Dynamic Asset Pricing Homework 1

Due 04/13

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Contents

Note: In all exercises, $(\Omega, \mathcal{F}, \mathbb{P})$ represents a probability space, $(W_t)_{t\geq 0}$ represents a standard Brownian motion on that probability space, and $(\mathcal{F}_t)_{t\geq 0}$ represents the filtration it generates.

1 Topic 1: Reflection principle for Brownian motion

Problem 1.1. *Define the stochastic process for $t \in \mathbb{R}_+$:*

$$M_t = \max_{0\leq s\leq t} W_s$$

*which is the **running maximum** of a Brownian motion. Also define the random variable for $b \in \mathbb{R}$:*

$$\tau_b = \inf\{t \geq 0 : W_t = b\}$$

*which gives the first time W reaches the level b. It's normally called the **first passage time**.*

The goal is to derive their distribution. To do so, we will first compute:

$$F(a, b) = \mathbb{P}(W_t \leq a, M_t \geq b)$$

*by the so called '**Reflection Principle**'.*

(a) Let $0 \leq a \leq b, t \in \mathbb{R}_+$ and define the stochastic process for $s \in [0, t]$:

$$\widetilde{W}_s = \begin{cases} W_s, & \text{if } s \leq \tau_b \\ 2b - W_s, & \text{if } s \geq \tau_b \end{cases}$$

Plot a Brownian path W on $[0, t]$, satisfying: $\tau_b < t$ and $W_t \leq a$. Plot the corresponding path (i.e. the same 'ω') for $\widetilde{W}$ on $[0, t]$. In what interval does $\widetilde{W}_t$ end up?

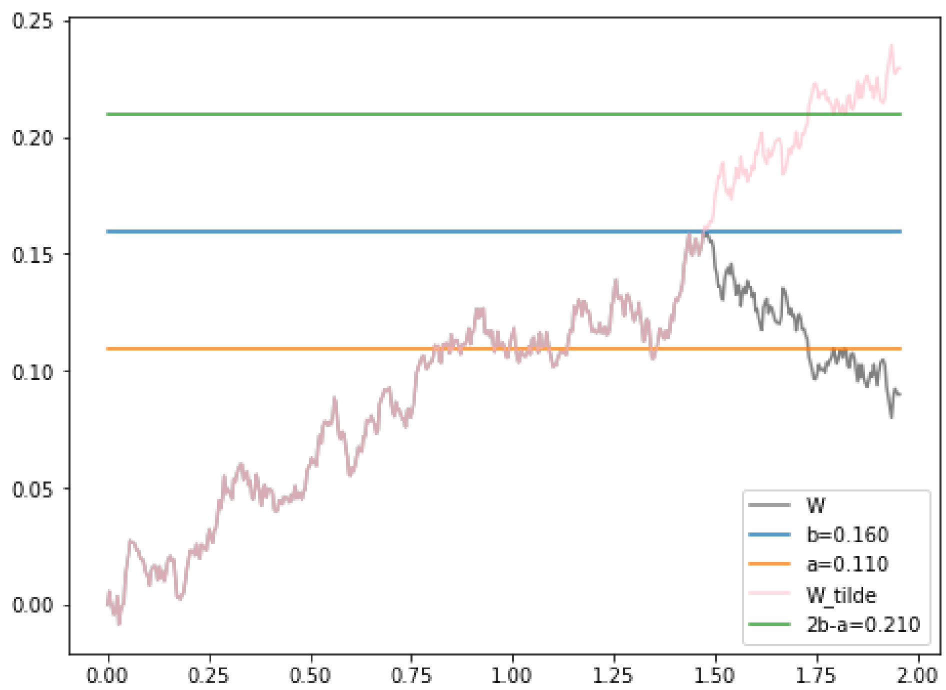

Figure 1: Plot of W_s and $\widetilde{W}_S$.

Solution: Set $t = 1.96$, $b = 0.160$, $a = 0.110$, numpy random seed$= 66$, we have $\tau_b = 1.472$ and yield Figure 1 above. $\widetilde{W}_t$ should end up in the interval $[2b - a, \infty)$.

(b) Let's admit that $\widetilde{W}$ is still a Brownian motion [1], *and hence*

$$F(a,b) = \mathbb{P}\left(\widetilde{W}_t \le a, \max_{0\le s\le t} \widetilde{W}_s \ge b\right)$$

By using the definition of $\widetilde{W}$, and by noting that:

$$\left\{\omega \in \Omega : \max_{0\le s\le t} \widetilde{W}_s \ge b\right\} = \{\omega \in \Omega : \tau_b \le t\} \tag{1}$$

show that:

$$F(a,b) = \mathbb{P}\left(W_t \ge 2b - a\right)$$

Solution: We have

$$\begin{aligned}
F(a,b) &= \mathbb{P}\left(\widetilde{W}_t \le a, \max_{0\le s\le t} \widetilde{W}_s \ge b\right)\\
\text{Equation (1)} \implies &= \mathbb{P}\left(\widetilde{W}_t \le a, \tau_b \le t\right)\\
\text{Definition of } \widetilde{W}_t \implies &= \mathbb{P}\left(2b - W_t \le a\right)\\
&= \mathbb{P}\left(W_t \ge 2b - a\right)
\end{aligned}$$

Therefore, we must have

$$F(a,b) = \mathbb{P}\left(W_t \ge 2b - a\right)$$

(c) Deduce $\mathbb{P}\left(M_t \ge b\right)$(Hint [2]*), $\mathbb{P}\left(\tau_b \le t\right)$, the densities of M_t, τ_b as well as the joint distribution of (W_t, M_t).*

Solution:

(i) By Equation (1), we have

$$\begin{aligned}
\mathbb{P}\left(\tau_b \le t\right) = \mathbb{P}\left(M_t \ge b\right) &= \mathbb{P}\left(M_t \ge b, W_t \le b\right) + \mathbb{P}\left(M_t \ge b, W_t \ge b\right)\\
(b) \text{ and } \{M_t \ge b\} \subset \{W_t \ge b\} \implies &= F(b,b) + \mathbb{P}\left(W_t \ge b\right)\\
&= 2\mathbb{P}\left(W_t \ge b\right)
\end{aligned}$$

Therefore, we have $\mathbb{P}\left(\tau_b \le t\right) = \mathbb{P}\left(M_t \ge b\right) = 2\mathbb{P}\left(W_t \ge b\right)$.

