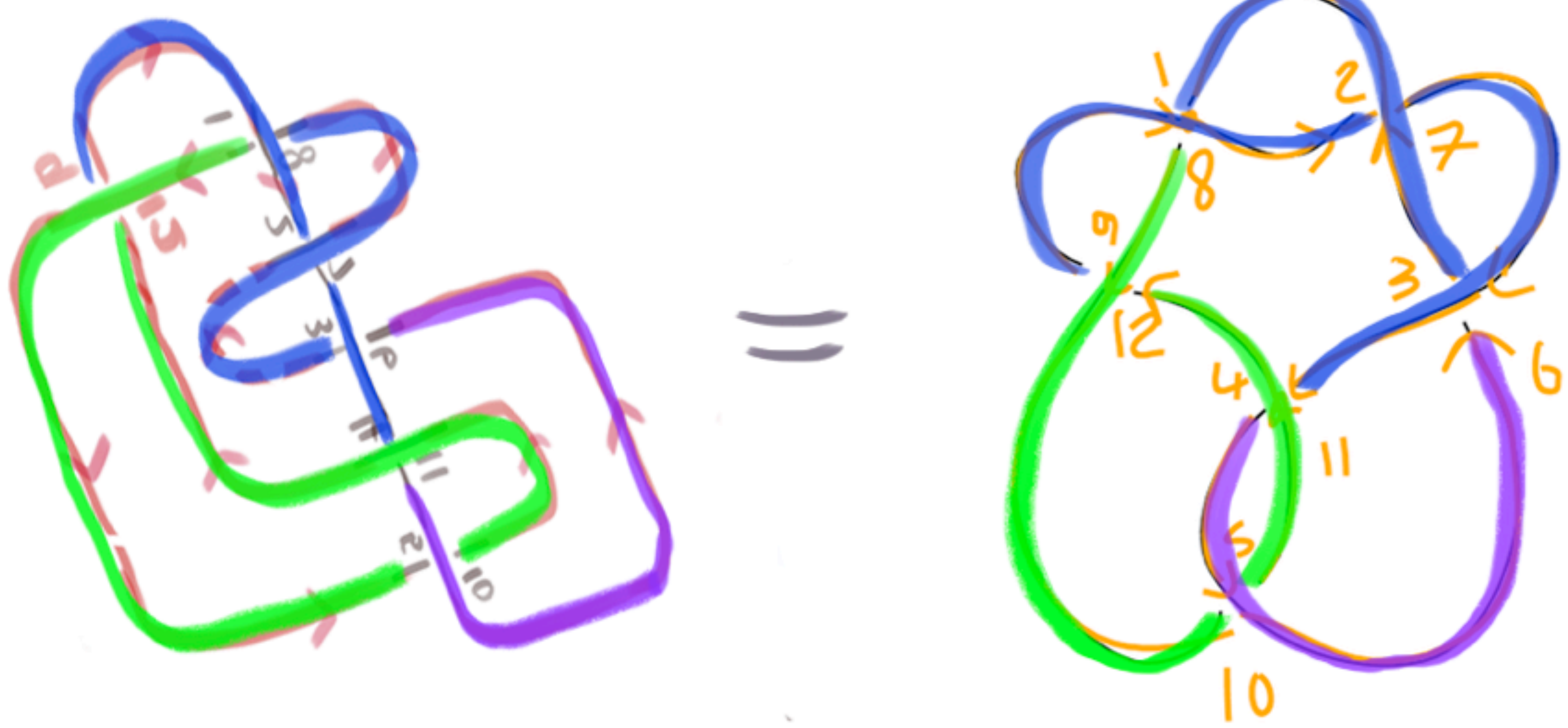




$\overset{?}{=}$





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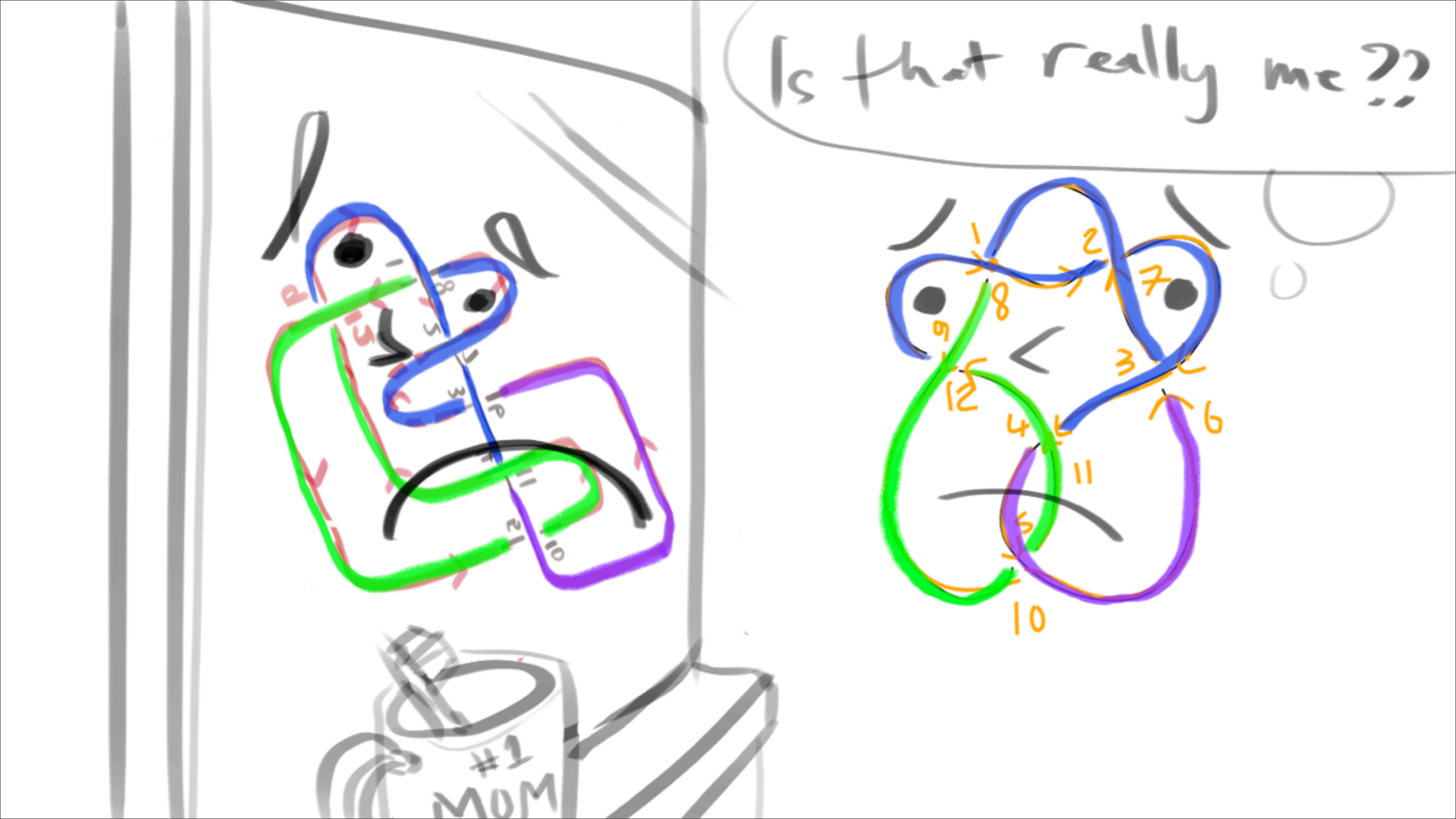
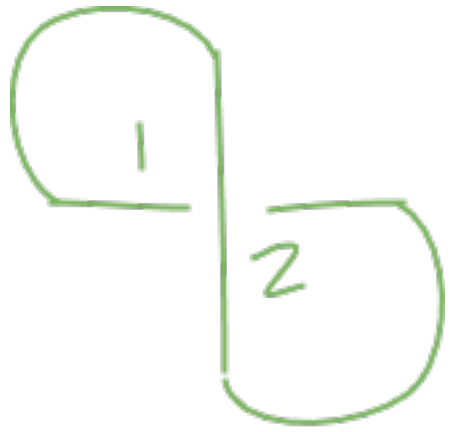

Is that really me??
#1 MOM


All Dowkers
All Knots
!

Enumerating with Dowker

```haskell
-- given # crossings, produce all knots (Dowker) with that #
crossings
-- includes duplicates and undrawable notations; ignores
non-alternating knots

dowkers :: Int -> [[(Int, Int)]]
dowkers n
 | n <= 0 = []
 | otherwise =
 let (odds, evens) = (take n [1, 3 ..], take n [2, 4 ..]) in
 map (zip odds) (permutations evens)
```

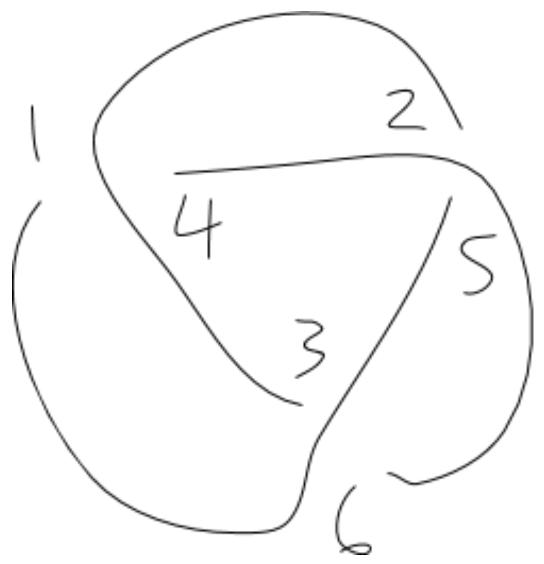
```
λ> dowkers 1
[[(1,2)]]

λ> dowkers 2
[[(1,2),(3,4)],[(1,4),(3,2)]]

λ> dowkers 3
[[(1,2),(3,4),(5,6)],
[(1,4),(3,2),(5,6)],
[(1,6),(3,4),(5,2)],
[(1,4),(3,6),(5,2)],
[(1,6),(3,2),(5,4)],
[(1,2),(3,6),(5,4)]]
```



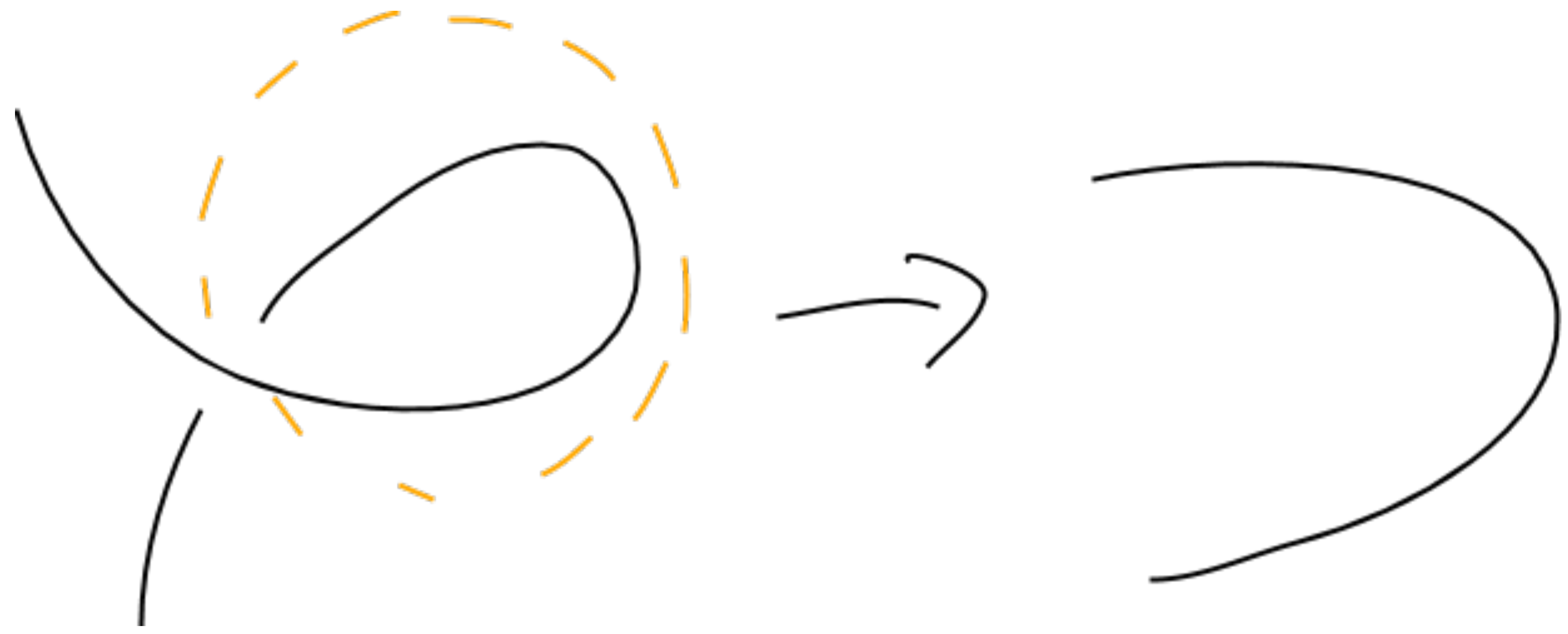
2015!

```
λ> dowkers 1
[[(1,2)]]

λ> dowkers 2
[[(1,2),(3,4)],[(1,4),(3,2)]]

λ> dowkers 3
[[(1,2),(3,4),(5,6)],
[(1,4),(3,2),(5,6)],
[(1,6),(3,4),(5,2)],
[(1,4),(3,6),(5,2)],
[(1,6),(3,2),(5,4)],
[(1,2),(3,6),(5,4)]]
```

```
λ> dowkers 1
[[(1,2)]]

λ> dowkers 2
[[(1,2),(3,4)],[(1,4),(3,2)]]

λ> dowkers 3
[[(1,2),(3,4),(5,6)],
[(1,4),(3,2),(5,6)],
[(1,6),(3,4),(5,2)],
[(1,4),(3,6),(5,2)],
[(1,6),(3,2),(5,4)],
[(1,2),(3,6),(5,4)]]
```



4 6 2



crossings labeled with “n” and “n+1” can always be removed

Outline

- Crash course in knot theory
- Alexander-Briggs notation
- Dowker notation
- **Intro to enumeration**
- Conway notation
- Modern tabulations
- Lessons


nerds
1870
Kelvin
Pb?
Tait

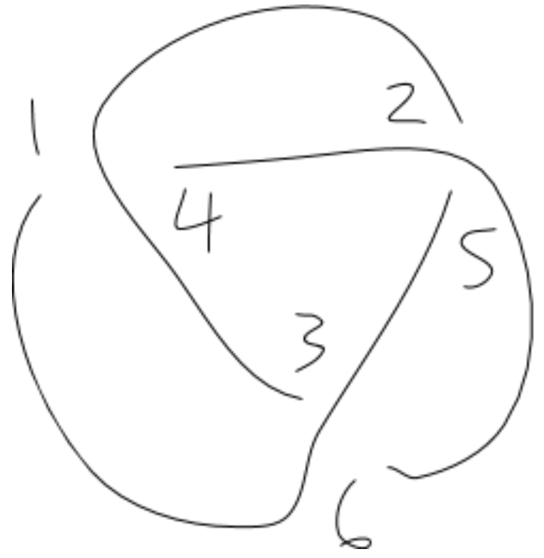
Name	Picture	Alexander- Briggs-Rolfsen	Dowker-Thistlethwaite	Dowker notation	Conway notation
Unknot		0_1	0a1	—	—
Trefoil knot		3_1	3a1	4 6 2	[3]
Figure-eight knot		4_1	4a1	4 6 8 2	[22]
Cinquefoil knot		5_1	5a2	6 8 10 2 4	[5]
Three-twist knot		5_2	5a1	4 8 10 2 6	[32]
Stevedore knot		6_1	6a3	4 8 12 10 2 6	[42]
6_2 knot		6_2	6a2	4 8 10 12 2 6	[312]
6_3 knot		6_3	6a1	4 8 10 2 12 6	[2112]

We hope that the census will serve as a rich source of examples and counterexamples and as a general testing ground for our collective intuition.

