Dynamic Asset Pricing Homework 2

Due May 4th

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Contents

1 Topic: Option pricing

Problem 1.1. *In this exercise S_t represents the price of a stock. We would like to price a down-and-out call; Given a maturity time T, a strike K and a barrier B, the option pays $(S_T - K)_+$ only if $S_u \geq B$* ***for all*** *$u \in (t, T)$. The interest rate r is constant, S_t starts at $x > B$ and follows a Geometric Brownian Motion in the* ***risk neutral*** *measure:*

$$dS_t = rS_t dt + \sigma S_t dW_t$$

The following questions will present the two main numerical approaches to price options.

In all the numerical experiments, we will take $t = 0$, $T = 1$ year, $r = 0.02$ (2% per year) and $\sigma = 0.20$ (20% per year), $x = 100\$$, $B = 80\$$ and $K = 110\$$.

1.1 Using Monte Carlo

We will start by using a Monte-Carlo approach to price the option:

(a) *Explain that the value of the option should be*

$$v_t = e^{-r(T-t)}\mathbb{E}\left[(S_T - K)_+ \mathbb{1}_{S_u \geq B, \forall u \in (t,T)} \mid \mathcal{F}_t\right]$$

where $\mathbb{E}$ is in the risk neutral measure.

Solution: The payoff of this a down-and-out call option is

$$V(S_T) = \begin{cases} (S_T - K)_+ & \text{if } S_u \geq B, \forall u \in (t, T) \\ 0 & \text{if } S_u < B, \forall u \in (t, T) \end{cases} = (S_T - K)_+ \mathbb{1}_{S_u \geq B, \forall u \in (t,T)}$$

Therefore, if $\mathbb{E}$ is in the risk neutral measure, by the risk-neutral pricing formula, we obtain

$$v_t = \mathbb{E}\left[e^{-r(T-t)}V(S_T) \mid \mathcal{F}_t\right] = e^{-r(T-t)}\mathbb{E}\left[(S_T - K)_+ \mathbb{1}_{S_u \geq B, \forall u \in (t,T)} \mid \mathcal{F}_t\right].$$

(b) *Using the SDE, write a program that computes one trajectory of S_t for $t < T$; it should return a list of values $S_0, S_{\Delta t}, S_{2\Delta t}\ldots, S_T$ for $\Delta T = \frac{T}{N}$ where N is the number of points (you can choose $N = 252$ for example).*

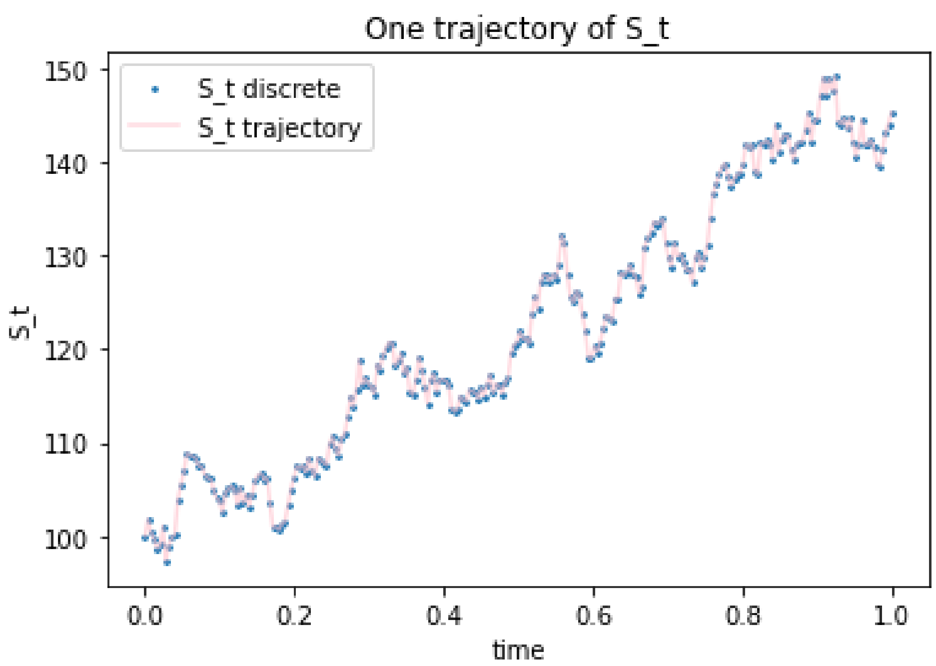

Figure 1: One trajectory of S_t using Monte Carlo.

Solution: See Jupyter notebook.

(c) *Write a program that takes the list of values computed in the previous question and returns the payoff of the option* $\phi(S) = (S_T - K)_+$ *if all* $S_{k\Delta t} > B$*, and 0 otherwise.*

Solution: See Jupyter notebook.

(d) *Using your previous code, write a program that generates* $N_p = 10^5$ *trajectories of* S*, and computes the average*

$$\frac{e^{-r(T-t)}}{N_p} \sum_{j=1}^{N_p} \phi\left(S^{(j)}\right)$$

where $S^{(j)}$ *is the* j^{th} *trajectory. Deduce the price of the option.*

Solution: The price of the option is $\$4.893809024339967$.
For the code, see Jupyter notebook.