(ii) Then, we have

$$\begin{aligned}
f_{\tau_b}(t) = \frac{\partial}{\partial t} F_{\tau_b}(t) = \frac{\partial}{\partial t}\mathbb{P}\left(\tau_b \le t\right) &= \frac{\partial}{\partial t} 2\mathbb{P}\left(W_t \ge b\right)\\
&= \frac{\partial}{\partial t} 2\int_b^\infty \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}\,dx\\
y = \frac{x}{\sqrt{t}} \implies &= \frac{2}{\sqrt{2\pi}} \frac{\partial}{\partial t} \int_{\frac{b}{\sqrt{t}}}^\infty e^{-\frac{y^2}{2}}\,dy\\
\text{Leibniz integral rule } \implies &= \frac{2}{\sqrt{2\pi}}\left(0 - e^{-\frac{b^2}{2t}} \frac{\partial}{\partial t}\frac{b}{\sqrt{t}} + \int_{\frac{b}{\sqrt{t}}}^\infty \frac{\partial}{\partial t} e^{-\frac{y^2}{2}}\,dy\right)\\
&= \frac{b}{t\sqrt{2\pi t}} e^{-\frac{b^2}{2t}}
\end{aligned}$$

and

$$\begin{aligned}
f_{M_t}(b) = \frac{\partial}{\partial b} F_{M_t}(b) = \frac{\partial}{\partial t}\mathbb{P}\left(M_t \le b\right) &= \frac{\partial}{\partial b}\left(1 - 2\mathbb{P}\left(W_t \ge b\right)\right)\\
&= \frac{\partial}{\partial b}\left(1 - 2\int_b^\infty \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}\,dx\right)\\
&= -\frac{\partial}{\partial b} 2\int_b^\infty \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}\,dx\\
\text{Leibniz integral rule } \implies &= -2\left(\frac{1}{\sqrt{2\pi t}} e^{-\frac{\infty^2}{2t}} \frac{\partial \infty}{\partial b} - \frac{1}{\sqrt{2\pi t}} e^{-\frac{b^2}{2t}} \frac{\partial b}{\partial b} + \int_b^\infty \frac{\partial}{\partial b} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}\,dx\right)\\
&= \frac{2}{\sqrt{2\pi t}} e^{-\frac{b^2}{2t}}
\end{aligned}$$

Therefore, we have $f_{\tau_b}(t) = \frac{b}{t\sqrt{2\pi t}} e^{-\frac{b^2}{2t}}$, $t > 0$, and $f_{M_t}(b) = \frac{2}{\sqrt{2\pi t}} e^{-\frac{b^2}{2t}}$, $b \ge 0$.

(iii) From (b), we know that

$$F(a,b) = \mathbb{P}\left(W_t \le a, M_t \ge b\right) = \mathbb{P}\left(W_t \ge 2b - a\right)$$

For the joint distribution f_{W_t, M_t}, we obtain

$$\mathbb{P}\left(W_t \le a, M_t \ge b\right) = \int_b^\infty \int_{-\infty}^a f_{W_t,M_t}(x,y)\,dx\,dy = \mathbb{P}\left(W_t \ge 2b - a\right) = \int_{2b-a}^\infty \frac{1}{\sqrt{2\pi t}} e^{-\frac{z^2}{dt}}\,dz$$

[1] a consequence of the independence of W_{τ_b} and $W_s - W_{\tau_b}$ for $s \ge \tau_b$, and of $u \mapsto W_{\tau_b + u} - W_{\tau_b}$ being a Brownian Motion

[2] Hint: $\mathbb{P}\left(M_t \ge b\right) = \mathbb{P}\left(M_t \ge b, W_t \le b\right) + \mathbb{P}\left(M_t \ge b, W_t \ge b\right)$

Differentiate both sides with respect to b, we get

$$\begin{aligned} &\implies \frac{\partial}{\partial b}\int_{b}^{\infty}\int_{-\infty}^{a} f_{W_t,M_t}(x,y)dxdy = \frac{\partial}{\partial b}\int_{2b-a}^{\infty}\frac{1}{\sqrt{2\pi t}}e^{-\frac{z^2}{2t}}dz \\ \text{Leibniz integral rule } &\implies -\frac{\partial b}{\partial b}\int_{-\infty}^{a} f_{W_t,M_t}(x,b)dx = -\frac{\partial 2b-a}{\partial b}\frac{1}{\sqrt{2\pi t}}e^{-\frac{(2b-a)^2}{2t}} \end{aligned}$$

Then, differentiate both sides with respect to a, we get

$$\begin{aligned} &\implies -\frac{\partial}{\partial a}\int_{-\infty}^{a} f_{W_t,M_t}(x,b)dx = -\frac{\partial}{\partial a}\frac{2}{\sqrt{2\pi t}}e^{-\frac{(2b-a)^2}{2t}}dz \\ \text{Leibniz integral rule } &\implies -\frac{\partial a}{\partial a}f_{W_t,M_t}(a,b) = -\frac{\partial}{\partial a}\frac{2}{\sqrt{2\pi t}}e^{-\frac{(2b-a)^2}{2t}} \\ &\implies f_{W_t,M_t}(a,b) = \frac{2(2b-a)}{t\sqrt{2\pi t}}e^{-\frac{(2b-a)^2}{2t}} \end{aligned}$$

Therefore, we have $f_{W_t,M_t}(a,b) = \frac{2(2b-a)}{t\sqrt{2\pi t}}e^{-\frac{(2b-a)^2}{2t}}, a \leq b, b > 0$.

2 Topic 2: Time independent boundary value problems

Problem 2.1. *Let $D = [a, b]$ and consider the stochastic process:*

$$dX_t = \alpha(X_t)\,dt + \beta(X_t)\,dW_t$$

Note that α, β are deterministic functions that do not depend on time. Define

$$u(x) = \mathbb{E}\left[\int_0^{\tau_x} f(X_s)\,ds + g(X_{\tau_x})\,\Big|\, X_0 = x\right] \tag{2}$$

for $x \in D$, where

$$\tau_x = \{\inf t \geq 0 : X_t \notin D\}$$

*Note that τ_x depends on x due to the starting point $X_0 = x$. f, g are deterministic functions, that represent respectively a **running payoff** and a **final time payoff**.*

In other words, we are playing a game where we receive (or pay) $f(X_t)\,dt$ for each unit of time dt as long as X_t remains in D. As soon as X_t exits D, we get (or pay) $g(X_t)$. $u(x)$ represents our expected payoff from this game.

The goal is to show that u solves the ODE (becomes a PDE if $x \in \mathbb{R}^n$):

$$\begin{cases} \alpha(x)\frac{d}{dx}u(x) + \frac{\beta^2(x)}{2}\frac{d^2}{dx^2}u(x) + f(x) = 0, \quad x \in D \\ u(a) = g(a), u(b) = g(b) \end{cases}$$

(a) *Apply Ito's lemma to $u(X_t)$ and integrate both sides of the equation between 0 and τ_x.*

Solution: By Itô's lemma, we have

$$\begin{aligned} du(X_t) &= u_t dt + u_x dX_t + \frac{1}{2}u_x x(dX_t)^2 \\ &= 0 \cdot dt + u_x(\alpha(X_t)\,dt + \beta(X_t)\,dW_t) + \frac{1}{2}u_{xx}(\alpha(X_t)\,dt + \beta(X_t)\,dW_t)^2 \\ &= \left(u_x\alpha(X_t) + \frac{1}{2}u_{xx}\beta^2(X_t)\right)dt + u_x\beta(X_t)\,dW_t \end{aligned}$$

Integrate both sides of the equation between 0 and τ_x, we get

$$u(X_{\tau_x}) - u(X_0) = \int_0^{\tau_x}\left(u_x\alpha(X_t) + \frac{1}{2}u_{xx}\beta^2(X_t)\right)dt + \int_0^{\tau_x} u_x\beta(X_t)\,dW_t$$

(b) *Assume that u does indeed solves the ODE above. Deduce that*

$$u(x) = \mathbb{E}\left[\int_0^{\tau_x} f(X_s)\,ds + g(X_{\tau_x}) \mid X_0 = x\right]$$

Hint: you can assume that $\mathbb{E}\left[\int_0^{\tau_x} h(X_t)\,dW_t \mid X_0 = x\right] = 0$ for any function h. This only holds if $\mathbb{E}[\tau_x \mid X_0 = x] < +\infty$, which is not hard to prove here (you are not asked to do this but will get bonus points if you do).