Hoste

Strategy

```
λ> dowkers 1
[[(1,2)]]

λ> dowkers 2
[[(1,2),(3,4)],[(1,4),(3,2)]]

λ> dowkers 3
[[(1,2),(3,4),(5,6)],
[(1,4),(3,2),(5,6)],
[(1,6),(3,4),(5,2)],
[(1,4),(3,6),(5,2)],
[(1,6),(3,2),(5,4)],
[(1,2),(3,6),(5,4)]]
```



4 6 2

Schools of notation

Break down:
Gauss (1800), Tait, Dowker

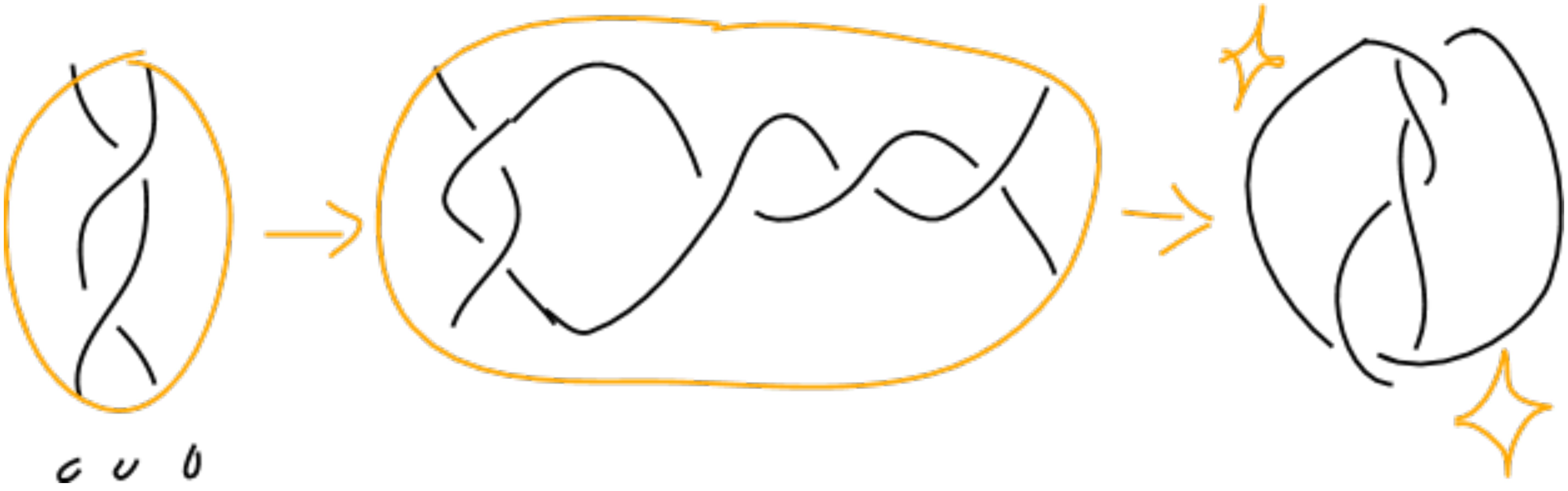
Build up:
Kirkman (1880), Conway, Caudron

Outline

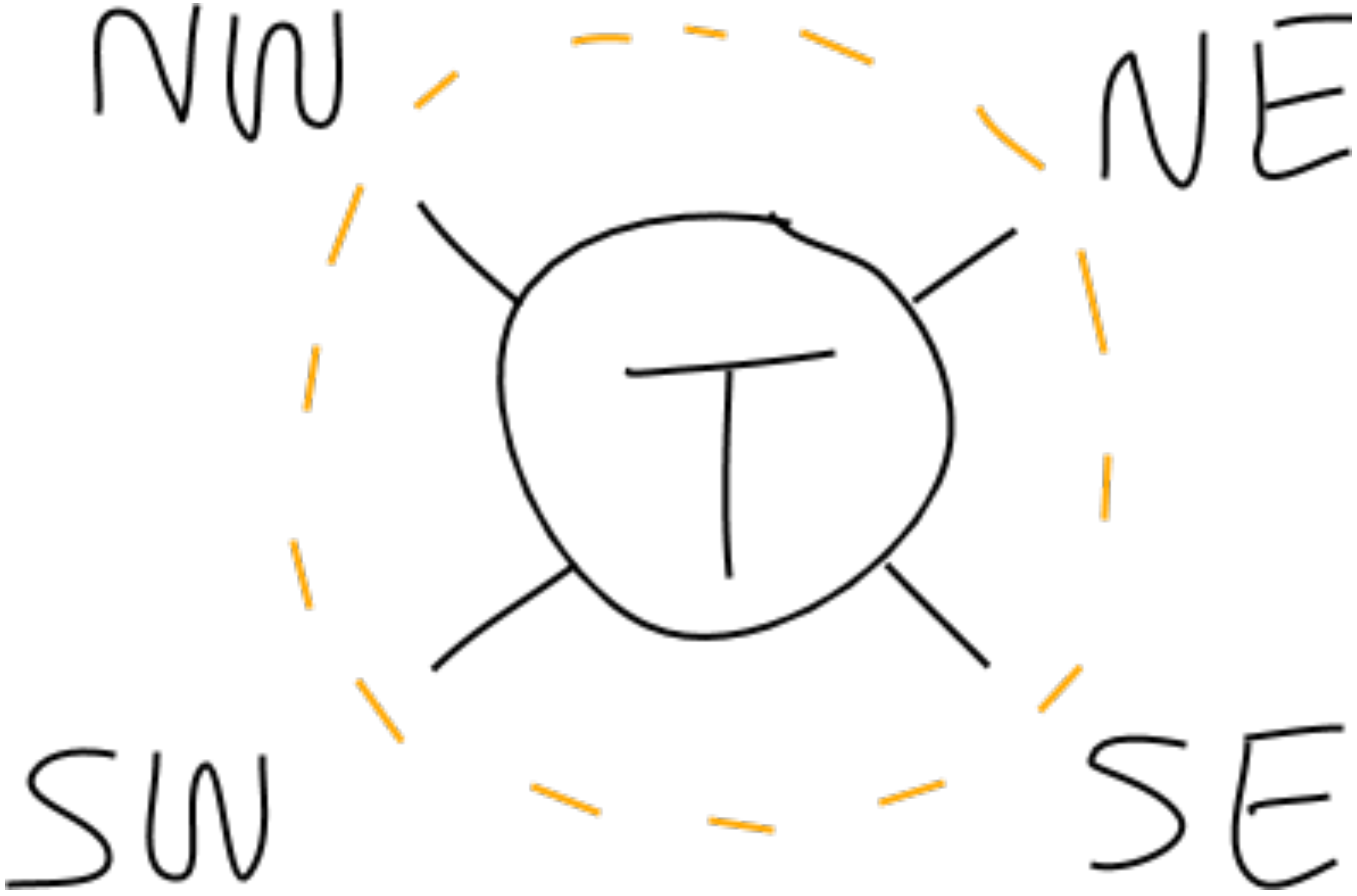
- Crash course in knot theory
- Alexander-Briggs notation
- Dowker notation
- Intro to enumeration
- **Conway notation**
- Modern tabulations
- Lessons

Little tells us that the enumeration of the 54 knots of [6] took him 6 years—from 1893 to 1899—the notation we shall soon describe made this just **one afternoon's work!**

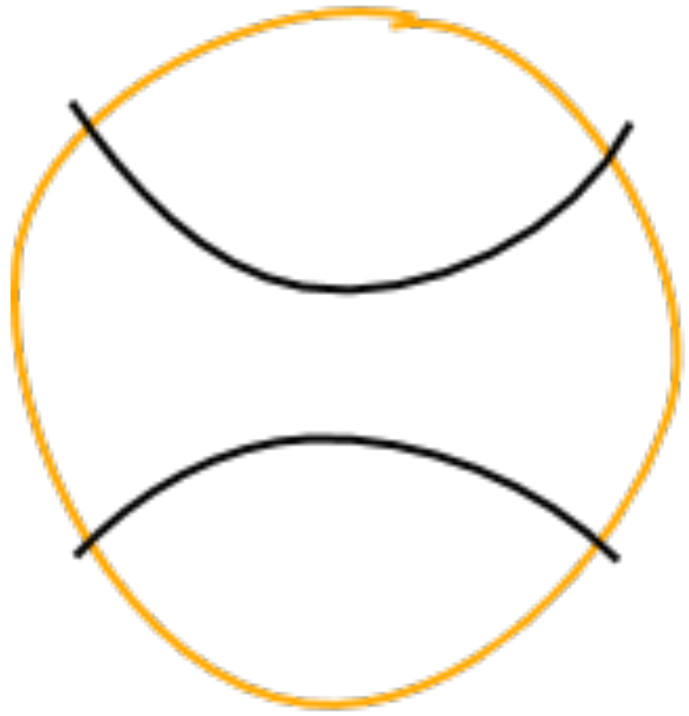
Twists Tangles Knots



The Knot Book



NW
NE
T
SW
SE



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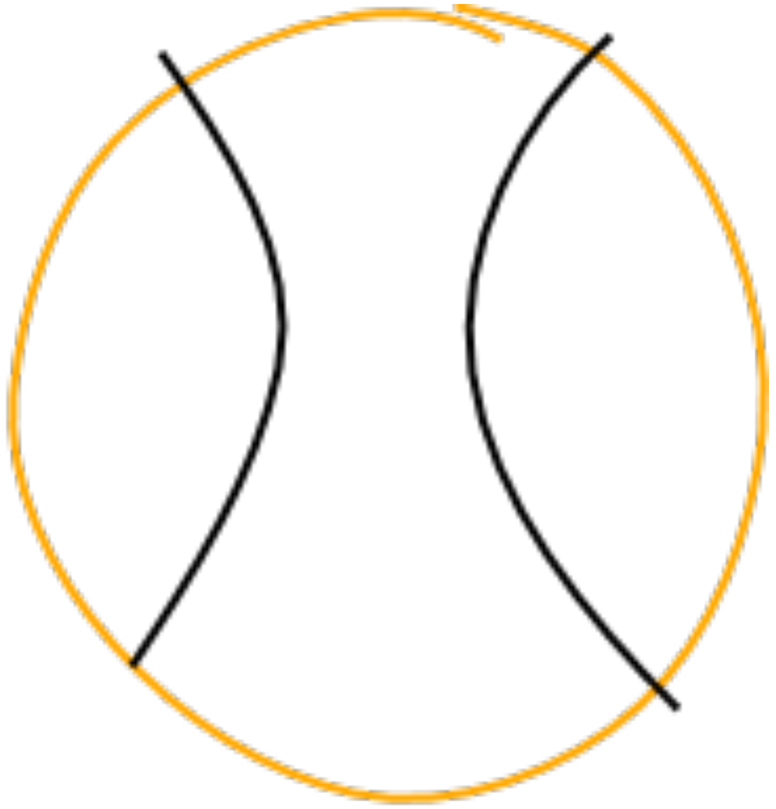

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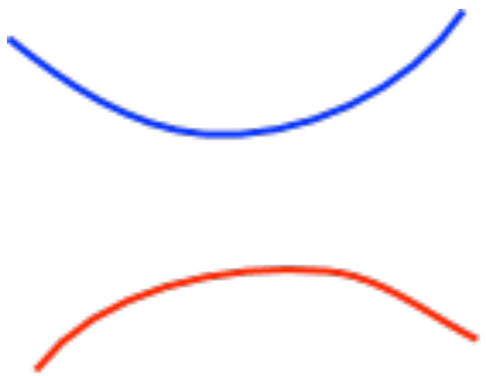
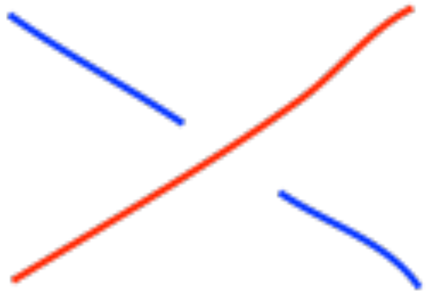
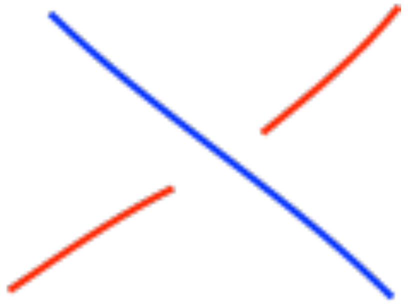