1.2 Using the PDE

Now we will do the same using a PDE approach. Let's rederive Feynman-Kac's result in the case of stopping times:

(e) *Define the stopping time*

$$\tau = \min\left\{\inf\left\{u \geq t \mid S_u = B\right\}, T\right\}$$

which is the first time you hit the barrier if it is less than T*, and* T *otherwise. Also define the function* ϕ *to be*

$$\phi(y, s) = \begin{cases} (y-K)_+ & \text{if } s = T \\ 0 & \text{if } s < T \end{cases}$$

Explain (without proof) that the price of the option is given by

$$\mathbb{E}\left[e^{-r(\tau-t)}\phi\left(S_\tau, \tau\right) \mid S_t = x\right]$$

Solution: Since the payoff will be zero once the underlying asset's price hits B, the option can be viewed as one with maturity τ instead of T. As such, according to the risk-neutral pricing formula, we have:

$$v_t = \mathbb{E}\left[e^{-r(\tau-t)}\phi\left(S_\tau, \tau\right) \mid S_t = x\right].$$

Let's prove that this price solves a PDE in the following questions;

(f) *Assume that we know that there is a function* v *that solves the following PDE:*

$$-rv(t,x) + v_t(t,x) + rxv_x(t,x) + \frac{1}{2}\sigma^2 x^2 v_{xx}(t,x) = 0 \tag{1}$$

for $t < T$ *and* $B < x$*, with the final time condition* $v(T,x) = (x-K)_+$ *for all* $x > B$ *and the boundary condition* $v(t,B) = 0$ *for all* $t < T$*.*
Show that

$$d\left(e^{-rt}v\left(t, S_t\right)\right) = e^{-rt}\sigma S_t v_x\left(t, S_t\right)dW_t \tag{2}$$

Solution: Applying the Itô formula, we obtain

$$\begin{aligned} dv(t,S_t) &= v_t(t,S_t)dt + v_x(t,S_t)dS_t + \frac{1}{2}v_{xx}(t,S_t)(dS_t)^2 \\ &= v_t(t,S_t)dt + rS_t v_x(t,S_t)dt + \sigma S_t v_x(t,S_t)dW_t + \frac{1}{2}\sigma^2 S_t^2 v_{xx}(t,S_t)dt \\ &= \left(v_t(t,S_t) + rS_t v_x(t,S_t) + \frac{1}{2}\sigma^2 S_t^2 v_{xx}(t,S_t)\right)dt + \sigma S_t v_x(t,S_t)dW_t \\ \text{Equation (1)} \implies &= rv_t(t,S_t)dt + \sigma S_t v_x(t,S_t)dW_t \end{aligned}$$

Rearranging the above equation, we obtain

$$\begin{aligned} &\implies dv(t,S_t) - rv_t(t,S_t)dt = \sigma S_t v_x(t,S_t)dW_t \\ &\implies d\left(e^{-rt}v\left(t,S_t\right)\right) = e^{-rt}\sigma S_t v_x\left(t,S_t\right)dW_t \end{aligned}$$

The proposition is then proved.

(g) Deduce that

$$e^{-r\tau}v(\tau,S_\tau)-e^{-rt}v(t,S_t)=\int_t^\tau \sigma e^{-rs}S_sv_x(s,S_s)\,dW_s$$

Solution: Integrating Equation (2) from t to τ, we get

$$\int_t^\tau d\left(e^{-rt}v(t,S_t)\right)=\int_t^\tau e^{-rt}\sigma S_tv_x(t,S_t)\,dW_t$$
$$\implies e^{-r\tau}v(\tau,S_\tau)-e^{-rt}v(t,S_t)=\int_t^\tau \sigma e^{-rs}S_sv_x(s,S_s)\,dW_s$$

(h) By using the following formula (called Dynkin's theorem, or just an application of Doob's optional stopping time theorem):

$$\mathbb{E}\left[\int_t^\tau \sigma e^{-rs}S_sv_x(s,S_s)\,dW_s \mid S_t=x\right]=0 \tag{3}$$

show that

$$v(t,x)=\mathbb{E}\left[e^{-r(\tau-t)}\phi(S_\tau,\tau)\mid S_t=x\right]$$

Solution: Taking the conditional expectation on both sides of the equation above and using Equation (3), we obtain

$$\mathbb{E}\left[e^{-r\tau}v(\tau,S_\tau)-e^{-rt}v(t,S_t)\mid S_t=x\right]=\mathbb{E}\left[\int_t^\tau \sigma e^{-rs}S_sv_x(s,S_s)\,dW_s\mid S_t=x\right]=0$$

Rearranging the equation, we get

$$\implies \mathbb{E}\left[e^{-r\tau}v(\tau,S_\tau)\mid S_t=x\right]=\mathbb{E}\left[e^{-rt}v(t,S_t)\mid S_t=x\right]=e^{-rt}v(t,x)$$

$$\implies v(t,x)=\mathbb{E}\left[e^{-r(\tau-t)}v(\tau,S_\tau)\mid S_t=x\right] \tag{4}$$

Now, if $\tau<T$, then $S_\tau=B$, and, by the given boundary condition, we have that $v(\tau,S_\tau)=v(\tau,B)=0$. On the other hand, if $\tau=T$, then $S_\tau=S_T$, and, by the given boundary condition, we have $v(\tau,S_\tau)=v(T,S_T)=(S_T-K)_+=(S_\tau-K)_+$. Therefore, essentially, we have

$$v(\tau,S_\tau)=\begin{cases}(S_\tau-K)_+ & \text{if } \tau=T\\ 0 & \text{if } \tau<T\end{cases} = \phi(S_\tau,\tau)$$