Solution: For $x \in D$, by the ODE, we have

$$u(X_{\tau_x}) - u(X_0) = \int_0^{\tau_x} -f(X_t)dt + \int_0^{\tau_x} u_x\beta(X_t)\,dW_t$$

Rearrange the above equation, we obtain

$$u(X_0) = \int_0^{\tau_x} f(X_t)dt + u(X_{\tau_x}) - \int_0^{\tau_x} u_x\beta(X_t)\,dW_t$$

Since $x \in D$ and $\tau_x = \{\inf t \geq 0 : X_t \notin D\}$ represents the time where X_t exits D the first time, X_{τ_x} must be at the boundary of D. That is, $X_{\tau_x} = a$ or b.

Therefore, $u(X_{\tau_x}) = u(a) = g(a)$, or $u(X_{\tau_x}) = u(b) = g(b)$; that is, $u(X_{\tau_x}) = g(X_{\tau_x})$. Plug this in the above equation, we obtain

$$u(X_0) = \int_0^{\tau_x} f(X_t)dt + g(X_{\tau_x}) - \int_0^{\tau_x} u_x\beta(X_t)\,dW_t$$

Take conditional expectation with respect to $X_0 = x$ on both sides yields

$$\mathbb{E}[u(X_0)|X_0 = x] = \mathbb{E}\left[\int_0^{\tau_x} f(X_t)dt + g(X_{\tau_x})\Big|X_0 = x\right] - \mathbb{E}\left[\int_0^{\tau_x} u_x\beta(X_t)\,dW_t\Big|X_0 = x\right]$$

Since Itô integrals are martingales, it simplifies to

$$u(x) = \mathbb{E}\left[\int_0^{\tau_x} f(X_t)dt + g(X_{\tau_x})\Big|X_0 = x\right]$$

We just showed that if there exists a solution $u \in C^2(D)$ to the ODE, then it is necessarily given by Equation (2). Existence is given by the theory of ODEs or PDEs (under some technical assumptions on α, β, f, g) and is out of the scope of the class.

(c) *Application 1: Let $dX_t = dW_t$ and define*

$$p_x = \mathbb{P}(X_{\tau_x} = b)$$

Show that

$$p_x = \frac{x-a}{b-a}$$

Hint: $p_x = \mathbb{E}\left[\mathbb{1}_{X_{\tau_x}=b} \mid X_0 = x\right]$

Solution: Since $p_x = \mathbb{P}(X_{\tau_x} = b) = \mathbb{E}\left[\mathbb{1}_{X_{\tau_x}=b} \mid X_0 = x\right]$, set $f \equiv 0$, $g(X_{\tau_x}) = \mathbb{1}_{X_{\tau_x}=b}$, we know $p(x)$ solves the ODE.
Since $dX_t = dW_t$, we have $\alpha \equiv 0$ and $\beta \equiv 1$. Then, $p(x)$ must solve

$$\begin{cases} \frac{1}{2}p_{xx}(x) = 0, \quad x \in D \\ p(a) = \mathbb{1}_{a=b} = 0, p(b) = \mathbb{1}_{b=b} = 1 \end{cases}$$

It follows that

$$p(x) = Ax + B, \quad p(a) = 0, \quad p(b) = 1$$

Solve this linear equation, we get $A = \frac{1}{b-a}$, $B = \frac{-a}{b-a}$.
Hence,

$$p(x) = \frac{x-a}{b-a}$$

(d) *Application 2: Let $dX_t = dW_t$ and define*

$$\bar{t}_{[a,b]}(x) = \mathbb{E}\left[\tau_x \mid X_0 = x\right]$$

Show that

$$\bar{t}_{[a,b]}(x) = (b-x)(x-a)$$

Solution: Since $\bar{t}_{[a,b]}(x) = \mathbb{E}\left[\tau_x \mid X_0 = x\right]$, set $f \equiv 1$, $g \equiv 0$, we know $\bar{t}_{[a,b]}(x)$ solves the ODE.
Since $dX_t = dW_t$, we have $\alpha \equiv 0$ and $\beta \equiv 1$. Then, $\bar{t}_{[a,b]}(x)$ must solve

$$\begin{cases} \frac{1}{2}\bar{t}_{[a,b]}(x) + 1 = 0, \quad x \in D \\ \bar{t}_{[a,b]}(a) = 0, \bar{t}_{[a,b]}(b) = 0 \end{cases}$$

It follows that

$$\bar{t}_{[a,b]}(x) = -x^2 + Ax + B, \quad \bar{t}_{[a,b]}(a) = 0, \quad \bar{t}_{[a,b]}(b) = 0$$

Solve this linear equation, we get $A = a + b$, $B = -ab$.
Hence,

$$\bar{t}_{[a,b]}(x) = -x^2 + (a+b)x - ab = (b-x)(x-a)$$

3 Topic 3: Pricing with the Black-Scholes formula and beyond

Problem 3.1. *Assume that we are in the Black-Scholes setting, that is the stock price is given in the risk neutral measure by:*

$$dS_t = rS_t dt + \sigma S_t dW_t$$

with some constant interest rate $r > 0$ and volatility $\sigma > 0$, and W_t a Brownian motion in the risk neutral measure. Assume that today's price $S_t = s > 0$.

We saw in class that the price at time t of a European Call option with strike K and maturity T, that is an option with payoff $(S_T - K)_+$ at time T, is given by:

$$C(t, s; K, T, \sigma, r) = sN(d_+) - Ke^{-r\tau}N(d_-)$$

with $\tau = T - t$, $N(z) = \int_{-\infty}^{z} \frac{\exp(-y^2/2)}{\sqrt{2\pi}} dy$ and

$$d_- = d_+ - \sigma\sqrt{\tau}, \quad d_+ = \frac{1}{\sigma\sqrt{\tau}}\left(\log(s/K) + \left(r + \frac{1}{2}\sigma^2\right)\tau\right)$$

3.1 Put-Call parity

(a) A forward contract with strike K and maturity T pays $S_T - K$ at time T. That is, we agree exchanging the stock at the price K, at time T. The price today of such a contract is given by the risk neutral formula:

$$F(t, s; K, T, r) = \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}(S_T - K) \mid S_t = s\right]$$

give a one line justification that proves:

$$F(t, s; K, T, r) = s - e^{-r(T-t)}K$$

Solution: We have

$$F(t, s; K, T, r) = \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma W_{T-t}} \mid S_t = s\right] - e^{-r(T-t)}K = s - e^{-r(T-t)}K$$

(b) We could do a computation similar to the one for the Call to get the price of a European put option, that is for an option that pays $(K - S_T)_+$ at time T. Instead of computing ugly integrals, use the fact that for any $x, K \in \mathbb{R}$:

$$x - K = (x - K)_+ - (K - x)_+ \tag{3}$$

to compute the price at time t of a European put option with strike K and maturity T.