0



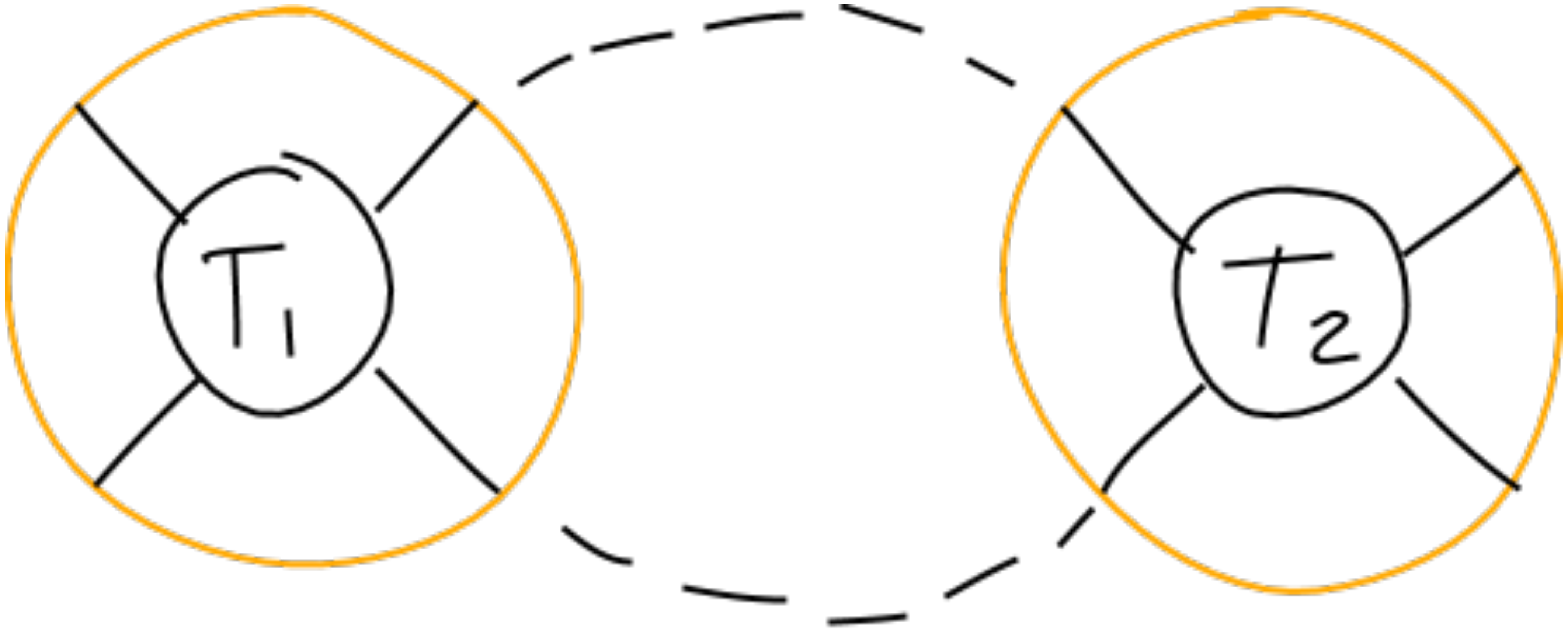
1



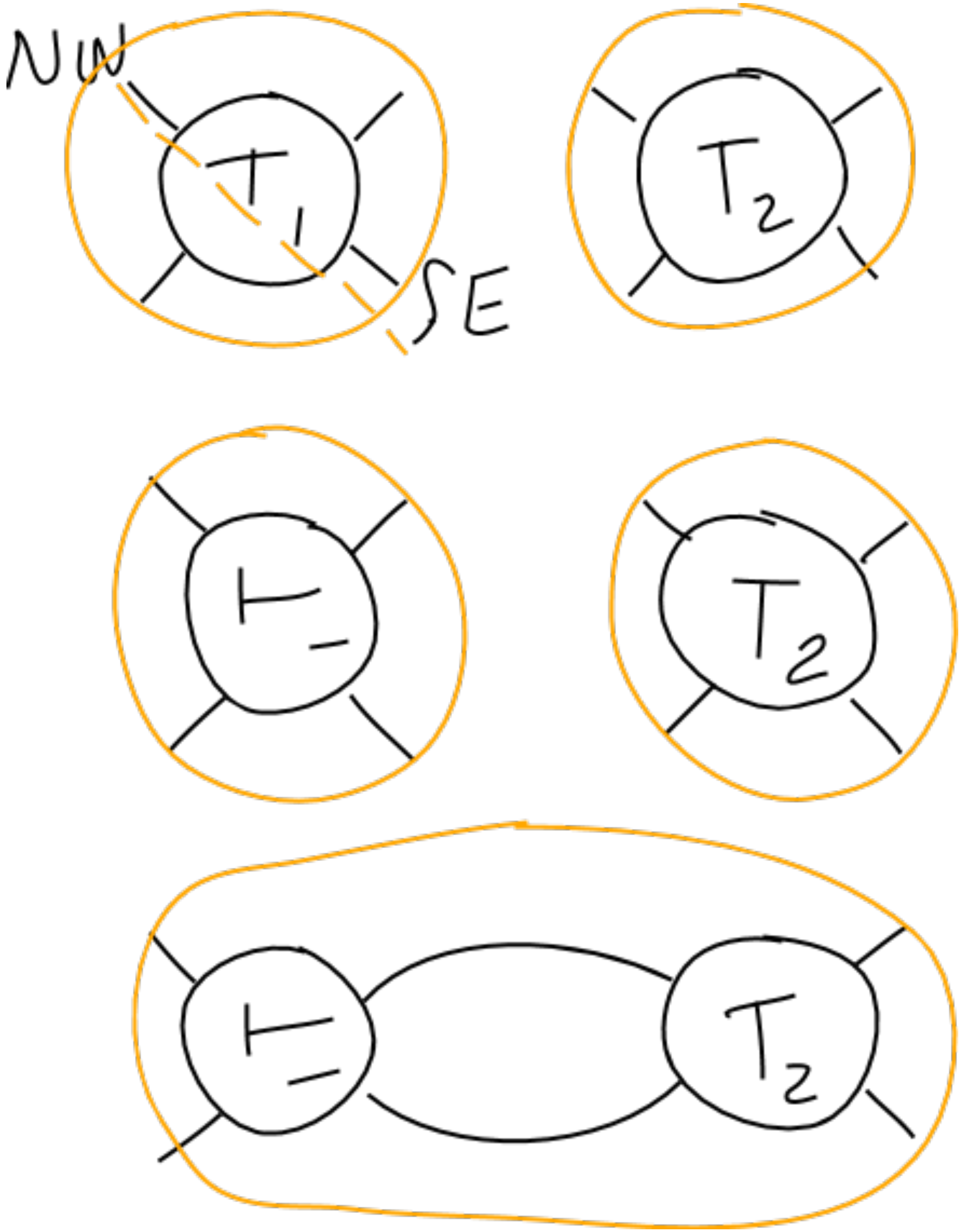
-1

$1 + 1 + 1$

$= 3$



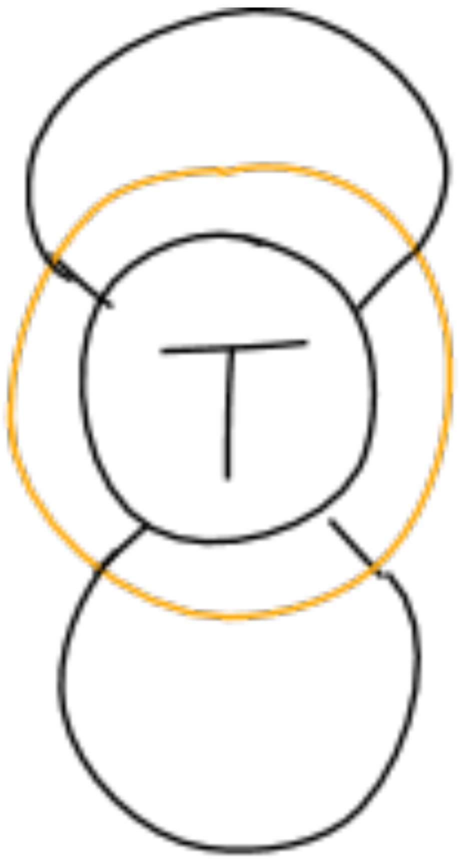
$$T_1 + T_2$$



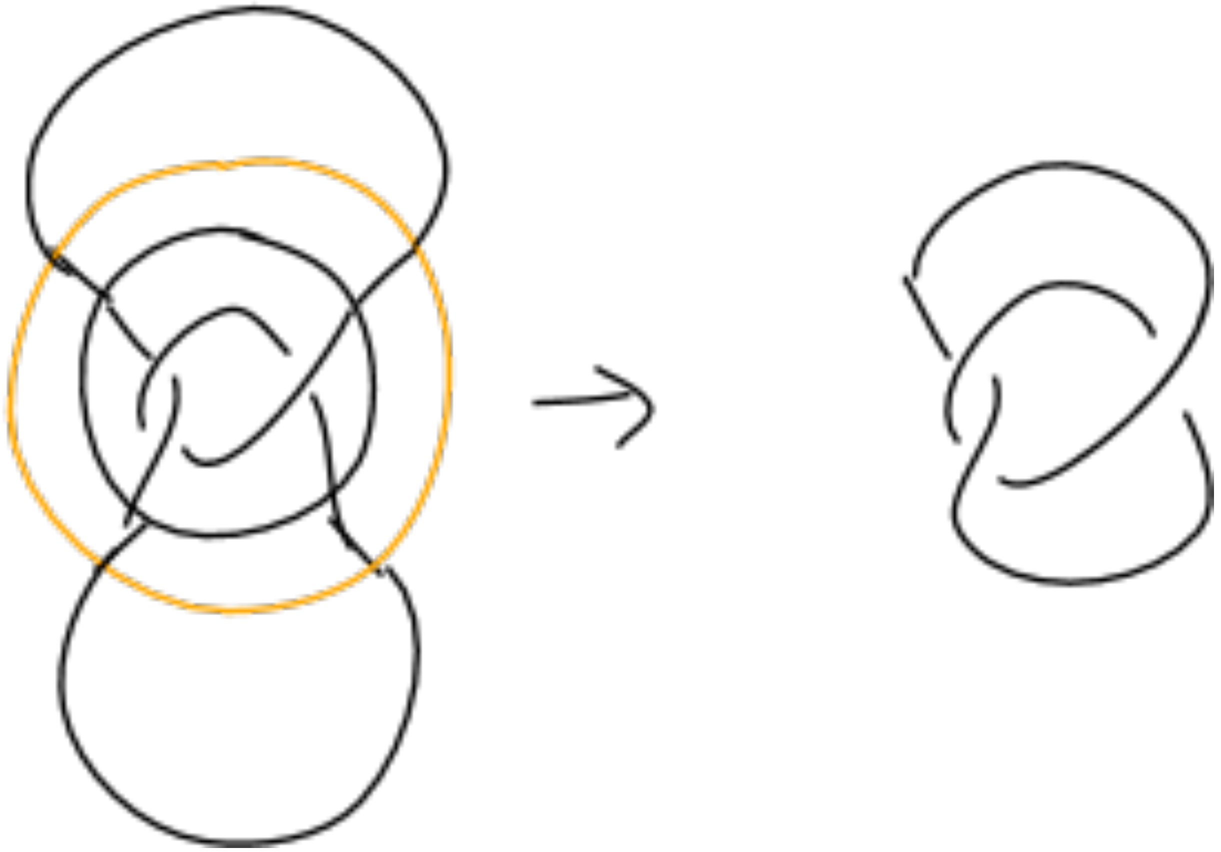
Multiplication =
flip, then add!

$T_1 * T_2$

1^*



2 1
knot



Weird? Arbitrary?
Well, yes and no...

(3) 2 -4:

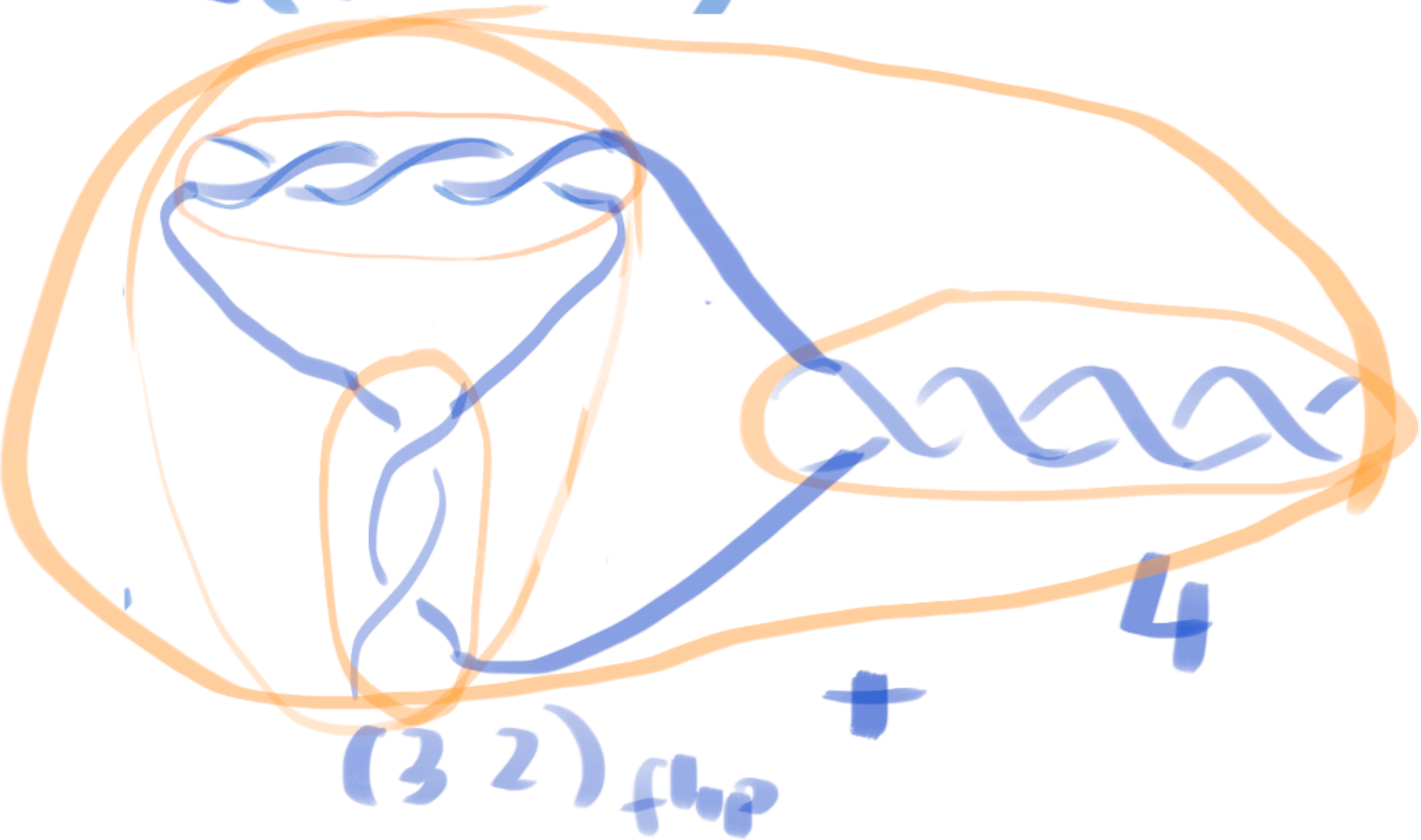
$((3)\ 2)-4:$

$((3)\ 2) - 4$:

