Dynamic Asset Pricing HW1 (Prof.Dupire)

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1 Maturity Skew – Intra maturity no-arbitrage conditions

Problem 1.1. *(a) For a given maturity T, what are the conditions on the European Call prices $C(K)$ as a function of strike $K \geq 0$ to prevent arbitrage?*

Solution: Suppose the current time is $t < T$, the risk-free rate is $r \geq 0$ and there is 0 dividend and borrowing cost.

1. **Call option prices must be non-negative:** $C(K) \geq 0, \forall K$.
   If $C(K) < 0$ for some K, then we have the following arbitrage strategy:
   At the current time t, the investor could buy the call option, receive the positive premium $-C(K)$, and reinvest the money in a risk-free asset. Suppose the investor holds this portfolio until maturity T. Then, the final payoff of the portfolio at time T is

$$\begin{aligned} \Phi_T &= -e^{r(T-t)}C(K) + C(K,T,S_T) \\ &= -e^{r(T-t)}C(K) + (S_T - K)_+ \\ C(K) < 0 \implies &\geq 0 \end{aligned}$$

   Since the final payoff of this strategy is always positive, we have an arbitrage.
   Therefore, we must have $C(K) \geq 0$ for all K.

2. **Monotonicity with respect to the strike price:** $\frac{\partial}{\partial K} C(K) \leq 0$.
   If there exists $C(K_1) < C(K_2)$ for some $K_1 < K_2$, then the investor could perform a bear call spread arbitrage strategy:
   At the current time t, the investor can buy the call option with strike price K_1 and sell the call option with strike price K_2 and receive a net premium $C(K_2) - C(K_1)$. Suppose the investor invests all of his premium from initiating the strategy into a risk-free asset and holds the portfolio until maturity T. Then, the final payoff of the portfolio at time T is

$$\begin{aligned} \Phi_T &= e^{r(T-t)}\left(C(K_2) - C(K_1)\right) + C(K_1,T,S_T) - C(K_2,T,S_T) \\ C(K_1) < C(K_2) \implies &> (S_T - K_1)_+ - (S_T - K_2)_+ \\ &= \begin{cases} 0 & \text{if } S_T < K_1 \\ S_T - K_1 \geq 0 & \text{if } K_1 \leq S_T < K_2 \\ K_2 - K_1 > 0 & \text{if } K_2 \leq S_T \end{cases} \end{aligned}$$

   Since the final payoff of this strategy is always positive, we have an arbitrage.
   Therefore, the call option price should decrease as the strike price increases: $\frac{\partial}{\partial K} C(K) \leq 0$.

3. **Convexity with respect to the strike price:** $\frac{\partial^2}{\partial K^2} C(K) \geq 0$.
   If $\frac{\partial^2}{\partial K^2} C(K) < 0$, an investor could perform a butterfly spread arbitrage strategy:
   Suppose $K_1 < K_2 < K_3$ and $2K_2 = K_1 + K_3$. At the current time t, the investor could buy one call option with strike price K_1, sell two call options with strike price K_2, and buy one call option with strike price K_3. Suppose the investor reinvests the total premium, $(-C(K_1) + 2C(K_2) - C(K_3)) > 0$, in the risk-free asset and holds the portfolio until maturity T.

The total payoff of his portfolio at the maturity time T is:

$$\begin{aligned}
\Phi_T &= e^{r(T-t)}\left(-C(K_1)+2C(K_2)-C(K_3)\right)+C(K_1,T,S_T)-2C(K_2,T,S_T)+C(K_3,T,S_T)\\
&> (S_T-K_1)_+ - 2(S_T-K_2)_+ + (S_T-K_3)_+ \quad \Longleftarrow \frac{\partial^2}{\partial K^2}C(K)<0\\
&= \begin{cases} 0 & \text{if } S_T<K_1\\ S_T-K_1\geq 0 & \text{if } K_1\leq S_T<K_2\\ 2K_2-S_T-K_1=K_3-S_T>0 & \text{if } K_2\leq S_T<K_3\\ 2K_2-K_3-K_1=0 & \text{if } K_3\leq S_T\end{cases}
\end{aligned}$$

Since the final payoff of this strategy is always positive, we have an arbitrage.
Therefore, we must have $\frac{\partial^2}{\partial K^2}C(K)\geq 0$.