Solution: We have

$$\begin{aligned}
P(t, s; K, T, \sigma, r) &= \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}(K - S_T)_+ \mid S_t = s\right] \\
\text{Equation (3)} \Longrightarrow &= \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}((S_T - K)_+ - (S_T - K)) \mid S_t = s\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}(S_T - K)_+ \mid S_t = s\right] - \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}(S_T - K) \mid S_t = s\right] \\
&= C(t, s; K, T, \sigma, r) - \left(s - e^{-r(T-t)}K\right) \\
&= sN(d_+) - Ke^{-r(T-t)}N(d_-) - s + e^{-r(T-t)}K \\
&= s(N(d_+) - 1) + Ke^{-r(T-t)}(1 - N(d_-)) \\
&= -sN(-d_+) + Ke^{-r(T-t)}N(-d_-)
\end{aligned}$$

Therefore, we obtain

$$P(t, s; K, T, \sigma, r) = -sN(-d_+) + Ke^{-r(T-t)}N(-d_-)$$

where $N(z) = \int_{-\infty}^{z} \frac{\exp(-y^2/2)}{\sqrt{2\pi}} dy$ and

$$d_- = d_+ - \sigma\sqrt{\tau}, \quad d_+ = \frac{1}{\sigma\sqrt{\tau}}\left(\log(s/K) + \left(r + \frac{1}{2}\sigma^2\right)\tau\right)$$

3.2 Payoff Decomposition (finite case)

(c) Let's extend this decomposition to other options; Using the Black-Scholes formula for the price of a European Call, give an analytical formula for the price of a Bull call spread which payoff is given by

$$V(S_T) = \begin{cases} B, & \text{if } S_T > B \\ \frac{B+A}{B-A}S_T - \frac{2AB}{B-A}, & \text{if } S_T \in [A, B] \\ -A, & \text{if } S_T < A \end{cases}$$

for some $0 < A < B$. Hint: Can this payoff be replicated with a combination of calls?

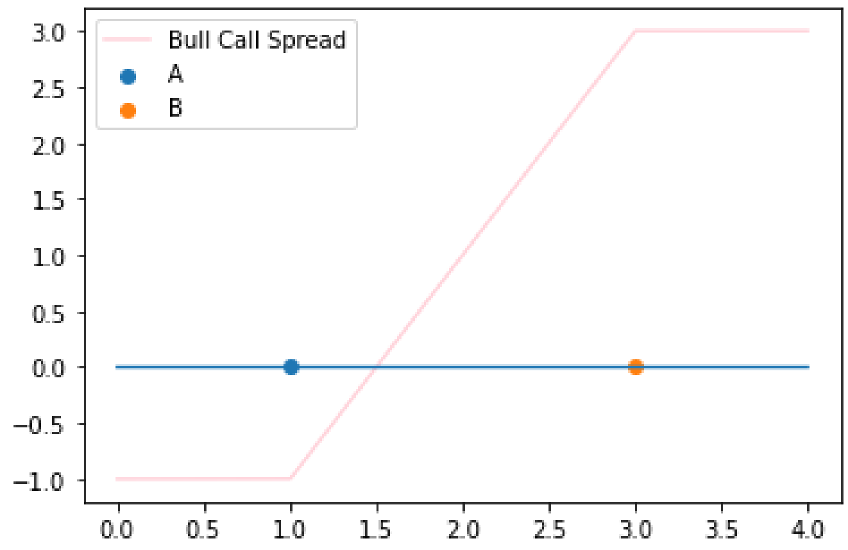

Figure 2: The payoff of a Bull Call spread

Solution: With the construction of a Bull Call spread, we want to replicate it with a combination of calls with different maturity dates. From Figure 2, our intuition tell us to consider replicating the payoff $V(S_T)$ as

$$V(S_T) = x(S_T - A)^+ + y(S_T - B)^+ + z$$

Then, we get a system of linear equations by considering $S_T > B$, $S_T \in [a, b]$, and $S_T < a$. That is,

$$\begin{cases} x(S_T - A) + y(S_T - B) + z & = B \\ x(S_T - A) + z & = \frac{B+A}{B-A}S_T - \frac{2AB}{B-A} \\ z & = -A \end{cases}$$

Solve it, we have $x = \frac{A+B}{A-B}$, $y = -\frac{A+B}{A-B}$, and $z = -A$. Therefore, we have

$$V(S_T) = \frac{A+B}{A-B}(S_T - A)^+ - \frac{A+B}{A-B}(S_T - B)^+ - A$$

By the pricing formula under the risk-neutral measure, we have

$$\begin{aligned} BC(t, s; K, T, r) &= \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}V(S_T) \mid S_t = s\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}\left(\frac{A+B}{A-B}(S_T - A)^+ - \frac{A+B}{A-B}(S_T - B)^+ - A\right)\Big| S_t = s\right] \\ &= \frac{A+B}{A-B}C(t, s; A, T, \sigma, r) - \frac{A+B}{A-B}C(t, s; B, T, \sigma, r) - Ae^{-r(T-t)} \\ &= \frac{A+B}{A-B}\left(sN\left(d_+^A\right) - Ae^{-r(T-t)}N\left(d_-^A\right)\right) - \frac{A+B}{A-B}\left(sN\left(d_+^B\right) - Be^{-r(T-t)}N\left(d_-^B\right)\right) - Ae^{-r(T-t)} \end{aligned}$$

Hence, the price of a Bull call spread is

$$BC(t, s; K, T, r) = \frac{A+B}{A-B}\left(sN\left(d_+^A\right) - Ae^{-r(T-t)}N\left(d_-^A\right)\right) - \frac{A+B}{A-B}\left(sN\left(d_+^B\right) - Be^{-r(T-t)}N\left(d_-^B\right)\right) - Ae^{-r(T-t)}$$

where $N(z) = \int_{-\infty}^{z} \frac{\exp(-y^2/2)}{\sqrt{2\pi}}dy$ and

$$d_-^A = d_+^A - \sigma\sqrt{\tau}, \quad d_+^A = \frac{1}{\sigma\sqrt{\tau}}\left(\log(s/A) + \left(r + \frac{1}{2}\sigma^2\right)\tau\right),$$
$$d_-^B = d_+^B - \sigma\sqrt{\tau}, \quad d_+^B = \frac{1}{\sigma\sqrt{\tau}}\left(\log(s/B) + \left(r + \frac{1}{2}\sigma^2\right)\tau\right).$$

(d) Do the same for a Butterfly spread which payoff is given by

$$V(S_T) = \begin{cases} 0, & \text{if } S_T < K - \delta \\ \frac{1}{\delta}\left(S_T - (K - \delta)\right), & \text{if } S_T \in [K - \delta, K) \\ -\frac{1}{\delta}\left(S_T - (K + \delta)\right), & \text{if } S_T \in [K, K + \delta] \\ 0, & \text{if } S_T > K + \delta \end{cases}$$

for some $K, \delta > 0$

Solution: With the construction of a Butterfly spread, we want to replicate it with a combination of calls with different maturity dates. From the definition of $V(S_T)$, our intuition tell us to consider replicating the payoff $V(S_T)$ as

$$V(S_T) = x(S_T - (K - \delta))^+ + y(S_T - K)^+ + z(S_T - (K + \delta))^+ + t$$

Then, we get a system of linear equations by considering $S_T < K - \delta$, $S_T \in [K - \delta, K)$, $S_T \in [K, K + \delta)$, and $S_T > K + \delta$. That is,

$$\begin{cases} t & = 0 \\ x(S_T - (K - \delta)) + t & = \frac{1}{\delta}\left(S_T - (K - \delta)\right) \\ x(S_T - (K - \delta)) + y(S_T - K) + t & = -\frac{1}{\delta}\left(S_T - (K + \delta)\right) \\ x(S_T - (K - \delta)) + y(S_T - K) + z(S_T - (K + \delta)) + t & = 0 \end{cases}$$