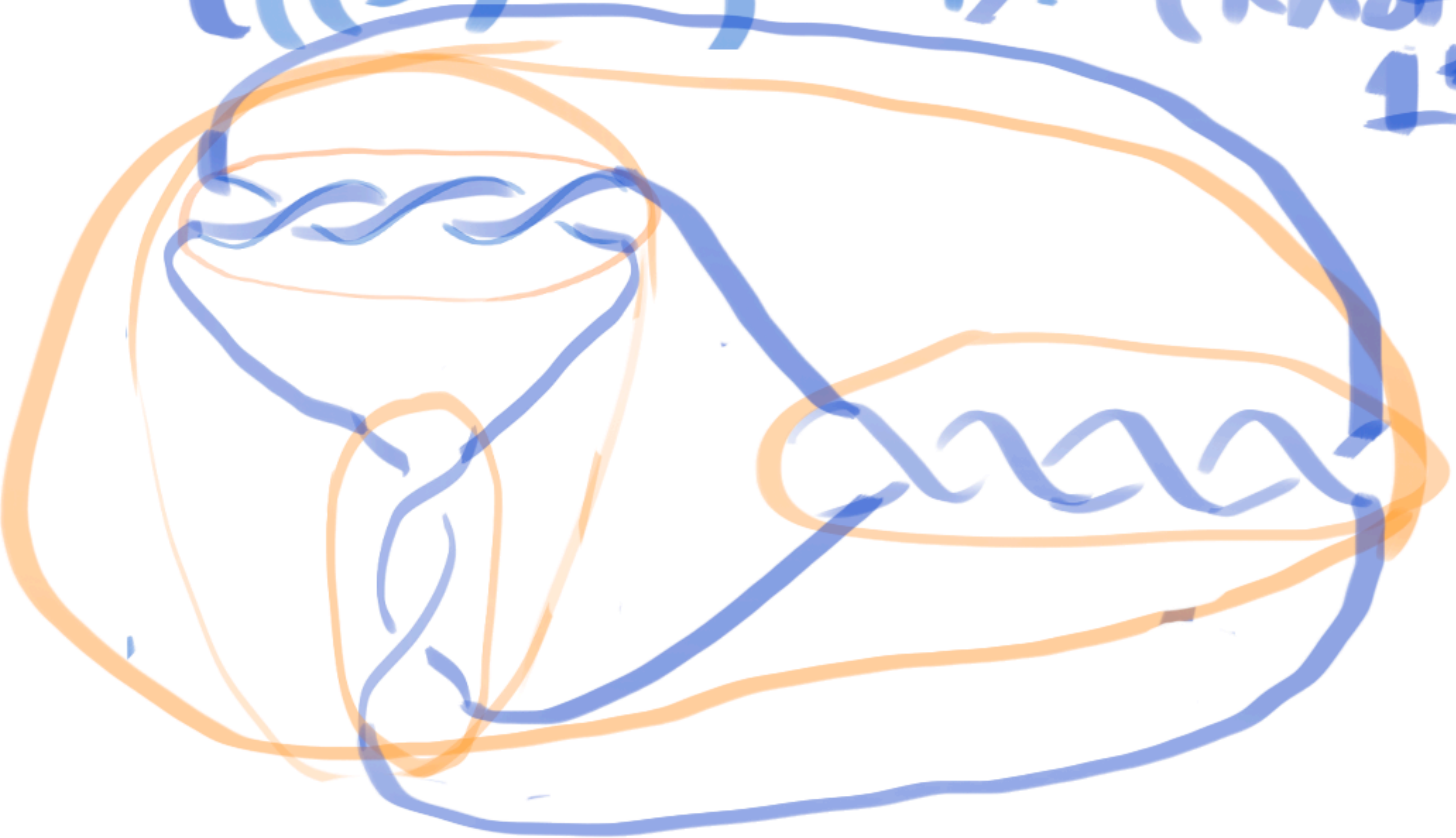
3 flip

$((3)\ 2) - 4$:

$(((3)\ 2)\ -4)$:



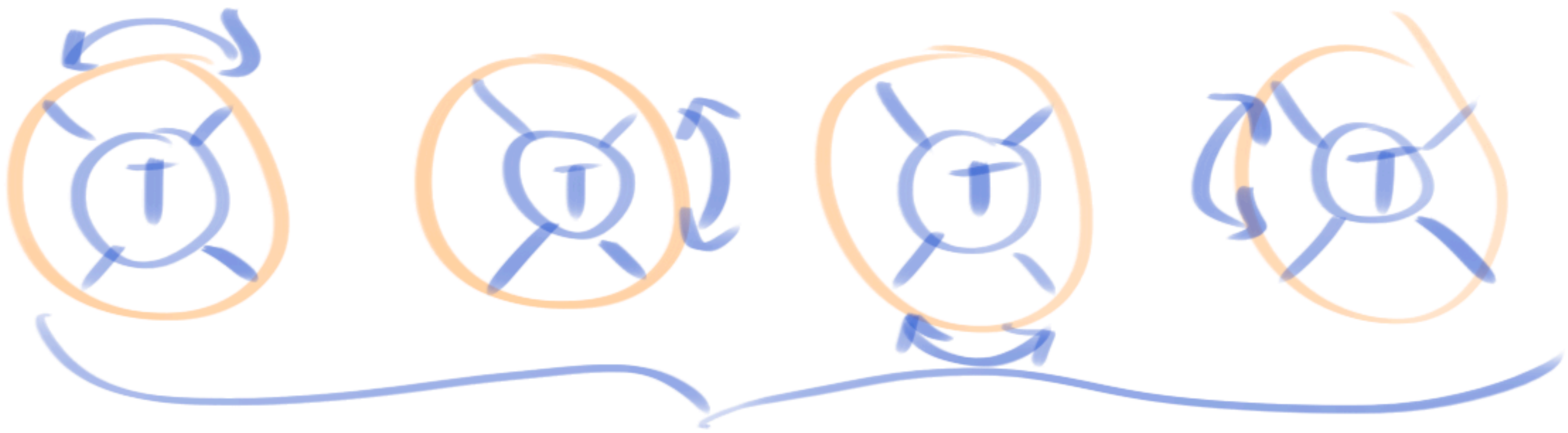
(((3) 2) −4): (Knot 1*)




3 2 -4

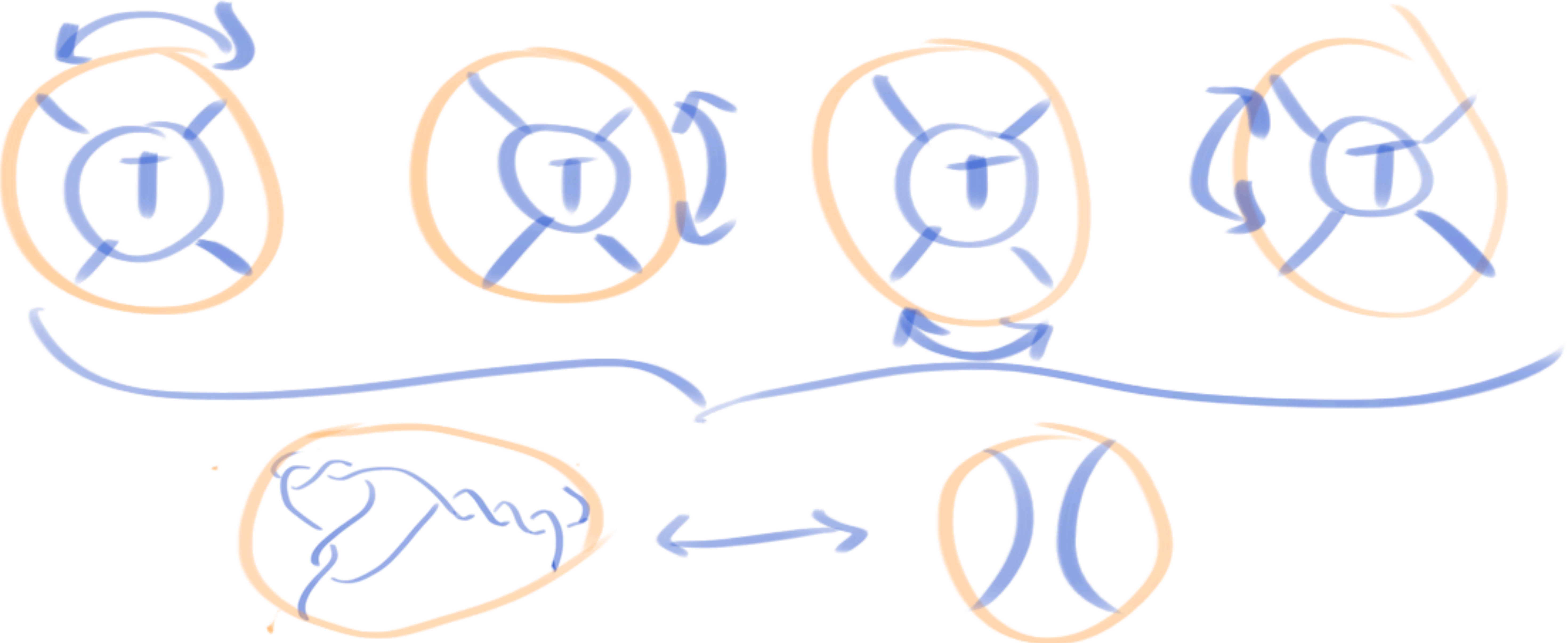
Rational Tangles :

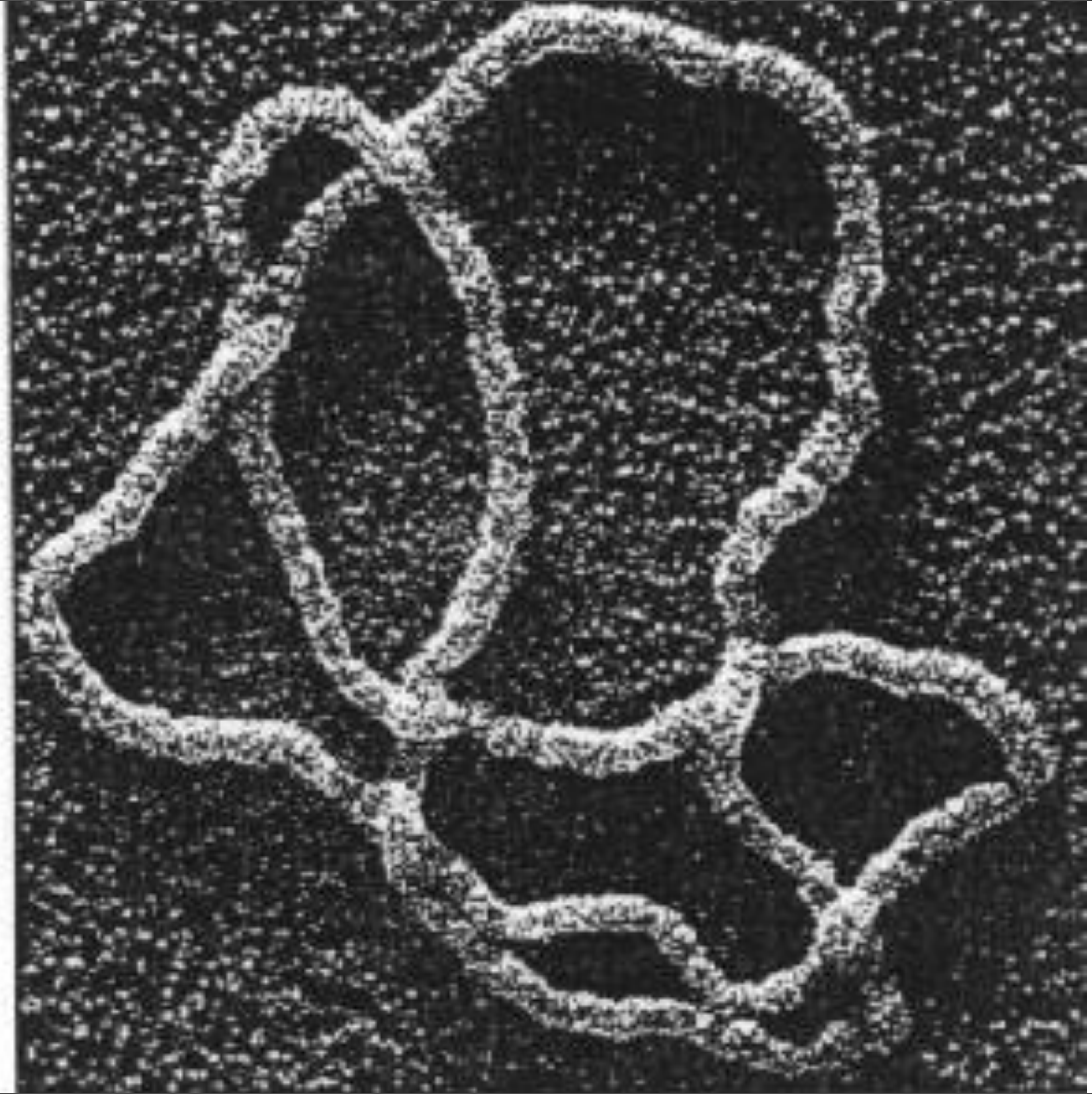
Do the twist/untwist!



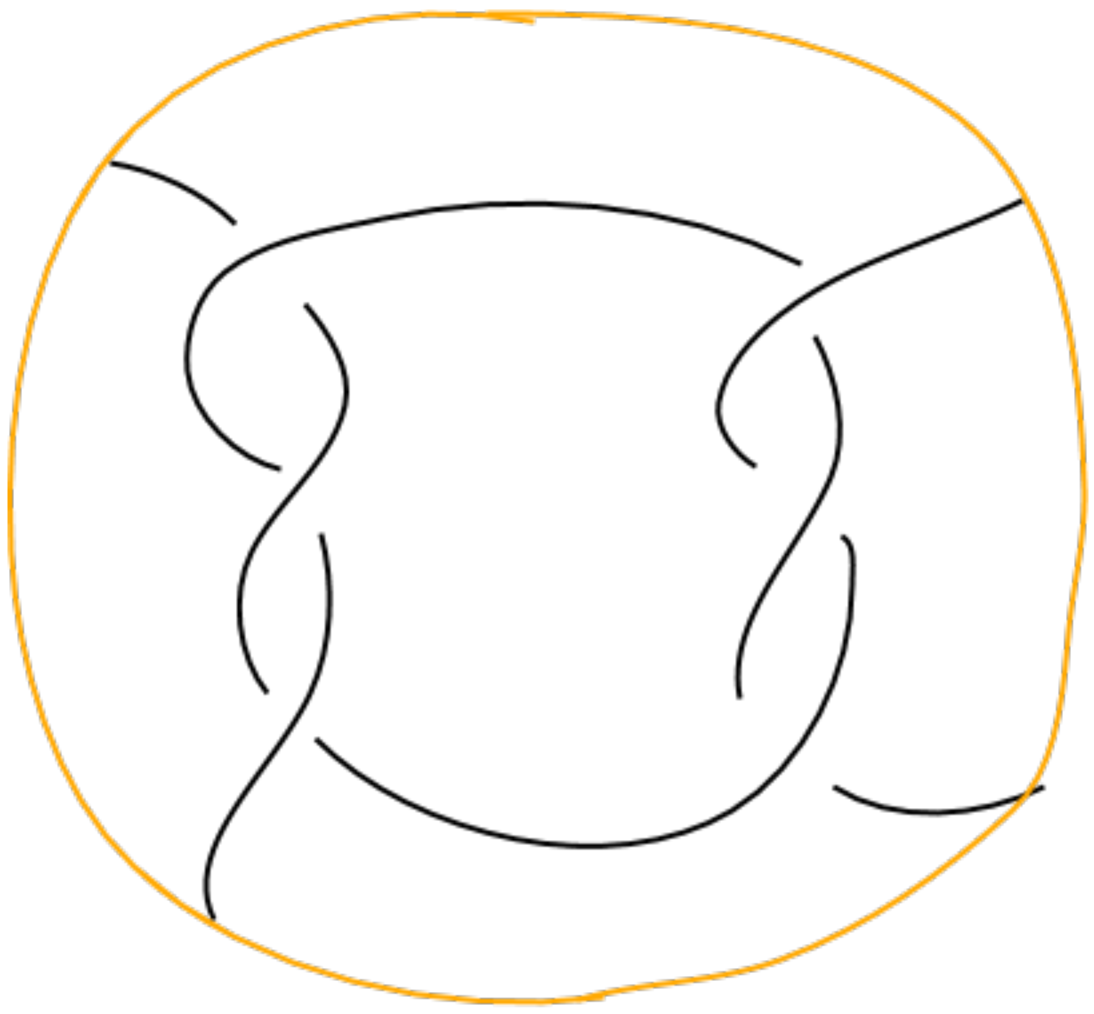
Rational Tangles;

Mult. ⟷ Do the twist/untwist!