Plug this into Equation (4), we obtain

$$v(t,x)=\mathbb{E}\left[e^{-r(\tau-t)}\phi(S_\tau,\tau)\mid S_t=x\right]$$

Remark 1.2. The boundary condition $v(t,B)=0,\quad$ for $t<T$ for this PDE is only true if the down-and-out option is knocked out as soon as the asset price hits the barrier line B. However, in the definition provided at the beginning of the homework, the option pays $(S_T-K)_+$ only if $S_u\geq B$ **for all** $u\in(t,T)$. The equality in $S_u\geq B$ allows the option not to be knocked out when the asset price is exactly at B. So there is a bit inconsistency here. △

This reasoning show that if a function v solves the PDE, it is necessarily equal to $\mathbb{E}\left[e^{-r(\tau-t)}\phi(S_\tau,\tau)\mid S_t=x\right]$. PDE theory tells us that the solution of this specific PDE exists and is unique, so we also have the converse statement. We just proved that $\mathbb{E}\left[e^{-r(\tau-t)}\phi(S_\tau,\tau)\mid S_t=x\right]$ solves the PDE:

$$\begin{cases}-rv(t,x)+v_t(t,x)+rxv_x(t,x)+\frac{1}{2}\sigma^2x^2v_{xx}(t,x)=0, & \text{for } t<T \text{ and } B<x\\ v(T,x)=(x-K)_+, & \text{for } B<x\\ v(t,B)=0, & \text{for } t<T\end{cases}$$

Technically, we need a growth condition to show uniqueness. In this case, it is

$$\lim_{x\to+\infty}\frac{v(t,x)}{x-e^{-r(T-t)}K}=1$$

(or more accurately $v \underset{x\to+\infty}{\sim} x-e^{-r(T-t)}K$)

(i) Explain that growth condition

Solution: Since $dS_t=rS_tdt+\sigma S_tdW_t$, we have $S_\tau=xe^{(r-\sigma^2/2)(\tau-t)+\sigma(W_\tau-W_t)}>\max\{B,K\}$ a.s. as $x\to+\infty$. Therefore, $\tau=T\iff \mathbb{1}_{\tau=T}=1$ a.s. and $S_\tau>K\iff \mathbb{1}_{S_\tau>K}=1$ a.s. Then, we have

$$\begin{aligned}v(t,x)&=\mathbb{E}\left[e^{-r(\tau-t)}\phi(S_\tau,\tau)\mid S_t=x\right]\\&=\mathbb{E}\left[e^{-r(T-t)}(S_T-K)\mathbb{1}_{\tau=T,S_\tau=S_T>K}\mid S_t=x\right]\\&\underset{x\to+\infty}{\sim}\mathbb{E}\left[e^{-r(T-t)}(S_T-K)\mid S_t=x\right]\\&=x-e^{-r(T-t)}K\end{aligned}$$

Also, we have

$$\lim_{x\to+\infty}\frac{v(t,x)}{x-e^{-r(T-t)}K}=\lim_{x\to+\infty}\frac{\mathbb{E}\left[e^{-r(T-t)}(S_T-K)\mathbb{1}_{\tau=T,S_\tau=S_T>K}\mid S_t=x\right]}{x-e^{-r(T-t)}K}\overset{\text{L'Hospital}}{=\!=}\lim_{x\to+\infty}\int e^{-\sigma^2/2(T-t)+\sigma(W_T-W_t)}\mathbb{1}_{\tau=T,S_\tau=S_T>K}d\mathbb{P}=1$$

Now we will turn to option pricing using the PDE version.

1.3 Numerical solution of the PDE

We will solve the PDE numerically instead.

We will solve only for $x \in (B, R)$ for some big constant $R = 300$. Let's choose $N_x = 2200$ to be the number of points of x and $N_t = 252$ to be the number of time points.

Define $\Delta x = \frac{R-B}{N_x}$, $\Delta t = \frac{T-t}{N_t}$, $x_k = B + k\Delta x$ for $k = 0, .., N_x$ and $t_j = t + j\Delta t$ for $j = 0, .., N_t$. In the following, we will define $v(t_j, x_k) = v_j^k$ for any function v.

(j) *Check that the final time and boundary conditions of the PDE can be numerically written as*

$$\begin{cases} v_{N_t}^k = (x_k - K)_+, & \text{for } k = 0, .., N_x \\ v_j^0 = 0, & \text{for } j = 0, .., N_t \\ v_j^{N_x} = R - e^{-r(T-t_j)}K, & \text{for } j = 0, .., N_t \end{cases}$$

Notice that there is no discontinuity at $t = T, x = B$ or $t = T, x = R$ (all the values specified are consistent).