4. **Upper bound: $C(K,T,S_t)\leq S_t$.**
   If there exists $C(K,T,S_t)>S_t$ for some K, we could perform the following arbitrage strategy:
   At the current time t, the investor could buy the underlying, sell the option, and receive the premium $C(K,T,S_t)-S_t>0$ from this transaction. Suppose the investor reinvest all of his premium from initiating the strategy in a risk-free asset and holds the portfolio until maturity T. The total payoff of his portfolio at the maturity time T is:

$$\begin{aligned}
\Phi_T &= e^{r(T-t)}\left(-S_t+C(K,T,S_t)\right)+S_T-C(K,T,S_T)\\
C(K,T,S_t)>S_t \Longrightarrow &> S_T-(S_T-K)_+ = \begin{cases} S_T>0 & \text{if } S_T<K\\ K\geq 0 & \text{if } K\leq S_T\end{cases}
\end{aligned}$$

   Since the final payoff of this strategy is always positive, we have an arbitrage.
   Therefore, we must have $C(K,T,S_t)\leq S_t$.
5. **Lower bound: $C(K,T,S_t)\geq S_t-Ke^{-r(T-t)}, \forall K, t\leq T$.**
   If there exists $C(K,T,S_t)<S_t-Ke^{-r(T-t)}$ for some K and $t\leq T$, then the investor could perform the following arbitrage strategy:
   At the current time t, the investor could buy the call option with strike price K, sell the underlying, and receive a net premium $S_t-C(K,T,S_t)>Ke^{-r(T-t)}\geq 0$. Suppose the investor reinvest all of his premium from initiating the strategy into a risk-free asset and holds the portfolio until maturity T. Then, the final payoff of the portfolio at time T is

$$\begin{aligned}
\Phi_T &= e^{r(T-t)}\left(S_t-C(K,T,S_t)\right)-S_T+C(K,T,S_T)\\
C(K,T,S_t)<S_t-Ke^{-r(T-t)} \Longrightarrow &> K-S_T+(S_T-K)_+\\
&= \begin{cases} K-S_T>0 & \text{if } S_T<K\\ 0 & \text{if } K\leq S_T\end{cases}
\end{aligned}$$

   Since the final payoff of this strategy is always positive, we have an arbitrage.
   Therefore, the european call option must have the lower bound $C(K,T,S_t)\geq S_t-Ke^{-r(T-t)}, \forall K, t\leq T$.

(b) Show that $C(K)$ converges to a non negative value as K goes to infinity.

Solution: If $C(K)<0$ for some K, then we have the following arbitrage strategy:

We buy the call option, receive the positive premium since $C(K)$ is negative, and reinvest the money in a risk-free asset. Since an option provides the right but not the obligation to buy the underlying asset, the PnL we will get from owning the option is going to be non-negative. Then, we will always end up having positive PnL ($\geq -e^{rT}C(K)$) after the trade.

Therefore, $C(K)\geq 0$ for all K.

Hence, $C(K)$ converges to a non negative value as K goes to infinity.

*(c) **(Bonus question)** Does the limit of $C(K)$ as K goes to infinity have to be 0? Assume $C(K)=\exp(-2K)$. Is it arbitrage free? If yes, prove it, if not show a trade that is always winning.*

Solution:

(a) Yes, the limit of $C(K)$ as K goes to infinity has to be 0.

For the sake of contradiction, suppose $\lim_{K\to\infty} C(K) = a > 0$, $a \in \mathbb{R}$. Then we have the following arbitrage strategy:

We sell the call option, receive the positive premium $C(K)$ and wait until maturity. Suppose the investor invest all of his premium from initiating the strategy into a risk-free asset. The total payoff of his portfolio at the maturity time T is:

$$\Phi_T = e^{r(T-t)}C(K) - (S_T - K)_+ \Longrightarrow \lim_{K\to\infty} \Phi_T = e^{r(T-t)}a > 0$$

Therefore, for large enough strike price K, the final payoff of this strategy is always positive; hence, we have an arbitrage.

(b) Since we don't know what the current price S_t of the underlying is, we cannot determine if it is arbitrage free or not.

For instance, if $C(K) > S_t$ for some K, the investor could perform an arbitrage strategy. The investor could buy the underlying, sell the option, and receive the premium $C(K) - S_t$. Suppose the investor invest all of his premium from initiating the strategy in a risk-free asset and holds this portfolio until maturity. The total payoff of his portfolio at the maturity time T is:

$$\Phi_T = e^{r(T-t)}\left(-S_t + C(K)\right) + S_T - (S_T - K)_+$$

$$C(K) > S_t \Longrightarrow \geq S_T - (S_T - K)_+ = \begin{cases} S_T > 0 & \text{if } S_T < K \\ K \geq 0 & \text{if } K \leq S_T \end{cases}$$

Since the final payoff of this strategy is always positive, we have an arbitrage.