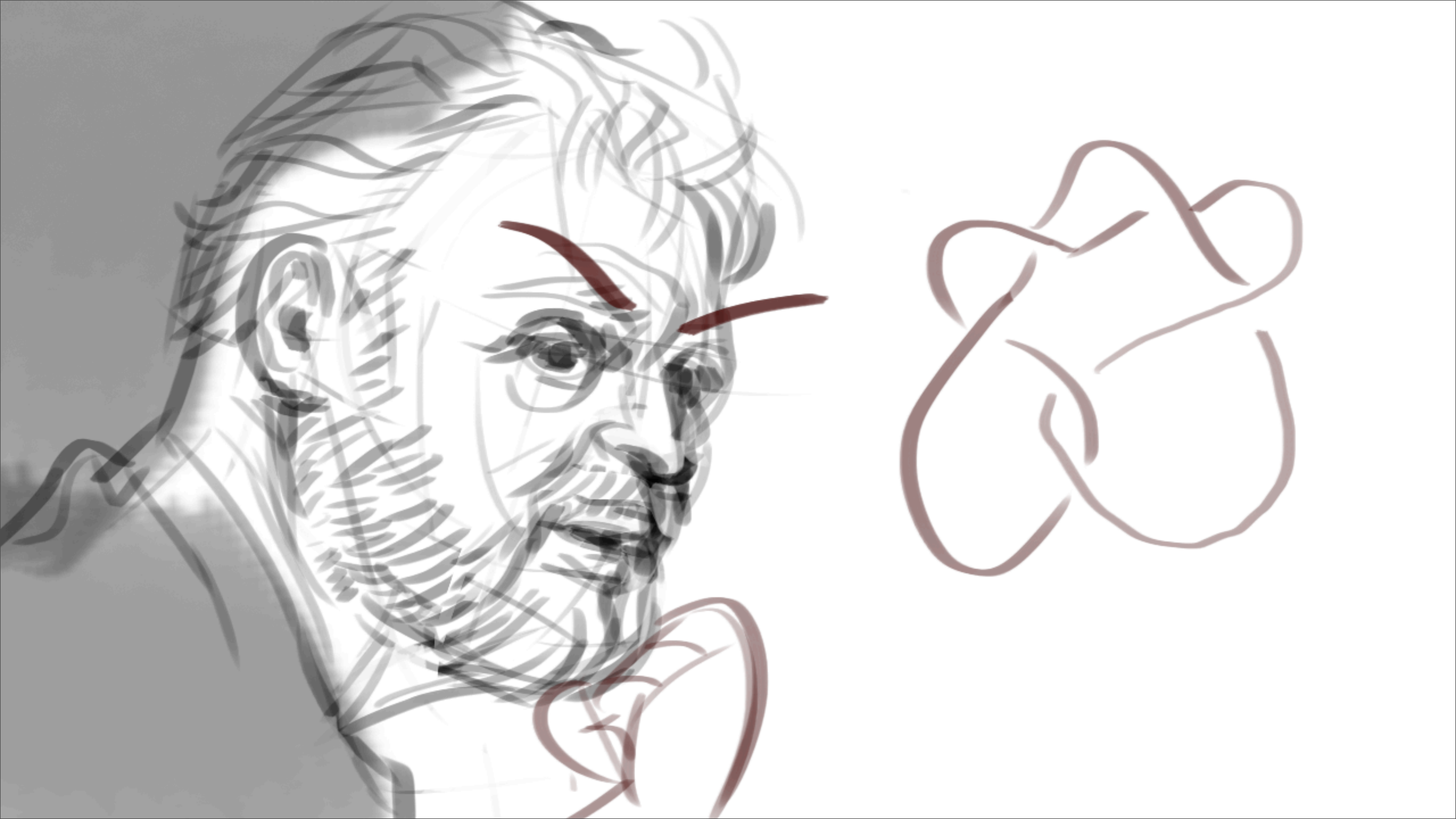


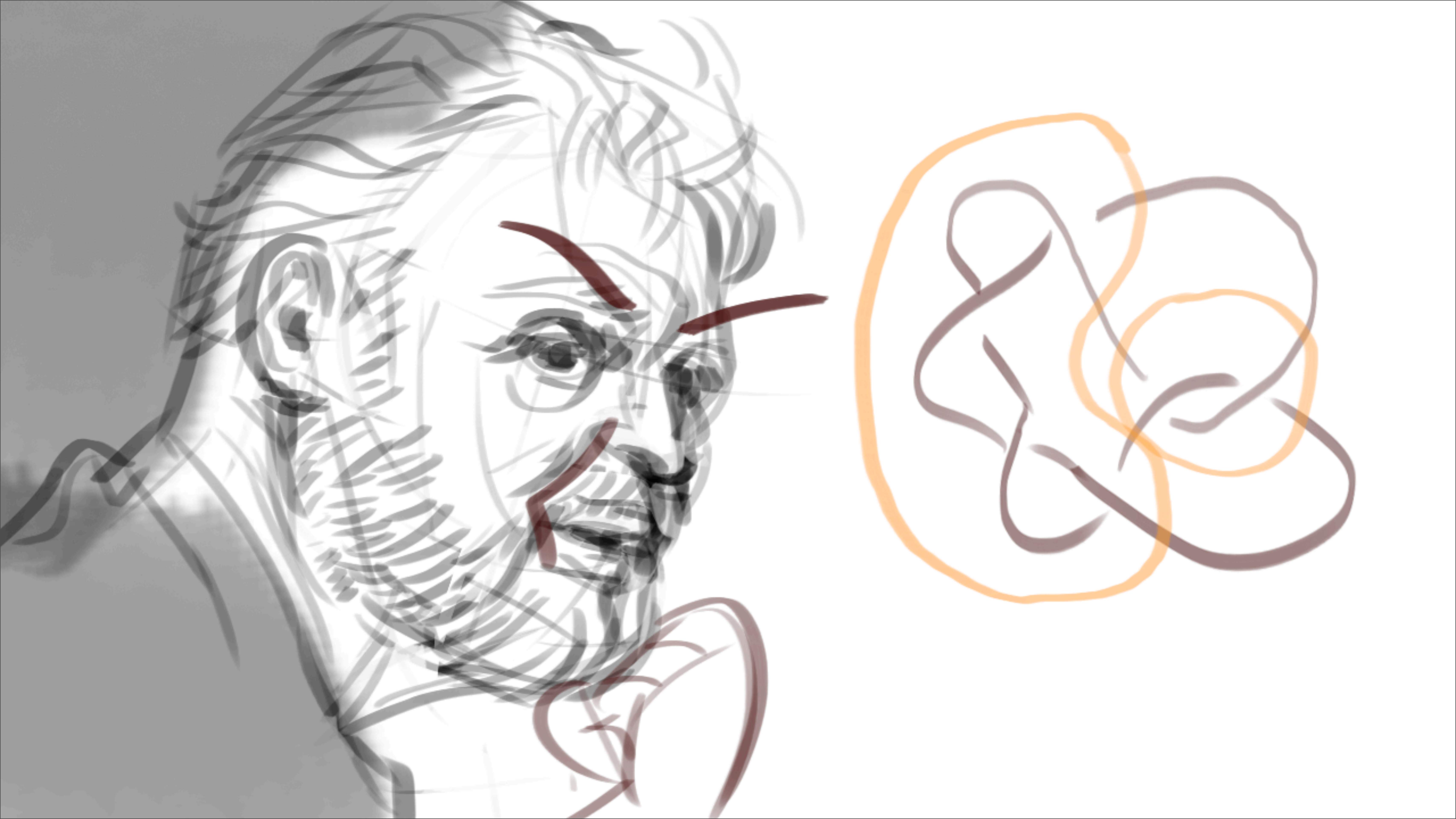


Algebraic tangles



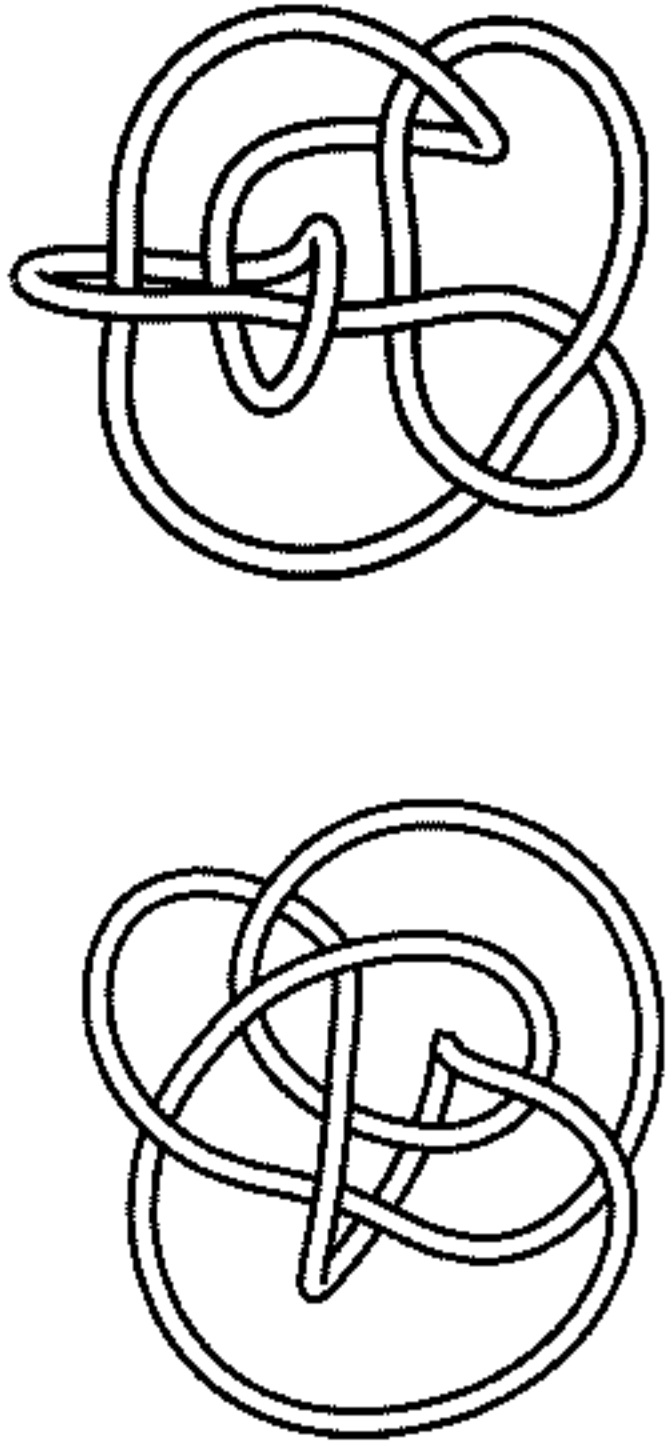
$(3, 3)$






4 2

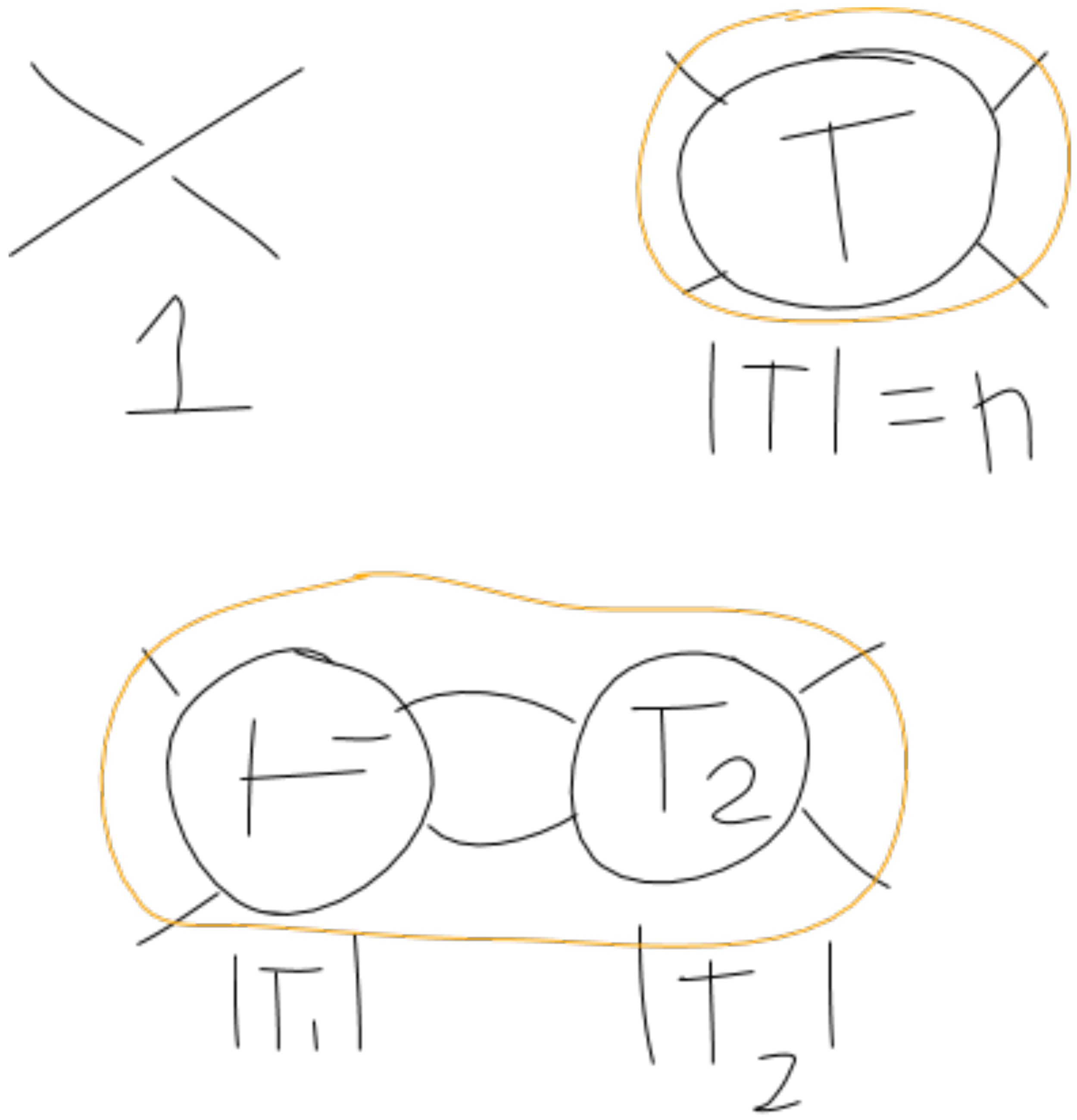

42



Bad news for enumeration?
Well, all “small” knots are rational!