Solution: From above, we know that $v(t, x)$ must satisfy the following boundary conditions:

$$\begin{cases} \text{Vertical Boundary: } v(T, x) = (x - K)_+, & \text{for } B < x \\ \text{Lower Boundary: } \quad v(t, B) = 0, & \text{for } t < T \\ \text{Upper Boundary: } \quad v(t, R) \approx R - e^{-r(T-t)}K, & \text{for } t < T \end{cases}$$

Then, converting this to finite time, we obtain

$$\begin{cases} \text{Vertical Boundary: } v(T, x_k) = v(t_{N_t}, x_k) = v_{N_t}^k = (x_k - K)_+, & \text{for } k = 0, .., N_x \\ \text{Lower Boundary: } \quad v(t_j, B) = v(t_j, x_0) = v_j^0 = 0, & \text{for } j = 0, .., N_t \\ \text{Upper Boundary: } \quad v(t_j, R) = v(t_j, x_{N_x}) = v_j^{N_x} \approx R - e^{-r(T-t)}K, & \text{for } j = 0, .., N_t \end{cases}$$

That is

$$\begin{cases} \text{Vertical Boundary: } v_{N_t}^k = (x_k - K)_+, & \text{for } k = 0, .., N_x \\ \text{Lower Boundary: } \quad v_j^0 = 0, & \text{for } j = 0, .., N_t \\ \text{Upper Boundary: } \quad v_j^{N_x} \approx R - e^{-r(T-t)}K, & \text{for } j = 0, .., N_t \end{cases}$$

(k) *Show that after discretization of the PDE we get*

$$v_j^k = \left(1 + r\Delta t + \sigma^2 \frac{\Delta t}{(\Delta x)^2} x_k^2\right) v_{j-1}^k - \Delta t \left(\frac{r x_k}{2\Delta x} + \frac{1}{2(\Delta x)^2}\sigma^2 x_k^2\right) v_{j-1}^{k+1} - \Delta t \left(-\frac{r x_k}{2\Delta x} + \frac{1}{2(\Delta x)^2}\sigma^2 x_k^2\right) v_{j-1}^{k-1}$$

for all $k = 1, \dots, N_x - 1$ and $j = 1, .., N_t$[1].

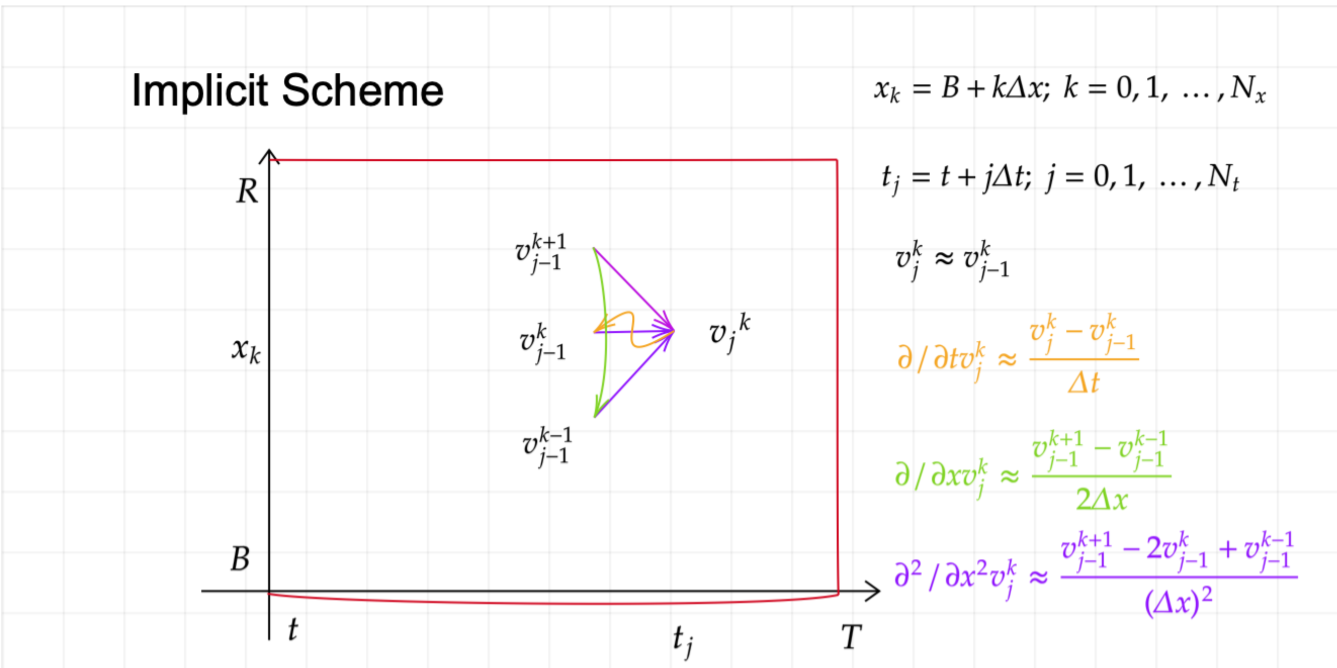

Figure 2: Implicit Scheme Graphical Version

Solution: We want to discretize the target PDE

$$-rv(t, x) + v_t(t, x) + rxv_x(t, x) + \frac{1}{2}\sigma^2 x^2 v_{xx}(t, x) = 0, \quad \text{for } t < T \text{ and } B < x$$