(d) ***(Bonus question)*** *What happens if the Black-Scholes implied volatility growth linearly in K $(ImpVol(K) = a + b \cdot K)$ with $a > 0$ and $b > 0$?*

Solution: If we have $ImpVol(K) = a + b \cdot K$, then by Roger Lee's Moments Formula, we must have

$$\lim_{k\to\infty} \sup \frac{T\widehat{\sigma^2(K)}}{\ln(K)} = \lim_{k\to\infty} \sup \frac{T(a+bK)^2}{\ln(K)} \to +\infty = \beta \in [1,2]$$

That's a contradiction. Therefore, if you linearly extrapolate implied volatility, there will be arbitrage.

2 Several Maturities – Inter maturity no-arbitrage conditions

Problem 2.1. *Assume no carry (0 interest rate, dividend, borrow cost).*

(a) *For $T_1 < T_2$ what is the no arbitrage condition between $C(K, T_1)$ and $C(K, T_2)$?*

Solution: For $T_1 < T_2$, the no-arbitrage condition between $C(K, T_1)$ and $C(K, T_2)$ is

$$C(K, T_1) \leq C(K, T_2).$$

In other words, the option with a longer maturity must have a higher price, all else being equal.

(b) *What is the trade to put in place if it is violated?*

Solution: If for some $T_1 < T_2$, we have $C(K, T_1) > C(K, T_2)$, then we have the following calendar spread arbitrage strategy:

At the current price t, the investor could sell option 1 with price $C(K, T_1, S_t)$, buy option 2 with price $C(K, T_2, S_t)$, and reinvest the premium $C(K, T_1, S_t) - C(K, T_2, S_t) > 0$ in a risk-free asset. Suppose the investor then holds this portfolio until time T_1, fulfills the obligation for option 1 by paying $C(K, T_1, S_{T_1}) = (S_{T_1} - K)_+$, and closes out the long position of option 2 that he is still holding to get $C(K, T_2, S_{T_1})$. Then, the final payoff of the portfolio at time T_1 is

$$\begin{aligned}\Phi_{T_1} &= e^{r(T_1-t)}\left(C(K, T_1, S_t) - C(K, T_2, S_t)\right) + C(K, T_2, S_{T_1}) - C(K, T_1, S_{T_1}) \\ &> C(K, T_2, S_{T_1}) - (S_{T_1} - K)_+\end{aligned}$$

If $S_{T_1} < K$, then $\Phi_{T_1} > C(K, T_2, S_{T_1}) - (S_{T_1} - K)_+ = C(K, T_2, S_{T_1}) \geq 0$.

If $S_{T_1} \geq K$, then since $T_2 > T_1$, we have

$$\Phi_{T_1} > C(K, T_2, S_{T_1}) - (S_{T_1} - K)_+ \geq S_{T_1} - Ke^{-r(T_2-T_1)} - S_{T_1} + K = K\left(1 - e^{-r(T_2-T_1)}\right) \geq 0$$

Since either way the final payoff of this strategy is always positive, we have an arbitrage.

Remark 2.2. It's important to note that selling an option you already own (closing out a long position) is different from writing or selling an option without owning it first (short position).

- When you closes out a long position in a European call option, you collect the premium and no longer have the right to buy the underlying asset at the strike price, nor do you have any obligation. The buyer of the option you sold will now have the right to buy the underlying asset at the strike price on the expiration date. You will have no further involvement in the transaction.
- When you write or sell an option without owning it first, you have an obligation to deliver the underlying asset at the strike price if the buyer chooses to exercise the option. In this case, you would collect the premium, but you also have the potential risk of having to fulfill your obligation if the option is exercised.

△

(c) *Suppose V is the implied variance: square of implied volatility times the maturity. What are the no arbitrage conditions on $V(K, T_1)$ and $V(K, T_2)$?*

Solution: Suppose the current time $t = 0$, then we have the implied variance equals square of implied volatility times the maturity.

The no-arbitrage conditions on $V(K, T_1)$ and $V(K, T_2)$ is:

Monotonicity with respect to maturity: $\frac{\partial}{\partial T}V(K, T) > 0$**. That is:** $V(K, T_2) > V(K, T_1)$**.**

If $V(K, T_2) < V(K, T_1)$ for $T_1 < T_2$, an arbitrage opportunity arises through a calendar spread strategy. The investor could:

Sell the option with maturity T_1 and strike price K Buy the option with maturity T_2 and strike price K If the implied variance for the longer maturity is lower, the option price should also be lower. Therefore, the net premium received for this trade should be positive. The investor can invest this premium in a risk-free asset. At maturity T_1, if the underlying price is below K, both options are out of the money, and the investor keeps the premium. If the underlying price is above K, the investor pays the intrinsic value for the short option but still holds the long option with maturity T_2. This long option should have a higher value because of the higher implied variance, which should lead to a profit.