Enumerating knots with Conway

Number of crossings adds!



$$|T_1 T_2| = |T_1| + |T_2|$$

Partitions

Partitions(o o o) =

o   o   o ,   3 groups of 1

o   oo ,

oo   o ,

ooo

Partitions(o o o) =

o o o) ← 3 groups of 1

o oo)

oo o)

ooo

Gives all notations
for knots of 3 crossings!

1 1 1

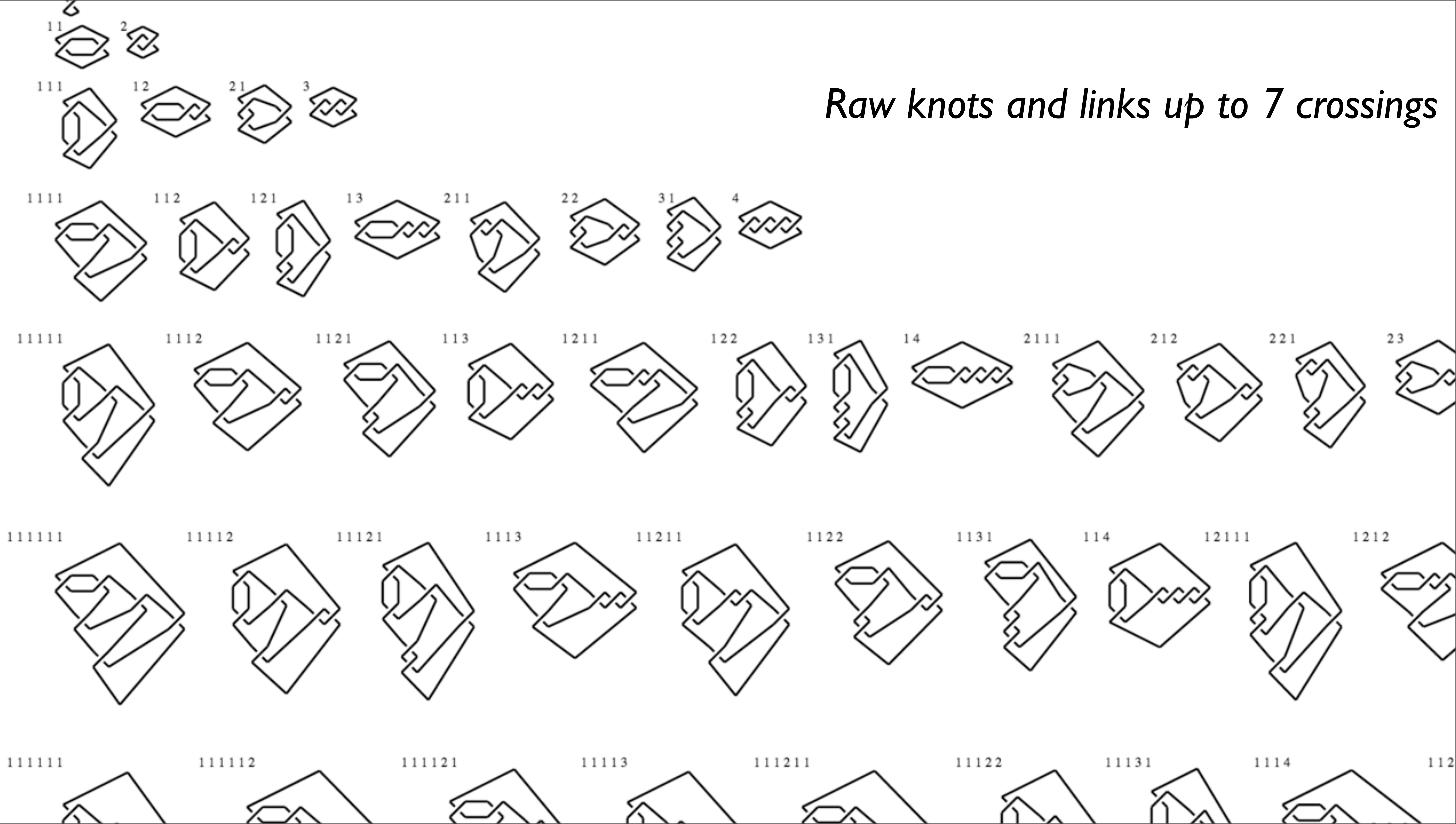
1 2

2 1

3

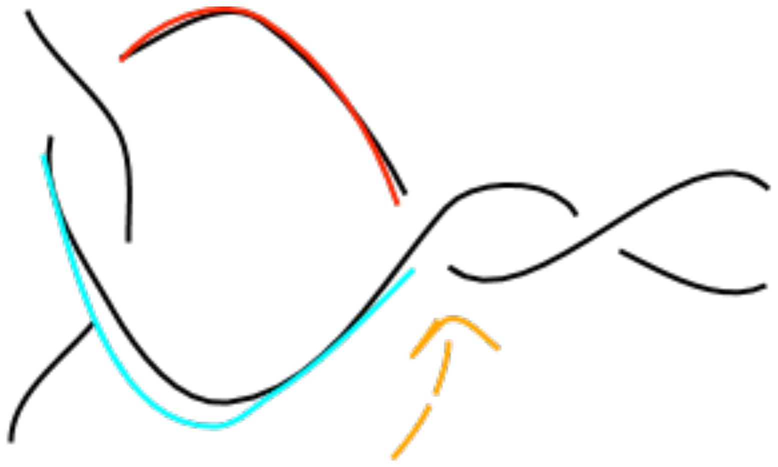
And we have an algorithmic way to draw diagrams, *so*...

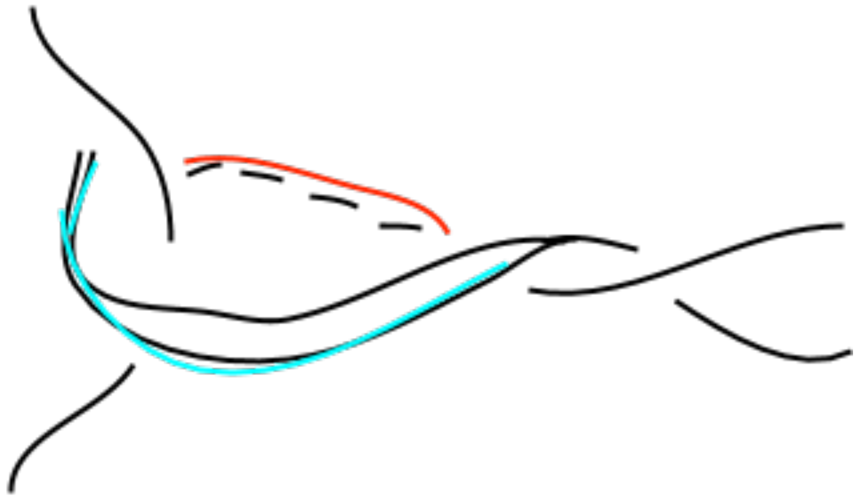
Raw knots and links up to 7 crossings



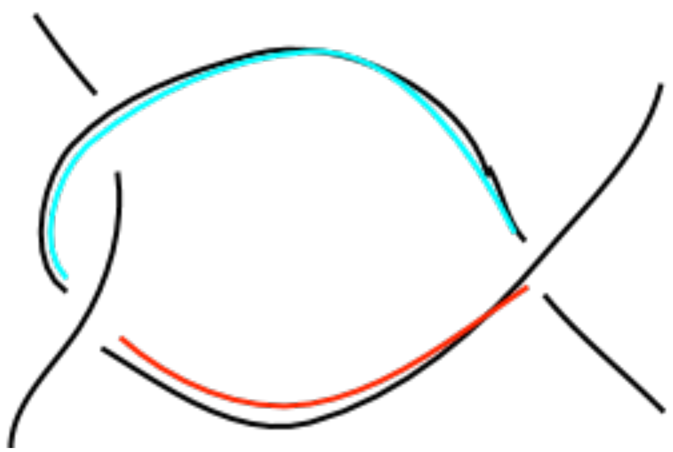
Finally, the answer: tangle/knot equivalence

$-2 \quad 2$





$2 \quad 1$



```
 2 1 = 1 + 1/2 = 3/2
-2 2 = 2 + 1/-2 = 3/2
```

Same!

HAS MATHEMATICS GONE TOO FAR?!

one weird trick to check knot equivalence

Put the tangle’s notation (all integers) “backward” into continued fraction

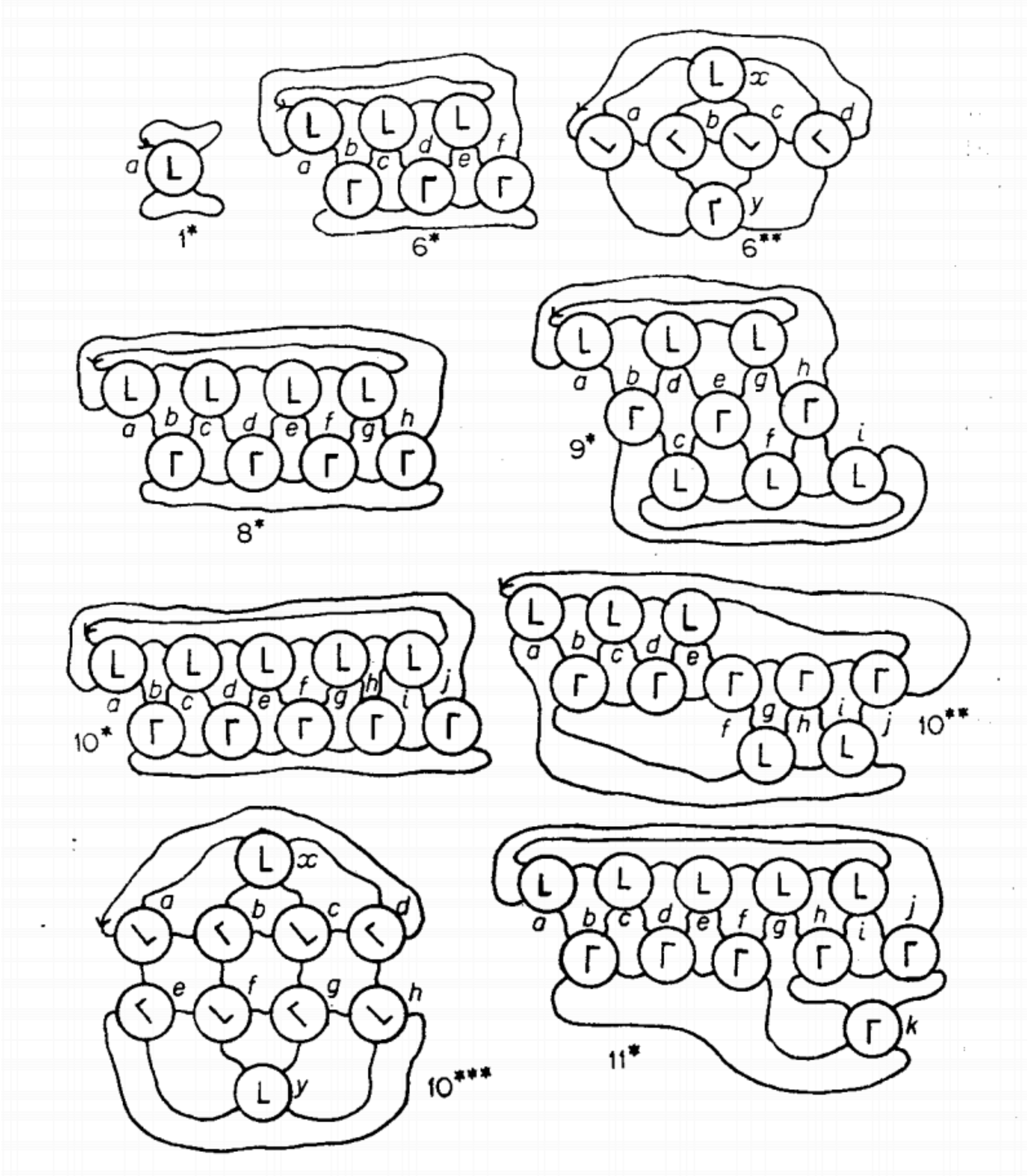
-> rational number

Rational tangles equivalent IFF
represented by
the same rational number!

???!!??!!?

Rational tangles equivalent
IFF represented by
the same rational number!

Algebraic -> polyhedra



How did Conway do it?

Powerful theorems about special cases!

Small knots are easier for humans.

Notation helped him discover new knot invariants

Outline

- Crash course in knot theory
- Alexander-Briggs notation
- Dowker notation
- Intro to enumeration
- Conway notation
- **Modern tabulations**
- Lessons

The First 1,701,936 Knots

What notation do **you** think they used?
You have all the same information!

I find it intriguing that the most successful modern tabulations don't use all the cleverness that Conway found, and instead use **Dowker notation,** which is a much more naive notation.

Dylan Thurston (email)

Dowker:
compact but hard to manipulate

Conway's scheme draws on a **large set of symbols arranged according to a rather large set of rules**, both of which grow with crossing number, and for this reason does not lend itself well to computer programming.