We use the four approximations below:

$$v_t(t_j, x_k) = \frac{\partial v(t_j, x_k)}{\partial t} \approx \frac{v(t_j, x_k) - v(t_{j-1}, x_k)}{\Delta t} = \frac{v_j^k - v_{j-1}^k}{\Delta t}, \quad v_x(t_j, x_k) \approx \frac{\partial v(t_{j-1}, x_k)}{\partial x} \approx \frac{v(t_{j-1}, x_{k+1}) - v(t_{j-1}, x_{k-1})}{2\Delta x} = \frac{v_{j-1}^{k+1} - v_{j-1}^{k-1}}{2\Delta x},$$

$$v(t_j, x_k) \approx v(t_{j-1}, x_k) = v_{j-1}^k, \quad v_{xx}(t_j, x_k) = \frac{\partial^2 v(t_j, x_k)}{\partial x^2} \approx \frac{v(t_{j-1}, x_{k+1}) - 2v(t_{j-1}, x_k) + v(t_{j-1}, x_{k-1})}{(\Delta x)^2} = \frac{v_{j-1}^{k+1} - 2v_{j-1}^k + v_{j-1}^{k-1}}{(\Delta x)^2}$$

Plug these into the target PDE, we obtain

$$-rv_{j-1}^k + \frac{v_j^k - v_{j-1}^k}{\Delta t} + rx_k \frac{v_{j-1}^{k+1} - v_{j-1}^{k-1}}{2\Delta x} + \frac{1}{2}\sigma^2 x_k^2 \frac{v_{j-1}^{k+1} - 2v_{j-1}^k + v_{j-1}^{k-1}}{(\Delta x)^2} = 0$$

[1] please let me know if there is a typo here!

Collecting terms, we get

$$\frac{1}{\Delta t}v_j^k - \left(\frac{1}{\Delta t} + r + \frac{\sigma^2 x_k^2}{(\Delta x)^2}\right)v_{j-1}^k + \left(\frac{rx_k}{2\Delta x} + \frac{1}{2(\Delta x)^2}\sigma^2 x_k^2\right)v_{j-1}^{k+1} + \left(-\frac{rx_k}{2\Delta x} + \frac{1}{2(\Delta x)^2}\sigma^2 x_k^2\right)v_{j-1}^{k-1} = 0$$

Rearranging the equation and multiplying Δt on both side, we attain

$$v_j^k = \left(1 + r\Delta t + \frac{\Delta t\sigma^2 x_k^2}{(\Delta x)^2}\right)v_{j-1}^k - \Delta t\left(\frac{rx_k}{2\Delta x} + \frac{1}{2(\Delta x)^2}\sigma^2 x_k^2\right)v_{j-1}^{k+1} - \Delta t\left(-\frac{rx_k}{2\Delta x} + \frac{1}{2(\Delta x)^2}\sigma^2 x_k^2\right)v_{j-1}^{k-1} \tag{5}$$

for all $k = 1, \ldots, N_x - 1$ and $j = 1, .., N_t$.

(l) *Define the* $(N_x+1)\times 1$ *vectors*

$$V_j = \begin{pmatrix} v_j^0 \\ \vdots \\ v_j^{N_x} \end{pmatrix}$$

Write the system in the form of

$$V_j = M \cdot V_{j-1} - C_{j-1}$$

or equivalently

$$V_{j-1} = M^{-1}(V_j + C_{j-1})$$

where you should explicitly define the matrix M*, and* C_{j-1} *is a correction term for the first and last entry of the vector* V_{j-1}[2]

Solution: Rearranging Equation (5), we have

$$v_j^k = a_k v_{j-1}^{k-1} + b_k v_{j-1}^k + c_k v_{j-1}^{k+1}, \text{ for all } k = 1, \ldots, N_x - 1 \text{ and } j = 1, .., N_t$$

where

$$a_k = -\Delta t\left(-\frac{rx_k}{2\Delta x} + \frac{1}{2(\Delta x)^2}\sigma^2 x_k^2\right), \quad b_k = \left(1 + r\Delta t + \frac{\Delta t\sigma^2 x_k^2}{(\Delta x)^2}\right), \quad c_k = -\Delta t\left(\frac{rx_k}{2\Delta x} + \frac{1}{2(\Delta x)^2}\sigma^2 x_k^2\right)$$

Since the boundary condition specifies

$$\begin{cases} v_{N_t}^k = (x_k - K)_+, & \text{for } k = 0, .., N_x \\ v_j^0 = 0, & \text{for } j = 0, .., N_t \\ v_j^{N_x} = R - e^{-r(T-t_j)}K, & \text{for } j = 0, .., N_t \end{cases}$$

We define M to be

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_1 & b_1 & c_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & a_2 & b_2 & c_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{N_x-1} & b_{N_x-1} & c_{N_x-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{(N_x+1)\times(N_x+1)}, \quad C_{j-1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ e^{-r(T-t_j))}K - e^{-r(T-t_{j-1}))}K \end{bmatrix} \in \mathbb{R}^{(N_x+1)}$$

That is

$$\begin{bmatrix} v_j^0 \\ v_j^1 \\ v_j^2 \\ \vdots \\ v_j^{N_x-1} \\ v_j^{N_x} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_1 & b_1 & c_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & a_2 & b_2 & c_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{N_x-1} & b_{N_x-1} & c_{N_x-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_{j-1}^0 \\ v_{j-1}^1 \\ v_{j-1}^2 \\ \vdots \\ v_{j-1}^{N_x-1} \\ v_{j-1}^{N_x} \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ e^{-r(T-t_j))}K - e^{-r(T-t_{j-1}))}K \end{bmatrix}$$

Then, we must have

$$V_{j-1} = M^{-1}(V_j + C_{j-1})$$

(m) *Write a program that computes* V_0 *if the initial price is* 100\$.