Remark 2.3. Intuitively, this condition arises because a longer time to maturity increases the uncertainty about the underlying asset's price movement, and therefore, the implied variance should be higher for options with longer maturities. △

Problem 2.4. *Assume you have skews at T_1 and T_2 with no intra/inter maturity arbitrage. We are looking at ways to interpolate between T_1 and T_2 without creating arbitrage. For a given strike K we consider 2 interpolations between T_1 and T_2:*

(a) $C(K, T)$ *linear in* T

(b) $V(K, T)$ *linear in* T

1. *Is the linear interpolation in C guaranteed to be arbitrage free? Prove it or find a counterexample.*

Solution: Linear interpolation in C IS guaranteed to be arbitrage free if $r = 0$. If $r > 0$, linear interpolation in C is NOT guaranteed to be arbitrage free. More specifically, it might violate the lower bound condition for European call options. Now, we try to prove it.

Assume we have $T_1 < T_2$ and call option prices $C(K, T_1)$ and $C(K, T_2)$ with no arbitrage, then, apparently, we have $C(K, T_1) < C(K, T_2)$ by Problem 2.1. If we use linear interpolation in C, the call option price $C(K, T)$ for $T_1 < T < T_2$ is:

$$C(K, T) = C(K, T_1) + (T - T_1)\frac{C(K, T_2) - C(K, T_1)}{T_2 - T_1} = \frac{T_2 - T}{T_2 - T_1}C(K, T_1) + \frac{T - T_1}{T_2 - T_1}C(K, T_2) \tag{1}$$

Now, select $T = \alpha T_1 + \beta T_2$ where $\alpha + \beta = 1$, $\alpha \neq 0$ and $\beta \neq 0$, then $T_1 < T < T_2$. Plugging T into Equation (1), we get that $C(K, T)$ can be represented as

$$C(K, T) = \alpha C(K, T_1) + \beta C(K, T_2) \tag{2}$$

Intuition: From Equation (2), it's easy to verify that for $T_1 < T < T_2$, $C(K,T)$ is guaranteed to satisfy the following condition:

- **Non-negative:** $C(K,T) = \alpha C(K,T_1) + \beta C(K,T_2) \geq 0$.
- **Monotonicity:** $\frac{\partial}{\partial K}C(K,T) = \alpha\frac{\partial}{\partial K}C(K,T_1) + \beta\frac{\partial}{\partial K}C(K,T_2) \leq 0$.
- **Convexity:** $\frac{\partial^2}{\partial K^2}C(K,T) = \alpha\frac{\partial^2}{\partial K^2}C(K,T_1) + \beta\frac{\partial^2}{\partial K^2}C(K,T_2) \geq 0$.
- **Upper bound:** $C(K,T) = \alpha C(K,T_1) + \beta C(K,T_2) \geq \alpha S_t + \beta S_t = S_t$.

However, it's not trivial to see the lower bound, and that is where things are starting to deviate.

- **Case 1:** $r = 0$.
  If $r = 0$, the lower bound is trivial. As we have

$$
\begin{aligned}
C(K,T) = \alpha C(K,T_1) + \beta C(K,T_2) &= \alpha(S_t - Ke^{-r(T_1-t)}) + \beta(S_t - Ke^{-r(T_2-t)}) \\
r = 0 \implies &= (\alpha+\beta)(S_t - K) = S_t - K = S_t - Ke^{-r(T-t)}
\end{aligned}
$$

  Hence, for $r = 0$, linear interpolation in C IS guaranteed to be arbitrage free.
- **Case 2:** $r > 0$.
  Inspired by the lower bound and the concave shape of $C(K,T_1) = S_t - Ke^{-r(T_1-t)}$ as presented in Figure 1, specifically, we pick $C(K,T_1) = S_t - Ke^{-r(T_1-t)}$ and $C(K,T_2) = S_t - Ke^{-r(T_2-t)}$ such that they are both greater than zero and $r > 0$. We can do this because the value of S_t can be any number. It's easy to verify that $C(K,T_1)$ and $C(K,T_2)$ constructed as such satisfy all of the conditions above. That is,
  - **Non-negative:** $C(K,T_1) \geq 0$.
  - **Monotonicity:** $\frac{\partial}{\partial K}C(K,T_1) = -e^{-r(T_1-t)} \leq 0$.
  - **Convexity:** $\frac{\partial^2}{\partial K^2}C(K,T_1) = 0 \geq 0$.
  - **Upper bound:** $C(K,T_1) = S_t - Ke^{-r(T_1-t)} \leq S_t$.
  - **Lower bound:** $C(K,T_1) = S_t - Ke^{-r(T_1-t)} \geq S_t - Ke^{-r(T_1-t)}$
  - **Monotonicity in T:** $C(K,T_1) - C(K,T_2) = S_t - Ke^{-r(T_1-t)} - S_t + Ke^{-r(T_1-t)} < 0$