Solve it, we have $x = \frac{1}{\delta}, y = -\frac{2}{\delta}, z = \frac{1}{\delta}$ and $t = 0$. Therefore, we have

$$V(S_T) = \frac{1}{\delta}(S_T - (K-\delta))^+ - \frac{2}{\delta}(S_T - K)^+ + \frac{1}{\delta}(S_T - (K+\delta))^+$$

By the pricing formula under the risk-neutral measure, we have

$$\begin{aligned}
&Butterfly(t,s;K,\delta,T,r) = \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}V\left(S_T\right) \mid S_t = s\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}\left(\frac{1}{\delta}(S_T-(K-\delta))^+ - \frac{2}{\delta}(S_T-K)^+ + \frac{1}{\delta}(S_T-(K+\delta))^+\right)\Big|S_t = s\right] \\
&= \frac{1}{\delta}C(t,s;K-\delta,T,\sigma,r) - \frac{2}{\delta}C(t,s;K,T,\sigma,r) + \frac{1}{\delta}C(t,s;K+\delta,T,\sigma,r) \\
&= \frac{1}{\delta}\left(sN\left(d_+^{K-\delta}\right) - (K-\delta)e^{-r(T-t)}N\left(d_-^{K-\delta}\right)\right) - \frac{2}{\delta}\left(sN\left(d_+^{K}\right) - Ke^{-r(T-t)}N\left(d_-^{K}\right)\right) + \frac{1}{\delta}\left(sN\left(d_+^{K+\delta}\right) - (K+\delta)e^{-r(T-t)}N\left(d_-^{K+\delta}\right)\right)
\end{aligned}$$

Hence, the price of a Bull call spread is

$$Butterfly(t,s;K,\delta,T,r) = \frac{1}{\delta}\left(sN\left(d_+^{K-\delta}\right) - (K-\delta)e^{-r(T-t)}N\left(d_-^{K-\delta}\right)\right) - \frac{2}{\delta}\left(sN\left(d_+^{K}\right) - Ke^{-r(T-t)}N\left(d_-^{K}\right)\right) + \frac{1}{\delta}\left(sN\left(d_+^{K+\delta}\right) - (K+\delta)e^{-r(T-t)}N\left(d_-^{K+\delta}\right)\right)$$

where $N(z) = \int_{-\infty}^{z} \frac{\exp(-y^2/2)}{\sqrt{2\pi}}dy$ and

$$\begin{aligned}
d_-^{K-\delta} = d_+^{K-\delta} - \sigma\sqrt{\tau}, \quad d_+^{K-\delta} &= \frac{1}{\sigma\sqrt{\tau}}\left(\log(s/(K-\delta)) + \left(r + \frac{1}{2}\sigma^2\right)\tau\right), \\
d_-^{K} = d_+^{K} - \sigma\sqrt{\tau}, \quad d_+^{K} &= \frac{1}{\sigma\sqrt{\tau}}\left(\log(s/K) + \left(r + \frac{1}{2}\sigma^2\right)\tau\right), \\
d_-^{K+\delta} = d_+^{K+\delta} - \sigma\sqrt{\tau}, \quad d_+^{K+\delta} &= \frac{1}{\sigma\sqrt{\tau}}\left(\log(s/(K+\delta)) + \left(r + \frac{1}{2}\sigma^2\right)\tau\right).
\end{aligned}$$

(e) *Explain the practical advantages of using such decompositions, as opposed to pricing directly using a PDE method or Monte-Carlo.*
Hint: What are the implications of these decompositions in terms of replication/hedging? Would you rather use the Δ from the PDE method?

Solution: The practical advantages of using such decompositions are:

(a) It's more accurate and less time-consuming than the PDE-Monte Carlo method. Analytical solutions are always preferred.

(b) Since we can replicate the spread, it's easier for us to hedge the risk by betting against the spread using a replicated portfolio, which helps us lock in the profits.

3.3 Payoff Decomposition (infinite case)

Some payoffs aren't a linear combination of calls, puts, forwards. etc., and hence we can't use a simple decomposition and linearity of expectation. We can still however use the butterfly spreads to approximate them by such a linear combination.

Set $v(x;K) = (x-K)_+, \forall x, K \in \mathbb{R}$.

(f) *Draw the shape of the function V defined in question (d), that is*

$$V(x;K,\delta) = \frac{v(x;K+\delta) - 2v(x;K) + v(x;K-\delta)}{\delta}$$

for $\delta \geq 0, K \in \mathbb{R}$.

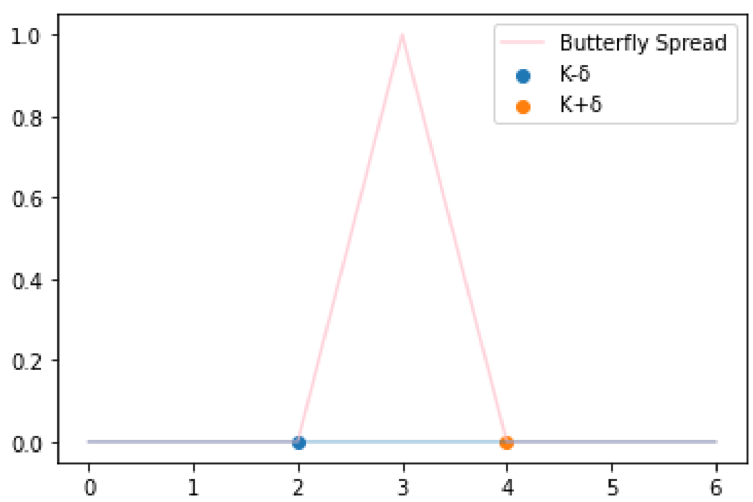

Figure 3: The payoff of a Butterfly spread

Solution: See Figure 3.

(g) *Let f be a C^0 function on an interval $[a,b]$. Show that the function f_N defined on $[a,b]$ for $N \in \mathbb{N}^*$ by:*

$$f_N(x) = \sum_{i=0}^{N} f\left(a + i\delta_N\right) V\left(x; a + i\delta_N, \delta_N\right)$$

for $\delta_N = \frac{b-a}{N}$, converges uniformly to f as $N \to +\infty$.

Hint: What kind of approximation of f is f_N ?

Solution: f_N is a piece-wise constant approximation of f.

Lemma 3.2. *For $\forall x \in H = [a + i\delta_N, a + (i+1)\delta_N]$, $i = 0, \dots, N-1$, we have*

$$V(x; a + i\delta_N, \delta_N) + V(x; a + (i+1)\delta_N, \delta_N) = 1$$

Proof (Lemma): By construction, we have

$$\begin{aligned} LHS &= \left(\frac{v(x; a + (i-1)\delta_N) - 2v(x; a + i\delta_N) + v(x; a + (i+1)\delta_N)}{\delta_N}\right) + \left(\frac{v(x; a + i\delta_N) - 2v(x; a + (i+1)\delta_N) + v(x; a + (i+2)\delta_N)}{\delta_N}\right) \\ x \in H \implies &= \frac{v(x; a + (i-1)\delta_N) - v(x; a + i\delta_N)}{\delta_N} \\ &= \frac{x - (a + (i-1)\delta_N) - (x - (a + i\delta_N))}{\delta_N} \\ &= \frac{\delta_N}{\delta_N} \\ &= 1 \end{aligned}$$

□

Back to the original question.