Hoste

Outline

- Crash course in knot theory
- Alexander-Briggs notation
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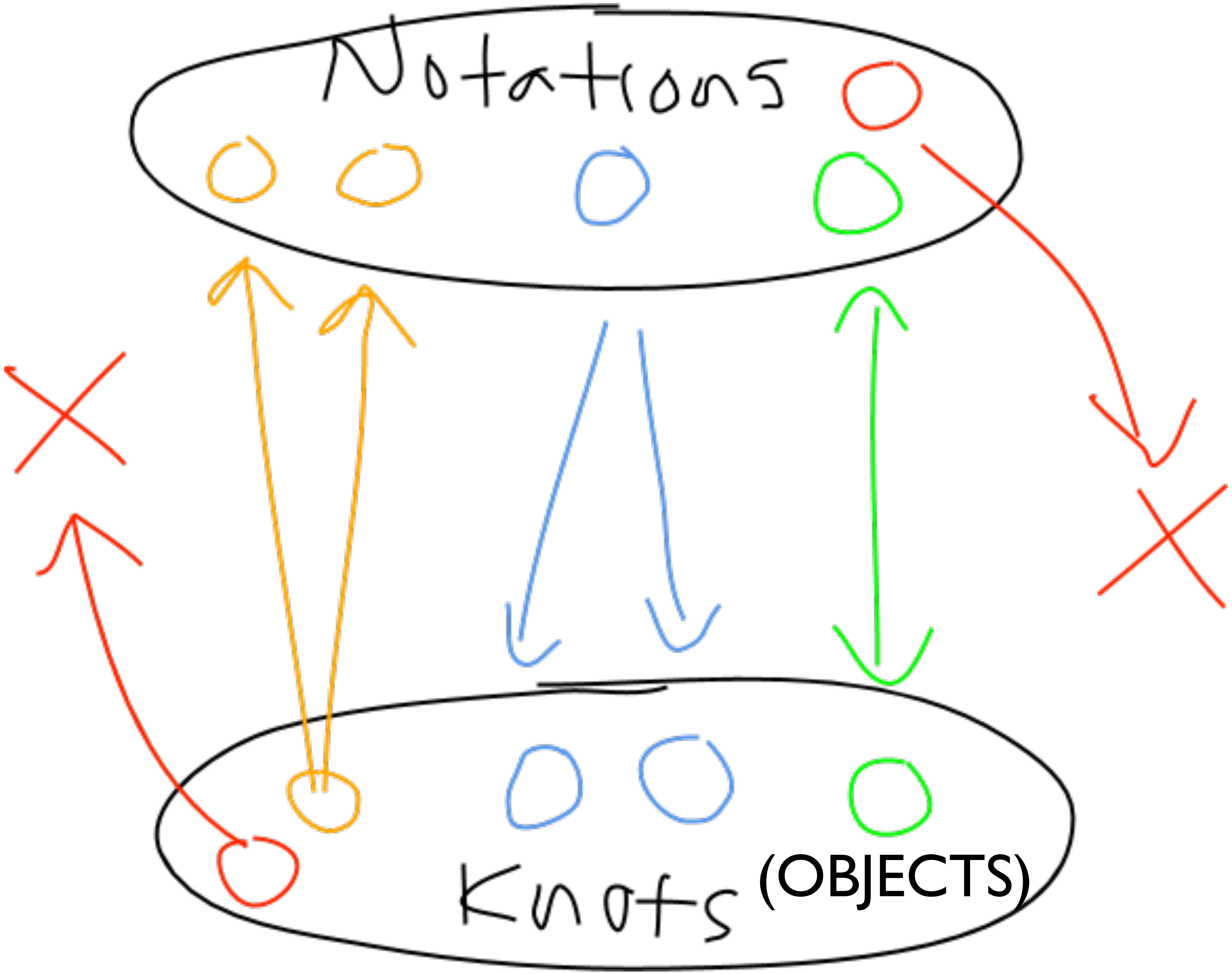
- Conway notation
- Modern tabulations
- **Lessons**

Axes of notation

Axes of notation

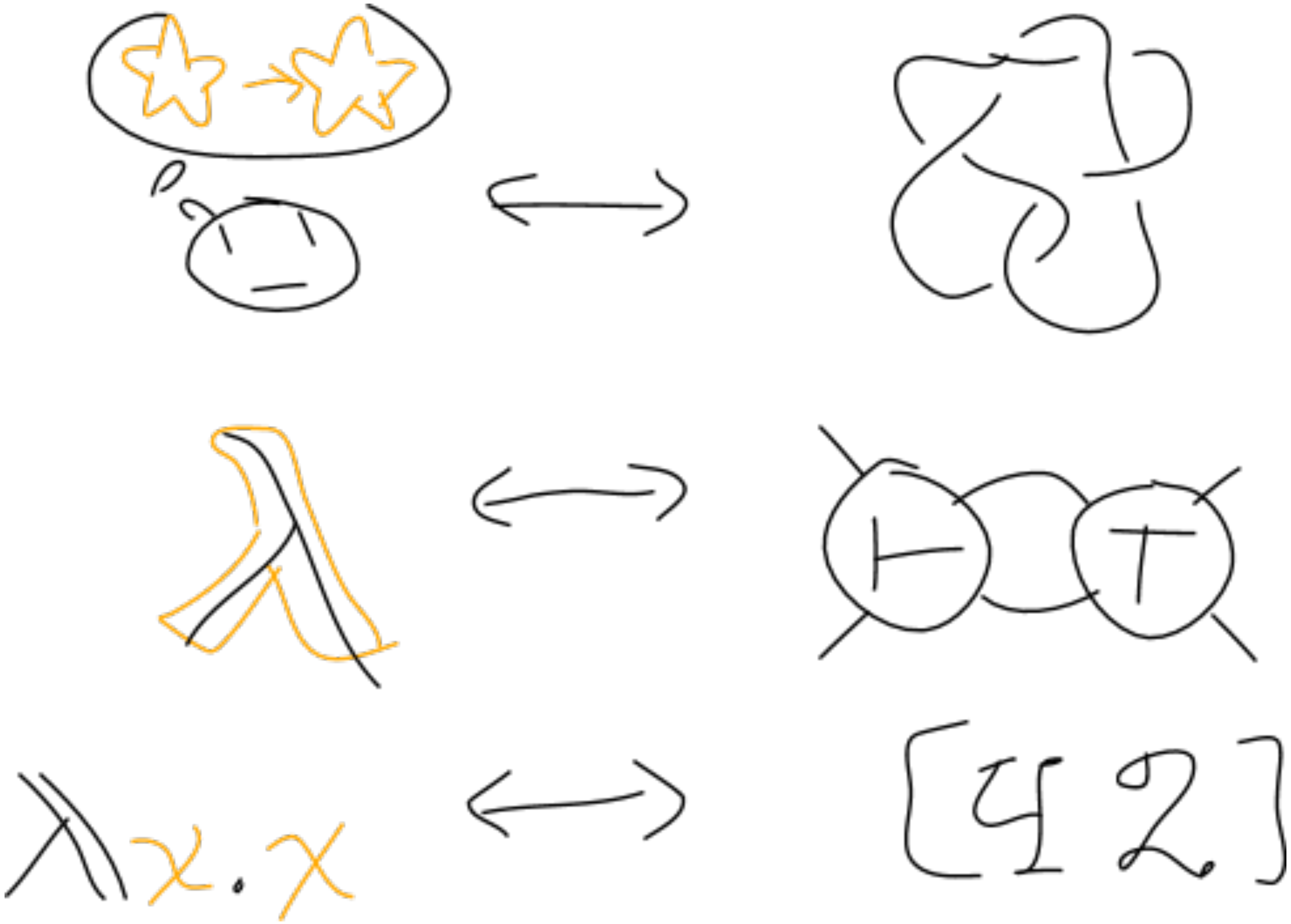
- Break down vs. build up
- Easy for humans to read and write?
- Easy for machines to read and write?
- Minimal?
- Notation to object correspondence?



Notations
Knots
(OBJECTS)

Axes of notation

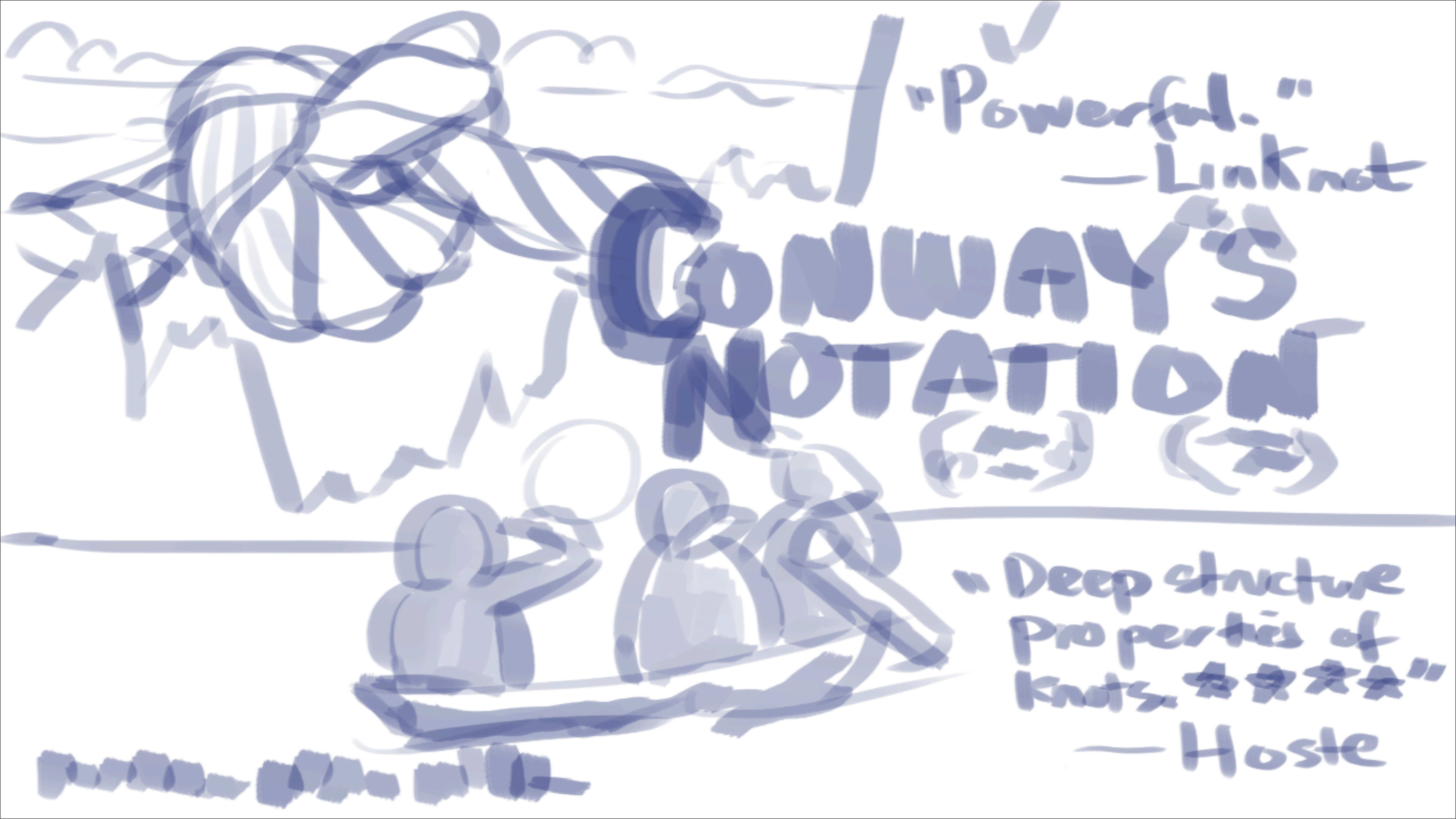
- Legibility
- Encoded insights
- Common operations
- **What new manipulations and ways of thinking does it encourage?**


λx.x
[42]

Program equivalence
$\leftrightarrow$ Knot equivalence

Don't get too attached to one notation or representation.

There can be something really useful, unexpected, or interdisciplinary waiting for you to discover or invent it!


"Powerful."
— LinKnot
CONWAY'S NOTATION
"Deep structure properties of knots. ★★★★★"
— Hoste

Sources

- "An enumeration of knots and links" (Conway's paper)
- *The Knot Book* (very accessible)
- "The First 1,701,936 Knots"
- "The enumeration and classification of knots and links"
- *Genius At Play: The Curious Mind of John Horton Conway*
- *LinKnot: Knot Theory By Computer*
- "Rational Tangles"
- *The Mystery of Knots: Computer Programming for Knot Tabulation*
- "Interactive Topological Drawing" (PhD thesis)
- So many undergraduate honors theses


THANKS!

Thanks!

Questions?

https://github.com/hypotext/knotation

@hypotext

XING FU BORDERS

Footnotes

1. But some of the Conway knots are actually links of 2 parts!

Yes... annoying but we can always figure it out by drawing it.

2. Isn't (1 2) the same Conway knot as (3), etc.?

Yep.

3. Intuition for rational tangles and continued fractions?

http://blog.sigfpe.com/2008/08/untangling-with-continued-fractions_23.html

Also see the paper "Rational Tangles"

4. More direct PL connections: C. C. Shan's monadic braids in Haskell; Dan Piponi's monad for knot drawing

[1] http://homes.soic.indiana.edu/ccshan/crossing/

[2] https://dl.acm.org/citation.cfm?id=1596553

3. Knot equivalence is an open problem; hasn't been proved hard or undecidable

4. Drawing Dowker in a computer:
Dowker → extract 4-valent graph →
check planarity → algorithm to draw (isn't pretty)

See "Interactive Topological Drawing" for the results of this algorithm

5. No, I did not enumerate the Conway knots up to equivalence (but it’s doable). Pull requests welcome

6. Conway's notation helped him discover new knot invariants: see the end of his paper

7. Applications: braids in quantum physics; knots as processes; chemical reactions; untangling DNA

Knots as processes: http://arxiv.org/abs/1009.2107

See *The Knot Book* for a detailed treatment of knots and DNA/reactions

8. Why are knots so hard to deal with, but graphs aren't?
IMO, graph proofs not visual unless planarity is involved.
(Graph isomorphism is hard too! NP-complete.)

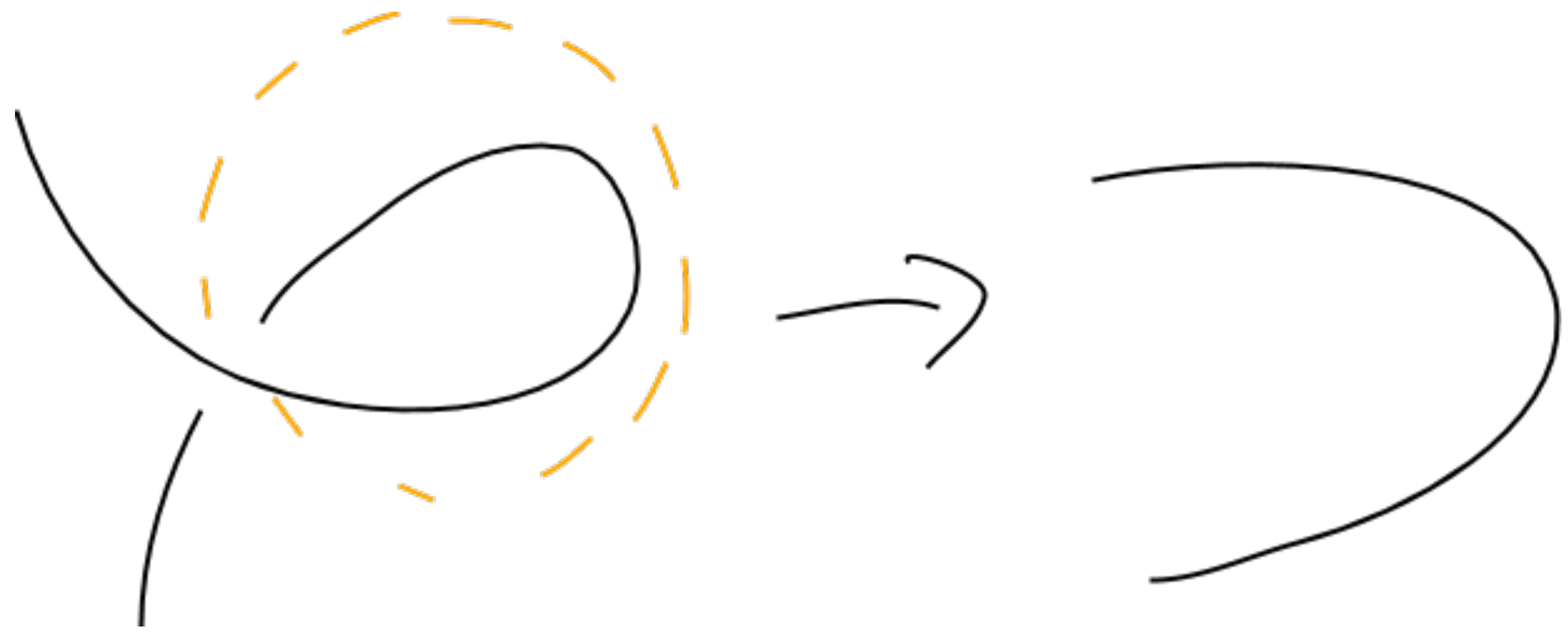
8. Yes, Conway notation does NOT cover all knots. It's hard to find a source online for this. I found a quote in the footnotes of *Genius at Play*, a Conway biography, where he says, "some knots aren't good enough to be written in my notation, and I'm not sure what to do with them!" A little egoistic but you get the point.