Solution: The value of V_0 if the initial price is 100\$ is \$4.877353434414326.

For the code, see Jupyter notebook.

1.4 Analytical solution of the PDE

(n) *If we solved the PDE analytically instead of numerically, we would have obtained the formula:*

$$v(t,x) = c_K(t,x) - \left(\frac{x}{B}\right)^{2\alpha} c_K\left(t, \frac{B^2}{x}\right) \tag{6}$$

where $c_K(t,x)$ *is the Black-Scholes value of a European call of strike* K *and maturity* T *if* $S_t = x$ *and if the interest rate is* r*, and* $\alpha = \frac{1}{2}\left(1 - \frac{2r}{\sigma^2}\right)$*. Check that this formula solves the PDE and the boundary/final time conditions given above.*

[2] it depends on how M was defined, but if you did like we did in class you can use 0's everywhere and the relevant boundary condition on the first and last entry. In that case, M has $1, 0, \ldots, 0$ as its first row and $0, \ldots, 0, 1$ in the last row.

Solution: Our target is to prove that Equation (6) satisfies the PDE below:

$$\begin{cases} -rv(t,x) + v_t(t,x) + rxv_x(t,x) + \frac{1}{2}\sigma^2 x^2 v_{xx}(t,x) = 0, & \text{for } t < T \text{ and } B < x \\ v(T,x) = (x-K)_+, & \text{for } B < x \\ v(t,B) = 0, & \text{for } t < T \end{cases}$$

For $B < x$ and $t < T$, since $C_K(t,x)$ is the Black-Scholes value of a European call of strike K and maturity T if $S_t = x$ and if the interest rate is r, it must satisfies the Black-Scholes PDE and the corresponding boundary condition; that is

$$-rc_K(t,x) + \frac{\partial}{\partial t}c_K(t,x) + rx\frac{\partial}{\partial x}c_K(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}c_K(t,x) = 0, \quad c_K(T,x) = (x-K)_+ \tag{7}$$

Substitute x with $z(x) := \frac{B^2}{x}$, we obtain

$$-rc_K(t,z) + \frac{\partial}{\partial t}c_K(t,z) + rz\frac{\partial}{\partial z}c_K(t,z) + \frac{\sigma^2}{2}z^2 \frac{\partial^2}{\partial z^2}c_K(t,z) = 0, \quad c_K(T,z) = (z-K)_+ \tag{8}$$

First, we prove $v(t,x)$ given in Equation (6) satisfies the target PDE.

Plug $v(t,x)$ into the target PDE and use $\frac{\partial}{\partial x}c_K(t,z) = \frac{\partial}{\partial z}c_K(t,z)\frac{\partial z}{\partial x} = -\frac{B^2}{x^2}\frac{\partial}{\partial z}c_K(t,z)$, we obtain

$$\begin{aligned}
0 &= -rv(t,x) + v_t(t,x) + rxv_x(t,x) + \frac{1}{2}\sigma^2 x^2 v_{xx}(t,x) \\
&= -r\left(c_K(t,x) - \left(\frac{x}{B}\right)^{2\alpha} c_K(t,z)\right) + \left(\frac{\partial}{\partial t}c_K(t,x) - \left(\frac{x}{B}\right)^{2\alpha}\frac{\partial}{\partial t}c_K(t,z)\right) \\
&\quad + rx\left(\frac{\partial}{\partial x}c_K(t,x) + \left(\frac{x}{B}\right)^{2\alpha-2}\frac{\partial}{\partial z}c_K(t,z) - 2\alpha\left(\frac{x^{2\alpha-1}}{B^{2\alpha}}\right)c_K(t,z)\right) \\
&\quad + \frac{\sigma^2 x^2}{2}\left(\frac{\partial^2}{\partial x^2}c_K(t,x) - \left(\frac{x}{B}\right)^{2\alpha-4}\frac{\partial^2}{\partial z^2}c_K(t,z) + (2\alpha-2)\left(\frac{x^{2\alpha-3}}{B^{2\alpha-2}}\right)\frac{\partial}{\partial z}c_K(t,z) + 2\alpha\left(\frac{x^{2\alpha-3}}{B^{2\alpha-2}}\right)\frac{\partial}{\partial z}c_K(t,z) - 2\alpha(2\alpha-1)\left(\frac{x^{2\alpha-2}}{B^{2\alpha}}\right)c_K(t,z)\right) \\
&= \left(-rc_K(t,x) + \frac{\partial}{\partial t}c_K(t,x) + rx\frac{\partial}{\partial x}c_K(t,x) + \frac{1}{2}\sigma^2 x^2\frac{\partial^2}{\partial x^2}c_K(t,x)\right) \\
&\quad + \left(\frac{x}{B}\right)^{2\alpha}\left(rc_K(t,z) - \frac{\partial}{\partial t}c_K(t,z) + r\frac{B^2}{x}\frac{\partial}{\partial z}c_K(t,z) - \frac{\sigma^2}{2}\frac{B^4}{x^2}\frac{\partial^2}{\partial z^2}c_K(t,z) + \left[\sigma^2(\alpha-1)\frac{B^2}{x}\frac{\partial}{\partial z}c_K(t,z) + \sigma^2\alpha\frac{B^2}{x}\frac{\partial}{\partial z}c_K(t,z)\right]\right. \\
&\qquad \left. - \left[2\alpha rc_K(t,z) + \sigma^2\alpha(2\alpha-1)c_K(t,z)\right]\right)
\end{aligned}$$