Let's first fix $\varepsilon > 0$. Since f is continuous in $[a, b]$, it is uniformly continuous, and there is a $\delta > 0$ such that for all $x, y \in [a, b]$ with $|x - y| \leq \delta$ we have $|f(x) - f(y)| \leq \varepsilon$. Since $\delta_N \to 0$, there is a N_ε s.t. for all $N \geq N_\varepsilon$, we have $\delta_N \leq \delta$. This means that

$$|f(a + i\delta_N) - f(x)| \leq \varepsilon, \quad \forall x \in [a + (i-1)\delta_N, a + (i+1)\delta_N], \forall i = 0, \dots, N-1, N \geq N_\varepsilon$$

Notice, by construction, $V(x; a + i\delta_N, \delta_N) \geq 0, \forall i$. Now, suppose $x \in [a + j\delta_N, a + (j+1)\delta_N]$, $j \in \{0, 1, \dots, N-1\}$, then, by construction, $V_x^i = 0$ if $i \leq j - 1$ or $i \geq j + 2$. Therefore, we have

$$\begin{aligned} |f(x) - f_N(x)| &= |f(x) - f(a + j\delta_N, \delta_N) V(x; a + j\delta_N, \delta_N) - f(a + (j+1)\delta_N) V(x; a + (j+1)\delta_N, \delta_N)| \\ Lemma\ 3.2 \implies &= |(f(x) - f(a + j\delta_N)) V(x; a + j\delta_N, \delta_N) + (f(x) - f(a + (j+1)\delta_N)) V(x; a + (j+1)\delta_N, \delta_N)| \\ &\leq |(f(x) - f(a + j\delta_N))| V(x; a + j\delta_N, \delta_N) + |(f(x) - f(a + (j+1)\delta_N))| V(x; a + (j+1)\delta_N, \delta_N) \\ &\leq \varepsilon V(x; a + j\delta_N, \delta_N) + \varepsilon V(x; a + (j+1)\delta_N, \delta_N) \\ &= \varepsilon (V(x; a + j\delta_N, \delta_N) + V(x; a + (j+1)\delta_N, \delta_N)) \\ Lemma\ 3.2 \implies &= \varepsilon \end{aligned}$$

Since this is ture for any $j \in \{0, 1, \dots, N-1\}$, we must have

$$|f(x) - f_N(x)| \leq \varepsilon, \quad \forall N \geq N_\varepsilon, \forall x \in [a, b]$$

Since $\varepsilon > 0$ is arbitrary, we have $f_N \to f$ uniformly in $[a, b]$.

(h) *Explain how you can approximately price an option with an arbitrary payoff using Butterfly spreads (or calls).*

Note that you don't theoretically need the Black-Scholes formula to price this way; 'just' observe the call prices on the market. On a practical side, you may not have all prices available for all strikes, and thus need to rely on an interpolation. This interpolation needs to be carefully implemented, otherwise you might introduce arbitrage opportunities.

Solution: Suppose we have an option with arbitrary payoff $f(S_T)$, then its price is given by

$$Price = \mathbb{E}_\mathbb{Q}\left[e^{-r(T-t)} f(S_T) \mid S_t = s\right]$$

By the approximation provided above, we know that $f(S_T)$ can be approximated by a linear combination of European call option; that is,

$$\begin{aligned} f(S_T) &\approx f_N(S_T) \\ &= \sum_{i=0}^{N} f(a + i\delta_N) \left(\frac{v(S_T; a + (i+1)\delta_N) - 2v(S_T; a + i\delta_N) + v(S_T; a + (i-1)\delta_N)}{\delta_N}\right) \end{aligned}$$

Then, the $Price$ can be represented as a linear combination of European call option prices.

$$\begin{aligned} Price &\approx \mathbb{E}_\mathbb{Q}\left[e^{-r(T-t)} f_N(S_T) \mid S_t = s\right] \\ &= \sum_{i=0}^{N} f(a + i\delta_N) \left(\frac{C(s; a + (i+1)\delta_N) - 2C(s; a + i\delta_N) + C(s; a + (i-1)\delta_N)}{\delta_N}\right) \end{aligned}$$

Since the price for the call option can be directly observed on the market or generated via careful interpolation, we can get a fine approximation for the $Price$ of the target option.

3.4 Density of S_T in the risk neutral measure

Let's examine closely the formula in question (f); we can formally write it as

$$
\begin{aligned}
f(x) &\approx \sum_{i=0}^{N} f\left(a+i\delta_N\right) \frac{v\left(x; a+(i+1)\delta_N\right)-2v\left(x; a+i\delta_N\right)+v\left(x; a+(i-1)\delta_N\right)}{\delta_N^2}\delta_N \\
&= \int_a^b f(K)\frac{\partial^2}{\partial K^2}v(x;K)dK
\end{aligned}
$$

where the last approximation comes from the Riemann sum definition of an integral. Using linearity of the expectation, this would imply that the price P at time t of an option with payoff $f\left(S_T\right)$ is given by

$$
P\left(t, S_t=s\right)=\int f(K)\left[\frac{\partial^2}{\partial K^2}C(t,s;K,T,\sigma,r)\right]dK
$$

This is to be contrasted with the risk neutral formula;

$$
P\left(t, S_t=s\right)=\mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}f\left(S_T\right) \mid S_t=s\right]=\int f(y)\left[e^{-r(T-t)}p(T,y;t,s)\right]dy
$$

where $p(T,y;t,s)$ is the density at time T of S_T given that $S_t=s$. This suggests that

$$
p(T,K;t,s)=e^{r(T-t)}\frac{\partial^2}{\partial K^2}C(t,s;K,T,\sigma,r) \tag{$\star$}
$$

The issue with this reasoning is that $v(x;K)$ is not differentiable at $K=x$

There is however a simple way to show that:

(i) Use the risk neutral formula on a European call:

$$
C(t,s;K,T,\sigma,r)=\mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}\left(S_T-K\right)_+ \mid S_t=s\right]=\int e^{-r(T-t)}(y-K)_+p(T,y;t,s)dy
$$

to show ($\star$).

Hint: Take $\partial/\partial K$ twice.

Solution: Let's take $\frac{\partial^2}{\partial K^2}$ on both sides of the equation above. Then, we get

$$
\begin{aligned}
\frac{\partial^2}{\partial K^2}C(t,s;K,T,\sigma,r) &= \frac{\partial^2}{\partial K^2}\int_K^\infty e^{-r(T-t)}(y-K)_+p(T,y;t,s)dy \\
\text{Leibniz integral rule} \implies &= \frac{\partial}{\partial K}\left(e^{-r(T-t)}(\infty-K)_+p(T,\infty;t,s)\frac{\partial\infty}{\partial K}-e^{-r(T-t)}(K-K)_+p(T,K;t,s)\frac{\partial K}{\partial K}+\int_K^\infty e^{-r(T-t)}\left[\frac{\partial}{\partial K}(y-K)_+\right]p(T,y;t,s)dy\right) \\
&= \frac{\partial}{\partial K}\left(-\int_K^\infty e^{-r(T-t)}p(T,y;t,s)dy\right) \\
\text{Leibniz integral rule} \implies &= -\frac{\partial}{\partial K}\left(e^{-r(T-t)}p(T,\infty;t,s)\frac{\partial\infty}{\partial K}-e^{-r(T-t)}p(T,K;t,s)\frac{\partial K}{\partial K}+\int_K^\infty\frac{\partial}{\partial K}e^{-r(T-t)}p(T,y;t,s)dy\right) \\
&= e^{-r(T-t)}p(T,K;t,s)
\end{aligned}
$$

Therefore, we must have

$$
\frac{\partial^2}{\partial K^2}C(t,s;K,T,\sigma,r)=e^{-r(T-t)}p(T,K;t,s)
$$

Rearrange a bit, we obtain

$$
p(T,K;t,s)=e^{r(T-t)}\frac{\partial^2}{\partial K^2}C(t,s;K,T,\sigma,r)
$$

Remark 3.3. We just showed that regardless of the model used (Black-Scholes or not), we can always use the prices of European calls to deduce the density of the stock price in the risk neutral measure. This allows us to deduce prices that are consistent with these options: any inconsistency can be taken advantage of using a static hedge of Calls. Of course the portfolio needs to be infinite, and hence an exact hedge isn't always feasible. △

4 Topic 4: Local Volatility Model

Problem 4.1. *We saw in class that assuming a Black-Scholes dynamic for the stock price isn't a realistic model for option pricing; in the case of European Calls for example, the volatility σ would have to depend on the strike K. We labeled this volatility 'implied volatility', as it is the one consistent with the observed prices.*

Can we devise a consistent model for the stock price dynamics that would recover the observed prices for any strike?