9. I ignored non-alternating Dowkers and dealing with Conways with negative twists. For negative twists and partitions, just take the absolute value.

Cut section #1: special cases for easy tabulation, and flypes

In tabulation, we focus on prime alternating knots of small crossing number, often with reduced diagrams.

Reduced = no trivial untwists



One reason: Menasco's theorem and flypes

Menasco: a knot diagram is composite if and only if it looks composite!

Prime

Composite

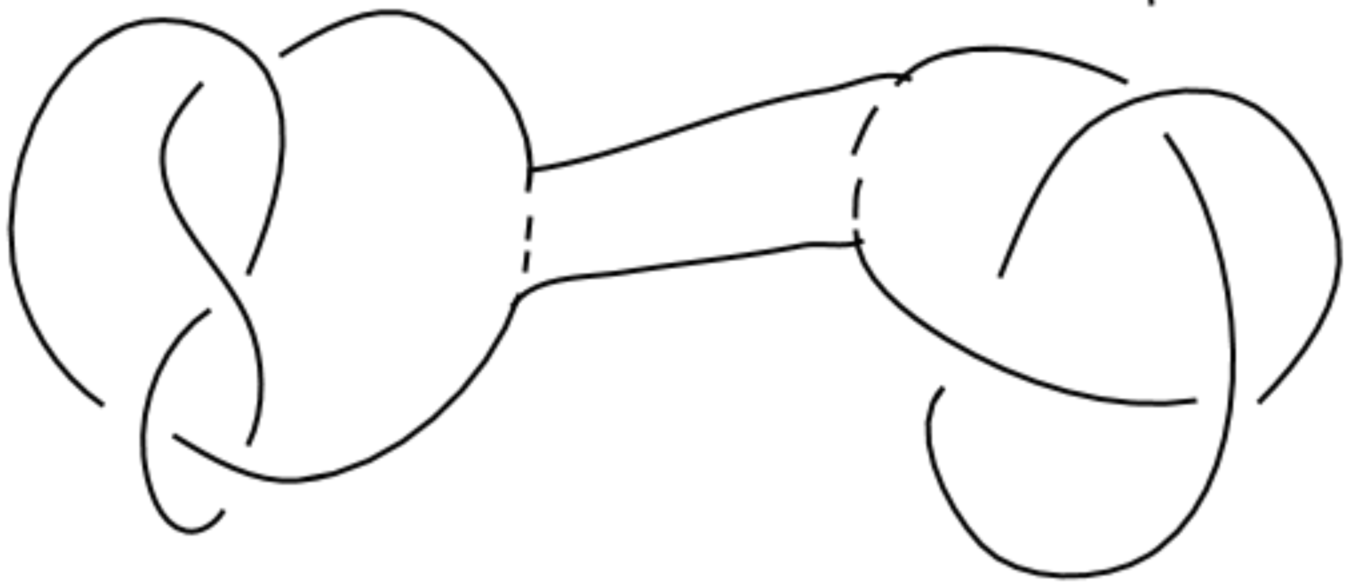
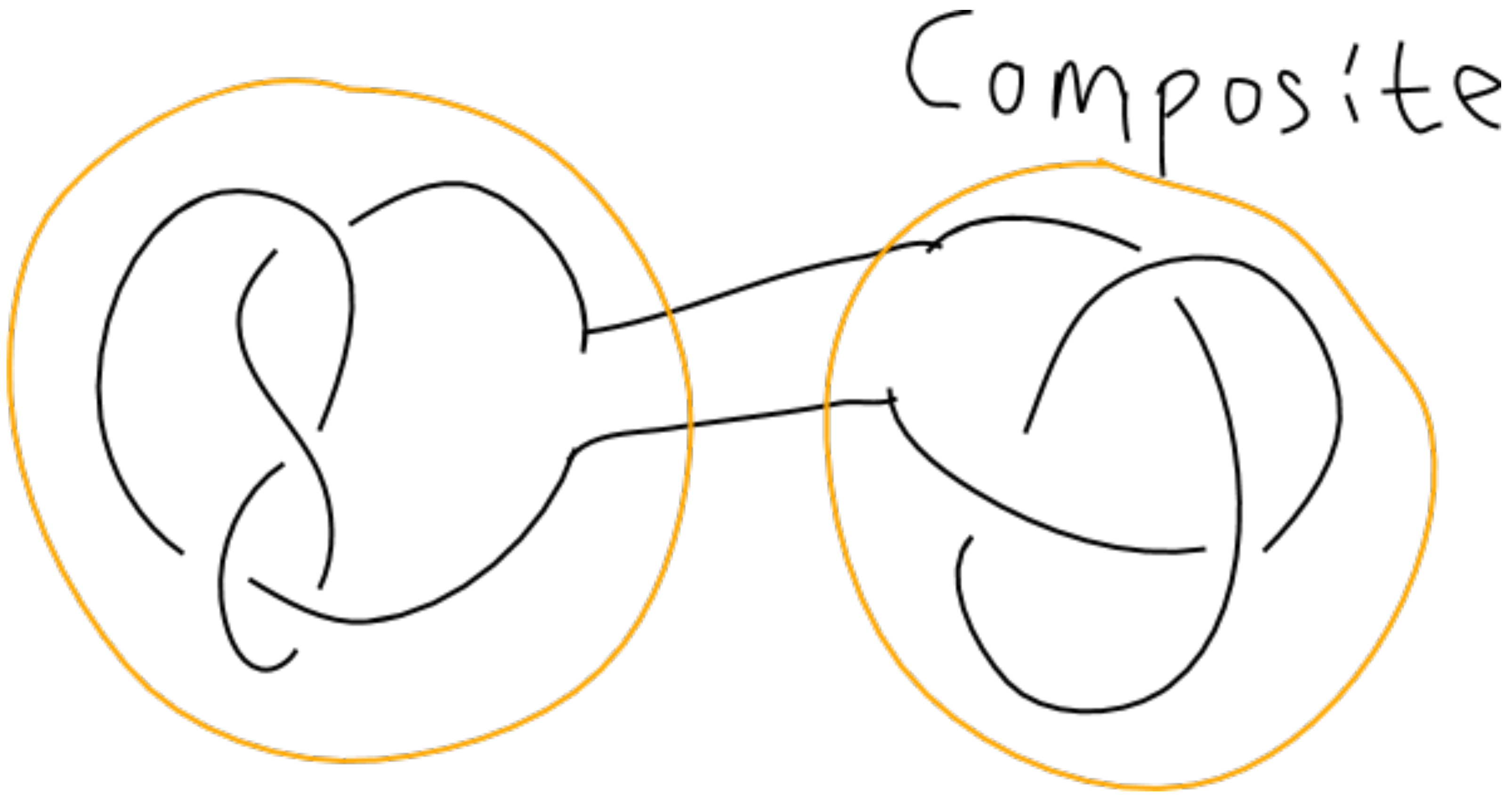
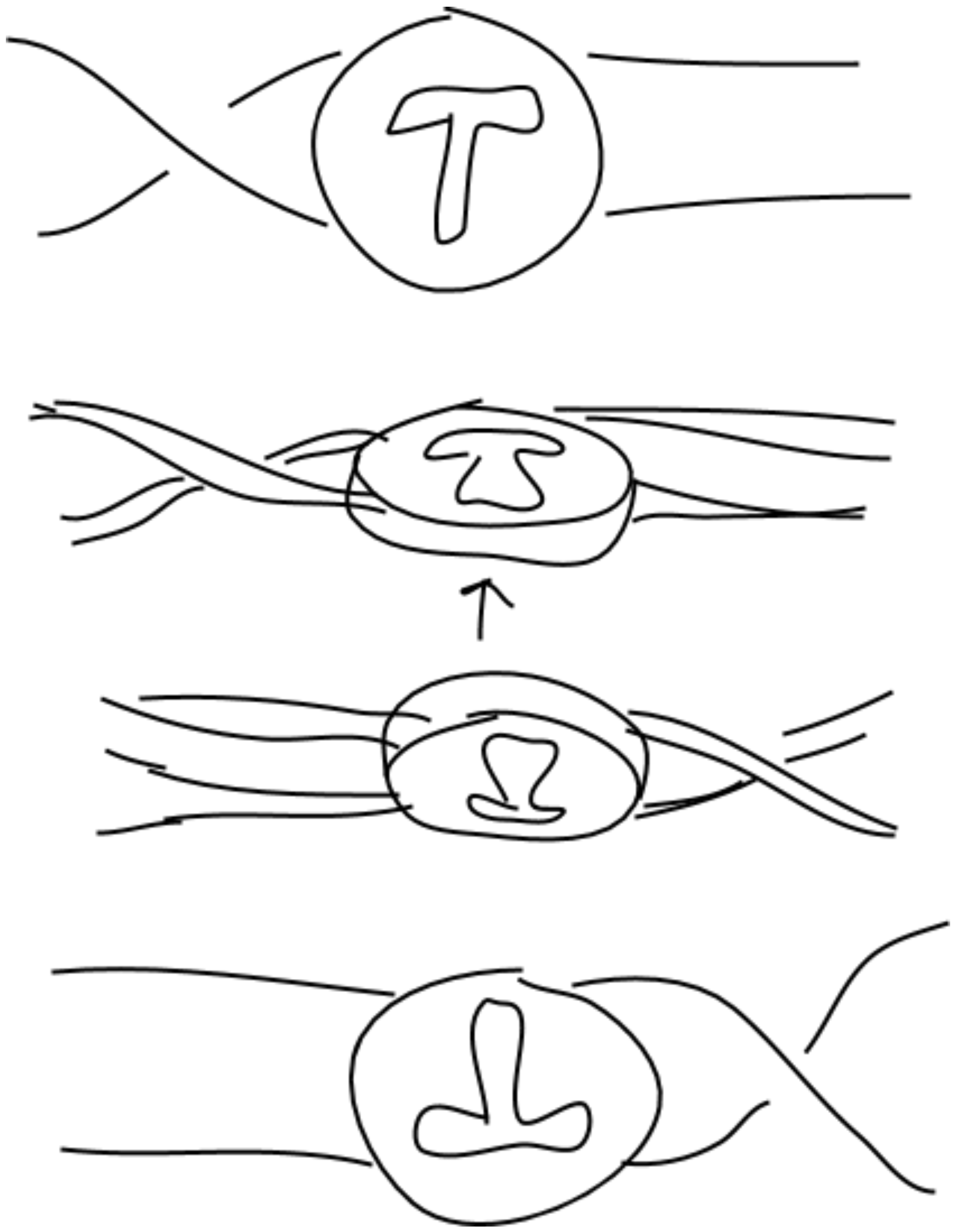


Fig.8 # Trefoil


Composite

and the Tait Flyping theorem

Flype: rotate tangle out of page, moving crossing to other side



By applying all possible flypes to a reduced prime alternating diagram, we get exactly all the reduced alternating diagrams that it's equivalent to.

Tait's theorem makes checking equivalence much more algorithmic, whereas Reidemeister moves are a good basic definition, but can increase the crossing number of the knot.

Cut section #2:
Conway's notation encodes equivalence by flype

Conway flype equivalence

What do we do with algebraic tangles? We can't use the nifty rational fraction/number method. But we CAN use another equivalence checking technique built into Conway's notation.

"The reader should now be able to interpret any knot name taken from our table, but she will not yet appreciate the reasons which make our ragbag of conventions so suspiciously efficient at naming small knots. Much of this efficiency arises from the fact that the notation absorbs Tait's 'flyping' notation (Fig. 7), which replaces 1 + t by t_h + 1, or 1bar + t by t_h + 1bar."

I won't go into this in detail, but in short, Conway's notation lets us convert the numeric notation for ANY tangle (not necessarily a rational one) into a "standard form." Converting a tangle into its standard form cancels out pairs of flypes that happened in the opposite direction, so any knots that convert into the same standard form are equivalent. Conway doesn't give a general algorithm to enumerate all flype-equivalent knots, but this seems do-able, and is left as an exercise for the reader!

See his paper for more details.

Cut section #3:
more detail on the work done by the two teams in
"The First 1,701,936 Knots"

http://pzacad.pitzer.edu/~jhoste/HosteWebPages/downloads/HTW.pdf

The First 1,701,936 Knots

"The primary advantage of Dowker notation seems to be its brevity. With over 6 billion knots and links now in the tables, using as little computer memory as possible has obvious value.

On the other hand, Dowker sequences are not easily transformed directly under the types of operations that need to be applied to diagrams in the course of a tabulation. Instead, it is usually necessary to derive additional, attendant information about the diagram, such as the signs of the crossings..."

Returning to the teams' work with Dowker notation: one team took advantage of the fact that all the knots were of a kind called hyperbolic, which allowed easy equivalence checking, and the other team applied flypes and ad-hoc moves like double-passes and Perko moves, then computed invariants.

Here are some of the heuristics they used to group equivalent knots in Dowker notation:

* twist/untwist (nugatory crossings). For example, a crossing labeled "x x+1" in a knot of n crossings can always be untwisted into a knot of n-1 crossings, so throw away all such knots)
* throw out all knots whose notations are cyclic permutations (same knots) or moving in the other direction
* easy to detect composite knots (sub-adjacent permutations)

Guess how long it took for their program to tabulate knots of 16 crossings? It took them 1-2 weeks! (p39) Their paper was published in 1998, but that still seems astoundingly slow.

Listeners and readers—can YOU do better?

Thanks for reading the footnotes!