Plug $\alpha = \frac{1}{2}\left(1 - \frac{2r}{\sigma^2}\right)$ and $z = \frac{B^2}{x}$ into the above equation, we obtain

$$\begin{aligned}
0 &= \left(-rc_K(t,x) + \frac{\partial}{\partial t}c_K(t,x) + rx\frac{\partial}{\partial x}c_K(t,x) + \frac{1}{2}\sigma^2 x^2\frac{\partial^2}{\partial x^2}c_K(t,x)\right) + \left(\frac{x}{B}\right)^{2\alpha}\left(rc_K(t,z) - \frac{\partial}{\partial t}c_K(t,z) - r\frac{B^2}{x}\frac{\partial}{\partial z}c_K(t,z) - \frac{\sigma^2}{2}\frac{B^4}{x^2}\frac{\partial^2}{\partial z^2}c_K(t,z)\right) \\
&= \left(-rc_K(t,x) + \frac{\partial}{\partial t}c_K(t,x) + rx\frac{\partial}{\partial x}c_K(t,x) + \frac{1}{2}\sigma^2 x^2\frac{\partial^2}{\partial x^2}c_K(t,x)\right) + \left(\frac{x}{B}\right)^{2\alpha}\left(rc_K(t,z) - \frac{\partial}{\partial t}c_K(t,z) - rz\frac{\partial}{\partial z}c_K(t,z) - \frac{\sigma^2}{2}z^2\frac{\partial^2}{\partial z^2}c_K(t,z)\right)
\end{aligned}$$

This equation right above is justified by Equation (7) and Equation (8).

Now, we prove that for $B < x$ and $t < T$, $v(t,x)$, as given in Equation (6), satisfies the boundary/final time conditions.

Since $B < x$, we have $\frac{B^2}{x} < B \leq K$. It follows that

$$\begin{aligned}
v(T,x) &= c_K(T,x) - \left(\frac{x}{B}\right)^{2\alpha} c_K\left(T, \frac{B^2}{x}\right) = (x-K)_+ - \left(\frac{x}{B}\right)^{2\alpha}\left(\frac{B^2}{x} - K\right)_+ = (x-K)_+ \\
v(t,B) &= c_K(t,B) - \left(\frac{x}{B}\right)^{2\alpha} c_K\left(t, \frac{B^2}{B}\right) = 0 \text{ for all } t < T
\end{aligned}$$

Therefore, for $B < x$ and $t < T$, the analytical solution $v(t,x)$, as given in Equation (6), also satisfies the boundary/final time conditions.

Hence, the analytical solution $v(t,x)$, as given in Equation (6), solves the PDE and the boundary/final time conditions given above.

Remark 1.3. We have to set $B < K$ or $B \leq K$ otherwise for the given analytical solution, the boundary condition $v(T,x) = c_K(T,x) - \left(\frac{x}{B}\right)^{2\alpha} c_K\left(T, \frac{B^2}{x}\right) = (x-K)_+ - \left(\frac{x}{B}\right)^{2\alpha}\left(\frac{B^2}{x} - K\right)_+ = (x-K)_+$ is not satisfied since we need $\frac{B^2}{x} < B < K$ to make the latter term zero. △

(o) Compare the prices (accuracy and speed) obtained numerically by Monte Carlo and the PDE method, with theoretical (given by the analytical solution) price from the PDE.

Solution: We have the following results:

	Price	Speed	Accuracy (RMSE)
Monte Carlo	4.906372599084435	2 min 1 s	0.013884209135937 02
Finite Differece PDE	4.877353434414326	1 min 1 s	0.042903373806045586
THEORETICAL SOLUTION	4.920256808220372	1.58 ms	0

The Monte Carlo method is the most time-consuming; however, it achieves relatively higher accuracy compared to the finite difference PDE method. The finite difference PDE method takes relatively less time but has lower accuracy compared to the Monte Carlo method. The analytical/theoretical solution achieves the fastest speed among all given methods.

2 Bonus Question

In the case where $r = 0, x = 100\$, B = 80\$$ and $K = 80\$$, and regardless of σ, find a replicating strategy for the barrier call option. Deduce its price (verify it by one of the methods above).