This is what the local volatility (Dupire 1994) model is about: Let $C(T,K)$ be the observed price at time t of European Calls of strike K and maturity T. Assume that C is a $C^{1,2}\left(\mathbb{R}_{+}\times\mathbb{R}_{+}\right)$ function, that is continuously differentiable w.r.t. T and twice continuously differentiable w.r.t. K.

The goal is to show that there exists a unique function $\sigma_{LV}(t,x)$ such that the stock price defined by the Markovian SDE:

$$dS_t = r_t S_t dt + \sigma_{LV}\left(t, S_t\right) S_t dW_t$$

for deterministic r_t and W_t a Brownian motion under some measure $\mathbb{Q}$, satisfies the risk neutral pricing formula:

$$C(T,K) = \mathbb{E}_{\mathbb{Q}}\left[\frac{D_T}{D_t}\left(S_T-K\right)_{+} \mid S_t = s\right] \tag{4}$$

for $D_t = e^{-\int_0^t r_s ds}$.

The previous exercise shows that having a consistent model for European calls yields a consistent model for arbitrary vanilla options (with payoff of the type $f\left(S_T\right)$), hence the focus is on calls (besides the fact that calls and puts are far more liquid than other options).

(a) *Assume such a σ_{LV} exists. Let's find some necessary conditions; write the forward PDE satisfied by the density $p(T,S;t,s)$ (or $p(T,S)$ in short) of the stock price S_T under such a model.*

Solution: Given the dynamic of S_t above, the forward PDE (Fokker-Planck equation) for the probability density $p(T,S)$ of the random variable S_t is

$$\frac{\partial}{\partial T}p(T,S) = -\frac{\partial}{\partial S}[r_T S p(T,S)] + \frac{\partial^2}{\partial S^2}\left[\frac{1}{2}\sigma_{LV}^2\left(T,S\right)S^2 p(T,S)\right]$$

(b) *Write Equation (5) in terms of p. Take ∂_T (assuming you can exchange integral and derivative) and use the forward equation to replace $\partial_T p$ by its spatial derivatives.*

Solution: We have

$$\begin{aligned}
\frac{\partial}{\partial T}C(T,K) &= \frac{\partial}{\partial T}\mathbb{E}_{\mathbb{Q}}\left[\frac{D_T}{D_t}\left(S_T-K\right)_{+} \mid S_t = s\right] \\
&= \frac{\partial}{\partial T}\int_K^{\infty}\frac{D_T}{D_t}\left(S-K\right)_{+} p(T,S)dS \\
&= \int_K^{\infty}\frac{D_T}{D_t}\left(S-K\right)_{+}\frac{\partial}{\partial T}p(T,S) + \frac{\partial D_T}{\partial T}\frac{1}{D_t}\left(S-K\right)_{+}p(T,S)dS \\
&= \int_K^{\infty}\frac{D_T}{D_t}\left(S-K\right)_{+}\frac{\partial}{\partial T}p(T,S) - r_T\frac{D_T}{D_t}\left(S-K\right)_{+}p(T,S)dS
\end{aligned}$$

Use the forward equation to replace $\partial_T p$ by its spatial derivatives, we get

$$\begin{aligned}
\frac{\partial}{\partial T}C(T,K) &= \int_K^{\infty}\frac{D_T}{D_t}\left(S-K\right)_{+}\left(-\frac{\partial}{\partial S}[r_T S p(T,S)] + \frac{\partial^2}{\partial S^2}\left[\frac{1}{2}\sigma_{LV}^2\left(T,S\right)S^2 p(T,S)\right]\right) - r_T\frac{D_T}{D_t}\left(S-K\right)_{+}p(T,S)dS \\
&= -\int_K^{\infty}\frac{D_T}{D_t}\left(S-K\right)_{+}\frac{\partial}{\partial S}[r_T S p(T,S)]dS + \int_K^{\infty}\frac{D_T}{D_t}\left(S-K\right)_{+}\frac{\partial^2}{\partial S^2}\left[\frac{1}{2}\sigma_{LV}^2\left(T,S\right)S^2 p(T,S)\right]dS - \int_K^{\infty}r_T\frac{D_T}{D_t}\left(S-K\right)_{+}p(T,S)dS
\end{aligned}$$

$$\Longrightarrow \frac{\partial}{\partial T}C(T,K) = -\int_K^{\infty}\frac{D_T}{D_t}\left(S-K\right)_{+}\frac{\partial}{\partial S}[r_T S p(T,S)]dS + \int_K^{\infty}\frac{D_T}{D_t}\left(S-K\right)_{+}\frac{\partial^2}{\partial S^2}\left[\frac{1}{2}\sigma_{LV}^2\left(T,S\right)S^2 p(T,S)\right]dS - \int_K^{\infty}r_T\frac{D_T}{D_t}\left(S-K\right)_{+}p(T,S)dS$$

(c) *Integrate by parts to get rid of any spatial derivative on p. You can assume that the boundary terms vanish [3].*

Solution: Apply IBP, we have

$$\begin{aligned}
\int_K^{\infty}\frac{D_T}{D_t}\left(S-K\right)_{+}\frac{\partial}{\partial S}[r_T S p(T,S)]dS &= \int_K^{\infty}\frac{D_T}{D_t}\left(S-K\right)_{+}d(r_T S p(T,S)) \\
\text{IBP} \Longrightarrow &= \left[\frac{D_T}{D_t}\left(S-K\right)_{+}r_T S p(T,S)\right]_K^{\infty} - \int_K^{\infty}r_T S p(T,S)d\left(\frac{D_T}{D_t}\left(S-K\right)_{+}\right) \\
&= -\int_K^{\infty}r_T\frac{D_T}{D_t}S p(T,S)dS
\end{aligned}$$

[3] let's assume that this SDE yields densities in the Schwartz class, that is densities (or its derivatives) which decay faster than any polynomials