  Now, we show that we must have $C(K,T) < S_t - Ke^{-r(T-t)}$ for $T_1 < T < T_2$.

  Lemma 2.5. *Prove that $e^x \geq x + 1$ for all real x.*

  Proof: Bernoulli's Inequality: for any $n \in \mathbb{N}$

$$
1 + x \leq \left(1 + \frac{x}{n}\right)^n \xrightarrow[n\to\infty]{} e^x
$$

  The inequality above is true for $x \geq -1$, and since the wanted inequality is trivial for $x < -1$ we're done.
  Back to our original question, from Equation (2), we get

$$
\begin{aligned}
C(K,T) &= \alpha C(K,T_1) + \beta C(K,T_2) \\
&= S_t - K\left(\alpha e^{-r(T_1-t)} + \beta e^{-r(T_2-t)}\right) \\
&= S_t - Ke^{-r(T-t)}\left(\alpha e^{r(T-T_1)} + \beta e^{r(T-T_2)}\right) \\
&= S_t - Ke^{-r(T-t)}\left(\alpha e^{r(T_2-T_1)\beta} + \beta e^{-r(T_2-T_1)\alpha}\right) \\
\text{Lemma 2.5}, \alpha,\beta \neq 0, r > 0 \implies &< S_t - Ke^{-r(T-t)}\Big(\alpha(1 + r(T_2-T_1)\beta) + \beta(1 - r(T_2-T_2)\alpha)\Big) \\
\alpha + \beta = 1 \implies &= S_t - Ke^{-r(T-t)}
\end{aligned}
$$

  Hence, for $r > 0$, linear interpolation in C is NOT guaranteed to be arbitrage free.

2. ***Bonus question**: same question for the linear interpolation in V.*

Solution: Similar to the case for C, we select $T = \alpha T_1 + \beta T_2$ where $\alpha + \beta = 1$, $\alpha \neq 0$ and $\beta \neq 0$, then $T_1 < T < T_2$, and

$$
V(K,T) = \alpha V(K,T_1) + \beta V(K,T_2) \tag{3}
$$

where $V(K,T_1)$ and $V(K,T_2)$ satisfies the no-arbitrage condition. Then, we must have $V(K,T_1) < V(K,T_2)$. It's easy to see that

$$
V(K,T_1) < \alpha V(K,T_1) + \beta V(K,T_2) < V(K,T_2)
$$

Therefore, we have $V(K,T_1) < V(K,T) < V(K,T_2)$, which means $V(K,T)$ satisfies the no-arbitrage condition.

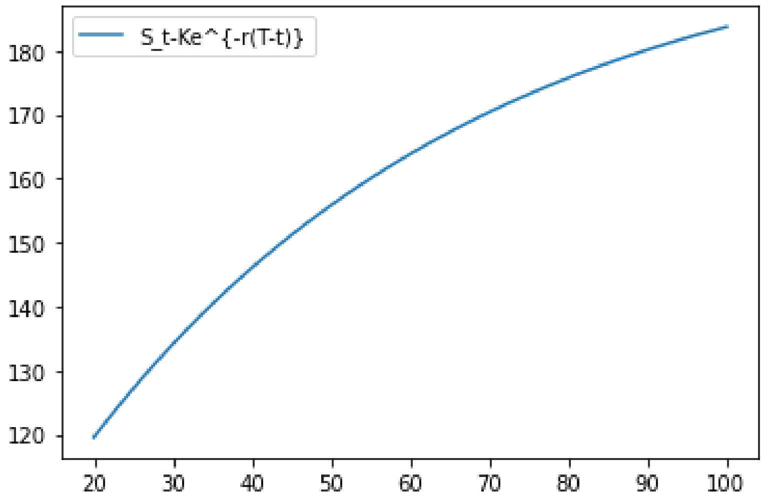

Figure 1: Caption

Thus, linear interpolation in V IS guaranteed to be arbitrage free.