Solution: Since $r = 0$, we have $\alpha = \frac{1}{2}(1 - \frac{2r}{\sigma^2}) = \frac{1}{2}$. The pricing formula, or from another perspective, the replicating portfolio for the vanilla European call option is

$$C(t, s; K, T, \sigma, r) = sN(d_+) - Ke^{-r(T-t)}N(d_-)$$

with $N(z) = \int_{-\infty}^{z} \frac{\exp(-y^2/2)}{\sqrt{2\pi}} dy$ and

$$d_- = d_+ - \sigma\sqrt{T-t}, \quad d_+ = \frac{1}{\sigma\sqrt{T-t}}\left(\log(s/K) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right)$$

Plug it in Equation (6) with $K = B$, we have

$$v(t,x) = xN(d_+) - Ke^{-r(T-t)}N(d_-) - \frac{x}{B}\left(\frac{B^2}{x}N(d'_+) - Ke^{-r(T-t)}N(d'_-)\right) = \left(e^{-r(T-t)}N(d'_-) + N(d_+)\right)x - \left(e^{-r(T-t)}N(d_-) + N(d'_+)\right)B$$

where

$$d_- = d_+ - \sigma\sqrt{T-t}, \quad d_+ = \frac{1}{\sigma\sqrt{T-t}}\left(\log(x/K) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right); d'_- = d'_+ - \sigma\sqrt{T-t}, \quad d'_+ = \frac{1}{\sigma\sqrt{T-t}}\left(\log(B^2/xK) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right)$$

Simplifying it, we obtian

$$v(t,x) = \left(N(d'_-) + N(d_+)\right)x - \left(N(d_-) + N(d'_+)\right)B \tag{9}$$

where

$$d_- = d_+ - \sigma\sqrt{T-t}, \quad d_+ = \frac{1}{\sigma\sqrt{T-t}}\left(\log(x/B) + \frac{1}{2}\sigma^2(T-t)\right); d'_- = d'_+ - \sigma\sqrt{T-t}, \quad d'_+ = \frac{1}{\sigma\sqrt{T-t}}\left(\log(B/x) + \frac{1}{2}\sigma^2(T-t)\right)$$

Notice, we have

$$d_- = d_+ - \sigma\sqrt{T-t} = \frac{1}{\sigma\sqrt{T-t}}\left(\log(x/B) - \frac{1}{2}\sigma^2(T-t)\right) = -d'_+ \quad \text{and} \quad d'_- = d'_+ - \sigma\sqrt{T-t} = \frac{1}{\sigma\sqrt{T-t}}\left(\log(B/x) - \frac{1}{2}\sigma^2(T-t)\right) = -d_+$$

It follows that

$$N(d'_-) + N(d_+) = N(-d_+) + N(d_+) = 1 \quad \text{and} \quad N(d_-) + N(d'_+) = N(d_-) + N(-d_-) = 1$$

Plug these into Equation (9), we attain

$$v(t,x) = x - B \iff v(t, S_t) = S_t - B, \quad \forall t \in [0,T]$$

The $\iff$ sign is valid since x is merely a value assigned to S_t. Therefore, the replicating strategy for this barrier option is to long 1 unit of underlying asset S_t and short B unit of risk-free asset at any time $t \in [0,T]$.

Specifically, at time t, the price of this barrier option is $S_t - B = x - B = \$20$.

Applying the analytical solution method above, we get Figure 3 which verifies that, indeed, $v(t,x) \equiv 20$ at time t. Besides, we can plot the prices given by analytical solution and prices given by replicating portfolio and see if they are identical. Indeed, from Figure 4, we see that prices given by analytical solution and prices given by replicating portfolio overlap and are exactly the same.

```
The the price of the option given by the closed formula at t=0 is: $ 20.000000000000036
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Figure 3: Price for the given barrier option at time t.

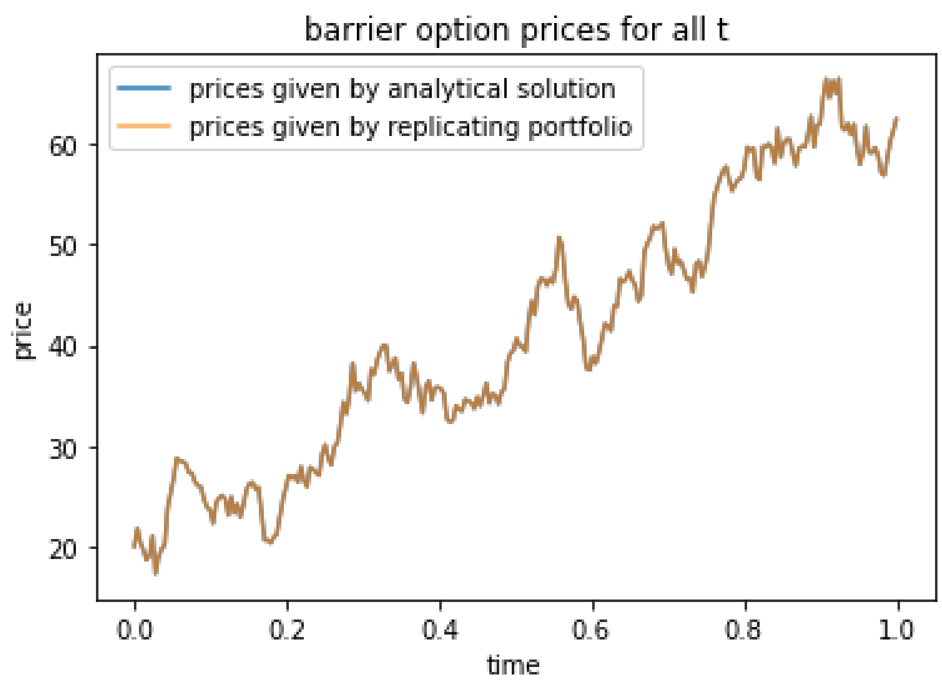

Figure 4: Prices given by analytical solution and prices given by replicating portfolio at all times t.