Again, apply IBP, we obtain

$$\begin{aligned}
\int_K^\infty \frac{D_T}{D_t}(S-K)_+ \frac{\partial^2}{\partial S^2}\left[\frac{1}{2}\sigma_{LV}^2(T,S)S^2p(T,S)\right]dS &= \int_K^\infty \frac{D_T}{D_t}(S-K)_+ d\left(\frac{\partial}{\partial S}\left[\frac{1}{2}\sigma_{LV}^2(T,S)S^2p(T,S)\right]\right)\\
\text{IBP} \implies &= \left[\frac{D_T}{D_t}(S-K)_+\frac{\partial}{\partial S}\left[\frac{1}{2}\sigma_{LV}^2(T,S)S^2p(T,S)\right]\right]_K^\infty - \int_K^\infty \frac{\partial}{\partial S}\left[\frac{1}{2}\sigma_{LV}^2(T,S)S^2p(T,S)\right]d\left(\frac{D_T}{D_t}(S-K)_+\right)\\
&= -\int_K^\infty \frac{D_T}{D_t}\frac{\partial}{\partial S}\left[\frac{1}{2}\sigma_{LV}^2(T,S)S^2p(T,S)\right]dS\\
&= -\left[\frac{D_T}{D_t}\left[\frac{1}{2}\sigma_{LV}^2(T,S)S^2p(T,S)\right]\right]_{S=K}^\infty\\
&= -0+\frac{D_T}{D_t}\left[\frac{1}{2}\sigma_{LV}^2(T,K)K^2p(T,K)\right]\\
&= \frac{1}{2}\frac{D_T}{D_t}\sigma_{LV}^2(T,K)K^2p(T,K)
\end{aligned}$$

Therefore, plug these in the equation for $\frac{\partial}{\partial T}C(T,K)$, we have

$$\begin{aligned}
\frac{\partial}{\partial T}C(T,K) &= \int_K^\infty r_T\frac{D_T}{D_t}Sp(T,S)dS + \frac{1}{2}\frac{D_T}{D_t}\sigma_{LV}^2(T,K)K^2p(T,K) - \int_K^\infty r_T\frac{D_T}{D_t}(S-K)_+p(T,S)dS\\
&= \int_K^\infty r_T\frac{D_T}{D_t}Kp(T,S)dS + \frac{1}{2}\frac{D_T}{D_t}\sigma_{LV}^2(T,K)K^2p(T,K)\\
&= r_T\frac{D_T}{D_t}K\int_K^\infty p(T,S)dS + \frac{1}{2}\frac{D_T}{D_t}\sigma_{LV}^2(T,K)K^2p(T,K)
\end{aligned}$$

Hence, we finally obtain

$$\frac{\partial}{\partial T}C(T,K) = r_T\frac{D_T}{D_t}K\int_K^\infty p(T,S)dS + \frac{1}{2}\frac{D_T}{D_t}\sigma_{LV}^2(T,K)K^2p(T,K)$$

(d) Similarly, compute $\partial_K C$ and $\partial_{KK}C$ from Equation (5), and use them to replace all terms involving p in your answer to (c).

Solution: We first compute $\partial_K C$:

$$\begin{aligned}
\frac{\partial}{\partial K}C(T,K) &= \frac{\partial}{\partial K}\mathbb{E}_{\mathbb{Q}}\left[\frac{D_T}{D_t}(S_T-K)_+ \mid S_t=s\right]\\
&= \frac{\partial}{\partial K}\int_K^\infty \frac{D_T}{D_t}(S-K)_+p(T,S)dS\\
\text{Leibniz integral rule} \implies &= \frac{D_T}{D_t}(\infty-K)_+p(T,\infty)\frac{\partial\infty}{\partial K} - \frac{D_T}{D_t}(K-K)_+p(T,K)\frac{\partial K}{\partial K} + \int_K^\infty \frac{D_T}{D_t}\left[\frac{\partial}{\partial K}(S-K)_+\right]p(T,S)dS\\
S\geq K \implies &= 0-0+\int_K^\infty -\frac{D_T}{D_t}p(T,S)dS\\
&= -\frac{D_T}{D_t}\int_K^\infty p(T,S)dS
\end{aligned}$$

Therefore, we have

$$\frac{\partial}{\partial K}C(T,K) = -\frac{D_T}{D_t}\int_K^\infty p(T,S)dS \tag{5}$$

Then, we compute $\partial_{KK}C$:

$$\begin{aligned}
\frac{\partial^2}{\partial K^2}C(T,K) &= \frac{\partial^2}{\partial K^2}\mathbb{E}_{\mathbb{Q}}\left[\frac{D_T}{D_t}(S_T-K)_+ \mid S_t=s\right]\\
&= \frac{\partial}{\partial K}\left(\frac{\partial}{\partial K}\int_K^\infty \frac{D_T}{D_t}(S-K)_+p(T,S)dS\right)\\
\text{Equation (5)} \implies &= \frac{\partial}{\partial K}\left(-\frac{D_T}{D_t}\int_K^\infty p(T,S)dS\right)\\
&= -\frac{D_T}{D_t}\frac{\partial}{\partial K}\int_K^\infty p(T,S)dS\\
\text{Leibniz integral rule} \implies &= -\frac{D_T}{D_t}\left(p(T,\infty)\frac{\partial\infty}{\partial K} - p(T,K)\frac{\partial K}{\partial K} + \int_K^\infty \frac{\partial}{\partial K}p(T,S)dS\right)\\
&= \frac{D_T}{D_t}p(T,K)
\end{aligned}$$

Therefore, we can conclude that

$$\begin{cases} \frac{\partial}{\partial K}C(T,K) &= -\frac{D_T}{D_t}\int_K^\infty p(T,S)dS\\ \frac{\partial^2}{\partial K^2}C(T,K) &= \frac{D_T}{D_t}p(T,K)\end{cases}$$

(e) Deduce that one necessarily has:

$$\sigma_{LV}^2(T,K) = \frac{\partial_T C(T,K) + r_T K\partial_K C(T,K)}{\frac{1}{2}K^2\partial_{KK}C(T,K)}$$

Solution: From (c), we have

$$\frac{\partial}{\partial T}C(T,K) = r_T\frac{D_T}{D_t}K\int_K^\infty p(T,S)dS + \frac{1}{2}\frac{D_T}{D_t}\sigma_{LV}^2\left(T,K\right)K^2p(T,K)$$

and from (d), we obtain

$$\begin{cases} \frac{\partial}{\partial K}C(T,K) & = -\frac{D_T}{D_t}\int_K^\infty p(T,S)dS \\ \frac{\partial^2}{\partial K^2}C(T,K) & = \frac{D_T}{D_t}p(T,K) \end{cases}$$

Plug them all in this formula, we get

$$\begin{aligned} \frac{\partial_T C(T,K) + r_T K\partial_K C(T,K)}{\frac{1}{2}K^2\partial_{KK}C(T,K)} =& \frac{r_T\frac{D_T}{D_t}K\int_K^\infty p(T,S)dS + \frac{1}{2}\frac{D_T}{D_t}\sigma_{LV}^2\left(T,K\right)K^2p(T,K) - r_TK\frac{D_T}{D_t}\int_K^\infty p(T,S)dS}{\frac{1}{2}K^2\frac{D_T}{D_t}p(T,K)} \\ =& \frac{\frac{1}{2}\frac{D_T}{D_t}\sigma_{LV}^2\left(T,K\right)K^2p(T,K)}{\frac{1}{2}K^2\frac{D_T}{D_t}p(T,K)} \\ =& \sigma_{LV}^2\left(T,K\right) \end{aligned}$$

Hence, we necessarily have:

$$\frac{\partial_T C(T,K) + r_T K\partial_K C(T,K)}{\frac{1}{2}K^2\partial_{KK}C(T,K)} = \sigma_{LV}^2(T,K)$$

Remark 4.2.

(a) One can show that the above formula gives a sufficient condition on the evolution on the stock dynamics to be consistent with Call options' prices.

(b) Note that σ_{LV} implicitly depends on t and S_t(from $C(T,K)$).

△